Homework Assignment 5

Due Wednesday October 12, 9:00 am, at the start of the Final Exam
This homework set consists of 6 exercises. No Case Report due for this homework assignment.

Exercise 1. *(15 marks)*
Consider the following nonlinear optimization problem: find $x = (x_1, x_2)^T \in \Re^2$ to

$$\begin{aligned} &\text{minimize} \quad x_1^2 - 24\,x_1 + x_2^2 - 10\,x_2 \\ &\text{subject to} \quad 0 \le x_1 \le 8 \text{ and } 0 \le x_2 \le 7 \end{aligned}$$

(a) Is this a convex programming problem?
(b) Write the KKT optimality conditions for this problem. Are these conditions necessary for an optimum solution? Are they sufficient?
(c) Show that this problem can be decomposed into two independent problems, one for each variable. Solve each of these two single-variable problems and find corresponding Lagrange multipliers (shadow prices) for the bound constraints.
(d) Use your answer in **(c)** above to find a solution to the KKT conditions of question **(b)**.

Exercise 2. *(15 marks)*
Consider the following nonlinear optimization problem: find $x = (x_1, x_2)^T \in \Re^2$ to

$$\begin{aligned} &\text{minimize} \quad x_1^2 - \ln(x_2 + 1) \\ &\text{subject to} \quad 2\,x_1 + x_2 \le 3 \\ &\qquad\qquad\quad x_1,\ x_2 \ge 0 \end{aligned}$$

where *ln* denotes the natural logarithm.

(a) Is this a convex programming problem?
(b) Write the KKT optimality conditions for this problem. Are these conditions necessary for an optimum solution? Are they sufficient?
(c) Use the KKT conditions to prove that $x = (1, 1)^T$ is *not* an optimum solution.
(d) Use the KKT conditions to find an optimum solution and corresponding shadow price and reduced costs.

Exercise 3. *(15 marks)*
Consider the following linearly constrained programming problem:

$$\begin{aligned} &\text{minimize} \quad x_1^3 + 4\,x_2^2 + 16\,x_3 \\ &\text{subject to} \quad x_1 + x_2 + x_3 = 5 \\ &\qquad\qquad\quad x_1 \ge 1, \quad x_2 \ge 1, \quad x_3 \ge 1 \end{aligned}$$

(a) Write the KKT conditions for this problem.
(b) Use the KKT conditions to check whether $(x_1, x_2, x_3) = (2, 1, 2)$ is optimal.

Exercise 4. *(15 marks)*
Problem 13.7-7 in H&L: The management of the Albert Hanson Company is trying to determine the best product mix for two new products. Because these products would share the same production facilities, the total number of units produced of the two products combined cannot exceed two per hour. Because of uncertainty about how well these products will sell, the profit from producing each product provides decreasing marginal returns as the production rate is increased. In particular, with a production rate of R_1 units per hour, it is estimated that Product 1 would provide a profit per hour of $200\,R_1 - 100\,R_1^2$. If the production rate of Product 2 is R_2 units per hour, its estimated profit per hour would be $300\,R_1 - 100\,R_1^2$.

- **(a)** Formulate a quadratic programming model in algebraic form for determining the product mix that maximizes the total profit per hour.
- **(b)** Solve this model using MS Excel and its GRG Nonlinear solving method.
- **(c)** Is the solution found in part **(b)** optimal? How do you know that you have really solved the problem? Explain.

Exercise 5. *(20 marks)*
Problem 13.8-1: The MFG Corporation is planning to produce and market three different products. Let x_1, x_2 and x_3 denote the number of units of the three respective products to be produced. The preliminary estimates of their potential profitability are as follows.

For the first 15 units produced of Product 1, the unit profit would be approximately \$360. The unit profit would be only \$30 for any additional units of Product 1. For the first 20 units produced of Product 2, the unit profit is estimated at \$240. The unit profit would be \$120 for each of the next 20 units and \$90 for any additional units. For the first 10 units of Product 3, the unit profit would be \$450. The unit profit would be \$300 for each of the next 5 units and \$180 for any additional units.

Certain limitations on the use of needed resources impose the following constraints on the production of the three products:

$$
\begin{aligned}
x_1 + x_2 + x_3 &\le 60\\
3x_1 + 2x_2 &\le 200\\
x_1 + 2x_3 &\le 70
\end{aligned}
$$

Management wants to know what values of x_1, x_2 and x_3 should be chosen to maximize the total profit.

- **(a)** Plot the profit graph for each of the three products.
- **(b)** Formulate a <u>linear</u> programming model for this problem.
- **(c)** Solve the model using MS Excel. What is the resulting recommendation to management about the values of x_1, x_2 and x_3 to use?
- **(d)** Now suppose that there is an additional constraint that the profit from products 1 and 2 must total at least \$12,000. Add this constraint to the model formulated in part **(b)** and repeat part **(c)**.

Exercise 6. *(20 marks)*
Problem 13.8-7: The MFG Corporation also produces a certain subassembly in each of two separate plants. These subassemblies are then brought to a third nearby plant where they are used in the production of a certain product. The peak season of demand for this product is approaching, so to maintain the production rate within a desired range, it is necessary to use temporarily some overtime in making the subassemblies. The cost per subassembly on regular time (RT) and on overtime (OT) is shown in the following table for both plants, along with the maximum number of subassemblies that can be produced on RT and on OT each day.

	Unit Cost		Capacity	
	RT	**OT**	**RT**	**OT**
Plant 1	$15	$25	2,000	1,000
Plant 2	$16	$24	1,000	500

Let x_1 and x_2 denote the total number of subassemblies produced per day at plants 1 and 2, respectively. The objective is to maximize $x_1 + x_2$ subject to the constraint that the total daily cost does not exceed $60,000.

(a) Formulate a linear programming model for this problem.

(b) Solve the model using MS Excel.

(c) Explain why the logic of separable programming also applies here to guarantee that an optimal solution for the model formulated in part **(a)** never uses OT unless the RT capacity at that plant has been fully used.

(Total: 100 marks)