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1) INTRODUCTION

The logistic map is a polynomial mapping (equivalently, recurrence relation) of degree 2, often cited as an archetypal example of how complex, chaotic behavior can arise from very simple non-linear dynamical equations. The map was popularized in a 1976 paper by the biologist Robert May in part as a discrete-time demographic model analogous to the logistic equation written down by Pierre François Verhulst. Mathematically, the logistic map is written:

$$x_{n+1} = rx_n(1 - x_n)$$

where x_n is a number between zero and one that represents the ratio of existing population to the maximum possible population. The values of interest for the parameter r (sometimes also denoted μ) are those in the interval [0,4], so that x_n remains bounded on [0,1]. This nonlinear difference equation is intended to capture two effects:

- reproduction where the population will increase at a rate proportional to the current population when the population size is small.
- starvation (density-dependent mortality) where the growth rate will decrease at a rate proportional to the value obtained by taking the theoretical "carrying capacity" of the environment less the current population.

However, as a demographic model the logistic map has the pathological problem that some initial conditions and parameter values (for example, if r > 4) lead to negative population sizes. This problem does not appear in the older Ricker model, which also exhibits chaotic dynamics.

The r = 4 case of the logistic map is a nonlinear transformation of both the bit-shift map and the μ = 2 case of the tent map.

Characteristics of the map

<u>Behavior dependent on r:</u>

The images below represents the amplitude and frequency content of some logistic map iterates for parameter values of 2.195 and 3.91.

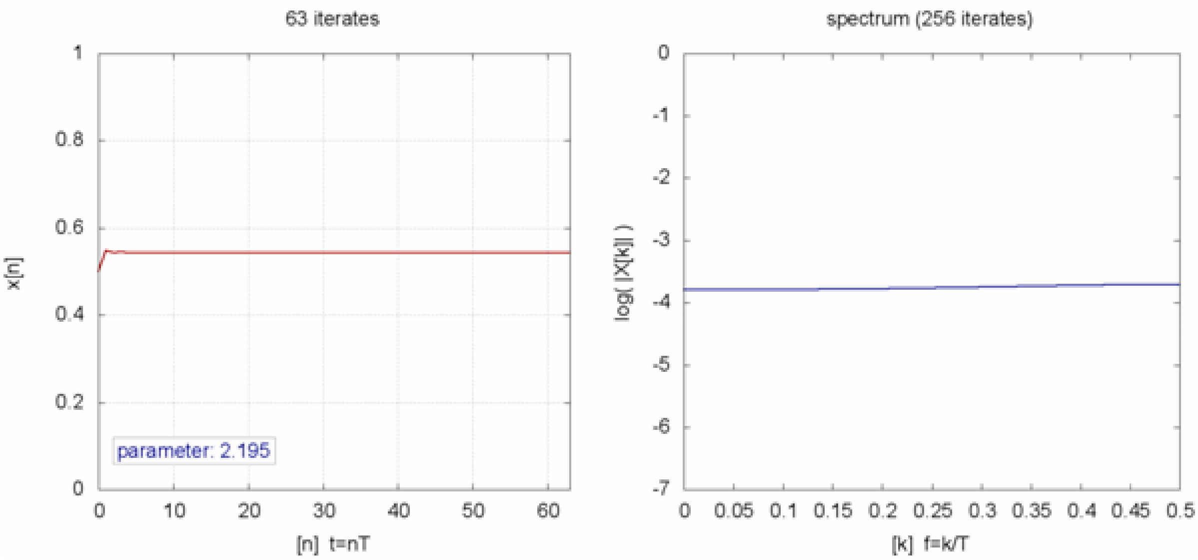

Figure.1: The representation of the amplitude and frequency content of some logistic map for parameter 2.195

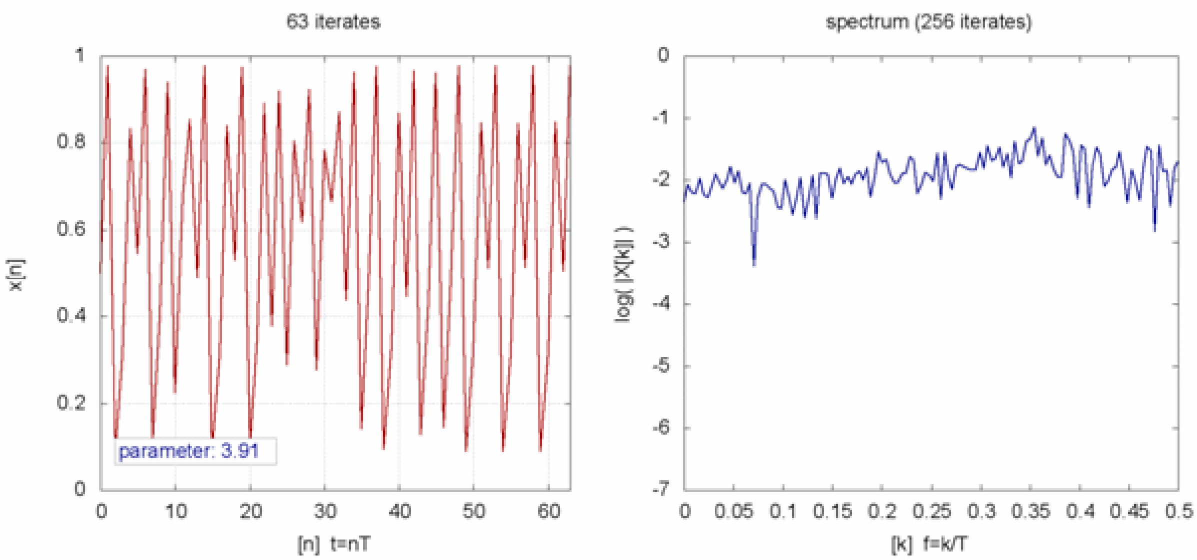

Figure.2: The representation of the amplitude and frequency content of some logistic map for parameter3.91

As a result, as the parameter size increases, the amplitude changes become more glaring, causing the graph to be fuzzier looking.

By varying the parameter r, the following behavior is observed:

- With r between 0 and 1, the population will eventually die, independent of the initial population.
- With r between 1 and 2, the population will quickly approach the value $\frac{r-1}{r}$, independent of the initial population.
- With r between 2 and 3, the population will also eventually approach the same value $\frac{r-1}{r}$, but first will fluctuate around that value for some time. The rate of convergence is linear, except for r = 3, when it is dramatically slow, less than linear.
- With r between 3 and 1 + √6 ≈ 3.44949, from almost all initial conditions the population will approach permanent oscillations between two values. These two values are dependent on r.
- With r approximately between 3.44949 and 3.54409, from almost all initial conditions the population will approach permanent oscillations among four values. The latter number is a root of a 12th degree polynomial.
- With r increasing beyond 3.54409, from almost all initial conditions the population will approach oscillations among 8 values, then 16, 32, etc. The lengths of the parameter intervals that yield oscillations of a given length decrease rapidly; the ratio between the lengths of two successive bifurcation intervals approaches the Feigenbaum constant δ ≈ 4.66920. This behavior is an example of a period-doubling cascade.
- At r ≈ 3.56995 is the onset of chaos, at the end of the period-doubling cascade. From almost all initial conditions, we no longer see oscillations of finite period. Slight variations in the initial population yield dramatically different results over time, a prime characteristic of chaos.
- Most values of r beyond 3.56995 exhibit chaotic behavior, but there are still certain isolated ranges of r that show non-chaotic behavior; these are sometimes called islands of stability. For instance, beginning at 1 + √8 there is a range of parameters r that show oscillation among three values, and for slightly higher values of r oscillation among 6 values, then 12 etc.
- The development of the chaotic behavior of the logistic sequence as the parameter r varies from approximately 3.56995 to approximately 3.82843 is sometimes called the Pomeau–Manneville scenario, characterized by a periodic phase interrupted by bursts of aperiodic behavior. Such a scenario has an application in semiconductor devices. There are other ranges that yield oscillation among 5 values etc.; all oscillation periods occur for some values of r. A period-doubling window with parameter c is a range of r-values consisting of a succession of subranges. The *kth* subrange contains the values of r for which there is a stable cycle, a cycle that attracts a set of initial points of unit measure of period $2^k c$. As r rises there is a succession of new windows with different c values. The first one is for c = 1; all subsequent windows involving odd c occur in decreasing order of c starting with arbitrarily large c.
- Beyond r = 4, almost all initial values eventually leave the interval [0,1] and diverge.
- For r between -2 and -1 the logistic sequence also features chaotic behavior.

For any value of r there is at most one stable cycle. If a stable cycle exists, it is globally stable, attracting almost all points. Some values of r with a stable cycle of some period have infinitely many unstable cycles of various periods.

In mathematics, particularly in dynamical systems, a bifurcation diagram shows the values visited or approached asymptotically (fixed points, periodic orbits, or chaotic attractors) of a system as a function of a bifurcation parameter in the system. It is usual to represent stable values with a solid line and unstable values with a dotted line, although often the unstable points are omitted.

The bifurcation diagram below summarizes this. The horizontal axis shows the possible values of the parameter r while the vertical axis shows the set of values of x visited asymptotically from almost all initial conditions by the iterates of the logistic equation with that r value.

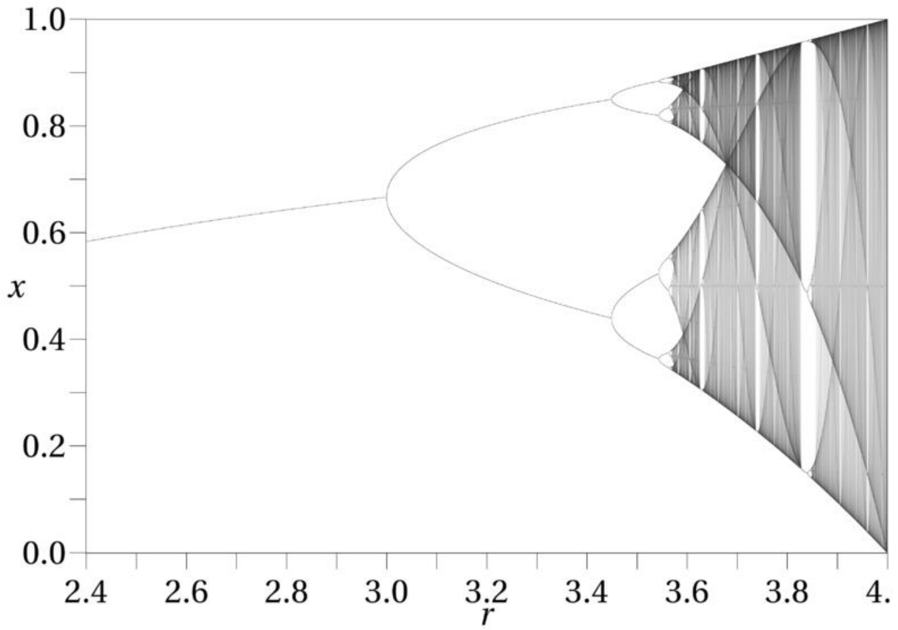

Figure.3: Bifurcation diagram of the logistic map. The attractor for any value of the parameter r is shown on the vertical line at that r.

The bifurcation diagram is a self-similar: if one zooms in on the above-mentioned value $r \approx 3.82843$ and focus on one arm of the three, the situation nearby looks like a shrunk and slightly distorted version of the whole diagram. The same is true for all other non-chaotic points. This is an example of the deep and ubiquitous connection between chaos and fractals.

According to the bifurcation diagram above, by varying the parameter r, the following behavior is observed:

- With r between 0 and 1, the population will eventually die, independent of the initial population.
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- At $r \approx 3.56995$ is the onset of chaos, at the end of the period-doubling cascade. From almost all initial conditions, we no longer see oscillations of finite period. Slight variations in the initial population yield dramatically different results over time, a prime characteristic of chaos.
- Most values of r beyond 3.56995 exhibit chaotic behavior, but there are still certain isolated ranges of r that show non-chaotic behavior; these are sometimes called islands of stability. For instance, beginning at $1 + \sqrt{8}$ there is a range of parameters r that show oscillation among three values, and for slightly higher values of r oscillation among 6 values, then 12 etc.
- The development of the chaotic behavior of the logistic sequence as the parameter r varies from approximately 3.56995 to approximately 3.82843 is sometimes called the Pomeau–Manneville scenario, characterized by a periodic phase interrupted by bursts of aperiodic behavior. Such a scenario has an application in semiconductor devices. There are other ranges that yield oscillation among 5 values etc.; all oscillation periods occur for some values of r. A period-doubling window with parameter c is a range of r-values consisting of a succession of subranges. The *kth* subrange contains the values of r for which there is a stable cycle, a cycle that attracts a set of initial points of unit measure of period $2^k c$. As r rises there is a succession of new windows with different c values. The first one is for c = 1; all subsequent windows involving odd c occur in decreasing order of c starting with arbitrarily large c.
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For any value of r there is at most one stable cycle. If a stable cycle exists, it is globally stable, attracting almost all points. Some values of r with a stable cycle of some period have infinitely many unstable cycles of various periods.

2) METHODS-DISCUSSION

The logistic map is a one-dimensional discrete-time map that, despite its formal simplicity, exhibits an unexpected degree of complexity. Historically it has been one of the most important and paradigmatic systems during the early days of research on deterministic chaos. It is an example of a 1-D map.

By 1-D map, the emphasized thing is as follows:

$$x_{n+1} = f(x)$$

Therefore a $1 - D$ flow would be: $\frac{dx}{dt} = f(x)$

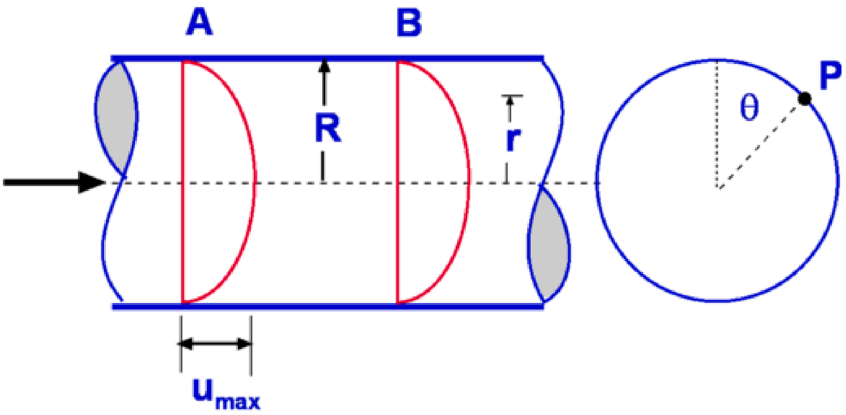

Figure.4: A schematic of 1-D (one dimensional) flow

Radioactive decay can be given as an example of 1-D flow.

$\frac{dN}{dt} = -\lambda N, \textit{where}\ \lambda$ is the decay constant, N is the total number of particles in the sample.

Referring the first equation given above ($x_{n+1} = f(x)$), $f(x)$ is $x_{n+1} = rx_n(1 - x_n)$

In the equation above, rx_n is a Malthus part. Malthus is someone who is concerned about the population growth. He declares that the population growth grows exponentially and in time, it will be hard to deal with this huge population. It is shown in formula as follows:

$$\frac{dN}{dt} = rN, \text{where } N = N_0 e^{rt}$$

$x_{n+1} = rx_n(1 - x_n)$

To give an example; if $r = 2.5$ and $x_0 = 0.1$, then $x_1 = 2.5 * 0.1 * (1 - 0.1) = 0.225$.

With a further improvement to Malthus equation, it becomes:

$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$, where K is the carrying capacity of the system.

When $N = K$, the system does not change, it stays there

With an arrangement in the equation:

$$\frac{dN}{dt} = K * r\frac{N}{K}\left(1 - \frac{N}{K}\right), \textit{if Kr is } \tilde{r}, \textit{ then the equation becomes } \frac{dN}{dt} = \tilde{r}\,\frac{N}{K}\left(1 - \frac{N}{K}\right)$$

This means that the population cannot go beyond 1 because of the term $\left(1 - \frac{N}{K}\right)$ in it.

Looking at the formula $x_{n+1} = rx_n(1 - x_n)$, it can be seen that the next x value is dependent on the current x value and the r value. Therefore, all x values are dependent on the initial x value. Also, if r is very close to 0 ($r \approx 0$), then all x values will be set to very close to 0, no matter what they actually are.

There are 2 variables in the equation which are x and r. One can plot the variables of them with x changing from 0 to 1 and r changing from 0 to 4.

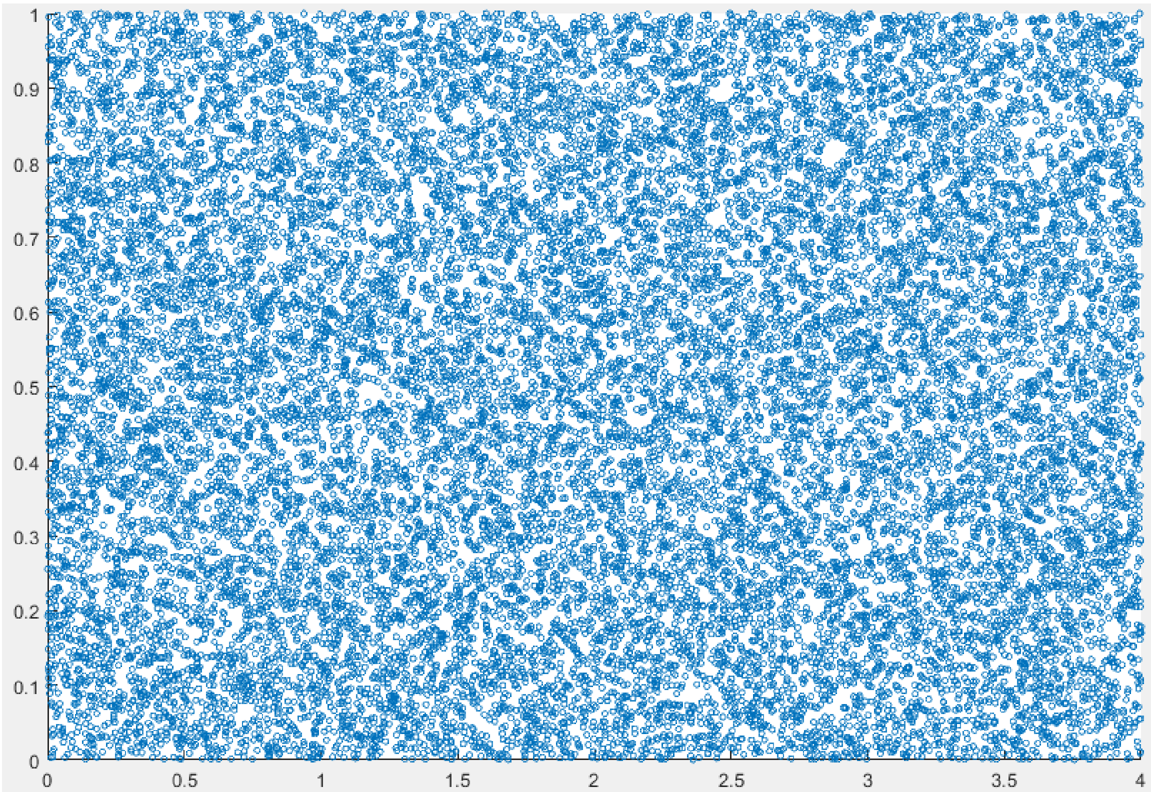

Figure.5: Plot of the ranges of r and x variables

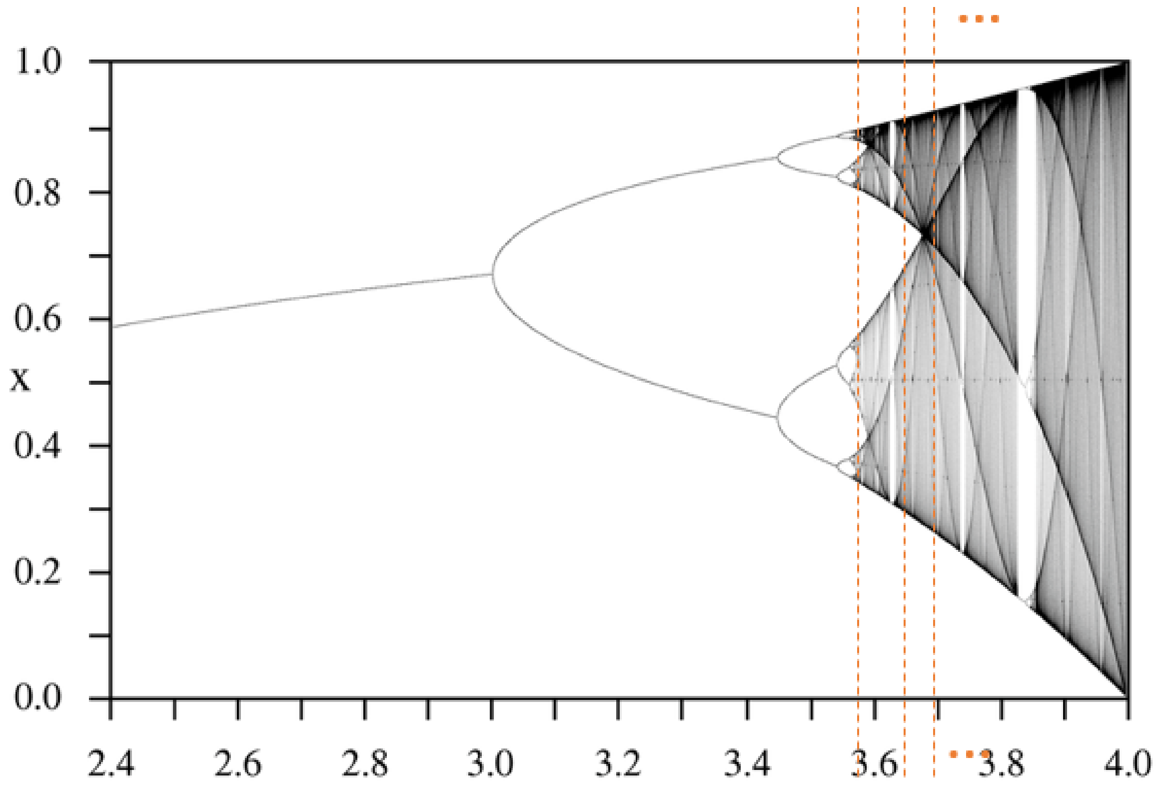

Figure.6: Bifurcation diagram with some of its chaos points

The diagram looks like a fracture, and it has repeating parts along the plot side. The plot looks like it is branching out. Chaos occurs at every branching and the system becomes a fracture that shrinks because of these points. The orange dashed lines in the graph represent chaos.

If the chaos points are called as r_c, the first r_c corresponds to a value very close to 3.57

To find what r values corresponds in a horizontal line, one can use the formula

$-\ln(\mathrm{r_c} - \mathrm{r})$, where $\mathrm{r} = 3.57 - \mathrm{e}^{-20}$ for the specific case of $\mathrm{r_c} = 3.57$

Then, $-\ln(\mathrm{r_c} - \mathrm{r})$ becomes $-\ln(3.57 - 3.57 - \mathrm{e}^{-20}) = 20$

This result can be shown as the following:

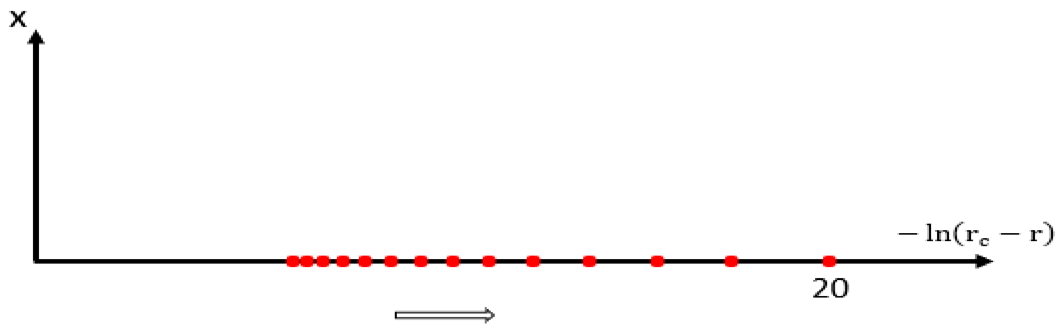

Figure.7: The corresponds of the bifurcation diagram chaos points in the logarithmic array. Only the horizontal axis values are shown, not the whole plot.

As one goes to the right in the bifurcation diagram, s/he will see that the chaos values will change slowly. Therefore, as s/he goes to the right in the logarithmic array of this bifurcation diagram, s/he will see that the values on that plot will start to change faster because of the minus (-) sign in the $-\ln(r_c - r)$.

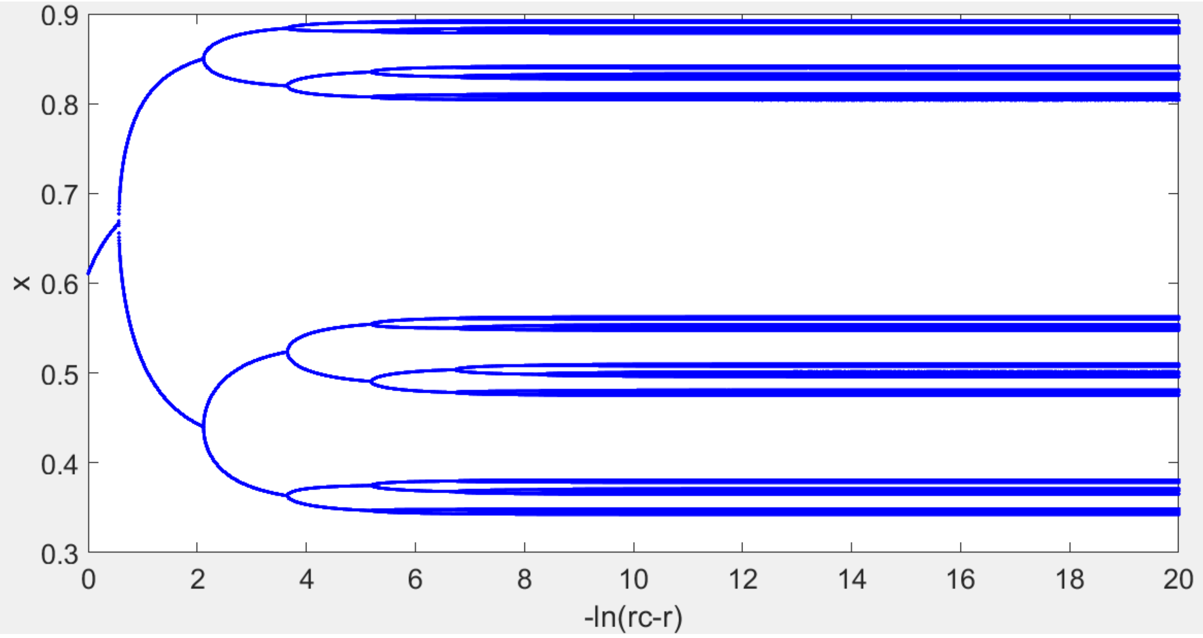

Figure.8: The corresponds of the bifurcation diagram chaos points in the logarithmic array. The whole plot is shown now.

The figure.8 is actually totally a fracture, which is that it is repeating itself with very small difference close to 0.

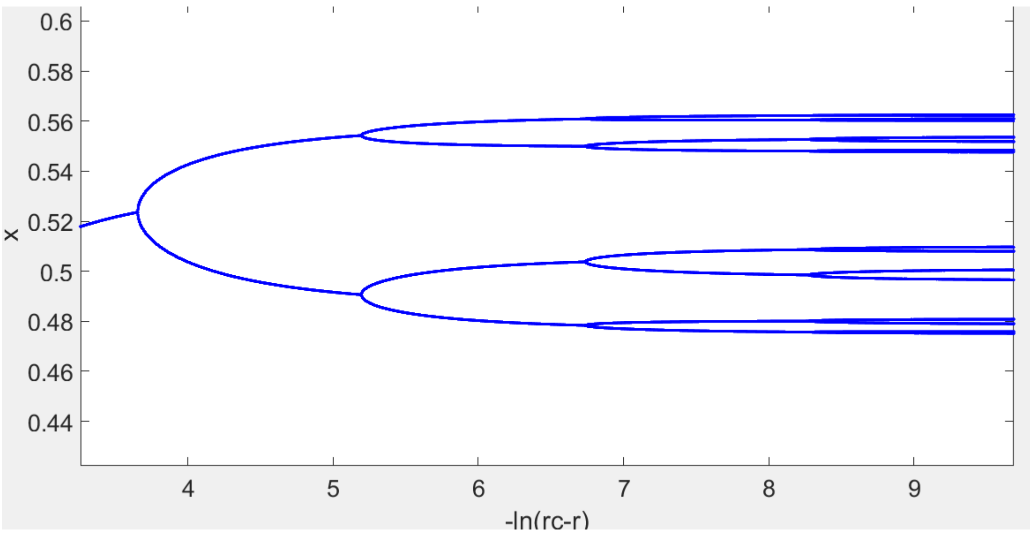

Figure.9: The once zoomed version of the figure.8

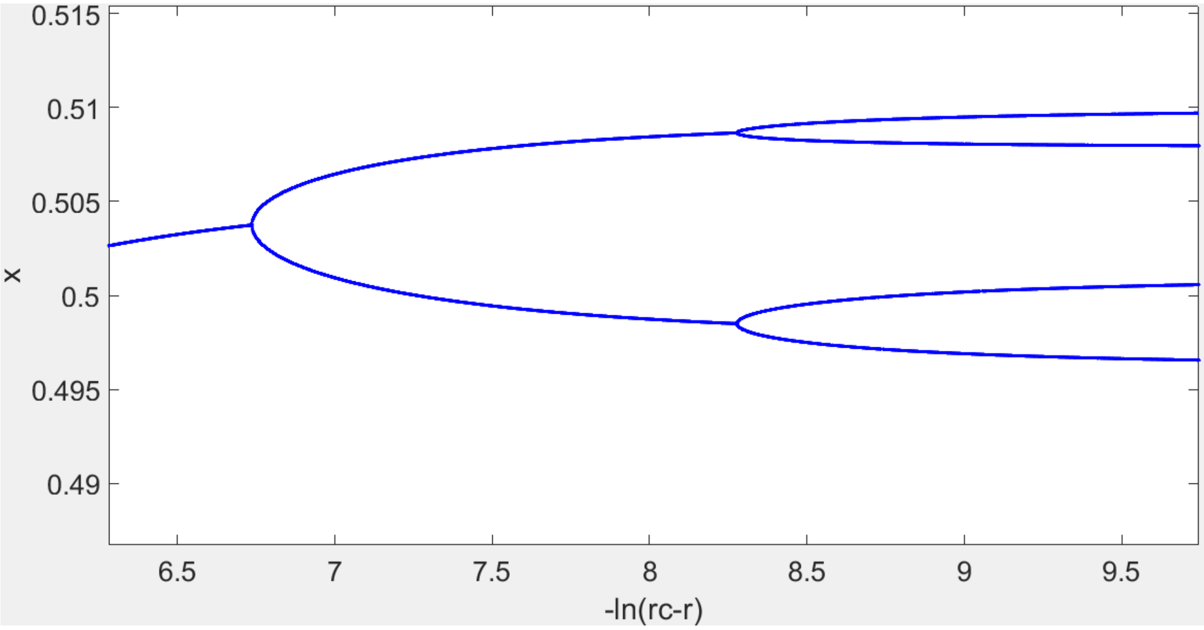

Figure.10: The twice zoomed version of the figure.8

As it can be seen from the figure.9 and figure.10, the portions of the logarithmic array plot are very similar to each other.

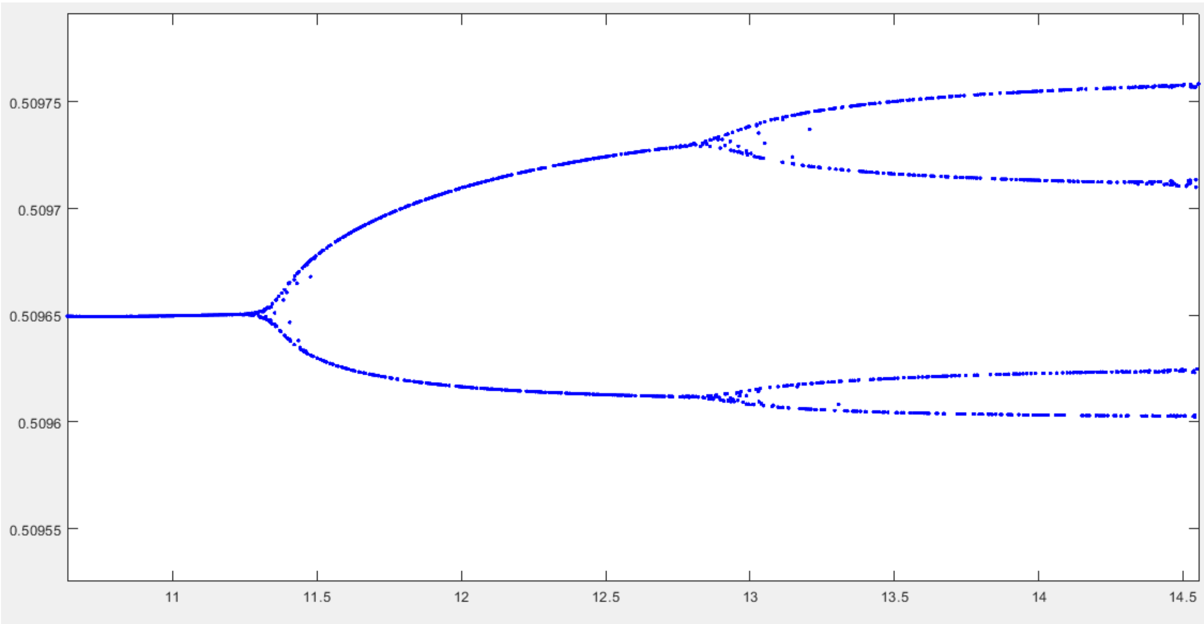

Figure.11: The six times zoomed version of the figure.8

After the sixth zoom, the appearance of the plot gets blurry.

After infinite number of zooms, the value gets closer to a constant which is approximately 4.669, called Feigenbaum constant. This situation can be shown as follows:

$$n \rightarrow \infty, \quad \frac{dn}{dt} \approx 4.669$$

Why this bifurcation occurs?

To answer this question, the so-called Cobweb concept must be investigated.

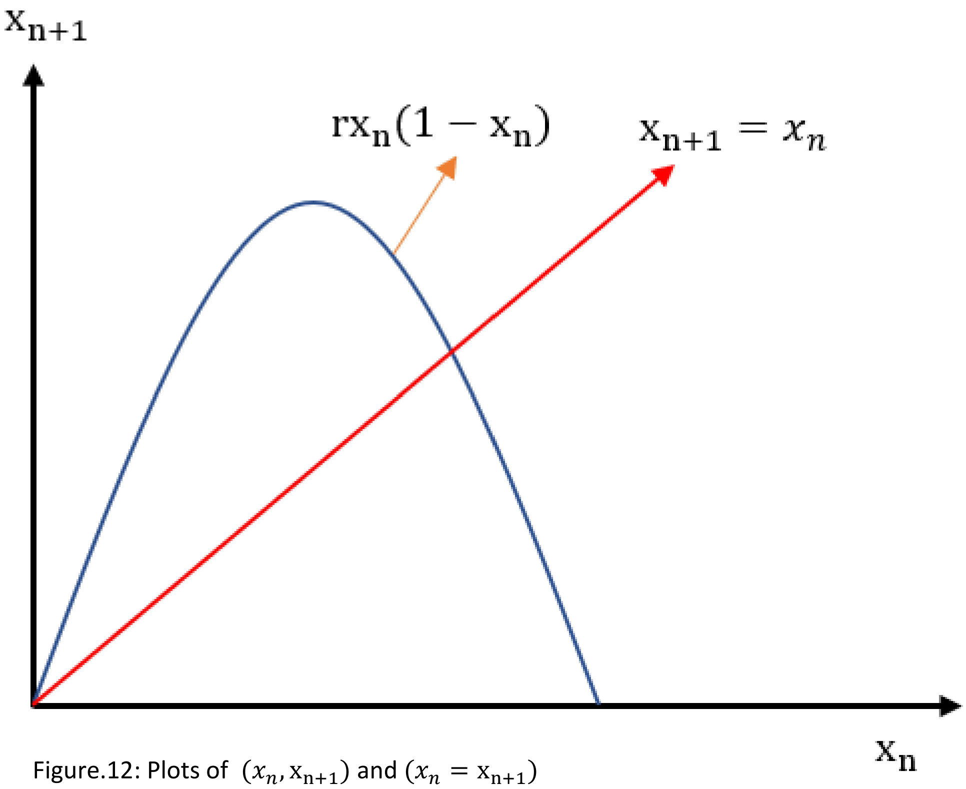

Figure.12: Plots of (x_n, x_{n+1}) and $(x_n = x_{n+1})$

By looking at the blue plot above, x_{n+1} can be found by using x_n. Then, this x_{n+1} becomes the new x_n according to the red plot. The process is repeated at each step. At one step, the blue plot and the red plot will intersect or at least will get very close to each other. This intersection point is called the fix point and it is changing with the r values because by changing r, the initial conditions are changing too. With increasing r value, after a specific value, oscillation is seen on the diagram like in the figure.12, the oscillation occurs around the intersection of the red and the blue plots. At some point around r = 3, firs bifurcation occurs, around 3.4 the second bifurcation occurs.

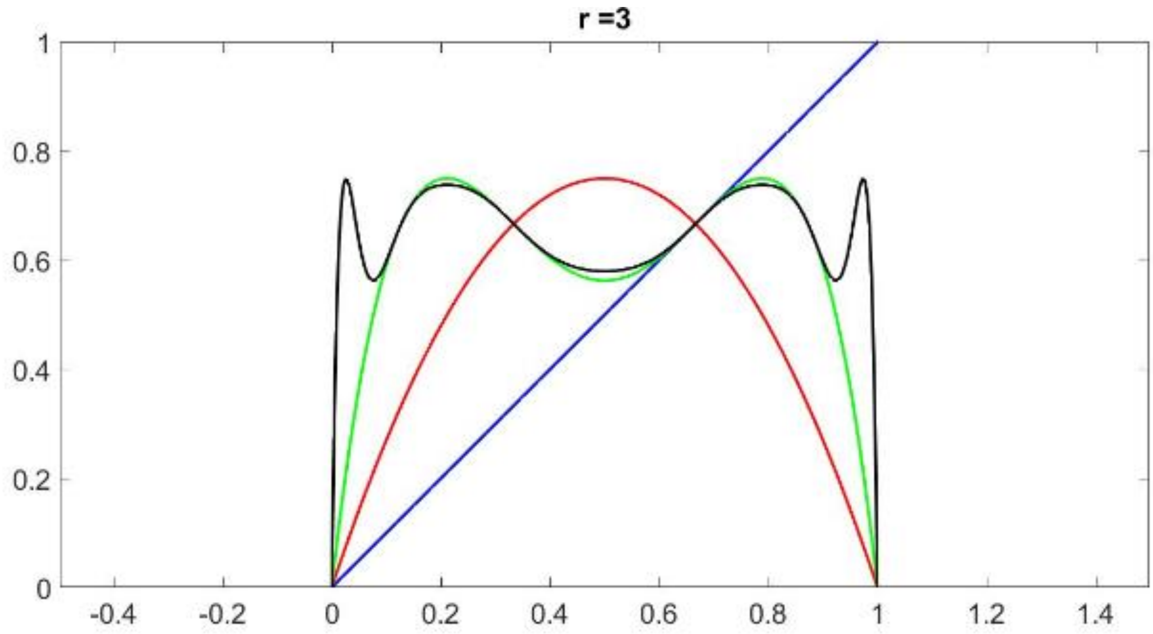

Figure.13: Plots of original mapping(red), double mapping(green) and the triple mapping(black)

The red plot above is the original mapping, the green plot is the double mapping. In the double mapping, the function $\mathrm{f(x) = rx_n(1 - x_n)}$ is used again which is $\mathrm{f(f(x)) = f(rx_n(1 - x_n))}$. The black plot, on the other hand, is representing triple mapping which is $\mathrm{f(f(f(x)))}$.

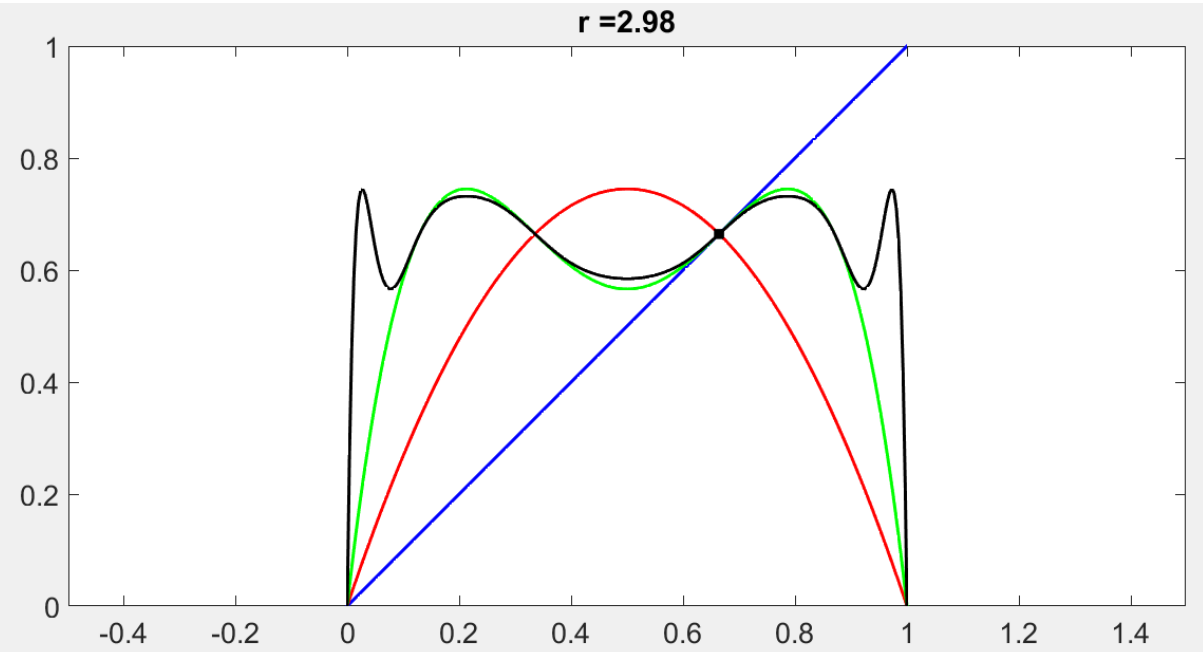

Figure.14: Schematic of the first bifurcation occurring

The second bifurcation occurs when the green plot function $\mathrm{f\big(f(x)\big)}$ becomes tangent to the blue plot, $\mathrm{x_{n+1}} = x_n$.

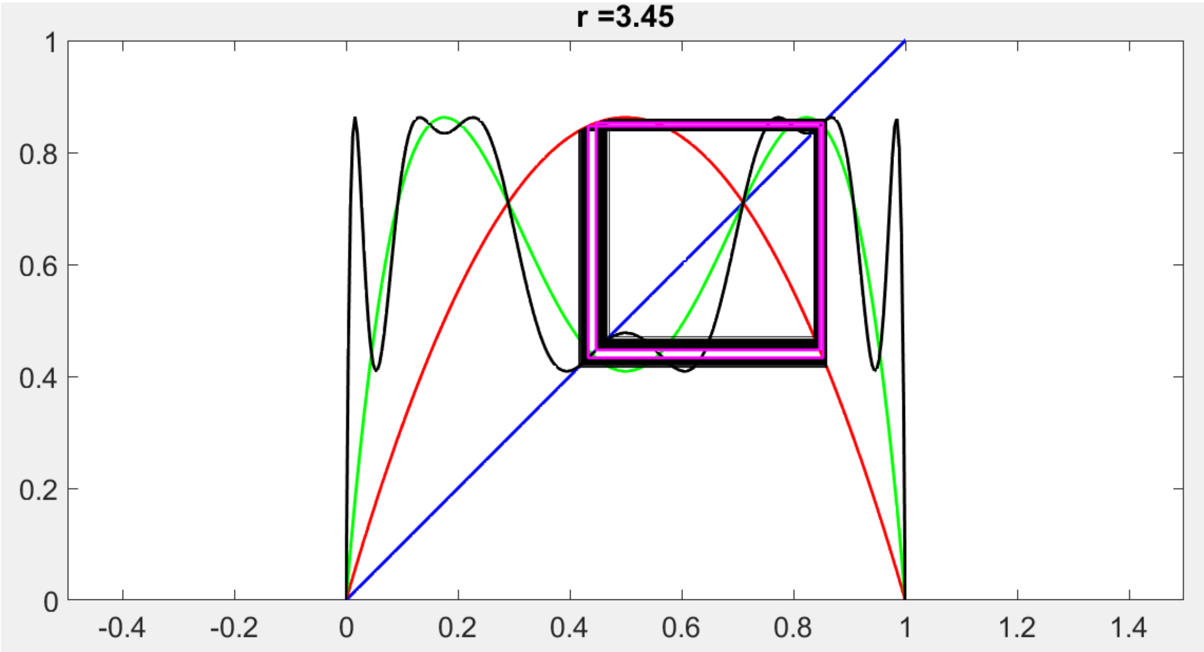

Figure.15: Schematic of the second bifurcation occurring

The second bifurcation occurs when the black plot function $f\left(f(f(x))\right)$ becomes tangent to the blue plot, $x_{n+1} = x_n$.

3) SOLUTION-RESULTS

$x_{n+1} = f_{\mu}(x_n)$

$f_{\mu}(x) = \mu \sin(\pi x)$ for $x \in [0,1], \mu > 0$

The critical value mentioned before was taken:

$$\mu_c = 3.5699456$$

To put the inside of the logarithmic function, finding derivative of $f_{\mu}(x)$ was necessary.

$$\frac{d f_{\mu}(x)}{dx} = \mu\pi(\cos(\pi x))$$

Number of 1000 iteration is used to get closer to the Feigenbaum constant. The error is less than 0.1%. If I tried to solve the system with a larger iteration number, I would be able to get the error almost 0.

I used 3 consecutive exponential numbers, which are -0.03815, -0.02869 and -0.03811.

$$\frac{e^{-0.03815} - e^{-0.02869}}{e^{-0.04023} - e^{-0.03809}} = 4.664402..$$

Feigenbaum constant = 4.669201..

$$\text{The error} = \frac{4.669201 - 4.664402}{4.669201} = 0.10278\ \%$$

4) CONCLUSION

When the system is iterated more, the result gets closer to Feigenbaum constant (4.669…). However, this increased iteration number means the less efficiency for the code to work. In theory, with the infinite iteration of the system, one can get the same as the Feigenbaum constant. It is not possible to solve such a code for sure.

5) CODE REVIEW

```
clear
N = 10000;
M = 100;
x = rand(N,M);
mu = repmat(linspace(0,1,N)',1,M);
```

Describing the arrays and initial values

```
for i = 1:1000
    plot(mu(:,1),x,'r');
    for t = 1 : 1000
        x = mu.*sin(pi.*x);
    end
    set(gca,'FontSize',15);
    xlabel('\mu');
    ylabel('x');
    drawnow;
end
```

Solving the bifurcation equation and plotting it

```
clear
N = 1001;
M = 1001;
x = rand(N,M);
muc = 3.5699456;
dmu = exp(linspace(-0.075,0,N)');
mu = repmat(muc-dmu,1,M);
L = zeros(N,M);
```

Changing the plot to log array to accumulate data using critical mu value

```
nL = 1;

for t = 1 : 1000
    x = mu.*sin(pi.*x);
    nL = nL + 1;
    L = L + log(abs(pi.*mu.*cos(pi.*x)));
end
figure(2)
plot(-log(dmu),sum(L,2)/M/nL);
```

Plotting the result

```
clear
N = 1001;
x = rand(N,1);
muc = 3.5699456;
dr = exp(linspace(-0.0398,-0.0396,N)');
mu = muc-dmu;
L = zeros(N,1);
T = 10^3;
for t = 1 : T
    x = mu.*sin(pi.*x);
    L = L + log(abs(pi.*mu.*cos(pi.*x)));
end
```

Iterating the system to get more accurate results (1000 times) and the peak desired

```
figure(3)
plot(-log(dmu),L/T,'b.-');
result = (exp(-0.03815)-exp(-0.02869))/(exp(-0.04023)-exp(-0.038091));
result
```

Plotting final result and getting the our Feigenbaum constant