Unit 17
The Theory of Linear Systems

Theorem 18.1. *Every system of linear equations $A\vec{x} = \vec{b}$ has either*

- *no solution,*

  or
- *exactly one solution,*

  or
- *or infinitely many solutions (i.e. parametric family of solutions)*

Proof: *We have seen examples of each kind, so we need only show that there are no more possibilities.*

Suppose $\vec{y}$ and $\vec{z}$ are two distinct solutions

$$\text{i.e.}\quad A\vec{y} = \vec{b} \text{ and } A\vec{z} = \vec{b} \text{ with } \vec{z} \neq \vec{y}$$

for $t \in \Re$ let $\vec{x} = (1-t)\vec{y} + t(\vec{z})$ (there are infinitely many such $\vec{x}$'s since t is arbitrary).

Then

$$\begin{aligned}
A\vec{x} &= A((1-t)\vec{y} + t\vec{z}) \\
&= A(\vec{y} + t(\vec{z} - \vec{y})) \\
&= A\vec{y} + t(A\vec{z} - A\vec{y}) \\
&= \vec{b} + t(\vec{b} - \vec{b}) \\
&= \vec{b} + \vec{0} = \vec{b}
\end{aligned}$$

Therefore, if we have more than one solution, we have infinitely many solutions.

Definition 18.2. The linear system $A\vec{x} = \vec{b}$ is said to be *homogeneous* if $\vec{b} = \vec{0}$. If $\vec{b} \neq \vec{0}$ then the system is said to be *non-homogeneous.*

Example 1. Identify the following systems as homogeneous or non-homogeneous.

(a) $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 6 & 3 & 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ homogeneous system.

(b) $\begin{array}{rcrcrcl} 3x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & 3x_2 & - & x_3 & = & 0 \\ 2x_1 & & & + & x_3 & = & 0 \end{array}$ homogeneous system.

(c) $\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ x_1 & & & - & 2x_3 & = & 0 \\ & & 5x_2 & + & x_3 & = & 1 \end{array}$ non-homogeneous system.

Definition 18.3. The zero vector $\vec{0}$ is a solution to every homogeneous system $A\vec{x} = \vec{0}$, since $A\vec{0} = \vec{0}$. The solution $\vec{x} = 0$ is called the *trivial solution,* any other solution is called a *nontrivial solution.*

Theorem 18.4. *Every homogeneous linear system $A\vec{x} = \vec{0}$ has either exactly one solution or infinitely many.*

Proof: *We showed that for any linear system there were only 3 options; no solution, one solution, or infinitely many solutions, and any homogeneous linear system has at least the trivial solution ($\vec{x} = 0$) so it can not have no solutions.*

Definition 18.5. The *rank* of a matrix A is equal to the number of non-zero rows when A is in row reduced echelon form and is denoted $r(A)$. Equivalently, $r(A)$ is equal to the number of leading 1's when A is in row reduced echelon form.

Definition 18.6. We say that an $m \times n$ matrix A has *full rank*, if $r(A) = n$, i.e. the row reduced form of A has a leading one in each column.

Example 2. Find $r(A)$ where $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix}$

The row reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has two leading ones (or equivalently, two non-zero rows). Therefore, the rank of matrix A is 2.

The rank of a matrix can tell us much about the solution(s) of the corresponding system of equations as the next theorem points out.

Theorem 18.7. *For the linear system $A\vec{x} = \vec{b}$ where A is $m \times n$ (* i.e. *has m equations (rows) and n variables (columns)), $[A|b]$ is the augmented matrix of the system.*

Let $p = r(A)$ and $q = r([A|b])$, (note that $(p \leq q)$ necessarily)

Then $A\vec{x} = \vec{b}$ has:

(a) No solution if $p < q$

(b) An unique solution if $p = q = n$

(c) Infinitely many solutions if $p = q$ and $p < n$.

We note what theorem 7 implies about homogeneous systems:

Corollary 18.8. *For an homogeneous system, $[A|0]$ is the augmented matrix so we have $r(A) = r([A|0])$ i.e. $p = q$ above. Therefore the system $A\vec{x} = \vec{0}$ has either:*

(a) An unique solution if $r(A) = n$.

or

(b) Infinitely many solutions if $r(A) < n$.

So there are only the two possibilities above for the number of solutions to an homogeneous linear system: exactly one solution or infinitely many solutions. Notice that if an homogeneous linear system has more unknowns than equations then necessarily $r(A) \leq m < n$ and so we must have an infinite number of solutions as the following corollary states.

Corollary 18.9. *If an homogeneous system has more unknowns than equations then the system will have infinitely many solutions .*

Example 3. By inspecting the following system, determine the number of solutions it must have.

$$\begin{array}{rrrl} 2x & +2y & -5z & = 0 \\ 23x & +14y & -1z & = 0 \end{array}$$

Solution: We see that this is an homogeneous system and so has either an unique solution (if the coefficient matrix A has full rank) or infinitely many solutions. The coefficient matrix here is

$$\begin{bmatrix} 2 & 2 & -5 \\ 23 & 14 & -1 \end{bmatrix}$$

In order for this matrix to have full rank (of 3 since we have three columns) we would need three leading 1's after reducing the matrix, but since we only have two rows we could not possibly do this. Thus, in the language of the above corollary $r(A) < n$ and so there are infinitely many solutions to the system.

Example 4. By inspecting the following system, determine the number of solutions it must have.

$$\begin{array}{rrrl} 2x & +2y & -5z & = 0 \\ 23x & +14y & -1z & = 1 \end{array}$$

Solution: This time we are not dealing with an homogeneous system. As above we can cross out the possibility of an unique solution to the system since the coefficient matrix could not have full rank. On other fronts, it is entirely possible that $r(A) = r([A|b])$ (meaning infinitely many solutions) or that $r(A) < r([A|b])$ (meaning no solution), so these are the two possibilities.

Example 5. By inspecting the following system, determine the number of solutions it must have.

$$\begin{array}{cccll} x_1 & & & +3x_4 & = 1 \\ & x_2 & & +5x_4 & = 3 \\ & & x_3 & -3x_4 & = 7 \end{array}$$

Solution: Here it is easy to see that by inspection $r(A) = 3$ and the augmented matrix $[A|b]$ also has rank 3. Thus we have $r(A) = r([A|b]) = 3 < n = 4$ and so there are infinitely many solutions.

Recall that if a system has infinitely many solutions, then there will be parameters involved in the solution. The rank of a matrix gives information on the number of parameters involved in the solutions to a given system.

Theorem 18.10. *Consider the system $A\vec{x} = \vec{b}$. If A is an $m \times n$ matrix with $r(A) = k = r([A|b])$ with $k < n$ (so the rank of A is less than the number of columns of A). Then the linear system of equations $A\vec{x} = \vec{b}$ will have an $(n-k)$-parameter family of solutions.*

Example 6. The system:

$$\begin{array}{cccll} x_1 & & & +3x_4 & = 1 \\ & x_2 & & +5x_4 & = 3 \\ & & x_3 & -3x_4 & = 7 \end{array}$$

has coefficient matrix

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

which is already in RREF, and so we see has rank 3.
The augmented matrix of the system

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 5 & 3 \\ 0 & 0 & 1 & -3 & 7 \end{array}\right]$$

is also in RREF and has rank 3.
Therefore by the above theorem with $n = 4$ we see that the system will have a $(4 - 3) = 1$-parameter family of solutions.

The following theorem will be added to in a later section and is an extremely important tool.

Theorem 18.11. *If A is a square $(n \times n)$ matrix then the following statements are equivalent to one another.*
(a) A is invertible (nonsingular)
(b) The system $A\vec{x} = \vec{b}$ has an unique solution (for any $\vec{b}$)
(c) $r(A) = n$ (A has full rank)
(d) The RREF of A is I (A is row equivalent to the identity matrix).