Case Report: Abbotsford Beads Company

Prepared for

Abbotsford Beads Company Management
Abbotsford Beads Company

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1 Executive Summary

Abbotsford Beads Company (ABC), ships specialty low-density beads to a distributor in Montreal at regular intervals using Canada Post. Initially, ABC shipped cubic parcels of size 60 cm along each side, at an annual shipping cost of $12,778.00. However, ABC's management learned their annual shipping cost could decrease since their box sizes were never previously optimized. Our services were requested **to reduce their mailing costs** by using an optimized parcel shape.

This report discusses the nonlinear programming models we formulated under various scenarios and our recommendations for ABC's optimal parcel specifications strategy. We first considered optimizing parcels in the shape of a right circular cylinder, because we proved the maximum volume of a parcel is obtained using one, then we considered rectangular parcels.

The key objective here for the consultation was **to reduce the mailing costs** by using an optimized parcel shape. Given the aforementioned goals, we determined a right circular cylinder with dimensions of 1 m length and a radius of 0.31831 m yields an optimal maximum volume 0.318 m^3. On the other hand, a maximum volume of 0.250 m^3 is obtained by using a rectangular parcel with dimensions of length equal to 1 m, and both the height and width equalling 0.5 m.

Per request of the Client, we estimated how much ABC would save in yearly shipping costs by adopting our solutions for its Montreal mailings. We determined that only 136 boxes, instead of 200 boxes, are required to be shipped if right circular cylinder-shaped parcels are used. Using such parcels instead of cubic parcels would meet the beads demand at an annual average cost of **$8689.04** to save ABC **$4,088.96** on shipment per year. On the other hand, shipping rectangular-shaped parcels is less optimal. For instance, we calculated that 173 boxes per year must be shipped to meet the beads demand which only saves ABC **$1,725.03** on shipment per year. The average cost of shipment increases by **$2,363.93** per year if ABC is restricted to using rectangular-shaped parcels over right circular cylinder-shaped parcels.

Our analysis was conducted for both right circular cylinder and rectangular-shaped parcels while meeting the following requirements:

1. **Max Length:** The length of the right circular cylinder or rectangular parcel cannot exceed 2 m.
2. **Girth**: The sum of the length and girth cannot exceed 3 m.
3. **Min Length:** The length of the right circular cylinder or rectangular parcel must be at least 7 cm.
4. **Width**: The width (i.e. 2 times the radius for right circular cylinder) must be at least 1 mm.
5. **Height**: The height (i.e. 2 times the radius right circular cylinder) must be at least 7 cm.
6. **Non-negativity**: Both the length, width and height must be positive numbers.

Furthermore, we made the following assumptions to conduct our analysis:

1. The beads are low density so the maximum weight is never reached.
2. There is no minimum weight per parcel.
3. All side issues such as packaging, handling, storage, and inventory costs can be ignored while determining the yearly savings in shipping cost.

Given the aforementioned considerations and findings, we concluded that ABC can meet all yearly bead demands by shipping 136 right circular cylinder-shaped parcels with dimensions of 1 m length and a radius of 0.31831 m. Using such parcels instead of cubic parcels would meet the beads demand at an annual average cost of **$8689.04** to save ABC **$4,088.96** on shipment per year.

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2 Introduction

Abbotsford Beads Company (ABC) ships specialty low-density beads to a distributor in Montreal at regular intervals using Canada Post. Initially, ABC shipped cubic parcels of size 60 cm along each side, at an annual shipping cost of $12,778.00. However, ABC's management learned their annual shipping cost could decrease since their box sizes were never previously optimized. Our services were requested **to reduce their mailing costs** by using an optimized parcel shape.

This report discusses the nonlinear programming models we formulated under various scenarios and our recommendations for ABC's optimal parcel specifications strategy. We first considered optimizing parcels in the shape of a right circular cylinder, then we considered rectangular-shaped parcels.

Our analysis was conducted for both right circular cylinder and rectangular-shaped parcels while meeting the following requirements:

1. **Max Length:** The length of the right circular cylinder or rectangular parcel cannot exceed 2 m.
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5. **Height**: The height (i.e. 2 times the radius right circular cylinder) must be at least 7 cm.
6. **Non-negativity**: Both the length, width and height must be positive numbers.

Furthermore, we made the following assumptions to conduct our analysis:

4. The beads are low density so the maximum weight is never reached.
5. There is no minimum weight per parcel.
6. All side issues such as packaging, handling, storage, and inventory costs can be ignored while determining the yearly savings in shipping cost.

2.1 Information Provided

The Abbotsford Beads Company's Yearly Shipping Cost

Abbotsford Beads Company (ABC), ships specialty low-density beads to a distributor in Montreal at regular intervals using Canada Post. Initially, ABC shipped cubic parcels of size 60 cm along each side, at an annual shipping cost of $12,778.00.

Canada Post’s Regular Parcel Specifications

Canada Post requires that parcels, as illustrated in Figure 1 below, stay within certain specifications. Both minimum and maximum size and weight requirements are listed in Table 1 below. In particular, Canada Post uses the length-plus-girth requirement instead of volume because it is easier and faster for a mail agent to determine using a single measuring tape and, for irregularly shaped parcels, it avoids complicated volume calculations.

FIGURE 1: Canada Post Regular Rectangular Parcel Specifications.

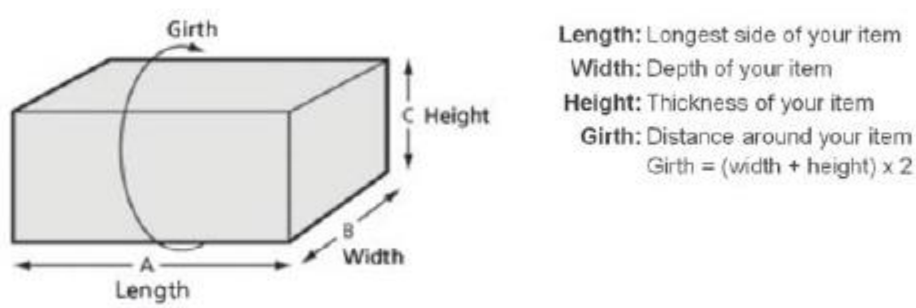


TABLE 1: Canada Post Regular Parcel Specifications.

	Dimensions			
	Length	Width	Height	Weight
Minimum	0.10 m	0.07 m	0.001 m	-
Maximum		Length = 2		30 kg
		Length + Girth = 3		

Canada Posts Regular Parcel Rates Charged to the Abbotsford Beads Company

The cost of shipping ABC’s beads, given they are low density items, is $63.89 per parcel independent of their weight. Table 2 below accounts for how this cost was tabulated.

TABLE 2: Canada Post regular parcel rates charged to Abbotsford Beads Company.

Cost	Option	Cost ($)
Base Price	-	56.21
Coverage	-	0.00
Fuel Surcharge	-	4.64
Options:	Collection on Delivery ($7.25)	0.00
(not used)	Mailing Tube ($1.50)	
	Delivery Confirmation (Incl.)	
	Signature Option ($1.50)	
	Unpackaged ($10.00)	
Tax	-	3.04
Total		**63.89**

Question (a) - Geometric Facts Proving the Maximum Volume Parcel is Right Circular Cylinder

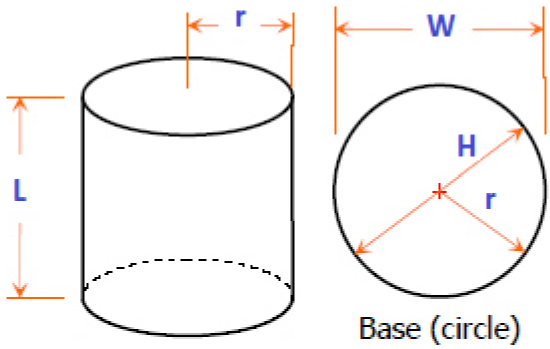
Using simple geometric facts, including the isoperimetric inequality, and reasoning, we demonstrated a maximum volume parcel satisfying Canada Post's parcel limits is a right circular cylinder. Our derivation is found in Appendix A.

3 Analysis for Right Circular Cylinder Parcels

3.1 Nonlinear Programming Model for Right Circular Cylinder

Since the maximum volume of a parcel satisfying Canada Post's parcel limits is a right circular cylinder, we formulated our nonlinear programming model to optimize such a parcels volume. Figure 3 illustrates the diagram of the right circular cylinder used in this nonlinear programming model.

FIGURE 2: Right circular cylinder diagram with variables length, L, and radius, r.



Decision Variables: For the coming year, let:

R	denote the radius of the right circular cylinder.	[m]
L	denote the height of the right circular cylinder.	[m]

Constraints: The following constraints define the nonlinear model to find the optimal solution:

1. **Max Length:** The length of the right cylinder cannot exceed 2 m.
   $L \leq 2$ [m]

2. **Girth**: The sum of the length and girth cannot exceed 3 m.
   $L + 2*\pi*R \leq 3$ [m]

3. **Min Length:** The length of the right cylinder must be at least 10 cm.
   $L \geq 0.10$ [m]

4. **Width**: The width must be at least 1 mm.
   $2*R \geq 0.001$ [m]

5. **Height**: The height must be at least 7 cm.
   $2*R \geq 0.07$ [m]

6. **Non-negativity**: Both the length and height must be positive numbers.
   R ≥ 0 [m]
   L ≥ 0 [m]

Objective Function:

The objective is to maximize the volume of the right circular cylinder. Therefore, we seek to:

max { π*R²*L } = V

The above data was entered in an Excel spreadsheet to facilitate manipulation of the model. A screenshot of the representation of the model in Excel model is provided in Appendix B.

3.2 Optimal Solution

Using the GRG Nonlinear method built-in to Excel's Solver add-on, we find that all aforementioned requirements can be met to produce as maximum volume of 0.318 m^3 for a right circular cylinder-shaped parcel. This can be achieved by setting the length to 1 m while the radius is 0.31831 m. The constructed nonlinear model can be found as an Excel spreadsheet is found in Figure 3 in Appendix B. Additionally, as per request by the ABC's Management, we generated an additional sensitivity output, of post-optimality analysis, which can be found in Appendix B as Figure 4.

3.3 Optimal Solution for Various Conditions

Question (c): Right Circular Cylinder has Convex Objective Function

The right circular cylinder's objective function can be proven to not be convex by the following steps:

1. The objective function f(x) is a function of 2 variables, f: $R^n \rightarrow R$
2. Determining whether the symmetric n x n matrix, of the Hessian Hf(x) is positive semi definite
   - this can be verified if and only if the n determinants of the square sub-matrices consisting of the first j rows and columns, for all j = 1, …, n, are all nonnegative. While it would be positive definite if all principled minors are strictly positive.
   - I determined the Hf(x) and it's determinants as follows:

$V(R,L) = \pi * R^2 * L$

$$\nabla V(R,L) = [\begin{matrix} \frac{d}{dR} (\pi * R^2 * L) \\ \frac{d}{dL} (\pi * R^2 * L) \end{matrix}] = [\begin{matrix} 2 * \pi * R * L \\ \pi * R^2 \end{matrix}]$$

$$\nabla^2 V(R,L) = [\begin{matrix} \frac{d}{dR} (2 * \pi * R * L) & \frac{d}{dL} (2 * \pi * R * L) \\ \frac{d}{dR} (\pi * R^2) & \frac{d}{dL} (\pi * R^2) \end{matrix}]$$

$$= [\begin{matrix} (2 * \pi * L) & (2 * \pi * R) \\ (2\pi * R) & 0 \end{matrix}]$$

$det \mid 2 * \pi * L \mid \; = \; 2 * \pi * L \geq 0$

$$det \begin{bmatrix} (2 * \pi * L) & (2 * \pi * R) \\ (2\pi * R) & 0 \end{bmatrix} = -(2 * \pi * R)^2 \leq 0$$

3. Second order characterization: if f is twice differentiable, then:
   - it is convex if and only if its Hessian Hf(x) is positive semi-definite for all x ε R^n
   - if its Hessian Hf(x) is positive definite for all x ε R^n then f is strictly convex
   - However, our function is not positive definite, therefore, the **function is not convex**.

The right circular cylinder's objective function can be proven to not be concave by the following steps:

1. The objective function f(x) is a function of 2 variables, g = -f: $R^n \rightarrow R$
2. Determining whether the symmetric n x n matrix, of the Hessian Hf(x) is positive semi definite
   - this can be verified if and only if the n determinants of the square sub-matrices consisting of the first j rows and columns, for all j = 1, …, n, are all nonnegative. While it would be positive definite if all principled minors are strictly positive.
   - I determined the Hf(x) and it's determinants as follows:

$V(R,L) = -\pi * R^2 * L$

$$\nabla V(R,L) = [\begin{matrix} \frac{d}{dR} (-\pi * R^2 * L) \\ \frac{d}{dL} (-\pi * R^2 * L) \end{matrix}] = [\begin{matrix} -2 * \pi * R * L \\ -\pi * R^2 \end{matrix}]$$

$$\nabla^2 V(R,L) = [\begin{matrix} \frac{d}{dR} (-2 * \pi * R * L) & \frac{d}{dL} (-2 * \pi * R * L) \\ \frac{d}{dR} (-\pi * R^2) & \frac{d}{dL} (-\pi * R^2) \end{matrix}]$$

$$= [\begin{matrix} (-2 * \pi * L) & (-2 * \pi * R) \\ (-2\pi * R) & 0 \end{matrix}]$$

$det \mid 2 * \pi * L \mid = -2 * \pi * L \leq 0$

3. Second order characterization: if f is twice differentiable, then:
    - it is concave if and only if its Hessian Hf(g(x)) is positive semi-definite for all x ε R^n
    - if its Hessian Hf(g(x)) is positive definite for all x ε R^n then f is strictly convex
    - However, our function is not positive definite, therefore, the **function is not concave**.

Question (d): Right Circular Cylinder – Karush-Kuhn-Tucker (KKT) Conditions

As per request by the Client, we were tasked to write out the Karush-Kuhn-Tucker (KKT) conditions our model as follows:

(NLP-min) $\min_x \{f(x) : g_i(x) \leq 0 \ \ \forall i = 1..m\}$:

$$\begin{cases} \nabla f(x^*) + \sum_{i=1..m} v_i^* \nabla g_i(x^*) = 0 & & \text{gradient condition} \\ v^* \geq 0 & & \text{dual feasibility} \\ g_i(x^*) \leq 0 & \text{for all } i = 1..m & \text{primal feasibility} \\ v_i^* \cdot g_i(x^*) = 0 & \text{for all } i = 1..m & \text{complementary slackness} \end{cases}$$

Primal Feasibility:

1. $g_1(x)$: L – 2 ≤ 0
2. $g_2(x)$: L + (2*π*R) – 3 ≤ 0
3. $g_3(x)$: 0.10 – L ≤ 0
4. $g_4(x)$: 0.07 – 2*R ≤ 0
5. $g_5(x)$: -R ≤ 0
6. $g_6(x)$: -L ≤ 0

Dual Feasibility:

$$v^* = \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \\ v_4^* \\ v_5^* \\ v_6^* \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Complementary Slackness:

1. v_1^* [L – 2] = 0
2. v_2^* [L + (2*π*R) – 3] = 0
3. v_3^* [0.10 – L] = 0
4. v_4^* [0.07 – 2*R] = 0
5. v_5^* [-R] = 0
6. v_6^* [-L] = 0

Gradient Condition:

- $\nabla g_1(x) = \begin{bmatrix} \frac{d}{dR}(L-2) \\ \frac{d}{dL}(L-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\nabla g_2(x) = \begin{bmatrix} \frac{d}{dR}(\mathrm{L} + (2*\pi*\mathrm{R}) - 3) \\ \frac{d}{dL}(\mathrm{L} + (2*\pi*\mathrm{R}) - 3) \end{bmatrix} = \begin{bmatrix} 2\pi \\ 1 \end{bmatrix}$
- $\nabla g_3(x) = \begin{bmatrix} \frac{d}{dR}(0.1-L) \\ \frac{d}{dL}(0.1-L) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
- $\nabla g_4(x) = \begin{bmatrix} \frac{d}{dR}(0.07 - 2*\mathrm{R}) \\ \frac{d}{dL}(0.07 - 2*\mathrm{R}) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$
- $\nabla g_5(x) = \begin{bmatrix} \frac{d}{dR}(-R) \\ \frac{d}{dL}(-R) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
- $\nabla g_6(x) = \begin{bmatrix} \frac{d}{dR}(-L) \\ \frac{d}{dL}(-L) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

1. $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\pi RL - 2\pi v_2^* + 2v_4^* + v_5^* \\ \pi R^2 - v_1^* - v_2^* + v_3^* + v_6^* \end{bmatrix}$

Using the GRG Nonlinear method built-in to Excel's Solver add-on, we find that all aforementioned requirements can be met to produce as maximum volume of **0.007697 m^3.** This can be achieved by setting the length of box to 2 m, the radius to 0.035 m, and v1 is found to be 0.0038. The constructed nonlinear model can be found as an Excel spreadsheet is found in Figure 5 in Appendix C. Additionally, we generated an additional sensitivity output, of post-optimality analysis, which can be found in Appendix C as Figure 6.

However, a value of 0.007697 m^3 is not the optimal solution. Although we determined this value using the KKT conditions, the necessary conditions required to be met by KKT were not satisfied. For example, as determined in Section 3.3, we proved the objective found is not convex or concave. Consequently, the optimal values R*, and L* cannot be optimum solutions using the KKT conditions. Thus, the original optimum solution yielding a maximum volume of 0.318 m^3 for a right circular cylinder-shaped parcel, with a length to 1 m while the radius is 0.31831 m, is the optimum solution.

Question (e): Right Circular Cylinder – Global Optimum and Yearly Shipping Cost Savings

Per request of the Client, we found an optimal solution and estimated how much ABC would save in yearly shipping costs by adopting our solution for its Montreal mailings. We determined that only 136 boxes, instead of 200 boxes, are required to be shipped to meet the beads demand at an annual average cost of **$8689.04**. This would save ABC **$4,088.96** on shipment per year. Our calculations are depicted in Appendix A.

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4 Analysis for Rectangular Parcels

4.1 Nonlinear Programming Model for Right Circular Cylinder

Decision Variables: For the coming year, let:

L	denote the height of the rectangular parcel.	[m]
W	denote the height of the rectangular parcel.	[m]
H	denote the height of the rectangular parcel.	[m]

Constraints: The following constraints define the nonlinear model to find the optimal solution:

1. **Max Length:** The length of the rectangular parcel cannot exceed 2 m.
   $L \leq 2$ [m]

2. **Girth**: The sum of the length and girth cannot exceed 3 m.
   $L + 2*(H + W) \leq 3$ [m]

3. **Min Length:** The length of the rectangular parcel must be at least 10 cm.
   $L \geq 0.10$ [m]

4. **Width**: The width must be at least 1 mm.
   $W \geq 0.001$ [m]

5. **Height**: The height must be at least 7 cm.
   $H \geq 0.07$ [m]

6. **Non-negativity**: Both the length and height must be positive numbers.
   $L \geq 0$ [m]
   $H \geq 0$ [m]
   $W \geq 0$ [m]

Objective Function:

The objective is to maximize the volume of the right circular cylinder. Therefore, we seek to:

max { L*W*H } = V

The above data was entered in an Excel spreadsheet to facilitate manipulation of the model. A screenshot of the representation of the model in Excel model is provided in Appendix D.

4.2 Optimal Solution

Using the GRG Nonlinear method built-in to Excel's Solver add-on, we find that all aforementioned requirements can be met to produce as maximum volume of 0.250 m^3. This can be achieved by setting the length to 1 m, and both the height and width to be 0.5 m. The constructed nonlinear model can be found as an Excel spreadsheet is found in Figure 7 in Appendix D. Additionally, as per request by the ABC's Management, we generated an additional sensitivity output, of post-optimality analysis, which can be found in Appendix D as Figure 8.

4.3 Optimal Solution for Various Conditions

Question (f-c): Rectangular Parcel has Convex Objective Function

The rectangular parcel's objective function can be proven to be convex by the following steps:

1. The objective function f(x) is a function of 2 variables, f: $R^n \rightarrow R$
2. Determining whether the symmetric n x n matrix, of the Hessian Hf(x) is positive semi definite
    - this can be verified if and only if the n determinants of the square sub-matrices consisting of the first j rows and columns, for all j = 1, …, n, are all nonnegative. While it would be positive definite if all principled minors are strictly positive.
    - I determined the Hf(x) and it's determinants as follows:

      $V(L, W, H) = L$*W*H

$$\nabla V(L, W, H) = [\begin{matrix} \frac{d}{dL}(L*W*H) \\ \frac{d}{dw}(L*W*H) \\ \frac{d}{dH}(L*W*H) \end{matrix}] = [\begin{matrix} WH \\ LH \\ LW \end{matrix}]$$

$$\nabla^2 V(L, W, H) = [\begin{matrix} \frac{d}{dL}(WH) & \frac{d}{dW}(WH) & \frac{d}{dH}(WH) \\ \frac{d}{dL}(LH) & \frac{d}{dW}(LH) & \frac{d}{dH}(LH) \\ \frac{d}{dL}(LW) & \frac{d}{dW}(LW) & \frac{d}{dH}(LW) \end{matrix}]$$

$$= [\begin{matrix} 0 & H & W \\ H & 0 & L \\ W & L & 0 \end{matrix}]$$

$$det\,|\,W\,| = W \geq 0$$

$$det \begin{bmatrix} H & W \\ 0 & L \end{bmatrix} = HL \geq 0$$

$$\det \begin{vmatrix} 0 & H & W \\ H & 0 & L \\ W & L & 0 \end{vmatrix} = 0 - \text{H}*(-L^2) + \text{W}*(\text{HL}) \geq 0$$

3. Second order characterization: if f is twice differentiable, then:
   - it is convex if and only if its Hessian Hf(x) is positive semi-definite for all x ε R^n
   - if its Hessian Hf(x) is positive definite for all x ε R^n then f is strictly convex
   - However, our function is not positive definite, therefore, the **function is convex**. **Consequently, the function cannot be concave**.

Question (f-d): Rectangular Parcel – Karush-Kuhn-Tucker (KKT) Conditions

As per request by the Client, we were tasked to write out the Karush-Kuhn-Tucker (KKT) conditions our model as follows:

$$\text{(NLP-min)} \quad \min_x \{ f(x) \; : \; g_i(x) \leq 0 \;\; \forall i = 1..m \} :$$

$$\begin{cases} \nabla f(x^*) + \sum_{i=1..m} v_i^* \; \nabla g_i(x^*) = 0 & & \text{gradient condition} \\ v^* \geq 0 & & \text{dual feasibility} \\ g_i(x^*) \leq 0 & \text{for all } i = 1..m & \text{primal feasibility} \\ v_i^* \cdot g_i(x^*) = 0 & \text{for all } i = 1..m & \text{complementary slackness} \end{cases}$$

Primal Feasibility:

1. $g_1(x)$: L – 2 ≤ 0
2. $g_2(x)$: L + 2*H + 2*W – 3 ≤ 0
3. $g_3(x)$: 0.10 – L ≤ 0
4. $g_4(x)$: 0.001 – W ≤ 0
5. $g_4(x)$: 0.07 – H ≤ 0

Dual Feasibility:

$$v^* = \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \\ v_4^* \\ v_5^* \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Complementary Slackness:

1. v_1^* [L – 2] = 0
2. v_2^* [L + (2*π*R) – 3] = 0
3. v_3^* [0.10 – L] = 0
4. v_4^* [0.07 – 2*R] = 0
5. v_5^* [-R] = 0
6. v_6^* [-L] = 0

Gradient Condition:

- $\nabla g_1(x) = \begin{bmatrix} \frac{d}{dL}(L-2) \\ \frac{d}{dH}(L-2) \\ \frac{d}{dW}(L-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- $\nabla g_2(x) = \begin{bmatrix} \frac{d}{dL}(\mathrm{L} + 2*\mathrm{H} + 2*\mathrm{W} - 3) \\ \frac{d}{dH}(\mathrm{L} + 2*\mathrm{H} + 2*\mathrm{W} - 3) \\ \frac{d}{dW}(\mathrm{L} + 2*\mathrm{H} + 2*\mathrm{W} - 3) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

- $\nabla g_3(x) = \begin{bmatrix} \frac{d}{dL}(0.10 - \mathrm{L}) \\ \frac{d}{dH}(0.10 - \mathrm{L}) \\ \frac{d}{dW}(0.10 - \mathrm{L}) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

- $\nabla g_4(x) = \begin{bmatrix} \frac{d}{dL}(0.001 - \mathrm{W}) \\ \frac{d}{dH}(0.001 - \mathrm{W}) \\ \frac{d}{dW}(0.001 - \mathrm{W}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

- $\nabla g_5(x) = \begin{bmatrix} \frac{d}{dL}(0.07 - \mathrm{H}) \\ \frac{d}{dH}(0.07 - \mathrm{H}) \\ \frac{d}{dW}(0.07 - \mathrm{H}) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$$\nabla V(L, W, H) = \begin{bmatrix} \frac{d}{dL}(L * W * H) \\ \frac{d}{dH}(L * W * H) \\ \frac{d}{dW}(L * W * H) \end{bmatrix} = \begin{bmatrix} WH \\ LW \\ LH \end{bmatrix}$$

1. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} WH - v_1^* - v_2^* + v_3^* \\ LW - 2 * v_2^* + v_5^* \\ LH - 2 * v_2^* + v_4^* \end{bmatrix}$

Using the GRG Nonlinear method built-in to Excel's Solver add-on, we find that all aforementioned requirements can be met to produce as maximum volume of **0.25 m^3**. This can be achieved by setting the length to 1 m, and both the height and width to 0.5 m. The constructed nonlinear model can be found as an Excel spreadsheet is found in Figure 9 in Appendix E. Additionally, we generated an additional sensitivity output, of post-optimality analysis, which can be found in Appendix E as Figure 10.

Question (f-g): Rectangular Parcel – Global Optimum and Yearly Shipping Cost Savings

Per request of the Client, we found an optimal solution and estimated how much ABC would save in yearly shipping costs by adopting our solution for its Montreal mailings. We determined that only 173 boxes, instead of 200 boxes, are required to be shipped to meet the beads demand at an annual average cost of **$11,052.97**. This would save ABC **$1,725.03** on shipment per year. The average cost of shipment increases by **$2,363.93** per year if ABC is restricted to using rectangular-shaped parcels over right circular cylinder-shaped parcels. Our calculations are depicted in Appendix A.

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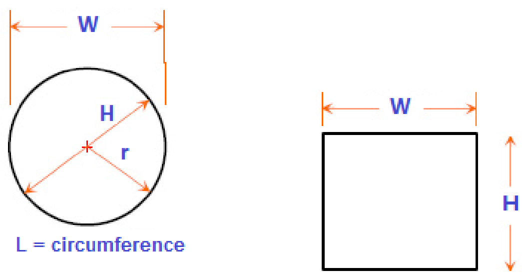
5 Appendices

Appendix A. Unit Conversions

The conclusion a maximum volume parcel satisfying Canada Post parcel limits is a right circular cylinder was derived by the following reasoning:

1. First, consider the geometry of the cross-sections of a right circular cylinder and a cubic parcel below:



2. For a circular parcel base:
   - $Perimeter = L = 2 * \pi * r$
   - $Area = \pi * r^2$
3. For a rectangular parcel base:
   - $Perimeter = L = 2 * (W + H)$
   - $Area = W * H$
4. The isoperimetric inequality states that the length L of a closed curve and area A that the 2D region encloses, will obey the following inequality if and only if the curve is a circles:

$$4\pi A \leq L^2$$

5. For a circular parcel base:

$$4 * \pi * (\pi * r^2) \leq (2 * \pi * r)^2 = 4 * \pi^2 * r^2$$

6. Where as, when comparing the areas and perimeters of a circular and rectangular base:
   - By setting the perimeters equal to each other, we get:

$$2 * \pi * r = 2 * (W + H)$$

$$\pi * r = W + H$$ (but H and W appear ~ 2*r)

$$\pi \neq 4$$

   - Therefore, the W and H must be smaller than 2*r
7. Consequently, the corresponding areas must adjust according to the isoperimetric inequality. Therefore, the **maximum volume parcel satisfying Canada Post parcel limits is a right circular cylinder.**

The following calculations were performed to determine the average volume of each of the 200 boxes originally shipped by ABC:

- Cubic parcels of size 60 cm along each side had an annual shipping cost = $\frac{\$\ 12{,}778}{year}$
    - L = H = W = 60 cm is still within the requirement specifications
- Total Volume per Box = $(60\text{ cm})^3$ = $\frac{216000\ cm^3}{box}$
- $\frac{\$\ 12{,}778}{year} * \frac{box}{\$63.89} = \frac{200\ boxes}{year}$
- $\frac{216000\ cm^3}{box} * \frac{200\ boxes}{year} * (\frac{1m}{100cm})^3 = \frac{43.2\ m^3}{year}$

The average cost of shipment per year using right circular cylinder parcels:

- Right Circular Cylinder Total Volume Per Parcel = $\frac{0.318309894\ m^3}{box}$
- Number of Parcels Per Year to Meet Volume = $\frac{43.2\ m^3}{year} * \frac{parcel}{0.318309894\ m^3} = 135.7167993\ \frac{parcels}{year}$
    - Must use whole boxes, so 136 parcels are required to meet the bead demand per year
- The Yearly Cost using Right Circular Cylinder Parcels = $136\ \frac{parcels}{year} * \frac{\$63.89}{\text{parcel}} = \frac{\$8689.04}{\text{year}}$
- Hence, the yearly savings are $12,778 - $8689.04 = $4,088.96 per year.

The average cost of shipment per year using rectangular-shaped parcels:

- Rectangular-shaped parcel total volume = $\frac{0.25\ m^3}{box}$
- Number of parcels required per year to meet demand = $\frac{43.2\ m^3}{year} * \frac{parcel}{0.25\ m^3} = 172.8\ \frac{parcels}{year}$
    - Must use whole boxes, so 173 parcels are required to meet the bead demand per year
- The yearly cost using rectangular-shaped parcels = $173\ \frac{parcels}{year} * \frac{\$63.89}{\text{parcel}} = \frac{\$11052.97}{\text{year}}$
- Hence, the yearly savings are $12,778 - $11,052.97 = $1,725.03 per year.

The average cost of shipment savings per year using right circular cylinder-shaped parcels over rectangular-shaped parcels:

- $11,052.97 - $8689.04 = $2,363.93 per year.

Appendix B. Representation of Right Circular Cylinder Model in Excel with the Optimal Solution

FIGURE 3: Maximum volume of 0.318 m^3 obtained for an optimized right circular cylinder as a parcel.

5	Data:	Length	Width	Height	Weight	Girth	
6	minimum	10 cm	7 cm	1 mm	-	-	
7	maximum	2 m	-	-	30 kg	3 m - Length	
8	Girth	2*(Length + Width)					
9							
10							
11							
12	Model:						
13	Decision Variables			L	R		
14	Values			1.00000	0.31831		
15							
16	Objective Function:		max { π*(R^2)*L }			0.318309894	m^3
17							

Constraints:					Slack
Max Length	L	1.00000	≤	2	-1
Girth	L + (2*pi()*R)	3.00000	≤	3	2.504E-08
Min Length	L	1.00000	≥	0.1	0.9
Width	R*2	0.63662	≥	0.001	0.6356198
Height	2*R	2	≥	0.07	1.9300001
Non-Negativity	R	0.31831	≥	0	0.3183099
	L	1.00000	≥	0	1

FIGURE 4: Sensitivity report corresponding to producing a maximum volume of 318 m^3 obtained for an optimized right circular cylinder as a parcel.

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [2016-10-01_A4 - Case.xlsx]Question 1-b
Report Created: 10/1/2016 8:57:31 PM

Variable Cells

Cell	Name	Final Value	Reduced Gradient
D14	Values L	1.000000034	0
E14	Values R	0.318309885	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
K6	L	1.000000034	0
K7	L + (2*pi()*R)	3.000000025	0.31831056
K8	L	1.000000034	0
K9	R*2	0.63661977	0
K10	2*R	2.000000068	0
K11	R	0.318309885	0
K12	L	1.000000034	0

Appendix C. Variants of the Right Circular Cylinder Model

FIGURE 5: Maximum volume of **0.007697** m^3 obtained for an optimized right circular cylinder as a parcel when taking Karush-Kuhn-Tucker conditions into account.

Data:	Length	Width	Height	Weight	Girth				
minimum	10 cm	7 cm	1 mm	-	-				
maximum	2 m	-	-	30 kg	3 m - Length				
Girth	2*(Length + Width)								
Model:									
Decision Variables		L	R	v1*	v2*	v3*	v4*	v5*	v6*
Values		2.00000	0.035	0.003848672	0.00000	0.00000	0.00000	0.00000	0.00000
Objective Function:	max { π*(R^2)*L }			0.007696902	m^3				

Constraints:					Slack
Primal Feasibility	L - 2	0.00000	≤	0	0.00
	L + (2*pi*R) - 3	-0.78009	≤	0	-0.78
	0.1 - L	-1.90000	≤	0	-1.90
	0.07 - 2*R	2.54E-14	≤	0	0.00
	-R	-0.03500	≤	0	-0.03
	-L	-2.00000	≤	0	-2.00
Complementary	v1*(L-2)	0.00000	=	0	0.00
Slackness	v2(L + 2*pi*R - 3)	0.00000	=	0	0.00
	v3*(0.1 - L)	0	=	0	0.00
	v5*(0.07 - 2*R)	0	=	0	0.00
	v6*(-R)	0.00000	=	0	0.00
	v7*(-L)	0.00000	=	0	0.00
Dual Feasibility	v1	0.00385	≥	0	0.00
	v2	0.00000	≥	0	0.00
	v3	0.00000	≥	0	0.00
	v5	0.00000	≥	0	0.00
	v6	0.00000	≥	0	0.00
	v7	0.00000	≥	0	0.00
Gradient	d/dR	0.439823	=	0	0.44
Condition	d/dL	-2.2E-07	=	0	0.00

FIGURE 6: Sensitivity report corresponding to producing a maximum volume of **0.007697** m^3 obtained for an optimized right circular cylinder as a parcel when taking Karush-Kuhn-Tucker conditions into account.

Variable Cells

Cell	Name	Final Value	Reduced Gradient
D10	Values L	2	0
E10	Values R	0.035	0
F10	Values v1*	0.003848672	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
D14	L - 2 L	0	-0.035
D15	L + (2*pi*R) - 3 L	-0.780088514	0
D16	0.1 - L L	-1.9	0
D17	0.07 - 2*R L	2.53964E-14	0.999999951
D18	-R L	-0.035	0
D19	-L L	-2	0
D20	v1*(L-2) L	0	0
D21	v2(L + 2*pi*R - 3) L	0	0
D22	v3*(0.1 - L) L	0	0
D23	v5*(0.07 - 2*R) L	0	0
D24	v6*(-R) L	0	0
D25	v7*(-L) L	0	0
D26	v1 L	0.003848672	0
D27	v2 L	0	0
D28	v3 L	0	0
D29	v5 L	0	0
D30	v6 L	0	0
D31	v7 L	0	0
D32	d/dR L	0.439822972	0
D33	d/dL L	-2.21005E-07	0

Appendix D. Representation of Rectangular Parcel Model in Excel with the Optimal Solution

FIGURE 7: Maximum volume of 0.250 m^3 obtained for an optimized rectangular-shaped parcel.

5	**Data:**	**Length**	**Width**	**Height**	**Weight**	**Girth**	
6	**minimum**	10 cm	7 cm	1 mm	-	-	
7	**maximum**	2 m	-	-	30 kg	3 m - Length	
8	**Girth**	2*(Length + Width)					
9							
10							
11							
12	**Model:**						
13	**Decision Variables**			**L**	**H**	**W**	
14	Values			1.00000	0.499999	0.50000	
15							
16	**Objective Function:**		max { L*H*W }			0.249999995	m^3
17							

Constraints:					**Slack**
Max Length	L	1.00000	≤	2	-1.0000013
Girth	L + 2*(H+W)	3.00000	≤	3	-1.855E-08
Min Length	L	1.00000	≥	0.1	0.8999987
Width	W	0.50000	≥	0.001	0.4990019
Height	H	0.499999	≥	0.07	0.4299988
Non-Negativity	W	0.50000	≥	0	0.5000019
	L	1.00000	≥	0	0.9999987
	H	0.50000	≥	0	0.4999988

FIGURE 8: Sensitivity report corresponding to producing a maximum volume of 0.250 m^3 obtained for an optimized rectangular-shaped parcel.

Microsoft Excel 15.0 Sensitivity Report
Worksheet: [2016-10-01_A4 - Case.xlsx]Rectangle Optimum
Report Created: 10/2/2016 7:20:04 PM

Variable Cells

Cell	Name	Final Value	Reduced Gradient
D14	Values L	0.999998699	0
E14	Values H	0.499998762	0
F14	Values W	0.50000188	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
K6	L	0.999998699	0
K7	L + 2*(H+W)	2.999999981	0.249999069
K8	L	0.999998699	0
K9	W	0.50000188	0
K10	H	0.499998762	0
K11	W	0.50000188	0
K12	L	0.999998699	0
K13	H	0.499998762	0

Appendix E. Variants of the Rectangular Parcel Model

FIGURE 9: Maximum volume of **0.25** $\boldsymbol{m^3}$ obtained for an optimized rectangular-shaped parcel when taking Karush-Kuhn-Tucker conditions into account.

Data:	Length	Width	Height	Weight	Girth
minimum	10 cm	7 cm	1 mm	-	-
maximum	2 m	-	-	30 kg	3 m - Length
Girth	2*(Length + Width)				

Model:								
Decision Variables	**L**	**H**	**W**	**v1**	**v2**	**v3**	**v4**	**v5**
Values	1.00001	0.499989	0.50001	0	0.249996	0	0	0

Objective Function:	max { L*H*W }	0.25	m^3

Constraints:					Slack
Primal Feasibility	L - 2	-0.99999	≤	0	-1.00
	L + 2*(H+W) - 3	0.00000	≤	0	0.00
	0.1 - L	-0.90001	≤	0	-0.90
	0.001 - W	-0.49901	≤	0	-0.50
	0.07 - H	-0.42999	≤	0	-0.43
Complementary	v1*(L-2)	0.00000	=	0	0.00
Slackness	v2(L + 2*(H+W)- 3	0.00000	=	0	0.00
	v3*(0.1 - L)	0	=	0	0.00
	v4*(0.001-1)	0	=	0	0.00
	v5*(0.7-H)	0.00000	=	0	0.00
Dual Feasibility	v1	0.00000	≥	0	0.00
	v2	0.25000	≥	0	0.25
	v3	0.00000	≥	0	0.00
	v4	0.00000	≥	0	0.00
	v5	0.00000	≥	0	0.00
Gradient	d/dL	2.09E-06	=	0	0.00
Condition	d/dH	1.8E-05	=	0	0.00
	d/dW	-1E-09	=	0	0.00

FIGURE 10: Sensitivity report corresponding to producing a maximum volume of **0.25** m^3 obtained for an optimized rectangular-shaped parcel when taking Karush-Kuhn-Tucker conditions into account.

Variable Cells

Cell	Name	Final Value	Reduced Gradient
D12	Values L	1.000006397	0
E12	Values H	0.499989419	0
F12	Values W	0.500007382	0
G12	Values v1	0	0
H12	Values v2	0.249996309	0
I12	Values v3	0	0
J12	Values v4	0	0
K12	Values v5	0	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
D17	L - 2 L	-0.999993603	0
D18	L + 2*(H+W) - 3 L	1.06848E-12	-0.499995342
D19	0.1 - L L	-0.900006397	0
D20	0.001 - W L	-0.499007382	0
D21	0.07 - H L	-0.429989419	0
D22	v1*(L-2) L	0	-0.999995094
D23	v2(L + 2*(H+W)- 3) L	2.67116E-13	-0.999982488
D24	v3*(0.1 - L) L	0	0
D25	v4*(0.001-1) L	0	0
D26	v5*(0.7-H) L	0	0
D27	v1 L	0	0.99999867
D28	v2 L	0.249996309	0
D29	v3 L	0	-0.99999867
D30	v4 L	0	0.999998074
D31	v5 L	0	-0.499998769
D32	d/dL L	2.09132E-06	0
D33	d/dH L	1.79625E-05	0
D34	d/dW L	-1.01536E-09	-0.999998037