# BAMS 506 – Optimal Decision Making

## Assignment 4

## Fall 2016

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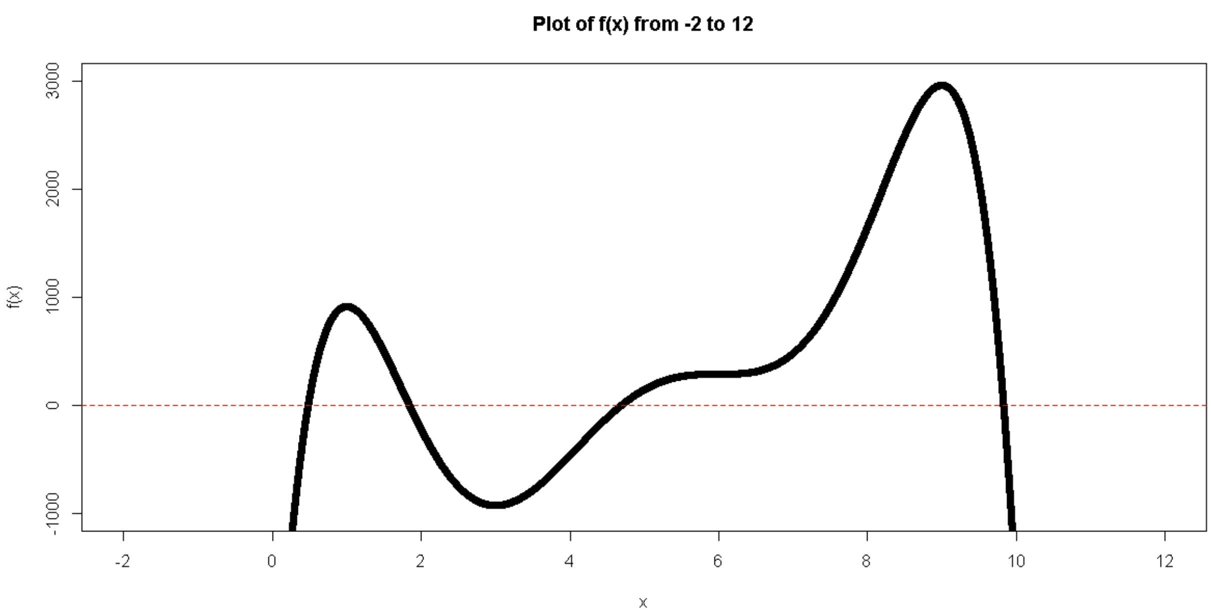
## Question 1:

(a) How many critical points can this function have? (Hint: note that this function is a degree-6 polynomial.) Find all critical points, their objective values, and identify all local minima, global minima, local maxima, global maxima, and inflection points.

f(x) = 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600

According to the fundamental theorem of algebra, a polynomial of degree n will have n roots. Therefore, our function f(x) has n = 6 since $x^6$ is in the function, and is therefore capable of having up to 6 roots. In addition, polynomial functions of 1 variable can have up to n-1 critical points. Consequently, **the function f(x) can have up to n-1 = 5 critical points.**

First, I plotted f(x) in R as shown below (all R codes are attached at the back of this document*):




I observed that f(x) is smooth (i.e. continuously differentiable). Then, I determined the zeros of f(x), using the uniroot function, to be:

1. 0.4798419
2. 1.840258
3. 4.69172
4. 9.830928

Next, I differentiated f(x) once to find f'(x) and again to find f''(x) as shown below:

f'(x) = 11,664 – (2*10,368*(x)) + (3*3,852*(x^2)) – (4*693*(x^3)) + (5*60*(x^4)) – (2*6*(x^5))
= 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ]

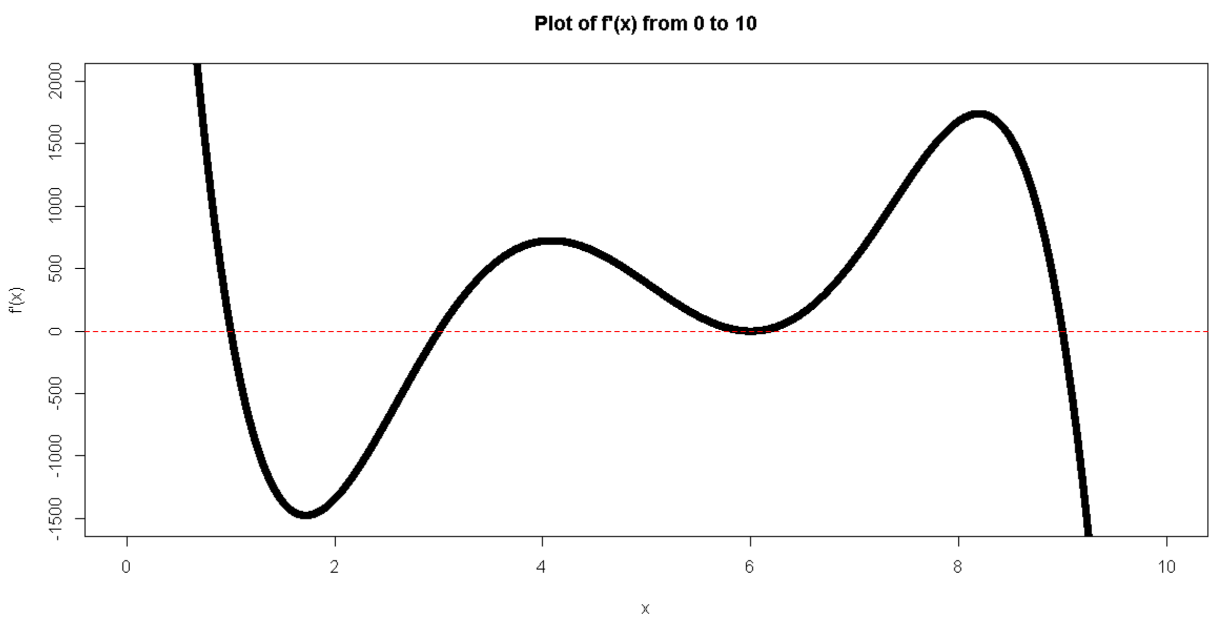
f''(x) = -20,736 + (2*11,556*(x)) – (3*2772*(x^2)) + (4*300*(x^3)) – (5*12*(x^4))
= -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ]

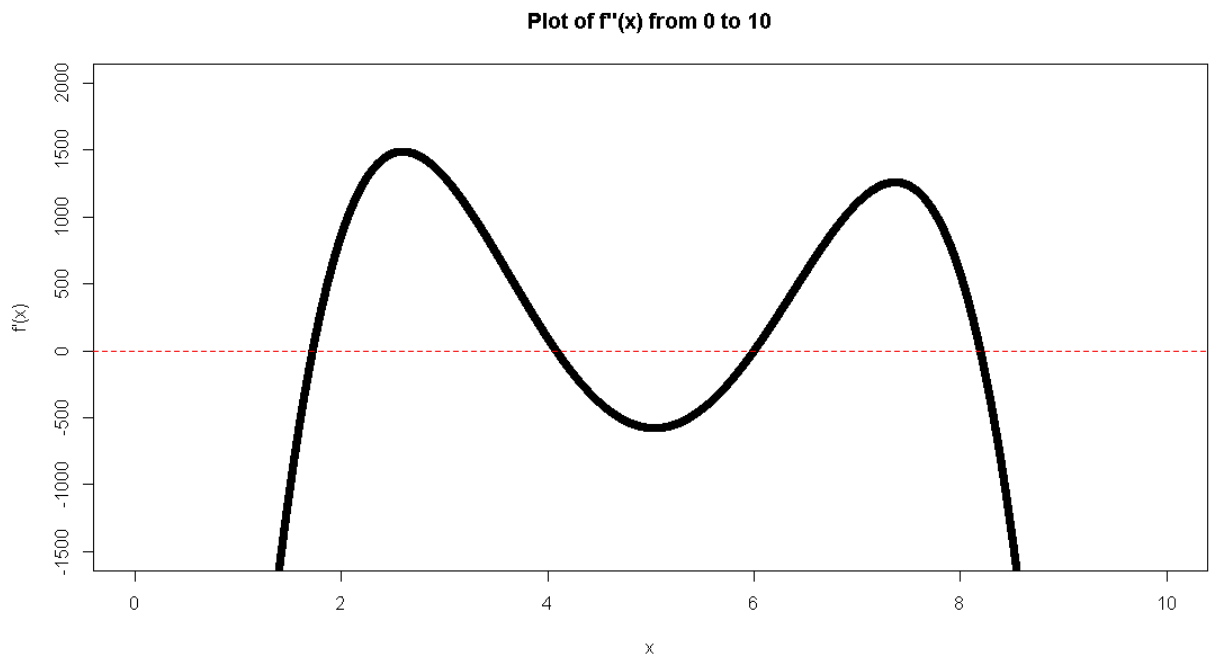
Afterwards, I used the same process to locate the zeros of each function. I determined the following roots as tabulated below (again, see the back of this document for my R-code):

| X | f(x) | f'(x) | f''(x) | Significance |
|---|---|---|---|---|
| 1 | 913 | 0 | - | local maximum (critical point) |
| 1.721144 | 174.6121 | - | 0 | inflection point |
| 3 | -927 | 0 | - | local minimum (critical point) |
| 4.083678 | -403.5292 | - | 0 | inflection point |
| 6 | 288 | 0 | - | inflection point |
| 8.195182 | 1983.554 | - | 0 | inflection point |
| 9 | 2961 | 0 | - | global maximum (critical point) |

There is no global minimum.

My functions for f'(x) and f''(x) are shown below:



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(b) Is the First-Order Condition f ’(x) = 0 necessary for x to be a global minimum of f (over the whole real line R)? Is it sufficient?

Yes, the first order condition f’(x) = 0 is necessary for x to be a global minimum of f over the whole real line. However, this condition alone is not sufficient. In addition, f(x) must be a concave function for condition to be sufficient. (Pages 552 and 563 in the textbook*). This means the following 2 conditions must be met:

- f(x*) ≤ f(y)          for all critical points y, and
- f(x*) ≤ lim $inf_{x\to\infty} f(x)$ and f(x*) ≤ lim $inf_{x\to-\infty} f(x)$

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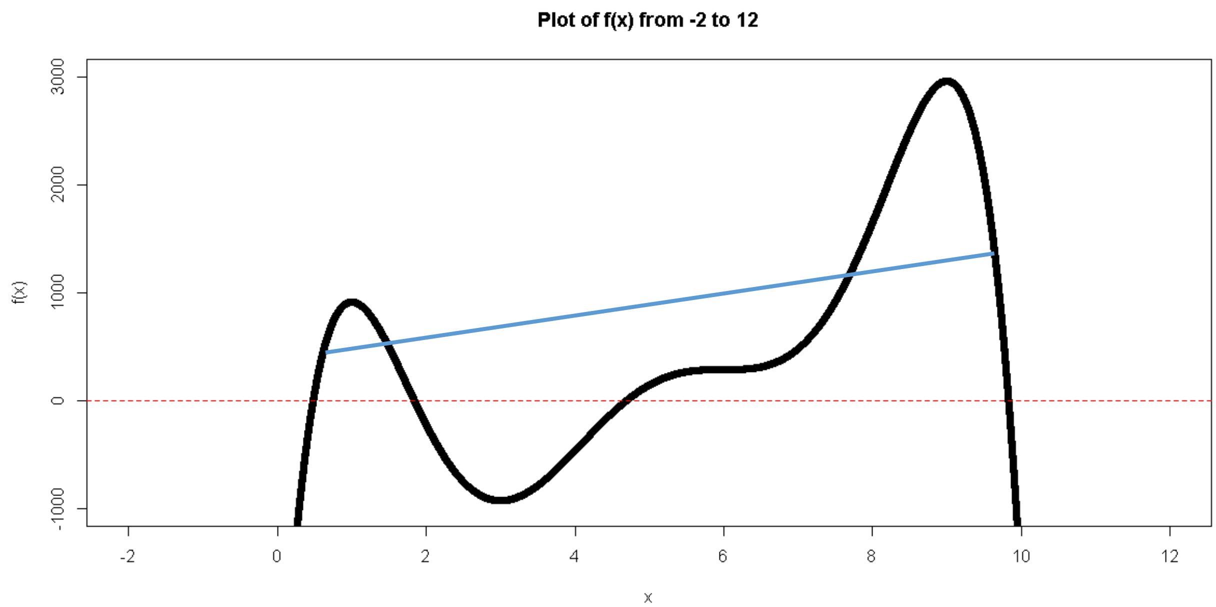
(c and d) Is function f convex (resp., strictly convex, concave, strictly concave) on the whole real line R? On the interval [0,10]?

No, the function f is not strictly convex or strictly concave on the whole real line R or on the interval [0,10]. For instance, f does not lie strictly below or above all cords as seen below:




Furthermore, given the inflection points I found in part (a), f does in fact curve down in some locations.

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(e) Consider the bound-constrained optimization problem min{ f (x) : 0 < x < 10}. Write the optimality conditions for this problem. Are these optimality conditions necessary for a global minimum over the interval [0, 10]? Are they sufficient?

**Decision Variables**: let:

x denote the value of x. [no units]

**Constraints**: The following constraints define the nonlinear model to find the optimal solution:

1. **Lower Bound:** The largest value of x allowed is greater than or equal to 0.
   $x \geq 0$ [no units]

2. **Upper Bound (non-negativity):** The largest value of x allowed is less than or equal to 10.
   $x \leq 10$ [no units]

**Objective Function**:
The objective is to minimize the function f(x). Therefore, we seek to:

**max { 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 } = f(x)**

The optimality conditions are necessary for global minimum over the interval [0, 10]. However, because f(x) is not convex then the necessary conditions are not sufficient for x* to be optimium.

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(f) Find all solutions (x, u, v) to these optimality conditions. (Hint: start with the Complementary Slackness conditions: for each CS condition one of its two factors must be zero.) For which of these solutions is x a local minimum, a global minimum, a local maximum or a global maximum over the interval [0, 10], or an inflection point?

**Necessary Optimality Conditions**:
Given the smooth function f: R → R and bounds 0 ≤ x ≤ 10
if x* is an optimum solution <u>Objective function</u>:

min{ f (x) : 0 < x < 10}
for f(x) = 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600

(i) x* satisfies 0 – x* ≤ 0 and x* - 10 ≤ 0 (i.e. x* is feasible) and
(ii) there exists numbers u and v satisfying
- the gradient condition: f'(x*) – u + v = 0
- the dual feasibility conditions: u ≥ 0, v ≥ 0
- the complementary slackness conditions: u(0 – x*) = 0 and v*(x* - 10) = 0

**<u>Note</u>**: the interpretation of u and v:
- u is the negative of the shadow price for constraint 0 – x* ≤ 0
- v is the negative of the shadow price for constraint x* -10 ≤ 0

We used Wolfram Mathematica to solve the values of x, u, and v. Please see our output below.

Input:

$$\{11\,664 - 20\,736\,x + 11\,556\,x^2 - 2772\,x^3 + 300\,x^4 - 12\,x^5 - u + v = 0,\ -u\,x = 0,\ v\,(x - 10) = 0\}$$

Alternate forms:

$$\{u + 12\,(x-9)\,(x-3)\,(x-1)\,(x-6)^2 = v,\ u\,x = 0,\ v\,x = 10\,v\}$$

$$\{u + 12\,(x^3 - 13\,x^2 + 39\,x - 27)\,(x-6)^2 = v,\ u\,x = 0,\ v\,x = 10\,v\}$$

$$\{v = 12\,x^5 - 300\,x^4 + 2772\,x^3 - 11\,556\,x^2 + 20\,736\,x + u - 11\,664,\ u = 0 \lor x = 0,\ v = 0 \lor x = 10\}$$

$e_1 \lor e_2 \lor \ldots$ is the logical OR function

Solutions: More solutions

$u = 0, \quad v = 0, \quad x = 1$

$u = 0, \quad v = 0, \quad x = 3$

$u = 0, \quad v = 0, \quad x = 6$

$u = 0, \quad v = 0, \quad x = 9$

$u = 0, \quad v = 12096, \quad x = 10$

Please note that for u and v both equal to 0, I calculated the solutions of x in question 1 (a) above. However, I must calculate the values for x when only one of u or v is equal to 0.

**When u = 0:**

$-u * x = 0 \rightarrow x \neq 0$

$v(x - 10) = 0$ when x = 10

f'(10) = 11,664 – (20,736*(10)) + (11,556*(10^2)) – (2772*(10^3)) + (300*(10^4)) – (12*(10^5)) – 0 + v = 0

**v = 12,096**

**When v = 0:**

$v * (x - 10) = 0 \rightarrow x \neq 10$

$-u * x = 0 \rightarrow x$

f'(0) = 11,664 – (20,736*(0)) + (11,556*(0^2)) – (2772*(0^3)) + (300*(0^4)) – (12*(0^5)) – u + v = 0

0 = 11,664 + 0 – u

**u = 11,664**

Next, I plugged in my values into the function f(x) to compute the solution of the constrained optimization problem for the interval x over [0, 10]

| X | f(x) | f'(x) | f''(x) | Significance |
|---|---|---|---|---|
| 0 | -3600 | - | - | Global minimum |
| 1 | 913 | 0 | - | Local maximum (critical point) |
| 3 | -927 | 0 | - | Local minimum (critical point) |
| 6 | 288 | 0 | - | Inflection point |
| 9 | 2961 | 0 | - | Global maximum (critical point) |
| 10 | −1760 | - | - | |

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(g) Solve the constrained optimization problem of question (e) above using the computer. Use the following initial values: x = 0.5; 2; 4; 6; 8; and 9.5. (In Excel, simply enter these values in the decision variable cell before calling the Solver). Compare the solutions and Lagrange Multipliers with the solutions and shadow prices found in question (f) above. (Note that the Excel Solver implicitly treats the lower and upper bound constraints on variables. To force it to treat them explicitly and produce their Lagrange Multipliers, simply add a cell containing 1*x and use it as the left hand-side of each of the two bound constraints.)

Using the GRG Nonlinear method built-in to Excel's Solver add-on, we find that the objective function is minimized and produces a value of -3600, when x = 0.

| | |
|---|---|
| **Data:** | |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] |
| 1 | |

| **Constraints:** | | | | | Slack |
|---|---|---|---|---|---|
| Upper Bound | x | 0 | ≤ | 10 | -10 |
| Lower Bound | x | 0 | ≥ | 0 | 0 |

| **Model:** | |
|---|---|
| **Decision Variables** | **x** |
| Values | 0.00000 |

**Objective Function:** min{ f (x) : 0 < x < 10} -3600

In addition, we generated an additional sensitivity output of post-optimality analysis which can be found below:

**Microsoft Excel 15.0 Sensitivity Report**
**Worksheet: [2016-10-01_A4-01.xlsx]Question 1-g**
**Report Created: 10/3/2016 6:45:30 PM**

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 0 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 0 | 0 |
| $M$7 | x | 0 | 11663.98973 |

Next, we solved the optimization problem for the following initial values: x = 0.5; 2; 4; 6; 8; and 9.5. For instance, we set the value of x to be one of the aforementioned values, solved the maximization of the objective function, and then the minimization of the objective function. My results are tabulated below with corresponding annotated Excel spreadsheets at the end of question 1.

**Table 1:** Results of solved optimization problem for initial values of x = 0.5, 2, 4, 6, 8, and 9.5.

| **x Value Set in Excel** | **Solver's x Value (local optimum)** | **min f(x)** | **max f(x)** | **u** | **v** | **Figure Numbers** |
|---|---|---|---|---|---|---|
| 0.5 | 0 | -3600 | ---- | 0 | **11663.99539** | 1, 2 |
| 0.5 | 1 | ---- | 913 | 0 | 0 | 3, 4 |
| 2 | 3 | -927 | ---- | 0 | 0 | 5, 6 |
| 2 | 9 | ---- | 2961 | 0 | 0 | 7, 8 |
| 4 | 3 | -927 | ---- | 0 | 0 | 9, 10 |
| 4 | 9 | ---- | 2961 | 0 | 0 | 11, 12 |
| 6 | 6 | 288 | ---- | 0 | 0 | 13, 14 |
| 6 | 6 | ---- | 288 | 0 | 0 | 15, 16 |
| 8 | 3 | -297 | ---- | 0 | 0 | 17, 18 |
| 8 | 9 | ---- | 2961 | 0 | 0 | 19, 20 |
| 9.5 | 10 | -1760 | ---- | **-12096.20645** | 0 | 21, 22 |
| 9.5 | 9 | ---- | 2961 | 0 | 0 | 23, 24 |

When I compared my solutions and Langrage Multiplies with my solutions, and shadow prices in question (f), I determined:

- u is the negative of the shadow price for constraint $0 - x^* \leq 0$ and equal to the Langrage Multiplier.
- v is the negative of the shadow price for constraint $x^* - 10 \leq 0$ however, the corresponding Langrage multiplier is the opposite sign.

**Figure 1:** Excel File for setting x = 0.5 for optimizing min f(x).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Data:** | | | | | | | |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | | | | | | |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | | | | | | |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | |
| 1 | | | | | | | |

| **Constraints:** | | | | | Slack |
|---|---|---|---|---|---|
| Upper Bound | x | 0 | ≤ | 10 | -10 |
| Lower Bound | x | 0 | ≥ | 0 | 0 |

| | |
|---|---|
| **Model:** | |
| **Decision Variables** | **x** |
| Values | 0.00000 |

| | |
|---|---|
| **Objective Function:** min/max { f (x) : 0 < x < 10} | -3600 |

**Figure 2:** Sensitivity report for setting x = 0.5 for optimizing min f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| \$D\$14 | Values x | 0 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| \$M\$6 | x | 0 | 0 |
| \$M\$7 | x | 0 | 11663.99539 |

**Figure 3:** Excel File for setting x = 0.5 for optimizing max f(x).

| | |
|---|---|
| **Data:** | |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] |
| 1 | |

| **Constraints:** | | | | | Slack |
|---|---|---|---|---|---|
| Upper Bound | x | 1 | ≤ | 10 | -9 |
| Lower Bound | x | 1 | ≥ | 0 | 1 |

| | |
|---|---|
| **Model:** | |
| **Decision Variables** | **x** |
| Values | 1.00000 |

| | |
|---|---|
| **Objective Function:** min/max { f (x) : 0 < x < 10} | 913 |

**Figure 4:** Sensitivity report for setting x = 0.5 for optimizing max f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 0.999999996 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 0.999999996 | 0 |
| $M$7 | x | 0.999999996 | 0 |

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**Figure 5:** Excel File for setting x = 2 for optimizing min f(x).

| | |
|---|---|
| **Data:** | |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] |
| 1 | |

| **Constraints:** | | | | | **Slack** |
|---|---|---|---|---|---|
| **Upper Bound** | x | 3 | ≤ | 10 | -7 |
| **Lower Bound** | x | 3 | ≥ | 0 | 3 |

| **Model:** | |
|---|---|
| **Decision Variables** | **x** |
| Values | 3.00000 |

| | |
|---|---|
| **Objective Function:** min/max { f (x) : 0 < x < 10} | -927 |

**Figure 6:** Sensitivity report for setting x = 2 for optimizing min f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 2.999999964 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 2.999999964 | 0 |
| $M$7 | x | 2.999999964 | 0 |

**Figure 7:** Excel File for setting x = 2 for optimizing max f(x).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Data:** | | | **Constraints:** | | | | | **Slack** |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | | **Upper Bound** | x | 9 | ≤ | 10 | -1 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | | **Lower Bound** | x | 9 | ≥ | 0 | 9 |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | | |
| | 1 | | | | | | | |
| **Model:** | | | | | | | | |
| **Decision Variables** | | **x** | | | | | | |
| Values | | 9.00000 | | | | | | |
| **Objective Function:** | min/max { f (x) : 0 < x < 10} | 2961 | | | | | | |

**Figure 8:** Sensitivity report for setting x = 2 for optimizing max f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 8.999999901 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 8.999999901 | 0 |
| $M$7 | x | 8.999999901 | 0 |

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**Figure 9:** Excel File for setting x = 4 for optimizing min f(x).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Data:** | | | **Constraints:** | | | | | **Slack** |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | | **Upper Bound** | x | 3 | ≤ | 10 | -7 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | | **Lower Bound** | x | 3 | ≥ | 0 | 3 |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | | |
| | 1 | | | | | | | |
| **Model:** | | | | | | | | |
| **Decision Variables** | | **x** | | | | | | |
| Values | | 3.00000 | | | | | | |
| **Objective Function:** | min/max { f (x) : 0 < x < 10} | -927 | | | | | | |

**Figure 10:** Sensitivity report for setting x = 4 for optimizing min f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 2.999999938 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 2.999999938 | 0 |
| $M$7 | x | 2.999999938 | 0 |

**Figure 11:** Excel File for setting x = 4 for optimizing max f(x).

**Data:**
f(x) = 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600
f'(x) = 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ]
f''(x) = -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ]
1

| Constraints: | | | | | Slack |
|---|---|---|---|---|---|
| Upper Bound | x | 9 | ≤ | 10 | -1 |
| Lower Bound | x | 9 | ≥ | 0 | 9 |

**Model:**

| Decision Variables | x |
|---|---|
| Values | 9.00000 |

**Objective Function:** min/max { f (x) : 0 < x < 10} 2961

**Figure 12:** Sensitivity report for setting x = 4 for optimizing max f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 8.999999998 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 8.999999998 | 0 |
| $M$7 | x | 8.999999998 | 0 |

**Figure 13:** Excel File for setting x = 6 for optimizing min f(x).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Data:** | | **Constraints:** | | | | | | **Slack** |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | **Upper Bound** | x | 6 | ≤ | 10 | -4 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | **Lower Bound** | x | 6 | ≥ | 0 | 6 |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | |
| | 1 | | | | | | |

| | | |
|---|---|---|
| **Model:** | | |
| **Decision Variables** | | **x** |
| Values | | 6.00000 |

| | | |
|---|---|---|
| **Objective Function:** | min/max { f (x) : 0 < x < 10} | 288 |

**Figure 14:** Sensitivity report for setting x = 6 for optimizing min f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 6 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 6 | 0 |
| $M$7 | x | 6 | 0 |

**Figure 15:** Excel File for setting x = 6 for optimizing max f(x).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Data:** | | **Constraints:** | | | | | **Slack** |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | **Upper Bound** | x | 6 | ≤ | 10 | -4 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | **Lower Bound** | x | 6 | ≥ | 0 | 6 |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | |
| | 1 | | | | | | |

| | | |
|---|---|---|
| **Model:** | | |
| **Decision Variables** | | **x** |
| Values | | 6.00000 |

| | | |
|---|---|---|
| **Objective Function:** | min/max { f (x) : 0 < x < 10} | 288 |

**Figure 16:** Sensitivity report for setting x = 6 for optimizing max f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 6 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 6 | 0 |
| $M$7 | x | 6 | 0 |

**Figure 17:** Excel File for setting x = 8 for optimizing min f(x).

| **Data:** | |
|---|---|
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] |
| 1 | |

| **Constraints:** | | | | | Slack |
|---|---|---|---|---|---|
| **Upper Bound** | x | 3 | ≤ | 10 | -7 |
| **Lower Bound** | x | 3 | ≥ | 0 | 3 |

| **Model:** | |
|---|---|
| **Decision Variables** | **x** |
| Values | 3.00000 |

| **Objective Function:** min/max { f (x) : 0 < x < 10} | -927 |
|---|---|

**Figure 18:** Sensitivity report for setting x = 8 for optimizing min f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 2.999999992 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 2.999999992 | 0 |
| $M$7 | x | 2.999999992 | 0 |

**Figure 19:** Excel File for setting x = 8 for optimizing max f(x).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Data:** | | Constraints: | | | | | | Slack |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | Upper Bound | x | 9 | ≤ | 10 | | -1 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | Lower Bound | x | 9 | ≥ | 0 | | 9 |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | | |
| 1 | | | | | | | | |

| | | |
|---|---|---|
| **Model:** | | |
| **Decision Variables** | | **x** |
| Values | | 9.00000 |

| | | |
|---|---|---|
| **Objective Function:** | min/max { f (x) : 0 < x < 10} | 2961 |

**Figure 20:** Sensitivity report for setting x = 8 for optimizing max f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 8.999999509 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 8.999999509 | 0 |
| $M$7 | x | 8.999999509 | 0 |

---

**Figure 21:** Excel File for setting x = 9.5 for optimizing min f(x).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Data:** | | Constraints: | | | | | | Slack |
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 | Upper Bound | x | 10 | ≤ | 10 | | 0 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] | Lower Bound | x | 10 | ≥ | 0 | | 10 |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] | | | | | | | |
| 1 | | | | | | | | |

| | | |
|---|---|---|
| **Model:** | | |
| **Decision Variables** | | **x** |
| Values | | 10.00000 |

| | | |
|---|---|---|
| **Objective Function:** | min/max { f (x) : 0 < x < 10} | -1760 |

**Figure 22:** Sensitivity report for setting x = 9.5 for optimizing min f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 10 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 10 | -12096.20645 |
| $M$7 | x | 10 | 0 |

**Figure 23:** Excel File for setting x = 9.5 for optimizing max f(x).

**Data:**

| | |
|---|---|
| f(x) = | 11,664*x – 10,368*(x^2) + 3,852*(x^3) – 693*(x^4) + 60*(x^5) – 2*(x^6) – 3,600 |
| f'(x) = | 11,664 – [ 20,736*(x) ] + [ 11,556*(x^2) ] – [ 2772*(x^3) ] + [ 300*(x^4) ] – [ 12*(x^5) ] |
| f''(x) = | -20,736 + [ 23,112*(x) ] – [ 8,316*(x^2) ] + [ 1200*(x^3) ] – [ 60*(x^4) ] |
| 1 | |

| **Constraints:** | | | | | **Slack** |
|---|---|---|---|---|---|
| **Upper Bound** | x | 9 | ≤ | 10 | -1 |
| **Lower Bound** | x | 9 | ≥ | 0 | 9 |

| **Model:** | |
|---|---|
| **Decision Variables** | **x** |
| Values | 9.00000 |

**Objective Function:** min/max { f (x) : 0 < x < 10} 2961

**Figure 24:** Sensitivity report for setting x = 9.5 for optimizing max f(x).

Variable Cells

| Cell | Name | Final Value | Reduced Gradient |
|---|---|---|---|
| $D$14 | Values x | 8.999999832 | 0 |

Constraints

| Cell | Name | Final Value | Lagrange Multiplier |
|---|---|---|---|
| $M$6 | x | 8.999999832 | 0 |
| $M$7 | x | 8.999999832 | 0 |

# Question 2:

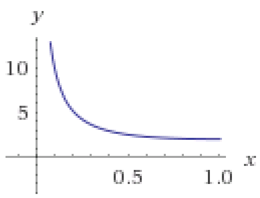
(a) Consider the single-variable function f (x) = ax + b/x where a and b are two given positive constants. Is function f convex (resp., strictly convex, concave, strictly concave) on the open interval (0, +∞) (i.e., for x > 0)? Find the global minimum x* over this interval (0, +∞).

f(x) = a*x + (b/x)

f'(x) = a - $\frac{b}{x^2}$

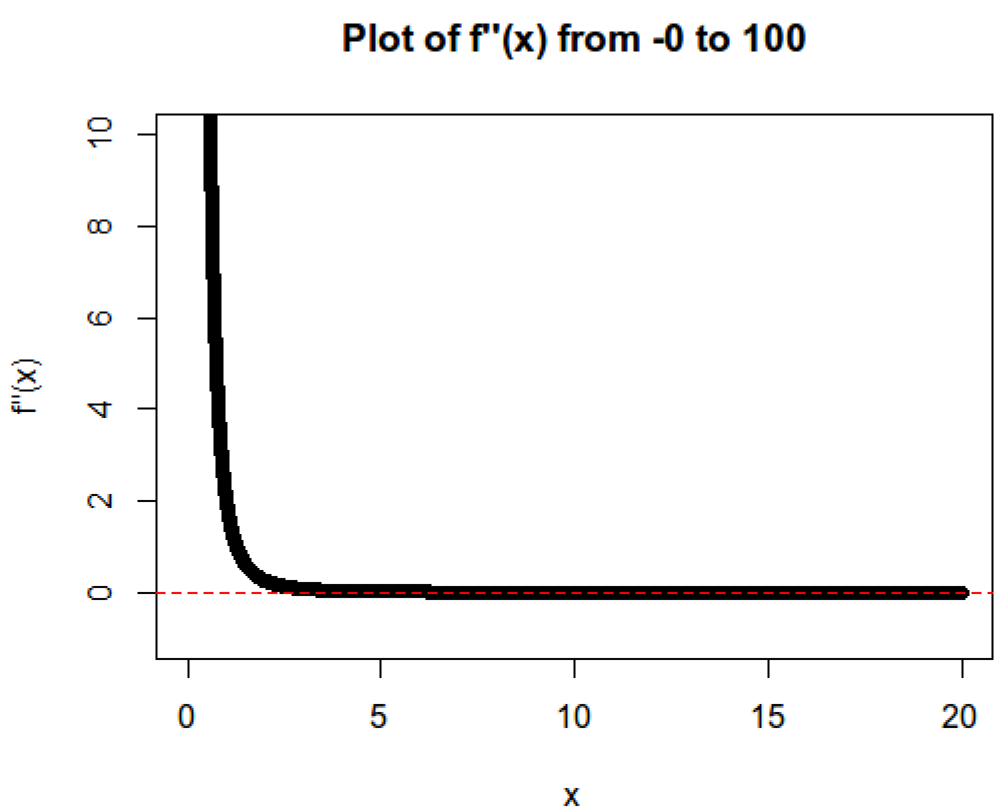
f''(x) = $\frac{2b}{x^3}$

If we set a and b to be 1, on interval of (0, +∞) then f(x) has the shape below:




Varying a and b to other positive constants, gives the same similar shape. From the graph, we can draw any cord and the function f : R → R is convex because "it lies below all its cords." For instance, f "does not curve down" anywhere.

In addition, f''(x) > 0 for all values of x from 0 to +∞ as graphed below. Hence, the function is strictly convex.



Next, we calculated the global minimum, x*, over the interval (0, +∞) as follows:

f'(x*) = 0

f'(x*) = a - $\frac{b}{x*^2}$ = 0

a*$x^{*2}$ = b

x* = $\sqrt{\frac{b}{a}}$

(b) For any x > 0 express f (x) as a function of the ratio x/x* and the optimal value f (x*) (but not of the parameters a and b) (Hint: prove that ax* = b/x* and then use this fact). For example, what are f (2x*) and f (x*/2) as functions of f (x*)?

Since x* = $\sqrt{\frac{b}{a}}$ then f($x^*$)= a*$x^*$ + ($\frac{b}{x^*}$)=a*$\sqrt{\frac{b}{a}}$ +$\frac{b}{\sqrt{\frac{b}{a}}}$ =$\sqrt{ab}$+$\sqrt{ab}$=2$\sqrt{ab}$

f(x)= a*x + ($\frac{b}{x}$)

=a$x^*$*($\frac{x}{x*}$)+$\frac{\frac{b}{x^*}}{\frac{x}{x^*}}$ = $\sqrt{ab}$($\frac{x}{x*}$+$\frac{1}{\frac{x}{x^*}}$ )=$\frac{1}{2}$ f($x^*$)( $\frac{x}{x*}$+$\frac{1}{\frac{x}{x^*}}$)

The following is a (simplified) model for managing the inventory of a divisible product, say, a fluid. Demand (or consumption) arises continuously in time, at rate D units per year. (For example, if a year has 365 days, then daily demand is D/365 units, hourly demand is D/(365*24) units, etc.) Demand has to be satisfied from inventory at every point in time, without any shortage. Inventory can be replenished at a fixed cost of K dollars for each replenishment. Let I(t) denote the inventory level at time t, so I(t) is piecewise linear with slope I'(t) = –D between replenishments and a jump of +Q at each replenishment of Q units (i.e., a replenishment instantaneously increases the inventory level by the replenishment quantity). Initial inventory I(0–) is zero. Inventory accrues holding cost continuously in time at a rate of h dollars per unit in inventory per year. The total cost of a replenishment policy over any (half-open) time interval [u, v) is the sum of the replenishment fixed costs, i.e., rK if there are r replenishments during this interval, and the inventory holding costs ( ) v t u h I t dt = ∫ during the interval. We seek a replenishment policy P which minimizes the long-run-average total cost per year. It can be shown that an optimum replenishment policy is to use equal replenishment quantities Q > 0 at equal intervals T = Q/D in such a way that the inventory level just reaches zero when the replenishment is received.

- demand [D/year]
- daily demand = D/365
- hourly demand = D/(365*24)
- demand must satisfy inventory at every point in time
- I(t) = inventory level as function of time
    - replenished at fixed cost k
    - piecewise linear
    - jump of Q at each replenishment (a point in time) [Q]
    - I[0] = 0
- total cost of replenishment = h dollars/(unit*year)

(c) Express, as a function of the replenishment quantity Q, the total cost C(Q) over such a replenishment cycle [u, u+T) where u is the replenishment date. Find, in terms of the three parameters D, K and h, the order quantity Q* which minimizes the time-averaged cycle cost g(Q) = C(Q)/T. Find the corresponding values of the order interval T* = Q*/D and minimum cost per time unit g(Q*). Verify that the units are consistent in all these formulas.

The total cost of a replenishment= replenishment fixed cost+ the inventory holding cost
Now we only consider one period [u, u+T), so there is only a replenishment cycle.

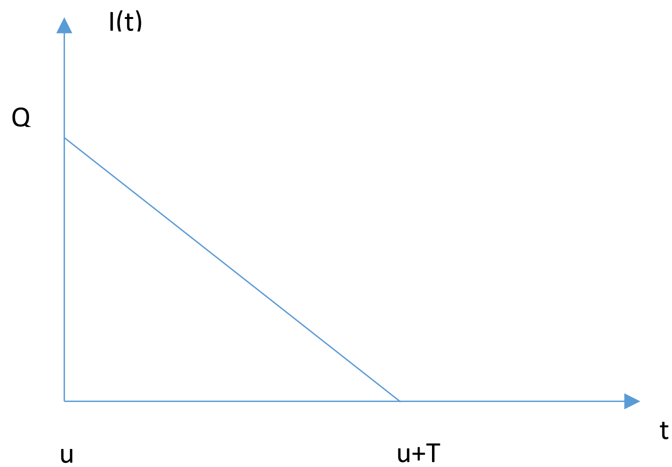
C(Q)= rK+$h\int_{\mathrm{u}}^{\mathrm{u+T}} I(t)dt$

In this problem, the variables or functions, and their units are as follows:

- C(Q) denotes the cost of replenishment over time T in dollars
- Q denotes the amount of units increase in a replenishment
- D denotes that the demand increases at rate D units per year
- K denotes the fixed cost of each replenishment in dollars
- H denotes the holding cost of inventory is at the rate of h dollars per unit per year
- r denotes the number of replenishments
- u denotes the replenishment date
- T denotes the the time interval from u
- I(t) denotes the inventory level at time t
- g(Q)n denotes the time-averaged cycle cost

We only consider one interval from time u to u+T for the inventory holding costfunction as:

$$h\int_{\mathrm{u}}^{\mathrm{u+T}} I(t)dt$$




$\int_{\mathrm{u}}^{\mathrm{u+T}} I(t)dt$ is the area below the line, which equals the area of the right triangle. So $\int_{\mathrm{u}}^{\mathrm{u+T}} I(t)dt = \frac{Q*T}{2}$

This graph was made by using the following assumptions

- R = 1, since only one replenishment cycle is considered, because each cycle will be identical to each other.

Next, we solved the following equations:

C(Q)= rK+$h\int_{u}^{u+T} I(t)dt$ =K+h*$\frac{Q*T}{2}$

Plug in T=$\frac{Q}{D}$, then C(Q)=K+$\frac{h}{2D}$*$Q^2$

Then the time-averaged cycle cost g(Q)=C(Q)/T=$\frac{DK}{Q}$+$\frac{h}{2}$Q

In order to find the quantity $Q^*$ to minimize g(Q), we take the first derivative and let it equal to zero

$g'(Q^*)$=-$\frac{DK}{Q^\wedge 2}$+$\frac{h}{2}$=0, then solve $Q^*$=$\sqrt{\frac{2DK}{h}}$.

The corresponding value of the order interval

$T^*$=$\frac{Q*}{D}$,=$\sqrt{\frac{2K}{hD}}$ is the optimal time interval to set for each replenishment.

$g(Q^*)$=$\sqrt{2DKh}$, which is the minimized cycle time per time unit.

Also we have verified that all units are consistent in all the formulas above.

(d) The mathematical solution Q* found in question (c) above is usually an irrational number, hardly useful in practice. Assume that order quantities Q are restricted to being integer powers-of-two multiples of a given base quantity β > 0, that is, Q can only be β, 2β, 4β, 8β, etc., or β/2, β/4, β/8, etc. Which property of the time-averaged cycle cost function g implies that, if Q* is between two successive integer power-of-two multiples of β, that is, if 2p β < Q* < 2p+1 β, then an optimum restricted value QR of Q is one of these two values 2p β or 2p+1 β?

In real world, the quantity should be an integer, so we usually use an optimum restricted value $Q^R$, which is one of these two values 2p β or 2p+1 β

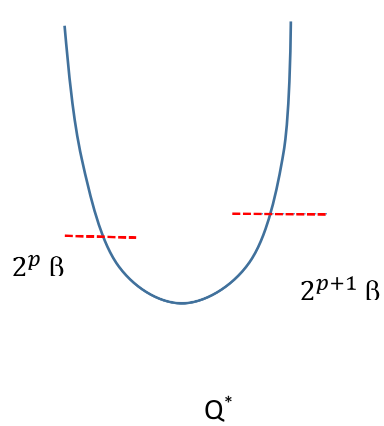
The cost function g(Q) = $\frac{C(Q)}{T} = \frac{DK}{Q} + \frac{hQ}{2}$

$$Q* = \sqrt{\frac{2DK}{h}}$$

$2^p$ β < Q* < $2^{p+1}$ β

$2^p$ β < $Q* = \sqrt{\frac{2DK}{h}}$ < $2^{p+1}$ β

So it is because the g(Q) function is in the form of f(x)=a*x + $\frac{b}{x}$, in which a=h/2, and b=DK. According to part (a), it shows the function is strictly convex for (0, +∞). The g(Q) is strictly convex as well.



In order for $Q^*$ to have optimum restricted values, the time-averaged cycle cost function should be convex.

(e) Assume $2^p$ β < Q* < $2^{p+1}$, where p is integer, as in question (d) above. For which values of Q* is the restricted optimum QR = 2p β, and for which ones is it $2^{p+1}$ β? (Hint: use the result in (b) above.) Conclude that the cost g(QR) of an optimum restricted policy is never more than 6% above the unrestricted optimum cost g(Q*).

According to the result in part b, $Q_R$

$f(x) )=\frac{1}{2} f(x^*)(\frac{x}{x*}+\frac{1}{\frac{x}{x^*}})$

so $g(Q_R)=\frac{1}{2} g(Q^*)(\frac{Q_R}{Q^*}+\frac{1}{\frac{Q_R}{Q^*}})$

suppose $Q_L=2^P\beta$ and $Q_R=2^{P+1}\beta$, then $g(Q_L)\leq g(Q_R)$

$$\frac{1}{2} g(Q^*)(\frac{2^p\beta}{Q^*}+\frac{1}{\frac{2^p\beta}{Q^*}}) \leq \frac{1}{2} g(Q^*)(\frac{2^{p+1}\beta}{Q^*}+\frac{1}{\frac{2^{p+1}\beta}{Q^*}})$$

$$(\frac{2^p\beta}{Q^*}+\frac{1}{\frac{2^p\beta}{Q^*}}) \leq (\frac{2^{p+1}\beta}{Q^*}+\frac{1}{\frac{2^{p+1}\beta}{Q^*}})$$

$$2^P\beta-2^{P+1}\beta\leq Q^2(\frac{1}{2^{p+1}\beta} - \frac{1}{2^p\beta})$$

$$Q^2\leq 2^{2p+1}\beta^2$$

Then if $-\sqrt{2}*2^P\beta \leq Q \leq \sqrt{2}*2^P\beta$, then $Q_R=2^P\beta$

$g(Q_L) \geq g(Q_R)$

follow the same step above

$Q \geq \sqrt{2}*2^P\beta$ ($Q \leq -\sqrt{2}*2^P\beta$ , we give up this because $Q\geq 0$)

In order to have $g(Q_R)$ to be no more than 6% of $g(Q^*)$, then we will find the boundary points that $g(Q_R) =1.06*g(Q^*)$

$\frac{1}{2} g(Q^*)(\frac{Q_R}{Q^*}+\frac{1}{\frac{Q_R}{Q^*}}) =1.06*g(Q^*)$

$\frac{Q_R}{Q^*}+\frac{1}{\frac{Q_R}{Q^*}}= 2.06$

then solve the function $\frac{Q_R}{Q^*} = 0.7 \text{ } or \text{ } 1.4$

so $Q_L=0.7\, Q^*$ and $Q_R=1.4\, Q^*$, which means $2Q_L=Q_R$.

If the range for the 6% boundary should be within ( $Q_R$, 2 $Q_R$ ) , so we can express that in the form of ($2^P\beta$ , $2^{P+1}\beta$ )