Matrix Completion Problems

Leslie Hogben
Iowa State University

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Partial Matrices

- A partial matrix is a square array in which some entries are specified and others are not.
- A completion of a partial matrix is a choice of values for the unspecified entries.

Example:

$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

B is a partial matrix and A is a completion of B.

Completions

- A matrix completion problem asks whether a partial matrix (or family of partial matrices with a given pattern of specified entries) has a completion of a specific type, such as a positive definite matrix.

Example:

$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Matrix A completes B to a positive semidefinite matrix.

Submatrices

- The submatrix $A[\alpha, \beta]$ consists of the entries in rows in α and columns in β.
- The submatrix $A[\alpha] := A[\alpha, \alpha]$ is principal.

Example:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad A[\{1,3\}] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A[\{1,3\},\{2,3,4\}] = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Partial X-Matrix

- All classes X of matrices discussed are hereditary, i.e. if A is an X-matrix then every principal submatrix of A is an X-matrix.
- If X is hereditary, in order for a partial matrix B to have an X-completion, it is necessary that every fully specified principal submatrix of B is an X-matrix, and any sign conditions on the entries are satisfied.
- These conditions are not usually sufficient to guarantee an X-completion.
- A partial matrix B is a partial X-matrix if every fully specified (principal) submatrix of B is an X-matrix, and any sign conditions on the entries are satisfied.

- All matrices discussed are real and square.

- All classes of matrices discussed are generalizations of the positive definite matrices.

- The following are equivalent:
  - A is symmetric and for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} > 0$ (positive definite).
  - A is symmetric and all eigenvalues are positive.
  - A is symmetric and all principal minors are positive.

- Analogous definition/characterizations for positive semidefinite

Classes of matrices to be discussed:

- positive definite matrices
- positive semidefinite matrices
- strictly copositive matrices: A is strictly copositive if A is symmetric and for all $\mathbf{x} \geq 0, \mathbf{x} \neq 0$, $\mathbf{x}^T A \mathbf{x} > 0$
- copositive matrices: A is copositive if A is symmetric and for all $\mathbf{x} \geq 0$, $\mathbf{x}^T A \mathbf{x} \geq 0$

Example:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix}$$

is strictly copositive but not positive definite (any positive matrix is strictly copositive).

Classes of matrices to be discussed:

- A is a P-matrix if all principal minors are positive.
- A is a P_0-matrix if all principal minors are nonnegative.
- A is a $P_{0,1}$-matrix if all principal minors are nonnegative and all diagonal elements are positive.
- And various sign conditions on entries.

Example:

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix} \text{ is a } P \text{ matrix.}$$

$$\det A = 14, \quad \det A[\{1,2\}] = \det \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix} = 5, \text{ etc.}$$

Classes of matrices to be discussed:

- A is totally positive if all minors are positive.
- A is totally nonnegative if all minors are nonnegative.

Example:

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix} \text{ is totally nonnegative.}$$

$$\det A = 16, \qquad \det A[\{1,2\}] = \det \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix} = 8$$

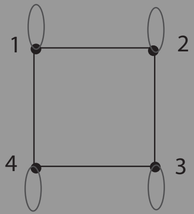
$$\det A[\{1,2\},\{1,3\}] = \det \begin{bmatrix} 5 & 0 \\ 2 & 2 \end{bmatrix} = 10, \quad \text{etc.}$$

Graph Theoretic Techniques

- Graphs are used for symmetric matrices; otherwise digraphs are used.
- Digraphs and graphs can have loops but not multiple edges or arcs in the same direction.
- The specified entries in partial matrix B are represented by edges in the graph $\mathcal{G}(B)$ or $\mathcal{D}(B)$.

  Example:

$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix}$$

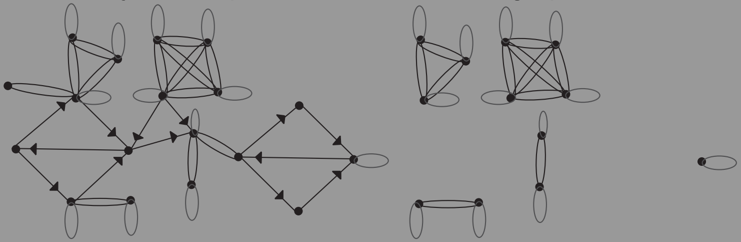


Permutation Similarity and Vertex Numbering

- All of the classes X of matrices discussed except totally positive and totally nonnegative are closed under permutation similarity.
- Applying a permutation similarity to a partial matrix B corresponds to renumbering the vertices of the digraph $\mathcal{D}(B)$
- If X is closed under permutation similarity, then unlabeled digraph diagrams can be used.
- For totally positive (nonnegative) matrices, labeled digraphs must be used.

- A digraph G has the X-completion property if every partial X-matrix B such that $\mathcal{D}(B) = G$ can be completed to an X-matrix.
- The problem often reduces to considering the sub(di)graph induced by the vertices with loops.

Example: If the X-completion problem reduces to the subdigraph induced by the looped vertices, the left digraph

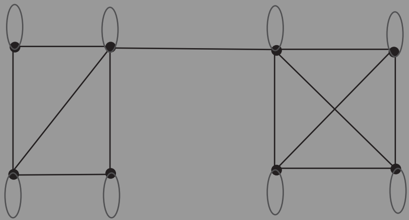


has the X-completion property.

Theorem (Grone, Johnson, Sá, Wolkowicz LAA 1984)

- A graph having a loop at every vertex has the positive definite completion property if and only if it is chordal (any cycle of length ≥ 4 has a chord).
- A graph has the positive definite completion property if and only if the subgraph induced by the vertices with loops has the positive definite completion property.

Example:



has the positive definite completion property.

Graph terminology

- A graph is connected if there is a path from any vertex to any other vertex.

- The undirected graph associated with the digraph D is obtained by replacing each arc (u, v) or pair $(u, v), (v, u)$ by edge $\{u, v\}$

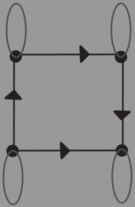
- A digraph is connected if its associated graph is connected.

For all the classes discussed:

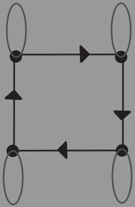
- If $A_1, A_2, \ldots, A_k$ are X-matrices then $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ is an X-matrix, i.e, X is closed under matrix direct sums.
- Let B be a partial matrix such that all specified entries are contained in diagonal blocks $B_1, B_2, \ldots, B_k$. The connected components of $\mathcal{D}(B)$ are the $\mathcal{D}(B_1), \ldots, \mathcal{D}(B_k)$.
- A graph or digraph G has the X-completion property if and only if every connected component of G has the X-completion property.

- A digraph is strongly connected if there is a path from any vertex to any other vertex.

Example



connected
but not strongly connected



strongly connected

- A class X has the triangular property if whenever A is a block triangular matrix and every diagonal block is an X-matrix, then A is an X matrix.

- If X has the triangular property, B is a partial matrix in block triangular form (as a pattern), and each diagonal block can be completed to an X-matrix, then B can be completed to an X-matrix.

- If X has the triangular property and is closed under permutation similarity, then a graph or digraph G has the X-completion property if and only if every strongly connected component of G has the X-completion property.

- A block of a graph or digraph is a maximal nonseparable sub(di)graph.
- A graph (respectively, digraph) is a clique if every vertex has a loop and for any two distinct vertices u, v, the edge $\{u, v\}$ is present (respectively, both arcs $(u, v), (v, u)$ are present).
- A graph or digraph is block-clique (also called 1-chordal) if every block is a clique.
- For many classes X, the completion problem reduces to the completion problem for the (graph) blocks.

The (Strictly) Copositive Matrix Completion Problems

- A is strictly copositive if A is symmetric and for all $\mathbf{x} \geq 0, \mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} > 0$.
- A is copositive if A is symmetric and for all $\mathbf{x} \geq 0$, $\mathbf{x}^T A\mathbf{x} \geq 0$.

Example:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

is copositive but not strictly copositive, and not positive semidefinite.

- The partial matrix B is a partial strictly copositive matrix if every fully specified principal submatrix of B is a strictly copositive matrix.

- The partial matrix B is a partial copositive matrix if every fully specified principal submatrix of B is a copositive matrix.

Example:

$$B = \begin{bmatrix} 3 & 1 & x_{13} & -1 \\ 1 & 1 & 2 & x_{24} \\ x_{13} & 2 & 1 & -1 \\ -1 & x_{24} & -1 & 2 \end{bmatrix}$$

is a partial strictly copositive matrix.

Theorem (Hogben, Johnson, Reams LAA 2005)

Let B be a partial copositive matrix with every diagonal entry specified. For each pair of unspecified off-diagonal entries, set $x_{ij} = x_{ji} = \sqrt{b_{ii}b_{jj}}$. The resulting matrix is copositive, and is strictly copositive if B is a partial strictly copositive matrix.

Example:

$$B = \begin{bmatrix} 3 & 1 & x_{13} & -1 \\ 1 & 1 & 2 & x_{24} \\ x_{13} & 2 & 1 & -1 \\ -1 & x_{24} & -1 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 1 & \sqrt{3} & -1 \\ 1 & 1 & 2 & \sqrt{2} \\ \sqrt{3} & 2 & 1 & -1 \\ -1 & \sqrt{2} & -1 & 2 \end{bmatrix}$$

A completes B to a strictly copositive matrix.

Theorem (Hogben 2005)

Let $B = \begin{bmatrix} x_{11} & \mathbf{b}^T \\ \mathbf{b} & B_1 \end{bmatrix}$ be a partial strictly copositive $n \times n$ matrix having all entries except the 1,1-entry specified. Let $\| \cdot \|$ be a vector norm. Complete B to a strictly copositive matrix by choosing a value for x_{11} as follows:

1. $\beta = \min_{\mathbf{y} \geq 0, ||\mathbf{y}||=1} \mathbf{b}^T \mathbf{y}$.
2. $\gamma = \min_{\mathbf{y} \geq 0, ||\mathbf{y}||=1} \mathbf{y}^T B_1 \mathbf{y}$.
3. $x_{11} > \frac{\beta^2}{\gamma}$.

Corollary Every partial strictly copositive matrix can be completed to a strictly copositive matrix.

Example The partial matrix

$$B = \begin{bmatrix} x_{11} & -5 & 1 & x_{14} & x_{15} & x_{16} \\ -5 & 1 & -2 & x_{24} & x_{25} & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ x_{14} & x_{24} & 1 & 1 & x_{45} & 1 \\ x_{15} & x_{25} & -1 & x_{45} & x_{55} & -1 \\ x_{16} & 1 & -1 & 1 & -1 & 3 \end{bmatrix}$$

is a partial strictly copositive matrix.

Select index 5. The only principal submatrices completed by a choice of b_{55} are $B[\{3,5\}]$ and $B[\{5,6\}]$.
Any value that makes $5x_{55} > (-1)^2$ and $3x_{55} > (-1)^2$ will work.
Choose $x_{55} = 1$.

Example The partial matrix

$$B = \begin{bmatrix} x_{11} & -5 & 1 & x_{14} & x_{15} & x_{16} \\ -5 & 1 & -2 & x_{24} & x_{25} & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ x_{14} & x_{24} & 1 & 1 & x_{45} & 1 \\ x_{15} & x_{25} & -1 & x_{45} & x_{55} & -1 \\ x_{16} & 1 & -1 & 1 & -1 & 3 \end{bmatrix}$$

is a partial strictly copositive matrix.
Select index 1. The only principal submatrices completed by a choice of b_{11} are principal submatrices of

$$B[\{1,2,3\}] = \begin{bmatrix} x_{11} & -5 & 1 \\ -5 & 1 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

.

$$B[\{1,2,3\}] = \begin{bmatrix} x_{11} & -5 & 1 \\ -5 & 1 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

Using $\|\cdot\|_1$:

1. $\beta = \min_{||\mathbf{y}||_1=1} \mathbf{b}^T \mathbf{y} = -5$.
2. $\gamma = \min_{||\mathbf{y}||_1=1} \mathbf{y}^T B[\{2,3\}]\mathbf{y} = \frac{1}{10}$.
3. Choose $x_{11} > \frac{\beta^2}{\gamma}$; choose $b_{11} = 256$.

$b_{11} = 256, \; b_{55} = 1.$ $$\begin{bmatrix} 256 & -5 & 1 & x_{14} & x_{15} & x_{16} \\ -5 & 1 & -2 & x_{24} & x_{25} & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ x_{14} & x_{24} & 1 & 1 & x_{45} & 1 \\ x_{15} & x_{25} & -1 & x_{45} & 1 & -1 \\ x_{16} & 1 & -1 & 1 & -1 & 3 \end{bmatrix}$$

Set $x_{ij} = x_{ji} = \sqrt{b_{ii} b_{jj}}$.

$$\begin{bmatrix} 256 & -5 & 1 & 16 & 16 & 16\sqrt{3} \\ -5 & 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ 16 & 1 & 1 & 1 & 1 & 1 \\ 16 & 1 & -1 & 1 & 1 & -1 \\ 16\sqrt{3} & 1 & -1 & 1 & -1 & 3 \end{bmatrix}$$

is a strictly copositive matrix.

It is not true that every partial copositive matrix can be completed to a copositive matrix.

Example $B = \begin{bmatrix} x_{11} & -1 \\ -1 & 0 \end{bmatrix}$ is a partial copositive matrix that cannot be completed to a copositive matrix.

Choose a value for x_{11}.

If $x_{11} = 0$, then with $\mathbf{x} = [1, 1]^T$, $\mathbf{x}^T B \mathbf{x} = -2$.

If $x_{11} > 0$, then for the vector $\mathbf{x} = [1, x_{11}]^T$, $\mathbf{x}^T B \mathbf{x} = -x_{11}$.

1. L. Hogben and A. Wangsness, Matrix Completion Problems, to appear in *Handbook of Linear Algebra*, CRC Press, Boca Raton, FL, 2006.
2. R. Grone, C. R. Johnson, E. M. Sá, and H. Wolkowicz, Positive definite completions of partial Hermitian matrices, *Linear Algebra and Its Applications*, 58:109–124, 1984.
3. L. Hogben, C. R. Johnson, R. Reams, The Copositive Matrix Completion Problem, *Linear Algebra and Its Applications*, 408 (2005) 207-211.
4. L. Hogben, The Copositive Matrix Completion Problem: Unspecified Diagonal, preprint.