2) INTRODUCTION

The Lorenz system is a system of ordinary differential equations first studied by Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system. In popular media the 'butterfly effect' stems from the real-world implications of the Lorenz attractor, i.e., that in any physical system, in the absence of perfect knowledge of the initial conditions (even the minuscule disturbance of the air due to a butterfly flapping its wings), our ability to predict its future course will always fail. This underscores that physical systems can be completely deterministic and yet still be inherently unpredictable even in the absence of quantum effects. The shape of the Lorenz attractor itself, when plotted graphically, may also be seen to resemble a butterfly.

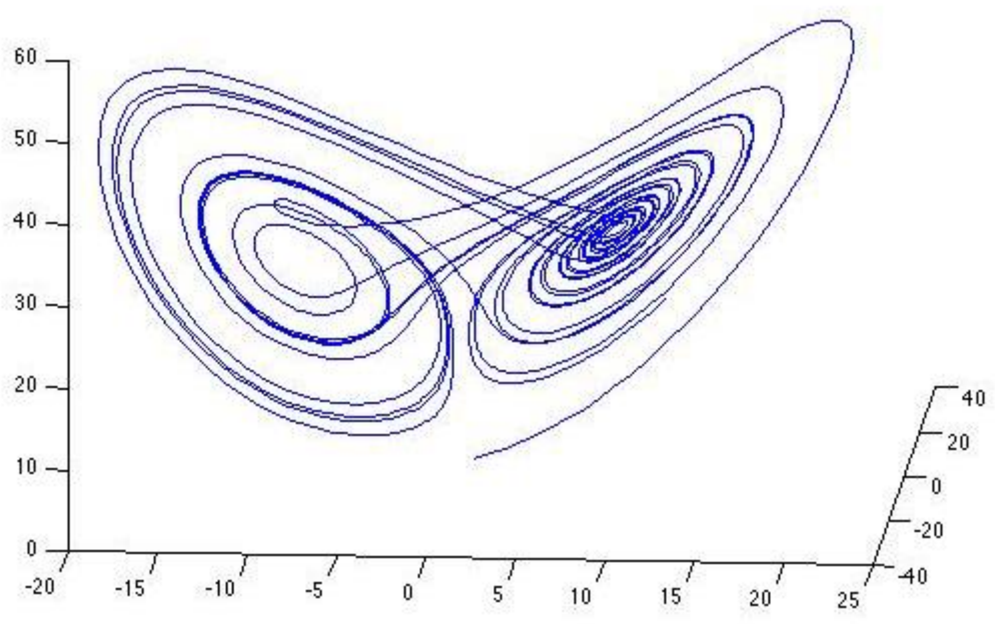

Figure.1: A representation for the Lorenz attractor; it is seen to resemble a butterfly.

In 1963, Edward Lorenz, with the help of Ellen Fetter, developed a simplified mathematical model for a 3D non-linear flow, atmospheric convection. A 3D non-linear flow is defined as a flow type which the hydrodynamic parameters are functions of three space coordinates and time. For example; wind flow in the atmosphere is a 3D non-linear flow since it is 3 dimensions in the space and it changes with time.

The Lorenz model is a system of three ordinary differential equations, since it is a 3D flow system, known as the Lorenz equations:

$$\frac{dx}{dt} = \mathrm{\sigma}(\mathrm{y} - \mathrm{x}),$$

$$\frac{dx}{dt} = \mathrm{x}(\mathrm{\rho} - \mathrm{z}) - y,$$

$$\frac{dx}{dt} = \mathrm{x}y - \mathrm{\beta z}$$

where the equations describe the rate of change of three quantities with respect to time: x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation. The constants σ, ρ and β are system parameters proportional to the Prandtl number, Rayleigh number, and certain physical dimensions of the layer itself. The non-linearity in the flow type comes from the multiplications of direction pairs, which are xz in the second equation and xy in the third equation.

One normally assumes that the parameters σ, ρ and β are positive (< 0). Lorenz used the values $\sigma = 10$, $\rho = 8/3$ and $\beta = 28$. The system exhibits chaotic behavior for these (and nearby) values.

If $\rho > 1$, then there is only one equilibrium point, which is at the origin. This point corresponds to no convection. All orbits converge to the origin, and this situation is called a global attractor, when $\rho > 1$. Lorenz used ρ in this way and as it is seen in the following figures, orbits converge to the origin as $\rho > 1$. The Lorenz attractor is difficult to analyze, but the action of the differential equation on the attractor is described by a fairly simple geometric model.

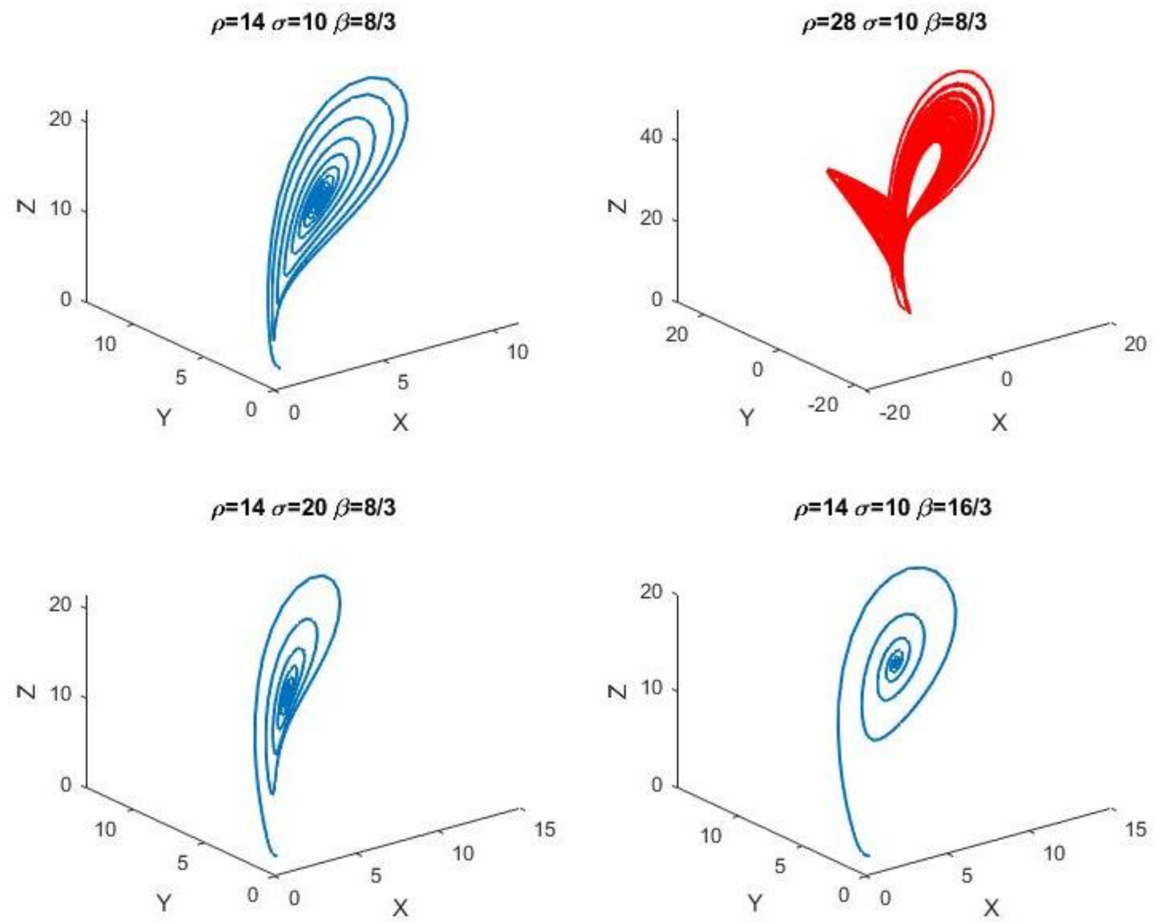

Figure.2: Different example solutions of the Lorenz system for different values of σ, ρ and β, orbits converge to the origin as $\rho > 1$.

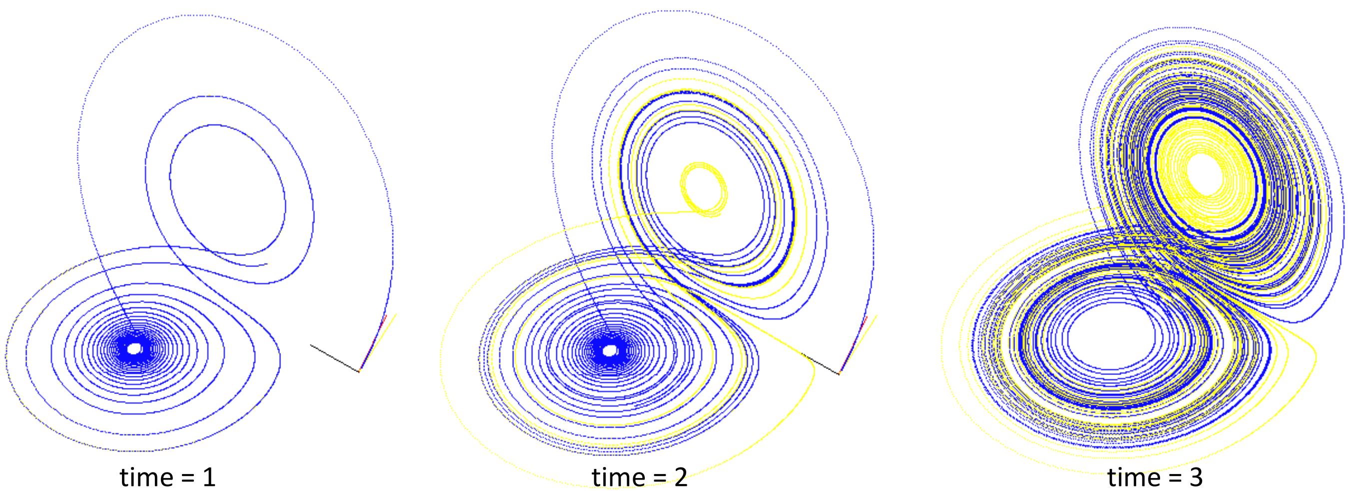

Figure.3: These figures, made using $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$, show three-time segments of the 3-D evolution of two trajectories (one in yellow, the other in blue) in the Lorenz attractor starting at two initial points that differ only by 10−5 in the x-coordinate. Initially, the two trajectories seem coincident (only the blue one can be seen, as it is drawn over the yellow one) but, after some time, the divergence is obvious.

3) METHODS – DISCUSSION

The Lorenz equations were originally derived by Saltzman (1962) as a 'minimalist' model of thermal convection in a box that has three dimensions which are x, y and z. The system of equations of 3D model of convection rolls in the atmosphere can represented as follows:

$\dot{x} = \sigma(y - x)$

$\dot{y} = rx - y - xz$

$\dot{z} = xy - bz$

where σ ("Prandtl number"), r ("Rayleigh number") and b are parameters (> 0). These equations also arise in studies of convection and instability in planetary atmospheres, models of lasers and dynamos etc. Willem Malkus also devised a water-wheel demonstration.

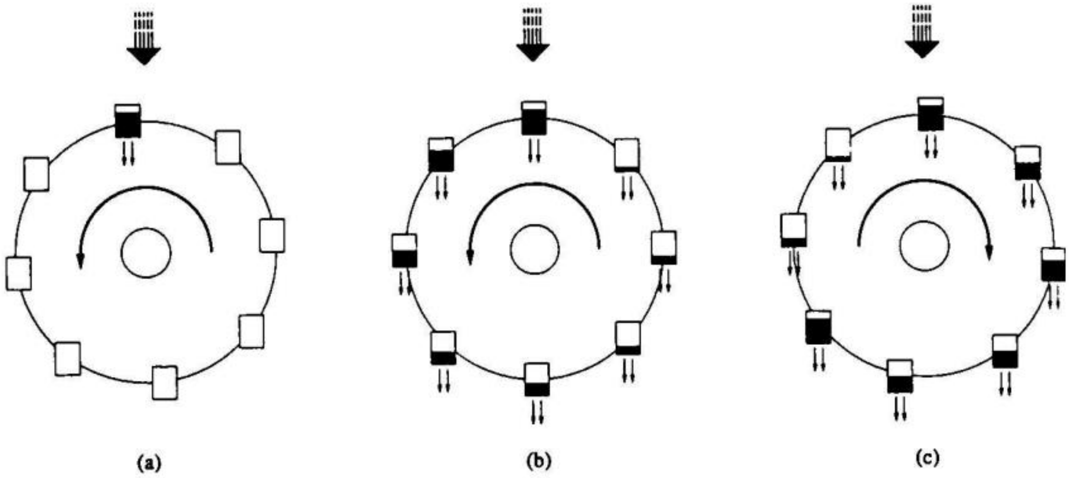

Figure.4: Water – wheel demonstration

The Malkus waterwheel, also referred to as the Lorenz waterwheel or chaotic waterwheel, is a mechanical model that exhibits chaotic dynamics. Its motion is governed by the Lorenz equations. While classical waterwheels rotate in one direction at a constant speed, the Malkus waterwheel exhibits chaotic motion where its rotation will speed up, slow down, stop, change directions, and oscillate back and forth between combinations of such behaviors in an unpredictable manner. This variant waterwheel was developed by Willem Malkus in the 1960s. As a pedagogic tool, the Malkus waterwheel became a paradigmatic realization of a chaotic system, and is widely used in the teaching of chaos theory. In addition to its pedagogic use, the Malkus waterwheel has been actively studied by researchers in dynamical systems and chaos.

In the Malkus waterwheel, a constant flow of water pours in at the top bucket of a simple circular symmetrical waterwheel and the base of the waterwheel has perforations to allow the outflow of water. At low rates of inflow, the wheel rolls permanently in the same direction.

At higher rates of inflow, the waterwheel enters a chaotic regime where it reverses its directionality, accelerates and decelerates in an apparently unpredictable way. A further increase in the incoming flow makes the waterwheel return to a periodic state, where it oscillates back and forth at fixed intervals. Since it was first proposed, many experimental and real-world applications have been related to the waterwheel dynamics.

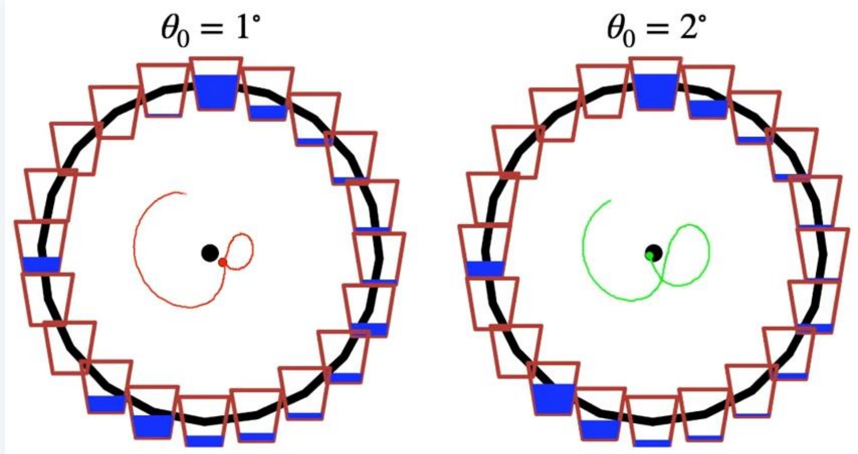

Figure.5: Computer simulation of two Malkus waterwheels with varying initial condition

Although the initial angle θ_0 differs between the wheels by only 1 degree, the trajectories of the plotted centers of mass diverge enormously in the long run.

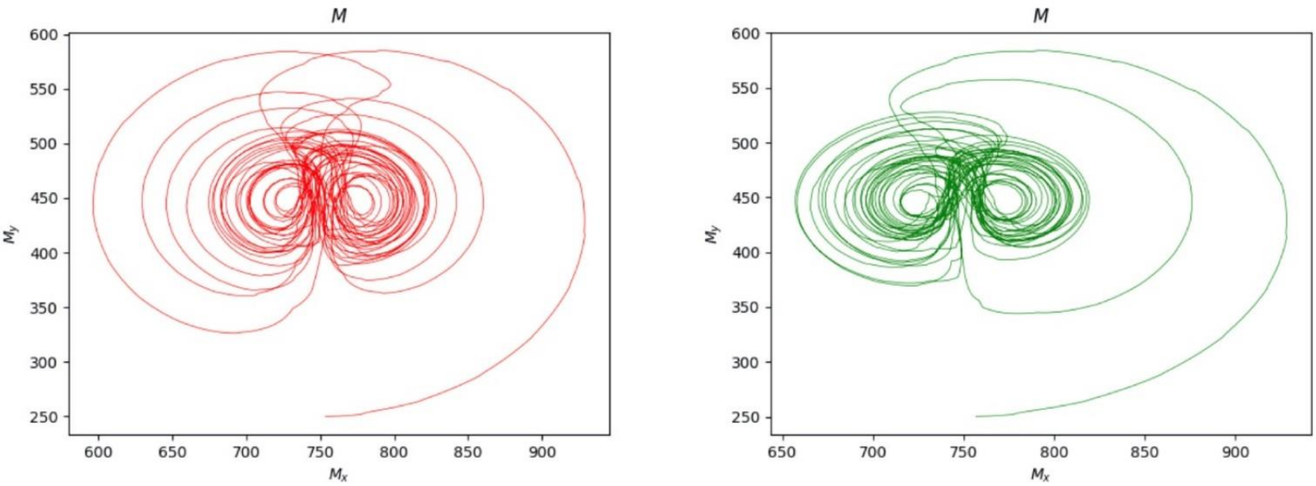

Figure.6: Plots of the trajectories of the centers of mass of the above two wheels, left and right respectively, after the same extended period of time.

In each plot, the center of mass $M \in R^2$ is the vector $(M_x, M_y)^T$, where M_x and M_y are the x and y components, respectively. The system indeed appears to exhibit a great dependence on initial conditions, a defining property of chaotic systems; moreover, two attractors of the system are seen in both plots.

Simple Properties of the Lorenz Equations

Nonlinearity: The two nonlinearities are xy and xz.

Symmetry: Equations are invariant under (x, y) → (−x, −y). Hence if (x(t), y(t), z(t)) is a solution, so is (−x(t), −y(t), z(t)).

Volume contraction: The Lorenz system is dissipative i.e. volumes in phase-space contract under the flow.

Fixed points: $(x^*, y^*, z^*) = (0, 0, 0)$ is a fixed point for all values of the parameters. For r > 1 there is also a pair of fixed points $C^{\pm}$ at $x^* = y^* = \pm\sqrt{b(r-1)}, z^* = r - 1$. These coalesce with the origin as r → 1+ in a pitchfork bifurcation.

Linear Stability of the Origin

Linearization of the original equations about the origin yields:

$\dot{x} = \sigma(y - x)$

$\dot{y} = rx - y$

$\dot{z} = -bz$

Hence, the z-motion decouples, leaving:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ with trace } \tau = -\sigma - 1 < 0 \text{ and determinant } \Delta = \sigma(1 - r).$$

For r > 1, origin is a saddle point since Δ < 0.

For r < 1, origin is a sink since $\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - r) = (\sigma - 1)^2 + 4\tau\sigma > 0 \rightarrow$ a stable node.

Actually, for r < 1 it can be shown that every trajectory approaches the origin as t → ∞ the origin is globally stable, hence there can be no limit cycles or chaos for r < 1.

Subcritical Hopf Bifurcation

Subcritical Hopf Bifurcation occurs when limit cycle is outside of the phase plane. At a Subcritical Hopf Bifurcation, trajectories must fly off to a distant attractor.

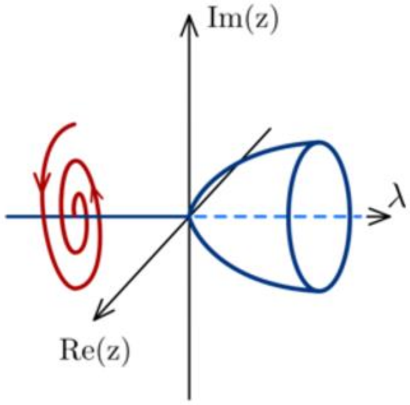

Figure.7: Representation of Subcritical Hopf Bifurcation, limit cycle is outside the phase plane and trajectories fly off to a distant attractor.

Subcritical Hopf bifurcation occurs at:

$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \equiv \mathrm{r_H} => 1, assuming\ that\ a - b - 1 > 0.$$

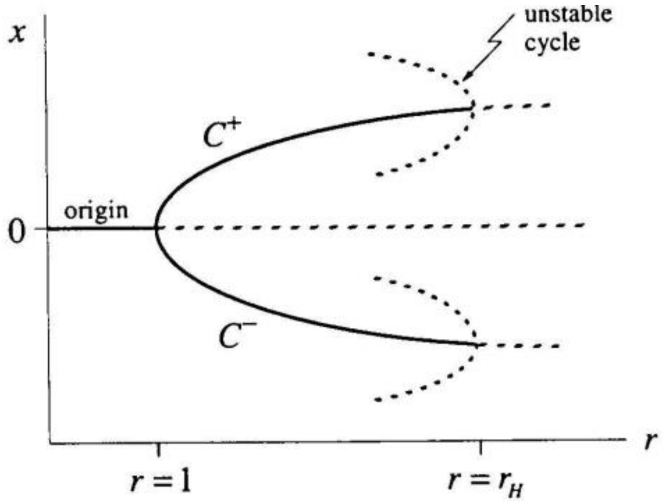

Figure.8: Partial bifurcation diagram

This 2nd attractor must have some strange properties, since any limit cycles for $r > r_H$ are unstable ("proof" by Lorenz). The trajectories for $r > r_H$ are therefore continually being repelled from one unstable object to another. At the same time, they are confined to a bounded set of zero volume, yet manage to move in this set forever without intersecting, which means a strange attractor occurs!

Chaos on a Strange Attractor

Lorenz considered the case σ = 10, b = 8/3, r = 28 with $(x_0, y_0, z_0) = (0, 1, 0)$.

Putting these numbers into the equation:

$r_H = \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \rightarrow r_H \approx 24.74$ hence $r > r_H$. The resulting solution $y(t)$ becomes:

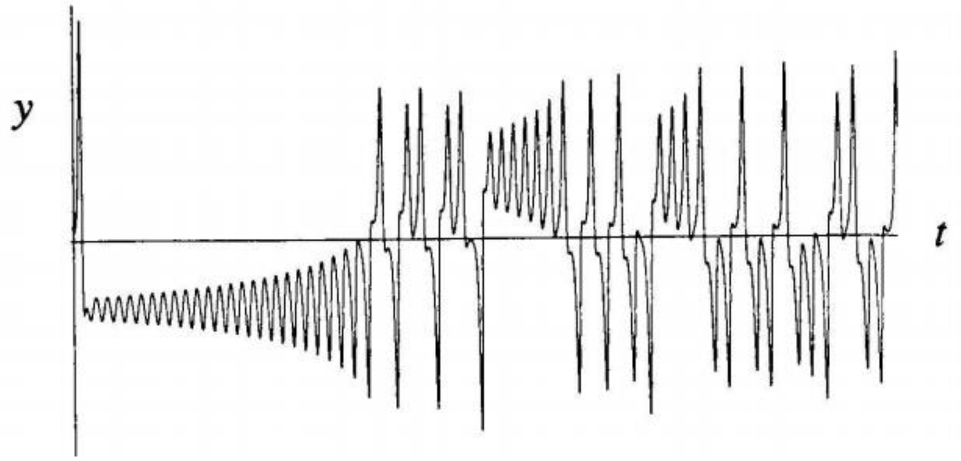

Figure.9: The chaos on the strange attractor; σ = 10, b =$8/3$, r = 28 with $(x_0, y_0, z_0) = (0, 1, 0)$, similar results would be gotten investigating x(t) - t or z(t) - t aspects.

After an initial transient, the solution settles into an irregular oscillation that persists as t → ∞ but never repeats exactly. The motion is aperiodic.

Lorenz discovered that a wonderful structure emerges if the solution is visualized as a trajectory in phase space. For instance, when x(t) is plotted against z(t), the famous butterfly wing pattern appears:

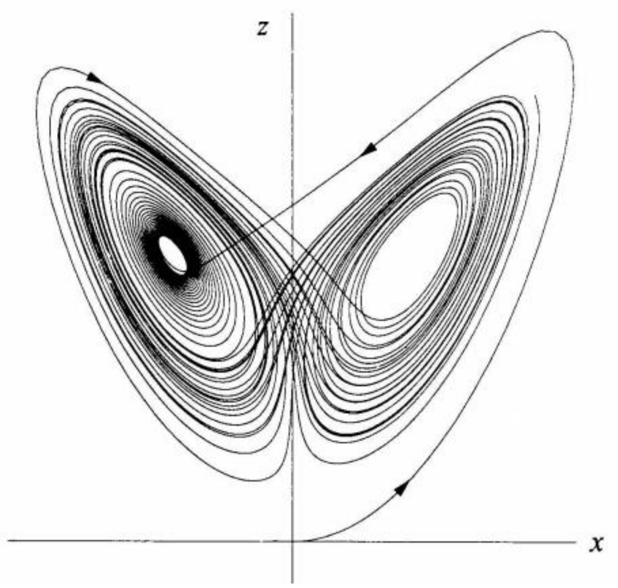

Figure.10: x(t) – z(t) graph at a strange attractor that looks like butterfly wings

The trajectory appears to cross itself repeatedly, but that's just an artifact of projecting the 3-dimensional trajectory onto a 2-dimensional plane. In 3-D no crossings occur.

The number of circuits made on either side varies unpredictably from one cycle to the next. The sequence of the number of circuits in each lobe has many of the characteristics of a random sequence.

When the trajectory is viewed in all 3 dimensions, it appears to settle onto a thin set that looks like a pair of butterfly wings. It is called a strange attractor and it can be shown schematically as follows:

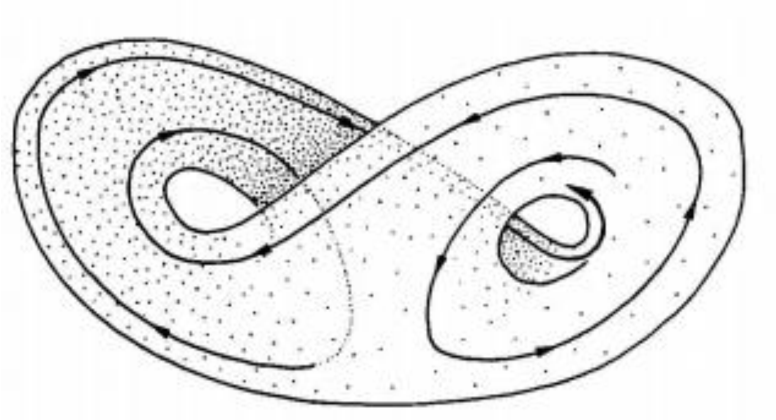

Figure.11: The trajectory resulted from a strange attractor that looks like a pair of butterfly wings

What is the geometric structure of the strange attractor?

The uniqueness theorem means that trajectories cannot cross or merge, hence the two surfaces of the strange attractor can only appear to merge. Lorenz concluded that "there is an infinite complex of surfaces" where they appear to merge. Today this "infinite complex of surfaces" would be called a fractal. A fractal is a set of points with zero volume but infinite surface area.

Exponential divergence of nearby trajectories

The motion on the attractor exhibits sensitive dependence on initial conditions. Two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures. The practical implication is that long-term prediction becomes impossible in a system like this, where small uncertainties are amplified enormously fast. Suppose we let transients to decay so that the trajectory is "on" the attractor. Suppose x(t) is a point on the attractor at time t, and consider a nearby point, x(t) + δ(t), where δ is a tiny separation vector of initial length $||\delta_0|| = 10^{-15}$.

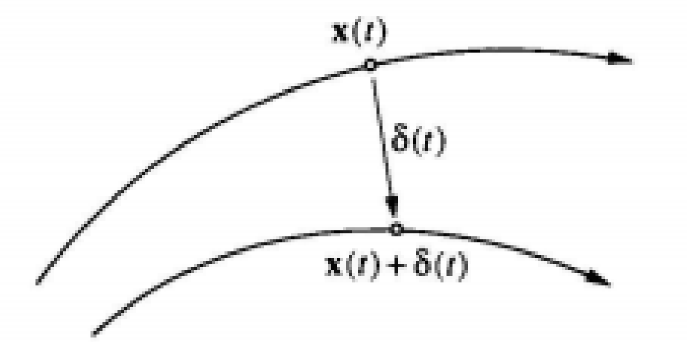

Figure.12: Representation of two different points of x separated by a tiny distance δ

This very tiny difference in the initial conditions results a huge difference in the behavior as follows:

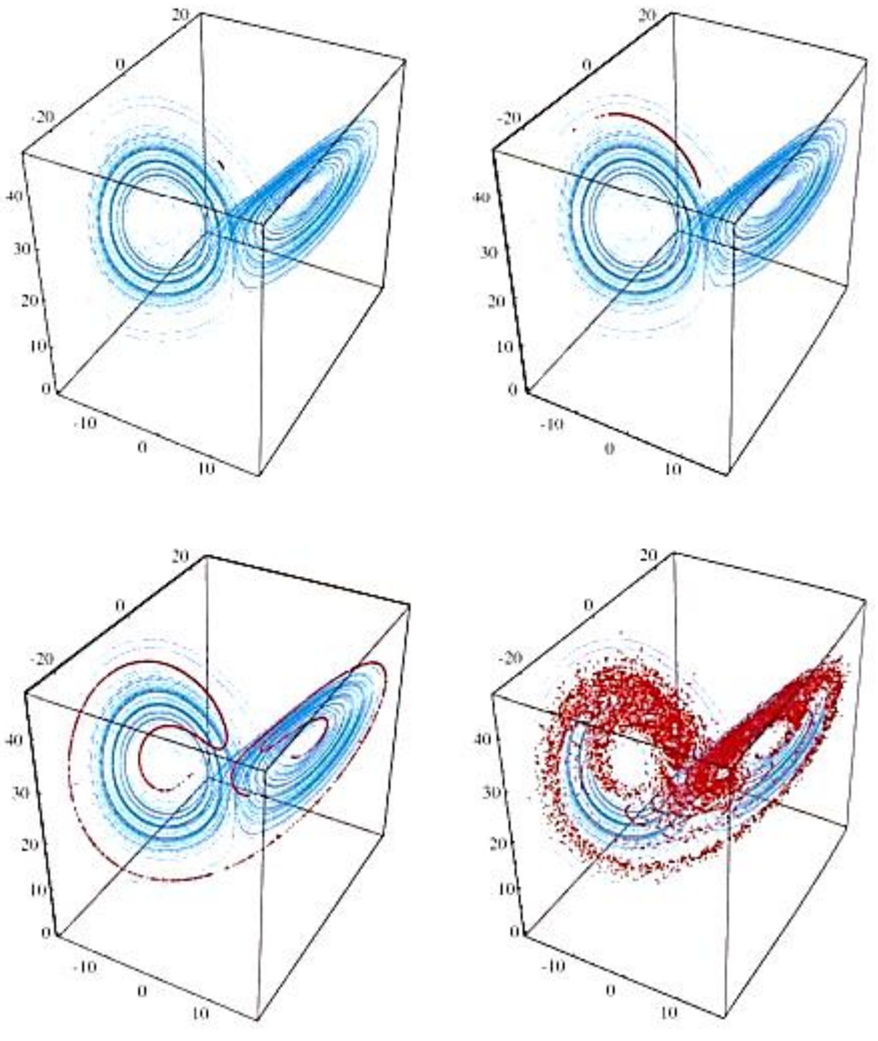

Figure.13: Sensitivity to initial conditions on Lorenz attractor

In numerical studies of the Lorenz attractor, one finds that $\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$, where $\lambda \approx 0.9$. Hence neighboring trajectories separate exponentially fast. Equivalently, if $\ln\|\delta(t)\|$ is plotted versus t, a curve that is close to a straight line with a positive slope λ is found.

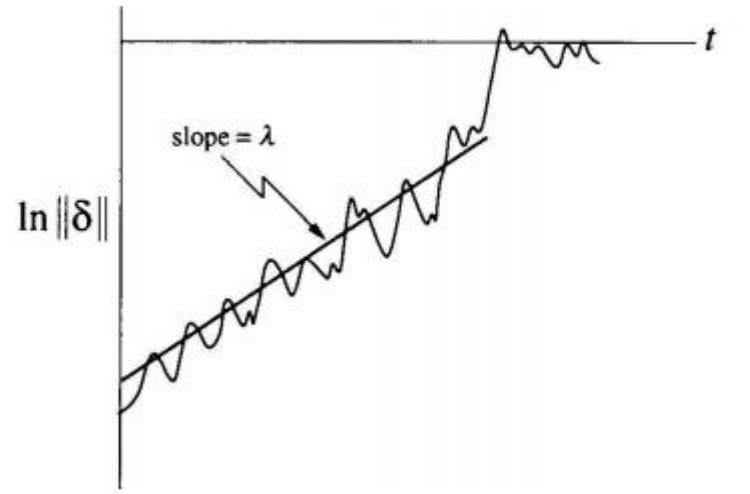

Figure.14: The plot of $\ln\|\delta(t)\|$ versus t

The curve is never straight, but has wiggles since the strength of exponential divergence varies somewhat along the attractor. The exponential divergence must stop when the separation is comparable to the “diameter” of the attractor - the trajectories cannot get any further apart (curve saturates for large t).

Defining Chaos

No definition of the term “chaos” is universally accepted, even now, but almost everyone would agree on the three ingredients used in the following working definition:

Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions:

- Aperiodic long-term behavior means that there are trajectories which do not settle down to fixed points, periodic or quasiperiodic orbits as $t \to \infty$.
- Deterministic means that the system has no random or noisy inputs or parameters. Irregular behavior arises solely from the system’s nonlinearity.
- Sensitive dependence on initial conditions means that nearby trajectories diverge exponentially fast.

Some people think that chaos is just a fancy word for instability. For example, the system $\dot{x} = x$ is deterministic and shows exponential separation of nearby trajectories. However, it should not be considered to be chaotic. Trajectories are repelled to infinity, and never return. Hence infinity is a fixed point of the system, and ingredient 1. above specifically excludes fixed points.

Defining "attractor" and "strange attractor"

The term attractor is also difficult to define in a rigorous way. Loosely, an attractor is a set of points to which all neighboring trajectories converge. Stable fixed points and stable limit cycles are examples. More precisely, an attractor is defined to be a closed set A with the following properties:

1) A is an invariant set: any trajectory x(t) that starts in A stays in A for all time.
2) A attracts an open set of initial conditions: there is an open set U containing A such that if $x(0) \in U$, then the distance from x(t) to A tends to zero as $t \to \infty$. Hence A attracts all trajectories that start sufficiently close to it. The largest such U is called the basin of attraction of A.
3) A is minimal: there is no proper subset of A that satisfies conditions 1. and 2.

To understand better what is going on it is beneficial to investigate an example:

$\dot{x} = \mu x - x^3$

$\dot{y} = -y$

Let I denote the interval $-1 \leq x \leq 1, y = 0$.

Is "I" an attractor?

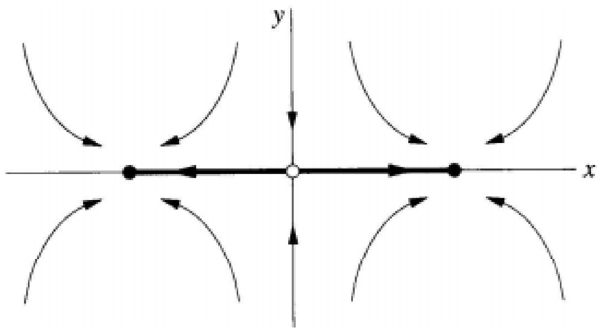

Figure.15: The interval $-1 \leq x \leq 1$ and $y = 0$

So, I is an invariant set (condition 1.). Also, "I" attracts an open set of initial conditions - it attracts all trajectories in the xy-plane. But "I" is not an attractor because it is not minimal. The stable fixed points (±1, 0) are proper subsets of I that also satisfy conditions 1. and 2. These points are the only attractors for the system.

A definition can be made a strange attractor to be an attractor that exhibits sensitive dependence on initial conditions.

Strange attractors were originally called strange because they are often (but not always) fractal sets. Nowadays, this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions. The terms "chaotic attractor" and "fractal attractor" are used when one wishes to emphasize one or other of these aspects.

The Lorenz Map

Consider the following view of the Lorenz strange attractor:

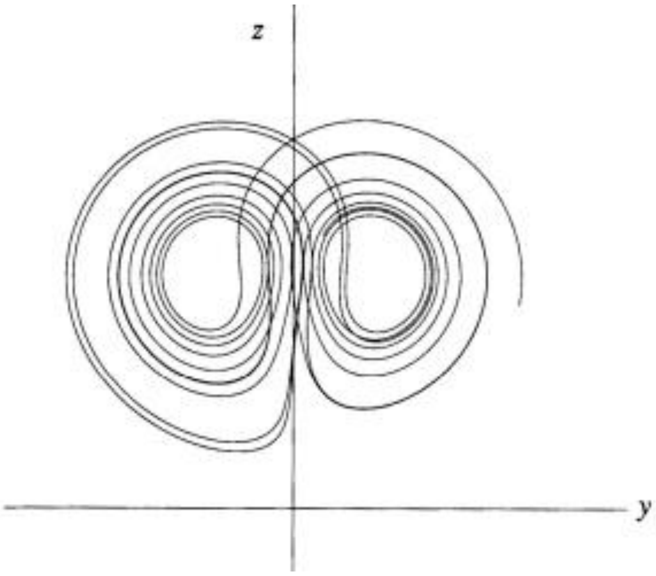

Figure.16: An example of a view of the Lorenz strange attractor

Lorenz wrote that "the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit."

The "single feature" he focused on was z_n, the n^{th} local maximum of z(t).

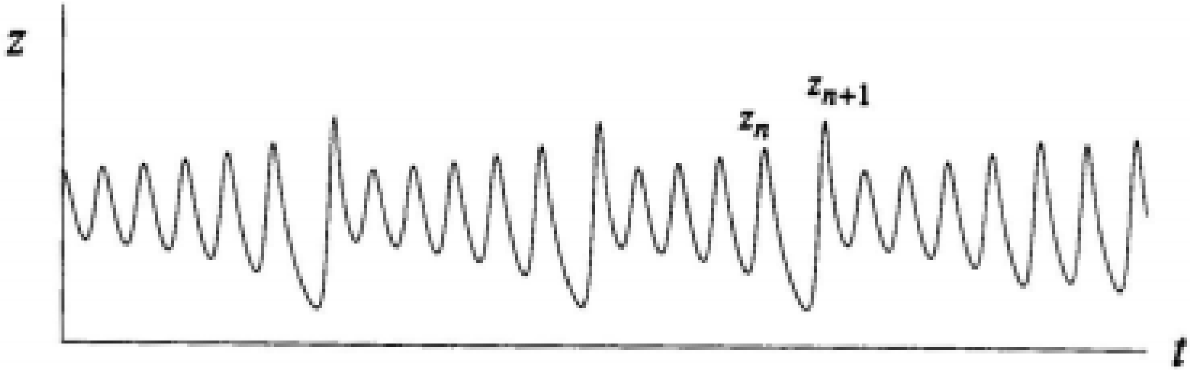

Figure.17: Representation of n^{th} local maximum of z(t).

Lorenz's idea is that z_n should predict z_{n+1}. He checked this by numerical integration. The plot of z_{n+1} vs z_n looks like:

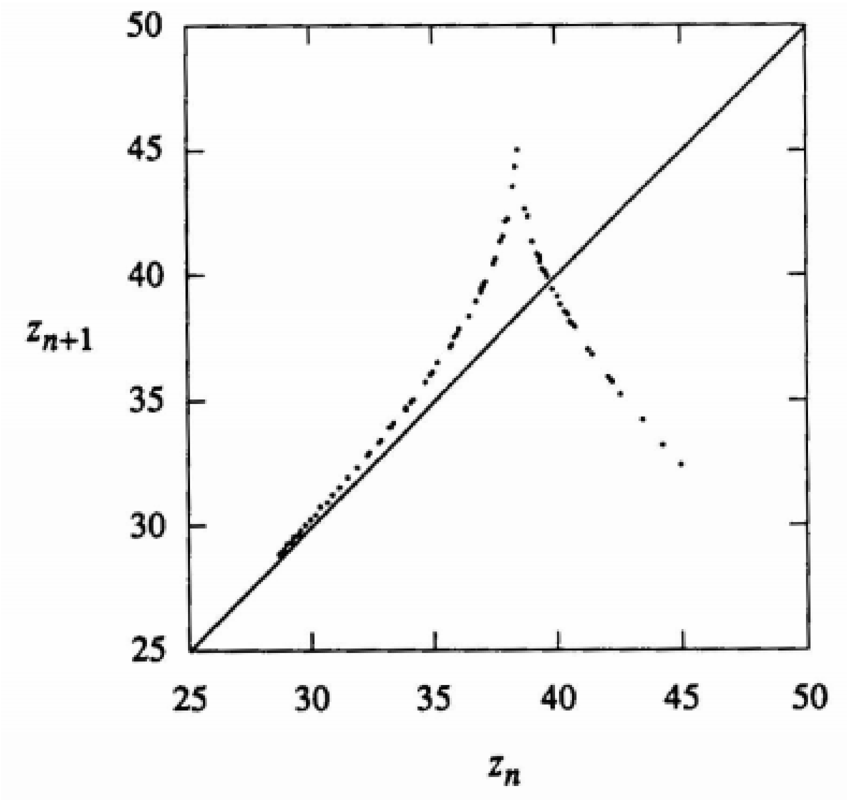

Figure.18: The plot of z_{n+1} vs z_n

The data from the chaotic time series appears to fall neatly on a curve, there is no "thickness" to the graph. Hence Lorenz was able to extract order from chaos.

The function $z_{n+1} = f(z_n)$ is now called the Lorenz Map. Some uncertainty on the map:

- The graph is not actually a curve, it does have some thickness so, f(z) is not a perfectly-defined function.
- The Lorenz Map reminds us of the Poincare map, but there is a distinction, the Poincare map tells us how the two coordinates of a point on a surface change after first return to the surface, while the Lorenz Map characterizes the trajectory by only one number. This approach only works if the attractor is very "flat" (close to 2-D).

Exploring Parameter Space

So far it has been concentrated mainly on the case σ = 10, b = 8/3, r = 28 as in Lorenz's original work published in 1963.

Changing parameters is like a walk through the jungle! One finds many sorts of exotic behaviors, such as exotic limit cycles tied in knots, pairs of limit cycles linked through each other, intermittent chaos, noisy periodicity, as well as strange attractors. In fact, much remains to be discovered.

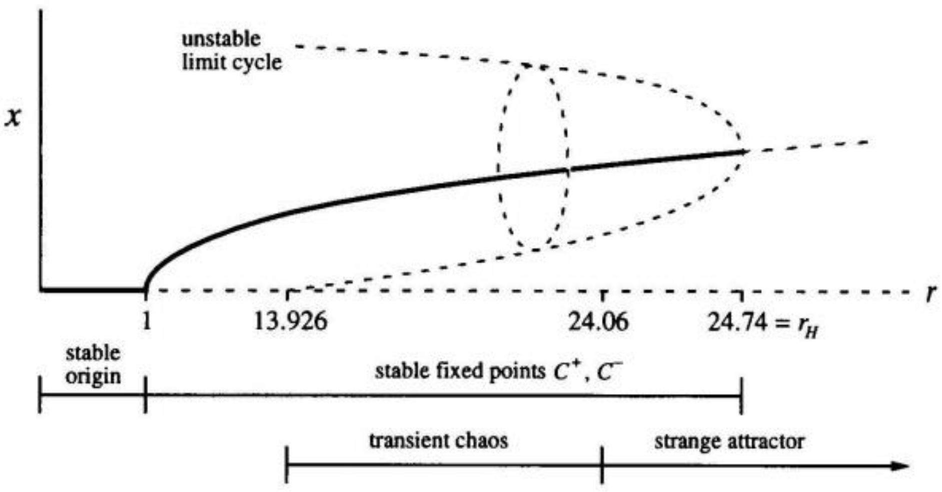

Figure.19: Changing unstable limit cycle with changing r

To give an example to clarify:

Consider r = 21, with σ = 10 and b = 8/3 as usual. The solution exhibits transient chaos, in which the trajectory seems to be tracing a strange attractor, but eventually spirals towards C^+ or C^- which are the stable fixed points.

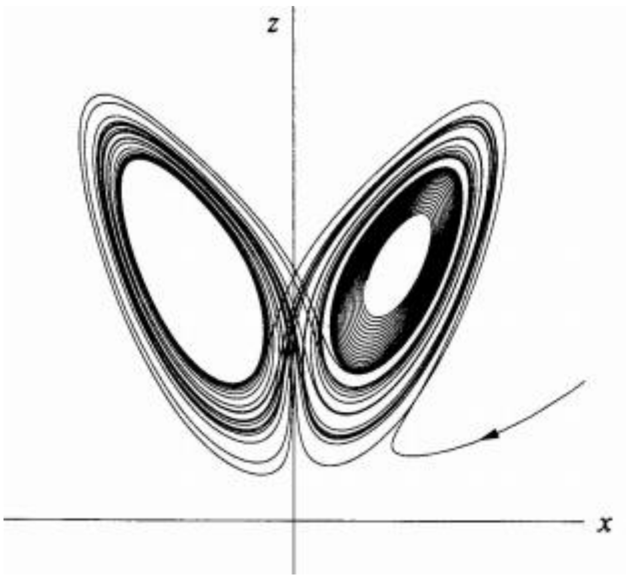

Figure.20: Strange attractor with r = 21, σ = 10 and b = 8/3

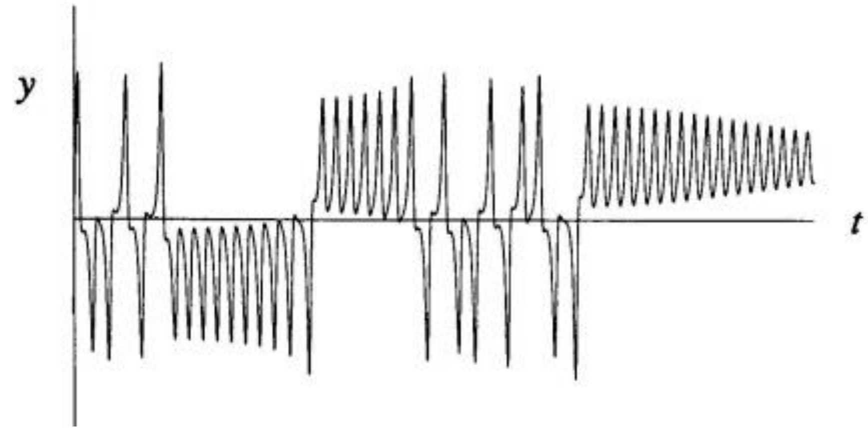

Figure.21: The chaos on the strange attractor; σ = 10, b =8/3 , r = 21 with $(x_0, y_0, z_0) = (0, 1, 0)$, similar results would be gotten investigating x(t) - t or z(t) - t aspects.

The long-term behavior is not aperiodic; hence the dynamics are not chaotic. However, the dynamics do exhibit sensitive dependence on initial conditions, the trajectory could end up on either C^+ or C^-, hence the system's behavior is unpredictable. Transient chaos shows that a deterministic system can be unpredictable, even if its final states are very simple. This is familiar from everyday experience, many games of "chance" used in gambling are essentially demonstrations of transient chaos, e.g., a rolling dice.

For large values of r, the Lorenz Map becomes symmetrical from the looks of x-y, x-z and y-z as follows:

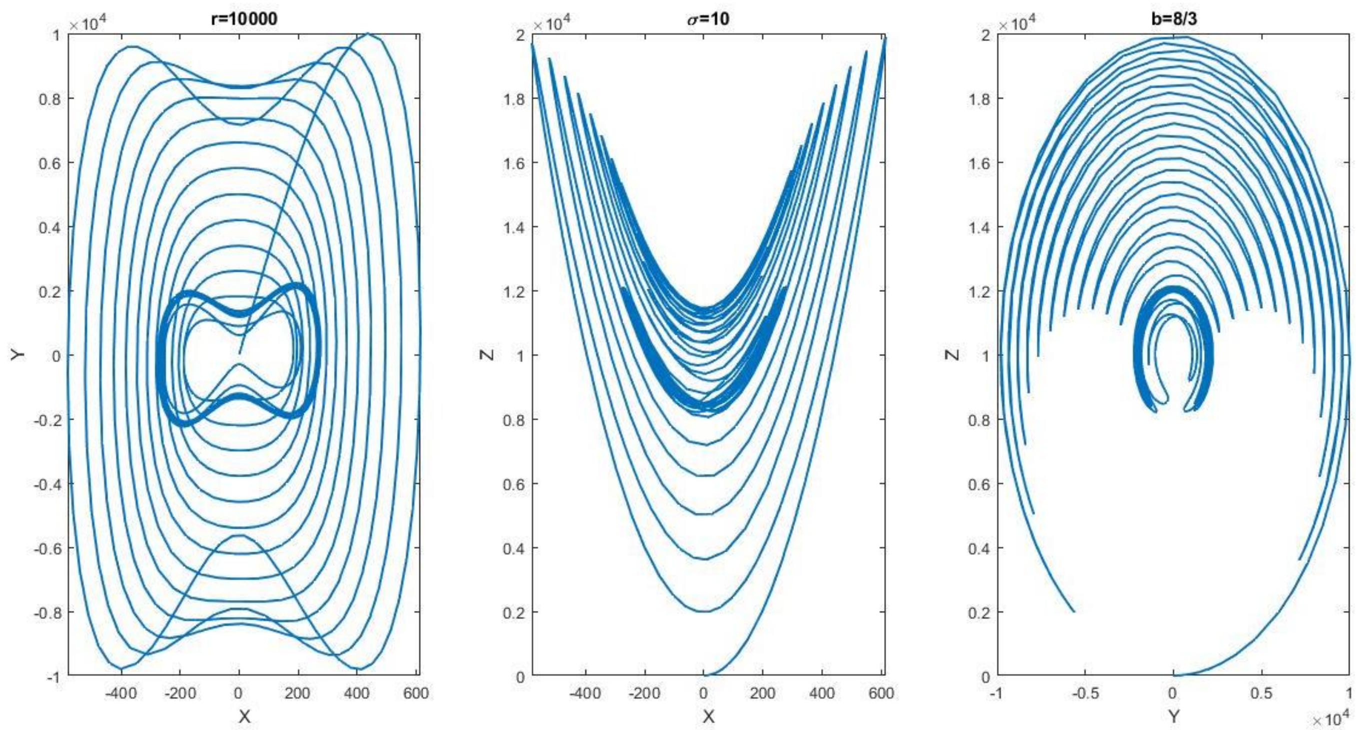

Figure.22: σ = 10, b =8/3 and very large r = 10000

4) SOLUTION – RESULTS

The Rössler attractor is the attractor for the Rössler system, a system of three non-linear ordinary differential equations originally studied by Otto Rössler in the 1970's. These differential equations define a continuous-time dynamical system that exhibits chaotic dynamics associated with the fractal properties of the attractor. Some properties of the Rössler system can be deduced via linear methods such as eigenvectors, but the main features of the system require non-linear methods such as Poincaré maps and bifurcation diagrams. The original Rössler paper states the Rössler attractor was intended to behave similarly to the Lorenz attractor, but also be easier to analyze qualitatively. An orbit within the attractor follows an outward spiral close to the x, y plane around an unstable fixed point. Once the graph spirals out enough, a second fixed point influences the graph, causing a rise and twist in the z-dimension. In the time domain, it becomes apparent that although each variable is oscillating within a fixed range of values, the oscillations are chaotic. This attractor has some similarities to the Lorenz attractor, but is simpler and has only one manifold.

The defining equations of the Rössler system are:

$$\frac{dx}{dt} = -\mathrm{y} - \mathrm{z},$$

$$\frac{dx}{dt} = \mathrm{x} + ay,$$

$$\frac{dx}{dt} = \mathrm{b} + \mathrm{z}(\mathrm{x} - \mathrm{c}), \qquad \text{where a, b and c are parameters}$$

Rössler studied the chaotic attractor with a = 0.2, b = 0.2 and c = 5.7, though properties of a = 0.2, b = 0.2 and c = 14 have been more commonly used since. Another line of the parameter space was investigated using the topological analysis. It corresponds to b = 2, c = 4, and a was chosen as the bifurcation parameter.

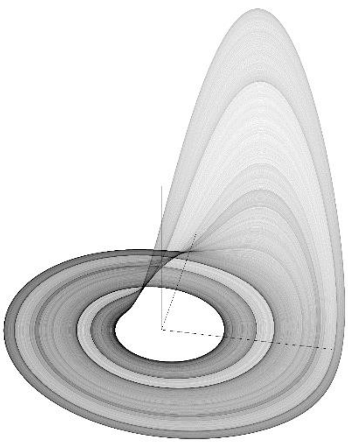

Figure.23: Strange attractor of the Rössler System

PART A:

$$\frac{dx}{dt} = -\mathrm{y} - \mathrm{z},$$

$$\frac{dx}{dt} = \mathrm{x} + ay,$$

$$\frac{dx}{dt} = \mathrm{b} + \mathrm{z}(\mathrm{x} - \mathrm{c}),$$

In the above system of equations, the parameters are taken as a = 0.2, b = 0.2 and c = 5. The time span is chosen from 0 to 1000 and increasing with 0.01 step to step. Initial values are taken as 3. The system of equation of the Rössler system is solved by using Runge-Kutta 4/5 method. Then, the corresponding strange attractor becomes as follows:

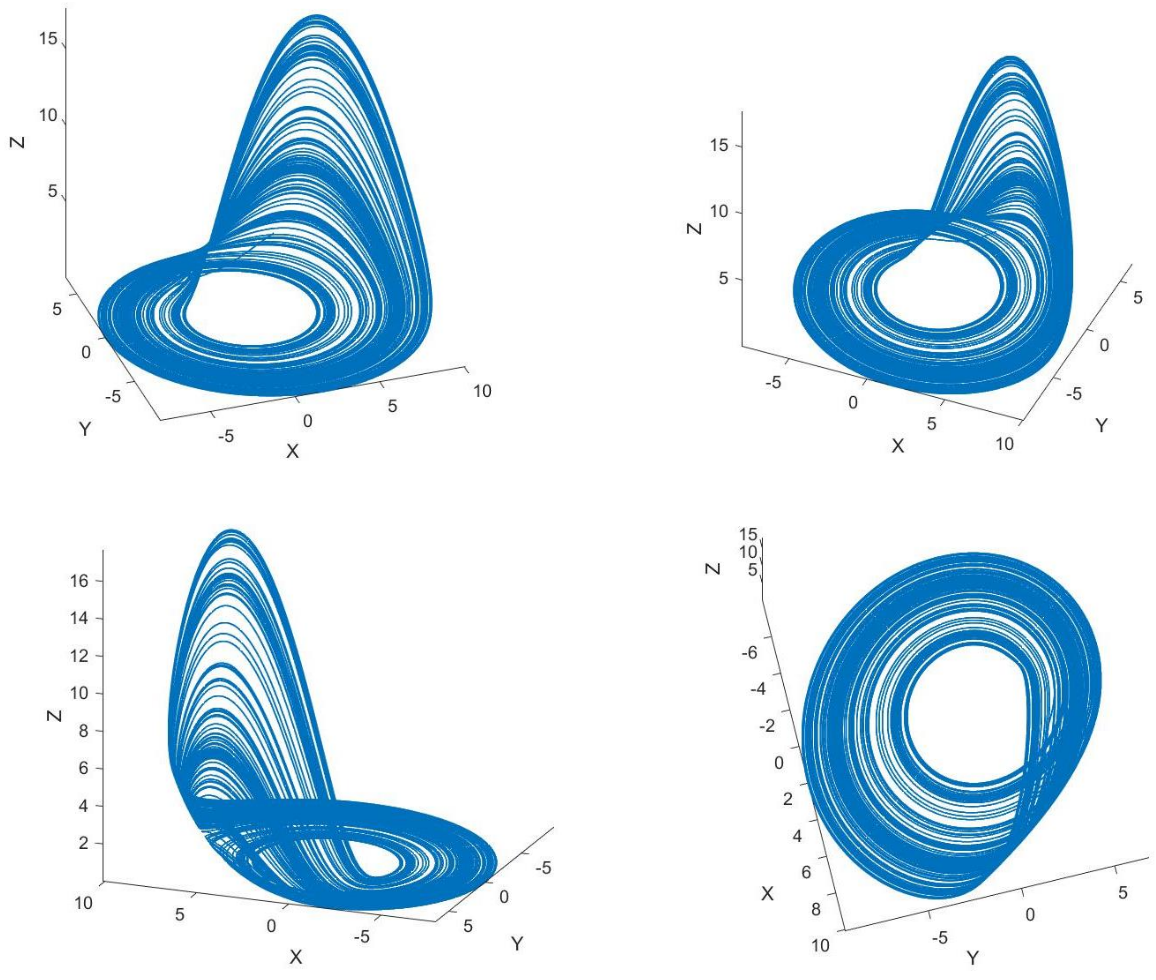

Figure.24: Different views of the strange attractor of the Rössler System with parameters: a = 0.2, b = 0.2 and c = 5.

The strange attractor takes its shape while time elapses. The shape starts to increase its number of periods with time until a certain point, after that, chaos occurs and continuous to grow as number of periods increases and accumulates which results in chaos.

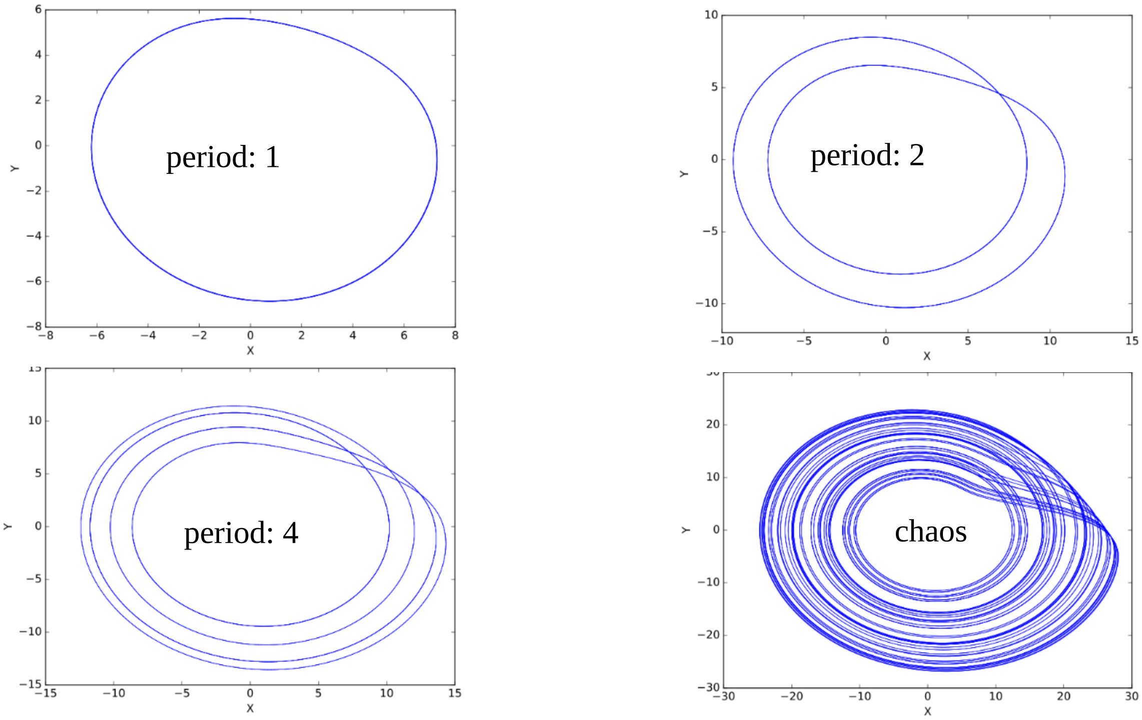

Figure.25: The shape of the strange attractor according to time: after a while chaos occurs and grows.

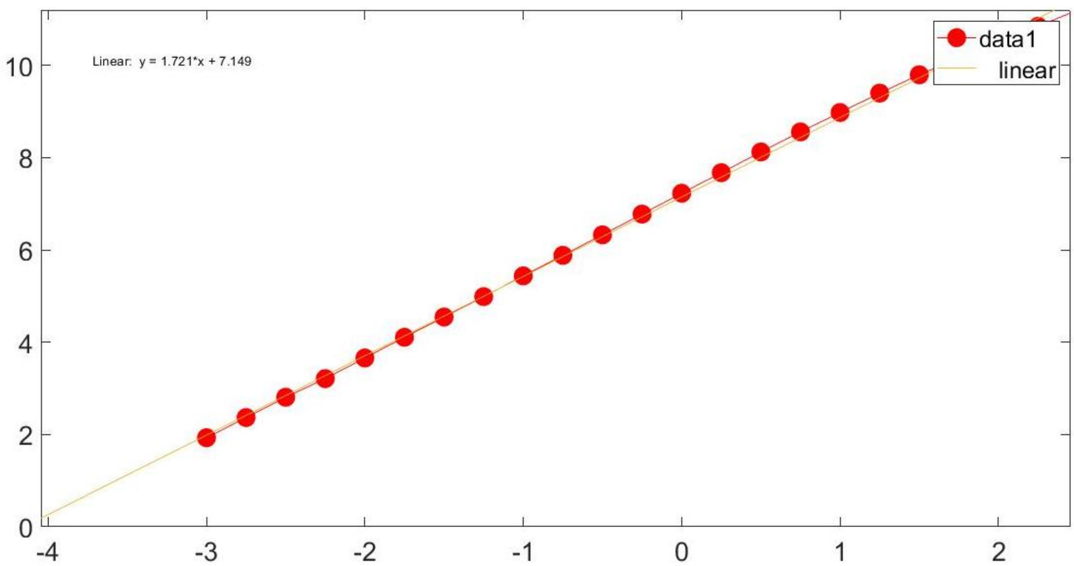

Figure.26: The fractal dimension plot of the strange attractor of the Rössler System

The best-fit shown with orange line has an equation of y = 1.721x + 7.149 and it shows how the periods of the strange attractors are increasing and aligned. It looks like the periods of the strange attractor are well aligned since the red plot doesn't deviate too much from the orange best line.

PART B:

Even very small changes in the size may occur huge differences in the strange attractors of the systems. The difference is shown below, in the one of the axes, z.

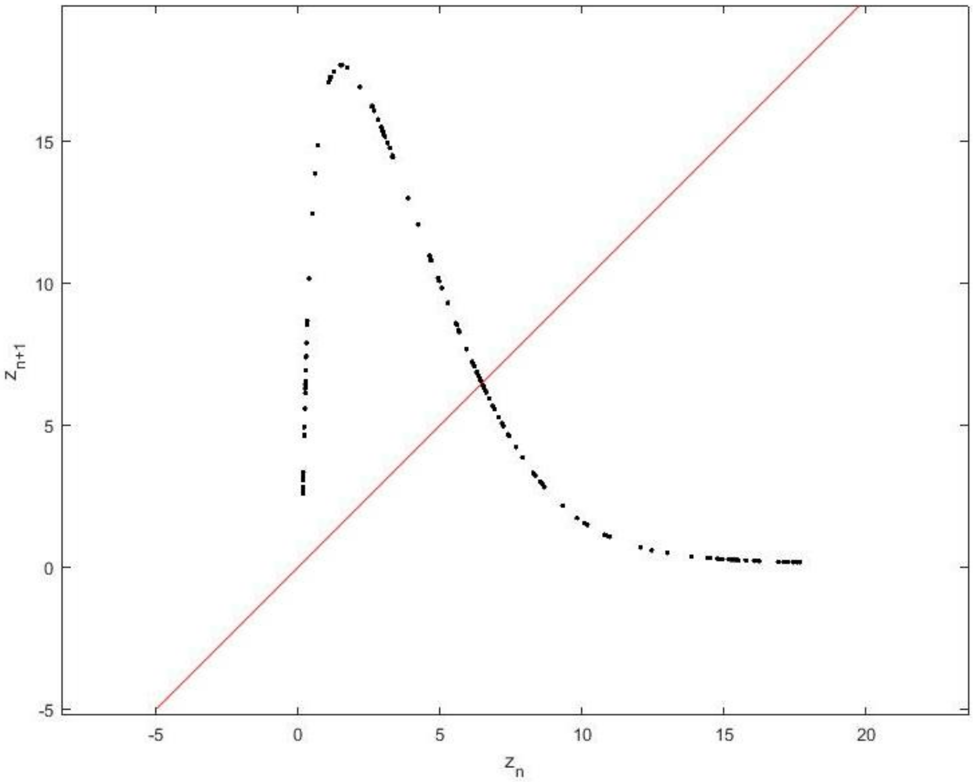

Figure.27: z_n versus z_{n+1} plot.

While the black points representing the z_n versus z_{n+1} plot look like creating a continuous line, the real case is not so due to the occurring chaos. The zoomed part of the one portion of the plot above is as follows:

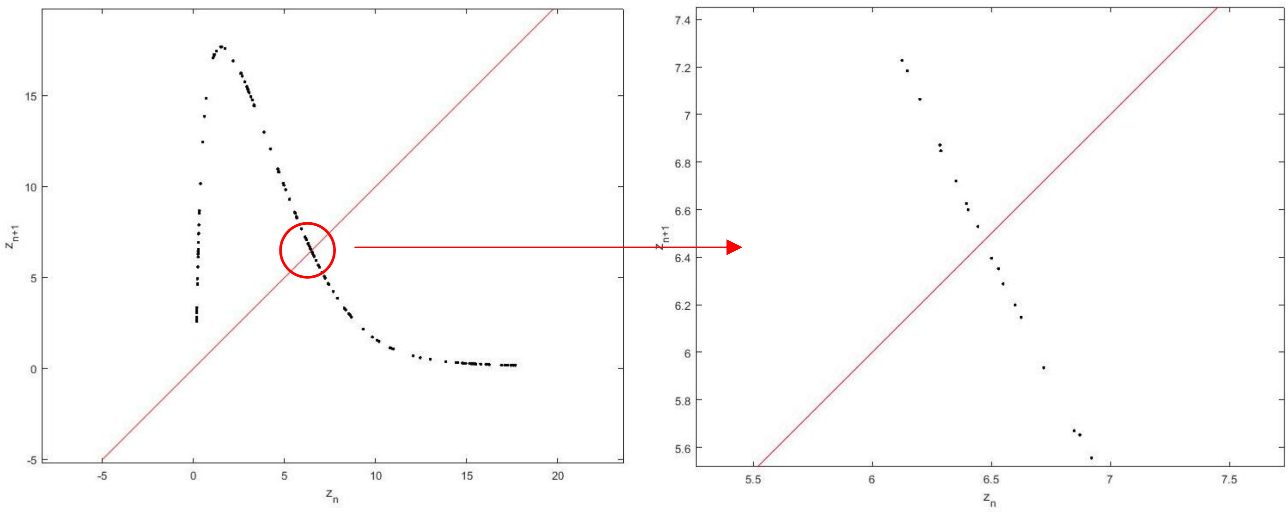

Figure.28: Zoomed representation of one portion of the z_n vs z_{n+1} plot: the points are not perfectly linear indeed.

PART C:

$$\frac{dx}{dt} = -\mathrm{y} - \mathrm{z},$$

$$\frac{dx}{dt} = \mathrm{x} + ay,$$

$$\frac{dx}{dt} = \mathrm{b} + \mathrm{z}(\mathrm{x} - \mathrm{c})$$

The parameters are initially set to be a = 0.2, b = 0.2 and c = 2 but in this case, c is increased with time and a bifurcation diagram is construct according to this change.

The time span is chosen from 0 to 100 and increasing with 0.001 step to increase the proximity of the data points to each other and make them look like a plot. The system of equation of the Rössler system is solved by using Runge-Kutta 4/5 method. c is increased with each time step by 0.01 from 2 to 6. Then, the corresponding bifurcation diagram becomes as follows:

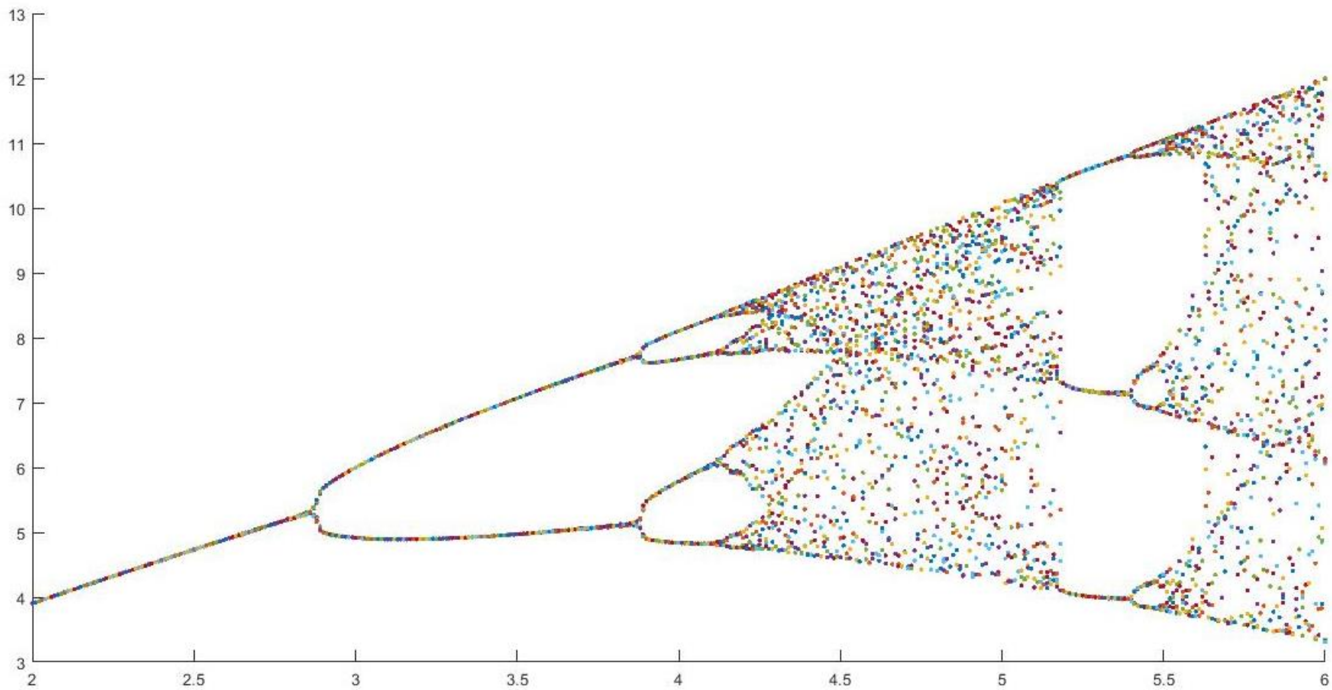

Figure.29: The bifurcation diagram caused by the time-uniformly changed c parameter from 2 to 6

Number of 1000 iteration is used to get closer to the Feigenbaum constant. If I tried to solve the system with a larger iteration number, I would be able to get the error closer to 0.

I used 3 consecutive exponential numbers, which are -0.04344, -0.02961 and -0.04104.

$$\frac{e^{-0.04104} - e^{-0.02961}}{e^{-0.04343} - e^{-0.04104}} = 4.8156$$

Actual Feigenbaum constant = 4.6692...

$$\text{The error} = \frac{4.8156 - 4.6692}{4.6692} = 3.14\ \%$$

This error can be decreased using more iteration number but, in that case, the code will be less efficient to run.

5) CONCLUSION

In part C, a = b = 0.2 and c changes between 2 and 6 increasingly. The bifurcation diagram reveals that low values of c are periodic, but quickly become chaotic as c increases. This pattern repeats itself as c increases – there are sections of periodicity interspersed with periods of chaos, and the trend is towards higher-period orbits as c increases. For example, the period one orbit only appears for values of c around 4 and is never found again in the bifurcation diagram. The same phenomenon is seen with period 3; until c becomes 12. period three orbits can be found, but thereafter, they do not appear.

When the system is iterated more, the result gets closer to Feigenbaum constant (4.6692…). However, this increased iteration number means the less efficiency for the code to work. In theory, with the infinite iteration of the system, one can get the same as the Feigenbaum constant. It is not possible to solve such a code for sure.

6) CODE REVIEW

A SMALL WARNING: WHEN YOU RUN THE CODE, PLEASE DO NOT PLAY WITH PLOTS OR CHANGE THEIR POSITIONS UNTIL RUNNING IS DONE SINCE IT CAN HAVE PROBLEM WHILE OPERATING!!!

```
clear;
clc;
clf;

a = 0.2; b= 0.2; c= 5; %defining parameters
x0 = [3 3 3]; %defining the initial values
tspan = [0:0.01:1000]; %defining the range and iteration
```

```

[t,x] = ode45(@(t,x) odefcn(x,a, b, c), tspan, x0); %solution with Runge-Kutta 4/5
figure(1) % to open the result in a seperate window
plot3(x(:,1),x(:,2),x(:,3),'LineWidth',1.2); % to plot the solution "Map of the Rössler system"
xlabel("X"); %x axis name
ylabel("Y"); %y axis name
zlabel("Z"); %z axis name
axis equal;  % equating the axis lengths
set(gca,'FontSize',15); %defining the font size
```

```

d = exp(-3:0.25:2.5)';
C(1:size(d,1),1) = 0;
N = size(x,1);
Nc = 1000;
r = randperm(N);
for n = 1 : Nc
    for m = 1 : size(d,1)
        C(m,1) = C(m,1) + sum((x(:,1)-x(r(n),1)).^2 ...
            + (x(:,2)-x(r(n),2)).^2 + (x(:,3)-x(r(n),3)).^2 < d(m,1)^2);
    end
end
C = C / Nc;
figure(2); % RESULT FOR PART A
plot(log(d),log(C),'.-r','MarkerSize',50); %fitting the fracture
set(gca,'FontSize',20);
```

```



dz = (x(3:N,3) - x(2:N-1,3)).*(x(2:N-1,3) - x(1:N-2,3)); % finding extremum
ddz = x(3:N,3) + x(1:N-2,3) - 2*x(2:N-1,3); % finding maximum
zn = x((dz<0).*(ddz<0)==1,3); %defining z_n

figure(3) % RESULT FOR PART B
plot(zn(1:end-1),zn(2:end),'.k'); % plotting z_n vs z_n_+_1
xlabel('z_n') %x axis name
ylabel('z_n_+_1') %y axis name
line([-5 20],[-5 20],'Color','r'); %drawing the middle line
axis equal;
```

```
a = 0.2; b= 0.2; %defining parameters a and b
tspan = [0:0.001:100]; %defining the range and iteration with increase 0.001

figure(4) % RESULT FOR PART C
hold on;
c_value = (2:0.01:6); % changing c from 2 to 6 with 0.01 increase at each step

for nc = 1 : size(c_value,2) % solution via Runge-Kutta 4/5 at each step
    c = c_value(nc);
    [t,x] = ode45(@(t,x) odefcn(x,a,b,c), tspan, x0);
    x0 = x(end,:);
    [t,x] = ode45(@(t,x) odefcn(x,a,b,c), tspan, x0);
    N = size(x,1);
    dz = (x(3:N,1) - x(2:N-1,1)).*(x(2:N-1,1) - x(1:N-2,1)); % find extremum
    ddz = x(3:N,1) + x(1:N-2,1) - 2*x(2:N-1,1); % find maximum
    zn = x((dz<0).*(ddz<0)==1,1);
    plot(c,zn(1:end),'.');
    drawnow; % adding each step on and on continuously
    x0 = x(end,:); %changing the initial value to the end value of result of the previous step
end


result = (exp(-0.04104)-exp(-0.02961))/(exp(-0.04343)-exp(-0.04104));
result % Feigenbaum constant is found

function dxdt = odefcn(x,a,b,c) %defining the equation system
dxdt = zeros(2,1);
dxdt(1) = -x(2) - x(3);
dxdt(2) = x(1) + a*x(2);
dxdt(3) = b + x(3)*(x(1) - c);
end
```