Homework Assignment 3

Due Wednesday September 28, 9:00 am

***Note**: All models in this homework set can be formulated as linear or smooth nonlinear programs. Thus you should <u>not</u> use integer or binary variables; nor discontinuous or non-smooth functions (such as MAX, MIN, ABS, etc.,) or logical operators (such as IF, OR, etc.), when their arguments depend on variables.*

1. Estimating Learning Effects (*30 marks*)
The production of a new product usually involves some learning: the larger the accumulated production x of the product, and the more experienced the workers will be in manufacturing the product. Let $t(x)$ denote the total time (hours) it takes to produce the first x units of the product. A frequently used model for such learning effects is a power function $t(x) = ax^p$. The following five observations of cumulative production times for Widgets are available:

Observation i	1	2	3	4	5
Cumulative production x_i	1	30	60	120	240
Cumulative time t_i	6	60	100	150	250

(a) Define an unconstrained optimization model to find the parameters a and p to minimize the sum of the squared deviations between the observed values t_i and the predicted values ax_i^p. Is the objective function a convex function of a and p?

(b) Use the computer to solve your model in question **(a)**. Try different initial values for a and p, say, ranging from 0.1 to 1,000 for a, and from 0.1 to 10 for p. What do you observe?

(c) For estimating such power functions, statisticians recommend the use of a logarithmic transformation, that is, $\ln(t(x)) = \ln(a) + p\ln(x)$. Define an unconstrained optimization model to find the parameters b (which is meant to represent $\ln(a)$) and p to minimize the sum of the squared deviations between the logarithms $\ln(t_i)$ of the observed times and the predicted values $b + p\ln(x_i)$. Is the objective function a convex function of b and p?

(d) Use the computer to solve your model in question **(c)**. Compare the results with those obtained in question **(b)**. What do you think of the statisticians' recommendation?

2. Maple Products (Part 1)[1] (*30 marks*)

Maple Products is planning the production and pricing of its three products, Chairs, Tables and Stools. Each Chair requires 0.1 Assembly hours, 0.4 Finishing hours, and has a unit production cost (materials and labour) of $13.50; the corresponding figures are 0.2 Assembly hours, 0.3 Finishing hours, and $16.50 for Tables; and 0.15 Assembly hours, 0.2 Finishing hours, and $7.50 for Stools. Every week, Maple Products has access to limited resources: 130 Assembly hours; 370 Finishing hours; and 500 Tabletops (Tabletops are only used for Tables, at the rate of one Tabletop per Table). At the end of the week, unused amount of the resources are lost. Maple Products is incurring a $3,800 overhead (sunk cost) per week.

The demand for each product *i* (Chair, Table, or Stool; abbreviated as C, T, S, resp.) depends linearly on its selling price as follows: there is a market of size M_i units; demand is M_i units if product *i* is sold at its unit production cost (i.e., under *marginal cost pricing*, e.g., at $16.50 for *i* = T); and demand drops to zero when the selling price reaches 64% above the unit production cost (e.g., at 1.64*16.50 = $27.06 for Tables), i.e., for a 64% margin (or "mark-up"). The current market sizes are M_C = 1,000 Chairs, M_T = 500 Tables, and M_S = 350 Stools. As in the lectures, ignore any integrality requirements or consideration, and assume that demand for each product is not affected by the price of the other products, i.e., there is no demand interaction (in particular, no demand substitutability or complementarity).

(a) Formulate an optimization model and determine prices and production amounts for the coming week (week 1) to maximize Maple Products' total profit. Which resource constraints are binding?

(b) Actually, the given market sizes were those of last week (week 0), and this week's market size for Tables is uncertain due to volatile economic conditions. Assume that, with probability 0.6, the market size for Tables will increase by 20% ("Up"); else it will decrease by 20% ("Down"). On the other hand, in this question **(b)** assume that there is no uncertainty on the market sizes for Chairs and for Stools, which are the same as in week 0. As sales occur at the end of the week, Maple Products have to determine the production amounts and prices *before* knowing the market size for Tables. Consider a one-week horizon: at the end of week 1 all unsold amounts of the products can be sold ("*salvaged*") on a secondary market at 10% below their unit production costs; on the other hand, unsatisfied demand is lost, with no further consequence. Formulate an optimization model and determine selling prices and production, realized sales and salvaged amounts for the coming week (week 1) to maximize Maple Products' *expected* total profit. (*Modelling hint*: You may want to use additional variables to represent salvaged amounts of each product for each market size.) Which resource constraints are now binding? What is the probability distribution of total profits for your solution?

[1] This example is similar to the Oak Doors example seen in class, with somewhat different data. Part 2 will be addressed in Homework Set 5.

(c) Assume now that the market sizes for all three products are uncertain and (for simplicity) perfectly correlated: with probability 0.6, all three market sizes will simultaneously increase by 20% ("Up"); else they will simultaneously decrease by 20% ("Down"). Formulate an optimization model and determine selling prices and production, realized sales and salvaged amounts for the coming week (week 1) to maximize Maple Products' expected total profit. (*Modelling hint*: Using additional indices, put together three single-product models as in question **(b)**, linked by the shared resource constraints and the shared weekly overhead cost.) Which resource constraints are now binding? What is the probability distribution of total profits for your solution?

3. International Investments case *(40 marks)*

Write a **Case Report** for the *International Investments* case (Case 12.2 [or 13.2 in the 7th edition] in the Hillier & Lieberman text; copy of this case attached). Include brief answers to all Parts (a) to (g) in your one-page Executive Summary. Your optimization models, which could be linear programs or smooth convex programs, will be explained in detail in the main body of your report. Present and explain, in appendices, all relevant computer printouts (your model, solution and/or, where appropriate, the sensitivity analysis) for each question.

Remarks:

1. In question (a), replace the word "chapter" with "course".
2. All investments (B-bonds, CD) are perfectly divisible: Charles can invest any nonnegative amount in them (not necessarily integer multiples of 100 DM).
3. The 3.6% interest on US Municipal Bonds is annual, and is compounded annually.
4. A CD (certificate of deposit) is a one-year security. If it is kept for more than one year then its interest is also compounded annually.
5. The interest from a CD investment is also taxable. The 6,100 DM allowance and the 30% tax rate apply to the *total* revenue from all investments.
6. Interest tax is paid the same year the investment is redeemed. (For example, it may be withheld at the source).
7. In each of questions (d) to (g), clearly state each corresponding optimization model.
8. The changes in questions (e) to (g) are to be considered independently (i.e., non-cumulatively).

(Total: 100 marks)

CASE 12.2 INTERNATIONAL INVESTMENTS

Charles Rosen relaxes in a plush, overstuffed recliner by the fire, enjoying the final vestiges of his week-long winter vacation. As a financial analyst working for a large investment firm in Germany, Charles has very few occasions to enjoy these private moments, since he is generally catching red-eye flights around the world to evaluate various investment opportunities. Charles pats the loyal golden retriever lying at his feet and takes a swig of brandy, enjoying the warmth of the liquid. He sighs and realizes that he must begin attending to his own financial matters while he still has the time during the holiday. He opens a folder placed conspicuously on the top of a large stack of papers. The folder contains information about an investment Charles made when he graduated from college four years ago. . . .

Charles remembers his graduation day fondly. He obtained a degree in business administration and was full of investment ideas that were born while he had been daydreaming in his numerous finance classes. Charles maintained a well-paying job throughout college, and he was able to save a large portion of the college fund that his parents had invested for him.

Upon graduation, Charles decided that he should transfer the college funds to a more lucrative investment opportunity. Since he had signed to work in Germany, he evaluated investment opportunities in that country. Ultimately, he decided to invest 30,000 German marks (DM) in so-called B-Bonds that would mature in 7 years. Charles purchased the bonds just 4 years ago last week (in early January of what will be called the "first year" in this discussion). He considered the bonds an excellent investment opportunity, since they offered high interest rates (see Table 1) that would rise over the subsequent 7 years and because he could sell the bonds whenever he wanted after the first year. He calculated the amount that he would be paid if he sold bonds originally worth DM 100 on the last day of any of the 7 years (see Table 2). The amount paid included the principal plus the interest. For example, if he sold bonds originally worth DM 100 on December 31 of the sixth year, he would be paid DM 163.51 (the principal is DM 100, and the interest is DM 63.51).

Charles did not sell any of the bonds during the first four years. Last year, however, the German federal government introduced a capital gains tax on interest income. The German government designated that the first DM 6,100 a single individual earns in interest per year would be tax-free. Any interest income beyond DM 6,100 would be taxed at a rate of 30 percent. For example, if Charles earned interest income of DM 10,100, he would be required to pay 30 percent of DM 4,000 (DM 10,100 − DM 6,100) in taxes, or DM 1,200. His after-tax income would therefore be DM 8,900.

Because of the new tax implemented last year, Charles has decided to reevaluate the investment. He knows that the new tax affects his potential return on the B-Bonds, but he also knows that most likely a strategy exists for maximizing his return on the bonds. He might be able to decrease the tax he has to pay on interest income by selling portions of his bonds in different years. Charles considers his strategy viable because the government requires investors to pay taxes on interest income only when they sell their B-Bonds. For example, if Charles were to sell one-third of his B-Bonds on

TABLE 1 Interest rates over the 7 years

Year	Interest Rate	Annual Percentage Yield
1	7.50%	7.50%
2	8.50%	8.00%
3	8.50%	8.17%
4	8.75%	8.31%
5	9.00%	8.45%
6	9.00%	8.54%
7	9.00%	8.61%

TABLE 2 Total return on 100 DM

Year	DM
1	107.50
2	116.64
3	126.55
4	137.62
5	150.01
6	163.51
7	178.23

December 31 of the sixth year, he would have to pay taxes on the interest income of DM 251 (DM 6,351 − DM 6,100).

Charles asks himself several questions. Should he keep all the bonds until the end of the seventh year? If so, he would earn 0.7823 times DM 30,000 in interest income, but he would have to pay very substantial taxes for that year. Considering these tax payments, Charles wonders if he should sell a portion of the bonds at the end of this year (the fifth year) and at the end of next year.

If Charles sells his bonds, his alternative investment opportunities are limited. He could purchase a certificate of deposit (CD) paying 4.0 percent interest, so he investigates this alternative. He meets with an investment adviser from the local branch of a bank, and the adviser tells him to keep the B-Bonds until the end of the seventh year. She argues that even if he had to pay 30 percent in taxes on the 9.00 percent rate of interest the B-Bonds would be paying in their last year (see Table 1), this strategy would still result in a net rate of 6.30 percent interest, which is much better than the 4.0 percent interest he could obtain on a CD.

Charles concludes that he would make all his transactions on December 31, regardless of the year. Also, since he intends to attend business school in the United States in the fall of the seventh year and plans to pay his tuition for his second, third, and fourth semester with his investment, he does not plan to keep his money in Germany beyond December 31 of the seventh year.

(For the first three parts, assume that if Charles sells a portion of his bonds, he will put the money under his mattress earning zero percent interest. For the subsequent parts, assume that he could invest the proceeds of the bonds in the certificate of deposit.)

(a) Identify one of the model types described in this chapter that is applicable to this problem, and then formulate a model of this kind to be used in the following parts.

(b) What is the optimal investment strategy for Charles?

(c) What is fundamentally wrong with the advice Charles got from the investment adviser at the bank?

(d) Now that Charles is considering investment in the certificate of deposit, what is his optimal investment strategy?

(e) What would his optimal investment strategy for the fifth, sixth, and seventh years have been if he had originally invested DM 50,000?

(f) Charles and his fiancée have been planning to get married after his first year in business school. However, Charles learns that for married couples, the tax-free amount of interest earnings each year is DM 12,200. How much money could Charles save on his DM 30,000 investment by getting married this year (the fifth year for his investment)?

(g) Due to a recession in Germany, interest rates are low and are expected to remain low. However, since the American economy is booming, interest rates are expected to rise in the United States. A rise in interest rates would lead to a rise of the dollar in comparison to the mark. Analysts at Charles' investment bank expect the dollar to remain at the current exchange rate of DM 1.50 per dollar for the fifth year and then to rise to DM 1.80 per dollar by the end of the seventh year. Therefore, Charles is considering investing at the beginning of the sixth year in a 2-year American municipal bond paying 3.6 percent tax-exempt interest to help pay tuition. How much money should he plan to convert into dollars by selling B-Bonds for this investment?