BAMS 508 – Discrete Optimization

Assignment 4

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Question 2

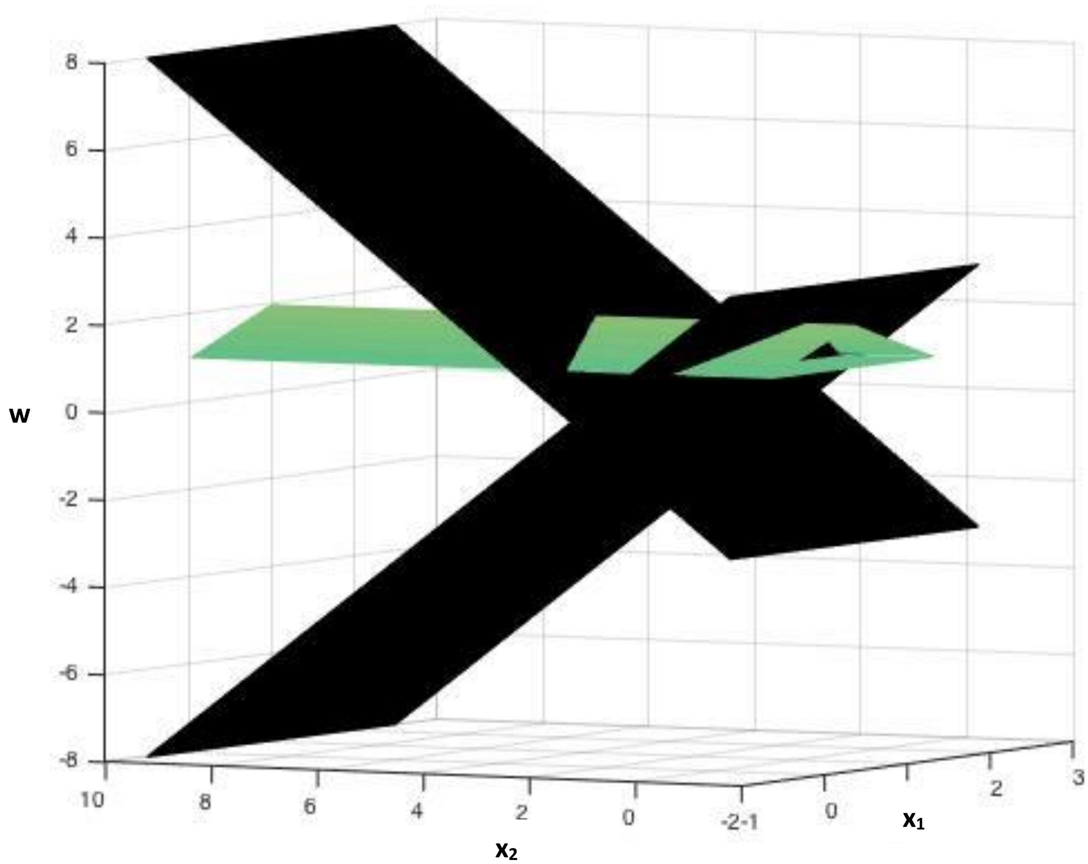
2. Projection. Let $P = \{ (x_1, x_2, w) \in \mathbf{R}^3 :$

$$\begin{array}{lrcl} (1) & x_1 \quad\quad + w & \le & 3 \\ (2) & -x_1 \quad\quad + w & \le & 1 \\ (3) & -w & \le & -1 \\ (4) & -x_2 - w & \le & -2 \\ (5) & x_2 - w & \le & 2 \;\}. \end{array}$$

(a) Construct a graphical representation of P in the (x_1, x_2, w)-space, with w as the "vertical" dimension. (Hint: the first three constraints, which do not involve x_2, define a "horizontal", unbounded triangular prism; the other two constraints cut this prism sideways.) List all 6 extreme points of P.

(b) Construct the projection $Q = proj_x P$ onto the subspace of the (x_1, x_2) coordinates. List the 6 constraints that define Q. Show how each of these 6 inequalities can be obtained as a nonnegative combination of the 5 inequalities defining P.

a. A graphical representation of *P* can be found below. It is the region inside the triangular prism and between the two intersecting planes.



To find the extreme points, these are points of intersection among two of the planes from the triangular prism and one of the two planes that cut the prism sideways.

Given the constraints below, solve the 6 combinations of constraints as described (i.e. systems of equations) to get the 6 extreme points.

$$\begin{cases} (1) \quad x_1 \qquad\qquad + w \leq \ \ 3 \\ (2) \ - x_1 \qquad\qquad + w \leq \ \ 1 \\ (3) \qquad\qquad\qquad - w \leq -1 \\ (4) \qquad\qquad - x_2 - w \leq -2 \\ (5) \qquad\qquad\quad x_2 - w \leq \ \ 2 \end{cases}$$

Point 1: $(1), (2), (4) \leftrightarrow x_1 = \mathbf{1}, x_2 = \mathbf{0}, w = \mathbf{2}$

Point 2: $(1), (2), (5) \leftrightarrow x_1 = \mathbf{1}, x_2 = \mathbf{4}, w = \mathbf{2}$

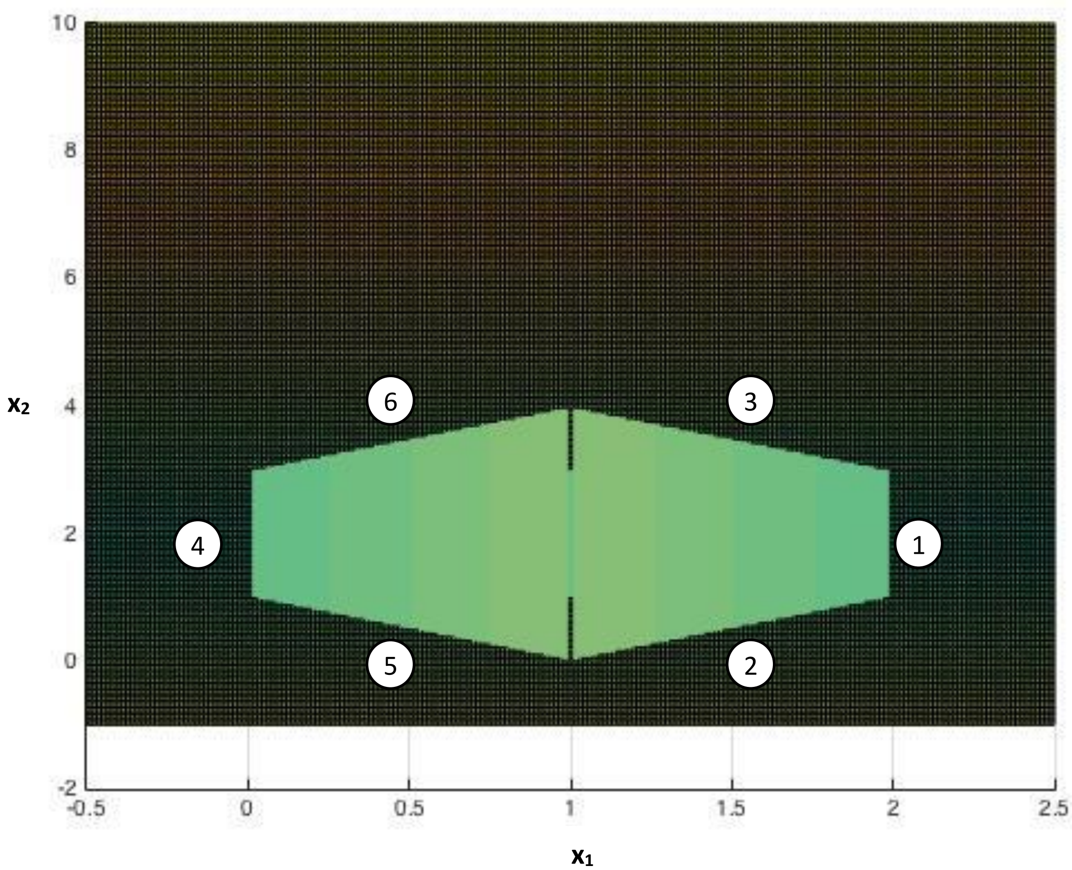
Point 3: $(1), (3), (4) \leftrightarrow x_1 = \mathbf{2}, x_2 = \mathbf{1}, w = \mathbf{1}$

Point 4: $(1), (3), (5) \leftrightarrow x_1 = \mathbf{2}, x_2 = \mathbf{3}, w = \mathbf{1}$

Point 5: $(2), (3), (4) \leftrightarrow x_1 = \mathbf{0}, x_2 = \mathbf{1}, w = \mathbf{1}$

Point 6: $(2), (3), (5) \leftrightarrow x_1 = \mathbf{0}, x_2 = \mathbf{3}, w = \mathbf{1}$

b. A projection of P on the (x_1, x_2) plane is as follows:



The constraints that define Q are basically linear inequalities that are intersections of 2 planes that form each side of the projection. These can be obtained by adding the inequalities that define these 2 planes.

Given the same constraints in (a),

Constraint 1: $(1) + (3) \leftrightarrow \boldsymbol{x_1 \leq 2}$

Constraint 2: $(1) + (4) \leftrightarrow \boldsymbol{x_1 - x_2 \leq 1}$

Constraint 3: $(1) + (5) \leftrightarrow \boldsymbol{x_1 + x_2 \leq 5}$

Constraint 4: $(2) + (3) \leftrightarrow \boldsymbol{-x_1 \leq 0}$

Constraint 5: $(2) + (4) \leftrightarrow \boldsymbol{-x_1 - x_2 \leq -1}$

Constraint 6: $(2) + (5) \leftrightarrow \boldsymbol{-x_1 + x_2 \leq 3}$

Question 3

<u>3. Valid inequalities and convex hulls.</u> For each of the three sets below, find a missing valid inequality and verify graphically that its addition to the formulation gives the convex hull of X

(i) $X = \{x \in \mathbf{B}^2 : 0 \leq x \leq 1;\ \ 3\,x_1 - 4\,x_2 \leq 1\}$

(ii) $X = \{(x, y) \in \mathbf{R} \times \mathbf{B} : 0 \leq x \leq 7;\ \ 0 \leq y \leq 1;\ \ x \leq 20\,y\}$

(iii) $X = \{(x, y) \in \mathbf{R} \times \mathbf{Z} : 0 \leq x \leq 16;\ y \geq 0;\ x \leq 6\,y\}$

(answers on following pages*)

3(i) $X = \{ x \in B^2 : 0 \leq x \leq 1 ;\ 3x_1 - 4x_2 \leq 1 \}$

when $x_1 = 0$

$-4x_2 \leq 1$

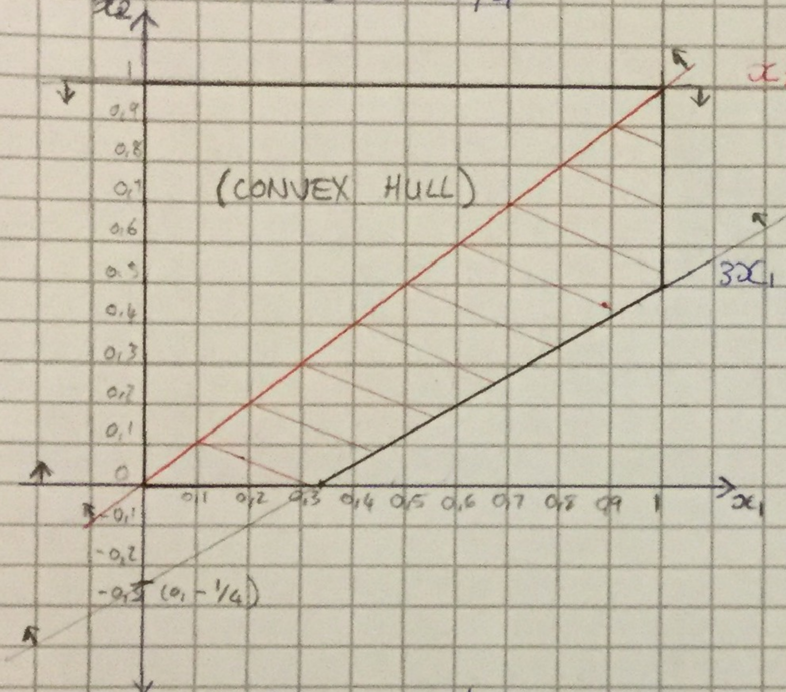
$-x_2 \leq 1/4$

$x_2 \geq -1/4$

when $x_2 = 0$

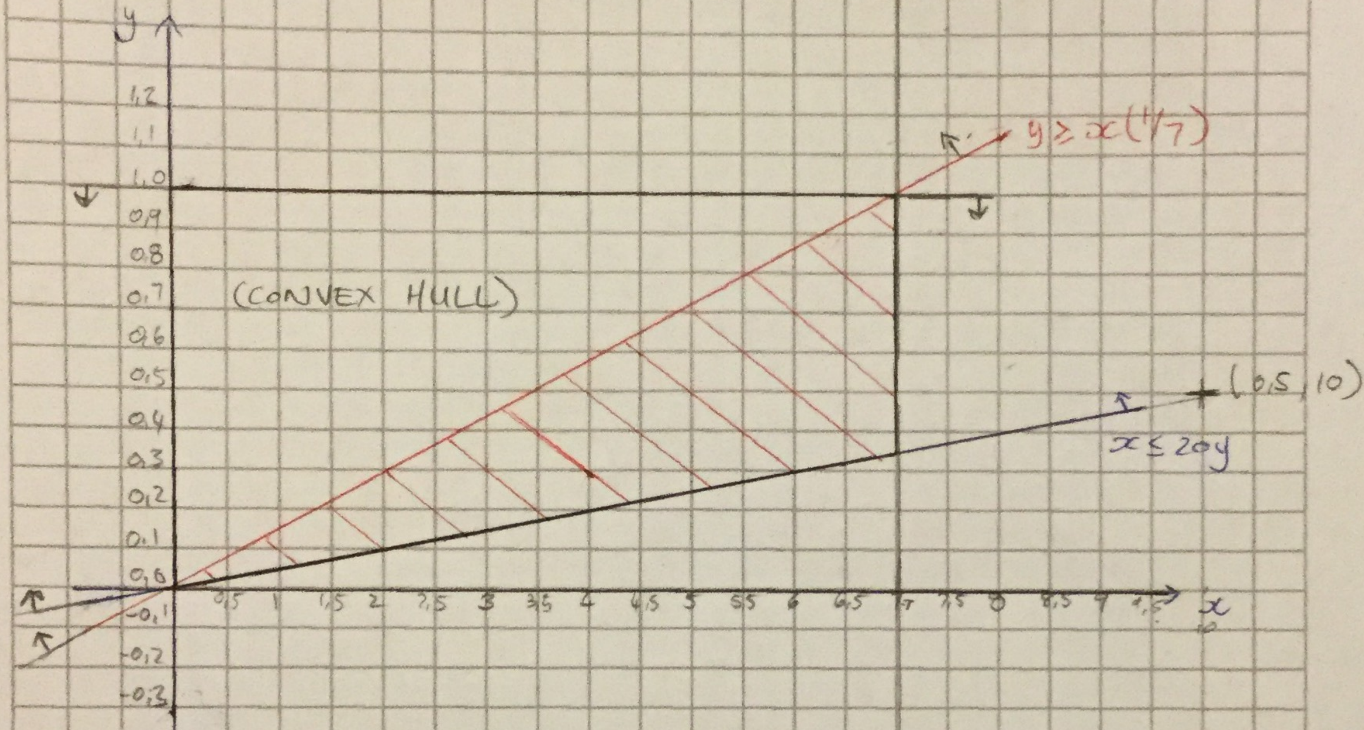
$3x_1 \leq 1$

$x_1 \leq 1/3$



The valid inequality is $x_2 \geq x_1$

3 (ii) $X = \{(x,y) \in R \times B : 0 \leq x \leq 7 ;\ 0 \leq y \leq 1 ;\ x \leq 20y\}$



when $x = 0$

$0 \leq y$

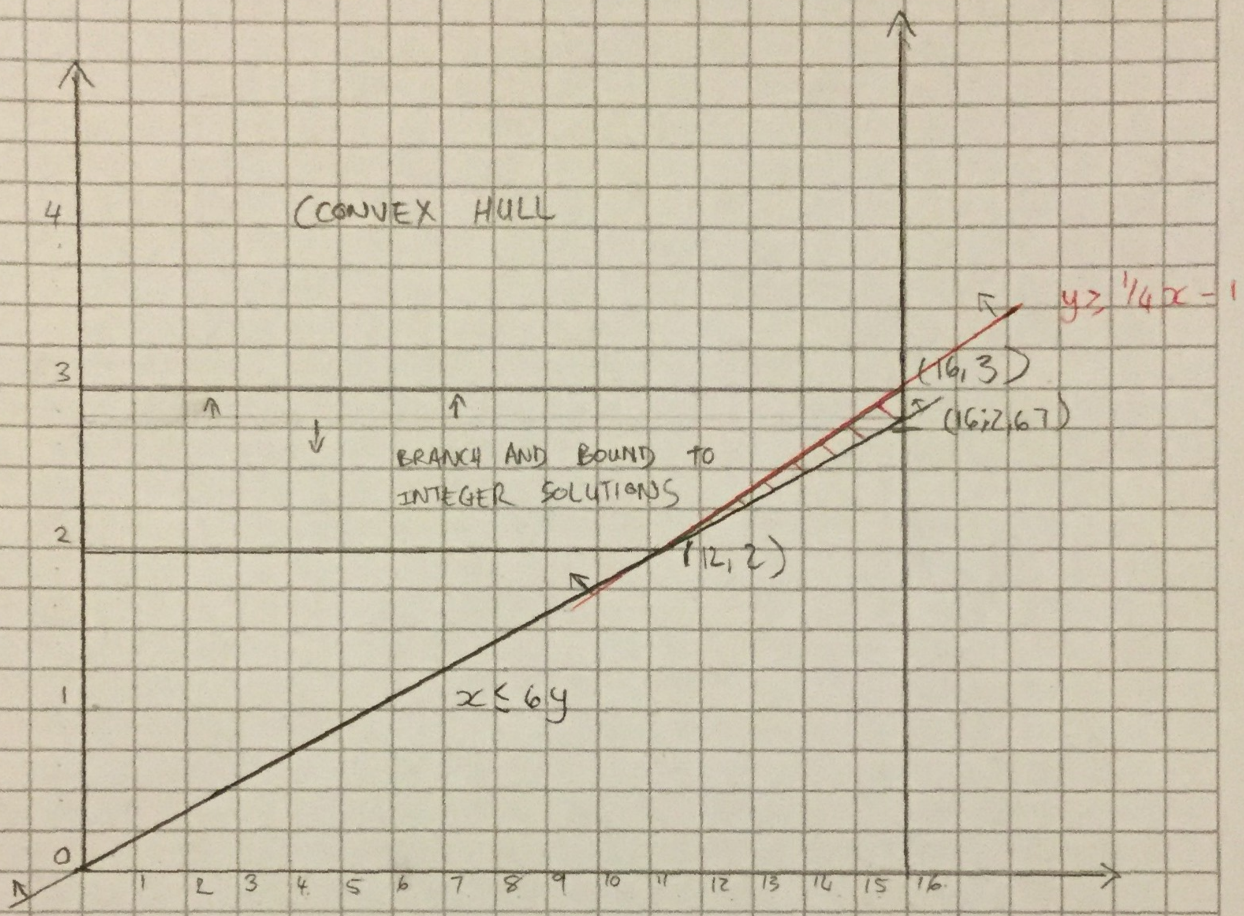
when $x = 10$

$10 \leq 20y$

$1/2 \leq y$

The valid inequality is $y \geq x(1/7)$

3(iii) $X = \{(x,y) \in R \times Z : 0 \le x \le 16 \; ; \; y \ge 0 \; ; \; x \le 6y\}$



when $x = 0$
$0 \le 6y$
$0 \le y$

when $x = 16$
$16 \le 6y$
$2{,}67 \le y$

when $y = 2$
$x = 6(2)$
$= 12$

Find the intercept

$$\frac{2-y}{12-0} = \frac{1}{4}$$

$$y = -1$$

The valid inequality is

$$y \geq 1/4\,x - 1$$

Question 4

4. Cutting planes vs. Branch-and-Bound. Consider the pure integer programming problem

$$\begin{aligned} &\text{maximize} \;\; 3x_1 - 3x_2 \\ &\text{subject to} \;\; 2x_1 - 2x_2 \le 1 \\ &\qquad\qquad x = (x_1, x_2) \ge 0 \;\text{ and integer.} \end{aligned}$$

(a) Try to solve this problem directly using the *Excel Solver* (standard version, with the default 1% Integer Optimality tolerance). What do you observe?

(b) Repeat question (a) with the *OpenSolver*.

(c) Find a Chvátal-Gomory inequality for this problem, such that its addition to the given initial formulation makes it trivial to solve, even by the *Excel Solver*.

(a) Integer Programming Formulation:

Decision Variables:

- x_i = denote the decision variables for i = 1 and 2.

Constraints:

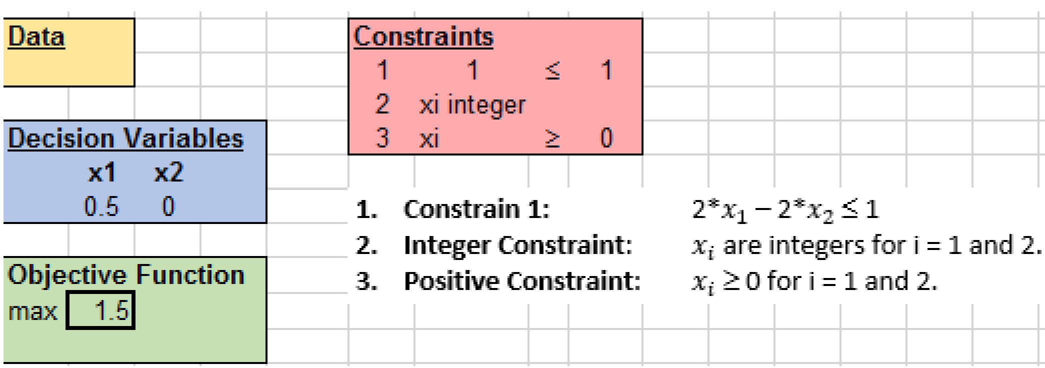
1. **Constraint 1:** $2{*}x_1 - 2{*}x_2 \le 1$
2. **Integer Constraint**: x_i are integers for i = 1 and 2.
3. **Positive Constraint**: $x_i \ge 0$ for i = 1 and 2.

Objective:

- To maximize the following:

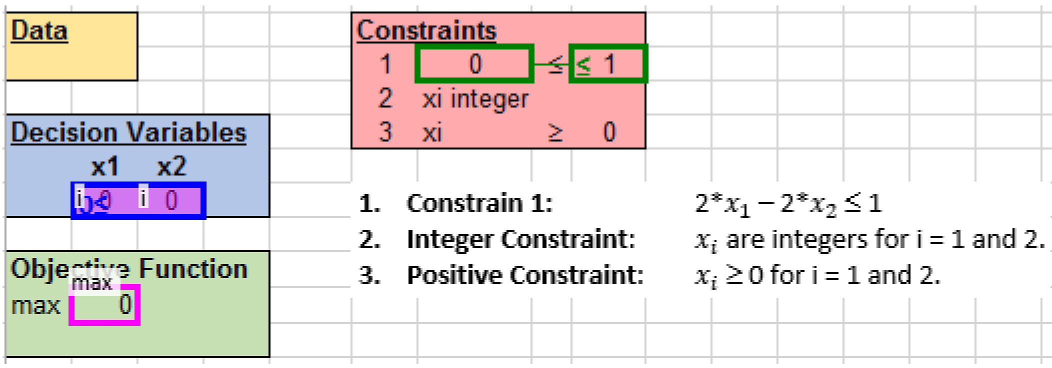
  max Z = { $3{*}x_1 - 3{*}x_2$ }

An annotated version of the Excel Solver spreadsheet is found below:



I observe that only variable x1 is used and takes on a value of 0.5 (i.e. not an integer) yielding a maximum objective function of 1.5. Also, it takes the solver about 15 seconds to find a solution.

(b) An annotated version of the Excel Open Solver spreadsheet, for the integer programming problem is found below:



I observe a different solution than that found in (a). Now both x1 and x2 appear to be 0.

(c) Find a Chvátal-Gomory inequality for this problem, such that its addition to the given initial formulation makes it trivial to solve, even by the Excel Solver

Because we cannot solve our MIP model, we want to add new valid linear inequalities (i.e. cutting planes) to tighten the formulation. To generate valid inequalities for the (Pure) Integer Set,

X = { x ε Z^n : Ax ≤ b, x > 0 } (A is a given m x n matrix)

Step 1: Construct a negative combination uAx ≤ ub of the given inequalities (with noninteger RHS ub) for some nonnegative u ε R^m

- giving the inequality $c^T x \leq c_o$ where c^T = uA and c_o = ub

$2^*x_1 - 2^*x_2 \leq 1$

A = | 2 -2 | x = $\left| \begin{matrix} x_1 \\ x_2 \end{matrix} \right|$ b = | 1 | u = $\frac{1}{2}$

uA = | 1 − 1 | c_o = | 1/2 |

Step 2: Relaxation of LHS: use the nonnegativity constraints x ≥ 0 to round down each LHS coefficient:

$\sum_j \lfloor c_j \rfloor * x_j \leq c_o$

$\lfloor 1 \rfloor^* x_1 - \lfloor 1 \rfloor^* x_2 \leq \frac{1}{2}$

Step 3: Since the LHS is integer for every x ε X apply the Rounding Principle on the inequality

$\sum_j \lfloor c_j \rfloor * x_j \leq \lfloor c_o \rfloor$ is valid for X

$\boldsymbol{x_1 - x_2 \leq 0}$ **(Chvátal-Gomory inequality for this problem)**

Since both x1 and x2 must be greater than or equal to 0 then their difference is equal to 0 by the Chvátal-Gomory inequality. Therefore, x1 must equal to x2.

An annotated version of the Excel Solver spreadsheet is found below:

Data					Constraints			
					1	0	≤	1
Decision Variables					2	xi integer		
	x1	x2			3	xi	≥	0
	0	0			4	0	≤	0

Objective Function	
max	0

1. **Constrain 1:** $2^*x_1 - 2^*x_2 \leq 1$
2. **Integer Constraint:** x_i are integers for i = 1 and 2.
3. **Positive Constraint:** $x_i \geq 0$ for i = 1 and 2.

$\boldsymbol{x_1 - x_2 \leq 0}$

Question 5

5. Cutting planes. In each of the examples below, a set X and a point x or (x, y) are given. Find a valid inequality for X cutting off the point, and present an *algebraic proof* (in contrast with a geometric argument based on drawing a picture) of its validity for the corresponding set X (by deriving it as a Chvátal-Gomory inequality, or using any other method, at your choice).

(i) $X = \{(x, y) \in \mathbf{R}^2 \times \mathbf{B} : 0 \le x \le 1;\ \ x_1 + x_2 \le 2y\};\ (x_1, x_2, y) = (1, 0, 0.5)$

(ii) $X = \{x \in \mathbf{Z}^4 : x \ge 0;\ 4x_1 + 8x_2 + 7x_3 + 5x_4 \le 33\};\ \ x = (0, 0, 33/7, 0)$

(iii) $X = \{x \in \mathbf{Z}^5 : x \ge 0;\ 9x_1 + 12x_2 + 8x_3 + 17x_4 + 13x_5 \ge 50\};$
$x = (0, 25/6, 0, 0, 0)$

(iv) $X = \{(x, y) \in \mathbf{R} \times \mathbf{Z} : 0 \le x \le 9;\ y \ge 0;\ x \le 4y\};\quad (x, y) = (9,\ 9/4)$

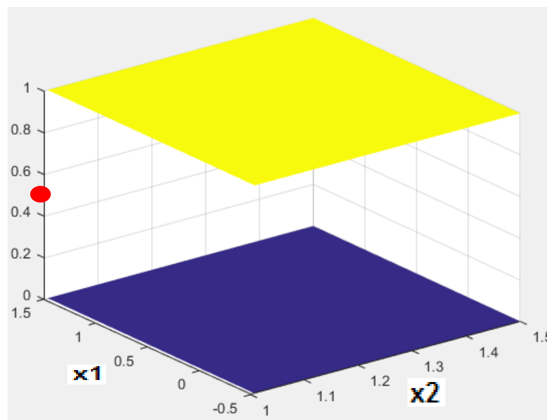
(v) $X = \{(x, y) \in \mathbf{R}^2 \times \mathbf{Z} : (x, y) \ge 0;\ x_1 + x_2 \le 25;\ x_1 + x_2 \le 8y\};$
$(x_1, x_2, y) = (20,\ 5,\ 25/8)$

(i) X = { (x,y) ε R^2 x B:

0 ≤ x ≤ 1;

x1 + x2 ≤ 2y;

(x1, x2, y) = (1, 0, 0.5)



First, I will validate we have a correct solution:

x1 + x2 ≤ 2y;

(1) + (0) ≤ 2*(0.5)

1 ≤ 1 → satisfied

y is binary so it can take on values of 0 or 1:

when y = 0

x1 + x2 ≤ 0;

this can only occur when x1 = x2 = 0

when y = 1

x1 + x2 ≤ 2

in order to avoid the point (x1, x2, y) = (1, 0, 0.5) then the following inequalities must exist:

y ≥ x1

Again, I will validate we have a correct solution by showing the point was cut:

y ≥ x1

0.5 ≥ 1 → unsatisfied

These inequality is valid for (x1, x2, y) values for the integer program, but not for the point (1, 0, 0.5) in the LP relaxation.

(ii) X = { (x) ε Z^4 :

0 ≤ x;

4*x1 + 8*x2 + 7*x3 + 5*x4 ≤ 33;

(x) = (0, 0, 33/7, 0)

First, I will validate we have a correct solution:

4*x1 + 8*x2 + 7*x3 + 5*x4 ≤ 33

4(0) + 8(0) + 7(33/7) + 5(0) ≤ 33

33 ≤ 33 → satisfied

Next, I will find a Chvátal-Gomory inequality for this problem, such that its addition to the given initial formulation makes it trivial to solve, even by the Excel Solver.

Because we cannot solve our MIP model, we want to add new valid linear inequalities (i.e. cutting planes) to tighten the formulation. To generate valid inequalities for the (Pure) Integer Set,

X = { x ε Z^n : Ax ≤ b, x > 0 } (A is a given m x n matrix)

Step 1: Construct a negative combination uAx ≤ ub of the given inequalities (with noninteger RHS ub) for some nonnegative u ε R^m

- giving the inequality $c^T x \leq c_o$ where c^T = uA and c_o = ub

4*x1 + 8*x2 + 7*x3 + 5*x4 ≤ 33

$\frac{4}{7}$*x1 + $\frac{8}{7}$*x2 + $\frac{7}{7}$*x3 + $\frac{5}{7}$*x4 ≤ $\frac{33}{7}$ where u = 1/7

Step 2: Relaxation of LHS: use the nonnegativity constraints x ≥ 0 to round down each LHS coefficient:

$\sum_j \lfloor c_j \rfloor * x_j \leq c_o$

$\lfloor\frac{4}{7}\rfloor$*x1 + $\lfloor\frac{8}{7}\rfloor$*x2 + $\lfloor\frac{7}{7}\rfloor$*x3 + $\lfloor\frac{5}{7}\rfloor$*x4 ≤ $\frac{33}{7}$

0*x1 + 1*x2 + 1*x3 + 0*x4 ≤ $\frac{33}{7}$

Step 3: Since the LHS is integer for every x ε X apply the Rounding Principle on the inequality

$\sum_j \lfloor c_j \rfloor * x_j \leq c_o$ is valid for X

0*x1 + 1*x2 + 1*x3 + 0*x4 ≤ $\lfloor \frac{33}{7} \rfloor$

0*x1 + x2 + x3 + 0*x4 ≤ 4 **(Chvátal-Gomory inequality for this problem)**

Again, I will validate we have a correct solution by showing the point was cut:

x1 + 2*x2 + x3 + x4 ≤ 4

0 + 0 + (33/7) + 0 ≤ 4

33/7 ≤ 4 → not satisfied

The inequality is valid for the integer program, but is violated by the point (x1, x2, x3, x4) = (0, 0, 33/7, 0) in the LP relaxation.

(iii) X = { x ε Z^5:

0 ≤ x

9*x1 + 12*x2 + 8*x3 + 17*x4 + 13*x5 ≥ 50

(x) = (0, 25/6, 0, 0, 0)

First, I will validate we have a correct solution:

9*x1 + 12*x2 + 8*x3 + 17*x4 + 13*x5 ≥ 50

9*(0) + 12*(25/6) + 8*(0) + 17*(0) + 13*(0) ≥ 50

50 ≥ 50 → satisfied

Next, I will find a Chvátal-Gomory inequality for this problem, such that its addition to the given initial formulation makes it trivial to solve, even by the Excel Solver.

Because we cannot solve our MIP model, we want to add new valid linear inequalities (i.e. cutting planes) to tighten the formulation. To generate valid inequalities for the (Pure) Integer Set,

X = { x ε Z^n : Ax ≤ b, x > 0 } (A is a given m x n matrix)

Step 1: Construct a negative combination uAx ≤ ub of the given inequalities (with noninteger RHS ub) for some nonnegative u ε R^m

- giving the inequality $c^T x \le c_o$ where c^T = uA and c_o = ub

9*x1 + 12*x2 + 8*x3 + 17*x4 + 13*x5 ≥ 50

$\frac{9}{12}$*x1 + $\frac{12}{12}$*x2 + $\frac{8}{12}$*x3 + $\frac{17}{12}$*x4 + $\frac{13}{12}$*x5 ≥ $\frac{50}{12}$ where u = 1/12

Step 2: Relaxation of LHS: use the nonnegativity constraints x ≥ 0 to round down each LHS coefficient:

$\sum_j \lfloor c_j \rfloor * x_j \le c_o$

$\lceil\frac{9}{12}\rceil$*x1 + $\lceil\frac{12}{12}\rceil$*x2 + $\lceil\frac{8}{12}\rceil$*x3 + $\lceil\frac{17}{12}\rceil$*x4 + $\lceil\frac{13}{12}\rceil$*x5 ≥ $\frac{50}{12}$

x1 + x2 + x3 + 2*x4 + 2*x5 ≥ $\frac{50}{12}$

Step 3: Since the LHS is integer for every x ε X apply the Rounding Principle on the inequality

$\sum_j \lfloor c_j \rfloor * x_j \le c_o$ is valid for X

x1 + x2 + x3 + 2*x4 + 2*x5 ≥ $\lceil\frac{50}{12}\rceil$

x1 + x2 + x3 + 2*x4 + 2*x5 ≥ 5 **(Chvátal-Gomory inequality for this problem)**

Again, I will validate we have a correct solution by showing the point was cut:

x1 + x2 + x3 + 2*x4 + 2*x5 ≥ 5

0 + (25/6) + 0 + 0 + 0 ≥ 5

(25/3) ≥ 5 → not satisfied

~ 4.16 is not ≥ 5

The inequality is valid for the integer program, but is violated by the point (x1, x2, x3, x4, x5) = (0, 25/6, 0, 0, 0) in the LP relaxation.

(iv) X = { (x,y) ε R x Z:

$0 \leq x \leq 9$;

$0 \leq y$;

$x \leq 4*y$;

(x, y) = (9, 9/4)

First, I will validate we have a correct solution:

$x \leq 4*y$;

$x - 4*y \leq 0$;

$9 - 4*(9/4) \leq 0$;

$0 \leq 0$ → satisfied

Second, I can add the two inequalities which are multiplied each by a factor in u

(1) u1* [$x \leq 4*y$]

(2) u2* [$x \leq 9$]

(1) + (2)

$u1*x + u2*x \leq u1*4*y + u2*9$

Third, try different values of u1 and u2. I won't list all of the ones I tried here, but I got the inequality not to work by letting u1 = 1/3, and u2 = 2/3:

$u1*x + u2*x \leq u1*4*y + u2*9$

$(1/3)*x + (2/3)*x \leq (1/3)*4*y + (2/3)*9$

$x \leq (4/3)*y + 6$

Fourth, use the mixed-integer rounding principle:

$x \leq \lfloor \frac{4}{3} \rfloor * y + 6$

$x \leq y + 6$

Again, I will validate we have a correct solution by showing the point was cut:

$x \leq y + 6$

$9 \leq (9/4) + 6$

$9 \leq 8.25$ → not satisfied

The inequality is valid for the integer program, but is violated by the point (x, y) = (0, 25/6, 0, 0, 0) in the LP relaxation.

(v) X = { (x,y) ε R^2 x Z:

$(x, y) \geq 0$;

$x1 + x2 \leq 25$;

$x1 + x2 \leq 8*y$;

$(x1, x2, y) = (20, 5, 25/8)$

First, I will validate we have a correct solution:

$x1 + x2 \leq 25;$
$(20) + (5) \leq 25;$
$25 \leq 25$ → satisfied

$x1 + x2 \leq 8*y;$
$(20) + (5) \leq 8*(25/8)$
$25 \leq 25$ → satisfied

Second, I can add the two inequalities which are multiplied each by a factor in u

(1) $u1* [x1 + x2 \leq 25]$
(2) $u2* [x1 + x2 \leq 8*y]$

(1) + (2)
$u1*(x1 + x2) + u2*(x1 + x2) \leq u1*25 + u2*8*y$

Third, try different values of u1 and u2. I won't list all of the ones I tried here, but I got the inequality not to work by letting u1 = 1/2, and u2 = 1/2:

$u1*(x1 + x2) + u2*(x1 + x2) \leq u1*25 + u2*8*y$
$(½)*(x1 + x2) + (1/2)*(x1 + x2) \leq (1/2)*25 + (1/2)*8*y$
$x1 + x2 \leq \frac{25}{2} + 4*y$

Fourth, use the mixed-integer rounding principle:

$x1 + x2 \leq \lfloor \frac{25}{2} \rfloor + \lfloor 4 \rfloor *y$

$x1 + x2 \leq 12 + 4*y$

Again, I will validate we have a correct solution by showing the point was cut:

$x1 + x2 \leq 12 + 4*y$
$20 + 5 \leq 12 + 4*(25/8)$
$25 \leq 24.5$

The inequality is valid for the integer program, but is violated by the point (x1, x2, y) = (20, 5, 25/8) in the LP relaxation.