Homework Assignment 4

Due Wednesday October 5, 9:00 am

Exercise 1. *(33 marks)*
Consider the single-variable function

$$f(x) = 11{,}664\,x - 10{,}368\,x^2 + 3{,}852\,x^3 - 693\,x^4 + 60\,x^5 - 2\,x^6 - 3{,}600$$

(a) How many critical points can this function have? (*Hint:* note that this function is a degree-6 polynomial.) Find all critical points, their objective values, and identify all local minima, global minima, local maxima, global maxima, and inflection points.

(b) Is the First-Order Condition $f'(x) = 0$ necessary for x to be a global minimum of f (over the whole real line $\Re$)? Is it sufficient?

(c) Is function f convex (resp., strictly convex, concave, strictly concave) on the whole real line $\Re$?

(d) Is function f convex (resp., strictly convex, concave, strictly concave) on the interval [0, 10]?

(e) Consider the bound-constrained optimization problem $\min\{f(x) : 0 \le x \le 10\}$. Write the optimality conditions for this problem. Are these optimality conditions necessary for a global minimum over the interval [0, 10]? Are they sufficient?

(f) Find all solutions (x, u, v) to these optimality conditions. (*Hint:* start with the Complementary Slackness conditions: for each CS condition one of its two factors must be zero.) For which of these solutions is x a local minimum, a global minimum, a local maximum or a global maximum over the interval [0, 10], or an inflection point?

(g) Solve the constrained optimization problem of question **(e)** above using the computer. Use the following initial values: x = 0.5; 2; 4; 6; 8; and 9.5. (In Excel, simply enter these values in the decision variable cell before calling the Solver). Compare the solutions and Lagrange Multipliers with the solutions and shadow prices found in question **(f)** above. (Note that the Excel Solver *implicitly* treats the lower and upper bound constraints on variables. To force it to treat them *explicitly* and produce their Lagrange Multipliers, simply add a cell containing 1*x and use it as the left hand-side of each of the two bound constraints.)

Exercise 2. *(32 marks)*

(a) Consider the single-variable function $f(x) = ax + b/x$ where a and b are two given positive constants. Is function f convex (resp., strictly convex, concave, strictly concave) on the open interval $(0, +\infty)$ (i.e., for $x > 0$)? Find the global minimum x^* over this interval $(0, +\infty)$.

(b) For any $x > 0$ express $f(x)$ as a function of the ratio x/x^* and the optimal value $f(x^*)$ (but <u>not</u> of the parameters a and b) (*Hint:* prove that $ax^* = b/x^*$ and then use this fact). For example, what are $f(2x^*)$ and $f(x^*/2)$ as functions of $f(x^*)$?

The following is a (simplified) model for managing the inventory of a divisible product, say, a fluid. Demand (or consumption) arises continuously in time, at rate D units per year. (For example, if a year has 365 days, then daily demand is $D/365$ units, hourly demand is $D/(365{*}24)$ units, etc.) Demand has to be satisfied from inventory at every point in time, without any shortage. Inventory can be replenished at a fixed cost of K dollars for each replenishment. Let $I(t)$ denote the inventory level at time t, so $I(t)$ is piecewise linear with slope $I'(t) = -D$ between replenishments and a jump of $+Q$ at each replenishment of Q units (i.e., a replenishment instantaneously increases the inventory level by the replenishment quantity). Initial inventory $I(0^-)$ is zero. Inventory accrues holding cost continuously in time at a rate of h dollars per unit in inventory per year. The total cost of a replenishment policy over any (half-open) time interval $[u, v)$ is the sum of the replenishment fixed costs, i.e., rK if there are r replenishments during this interval, and the inventory holding costs $h\int_{t=u}^{v} I(t)\,dt$ during the interval. We seek a replenishment policy P which minimizes the long-run-average total cost per year. It can be shown that an optimum replenishment policy is to use equal replenishment quantities $Q > 0$ at equal intervals $T = Q/D$ in such a way that the inventory level just reaches zero when the replenishment is received.

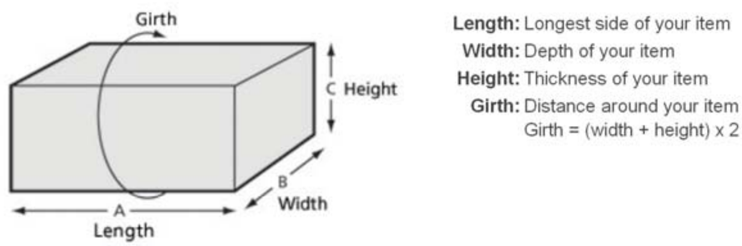
(c) Express, as a function of the replenishment quantity Q, the total cost $C(Q)$ over such a replenishment cycle $[u, u+T)$ where u is the replenishment date. Find, in terms of the three parameters D, K and h, the order quantity Q^* which minimizes the time-averaged cycle cost $g(Q) = C(Q)/T$. Find the corresponding values of the order interval $T^* = Q^*/D$ and minimum cost per time unit $g(Q^*)$. Verify that the units are consistent in all these formulas.

(d) The mathematical solution Q^* found in question **(c)** above is usually an irrational number, hardly useful in practice. Assume that order quantities Q are restricted to being integer powers-of-two multiples of a given base quantity $\beta > 0$, that is, Q can only be β, 2β, 4β, 8β, etc., or $\beta/2$, $\beta/4$, $\beta/8$, etc. Which property of the time-averaged cycle cost function g implies that, if Q^* is between two successive integer power-of-two multiples of β, that is, if $2^p\beta < Q^* < 2^{p+1}\beta$, then an optimum restricted value Q_R of Q is one of these two values $2^p\beta$ or $2^{p+1}\beta$?

(e) Assume $2^p\beta \le Q^* < 2^{p+1}\beta$, where p is integer, as in question **(d)** above. For which values of Q^* is the restricted optimum $Q_R = 2^p\beta$, and for which ones is it $2^{p+1}\beta$? (*Hint*: use the result in (b) above.) Conclude that the cost $g(Q_R)$ of an optimum restricted policy is never more than 6% above the unrestricted optimum cost $g(Q^*)$.

3. Case Report: The Abbotsford Beads Company *(35 marks)*

The Abbotsford Beads Company (ABC) produces specialty beads used by jewelers and interior decorators. ABC sends part of its production to a distributor in Montreal by sending by mail, at regular time intervals, cubic parcels of size 60 cm along each side, at an annual shipping cost of $12,778.00. Canada Post requires that parcels not exceed a maximum length of 200 cm and a total length-plus-girth of 300 cm, where the *length* is the longest dimension of the parcel and the *girth* is the maximum, along the length of the item, of the total distance around the parcel cross section perpendicular to its length, see Figure 1. (The total length-plus-girth is used, instead of volume, because it is easier and faster for a mail agent to determine using a single measuring tape and, for irregularly shaped parcels, it avoids complicated volume calculations.)



	Length	Width	Height	Weight
Minimum	10 cm	7 cm	1 mm	-
Maximum	Length = 2 m Length + Girth = 3 m			30 kg

Figure 1: Canada Post *Regular Parcel* specifications

Since beads are low density items, the cost of one ABC parcel satisfying these length and extent limits is $63.89, independent of its weight (Figure 2). As an analyst recently hired by ABC, you realize that they could reduce their mailing costs by using an optimized parcel shape.

Regular Parcel™ 7 business days **$63.89**

Options	
☐ Collection on Delivery	$7.25
☐ Mailing Tube	$1.50
☑ Delivery Confirmation	Incl.
☐ Signature Option	$1.50
☐ Unpackaged	$10.00

Rate Summary	
Base Price	$56.21
Coverage* $ 0 Update	$0.00
Fuel Surcharge	$4.64
Options	$0.00
Tax	$3.04
Total	**$63.89**

*Coverage Details: $0.00 included. Maximum $5,000.00 @ $1.80 per $100.00.

Figure 2: Canada Post *Regular Parcel* rates charged to ABC

Write a case report to ABC management. Include brief answers to questions **(e)** and **(f)** below and your recommendations in the Executive Summary.

(a) Using simple geometric facts (such as the isoperimetric inequality) and reasoning, prove that a maximum volume parcel satisfying Canada Post parcel limits is a right circular cylinder.

(b) Formulate a nonlinear programming model with 2 decision variables for the problem of determining the dimensions of a right circular cylindrical parcel satisfying Canada Post parcel limits and with maximum volume.

(c) In your model, is the objective function convex? Is it concave?

(d) Write the Karush-Kuhn-Tucker (KKT) conditions for your model. Are these conditions necessary for a global optimum? Are they sufficient?

(e) Find a global optimum by solving the KKT conditions. Estimate how much ABC would save in yearly shipping cost by adopting your solution for its Montreal mailings. (Ignore all side issues such as packaging, handling, storage and inventory costs.)

(f) Actually, ABC does not like rounded parcels, and they only want to use rectangular (i.e., parallelepiped) parcels. Repeat questions **(b)** (suitably modified) to **(e)** for the problem of determining the dimensions of a *rectangular* parcel satisfying Canada Post parcel limits and with maximum volume. (Your model should now have 3 decision variables.) How much would this restriction to rectangular, instead of cylindrical, parcels cost ABC per year?