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1) INTRODUCTION

An orbit is called closed if a point of the orbit evolves to itself. This means that the orbit will repeat itself. Such orbits are also called periodic. The simplest closed orbit is a fixed point, where the orbit is a single point.

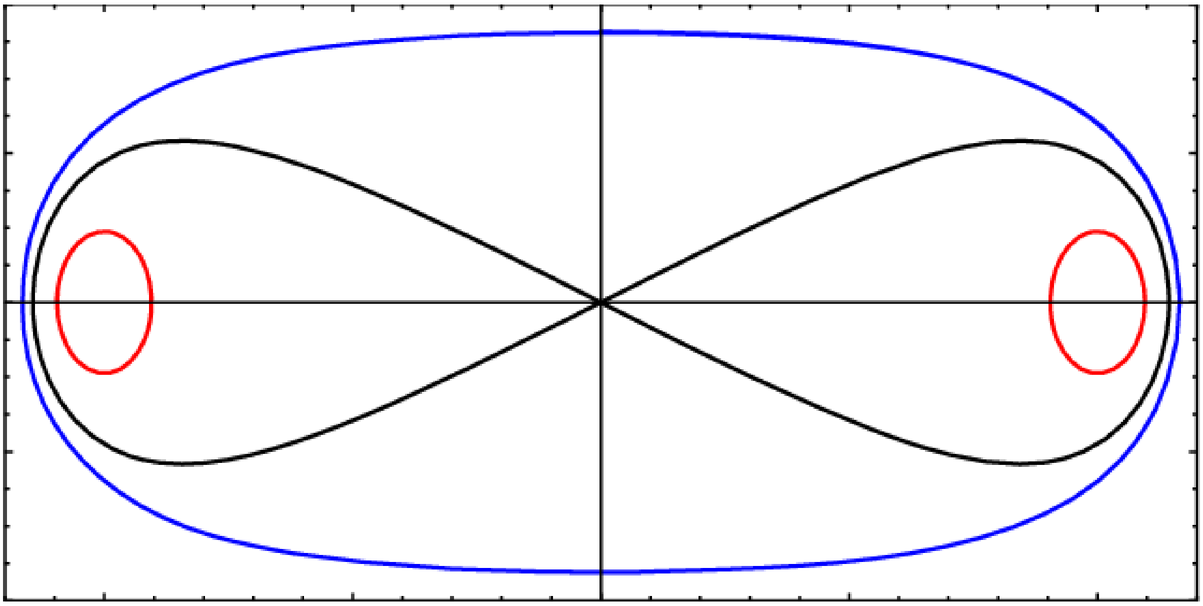


Figure.1: An example of closed orbits

It has been a matter of wonder if closed orbits exist in particular systems and if so, in which extend. There are different theorems that are useful for finding methods to establish that closed orbits exist in particular systems. Poincare-Bendixson method is one of the few results in this direction. It is also one of the key theoretical results in nonlinear dynamics, because it implies that chaos cannot occur in the phase plane, as will be discussed later on.

In mathematics and science, a nonlinear system is a system in which the change of the output is not proportional to the change of the input. Nonlinear problems are of interest to engineers, biologists, physicists, mathematicians, and many other scientists because most systems are inherently nonlinear in nature.[8] Nonlinear dynamical systems, describing changes in variables over time, may appear chaotic, unpredictable, or counterintuitive, contrasting with much simpler linear systems.

Typically, the behavior of a nonlinear system is described in mathematics by a nonlinear system of equations, which is a set of simultaneous equations in which the unknowns (or the unknown functions in the case of differential equations) appear as variables of a polynomial of degree higher than one or in the argument of a function which is not a polynomial of degree one. In other words, in a nonlinear system of equations, the equation(s) to be solved cannot be written as a linear combination of the unknown variables or functions that appear in them. Systems can be defined as nonlinear, regardless of whether known linear functions appear in the equations. In particular, a differential equation is linear if it is linear in terms of the unknown function and its derivatives, even if nonlinear in terms of the other variables appearing in it.

As nonlinear dynamical equations are difficult to solve, nonlinear systems are commonly approximated by linear equations (linearization). This works well up to some accuracy and some range for the input values, but some interesting phenomena such as solitons, chaos, and singularities are hidden by linearization. It follows that some aspects of the dynamic behavior of a nonlinear system can appear to be counterintuitive, unpredictable or even chaotic. Although such chaotic behavior may resemble random behavior, it is in fact not random. For example, some aspects of the weather are seen to be chaotic, where simple changes in one part of the system produce complex effects throughout. This nonlinearity is one of the reasons why accurate long-term forecasts are impossible with current technology.

2) METHODS-DISCUSSION

Fixed Points

In mathematics, a fixed point of a function is an element of the function's domain that is mapped to itself by the function. That is to say, c is a fixed point of the function f if f(c) = c. This means $f(f(\ldots f(c)\ldots)) = f^n(c) = c$, an important terminating consideration when recursively computing f. A set of fixed points is sometimes called a fixed set.

For example, if f is defined on the real numbers by

$$f(x) = x^2 - 3x + 4$$

then 2 is a fixed point of f because f(2)=2

Not all functions have fixed points: for example, f(x) = x + 1, has no fixed points, since x is never equal to x + 1 for any real number. In graphical terms, a fixed point x means the point (x, f(x)) is on the line y = x, or in other words the graph of f has a point in common with that line.

Points that come back to the same value after a finite number of iterations of the function are called periodic points. A fixed point is a periodic point with period equal to one. In projective geometry, a fixed point of a projectivity has been called a double point.

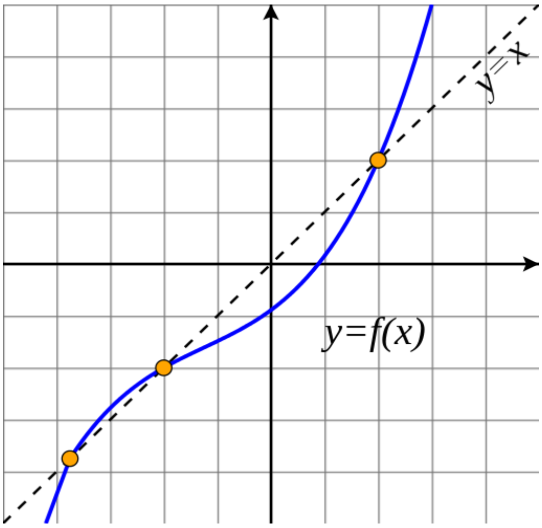


Figure.2: A function with 3 fixed points

An attracting fixed point of a function f is a fixed point x0 of f such that for any value of x in the domain that is close enough to x0, the iterated function sequence:

x, f(x), f(f(x)), f(f(f(x))),….

converges to x0. An expression of prerequisites.

The natural cosine function, "natural" means in radians, not degrees or other units has exactly one fixed point, which is attracting. In this case, "close enough" is not a stringent criterion at all—to demonstrate this, start with any real number and repeatedly press the cos key on a calculator (checking first that the calculator is in "radians" mode). It eventually converges to about 0.739085133, which is a fixed point. That is where the graph of the cosine function intersects the line y = x.

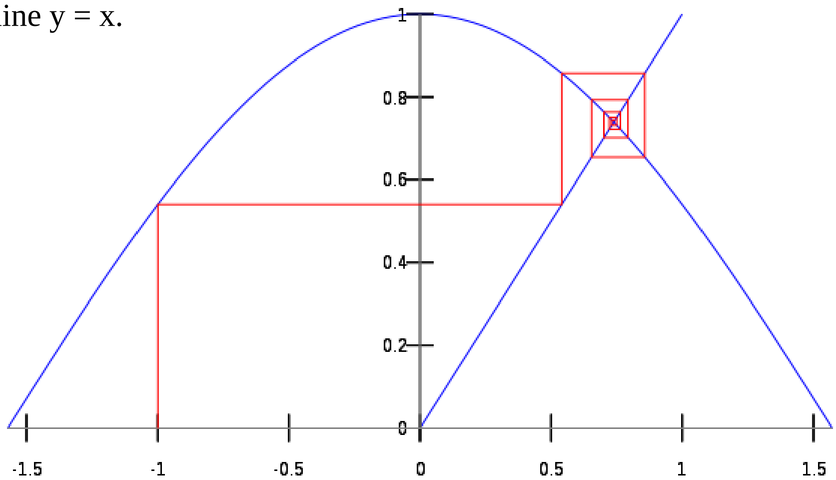


Figure.3: The fixed point iteration $x_{n+1} = \cos x_n$ with initial value $x_1 = -1$.

Not all fixed points are attracting. For example, x = 0 is a fixed point of the function f(x) = 2x, but iteration of this function for any value other than zero rapidly diverges. However, if the function f is continuously differentiable in an open neighborhood of a fixed point x_0, and $|f'(x_0)| < 1$, attraction is guaranteed.

Nullclines

Nullclines, sometimes called zero-growth isoclines, are encountered in a system of ordinary differential equations:

$x_1' = f_1(x_1, \ldots\ldots, x_n)$

$x_2' = f_2(x_1, \ldots\ldots, x_n)$

.

.

.

$x_n' = f_n(x_1, \ldots\ldots, x_n)$

Where x' here represents a derivative of x with respect to another parameter, such as time t. The j'th nullcline is the geometric shape for which $x_j' = 0$. The equilibrium points of the system are located where all of the nullclines intersect. In a two-dimensional linear system, the nullclines can be represented by two lines on a two-dimensional plot; in a general two-dimensional system they are arbitrary curves.

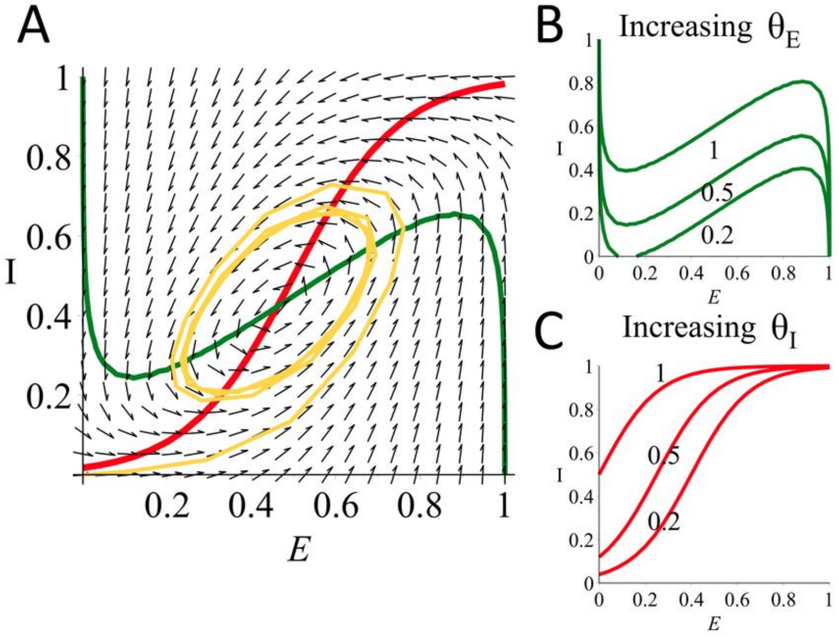


Figure.4: The nullcline representations of different systems

Trapping Region

In applied mathematics, a trapping region of a dynamical system is a region such that every trajectory that starts within the trapping region will move to the region's interior and remain there as the system evolves.

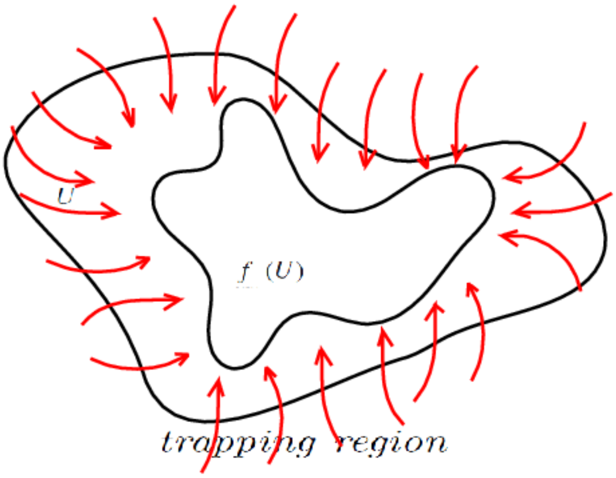


Figure.5: Trapping region of a dynamical system representation

Limit Cycle

A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity. In other words, the limit cycle is an isolated trajectory (isolated in the sense that neighboring trajectories are not closed, they spiral either toward or away from the limit cycle). If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting, that is, all the neighboring trajectories approach the limit cycle as time approaches infinity. Otherwise, the limit cycle is unstable, that is, all neighboring trajectories approach it as time approaches negative infinity.

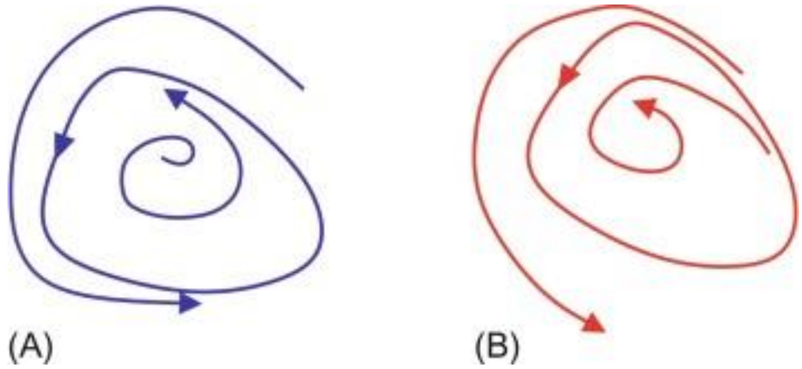


Figure.6: A depicts a stable limit cycle and B depicts an unstable limit cycle.

Limit cycle is an oscillation peculiar to nonlinear systems. The oscillatory behavior, unexplainable in terms of linear theory, is characterized by a constant amplitude and frequency determined by the nonlinear properties of the system. Limit cycles are distinguishable from linear oscillations in that their amplitude of oscillation is independent of initial conditions. For instance, if a system has a stable limit cycle, the system will tend to fall into the limit cycle, with the output approaching the amplitude of that limit cycle regardless of the initial condition and forcing function. A limit cycle is easily recognized in the phase plane as an isolated closed path as shown in Figure.7.

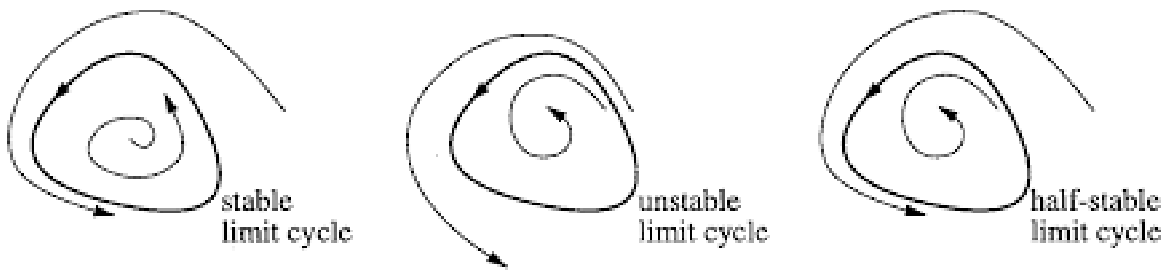


Figure.7: Limit cycle types with different stabilities

Phase Plane

In particular the context of nonlinear system analysis, a phase plane is a visual display of certain characteristics of certain kinds of differential equations; a coordinate plane with axes being the values of the two state variables, say (x, y), or (q, p) etc. (any pair of variables). It is a two-dimensional case of the general n-dimensional phase space.

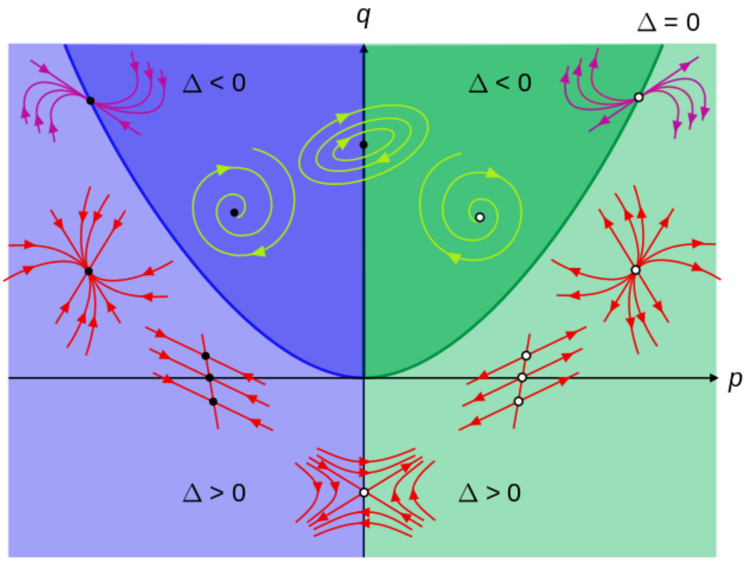
The phase plane method refers to graphically determining the existence of limit cycles in the solutions of the differential equation.

The solutions to the differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field. Vectors representing the derivatives of the points with respect to a parameter (say time t), that is (dx/dt, dy/dt), at representative points are drawn. With enough of these arrows in place the system behavior over the regions of plane in analysis can be visualized and limit cycles can be easily identified.

The entire field is the phase portrait, a particular path taken along a flow line (i.e. a path always tangent to the vectors) is a phase path. The flows in the vector field indicate the time-evolution of the system the differential equation describes.

In this way, phase planes are useful in visualizing the behavior of physical systems; in particular, of oscillatory systems such as predator-prey models. In these models the phase paths can "spiral in" towards zero, "spiral out" towards infinity, or reach neutrally stable situations called centres where the path traced out can be either circular, elliptical, or ovoid, or some variant thereof. This is useful in determining if the dynamics are stable or not.

Other examples of oscillatory systems are certain chemical reactions with multiple steps, some of which involve dynamic equilibria rather than reactions that go to completion. In such cases one can model the rise and fall of reactant and product concentration (or mass, or amount of substance) with the correct differential equations and a good understanding of chemical kinetics.



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$
$$\Delta = p^2 - 4q$$

Figure.8: Phase plane for a specific differential equation system shown above

Poincare-Bendixson Theorem, supposing that:

1) R is a closed, bounded subset of the plane;
2) $\dot{x} = f(x)$ is a continuously differentiable vector field on an open set containing R;
3) R does not contain any fixed points; and
4) There exist a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time as shown in Figure.9.

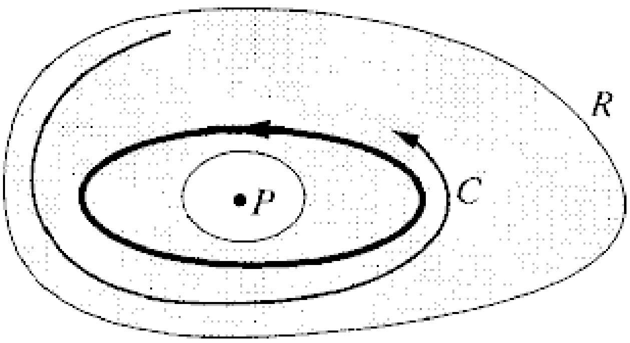


Figure.9: Trajectory(C) behavior in a subset(R) of the plane

C(trajectory) starts in R(subset) and stays in R all the times.

The either C is a closed orbit, or it spirals toward a closed orbit as t $\rightarrow\infty$. In either case, R contains a closed orbit shown as follows:

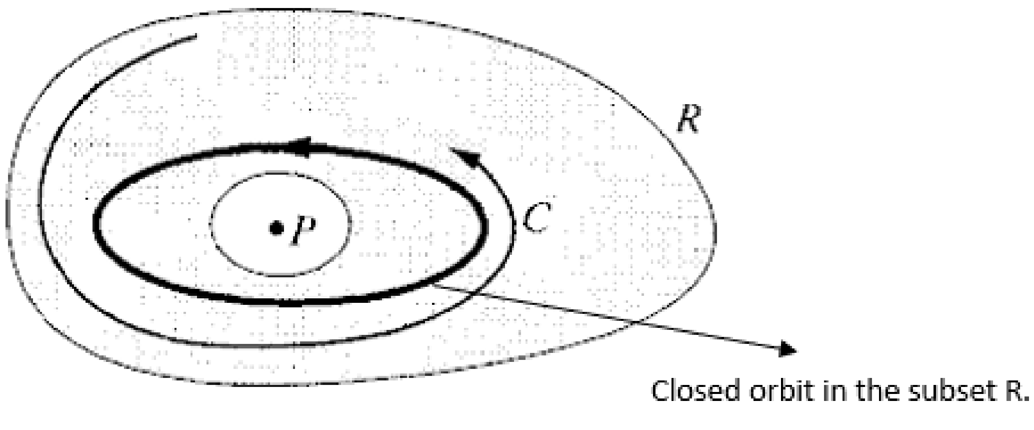


Figure.10: Representation of closed orbit in the subset R

In the figure above, R is drawn as a ring-shaped region because any closed orbit must encircle a fixed point (P) and no fixed points are allowed in R.

When applying the Poincare-Bendixson theorem, it is easy to satisfy conditions (1)-(3); condition (4) is a bit tougher than the others. How can one be sure that a confined trajectory C exists? The standard trick is to construct a trapping region R, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of R as shown in Figure.11. Then all trajectories in R are confined. If one can also arrange that there are no fixed points in R, then the Poincare-Bendexson theorem ensures that R contains a closed orbit.

The Poincare-Bendixson theorem can be difficult to apply in practice. One convenient case occurs when the system has a simple representation in polar coordinates.

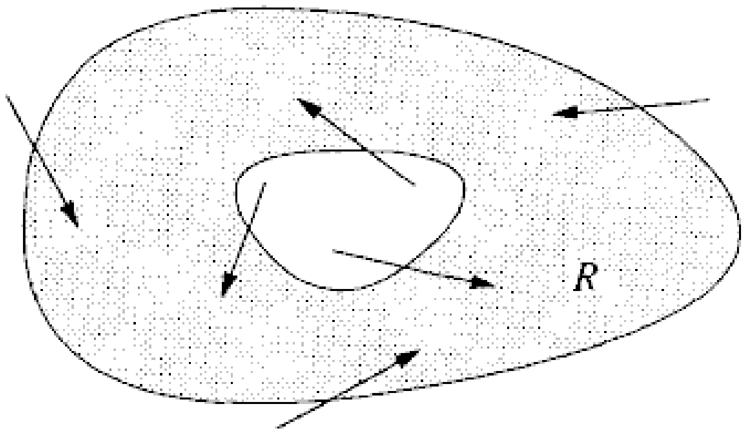


Figure.11: Vector field points “inward” everywhere on the boundary of R

To make a better understanding on the subject mentioned so far, it is useful to investigate an example:

Consider a system with properties:

$\dot{r} = r(1 - r^2) + \mu r\cos\theta$

$\dot{\theta} = 1$

When $\mu = 0$, there is a stable limit cycle at r = 1, since

$\dot{r} = 1(1 - 1^2) + 0 * 1 * \cos\theta = 0$

The task is here to show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small. To do so;

Two concentric circles with radii r_{min} and r_{max}, such that r < 0 on the outer circle and r > 0 on the inner circle. Then the annulus $0 < r_{min} \leq r \leq r_{max}$ will be the desired trapping region.

Note that there are no fixed points in the annulus since $\dot{\theta} > 0$; hence if r_{min} and r_{max} can be found, the Poincare-Bendixson theorem will imply the existence of a closed orbit.

To find r_{min}, it is required that

$\dot{r} = r(1 - r^2) + \mu r\cos\theta > 0$ for all θ. Since $\cos\theta \geq -1$ a sufficient condition for r_{min} is

$1 - r^2 - \mu > 0$. Hence any $r_{min} < \sqrt{1-\mu}$ will work, as long as $\mu < 1$ so that the square root makes sense. r_{min} should be chosen as large as possible, to hem in the limit cycle as tightly as it can be. For instance, $r_{min} = 0.999\sqrt{1-\mu}$ can be picked. Even $r_{min} = \sqrt{1-\mu}$ works, but more careful reasoning is required. By a similar argument, the flow is inward on the outer circle if $r_{max} = 1.001\sqrt{1+\mu}$.

Therefore, a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$.

The estimates used in this example are conservative. In fact, the closed orbit can exist even if $\mu \geq 1$. Figure.12 illustrates a computer-generated phase portrait of (1) for $\mu = 1$. It is also possible to obtain some analytical insight about the close orbit for small μ.

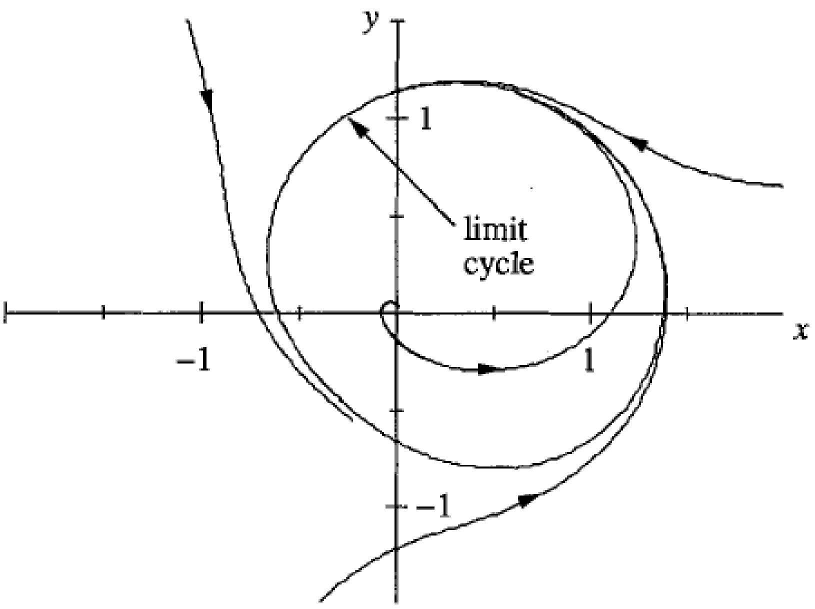


Figure.12: A computer-generated phase portrait of (1) for $\mu = 1$.

When a polar coordinates are inconvenient, an approximate trapping region may still be found by examining the system's nullclines, as in the next example.

Another example;

In the fundamental biochemical process called glycolysis, living cells obtain energy by breaking down sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis can proceed in an oscillatory fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. A simple model of these oscillations has been proposed by Sel'kov (1968). In dimensionless form, the equations are

$$\dot{x} = -x + ay + x^2 y$$

$$\dot{y} = b - ay - x^2 y$$

Where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate); a and b are kinetic parameters with a, b > 0. The main goal here is to construct a trapping region for this system.

Firstly, the nullclines should be found.

The first equation shows that $\dot{x} = 0$ on the curve $y = \dfrac{x}{a + x^2}$

The first equation shows that $\dot{x} = 0$ on the curve $y = \dfrac{b}{a + x^2}$

These nullclines are sketched in the figure below, along with some representative vectors.

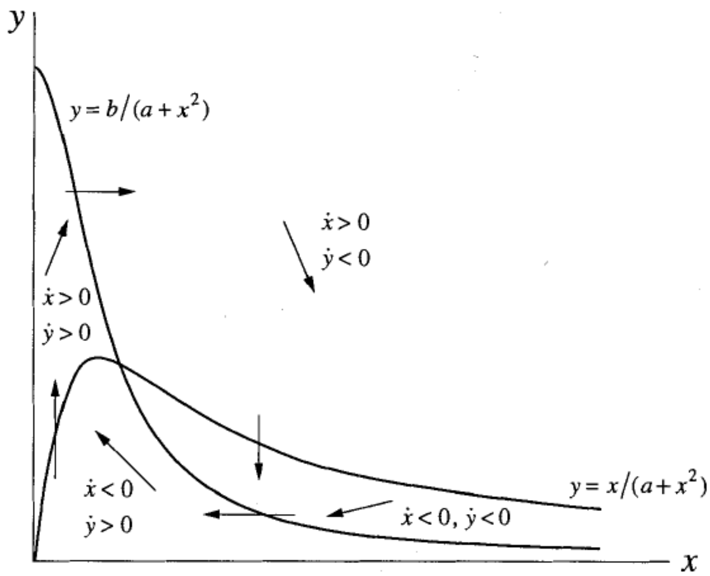


Figure.13: Nullclines for the equation system of

$\dot{x} = -x + ay + x^2 y$ and $\dot{y} = b - ay - x^2 y$

There are seen vectors on the plot above but how can one know how to plot them, their positions and their directions. By definition, the arrows are vertical on the $\dot{x} = 0$ nullcline, and horizontal on the $\dot{y} = 0$ nullcline. The direction of flow is determined by the signs of $\dot{x}$ and $\dot{y}$. For instance, in the region above both nullclines, the governing equations imply $\dot{x} > 0$ and $\dot{y} < 0$, so the arrows point down and to the right, as shown in Figure.13.

Now consider the region bounded by the dashed line shown in Figure 7. It is claimed that it's a trapping region. In order to verify this, one has to show that all the vectors on the boundary point into the box. On the horizontal and vertical sides, there' s no problem: the claim follows from Figure.13. The tricky part of the construction is the diagonal line of slope -1 extending from the point (b, b/a) to the nullcline y = x/$(a + x^2)$ Where did this come from?

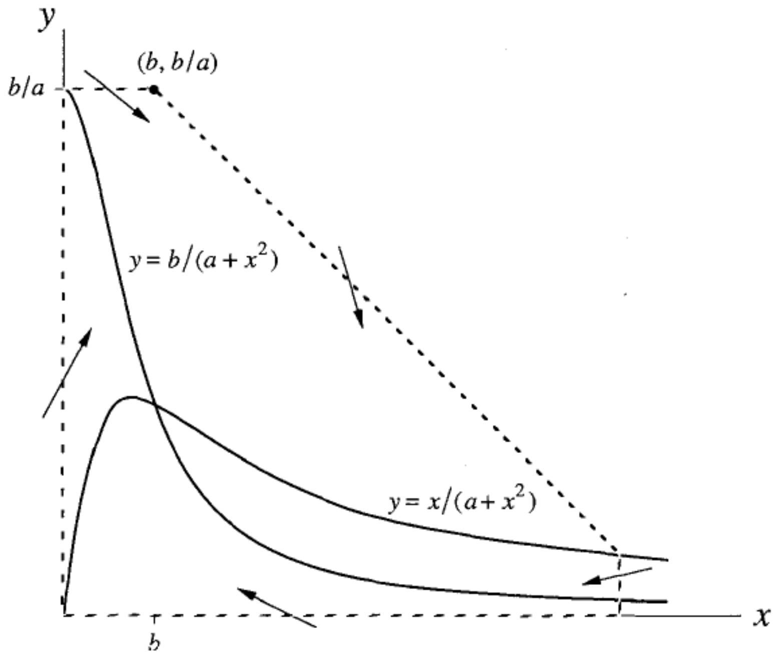


Figure.14. Nullclines for a trapping region and the vectors on the boundary point into the box

To get the right intuition, consider $\dot{x}$ and $\dot{y}$ in the limit of very large x. Then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$, so $\dot{y}/\dot{x} = dy/dx \approx -1$ along trajectories. Hence the vector field at large x is roughly parallel to the diagonal line. This suggests that in a more precise calculation, one should compare the sizes of $\dot{x}$ and $-\dot{y}$, for some sufficiently large x.

In particular, consider $\dot{x} - (-\dot{y})$. It is found:

$$\dot{x} - (-\dot{y}) = -x + ay + x^2y + (b - ay - x^2y) = b - x$$

Hence,

$$-\dot{y} > \dot{x} \text{ if } x > b.$$

This inequality implies that the vector field points inward on the diagonal line in Figure.7, because dy/dx is more negative than -1, and therefore the vectors are steeper than the diagonal line. Thus, the region is a trapping region, as claimed.

Can it be concluded that there is a closed orbit inside the trapping region? The answer is No. There is a fixed point in the region at the intersection of the nullclines, and so the conditions of the Poincare-Bendixson theorem are not satisfied. But if this fixed point is a repeller, then it can be proven the existence of a closed orbit by considering the modified "punctured" region shown in Figure.15.

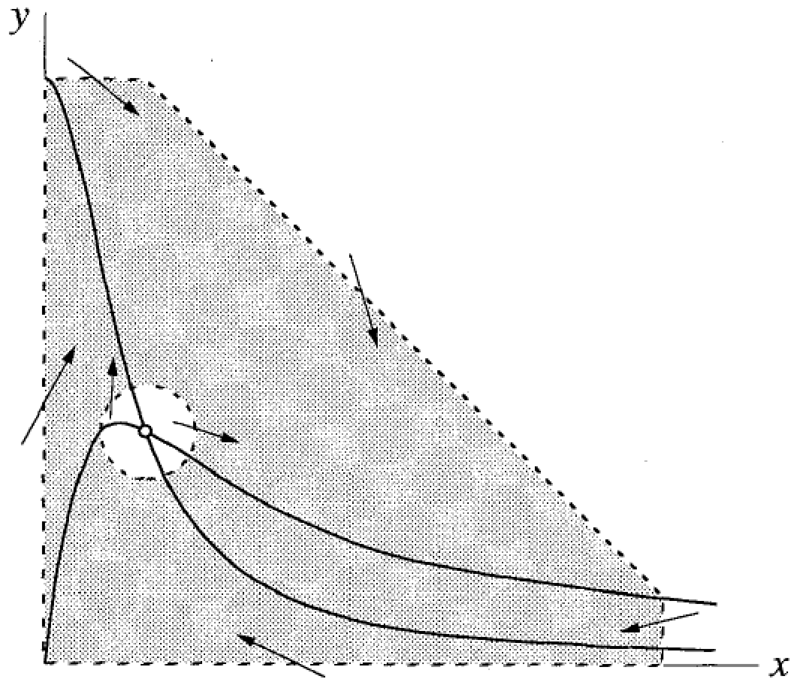


Figure.15. Representation of punctured region, although the hole is infinitesimal, it is drawn larger for clarity.

The repeller drives all neighboring trajectories into the shaded region, and since this region is free of fixed points, the Poincare-Bendixson theorem applies.

Now it is found that conditions under which the fixed point is a repeller.

Another example;

Once again, consider the glycolytic oscillator $\dot{x} = -x + ay + x^2y$, $\dot{y} = b - ay - x^2y$ of previous example. The goal is to prove that a closed orbit exists if a and b satisfy an appropriate condition, as before, a, b > 0.

By the argument above, it suffices to find conditions under which the fixed point is a repeller, i.e., an unstable node or spiral. In general, the Jacobian form is

$$A = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{pmatrix}$$

After some algebra, it is found that at the fixed point,

$$x^* = b, \qquad y^* = \frac{b}{a + b^2}$$

the Jacobian has determinant:

$\Delta = a + b^2 > 0$

and the trace:

$y^* = \frac{b}{a + b^2}$

$$\tau = -\frac{b^4 + (2a - 1)b^2 + (a + a^2)}{a + b^2}.$$

Hence the fixed point is unstable for $\tau > 0$, and stable for $\tau < 0$. The dividing line $\tau = 0$ occurs when

$$b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1 - 8a}\right).$$

This defines a curve in (a, b) space, as shown in Figure.16.

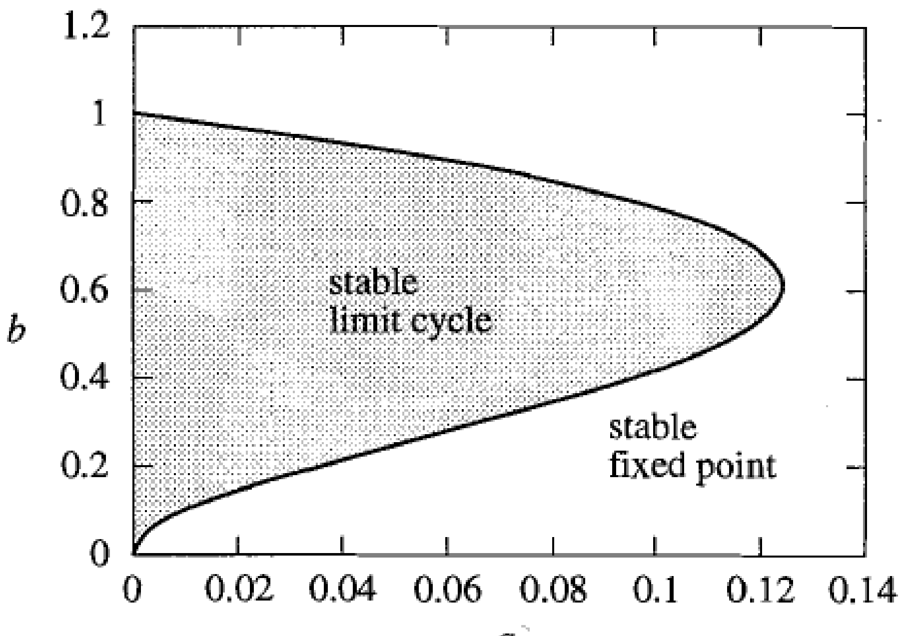


Figure.16: Stable limit cycle and stable fixed point separated by the border curve in (a, b) space

For parameters in the region corresponding to $\tau > 0$, it is guaranteed that the system has a closed orbit-numerical integration shows that it is actually a stable limit cycle. Figure.17 shows a computer-generated phase portrait for the typical case a = 0.08, b= 0.6.

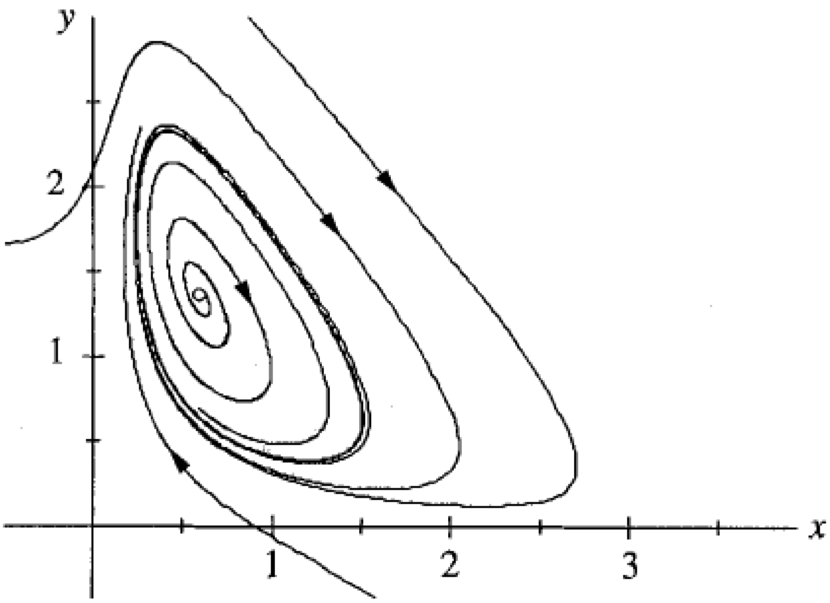


Figure.17: Computer-generated phase portrait for the typical case a = 0.08, b= 0.6

Conclusion about the Poincare-Bendixson theorem: No Chaos in the Phase Plane

The Poincare-Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit.

Nothing more complicated is possible. This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n \geq 3$), the Poincare-Bendixson theorem no longer applies, and something radically new can happen: trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a “strange attractor”, a fractal set on which die motion is aperiodic and sensitive to tiny changes in the initial conditions. This sensitivity makes the motion unpredictable in the long run. One is now face to face with chaos. All in all, the Poincare-Bendixson theorem implies that chaos can never occur in the phase plane.

3) SOLUTION-RESULTS

The differential functions given are:

$$\dot{x} = B - x - \frac{xy}{1 + qx^2}$$

$$\dot{y} = A - \frac{xy}{1 + qx^2}$$

where x and y are the levels of nutrient and oxygen, respectively, and A, B, q > 0 are parameters.

In this solution to test the effect of A, B and q, the parameters are defined as follows:

1) A = 1, B = 1, q = 1
2) A = 0.5, B = 1, q = 1
3) A = 1, B = 0.5, q = 1
4) A = 1, B = 1, q = 0.5

Using 1 and 2, the effect of A is investigated.

Using 1 and 3, the effect of B is investigated.

Using 1 and 4, the effect of q is investigated.

Note that, only one parameter is different in each pair of uses to understand better the effect of that parameter.

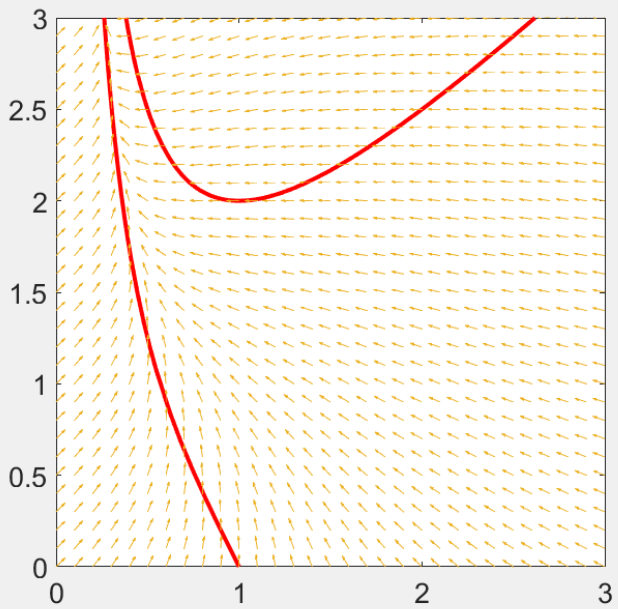
To find the nullclines, $\dot{x}$ and $\dot{y}$ goes to zero and y should be found individually to plot.

$$\dot{x} = B - x - \frac{xy}{1 + qx^2} \qquad 0 = B - x - \frac{xy}{1 + qx^2} \quad \longrightarrow \quad y = \frac{(B - x)(1 + qx^2)}{x}$$

$$\dot{y} = A - \frac{xy}{1 + qx^2} \qquad 0 = A - \frac{xy}{1 + qx^2} \quad \longrightarrow \quad y = \frac{(A)(1 + qx^2)}{x}$$

Using 1 and 2 (A = 1, B = 1, q = 1 and A = 0.5, B = 1, q = 1)

<u>A = 1, B = 1, q = 1:</u>



The red curves represent the nullclines that were defined above.

Figure.18: Result from using A=1, B=1, q=1

A = 0.5, B = 1, q = 1:

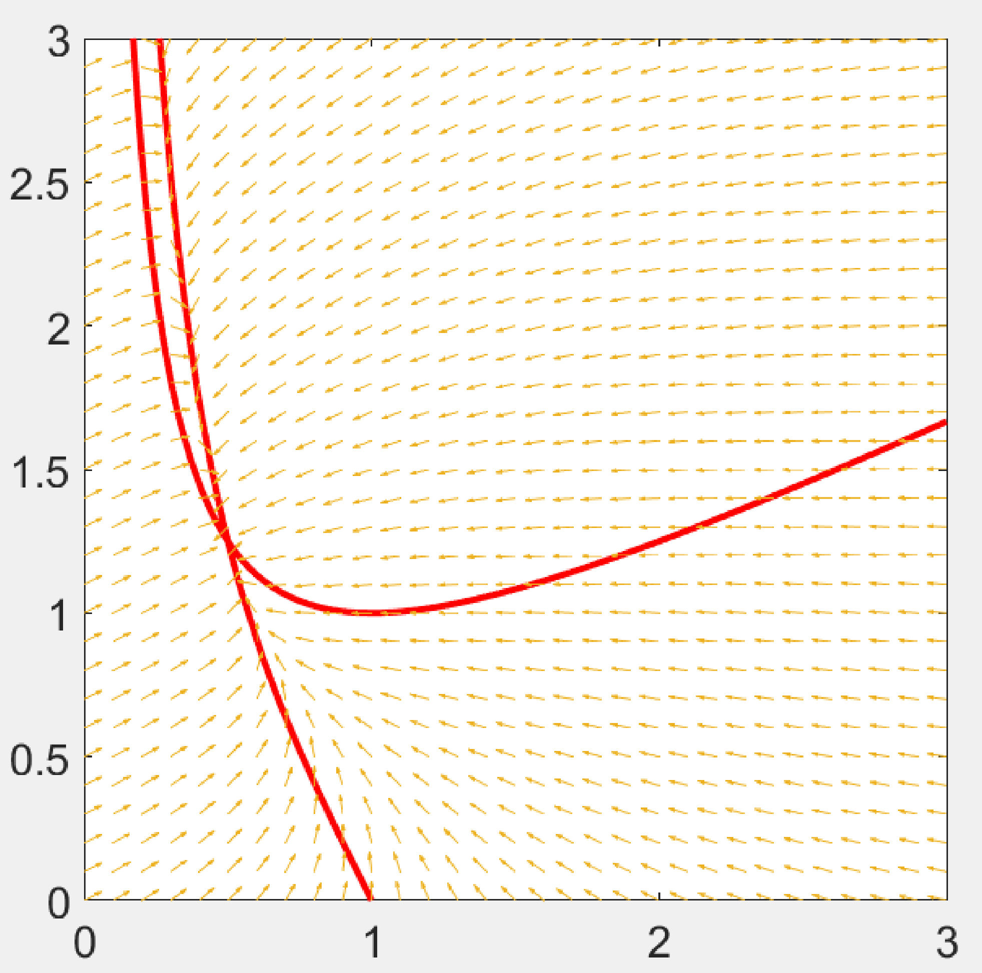


Figure.19: Result from using A=0.5, B=1, q=1

Comparison:

One of the curves did not change because it depends on the parameters B and q but since they are not changed, the curve is constant.

$$y = \frac{(B - x)(1 + qx^2)}{x}$$

On the other hand, other curve changed since it has a multiplier of A and A is changed to compare. The curve is now more diffuser.

$$y = \frac{(A)(1 + qx^2)}{x}$$

Also, it looks like the orange arrows tend to head to the intersection of two nullclines. This makes them to head to a lower point compared to the parameters A = 1, B = 1, q = 1.

A = 1, B = 0.5, q = 1:

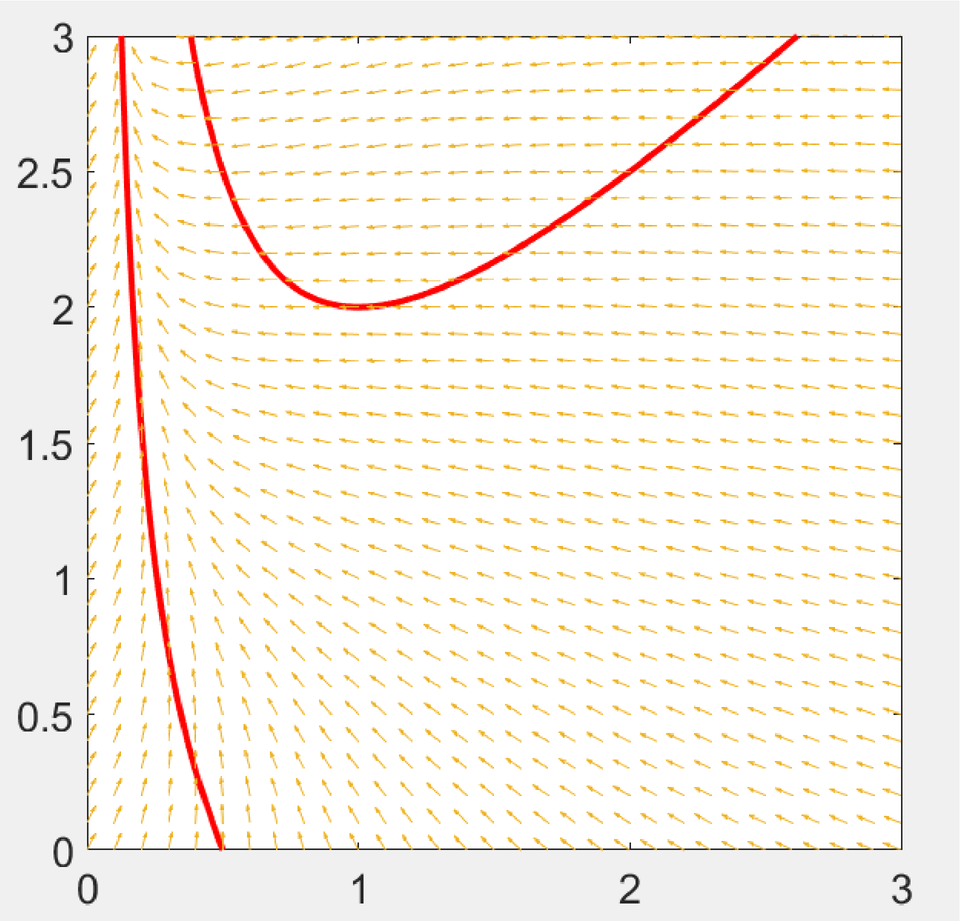


Figure.20: Result from using A=1, B=0.5, q=1

Comparison:

One of the curves did not change because it depends on the parameter A only but since is not changed, the said curve is constant.

$$y = \frac{(A)(1 + qx^2)}{x}$$

On the other hand, other curve changed since it has a multiplier of B and q while q is constant B is changed to compare. The curve is now steeper.

$$y = \frac{(B - x)(1 + qx^2)}{x}$$

In the appearance of the plot, since the orange arrows tend to follow the red curve on the left, they are headed steeper now compared to A = 1, B = 1, q = 1.

A = 1, B = 1, q = 0.5:

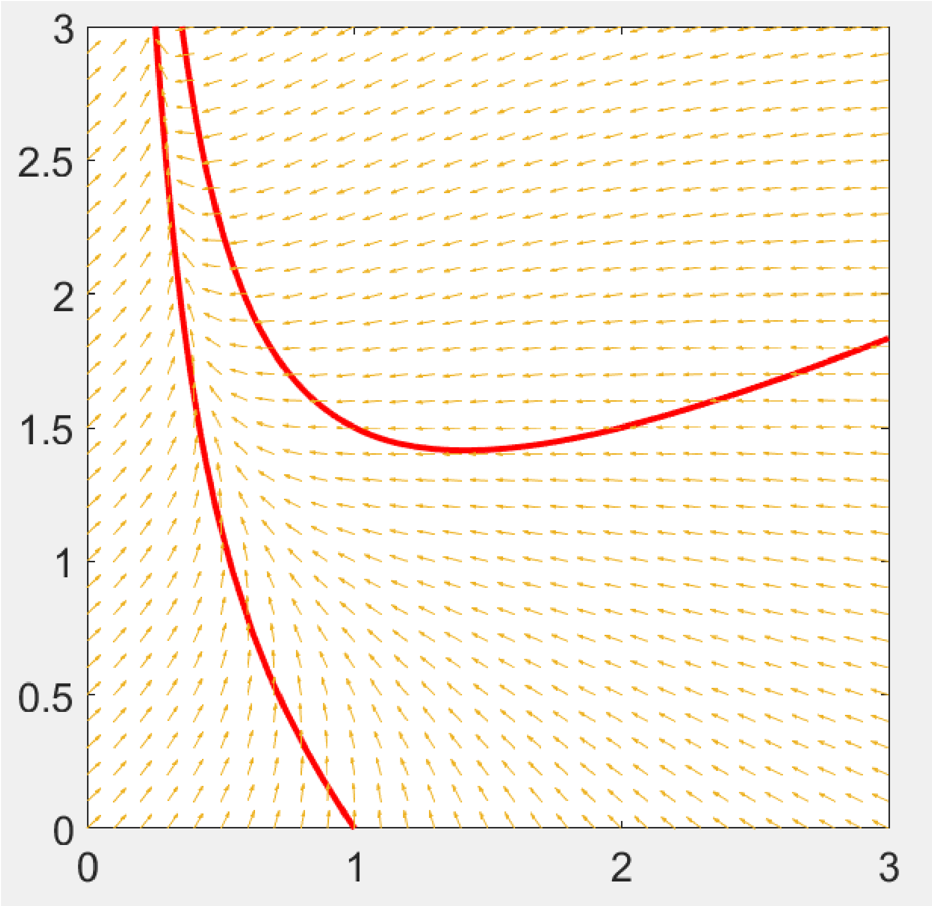


Figure.21: Result from using A=1, B=1, q=0.5

Comparison:

$$y = \frac{(A)(1 + qx^2)}{x} \qquad y = \frac{(B - x)(1 + qx^2)}{x}$$

Now, q is changed and A and B are hold constant. Since the both curve equations have a multiplier of x^2 on the numerator and the q value is decreased two times compared to A = 1, B = 1, q = 1, both nullclines are now diffuser in comparison to A = 1, B = 1, q = 1.

Different graphical results from using A = 1, B = 1, q = 1.

To show the following the arrows are correct, the users click on some point on the plot while it is running, then the results follow the orange arrows accordingly. Here are the graphical inputs and traces:

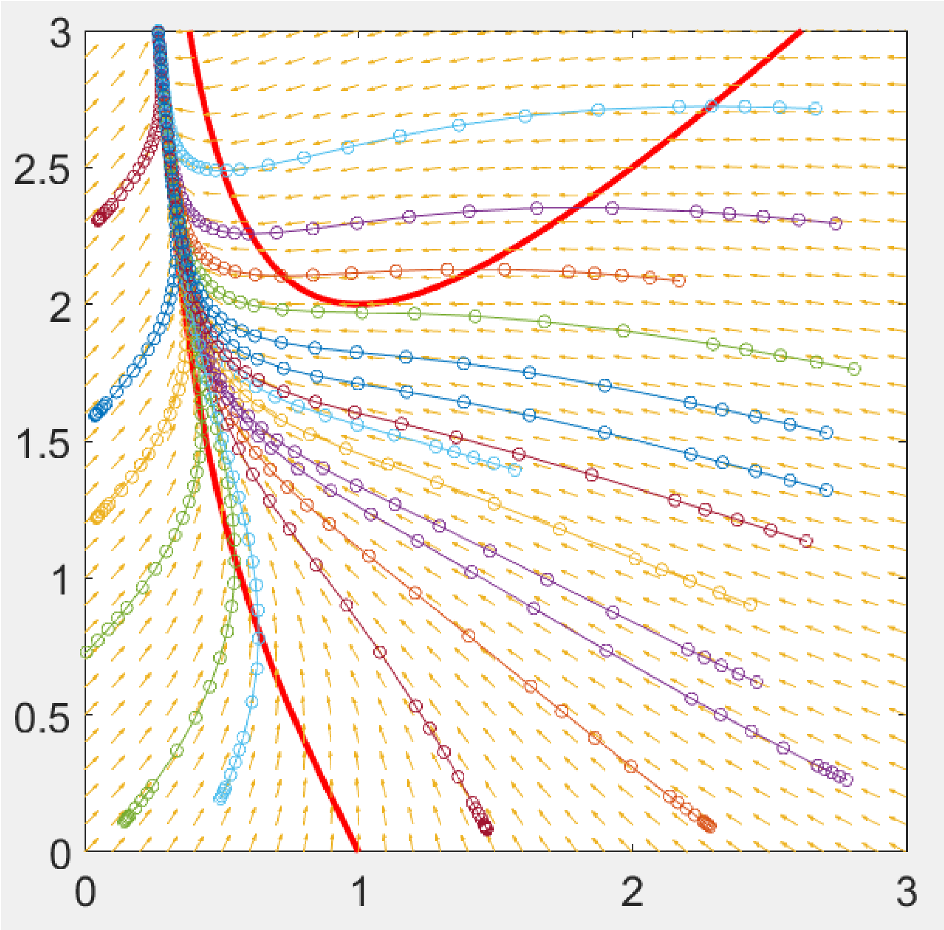


Figure.22: The colorful lines follow the small orange arrows depending on the initial point.

4) CONCLUSION

When the parameters are decreased, the results change accordingly because of the differential equations including the parameters.

$$\dot{x} = B - x - \frac{xy}{1 + qx^2}$$

$$\dot{y} = A - \frac{xy}{1 + qx^2}$$

To get the nullclines, the equations should be changed slightly to get y individually. To do so, one needs to make $\dot{x}$ and $\dot{y}$ zero. Then, the y is got as follows respectively:

$$y = \frac{(A)(1 + qx^2)}{x}$$

$$y = \frac{(B - x)(1 + qx^2)}{x}$$

Decreasing A alone will make the right “U” shaped curve diffuser. Decreasing B alone will make the left curve steeper. Decreasing q will make both of them diffuser. The reasons behind the behaviors were discussed above deeply.

Finally, the colorful lines follow the small orange arrows depending on the initial point that the user give.

5) CODE REVIEW

```
clear
[x1,x2] = meshgrid(0:0.1:3);
A = 1;
B = 1;
q = 1;

dx1 = B - x1 - (x1.*x2)./(1+q*x1.*x1);
dx2 = A - (x1.*x2)./(1+q*x1.*x1);
```

Mesh (how dense the orange arrows) is defined. Parameters are defined (User chooses.) The differential equations are defined.

```
 % plots nullclines
 syms t1 t2 z1 z2;
 z1 = (B - t1)*(1+q*t1^2)/t1;
 fplot(z1 ,"r",'LineWidth',3);
 hold on
 z2 = (A)*(1+q*t2^2)/t2;
 fplot(z2 ,"r",'LineWidth',3);
```

The nullclines are plotted.

```
% plot arrows
r = sqrt(dx1.^2 + dx2.^2); % normalization
quiver(x1,x2,dx1./r,dx2./r,1/2,'LineWidth',1); % plot arrows
axis equal;
set(gca,'FontSize',20);
axis([0 3 0 3]);
```

Normalization is made and the previously defined arrows are plotted.

```
while (true)
    x0 = ginput(1); % take a graphical input
    tspan = [0 20];
    [t,x] = ode45(@(t,x) odefcn(x), tspan, x0);
    plot(x(:,1),x(:,2),'-o');
end
```

User is allowed to click on the plot to decide the initial point. Then the traces of the colorful plots occur.

```
function dxdt = odefcn(x)
A = 1;
B = 1;
q = 1;
dxdt = zeros(2,1);
dxdt(1) = B - x(1) - (x(1)*x(2))/(1+q*x(1)^2); % describes derivatives
dxdt(2) = A - (x(1)*x(2))/(1+q*x(1)^2);   % describes derivatives
end
```

In the function part, parameters should be defined again same as the previous inputs by the user.