Analysis of RLC Circuits

Guruprasad
Assistant Professor, MIT, Manipal.

A typical second order circuit consists of two energy storing elements. A series RLC circuit is an example for second order circuit. Figure 1 shows a series RLC circuit.

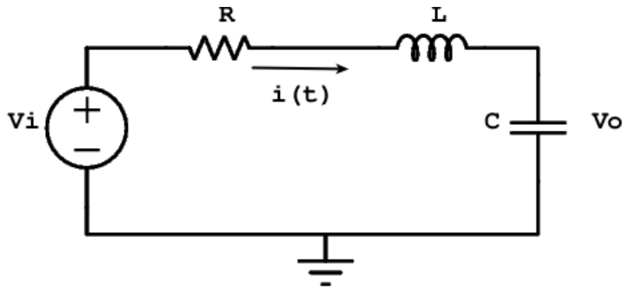

Figure 1: Series RLC Circuit

Analyzing RLC circuit involves the investigating circuit behavior to different time varying input signals.

Response to step input

Assume V_i as step input.

$$\begin{aligned} V_i &= 1 \quad for \quad t \geq 0 \\ &= 0 \quad for \quad t < 0 \end{aligned}$$

From KVL law

$$\begin{aligned} V_i &= i(t)R + L\frac{di(t)}{dt} + V_c(t) \\ &= RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c \\ \frac{V_i}{LC} &= \frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} \end{aligned} \tag{1}$$

This is second order ordinary homogeneous differential equation. One of the solutions is method of homogeneous and particular solution.

1. Find the Homogeneous solution.
2. Find the Particular solution
3. Total solution is sum of both.
4. Use initial condition to find constants.

$$V_c = V_p[ParticularSolution] + V_h[HomogenousSolution]$$

Homogeneous equation is the equation (1) with source set to zero.

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = 0$$

Assume a solution $V_c = Ae^{st}$

$$As^2e^{st} + \frac{R}{L}Ase^{st} + \frac{Ae^{st}}{LC} = 0$$

Avoiding trivial solutions $e^{st} = 0$ and A=0

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{2}$$

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

Equation (2) is called characteristic equation. If either $V_{c1} = A_1e^{s_1t}$ or $V_{c1} = A_2e^{s_2t}$ satisfies the characteristic equation then sum of two terms also satisfies . Hence $V_c = A_1e^{s_1t} + A_2e^{s_2t}$ is solution to homogeneous equation. where A_1 and A_2 are constants and found from initial condition.

Next step is to find particular solution.

$$\frac{d^2V_p}{dt^2} + \frac{R}{L}\frac{dV_p}{dt} + \frac{V_p}{LC} = \frac{V_i}{LC}$$

Let $V_p = K$ be the solution then

$$K = V_i$$

Hence $V_p = V_i$ is the particular solution. Total solution is

$$V_c = A_1e^{s_1t} + A_2e^{s_2t} + V_i \tag{3}$$

Lats step is to find the constants from initial condition. At t=0

$$V_{c,0} = A_1e^{s_1t} + A_2e^{s_2t} + V_{i,0}]_{t=0} \tag{4}$$

$$\frac{dV_{c,0}}{dt} = \frac{i_0}{C} = \frac{d}{dt}[A_1e^{s_1t} + A_2e^{s_2t} + V_{i,0}]_{t=0} \tag{5}$$

From equation (4) and (5), A_1 and A_2 can be determined.

Roots of Characteristic Equation

The second order system response is dependent on value and type of roots of characteristic equation. We shall use following notation for simplification.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$$

$$\alpha = \frac{R}{2L} \qquad \omega_n^2 = \frac{1}{LC}$$

1. **Roots are real and distinct**

   $$\alpha^2 > \omega_n^2$$

   Roots are sum of two exponentially decaying components. Hence the response is *over damped.*

2. **Roots are real and equal**

   $$\alpha^2 = \omega_n^2 \qquad s_1 = s_2 = s$$

   $$V_c = A_1e^{st} + A_2e^{st} = (A_1 + A_2)e^{st} = Ae^{st}$$

   However a second order system must have two constants so,

   $$V_c = A_1e^{st} + A_2te^{st}$$

   The system is critically damped.

3. **Roots are complex**

$$\alpha^2 < \omega_n^2$$

When roots are complex, they always occur in conjugate pair. i.e if $\alpha^2 = \omega_n^2 - \alpha^2$

$$s_1 = -\alpha + j\omega \quad s_2 = -\alpha - j\omega$$

So solution is

$$V_c = A_1 e^{(-\alpha+j\omega)t} + A_2 e^{(-\alpha-j\omega)t}$$

From Euler's theorem,

$$e^{j\theta} = cos\theta + jsin\theta$$

If V_c to be real, A_1 and A_2 must also be chosen as complex conjugates namely $A_2 = A_1^*$.

$$V_c = A_1 e^{(-\alpha+j\omega)t} + A_1^* e^{(-\alpha-j\omega)t}$$

$$= e^{-\alpha t}[A_1 cos(\omega t) + jA_1 sin(\omega t)] + e^{-\alpha t}[A_1^* cos(\omega t) - jA_1^* sin(\omega t)]$$

$$= e^{-\alpha t}[(A_1 + A_1^*)cos(\omega t) + (jA_1 - jA_1^*)sin(\omega t)]$$

$$V_c = e^{-\alpha t}[Acos(\omega t) + Bsin(\omega t)]$$

Where $A = A_1 + A_1^* = 2Re[A_1]$ and $B = jA_1 - jA_1^* = -2Im[A_1]$ But sum of sine and cosine must be another sine with a phase difference,

$$V_c = Ae^{-\alpha t} sin(\omega t + \theta)$$

The response is sum of decaying exponential component and sinusoidal. The resulting wave form is exponentially decaying sinusoidal.

Illustration

For the circuit shown in Figure 1, Assume V_{in}=5V, L=1mH, $R = 10\Omega$ and C=0.01mF. Determine V_c.

$$10^8 V_i = \frac{d^2 V_c}{dt^2} + 10^6 \frac{dV_c}{dt} + 10^8 V_c$$

1. Step 1 : Characteristic equation

$$s^2 + 10^6 s + 10^8 = 0$$

$$s_{1,2} = -5000 \pm j8660$$

Roots are complex hence the response is under-damped.

2. Step 2: Solution

$$V_c = e^{-\alpha t}[Acos\omega t + Bsin\omega t] + V_i$$

$$\alpha = \frac{R}{2L} = 5000 \quad \omega = 8660 \quad V_i = 5V$$

3. Step 3: Find Constants
At t=0

$$V_{c,0} = V_{i,0} + A$$

$$A = -5V$$

Taking derivative of V_c equation at t=0

$$\frac{dV_{c,0}}{dt} = \frac{i_0}{C} = \frac{d}{dt}[e^{-\alpha t}[Acos\omega t + Bsin\omega t] + V_i]$$

$$\frac{i_0}{C} = -\alpha A + \omega B$$

$$0 = 25000 + 8660B$$

$$B = -2.88$$

The complete solution is

$$V_c = e^{-5000t}[-5cos(8660t) - 2.88sin(8660t)] + 5$$

Figure 2 shows the plot of above equation using *GNU Octave*.

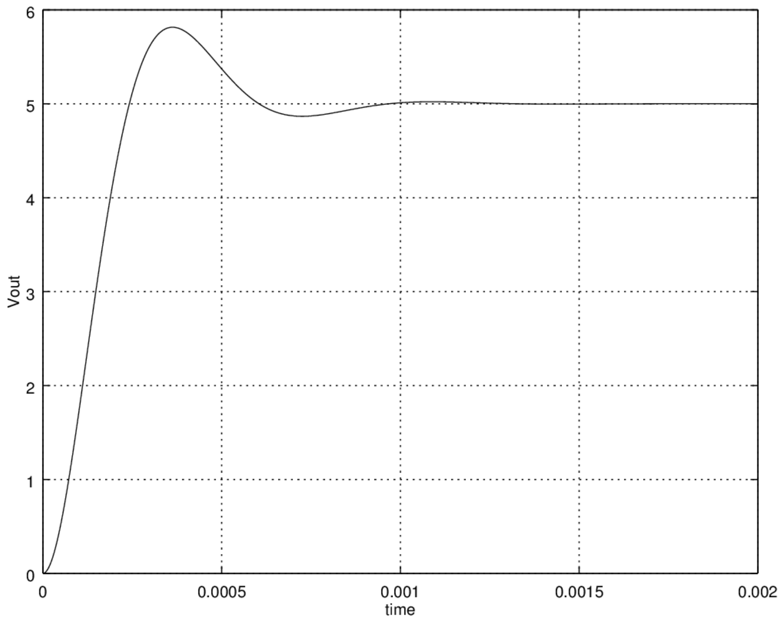

Figure 2: Output plot using Octave

The code for producing above plot is give below
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```
t=0:0.000001:0.002;
y=exp(-5000*t);
z=(-5*cos(8660*t))-(2.88*sin(8660*t));
p=y.*z+5;
plot (t,p,"k"):  xlabel( "time") :ylabel("Vout")
```

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The same problem can be solved and verified using Ngspice. The SPICE code is given below -
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```
RLC circuit Analysis
Vi 1 0 5
R1 1 2 10
L1 2 3 1m
C1 3 0 0.01m
.control
op
run
ic v(3)=0
tran 0.1m 10m uic
.endc
.end
```

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Figure 3 shows the plot obtained from Ngspice.

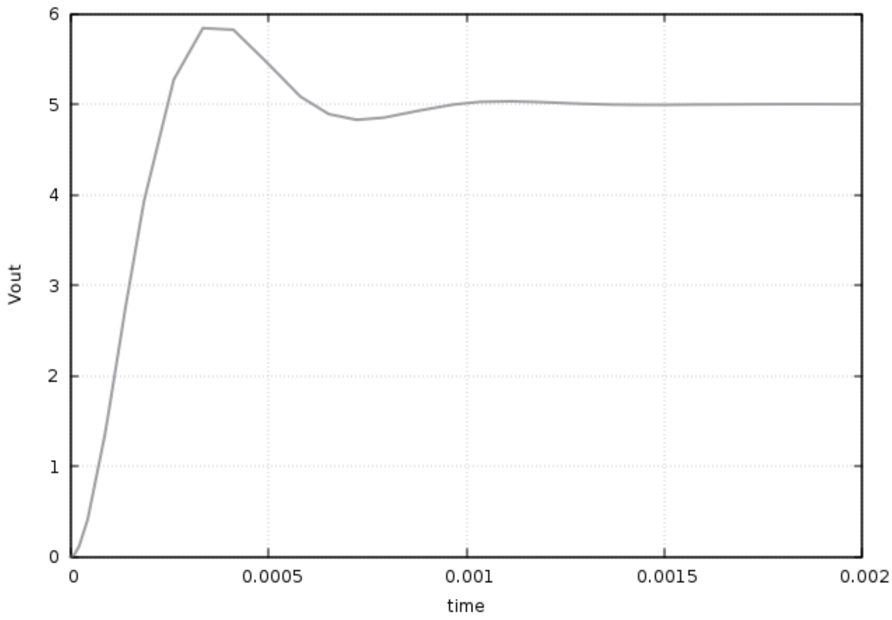

Figure 3: Ngspice Output

Exercise

For the above problem determine the response if i) $R = 500\Omega$ ii) $R = 0\Omega$ iii) Calculate the value of R for which the system response will be *Critically Damped*.

Reference

1. DeCarlo and Lin, *Linear Circuit Analysis*, Second edition, Oxford publisher, 2005.
2. Ananth Agarwal and Jeffrey H. Lang, *Foundations of Analog and Digital Electronic Circuits*, Elsevier- Morgan Kaufmann Publishers, 2016.
3. *GNU Octave* user Manual.
4. *Ngspice* user Manual.
5. *Gnu Plot* user Manual.