BAMS 508 – Discrete Optimization

Assignment 1

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Question 1

1. Bus drivers shift scheduling. The Urban Bus Co. (UBC) is considering the acquisition of the Suburban Bus Co. (SBC). As an optimization specialist for UBC you are being asked to evaluate possible labour cost savings that may result from this acquisition. UBC and SBC currently have very similar agreements with the bus drivers union, which will continue to be in effect for some time after the acquisition. These agreements specify the two types of bus drivers work shifts in effect at each company:

- A *normal shift*, consisting of 4 consecutive hours of work, followed by a one-hour paid break, immediately followed by 4 other consecutive hours of work;
- A *short shift*, consisting of only 4 consecutive hours of work.

The union has indicated that after the acquisition they will agree to a third shift type:

- A *dual shift*, consisting of 4 consecutive hours of work in one of the companies (UBC or SBC), followed by a one-hour paid break, immediately followed by 4 consecutive hours of work at the other company.

You have estimated the following daily minimum staffing requirements at each company:

From	To	Min. drivers req'd UBC	Min. drivers req'd SBC	From	To	Min. drivers req'd UBC	Min. drivers req'd SBC
5:00 AM	6:00 AM	3	5	3:00 PM	4:00 PM	12	12
6:00 AM	7:00 AM	10	12	4:00 PM	5:00 PM	10	15
7:00 AM	8:00 AM	12	20	5:00 PM	6:00 PM	8	21
8:00 AM	9:00 AM	12	20	6:00 PM	7:00 PM	6	20
9:00 AM	10:00 AM	10	15	7:00 PM	8:00 PM	5	12
10:00 AM	11:00 AM	10	8	8:00 PM	9:00 PM	5	6
11:00 AM	12:00 PM	15	6	9:00 PM	10:00 PM	5	4
12:00 PM	1:00 PM	19	6	10:00 PM	11:00 PM	4	3
1:00 PM	2:00 PM	18	6	11:00 PM	12:00 AM	3	2
2:00 PM	3:00 PM	15	8	12:00 AM	1:00 AM	2	2

(a) Define an integer programming (IP) model to determine a minimum cost daily staff schedule after the acquisition. Set up your model in an Excel spreadsheet and solve it with the Excel Solver. What do you observe about the mix of shift types in each company? Briefly explain why this is so.

Assumptions:

The cost per hour, regardless of which shift a driver performs, is the same. Hence, we can give a weight of 9 to shifts 1 and 3, and a weight of 4 for shift type 2 when forming the objective function.

Decision Variables:

Let x_{ijk} denotes the number of drivers who will start on the hour according to: *[driver]*

i = the starting hours { i = 5, 6, … 24 } where 5 means 5am and 24 is midnight.
j = the bus company { j = 1, 2 } where 1 is Urban Bus Co. (UBC) and 2 is Suburban Bus Co. (SBC).
k = the shift type { k = 1, 2, 3 which denotes normal, short, and dual, respectively}.

This means the number of decision variables is 120. For example, in Excel, we coded:

$x_{i,1,1}$ are variables x_1 through x_{20}
$x_{i,1,2}$ are variables x_{21} through x_{40}

$x_{i,1,3}$ are variables x_{41} through x_{60}
$x_{i,2,1}$ are variables x_{61} through x_{80}
$x_{i,2,2}$ are variables x_{81} through x_{100}
$x_{i,2,3}$ are variables x_{101} through x_{120}

<u>Objective</u>: To minimize the daily staffing cost schedule after acquisition according to:

minimize f(x_{ijk}) = ($\sum_{i=1}^{20} x_{i,1,1}$ + $\sum_{i=1}^{20} x_{i,2,1}$ + $\sum_{i=1}^{20} x_{i,1,3}$ + $\sum_{i=1}^{20} x_{i,2,3}$)*(9)
+ ($\sum_{i=1}^{20} x_{i,1,2}$ + $\sum_{i=1}^{20} x_{i,2,2}$)*(4) *[dollars]*

This objective function is linear.

<u>Constraints</u>:

1. **Non-negativity:** $x_{ijk} \geq 0$ *[drivers]*
   The number of drivers that start on the specified hours cannot be a negative number.

2. **Minimum Drivers Required between 5am-6am for UBC:** *[drivers]*
   The minimum number of drivers required is 3.
   $x_1 + x_{21} + x_{41} \geq 3$

3. **Minimum Drivers Required between 6am-7am for UBC:** *[drivers]*
   The minimum number of drivers required is 10.
   $x_1 + x_2 + x_{21} + x_{22} + x_{41} + x_{42} \geq 10$

4. **Minimum Drivers Required between 7am-8am for UBC:** *[drivers]*
   The minimum number of drivers required is 12.
   $x_1 + x_2 + x_3 + x_{21} + x_{22} + x_{23} + x_{41} + x_{42} + x_{43} \geq 12$

5. **Minimum Drivers Required between 8am-9am for UBC:** *[drivers]*
   The minimum number of drivers required is 12.
   $x_1 + x_2 + x_3 + x_4 + x_{21} + x_{22} + x_{23} + x_{24} + x_{41} + x_{42} + x_{43} + x_{44} \geq 12$

6. **Minimum Drivers Required between 9am-10am for UBC:** *[drivers]*
   The minimum number of drivers required is 10.
   $x_2 + x_3 + x_4 + x_5 + x_{22} + x_{23} + x_{24} + x_{25} + x_{42} + x_{43} + x_{44} + x_{45} \geq 10$

7. **Minimum Drivers Required between 10am-11am for UBC:** *[drivers]*
   The minimum number of drivers required is 10.
   $x_1 + x_3 + x_4 + x_5 + x_6 + x_{23} + x_{24} + x_{25} + x_{26} + x_{43} + x_{44} + x_{45} + x_{46} \geq 10$

8. **Minimum Drivers Required between 11am-12pm for UBC:** *[drivers]*
   The minimum number of drivers required is 15.
   $x_1 + x_2 + x_4 + x_5 + x_6 + x_7 + x_{24} + x_{25} + x_{26} + x_{27} + x_{44} + x_{45} + x_{46} + x_{47} \geq 15$

9. **Minimum Drivers Required between 12pm-1pm for UBC:** *[drivers]*
   The minimum number of drivers required is 19.
   $x_1 + x_2 + x_3 + x_5 + x_6 + x_7 + x_8 + x_{25} + x_{26} + x_{27} + x_{28} + x_{45} + x_{46} + x_{47} + x_{48} \geq 19$

10. **Minimum Drivers Required between 1pm-2pm for UBC:** *[drivers]*
    The minimum number of drivers required is 18.
    $x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9 + x_{26} + x_{27} + x_{28} + x_{29} + x_{46} + x_{47} + x_{48} + x_{49} \geq 18$

11. **Minimum Drivers Required between 2pm-3pm for UBC:** *[drivers]*
    The minimum number of drivers required is 15.
    $x_2 + x_3 + x_4 + x_5 + x_7 + x_8 + x_9 + x_{10} + x_{27} + x_{28} + x_{29} + x_{30} + x_{47} + x_{48} + x_{49} + x_{50} \geq 15$

12. **Minimum Drivers Required between 3pm-4pm for UBC:** *[drivers]*
    The minimum number of drivers required is 12.
    $x_3 + x_4 + x_5 + x_6 + x_8 + x_9 + x_{10} + x_{11} + x_{28} + x_{29} + x_{30} + x_{31} + x_{48} + x_{49} + x_{50} + x_{51} \geq 12$

13. **Minimum Drivers Required between 4pm-5pm for UBC:** *[drivers]*
    The minimum number of drivers required is 10.
    $x_4 + x_5 + x_6 + x_7 + x_9 + x_{10} + x_{11} + x_{12} + x_{29} + x_{30} + x_{31} + x_{32} + x_{49} + x_{50} + x_{51} + x_{52} \geq 10$

14. **Minimum Drivers Required between 5pm-6pm for UBC:** *[drivers]*
    The minimum number of drivers required is 8.
    $x_5 + x_6 + x_7 + x_8 + x_{10} + x_{11} + x_{12} + x_{13} + x_{30} + x_{31} + x_{32} + x_{33} + x_{50} + x_{51} + x_{52} \geq 8$

15. **Minimum Drivers Required between 6pm-7pm for UBC:** *[drivers]*
    The minimum number of drivers required is 6.
    $x_6 + x_7 + x_8 + x_9 + x_{11} + x_{12} + x_{31} + x_{32} + x_{33} + x_{34} + x_{51} + x_{52} \geq 6$

16. **Minimum Drivers Required between 7pm-8pm for UBC:** *[drivers]*
    The minimum number of drivers required is 5.
    $x_7 + x_8 + x_9 + x_{10} + x_{12} + x_{32} + x_{33} + x_{34} + x_{35} + x_{52} \geq 5$

17. **Minimum Drivers Required between 8pm-9pm for UBC:** *[drivers]*
    The minimum number of drivers required is 5.
    $x_8 + x_9 + x_{10} + x_{11} + x_{33} + x_{34} + x_{35} + x_{36} \geq 5$

18. **Minimum Drivers Required between 9pm-10pm for UBC:** *[drivers]*
    The minimum number of drivers required is 5.
    $x_9 + x_{10} + x_{11} + x_{12} + x_{34} + x_{35} + x_{36} + x_{37} \geq 5$

19. **Minimum Drivers Required between 10pm-11pm for UBC:** *[drivers]*
    The minimum number of drivers required is 4.
    $x_{10} + x_{11} + x_{12} + x_{35} + x_{36} + x_{37} \geq 4$

20. **Minimum Drivers Required between 11pm-12am for UBC:** *[drivers]*
    The minimum number of drivers required is 3.
    $x_{11} + x_{12} + x_{36} + x_{37} \geq 3$

21. **Minimum Drivers Required between 12am-1am for UBC:** *[drivers]*
    The minimum number of drivers required is 2.

$x_{12} + x_{37} \geq 2$

22. Minimum Drivers Required between 5am-6am for SBC: *[drivers]*

The minimum number of drivers required is 5.

$x_{61} + x_{81} + x_{101} \geq 5$

23. Minimum Drivers Required between 6am-7am for SBC: *[drivers]*

The minimum number of drivers required is 12.

$x_{61} + x_{62} + x_{81} + x_{82} + x_{101} + x_{102} \geq 12$

24. Minimum Drivers Required between 7am-8am for SBC: *[drivers]*

The minimum number of drivers required is 20.

$x_{61} + x_{62} + x_{63} + x_{81} + x_{82} + x_{83} + x_{101} + x_{102} + x_{103} \geq 20$

25. Minimum Drivers Required between 8am-9am for SBC: *[drivers]*

The minimum number of drivers required is 20.

$x_{61} + x_{62} + x_{63} + x_{64} + x_{81} + x_{82} + x_{83} + x_{84} + x_{101} + x_{102} + x_{103} + x_{104} \geq 20$

26. Minimum Drivers Required between 9am-10am for SBC: *[drivers]*

The minimum number of drivers required is 15.

$x_{62} + x_{63} + x_{64} + x_{65} + x_{82} + x_{83} + x_{84} + x_{85} + x_{102} + x_{103} + x_{104} + x_{105} \geq 15$

27. Minimum Drivers Required between 10am-11am for SBC: *[drivers]*

The minimum number of drivers required is 8.

$x_{41} + x_{61} + x_{63} + x_{64} + x_{65} + x_{66} + x_{83} + x_{84} + x_{85} + x_{86} + x_{103} + x_{104} + x_{105} + x_{106} \geq 8$

28. Minimum Drivers Required between 11am-12pm for SBC: *[drivers]*

The minimum number of drivers required is 6.

$x_{41} + x_{42} + x_{61} + x_{62} + x_{64} + x_{65} + x_{66} + x_{67} + x_{84} + x_{85} + x_{86} + x_{87} + x_{104} + x_{105} + x_{106} + x_{107} \geq 6$

29. Minimum Drivers Required between 12pm-1pm for SBC: *[drivers]*

The minimum number of drivers required is 6.

$x_{41} + x_{42} + x_{43} + x_{61} + x_{62} + x_{63} + x_{65} + x_{66} + x_{67} + x_{68} + x_{85} + x_{86} + x_{87} + x_{88} + x_{105} + x_{106} + x_{107} + x_{108} \geq 6$

30. Minimum Drivers Required between 1pm-2pm for SBC: *[drivers]*

The minimum number of drivers required is 6.

$x_{41} + x_{42} + x_{43} + x_{44} + x_{61} + x_{62} + x_{63} + x_{64} + x_{66} + x_{67} + x_{68} + x_{69} + x_{86} + x_{87} + x_{88} + x_{89} + x_{106} + x_{107} + x_{108} + x_{109} \geq 6$

31. Minimum Drivers Required between 2pm-3pm for SBC: *[drivers]*

The minimum number of drivers required is 8.

$x_{42} + x_{43} + x_{44} + x_{45} + x_{62} + x_{63} + x_{64} + x_{65} + x_{67} + x_{68} + x_{69} + x_{70} + x_{87} + x_{88} + x_{89} + x_{90} + x_{107} + x_{108} + x_{109} + x_{110} \geq 8$

32. Minimum Drivers Required between 3pm-4pm for SBC: *[drivers]*

The minimum number of drivers required is 12.

$x_{43} + x_{44} + x_{45} + x_{46} + x_{63} + x_{64} + x_{65} + x_{66} + x_{68} + x_{69} + x_{70} + x_{71} + x_{88} + x_{89} + x_{90} + x_{91} + x_{108} + x_{109} + x_{110} + x_{111} \geq 12$

33. **Minimum Drivers Required between 4pm-5pm for SBC:** *[drivers]*

    The minimum number of drivers required is 15.

    $x_{44} + x_{45} + x_{46} + x_{47} + x_{64} + x_{65} + x_{66} + x_{67} + x_{69} + x_{70} + x_{71} + x_{72} + x_{89} + x_{90} + x_{91} + x_{92} + x_{109} + x_{110} + x_{111} + x_{112} \geq 15$

34. **Minimum Drivers Required between 5pm-6pm for UBC:** *[drivers]*

    The minimum number of drivers required is 21.

    $x_{45} + x_{46} + x_{47} + x_{48} + x_{65} + x_{66} + x_{67} + x_{68} + x_{69} + x_{70} + x_{71} + x_{90} + x_{91} + x_{92} + x_{93} + x_{110} + x_{111} + x_{112} \geq 21$

35. **Minimum Drivers Required between 6pm-7pm for UBC:** *[drivers]*

    The minimum number of drivers required is 20.

    $x_{46} + x_{47} + x_{48} + x_{49} + x_{66} + x_{67} + x_{68} + x_{69} + x_{71} + x_{72} + x_{73} + x_{91} + x_{92} + x_{93} + x_{94} + x_{111} + x_{112} \geq 20$

36. **Minimum Drivers Required between 7pm-8pm for UBC:** *[drivers]*

    The minimum number of drivers required is 12.

    $x_{47} + x_{48} + x_{49} + x_{50} + x_{67} + x_{68} + x_{69} + x_{70} + x_{72} + x_{92} + x_{93} + x_{94} + x_{95} + x_{112} \geq 12$

37. **Minimum Drivers Required between 8pm-9pm for UBC:** *[drivers]*

    The minimum number of drivers required is 6.

    $x_{49} + x_{50} + x_{51} + x_{52} + x_{68} + x_{69} + x_{70} + x_{71} + x_{93} + x_{94} + x_{95} + x_{96} \geq 6$

38. **Minimum Drivers Required between 9pm-10pm for UBC:** *[drivers]*

    The minimum number of drivers required is 4.

    $x_{49} + x_{50} + x_{51} + x_{52} + x_{69} + x_{70} + x_{71} + x_{72} + x_{94} + x_{95} + x_{96} + x_{97} \geq 4$

39. **Minimum Drivers Required between 10pm-11pm for UBC:** *[drivers]*

    The minimum number of drivers required is 3.

    $x_{50} + x_{51} + x_{52} + x_{70} + x_{71} + x_{72} + x_{95} + x_{96} + x_{97} \geq 3$

40. **Minimum Drivers Required between 11pm-12am for UBC:** *[drivers]*

    The minimum number of drivers required is 2.

    $x_{51} + x_{52} + x_{71} + x_{72} + x_{96} + x_{97} \geq 2$

41. **Minimum Drivers Required between 12am-1am for UBC:** *[drivers]*

    The minimum number of drivers required is 2.

    $x_{52} + x_{72} + x_{97} \geq 2$

The constructed integer programming model can be found as an Excel spreadsheet below:

xijk	i =	starting hours { i = 5, 6, ... 24 n = 20	j = 1, k = 1	j = 2, k = 1
	j =	starting company { 1 = UBC, 2 = SBC }	j = 1, k = 2	j = 2, k = 2
	k =	shift types { k = 1 normal, 2 short, 3 dual}	j = 1, k = 3	j = 2, k = 3

From	To	From	To	1	2	3	4		6	7	8	9	10	11	12	13	14	15	16	17	18	19	#	21	#	#	#	#	#	#	#	#	#	31	#	#	#	#	#	#	#	#	#	41	#	#	#	#	#	#	#	#	#	51	#	#	#	#	#	#	#	#	#
5:00 AM	6:00 AM	5:00	6:00	1	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6:00 AM	7:00 AM	6:00	7:00	1	1	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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8:00 AM	9:00 AM	8:00	9:00	1	1	1	1		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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######	11:00 AM	10:00	11:00	1	0	1	1		1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11:00 AM	######	11:00	12:00	1	1	0	1		1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
######	1:00 PM	12:00	13:00	1	1	1	0		1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1:00 PM	2:00 PM	13:00	14:00	1	1	1	1		1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
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3:00 PM	4:00 PM	15:00	16:00	0	0	1	1		1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
4:00 PM	5:00 PM	16:00	17:00	0	0	0	1		1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
5:00 PM	6:00 PM	17:00	18:00	0	0	0	0		1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
6:00 PM	7:00 PM	18:00	19:00	0	0	0	0		1	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
7:00 PM	8:00 PM	19:00	20:00	0	0	0	0		0	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
8:00 PM	9:00 PM	20:00	21:00	0	0	0	0		0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9:00 PM	######	21:00	22:00	0	0	0	0		0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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11:00 PM	######	23:00	0:00	0	0	0	0		0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
######	1:00 AM	0:00	1:00	0	0	0	0		0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		5:00	6:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		6:00	7:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		7:00	8:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		8:00	9:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		9:00	10:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		10:00	11:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		11:00	12:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		12:00	13:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		13:00	14:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		14:00	15:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		15:00	16:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		16:00	17:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
		17:00	18:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
		18:00	19:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
		19:00	20:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
		20:00	21:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
		21:00	22:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
		22:00	23:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
		23:00	0:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
		0:00	1:00	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
		Decision Variables		0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	7	2	0	1	7	7	4	0	4	4	2	0	0	3	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

61	#	#	#	#	#	#	#	#	#	71	#	#	#	#	#	#	#	#	#	81	#	#	#	#	#	#	#	#	#	91	#	#	#	#	#	#	#	#	##	101	##	##	##	##	##	##	##	##	110	111	112	113	114	115	116	117	118	119	##	Min Drivers		slack	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	≥ 3	0	UBC
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10	≥ 10	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	≥ 12	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	≥ 12	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10	≥ 10	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10	≥ 10	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	≥ 15	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	19	≥ 19	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	18	≥ 18	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	≥ 15	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	≥ 12	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	10	≥ 10	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	10	≥ 8	2	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	6	≥ 6	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	5	≥ 5	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	5	≥ 5	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	7	≥ 5	2	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	7	≥ 4	3	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	4	≥ 3	1	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	2	≥ 2	0	
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	≥ 5	0	SBC
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	≥ 12	0	
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	≥ #	0	
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	≥ #	0	
0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	≥ 15	0	
1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	≥ 8	0	
1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	6	≥ 6	0	
1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	6	≥ 6	0	
1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	6	≥ 6	0	
0	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	8	≥ 8	0	
0	0	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	12	≥ 12	0	
0	0	0	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	18	≥ 15	3	
0	0	0	0	1	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	21	≥ 21	0	
0	0	0	0	0	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	21	≥ #	1	
0	0	0	0	0	0	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	12	≥ 12	0	
0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	≥ 6	0	
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	≥ 4	1	
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	≥ 3	0	
0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	≥ 2	0	
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	≥ 2	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	7	8	0	0	0	6	0	0	2	#	6	3	2	1	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	drivers		

Objective Function	{(x j = 1, k = 1)+(x j = 2, k = 1)}*(9) + {(x j = 1, k = 2)+(x j = 2, k = 3)}*(4) + (x j = 1, k = 1)+(x j = 2, k = 3)}*(9)
	(x variables 1-20 + 61-80)*(9) + (x variables 21-40 + 81-100)*(9) + (x variables 41-60 + 101-120)*(9)
	400

Hence, we determined that a minimum of 100 drivers are needed. More specifically, we found that:
$x_{21} = 3$
$x_{22} = 7$
$x_{23} = 2$
$x_{25} = 1$
$x_{26} = 7$
$x_{27} = 7$
$x_{28} = 4$
$x_{30} = 4$
$x_{31} = 4$
$x_{32} = 2$
$x_{35} = 3$
$x_{36} = 2$
$x_{37} = 2$

$x_{81} = 5$
$x_{82} = 7$
$x_{83} = 8$
$x_{87} = 6$
$x_{90} = 2$
$x_{91} = 10$
$x_{92} = 6$
$x_{93} = 3$
$x_{94} = 2$
$x_{95} = 1$
$x_{97} = 2$

Recall x_{ijk} denotes the number of drivers who will start on the hour according to:

i = the starting hours { i = 5, 6, … 24 } where 5 means 5am and 24 is midnight.
j = the bus company { j = 1, 2 } where 1 is Urban Bus Co. (UBC) and 2 is Suburban Bus Co. (SBC).
k = the shift type { k = 1, 2, 3 which denotes normal, short, and dual, respectively}.

This means the number of decision variables is 120. For example, in Excel, we coded:

$x_{i,1,1}$ are variables x_1 through x_{20}
$x_{i,1,2}$ are variables x_{21} through x_{40}
$x_{i,1,3}$ are variables x_{41} through x_{60}
$x_{i,2,1}$ are variables x_{61} through x_{80}
$x_{i,2,2}$ are variables x_{81} through x_{100}
$x_{i,2,3}$ are variables x_{101} through x_{120}

We observed that the solution only had short shifts. This means no regular or dual-hour shifts would be used in the optimal solution. Logically, this makes sense because there is no restriction on the number of employees per type of shift and both the normal and dual shifts include a 1-hour paid excess break.

(b) In fact, the current work agreements require that in each company at least 65% of the shifts must be normal shifts. The union has also indicated that after the acquisition they will agree to change these clauses to: "in each company at least 65% of the shifts must be normal or dual shifts". Modify your IP model of question (a) above, to take these two constraints into account. Try and solve with the Excel Solver, using the standard (default) version of the Solver. What happens?

Now, we will add the following 2 constraints to the integer programming model.

42. **Normal and Dual Shifts at UBC:** *[drivers]*
    UBC must assign at least 65% of the shifts as normal or dual.
    $\{ \sum_{i=1}^{20} x_i + \sum_{i=41}^{60} x_i \} \geq 0.65 * \{ \sum_{i=1}^{60} x_i \}$

43. **Normal and Dual Shifts at SBC:** *[drivers]*
    SBC must assign at least 65% of the shifts as normal or dual.
    $\{ \sum_{i=61}^{80} x_i + \sum_{i=101}^{120} x_i \} \geq 0.65 * \{ \sum_{i=61}^{120} x_i \}$

The constructed integer programming model can be found as an Excel spreadsheet below:

xijk	i =	starting hours {i = 5, 6, ... 24} n = 20	j = 1, k = 1	j = 2, k = 1
	j =	starting company {1 = UBC, 2 = SBC}	j = 1, k = 2	j = 2, k = 2
	k =	shift types {k = 1 normal, 2 short, 3 dual}	j = 1, k = 3	j = 2, k = 3

From	To	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	#	21	#	#	#	#	#	#	#	#	#	31	#	#	#	#	#	#	#	#	#	41	#	#	#	#	#	#	#	#	#	51	#	#	#	#	#	#	#	#	#
5:00	6:00	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6:00	7:00	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7:00	8:00	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8:00	9:00	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9:00	10:00	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10:00	11:00	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11:00	12:00	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
12:00	13:00	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
13:00	14:00	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
14:00	15:00	0	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
15:00	16:00	0	0	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
16:00	17:00	0	0	0	1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
17:00	18:00	0	0	0	0	1	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
18:00	19:00	0	0	0	0	0	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
19:00	20:00	0	0	0	0	0	0	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
20:00	21:00	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21:00	22:00	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22:00	23:00	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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6:00	7:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7:00	8:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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13:00	14:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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16:00	17:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
17:00	18:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
18:00	19:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
19:00	20:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
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21:00	22:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
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23:00	0:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
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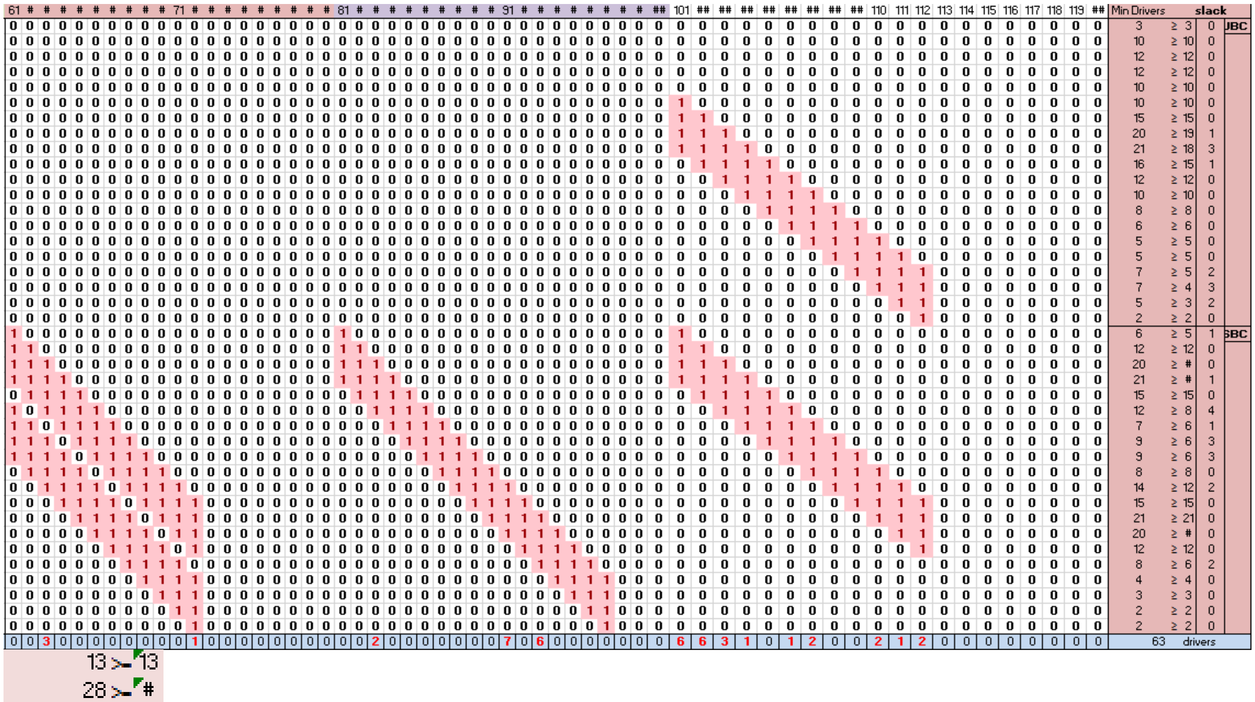
Decision Variables		0	0	2	0	1	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	1	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	0	1	1	0	1	1	0	1	0	0	0	0	0	0	0	0



Objective Function	{(x j = 1, k = 1)+(x j = 2, k = 1)}*(9) + {(x j = 1, k = 2)+(x j = 2, k = 3)}*(4) + (x j = 1, k = 1)+(x j = 2, k = 3)}*(9)
	(x variables 1-20 + 61-80)*(9) + (x variables 21-40 + 81-100)*(9) + (x variables 41-60 + 101-120)*(9)
	457

Hence, we determined that a minimum of 63 drivers are needed which work 457 work hours. More specifically, we found that:

$x_3 = 2$
$x_5 = 1$
$x_{11} = 2$
$x_{21} = 1$
$x_{22} = 6$
$x_{41} = 2$
$x_{42} = 1$
$x_{46} = 1$
$x_{47} = 1$
$x_{49} = 1$
$x_{50} = 1$
$x_{52} = 1$
$x_{63} = 3$
$x_{72} = 1$
$x_{83} = 2$
$x_{91} = 7$
$x_{93} = 6$
$x_{101} = 6$
$x_{102} = 6$
$x_{103} = 3$
$x_{104} = 1$
$x_{106} = 1$
$x_{107} = 2$

$x_{110} = 2$
$x_{111} = 1$
$x_{112} = 2$

Recall x_{ijk} denotes the number of drivers who will start on the hour according to:

i = the starting hours { i = 5, 6, … 24 } where 5 means 5am and 24 is midnight.
j = the bus company { j = 1, 2 } where 1 is Urban Bus Co. (UBC) and 2 is Suburban Bus Co. (SBC).
k = the shift type { k = 1, 2, 3 which denotes normal, short, and dual, respectively}.

This means the number of decision variables is 120. For example, in Excel, we coded:

$x_{i,1,1}$ are variables x_1 through x_{20}
$x_{i,1,2}$ are variables x_{21} through x_{40}
$x_{i,1,3}$ are variables x_{41} through x_{60}
$x_{i,2,1}$ are variables x_{61} through x_{80}
$x_{i,2,2}$ are variables x_{81} through x_{100}
$x_{i,2,3}$ are variables x_{101} through x_{120}

(c) Now try and solve your model of question (b) using the OpenSolver. What happens?

After running OpenSolver on the same model, I again obtained 63 drivers with 457 working hours. However, the decision variables have different values. More specifically, we obtained the following values:
$x_2 = 6$
$x_5 = 2$
$x_6 = 1$
$x_9 = 2$
$x_{10} = 1$
$x_{21} = 3$
$x_{22} = 2$
$x_{23} = 1$
$x_{31} = 2$
$x_{45} = 1$
$x_{52} = 2$
$x_{62} = 6$
$x_{63} = 3$
$x_{72} = 4$
$x_{82} = 1$
$x_{83} = 1$
$x_{91} = 6$
$x_{93} = 6$
$x_{101} = 5$
$x_{103} = 4$
$x_{111} = 2$
$x_{112} = 2$

Recall x_{ijk} denotes the number of drivers who will start on the hour according to:

i = the starting hours { i = 5, 6, … 24 } where 5 means 5am and 24 is midnight.
j = the bus company { j = 1, 2 } where 1 is Urban Bus Co. (UBC) and 2 is Suburban Bus Co. (SBC).
k = the shift type { k = 1, 2, 3 which denotes normal, short, and dual, respectively}.

This means the number of decision variables is 120. For example, in Excel, we coded:

$x_{i,1,1}$ are variables x_1 through x_{20}
$x_{i,1,2}$ are variables x_{21} through x_{40}
$x_{i,1,3}$ are variables x_{41} through x_{60}
$x_{i,2,1}$ are variables x_{61} through x_{80}
$x_{i,2,2}$ are variables x_{81} through x_{100}
$x_{i,2,3}$ are variables x_{101} through x_{120}

(d) Now set up and solve your models in questions (a) and (b) using a mathematical programming system with an algebraic modeling language. Compare the results with those in (a) and (c). Comment on the relative advantages and disadvantages of these two optimization systems (spreadsheet-based, vs. mathematical programming system with an algebraic modeling language) for setting up and solving these IP models.

Using AMPL, here is the data file:

```
param Nb_Requirements := 20;
      # number of requirements specified below

set SHIFTS := 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24;   # set of workshifts, indexed by start time
set shift_type := 1 2 3;
#set shift_type := normal short dual;
set company := 1 2;


param: From To UBC SBC :=
1       5   6    3         5
2       6   7    10        12
3       7   8    12        20
4       8   9    12        20
5       9   10   10        15
6       10  11   10        8
7       11  12   15        6
8       12  13   19        6
9       13  14   18        6
10      14  15   15        8
11      15  16   12        12
12      16  17   10        15
13      17  18   8         21
14      18  19   6         20
15      19  20   5         12
16      20  21   5         6
17      21  22   5         4
18      22  23   4         3
19      23  24   3         2
20      24  25   2         2;
```

Here is the model file:

```
#data

param Nb_Requirements;          # Requirements given in data file
param From{1..Nb_Requirements};     # start time
param To{1..Nb_Requirements};       # end time
param UBC{1..Nb_Requirements};
param SBC{1..Nb_Requirements};


#model
set SHIFTS; # set of workshifts, indexed by start time
#set shift_type;
set shift_type;
set company;
#set company := UBC SBC;

#var x{SHIFTS};
var x{i in SHIFTS, j in company, k in shift_type} integer >=0;
# minimize Total : sum{ j in SHIFTS}  Agents_Starting_At[j];
minimize total_labor_hours : sum{ i in SHIFTS, j in company} (9*x[i,j,1]+9*x[i,j,3]+4*x[i,j,2]);      #not sure if this works

subject to UBC_requirement {r in SHIFTS}:
sum { i in SHIFTS, j in company, k in shift_type: ((i>=5) and (i <= r) and (i > r-4) and (j=1) and (i<=24))
                                                or ((i>=5) and (i<=r-5) and (i>r-9) and (i<=24) and (j=1) and (k=1))
                                                or ((i>=5) and (i<=r-5) and (i>r-9) and (i<=24) and (j=2) and (k=3))}
    x[i,j,k]
   >=UBC[r-4];

subject to SBC_requirement {r in SHIFTS}:
sum { i in SHIFTS, j in company, k in shift_type: ((i>=5) and (i <= r) and (i > r-4) and (j=2) and (i<=24))
                                                or ((i>=5) and (i<=r-5) and (i>r-9) and (i<=24) and (j=2) and (k=1))
                                                or ((i>=5) and (i<=r-5) and (i>r-9) and (i<=24) and (j=1) and (k=3))}
    x[i,j,k]
   >=SBC[r-4];

subject to UBC_nd_shift :
sum{i in SHIFTS}
    (x[i,1,1]+x[i,1,3])
>=sum{i in SHIFTS, k in shift_type}
    (0.65*x[i,1,k]);

subject to SBC_nd_shift :
sum{i in SHIFTS}
    (x[i,2,1]+x[i,2,3])
>=sum{i in SHIFTS, k in shift_type}
    (0.65*x[i,2,k]);
```

And here is the run file:

```
reset;       # clears AMPL memory, to allow repeated runs
             # within the same AMPL session
model bus_model.mod;
data  bus_data.dat;

option display_1col 0;  # for compact display
option display_round 2; # output formatting (rounded to 2 decimal digits)

# Next, solve the IP model using the CPlex MIP solver:
option solver cplex;
solve;

display x;
```

The results it generates is as follows:

```
x [*,1,*]
:      1       2       3
5     2.00    1.00    0.00
6     0.00    2.00    5.00
7     0.00    2.00    0.00
8     0.00    0.00    0.00
9     0.00    0.00    1.00
10    0.00    0.00    0.00
11    0.00    0.00    0.00
12    0.00    0.00    0.00
13    0.00    0.00    2.00
14    0.00    0.00    2.00
15    3.00    0.00    0.00
16    0.00    3.00    0.00
17    0.00    0.00    0.00
18    0.00    0.00    0.00
19    0.00    0.00    0.00
20    0.00    0.00    0.00
21    0.00    0.00    0.00
22    0.00    0.00    0.00
23    0.00    0.00    0.00
24    0.00    0.00    0.00
```

```
 [*,2,*]
:      1       2       3
5     0.00    0.00    5.00
6     0.00    0.00    7.00
7     0.00    3.00    5.00
8     0.00    0.00    0.00
9     0.00    0.00    0.00
10    1.00    0.00    0.00
11    0.00    0.00    0.00
12    0.00    0.00    0.00
13    0.00    0.00    0.00
14    0.00    0.00    2.00
15    0.00    9.00    0.00
16    0.00    2.00    0.00
17    4.00    0.00    2.00
18    0.00    0.00    0.00
19    0.00    0.00    0.00
20    0.00    0.00    0.00
21    0.00    0.00    0.00
22    0.00    0.00    0.00
23    0.00    0.00    0.00
24    0.00    0.00    0.00
```

```
objective 457
```

It is slightly different combination but the objective is still the same as we get from part b and c.

(e) Use your results in (c) or (d) above to estimate the labour cost savings, if any, that may result from the use of dual shits in case of the SBC acquisition.

Before the acquisition, there is no dual shift. In order to see how many labour hours have been saved, we add the following constraint that all dual shifts are 0.

```
subject to no_dual :
sum{i in SHIFTS, j in company}
x[i,j,3]
=0;
```

This gives an objective of 510 hours in total, which means dual shifts save 510-457=53 hours.

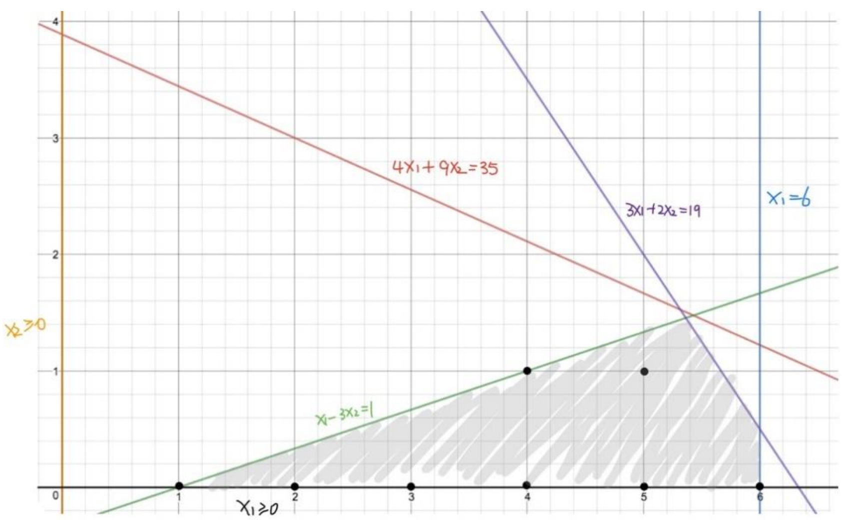
Question 2

2. LP-based Branch-and-Bound, graphical example. Solve the following two-variable integer program by the LP-based Branch-and-Bound method explained in class. (Note the direction of the third inequality). Solve each LP relaxation using the graphical method in the (x_1, x_2)-space. Detail all calculations and display the branch-and-bound tree as it evolves.

$$\begin{aligned} \text{maximize} \quad & 9x_1 + 5x_2 \\ \text{subject to} \quad & 4x_1 + 9x_2 \leq 35 \\ & x_1 \leq 6 \\ & x_1 - 3x_2 \geq 1 \\ & 3x_1 + 2x_2 \leq 19 \\ & x_1 \text{ and } x_2 \geq 0, \text{ integer.} \end{aligned}$$

Incumbent x*=(0,0), c(x*)=0

Solve original problem P:



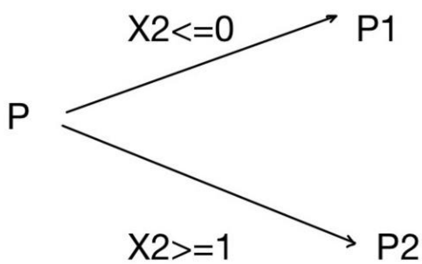
decision variables	x1	x2	
	6	0.5	
constraints			
4x1+9x2<=35	28.5	<=	35
x1<=6	6	<=	6
x1-3x2>=1	4.5	>=	1
3x1+2x2<=19	19	<=	19
nonnegativity			
objective function	9x1+5x2		
	56.5		

Initial dual bound z^{LP}=56.5 with LP solution x^{LP}=(6,0.5)

Subproblem(P1) add x2<=0

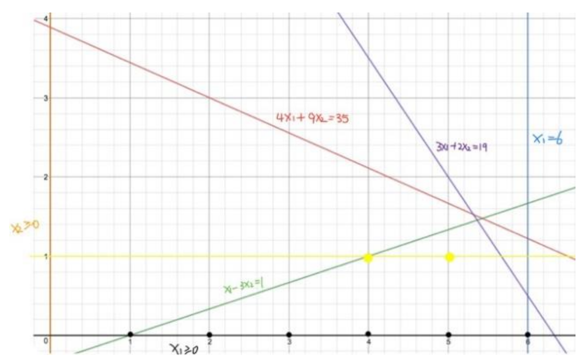
Subproblem(P2) add x2>=1

The branch and bound tree is



Solving P2:

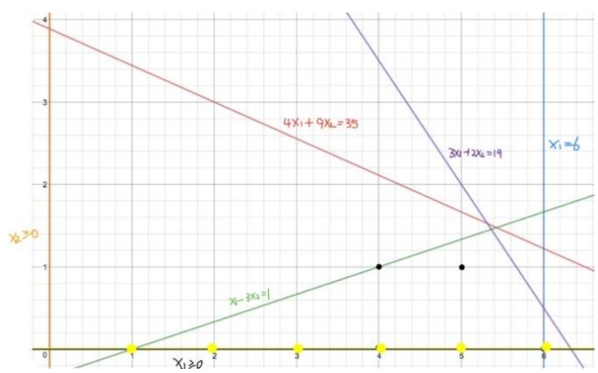
decision variables	x1	x2	
	5.666667	1	
constraints			
4x1+9x2<=35	31.66667	<=	35
x1<=6	5.666667	<=	6
x1-3x2>=1	2.666667	>=	1
3x1+2x2<=19	19	<=	19
x2>=1	1	>=	1
nonnegativity			
objective function	9x1+5x2		
	56		



Dual bound z^{LP2} =56, LP solution x^{LP1}=(5.666667,1)

Solving P1:

decision variables	x1	x2	
	6	0	
constraints			
4x1+9x2<=35	24	<=	35
x1<=6	6	<=	6
x1-3x2>=1	6	>=	1
3x1+2x2<=19	18	<=	19
x2<=0	0	<=	0
nonnegativity			
objective function	9x1+5x2		
	54		



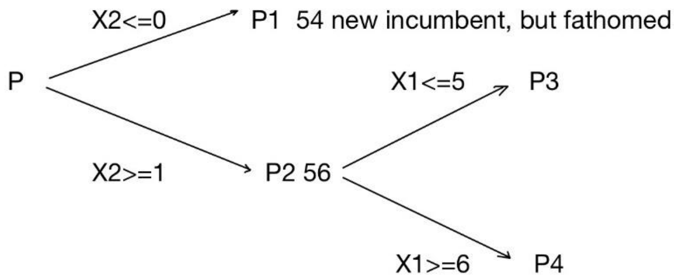
Dual bound z^{LP1} =54<56, LP solution x^{LP1}=(6,0) is integer. Fathomed by the bound.

New incumbent=x*=(6,0), c(x*)=54

Subproblem P3: add x1<=5

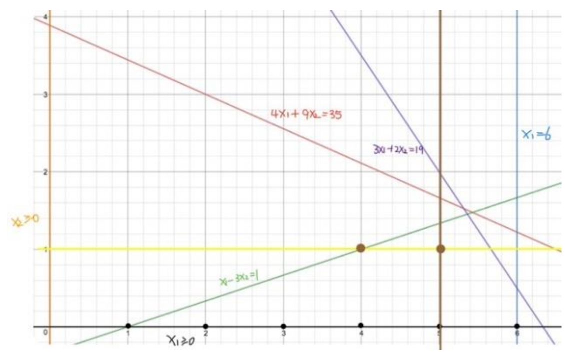
Subproblem P4: add x1>=6

The branch and bound tree becomes



Solving P3

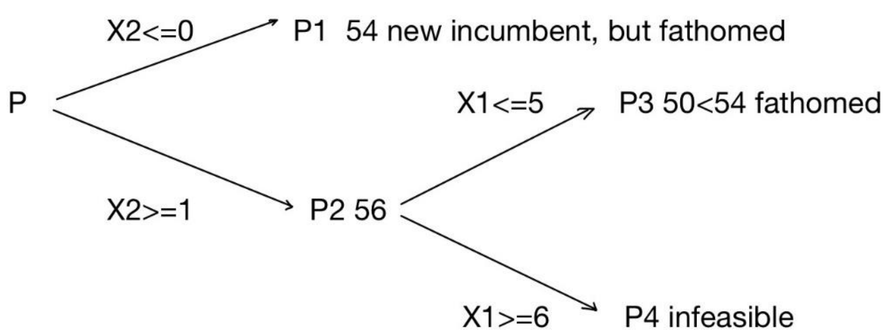
decision variables x1	x2		
	5	1	
constraints			
4x1+9x2<=35	29	<=	35
x1<=6	5	<=	6
x1-3x2>=1	2	>=	1
3x1+2x2<=19	17	<=	19
x2>=1	1	>=	1
x1<=5	5	<=	5
nonnegativity			
objective functior 9x1+5x2			
	50		



Dual bound z^{LP3} =50<54

P4 is infeasible.

The branch and bound tree becomes



Therefore, incumbent x*=(6,0) is optimal. The optimal value is 54.