ABSTRACT

The purpose of the present paper is to improve some results of R. Askey, P. Erdős, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegő and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem will (positively) be answered: is there any weight w with compact support such that for each $p > 2$ the Lagrange interpolating polynomials corresponding to w diverge in L_w^p for some continuous function f? Most of the paper deals with Christoffel functions and their applications. Many limit relations for orthogonal polynomials are found in the assumption that the coefficients in the recursion formula behave nicely.

AMS(MOS) Subject Classifications - 42A52, 33A65, 41A05, 41A10, 41A20, 41A25, 41A35, 41A55, 41A60, 42A04, 42A56, 42A62, 40A25, 26A75, 26A78, 26A84, 30A06, 30A84, 30A86, 65D30, 65F15, 65F35.

Key Words - Orthogonal polynomials, Quadrature processes, Fourier series, Interpolation, Positive operators, Toeplitz matrices, Christoffel functions.

Library of Congress Cataloging in Publication Data

Nevai, Paul G. 1948- 1987. — V, 185p. ; 26 cm.
Orthogonal polynomials.
(Memoirs of the American Mathematical Society ; no. 213 ISSN 0065-9290) 1. Polinomios ortogonales
"Volume 18."
Bibliography: p.
1. Orthogonal polynomials. I. Title. II. Series: American Mathematical Society. Memoirs ; no. 213.
QA3.A57 no. 213 [QA404.5] 510'.8s [515'.55] 78-32112
ISBN 0-8218-2213-6

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Reprinted 1987

This work is dedicated

to

Richard Askey

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ORTHOGONAL POLYNOMIALS

1. Introduction

The purpose of the present paper is to improve some results of R. Askey, P. Erdős, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegő and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem [1] will be answered: is there any weight w with compact support such that for each $p > 2$ the Lagrange interpolating polynomials corresponding to w diverge in L^p_w for some continuous function f ? R. Askey [1] conjectured that the answer was yes and the solution was given by the Pollaczek weight because the logarithm of the Pollaczek weight is not integrable. We will show that Askey's conjecture is right but for different reasons. In fact, there are many weights solving Turan's problem; some of them do have integrable logarithm, some of them do not.

Most of this paper deals with investigation of Christoffel functions and its generalization. The results and the methods are stronger than those of the above authors. The Christoffel functions play a very important role in the theory of orthogonal polynomials. Many results in orthogonal Fourier series and interpolation are based on estimates and asymptotics of Christoffel functions. We will show how successfully Christoffel functions can be applied in finding necessary conditions for weighted mean convergence of orthogonal Fourier series and Lagrange interpolation processes. Introducing generalized Christoffel functions we shall find a connection between different weighted L^p norms of polynomials. Especially interesting is the case when $0 < p < 1$.

Sponsored by
1) The United States Army under Contract No. DAAG29-75-C-0024,
2) The American Mathematical Society, Providence, RI 02940,
3) The National Science Foundation, under Grant No. MCS75-06687.
Received by the editors March 5, 1977 and in revised form January 21, 1978.

We will investigate a new kind of quadrature process which helps to find asymptotics for Christoffel functions and orthogonal polynomials outside the support of the weight function. Taking the recursion formula as a starting point and assuming properties on the coefficients in the recursion formula we will obtain results on orthogonal polynomials. In certain cases we will be able to calculate the weight function using the coefficients in the recursion formula. We will also find the $(C,1)$ limit of Turan type determinants under rather weak conditions.

I learned the theory of orthogonal polynomials from G. Freud who has been supervising my research for several years. Many of the methods I use in this paper can be found in his book on orthogonal polynomials which is a rich source of methods and unsolved research problems. I wish to express my deep feeling of gratitude to G. Freud as well as to R. Askey, L. Bers, M. Cwikel, J. Landin, G.G. Lorentz and W. Proxmire without the help of which this paper would never have been written. I am grateful to the American Mathematical Society, to the National Science Foundation and to the United States Army for sponsoring my research.

This whole work was born from the attempts to solve problems mentioned in R. Askey's paper [1]. I discussed my results with R. Askey several times. He read a draft version of the manuscript and made various suggestions to improve the presentation. I dedicate this work to Richard Askey because his generous support and help made it possible for me to carry out the research which led to this paper.

Finally, I would like to thank T.S. Chihara for reading the entire manuscript, for making almost a hundred corrections, for pointing out that some of my results had previously been known and for supplying me with additional references.

2. Notations

The function $\alpha\colon \mathbb{R} \to \mathbb{R}$ is called a weight function if it is nondecreasing, it has infinitely many points of increase and all the moments

$$\int_{-\infty}^{\infty} x^{2n} d\alpha(x) \quad (n = 0,1,\dots)$$

are finite*. For a given weight α the corresponding system of orthogonal polynomials $\{p_n(d\alpha)\}_{n=0}^{\infty}$ is defined by $p_n(d\alpha,x) = \gamma_n(d\alpha)x^n + \dots$, $\gamma_n(d\alpha) > 0$ and

$$\int_{-\infty}^{\infty} p_n(d\alpha,x)\, p_m(d\alpha,x)\, d\alpha(x) = \delta_{nm} .$$

If α happens to be absolutely continuous then we will usually write w and $p_n(w,x)$ instead of α' and $p_n(d\alpha,x)$ respectively. In the general case α can be written in the form

$$\alpha = \alpha_{ac} + \alpha_s + \alpha_j$$

where α_{ac} is absolutely continuous, α_s is singular and α_j is a jump function.

One of the basic properties of a system of orthogonal polynomials $\{p_n(d\alpha)\}$ is that the polynomials $p_n(d\alpha)$ satisfy the three term recurrence relation

$$xp_n(d\alpha,x) = \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)}\, p_{n+1}(d\alpha,x) + \alpha_n(d\alpha)\, p_n(d\alpha,x) + \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_{n-1}(d\alpha,x)$$

$(n = 0,1,\dots)$ where $p_{-1} \equiv 0$, $\gamma_{-1} = 0$ and

$$\alpha_n(d\alpha) = \int_{-\infty}^{\infty} tp_n^2(d\alpha,t)\, d\alpha(t) .$$

* Usually α is called a weight function if it is absolutely continuous. Otherwise α is a distribution function, measure or integral weight function. We will use the same terminology for both cases. If the weight is denoted by a greek letter then we mean that it is actually a distribution. Latin letters mean that we are dealing with absolutely continuous weights.

This recurrence relation will be one of our main points of interest. By a famous result of J. Favard if a system of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies the recurrence formula

$$xp_n(x) = \frac{\gamma_n}{\gamma_{n+1}} p_{n+1}(x) + \alpha_n p_n(x) + \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x)$$

for $n = 0,1,\ldots$ with $p_{-1} \equiv 0$, $\gamma_{-1} = 0$, $p_0 \equiv \gamma_0$, $\gamma_n > 0$ and $\alpha_n \in \mathbb{R}$ then $\{p_n(x)\}$ is orthogonal with respect to some weight α which may not uniquely be determined (see Freud, §II.1)*. In this paper we are going to deal with such cases when both $\{|\alpha_n|\}$ and $\{\gamma_{n-1}/\gamma_n\}$ are bounded and then α is uniquely defined.

The zeros of $p_n(d\alpha)$, which are real and distinct, will be denoted by $x_{kn}(d\alpha)$: $x_{1n}(d\alpha) > x_{2n}(d\alpha) > \ldots > x_{nn}(d\alpha)$. The Christoffel function $\lambda_n(d\alpha)$ corresponding to α is defined by

$$\lambda_n(d\alpha, z) = \min_{\substack{\pi \in \mathbb{P}_{n-1} \\ \pi(z) = 1}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t)$$

for $z \in \mathbb{C}$, $n = 1,2,\ldots$ where $\mathbb{P}_n$ is the set of polynomials of degree at most n. It is rather easy to see that

$$\lambda_n(d\alpha, z)^{-1} = \sum_{k=0}^{n-1} |p_k(d\alpha, z)|^2.$$

The numbers $\lambda_n(d\alpha, x_{kn}(d\alpha))$ are called Christoffel numbers and are usually denoted by $\lambda_{kn}(d\alpha)$. There are two important results involving Christoffel numbers which will often be used. The first of them is the Gauss-Jacobi mechanical quadrature formula:

$$\int_{-\infty}^{\infty} \pi(t)\, d\alpha(t) = \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \pi(x_{kn}(d\alpha))$$

* In the following books listed in the references will be referred by mentioning the name of their authors.

for each $\pi \in \mathbb{P}_{2n-1}$, the second one is the Markov-Stieltjes inequalities which can be expressed as

$$\sum_{k=i+1}^{n} \lambda_{kn}(d\alpha) \le \int_{-\infty}^{x_{in}(d\alpha)} d\alpha(t) \le \sum_{k=i}^{n} \lambda_{kn}(d\alpha)$$

$(i = 1,2,\ldots,n)$. The support of α, that is $\mathrm{supp}(d\alpha)$, is the set of points of increase of α. If $\mathrm{supp}(d\alpha)$ is bounded $\Delta(d\alpha)$ will denote the smallest closed interval containing $\mathrm{supp}(d\alpha)$. The symbols Δ and τ will always mean closed intervals, the interior part of Δ is denoted by Δ^0 and the length of Δ by $|\Delta|$. For a given τ the Chebyshev weight corresponding to τ will be written as v_τ. If $\tau = [-1,1]$ then we write v instead of v_τ. If $\tau = [a - b, a + b]$ then

$$v_\tau(x) = [b^2 - (x - a)^2]^{-1/2} .$$

For $f \in L^1_{d\alpha}$ ($\mathrm{supp}(d\alpha)$ is bounded) $S_n(d\alpha,f)$ denotes the nth partial sum of the orthogonal Fourier series of f. Hence for $x \in \mathbb{R}$

$$S_n(d\alpha,f,x) = \int_{-\infty}^{\infty} f(t)K_n(d\alpha,x,t)d\alpha(t)$$

where

$$K_n(d\alpha,x,t) = \sum_{k=0}^{n-1} p_k(d\alpha,x)\, p_k(d\alpha,t)$$

or by the Christoffel-Darboux formula

$$K_n(d\alpha,x,t) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha,t)\, p_n(d\alpha,x) - p_{n-1}(d\alpha,x)\, p_n(d\alpha,t)}{x - t} .$$

For a given function f the Lagrange interpolation polynomial $L_n(d\alpha,f)$ corresponding to α is defined to be the unique polynomial of degree at most $n - 1$ which agrees with f at the nodes $x_{kn}(d\alpha)$ $(k = 1,2,\ldots,n)$. If we denote by $\ell_{kn}(d\alpha)$ the fundamental polynomials of Lagrange interpolation $L_n(d\alpha,f)$ can be written as

$$L_n(d\alpha,f,x) = \sum_{k=1}^{n} f(x_{kn}(d\alpha))\, \ell_{kn}(d\alpha,x) .$$

It will be useful to remember that

$$\ell_{kn}(d\alpha,x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn}(d\alpha)\, p_{n-1}(d\alpha,x_{kn}) \frac{p_n(d\alpha,x)}{x - x_{kn}}$$

$(x_{kn} \equiv x_{kn}(d\alpha))$.

The Chebyshev polynomials $\cos n\theta$ $(x = \cos\theta)$ will always be denoted by $T_n(x)$. For a given set $\mathfrak{B}$ the characteristic function of $\mathfrak{B}$ is $1_{\mathfrak{B}}$ and $\mathfrak{B}(\epsilon)$ $(\epsilon > 0)$ means the ϵ-neighborhood of $\mathfrak{B}$. π_n and P_n denote polynomials belonging to $\mathbb{P}_n$. The letters $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural integers, real numbers and complex numbers respectively. $\mathbb{R}^+$ is the set of positive real numbers.

For $0 < p < \infty$, $\| \ \|_{d\alpha,p}$ is defined by

$$\|f\|_{d\alpha,p}^p = \int_{-\infty}^{\infty} |f(t)|^p \, d\alpha(t) \ .$$

(Of course, for $0 < p < 1$ this is not a norm.)

Sometimes we will omit unnecessary parameters in the formulas. (E.g., $x_k \equiv x_{kn}(d\alpha)$.)

We assume that the reader is familiar with methods in the theory of one-sided approximation and positive operators.

For the convenience of the reader we give an index where to find the definition of symbols used frequently. D.3.1.4 below means that see Definition 4 in Chapter 3.1 .

δ_t	- D.7.14,	p. 131
GJ	- D.9.28,	p. 169
$\Gamma(\theta)$	- D.4.2.4,	p. 41
JS	- D.10.17,	p. 181
$\lambda_n(d\alpha,p,x)$	- D.6.3.1,	p. 106
$\lambda_n^*(d\alpha,x)$	- Formula 4.1(5),	p. 28
$M(a,b)$	- D.3.1.6,	p. 10
Pollaczek weight	- D.6.2.12,	p. 80
$\rho(z)$	- D.4.1.8,	p. 30
S	- D.4.2.1,	p. 39
$u \equiv u^{(a,b)}$	- D.6.2.7,	p. 79
$U_n(x)$	- p. 8.	
u_n	- D.6.3.4,	p. 107
w_n	- D.9.28,	p. 169
$\sim$	- D.6.3.3,	p. 107

All those symbols which were not listed here were introduced in this chapter.

3. Basic Facts

3.1. The Generalized Recurrence Formula

Let $U_n(x)$ denote the Chebyshev polynomial of second kind, that is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta$$

$(0 \le \theta \le \pi, \ -1 \le x \le 1)$. The polynomials $U_n(x)$ satisfy the recurrence formula

$$(1) \qquad 2x\, U_{n-1}(x) = U_n(x) + U_{n-2}(x), \qquad n = 1,2,\ldots$$

where $U_{-1}(x) \equiv 0$ and $U_0(x) \equiv 1$.

<u>Theorem 1.</u> Let $0 \le k \le n$. For an arbitrary weight α, $p_n(d\alpha,x)$ can be expressed as

$$(2) \qquad p_n(d\alpha,x) = U_{n-k}(x)\, p_k(d\alpha,x) - U_{n-k-1}(x)\, p_{k-1}(d\alpha,x) + R_{n,k}(d\alpha,x)$$

where

$$(3) \qquad R_{n,k}(d\alpha,x) = \sum_{j=k+1}^{n} U_{n-j}(x)\,\{[1 - 2\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}]\, p_j(d\alpha,x) -$$

$$- 2\alpha_{j-1}(d\alpha)\, p_{j-1}(d\alpha,x) + [1 - 2\frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)}]\, p_{j-2}(d\alpha,x)\}.$$

<u>Proof.</u> We will prove (2) by induction. If $k = n$ then (2) and (3) give $p_n(d\alpha,x) = p_n(d\alpha,x)$. If $n \ge 1$ and $k + 1 = n$ then (2) and (3) coincide with the recurrence formula. Now fix n and let $n - 1 > k \ge 0$. Suppose that (2) and (3) hold if we replace their k by $k + 1$, that is

$$p_n = U_{n-k-1}\, p_{k+1} - U_{n-k-2}\, p_k + R_{n,k+1}\, .$$

Applying (2) and (3) to the case $n = k + 1 \ge 1$ we obtain

$$p_{k+1} = U_1\, p_k - U_0\, p_{k-1} + R_{k+1,k}\, .$$

Thus by (1)

$$p_n = (U_1 U_{n-k-1} - U_0 U_{n-k-2})p_k - U_{n-k-1} p_{k-1} + R_{n,k+1} + U_{n-k-1} R_{k+1,k}$$

$$= U_{n-k} p_k - U_{n-k-1} p_{k-1} + R_{n,k} ,$$

that is (2) and (3) hold also for k .

Remark 2. Putting $k = 0$ and $d\alpha$ = Chebyshev weight we obtain from Theorem 1

$$T_n(x) = x U_{n-1}(x) - U_{n-2}(x) = T_1(x) U_{n-1}(x) - T_0(x) U_{n-2}(x) .$$

For $k = 1$ (2) and (3) give the same as for $k = 0$. When $2 \leq k \leq n$ we get

$$T_n(x) = T_k(x) U_{n-k}(x) - T_{k-1}(x) U_{n-k-1}(x) .$$

This formula may easily be checked directly. In fact, the above formula suggested (2) and (3).

Theorem 3. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 \leq k \leq n$. Then

(4) $$p_n(d\alpha,x) = U_{n-k}(\frac{x-a}{b}) p_k(d\alpha,x) - U_{n-k-1}(\frac{x-a}{b}) p_{k-1}(d\alpha,x) + R^{a,b}_{n,k}(d\alpha,x)$$

where

(5) $$R^{a,b}_{n,k}(d\alpha,x) = \sum_{j=k+1}^{n} U_{n-j}(\frac{x-a}{b}) \cdot \{[1 - \frac{2}{b} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}] p_j(d\alpha,x) +$$

$$+ \frac{2}{b} [a - \alpha_{j-1}(d\alpha)] p_{j-1}(d\alpha,x) + [1 - \frac{2}{b} \frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)}] p_{j-2}(d\alpha,x)\} .$$

Proof. Let α^* be defined by $\alpha^*(t) = \alpha(bt + a)$. Then $p_n(d\alpha,x) = p_n(d\alpha^*,\frac{x-a}{b})$, $\alpha_n(d\alpha^*) = \frac{1}{b}[\alpha_n(d\alpha) - a]$ and $\gamma_{n-1}(d\alpha^*)/\gamma_n(d\alpha^*) = \frac{1}{b}[\gamma_{n-1}(d\alpha)/\gamma_n(d\alpha)]$. Now apply Theorem 1 to α^* and then return to α .

We will call (4) and (5) the generalized recurrence formula. It will help us to prove many properties of orthogonal polynomials in case the coefficients of the recurrence formula are convergent.

Definition 4. Let $a \in \mathbb{R}$, $b \geq 0$. Then

$$c_k^{a,b}(d\alpha) = |\alpha_k(d\alpha) - a| + \left|\frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} - \frac{b}{2}\right| + \left|\frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)} - \frac{b}{2}\right| .$$

Corollary 5. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 \leq k \leq n$. Then

$$(6) \qquad p_n(d\alpha,x) = U_{n-k}\left(\frac{x-a}{b}\right) p_k(d\alpha,x) - U_{n-k-1}\left(\frac{x-a}{b}\right) p_{k-1}(d\alpha,x) + $$

$$+ O(1) \frac{1}{\sqrt{b^2 - (x-a)^2}} \sum_{j=k-1}^{n} c_j^{a,b}(d\alpha) |p_j(d\alpha,x)|$$

for $x \in (a - b, a + b)$ where $|O(1)| \leq 2$.

Definition 6. Let $a \in \mathbb{R}$, $b \geq 0$. Then $\alpha \in M(a,b)$ if

$$\lim_{k\to\infty} c_k^{a,b}(d\alpha) = 0 .$$

Remark 7. When considering $M(a,b)$ we can always assume without loss of generality that either $a = 0$, $b = 1$ or $a = 0$, $b = 0$. For, if $\alpha \in M(a,b)$ with $b > 0$ then $\alpha^* \in M(0,1)$ where $\alpha^*(t) = \alpha(bt + a)$. If $\alpha \in M(a,0)$ then $\alpha^{**} \in M(0,0)$ where $\alpha^{**}(t) = \alpha(t + a)$.

Theorem 8. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 < \epsilon < 1$, $x \in [a - b, a + b]$. Then

$$[b^2 - (x-a)^2]\, \lambda_{n+1}(d\alpha,x)\, p_n^2(d\alpha,x) \leq 6\{\frac{b^2}{\epsilon n} + 2\sum_{n(1-\epsilon)\leq j\leq n} [c_j^{a,b}(d\alpha)]^2\}$$

for $n = 1,2,\ldots$

Proof. We obtain from (6) that for $[(1 - \epsilon)n] + 1 \leq k \leq n$

$$[b^2 - (x-a)^2]^{1/2} |p_n(x)| \leq b|p_k(x)| + b|p_{k-1}(x)| +$$

$$+ 2\{\sum_{j=[(1-\epsilon)n]}^{n} (c_j)^2 \lambda_{n+1}^{-1}(x)\}^{1/2} ,$$

that is

$$[b^2 - (x-a)^2]\, p_n^2(x) \le 3b^2\, p_k^2(x) + 3b^2\, p_{k-1}^2(x) + 12\lambda_{n+1}^{-1}(x) \sum_{j=[(1-\epsilon)n]}^{n} (c_j)^2 .$$

Thus

$$\sum_{k=[(1-\epsilon)n]+1}^{n} [b^2 - (x-a)^2]\, p_n^2(x) \le 6b^2 \lambda_{n+1}^{-1}(x) + 12\lambda_{n+1}^{-1}(x) \cdot$$

$$\sum_{j=[(1-\epsilon)n]}^{n} (c_j)^2 \sum_{k=[(1-\epsilon)n]+1}^{n} 1 .$$

The theorem follows from this inequality.

<u>Theorem 9</u>. Let $a \in M(a,b)$ with $b > 0$. Then

$$\lim_{n\to\infty} [b^2 - (x-a)^2]\, \lambda_{n+1}(d\alpha,x)\, p_n^2(d\alpha,x) = 0$$

uniformly for $x \in [a-b, a+b]$.

<u>Proof</u>. By Theorem 8 we have to show that we can choose $\epsilon = \epsilon_n \in (0,1)$ so that

$$\lim_{n\to\infty} 6(b^2+2)\Big\{\frac{1}{n\epsilon_n} + \sum_{(1-\epsilon_n)n \le j \le n} [c_j^{a,b}(d\alpha)]^2\Big\} = 0 .$$

Let

$$\epsilon_n = \frac{1}{n}\Big[\sup_{j \ge n/2} ([c_j]^2 + j^{-2})\Big]^{-1/2} .$$

Then $\epsilon_n \le \frac{1}{2}$. Thus

$$\frac{1}{n\epsilon_n} + \sum_{(1-\epsilon_n)n \le j \le n} [c_j]^2 \le \frac{1}{n\epsilon_n} + (\epsilon_n\, n+1) \sup_{j \ge n/2} [c_j]^2 \le$$

$$\le 2 \sup_{j \ge n/2} \{[c_j]^2 + j^{-2}\}^{1/2} + \sup_{j \ge n/2} [c_j]^2 \underset{n\to\infty}{\to} 0 .$$

<u>Remark 10</u>. It is useful to remember that if $\mathrm{supp}(d\alpha) = [-1,1]$ and $v \log \alpha' \in L^1$ then $\alpha \in M(0,1)$ (See e.g. Freud).

It might be interesting to compare Theorem 9 with (weaker) results of Geronimus (See Chapter III in his book).

Theorem 11. Let $\alpha \in M(a,b)$ with $b > 0$. Suppose that

$$\limsup_{n\to\infty} \sum_{j=n}^{2n} j C_j^{a,b}(d\alpha)^2 < \infty .$$

Let $\mathfrak{B} = \overline{\mathfrak{B}} \subset (a-b,a+b)$. Then the following two statements are equivalent.

(i) $\{p_n^2(d\alpha,x)\}$ is uniformly bounded for $x \in \mathfrak{B}$.

(ii) $\{n^{-1}\lambda_{n+1}^{-1}(d\alpha,x)\}$ is uniformly bounded for $x \in \mathfrak{B}$.

Proof. (i) $\Rightarrow$ (ii): $\{(n+1)^{-1}\lambda_{n+1}^{-1}(d\alpha,x)\}$ is the arithmetical mean of $\{p_n^2(d\alpha,x)\}$.

(ii) $\Rightarrow$ (i): Use Theorem 8.

Let us remark that if $C_j^{a,b}(d\alpha) = O(j^{-1})$ then the conditions of the theorem are satisfied. Example: Jacobi and Pollaczek polynomials.

Theorem 12. Let $\alpha \in M(a,b)$ with $b > 0$ and let

$$\sum_{j=0}^{\infty} C_j^{a,b}(d\alpha) < \infty .$$

If $\Delta \subset (a-b,a+b)$ then the sequence $\{|p_n(d\alpha,x)|\}$ is uniformly bounded for $x \in \Delta$.

Proof. Let, for simplicity, $\alpha \in M(0,1)$, $\Delta = [-\epsilon,\epsilon]$, $0 < \epsilon < 1$, $C_j^{0,1}(d\alpha) = C_j$. Let us fix $k = k(\epsilon)$ such that

$$\frac{2}{\sqrt{1-\epsilon^2}} \sum_{j=k-1}^{\infty} C_j < \frac{1}{2} .$$

If $m > k$ then by Corollary 5

$$|p_m(x)| \le \left[\frac{1}{\sqrt{1-\epsilon^2}} + \frac{1}{2}\right]\left[|p_k(x)| + |p_{k-1}(x)|\right] + \frac{2}{\sqrt{1-\epsilon^2}} \sum_{j=k+1}^{m} C_j |p_j(x)| .$$

Hence

$$|p_m(x)| \le \frac{1}{\sqrt{1-\epsilon^2}} + \frac{1}{2}][|p_k(x)| + |p_{k-1}(x)|] + \frac{1}{2} \max_{k+1 \le j \le m} |p_j(x)| .$$

Thus for $n > k$

$$\max_{k+1 \le m \le n} |p_m(x)| \le [\frac{1}{\sqrt{1-\epsilon^2}} + \frac{1}{2}][|p_k(x)| + |p_{k-1}(x)|] + \frac{1}{2} \max_{k+1 \le m \le n} \max_{k+1 \le j \le m} |p_j(x)| =$$

$$= [\frac{1}{\sqrt{1-\epsilon^2}} + \frac{1}{2}][|p_k(x)| + |p_{k-1}(x)|] + \frac{1}{2} \max_{k+1 \le m \le n} |p_m(x)| ,$$

in particular, for $n > k$

$$|p_n(x)| \le [\frac{2}{\sqrt{1-\epsilon^2}} + 1][|p_k(x)| + |p_{k-1}(x)|] .$$

Now remember that k does not depend on n.

Sometimes instead of Theorem 3 we will use the following generalization of the recurrence formula.

Theorem 13. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 \le n \le k$. Then

$$p_n(d\alpha,x) = U_{k-n}(\frac{x-a}{b})\, p_k(d\alpha,x) - U_{k-n-1}(\frac{x-a}{b})\, p_{k+1}(d\alpha,x) + \bar{R}^{a,b}_{n,k}(d\alpha,x)$$

where

$$\bar{R}^{a,b}_{n,k}(d\alpha,x) = \sum_{j=n}^{k-1} U_{j-n}(\frac{x-a}{b}) \cdot$$

$$\cdot \{[1 - \frac{2}{b}\frac{\gamma_j(d\alpha)}{\gamma_{j+2}(d\alpha)}]\, p_j(d\alpha,x) + \frac{2}{b}[a - \alpha_{j+1}(d\alpha)]\, p_{j+1}(d\alpha,x) +$$

$$+ [1 - \frac{2}{b}\frac{\gamma_{j+1}(d\alpha)}{\gamma_{j+2}(d\alpha)}]\, p_{j+2}(d\alpha,x)\} .$$

Proof. The theorem can be proved by induction in exactly the same way as Theorem 3.

Corollary 14. Let $k \ge 0$. Then

$$U_{k-1}(x)\, p_{k+1}(d\alpha,x) = U_k(x)\, p_k(x) +$$

$$+ \sum_{j=0}^{k-1} U_j(x)\, \{[1 - 2\frac{\gamma_j(d\alpha)}{\gamma_{j+1}(d\alpha)}]\, p_j(d\alpha,x) - 2\alpha_{j+1}(d\alpha)\, p_{j+1}(d\alpha,x) +$$

$$+ [1 - 2\frac{\gamma_{j+1}(d\alpha)}{\gamma_{j+2}(d\alpha)}]\, p_{j+2}(d\alpha,x)\} - \gamma_0(d\alpha)\ .$$

<u>Theorem 15</u>. Let $\theta = \theta_1 + i\theta_2$ with $\theta_2 \leq 0$. Then for $n = 0,1\ldots,$ $p_n(d\alpha,x)$ $(x = \cos\theta)$ can be represented as

$$(7) \quad \sin\theta\, p_n(d\alpha,\cos\theta) = |\varphi_{2n}(d\alpha,e^{i\theta})| \cdot \sin[(n+1)\theta - \arg\varphi_{2n}(d\alpha,e^{i\theta})]$$

when $\theta_2 = 0$ and

$$(8) \quad 2i\sin\theta\, p_n(d\alpha,\cos\theta) = e^{i(n+1)\theta}\, \varphi_{2n}(d\alpha,e^{-i\theta}) - e^{-i(n+1)\theta}\, \varphi_{2n}(d\alpha,e^{i\theta})$$

otherwise. Here

$$\varphi_{2n}(d\alpha,e^{i\theta}) = \sum_{j=0}^{n} a_j(d\alpha,\cos\theta)e^{ij\theta}\ ,$$

$$a_j(d\alpha,x) = [1 - 2\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}]\, p_j(d\alpha,x) - 2\alpha_{j-1}\, p_{j-1}(d\alpha,x) +$$

$$+ [1 - 2\frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)}]\, p_{j-2}(d\alpha,x)$$

$(j = 0,1,\ldots)$. Consequently, $\varphi_{2n}(d\alpha)$ is a polynomial of degree at most $2n$ with $\varphi_{2n}(d\alpha,0) = 2^{-n}\gamma_n(d\alpha)$ and $\varphi_{2n}(d\alpha,z^{-1}) = \overline{\varphi_{2n}(d\alpha,z)}$ if $|z| = 1$.

<u>Proof.</u> Let us write (7) in the form

$$p_n(d\alpha,\cos\theta) = \mathrm{Re}\,\varphi_{2n}(d\alpha,e^{i\theta})\, U_n(x) - \mathrm{Im}\,\varphi_{2n}(d\alpha,e^{i\theta})\, T_{n+1}(x)\, (1 - x^2)^{-1/2}\ .$$

Thus (7) means that

$$p_n(d\alpha,x) = R_{n,-1}(d\alpha,x) \equiv U_n(x)\, p_0(d\alpha,x) + R_{n,0}(d\alpha,x)$$

which is equivalent to (2) applied with $k = 0$. (8) obviously follows from (7).

Corollary 16. Let k be a nonnegative integer and let

$$\frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)} = \frac{1}{2}, \qquad \alpha_{j-1}(d\alpha) = 0$$

for $j > k$. Then for each $n > k$

$$\sin\theta\, p_n(d\alpha, \cos\theta) = |\varphi_{2k}(d\alpha, e^{i\theta})| \sin[(n+1)\theta - \arg\varphi_{2k}(d\alpha, e^{i\theta})] \qquad (\theta \in \mathbb{R}),$$

that is

$$p_n(d\alpha, x) = U_{n-k}(x)\, p_k(d\alpha, x) - U_{n-k-1}(x)\, p_{k-1}(d\alpha, x) .$$

Let us note that the conditions of corollary 16 are satisfied if $\operatorname{supp}(d\alpha) = [-1,1]$ and

$$\alpha(x) = \int_{-1}^{x} \frac{\sqrt{1-t^2}}{\pi(t)}\, dt \qquad (-1 \le x \le 1)$$

where π is a polynomial which is positive on $[-1,1]$. In this case

$$\frac{2}{\pi} |\varphi_{2k}(d\pi, e^{i\theta})|^2 = \pi(\cos\theta)$$

$(\theta \in \mathbb{R})$. (See Szegő, Chapter II.)

3.2. Modified Quadrature Processes

Lemma 1. Let $\alpha \in M(a,b)$ with $b > 0$. Let m be a nonnegative integer. If $n > m - 1$ then $x^m p_{n-1}(d\alpha,x)$ may be written in the form

$$(1) \qquad x^m p_{n-1}(d\alpha,x) = R^{d\alpha}_{m-1,n}(x)\, p_n(d\alpha,x) + a^{d\alpha}_{n-1,m}\, p_{n-1}(d\alpha,x) + \pi^{d\alpha}_{n-2,m}(x)$$

where $R^{d\alpha}_{m-1,n}$ and $\pi^{d\alpha}_{n-2,m}$ are polynomials of degree $m - 1$ and $n - 2$ respectively and $R^{d\alpha}_{m-1,n} \equiv 0$ if $m = 0$. Further

$$(2) \qquad \lim_{n\to\infty} a^{d\alpha}_{n-1,m} = \frac{2}{\pi b^2} \int_{a-b}^{a+b} t^m \sqrt{b^2 - (t-a)^2}\, dt .$$

Proof. Let, for simplicity, $\alpha \in M(0,1)$. For $m = 0$ and $m = 1$ the lemma is certainly true. Suppose that for $m > 1$ we have

$$x^{m-1} p_{n-1}(d\alpha,x) = R^{d\alpha}_{m-2,n}(x)\, p_n(d\alpha,x) + \sum_{k=n-m}^{n-1} a^{d\alpha,n}_{k,m-1}\, p_k(d\alpha,x)$$

with existing

$$\lim_{n\to\infty} a^{d\alpha,n}_{k,m-1} \qquad (k = n-m,\ n-m+1,\ \ldots,\ n-1)$$

which depends only on $M(0,1)$ and is independent of the particular $\alpha \in M(0,1)$. Using the recursion formula we see that

$$x^m p_{n-1}(d\alpha,x) = \left[x R^{d\alpha}_{m-2,n}(x) + a^{d\alpha,n}_{n-1,m-1} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\right] p_n(d\alpha,x) +$$

$$+ \left[a^{d\alpha,n}_{n-2,m-1} \frac{\gamma_{n-2}(d\alpha)}{\gamma_{n-1}(d\alpha)} + a^{d\alpha,n}_{n-1,m-1}\, \alpha_{n-1}(d\alpha)\right] p_{n-1}(d\alpha,x) +$$

$$\sum_{k=n-m+1}^{n-1} \left[a^{d\alpha,n}_{k-1,m-1} \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} + a^{d\alpha,n}_{k,m-1}\, \alpha_k(d\alpha) + a^{d\alpha,n}_{k+1,m-1} \frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)}\right] p_k(d\alpha,x) +$$

$$+ \left[a^{d\alpha,n}_{n-m,m-1}\, \alpha_{n-m}(d\alpha) + a^{d\alpha,n}_{n-m+1,m-1} \frac{\gamma_{n-m}(d\alpha)}{\gamma_{n-m+1}(d\alpha)}\right] p_{n-m}(d\alpha,x) +$$

$$+ a^{d\alpha,n}_{n-m,m-1} \frac{\gamma_{n-m-1}(d\alpha)}{\gamma_{n-m}(d\alpha)}\, p_{n-m-1}(d\alpha,x) .$$

This formula proves (1) and shows that $\lim_{n\to\infty} a^{d\alpha}_{n-1,m}$ exists and depends only on $M(0,1)$. To compute (2) we put in (1) α = Chebyshev weight. We have in this case $\frac{\pi}{2} p^2_{n-1}(v,x_{kn}(v)) = 1 - x^2_{kn}(v)$. Thus by the Gauss-Jacobi mechanical quadrature formula and by (1) we have for $m + 3 < 2n$

$$\frac{2}{\pi}\int_{-1}^{1} t^m \sqrt{1-t^2}\, dt = \frac{2}{\pi}\sum_{k=1}^{n} \lambda_{kn}(v)\ [1 - x^2_{kn}(v)]\ x^m_{kn}(v) =$$

$$= \sum_{k=1}^{n} \lambda_{kn}(v)\ p^2_{n-1}(v,x_{kn}(v))\ x^m_{kn}(v) =$$

$$= a^v_{n-1,m} \sum_{k=1}^{n} \lambda_{kn}(v)\ p^2_{n-1}(v,x_{kn}(v)) + \sum_{k=1}^{n} \lambda_{kn}(v)\ p_{n-1}(v,x_{kn}(v)) \cdot$$

$$\cdot\ \pi^v_{n-2,m}(x_{kn}(v)) = a^v_{n-1,m}\ .$$

<u>Lemma 2</u>. Let $\alpha \in M(a,b)$ with $b > 0$. Then $[a-b,a+b] \subset \Delta(d\alpha)$.

<u>Proof</u>. It follows rom Lemma 1 that if f is continuous on $\mathbb{R}$ and has compact support then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\ f(x_{kn})\ p^2_{n-1}(d\alpha,x_{kn}) = \frac{2}{\pi b^2}\int_{a-b}^{a+b} f(t)\sqrt{b^2-(t-a)^2}\, dt\ .$$

If $[a-b,a+b] \not\subset \Delta(d\alpha)$, then we can choose f so that $f(x_{kn}) = 0$ for every n and $k = 1,2,\dots,n$ and

$$\int_{a-b}^{a+b} f(t)\sqrt{b^2-(t-a)^2}\, dt < 0$$

which contradicts the above limit relation.

Let us note that Lemma 2 follows from results of O. Blumenthal [14].

<u>Theorem 3</u>. Let $\alpha \in M(a,b)$ with $b > 0$. Let f be a complex valued, bounded function $\Delta(d\alpha)$. If f is Riemann integrable on $[a-b,a+b]$ then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\ f(x_{kn})\ p^2_{n-1}(d\alpha,x_{kn}) = \frac{2}{\pi b^2}\int_{a-b}^{a+b} f(t)\sqrt{b^2-(t-a)^2}\, dt\ . \tag{3}$$

Proof. If f is a polynomial then the theorem follows immediately from Lemma 1. Otherwise we write

$$f = \mathrm{Re}(f)\,1_\Delta + \mathrm{Re}(f)\,1_{\Delta(d\alpha)\setminus\Delta} + i\,\mathrm{Im}(f)\,1_\Delta + i\,\mathrm{Im}(f)\,1_{\Delta(d\alpha)\setminus\Delta}$$

where $\Delta = [a-b, a+b]$. Let, for simplicity, $\Lambda_n(g) = \sum_{k=1}^{n} \lambda_k g(x_k)\, p_{n-1}^2(d\alpha, x_k)$.

Fix $\epsilon > 0$. We construct two polynomials π_1 and π_2 such that

$$\pi_1(x) \le \mathrm{Re}(f)(x)\,1_\Delta(x) \le \pi_2(x)$$

for $x \in \Delta(d\alpha)$ and

$$\frac{2}{\pi b^2}\int_{\Delta(d\alpha)} [\pi_2(t) - \pi_1(t)]dt < \epsilon .$$

We can do this because $\mathrm{Re}(f)\,1_\Delta$ is Riemann integrable on $\Delta(d\alpha)$ (See e.g., Szegő, 1.5). Hence

$$\lim_{n\to\infty} \Lambda_n(\mathrm{Re}(f)\,1_\Delta) = \frac{2}{\pi b^2}\int_{a-b}^{a+b} \mathrm{Re}(f)(t)\sqrt{b^2 - (t-a)^2}\,dt .$$

We have, further,

$$|\Lambda_n(\mathrm{Re}(f)\,1_{\Delta(d\alpha)\setminus\Delta})| \le \sup_{t\in\Delta(d\alpha)} |f(t)|\,\Lambda_n(1_{\Delta(d\alpha)\setminus\Delta})$$

and we can find a polynomial π such that

$$1_{\Delta(d\alpha)\setminus\Delta}(x) \le \pi(x) \qquad (x \in \Delta(d\alpha))$$

and

$$\frac{2}{\pi b^2}\int_{a-b}^{a+b} \pi(t)dt < \epsilon .$$

Thus

$$\lim_{n\to\infty} \Lambda_n(\mathrm{Re}(f)\,1_{\Delta(d\alpha)\setminus\Delta}) = 0 .$$

The limit of $\Lambda_n(\mathrm{Im}(f))$ can be found in the same way.

Theorem 4. Let $a \in \mathbb{R}$, $b > 0$. If for every polynomial π

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\,\pi(x_{kn})\,p_{n-1}^2(d\alpha, x_{kn}) = \frac{2}{\pi b^2}\int_{a-b}^{a+b} \pi(t)\sqrt{b^2-(t-a)^2}\,dt$$

then $\alpha \in M(a,b)$.

Proof. We have by the recursion formula

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, x_{kn}\, p_{n-1}^{2}(d\alpha, x_{kn}) = \alpha_{n-1}(d\alpha)$$

and

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, x_{kn}^{2}\, p_{n-1}^{2}(d\alpha, x_{kn}) = \alpha_{n-1}^{2}(d\alpha) + \frac{\gamma_{n-2}^{2}(d\alpha)}{\gamma_{n-1}^{2}(d\alpha)} .$$

The theorem follows immediately from the above identities.

3.3 The Support of $d\alpha$

In this section we are going to prove several properties of $\operatorname{supp}(d\alpha)$ for $\alpha \in M(a,b)$. Many results in this paper depend heavily on those properties. Let us note that practically all the results of this section are well known. For alternate proofs, we refer to Blumenthal [14], Chihara [15], Krein [13] and Sherman [16].

Supp$(d\alpha)$, that is the set of points of increase of α, is always closed. Hence $\operatorname{supp}(d\alpha)$ is compact iff it is bounded.

Lemma 1. The following three statements are equivalent. (i) $\operatorname{supp}(d\alpha)$ is compact. (ii) $\sup_{\substack{n\in\mathbb{N}\\ k=1,\dots,n}} |x_{kn}(d\alpha)| < \infty$. (iii) $\sup_{k\in\mathbb{N}} C_k^{0,0}(d\alpha) < \infty$.

Proof. Easy computation. Let us prove e.g., (iii) $\Rightarrow$ (ii). We have the following important identity by the Gauss-Jacobi mechanical quadrature formula:

$$(1)\qquad x_{kn}(d\alpha) = \lambda_{kn}(d\alpha)\int_{-\infty}^{\infty} x\, K_n^2(d\alpha,x,x_{kn})\,d\alpha(x) =$$

$$= \lambda_{kn}(d\alpha)\sum_{j=0}^{n-1}\alpha_j(d\alpha)\,p_j^2(d\alpha,x_{kn}) + \lambda_{kn}(d\alpha)\sum_{j=1}^{n-1} 2\,\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}\,p_{j-1}(d\alpha,x_{kn})\,p_j(d\alpha,x_{kn}).$$

Thus

$$|x_{kn}(d\alpha)| \le \max_{0\le j\le n-1}|\alpha_j(d\alpha)| + 2\max_{1\le j\le n-1}\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}\ .$$

Lemma 2. If $\operatorname{supp}(d\alpha)$ is compact, $x\in\operatorname{supp}(d\alpha)$ and $\epsilon>0$ then there exists a number $N = N(\epsilon,x)$ such that for every $n\ge N$, $p_n(d\alpha,t)$ has at least one zero in $[x-\epsilon,x+\epsilon]$, in particular,

$$\Delta(d\alpha) = [\lim_{n\to\infty} x_{nn}(d\alpha),\ \lim_{n\to\infty} x_{1n}(d\alpha)]\ .$$

Further, if α is constant on an interval Δ, then for every n, $p_n(d\alpha,t)$ has no more than one zero in Δ.

Proof. See Szegö, §6.1, and Freud, §I.2 .

Note, that if $\alpha(x)+\alpha(-x) = \text{const}$, $\alpha(x) = \text{const}$ on $(-1,1)$ then for every n , $p_{2n+1}(d\alpha,0) = 0$ but $p_{2n}(d\alpha,t)$ has no zero in $(-1,1)$.

Lemma 3. Let $\operatorname{supp}(d\alpha)$ be compact. Then

$$\Delta(d\alpha) \subset \Big[\inf_{j\geq 0} \alpha_j - 2 \sup_{j\geq 0} \frac{\gamma_j}{\gamma_{j+1}} , \sup_{j\geq 0} \alpha_j + 2 \sup_{j\geq 0} \frac{\gamma_j}{\gamma_{j+1}}\Big]$$

where $\alpha_j = \alpha_j(d\alpha)$ and $\gamma_j = \gamma_j(d\alpha)$.

Proof. Let $A = \inf_{j\geq 0} \alpha_j$, $B = \sup_{j\geq 0} \alpha_j$. Then by (1)

$$x_{kn} - \frac{A+B}{2} = \lambda_{kn} \sum_{j=0}^{n-1} \Big[\alpha_j - \frac{A+B}{2}\Big] p_j^2(d\alpha,x_{kn}) +$$

$$+ 2 \lambda_{kn} \sum_{j=1}^{n-1} \frac{\gamma_{j-1}}{\gamma_j} p_{j-1}(d\alpha,x_{kn})\, p_j(d\alpha,x_{kn}) .$$

Hence

$$\Big|x_{kn} - \frac{A+B}{2}\Big| \leq \frac{B-A}{2} + 2 \sup_{j\geq 0} \frac{\gamma_j}{\gamma_{j+1}} . \tag{2}$$

Put here $k = 1$ and let $n \to \infty$. By Lemma 2 we obtain

$$\Delta(d\alpha) \subset \Big(-\infty , B + 2 \sup_{j\geq 0} \frac{\gamma_j}{\gamma_{j+1}}\Big] .$$

If we put $k = n$ in (2) and let $n \to \infty$ then we get

$$\Delta(d\alpha) \subset \Big[A - 2 \sup_{j\geq 0} \frac{\gamma_j}{\gamma_{j+1}} , \infty\Big) .$$

Lemma 4. Let $\operatorname{supp}(d\alpha)$ be compact and let x be fixed. If for every $\epsilon > 0$, α takes infinitely many values in $(x-\epsilon, x+\epsilon)$ then there exists a sequence of natural integers $\{k_n\}_{n=1}^{\infty}$ such that $1 \leq k_n \leq n$ and

$$\lim_{n\to\infty} x_{k_n,n}(d\alpha) = x , \quad \lim_{n\to\infty} \lambda_{k_n,n}(d\alpha) = 0 . \tag{3}$$

Proof. Suppose, without loss of generality, that for every $\varepsilon > 0$, α takes infinitely many values in $(x-\varepsilon, x)$. Let for every n the number j_n be defined by $j_n = \{k\colon\ x_{kn}(d\alpha) < x \le x_{k-1,n}(d\alpha)\}$ with $x_{0n} = +\infty$. Let $k_n = j_n + 1$. We shall show that $\{k_n\}_{n=N}^{\infty}$ satisfies the requirements of the lemma. Because of Lemma 2 $k_n < n$ for n large. If we can show that

$$\lim_{n\to\infty} x_{k_n+1,n} = x \tag{4}$$

then

$$\lim_{n\to\infty} x_{k_n,n} = \lim_{n\to\infty} x_{j_n,n} = x$$

and by the Markov-Stieltjes inequalities

$$\lambda_{k_n,n} \le \int_{x_{k_n+1,n}}^{x_{k_n-1,n}} d\alpha(t) \ {}_{n\to\infty}\ \alpha(x-0) - \alpha(x-0) = 0 .$$

Suppose now that (4) does not hold. Then there exists an $\varepsilon > 0$ and a sequence $\{n_\ell\}$ such that $p_{n_\ell}(d\alpha, t)$ has no more than two zeros in $(x-\varepsilon, x)$ for $\ell = 1,2,\ldots$. Because α takes infinitely many values in $(x-\varepsilon, x)$ we can find three points $x_1, x_2, x_3 \in (x-\varepsilon, x) \cap \operatorname{supp}(d\alpha)$ and by Lemma 2, $p_{n_\ell}(d\alpha, t)$ must have zeros near each x_k for every ℓ large. Hence $p_{n_\ell}(\alpha, t)$ has at least three zeros in $(x-\varepsilon, x)$. This contradiction proves (4).

Lemma 5. Let $\operatorname{supp}(d\alpha)$ be compact and let $x \in \mathbb{R}$ be fixed. Suppose that for every $\varepsilon > 0$, α takes infinitely many values in $(x-\varepsilon, x+\varepsilon)$. Then for every $c \in \mathbb{R}$

$$|x - c| \le \limsup_{j\to\infty} |\alpha_j(d\alpha) - c| + 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} ,$$

in particular, if $a = \lim_{j\to\infty} \alpha_j(d\alpha)$ exists, then

$$x \in [a - 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}}{\gamma_j} ,\quad a + 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}}{\gamma_j}] .$$

Proof. Let us take $\{k_n\}$ from Lemma 4. Let $M \in \mathbb{N}$. Then by (1)

$$|x_{k_n,n} - c| \le \lambda_{k_n,n} \sum_{j=0}^{M-1} |\alpha_j - c|\, p_j^2(d\alpha, x_{k_n,n}) + \sup_{j \ge M} |\alpha_j - c| +$$

$$+ 2\lambda_{k_n,n} \sum_{j=1}^{M-1} \frac{\gamma_{j-1}}{\gamma_j} |p_{j-1}(d\alpha, x_{k_n,n})\, p_j(d\alpha, x_{k_n,n})| + 2 \sup_{j \ge M} \frac{\gamma_{j-1}}{\gamma_j} .$$

First let $n \to \infty$, then $M \to \infty$.

Lemma 6. Let $\alpha \in M(a,b)$ with $b > 0$. Then $[a-b, a+b] \subset \text{supp}(d\alpha)$.

Proof. If $[a-b,a+b] \not\subset \text{supp}(d\alpha)$ then $[a-b,a+b] \cap [\mathbb{R} \setminus [a-b,a+b] \cap \text{supp}(d\alpha)]$ contains an interval Δ_1. Let $\Delta \subset \Delta_1^0$. Then by Theorem 3.2.3

$$(5) \qquad \sum_{x_{kn} \in \Delta} \lambda_{kn}\, p_{n-1}^2(d\alpha, x_{kn}) \underset{n\to\infty}{\to} \frac{2}{\pi b^2} \int_{t \in \Delta} \sqrt{b^2 - (t-a)^2}\, dt > 0 .$$

On the other hand, by Lemma 2 Δ contains no more than one x_{kn} for every n since $\Delta \cap \text{supp}(d\alpha) = \emptyset$. Further $\Delta \subset (a-b, a+b)$ and by Theorem 3.1.9

$$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p_{n-1}^2(d\alpha, x) = 0$$

uniformly for $x \in \Delta$. Thus the left side of (5) converges to 0 when $n \to \infty$. This contradiction shows that $[a-b, a+b] \subset \text{supp}(d\alpha)$.

Theorem 7. Let $\text{supp}(d\alpha)$ be compact and let $\lim_{j\to\infty} \alpha_j(d\alpha) = a$ exist. Then

$$\text{supp}(d\alpha) = A \cup B , \quad A \cap B = \emptyset$$

where A is closed and belongs to

$$(6) \qquad [a - 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} , \quad a + 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}] .$$

B is at most denumerable, isolated, the only two possible limit points of B are the two endpoints of (6), if $x \in B$ then $\alpha_{ac} + \alpha_s$ is constant near x and α has an isolated jump at x, furthermore,

$$B \subset [\inf_{j \ge 0} \alpha_j - 2 \sup_{j \ge 1} \frac{\gamma_{j-1}}{\gamma_j} , \quad \sup_{j \ge 0} \alpha_j + 2 \sup_{j \ge 1} \frac{\gamma_{j-1}}{\gamma_j}] .$$

If $\alpha \in M(a,b)$ then A is the interval (6).

Proof. The theorem follows immediately from Lemmas 1 - 6. The only thing which we have to show is that if $\alpha \in M(a,0)$ then $a \in \operatorname{supp}(d\alpha)$. If $a \notin \operatorname{supp}(d\alpha)$ then $B = \operatorname{supp}(d\alpha)$ and hence B is closed. But B can be closed only if B is finite and then α has only finitely many points of increase, that is α is not a weight.

Theorem 8. Let $\alpha \in M(a,b)$. If $x \notin \operatorname{supp}(d\alpha)$ then there exist $\epsilon > 0$ and $N \geq 0$ such that for every $n \geq N$, $p_n(d\alpha,t)$ has no zeros in $[x-\epsilon, x+\epsilon]$.

Proof. Let, for simplicity, $a = 0$. By Theorem 7, $x \notin \operatorname{supp}(d\alpha)$ implies $x \notin [-b,b]$ (or $x \neq 0$ if $b = 0$). Suppose, without loss of generality, that $b < x < \infty$. If $x \notin \Delta(d\alpha)$ then the theorem says nothing since $x_{kn}(d\alpha) \in \Delta(d\alpha)$ for every n and $1 \leq k \leq n$. Now let $x \in \Delta(d\alpha) \cap (b,\infty)$. By Theorem 7 $(\frac{b+x}{2},\infty) \cap \operatorname{supp}(d\alpha)$ is finite and it is not empty since $x \in \Delta(d\alpha)$. Let $t_1 < t_2 < \ldots < t_m$ denote those points of $\operatorname{supp}(d\alpha)$ which belong to (x,∞). Let $\epsilon > 0$ be such that

$$x + \epsilon < t_1 - \epsilon < t_2 + \epsilon < \ldots < t_{m-1} + \epsilon < t_m - \epsilon < t_m + \epsilon$$

and $[x-\epsilon,x] \cap \operatorname{supp}(d\alpha) = \emptyset$. By Lemma 2 we can find $N = N(\epsilon,\{t_i\})$ such that for every $n \geq N(\epsilon)$, $p_n(d\alpha,t)$ has zeros in each $[t_i-\epsilon,t_i+\epsilon]$ $(i = 1,2,\ldots,m)$. Thus for $n \geq N(\epsilon)$ $p_n(d\alpha,t)$ has not less than m zeros in $[t_1 - \epsilon,\infty)$. On the other hand α takes exactly $m+1$ values in $(x-\epsilon,\infty)$ if we do not count the values of $\alpha(t_i)$. Further, $[t_m,\infty)$ does not contain zeros of $p_n(d\alpha,t)$ since $\Delta(d\alpha) \cap (t_m,\infty) = \emptyset$. Thus by Lemma 2 for every $n \geq 0$, $p_n(d\alpha,t)$ has no more than m zeros in $(x-\epsilon,\infty)$. Hence for $n > N$ both $(x-\epsilon,\infty)$ and $[t_1-\epsilon,\infty)$ contain exactly m zeros of $p_n(d\alpha,t)$, that is $[x-\epsilon,x+\epsilon]$ contains no zeros of $p_n(d\alpha,t)$ if $n \geq N$.

Let us note that without the assumption $\alpha \in M(a,b)$, Theorem 8 does not necessarily hold. (See Remark 4.1.6 .)

4. Limit Relations

4.1. Pointwise Limits

We begin with a simple result which we will not apply in the following but which is worth recording.

Theorem 1. For every weight α and $x \in \mathbb{R}$

$$(1) \qquad \sum_{k=0}^{\infty} \lambda_{k+1}^2(d\alpha,x)\, p_k^2(d\alpha,x) \le [1+\alpha(\infty)-\alpha(-\infty)]^2 ,$$

in particular, for every $x \in \mathbb{R}$

$$\lim_{n\to\infty} \lambda_{n+1}(d\alpha,x)\, p_n(d\alpha,x) = 0 .$$

Proof. Let x be fixed and let $\beta = \alpha + \delta_x$ where δ_x is the unit mass concentrated at x. Let us expand $p_n(d\alpha,t)$ in a Fourier series in $p_k(d\beta,t)$. We have

$$p_n(d\alpha,t) = \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)}\, p_n(d\beta,t) + K_{n+1}(d\beta,t,x)\, p_n(d\alpha,x) .$$

Putting $t = x$ we obtain

$$p_n(d\alpha,x) = \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)}\, p_n(d\beta,x) + \lambda_{n+1}^{-1}(d\beta,x)\, p_n(d\alpha,x) .$$

By an easy computation $\lambda_{n+1}(d\beta,x) = \lambda_{n+1}(d\alpha,x) + 1$, $\gamma_n(d\beta) \le \gamma_n(d\alpha)$ and $\lambda_{n+1}(d\alpha,x) \le \alpha(\infty) - \alpha(-\infty)$. Thus for every $n = 0,1,\ldots$

$$\lambda_{n+1}^2(d\alpha,x)\, p_n^2(d\alpha,x) \le p_n^2(d\beta,x)\, [1+\alpha(\infty)-\alpha(-\infty)]^2$$

and

$$\sum_{k=0}^{n-1} \lambda_{k+1}^2(d\alpha,x)\, p_k^2(d\alpha,x) \le \lambda_n^{-1}(d\beta,x)[1+\alpha(\infty)-\alpha(-\infty)]^2 \le [1+\alpha(\infty)-\alpha(-\infty)]^2.$$

Letting $n \to \infty$ we obtain (1).

<u>Lemma 2</u>. Let $\alpha \in M(a,b)$ with $b > 0$. Then for every $x \in [a-b, a+b]$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} = +\infty. \tag{2}$$

<u>Proof</u>. Let, for simplicity, $a \le x \le a+b$. Let $0 < x \le a+b$. Let $0 < \epsilon < b$. Then

$$\sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \ge \epsilon^{-2} \sum_{x-\epsilon \le x_{kn} \le x} \lambda_{kn} p_{n-1}^2(d\alpha, x_{kn})$$

and by Theorem 3.2.3 (take $f = 1_{[x-\epsilon, x]}$)

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \ge \frac{2}{\epsilon^2 \pi b^2} \int_{x-\epsilon}^{x} \sqrt{b^2 - (t-a)^2}\, dt .$$

If $x < a + b$ the right side is exactly of order ϵ^{-1}, if $x = a + b$ it is exactly of order $\epsilon^{-1/2}$. Letting $\epsilon \to 0$ we obtain (2).

<u>Theorem 3</u>. Let $\alpha \in M(a,b)$ with $b > 0$. Then for $x \in [a-b, a+b]$

$$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) = 0 \tag{3}$$

and the convergence is uniform for $x \in \Delta \subset (a-b, a+b)$.

<u>Proof</u>. To get (3) use Lemma 2 and the formula

$$\lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) = \frac{\gamma_n^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)} \left[\sum_{k=1}^{n} \frac{\lambda_{kn}(d\alpha)\, p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \right]^{-1}$$

which follows from $\lambda_n^{-1}(d\alpha, x) = \sum_{k=1}^{n} \frac{\ell_{kn}^2(d\alpha, x)}{\lambda_{kn}(d\alpha)}$. In order to show that the convergence in (3) is uniform for $x \in \Delta \subset (a-b, a+b)$ we will apply Theorem 3.1.9. By that theorem

$$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p_{n-1}^2(d\alpha, x) = 0$$

uniformly for $x \in \Delta \subset (a-b, a+b)$. Therefore

$$\lim_{n\to\infty} [\lambda_n(d\alpha, x)\, p_{n-1}^2(d\alpha, x) + \lambda_{n-1}(d\alpha, x)\, p_{n-2}^2(d\alpha, x)] = 0 \tag{3'}$$

also holds uniformly for $x \in \Delta$. Using the recurrence formula we obtain

$$p_n^2(d\alpha,x(\le C[p_{n-1}^2(d\alpha,x) + p_{n-2}^2(d\alpha,x)]$$

for $x \in \Delta(d\alpha)$. Since $\lambda_n(d\alpha,x) \le \lambda_{n-1}(d\alpha,x)$ we have

$$\lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) \le C[\lambda_n(d\alpha,x)\, p_{n-1}^2(d\alpha,x) + \lambda_{n-1}(d\alpha,x)\, p_{n-2}^2(d\alpha,x)]$$

for $x \in \Delta(d\alpha)$. Hence by (3') the limit relation (3) holds uniformly for $x \in \Delta \subset (a-b, a+b)$.

There are two possible ways to define the Christoffel functions for complex values of the argument. We can either put

$$\lambda_n(d\alpha,z) = [\sum_{k=0}^{n-1} p_k^2(d\alpha,z)]^{-1}$$

or

$$\lambda_n(d\alpha,z) = [\sum_{k=0}^{n-1} |p_k(d\alpha,z)|^2]^{-1} .$$

It is easy to see that the second definition coincides with

$$\lambda_n(d\alpha,z) = \min_{\pi_{n-2}} \int_{-\infty}^{\infty} |(1 + (z - t)\, \pi_{n-2}(t))|^2 \, d\alpha(t) .$$

To avoid confusion we shall write $\lambda_n^*(d\alpha,z)$ when we mean the first definition:

$$\lambda_n^*(d\alpha,z) =[\sum_{k=0}^{n-1} p_k^2(d\alpha,z)]^{-1} .$$

Let for $z,u \in \mathbb{C}$

$$K_n(d\alpha,z,u) = \sum_{k=0}^{n-1} p_k(d\alpha,z)\, \overline{p_k(d\alpha,u)} ,$$

$$k_n(d\alpha,z,u) = \sum_{k=0}^{n-1} p_k(d\alpha,z)\, p_k(d\alpha,u) .$$

<u>Properties 4</u>. λ_n is real valued, monotonic in n and positive, λ_n^* is meromorphic with $2n - 2$ poles.

$$\lambda_n^{-1}(z) = K_n(z,z) , \quad K_n(z,u) = \overline{K_n(u,z)} ,$$

$$\lambda_n^*(t)^{-1} = k_n(z,z) , \; k_n(z,u) = k_n(u,z) ,$$

$$K_n(d\alpha,z,u) = \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha,z)\,\overline{\ell_{kn}(d\alpha,u)}}{\lambda_{kn}(d\alpha)}$$

$$k_n(d\alpha,z,u) = \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha,z)\,\ell_{kn}(d\alpha,u)}{\lambda_{kn}(d\alpha)}$$

further

$$(4) \qquad \lambda_n^{-1}(d\alpha,z) = |p_n(d\alpha,z)|^2 \frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{|z-x_{kn}|^2}$$

and

$$(5) \qquad \lambda_n^*(d\alpha,z)^{-1} = p_n^2(d\alpha,z) \frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{(z-x_{kn})^2} .$$

We obtain immediately from (4) and (5) the following

Theorem 5. Let $\operatorname{supp}(d\alpha)$ be compact and let $z \notin \Delta(d\alpha)$. Then

$$\liminf_{n\to\infty} \lambda_n(d\alpha,z)\ |p_n^2(d\alpha,z)| > 0$$

and

$$\liminf_{n\to\infty} |\lambda_n^*(d\alpha,z)\ p_n^2(d\alpha,z)| > 0 .$$

Proof. If $\operatorname{supp}(d\alpha)$ is compact then

$$\gamma = \limsup_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} < \infty .$$

Since $x_{kn}(d\alpha) \in \Delta(d\alpha)$ for every $n = 1,2,\ldots$ and $k = 1,2,\ldots,n$ and $z \notin \Delta(d\alpha)$ the inequality

$$|z - x_{kn}|^{-2} \le c < \infty$$

holds for every x_{kn} . Therefore by (4)

$$\limsup_{n\to\infty} [\lambda_n(d\alpha,z)\ |p_n(d\alpha,z)|^2]^{-1} \le c\gamma^2 .$$

The second part of the theorem can be proved in the same way.

Remark 6. It is not true that Theorem 5 holds for every $z \notin \operatorname{supp}(d\alpha)$. For, if $\alpha(x) + \alpha(-x) = \text{const}$, $\operatorname{supp}(d\alpha)$ is compact and $\alpha(t) = \text{const}$ for $t \in [-\epsilon,\epsilon]$, $(\epsilon > 0)$ then $p_{2k+1}(d\alpha,0) = 0$ although $0 \notin \operatorname{supp}(d\alpha)$.

Theorem 7. Let $\alpha \in M(a,0)$. Then

(i) for every $z \notin \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \lambda_n^*(d\alpha,z)\, p_n^2(d\alpha,z) = \infty , \quad \lim_{n\to\infty} \lambda_n(d\alpha,z)\, p_n^2(d\alpha,z) = \infty .$$

(ii) for every $x \in \operatorname{supp}(d\alpha)\backslash a$

$$\lim_{n\to\infty} \lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) = 0 .$$

(iii) there exist two weights $\hat{\alpha}$ and $\hat{\hat{\alpha}}$ in $M(a,0)$ such that

$$\lim_{n\to\infty} \lambda_n(d\hat{\alpha},a)\, p_n^2(d\hat{\alpha},a) = 0$$

and

$$\liminf_{n\to\infty} \lambda_n(d\hat{\hat{\alpha}},a)\, p_n^2(d\hat{\hat{\alpha}},a) = 0 ,$$

$$\limsup_{n\to\infty} \lambda_n(d\hat{\hat{\alpha}},a)\, p_n^2(d\hat{\hat{\alpha}},a) = \infty .$$

Proof. (i) If z is complex use (4) and (5). Let now $z = x$ be real and $x \notin \operatorname{supp}(d\alpha)$. Since $\operatorname{supp}(d\alpha)$ is compact and α is constant in a neighborhood of x , we have $\lim_{n\to\infty} \lambda_n(d\alpha,x) = 0$. Suppose that there exists a sequence $n_1 < n_2 < \ldots$ such that

$$\lim_{k\to\infty} \lambda_{n_k}(d\alpha,x)\, p_{n_k}^2(d\alpha,x) < \infty .$$

We have by the recurrence formula

$$x\, \lambda_n^{-1}(x) = \sum_{k=0}^{n-1} \alpha_k\, p_k^2(x) + 2 \sum_{k=0}^{n-1} \frac{\gamma_k}{\gamma_{k+1}}\, p_k(x)\, p_{k+1}(x) + \frac{\gamma_{n-1}}{\gamma_n}\, p_{n-1}(x)\, p_n(x)$$

with $\lambda_n(x) = \lambda_n(d\alpha,x)$, $\alpha_k = \alpha_k(d\alpha)$ etc. Let M be a natural integer. Then

$$|x - a| \le \lambda_n(x) \sum_{k=0}^{M-1} |\alpha_k - a|\, p_k^2(x) + \sup_{k\ge M} |\alpha_k - a| +$$

$$+ 2\, \lambda_n(x) \sum_{k=0}^{M-1} \frac{\gamma_k}{\gamma_{k+1}} |p_k(x)\, p_{k+1}(x)| + 2 \sup_{k\ge M} \frac{\gamma_k}{\gamma_{k+1}} + \frac{\gamma_{n-1}}{\gamma_n} [\lambda_n(x)\, p_n^2(x)]^{1/2} .$$

Put $n = n_\ell$, first let $\ell \to \infty$ and then $M \to \infty$. We get $|x - a| \le 0$ that is $x = a$. By Theorem 3.37, $x \in \operatorname{supp}(d\alpha)$. This is a contradiction.

(ii) If $x \in \operatorname{supp}(d\alpha)\backslash a$ then by Theorem 3.3.7 x is a jump of α and consequently (ii) is true.

(iii) Let $\hat{\alpha}$ be defined by $\alpha_i(d\hat{\alpha}) \equiv a$ and $\gamma_i(d\hat{\alpha})/\gamma_{i+1}(d\hat{\alpha}) = (i+1)^{-1/4}$ for $i = 0,1,\ldots$. Then $\hat{\alpha}(a+x) + \hat{\alpha}(a-x) = \text{const}$ and thus $p_{2k+1}(d\hat{\alpha},a) = 0$ for $k = 0,1,\ldots$. Hence

(6) $$\lim_{k\to\infty} \lambda_{2k+1}(d\hat{\alpha},a)\, p^2_{2k+1}(d\hat{\alpha},a) = 0 \,.$$

Let us consider now $p_{2k}(d\hat{\alpha},a)$. By the recurrence formula

$$\frac{\gamma_{2k-1}}{\gamma_{2k}}\, p_{2k}(a) + \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\, p_{2k-2}(a) = 0 \,.$$

Hence

(7) $$p^2_{2k}(a) = \left(\frac{2k}{2k-1}\right)^{1/2} p^2_{2k-2}(a) \,.$$

By repeating application of (7) we obtain that for every $j = 1,2,\ldots,k$

$$p^2_{2k}(a) = \left[\frac{2k}{2k-1}\,\frac{2k-2}{2k-3}\cdots\frac{2k-2(j-1)}{2k-2(j-1)-1}\right]^{1/2} p^2_{2k-2j}(a) \,.$$

Thus for $j = 0,1,\ldots,k-1$

$$p^2_{2k}(a) \le \sqrt{2k}\; p^2_{2j}(a) \,,$$

that is

$$p^2_{2k}(a) \le \sqrt{\frac{2}{k}} \sum_{j=0}^{k-1} p^2_{2j}(a) = \sqrt{\frac{2}{k}}\; \lambda^{-1}_{2k}(a)$$

which together with (6) proves the first part of (iii). Let $\hat{\hat{\alpha}}$ be defined by $\alpha_i(d\hat{\hat{\alpha}}) \equiv a$ and $\gamma_i(d\hat{\hat{\alpha}})/\gamma_{i+1}(d\hat{\hat{\alpha}}) = \exp(-(i+1)^2)$ for $i = 0,1,\ldots$. Repeating the above argument we see that (6) holds if we replace $\hat{\alpha}$ by $\hat{\hat{\alpha}}$. Further, similarly to (7),

$$p^2_{2k}(d\hat{\hat{\alpha}},a) = e^{8k-2}\, p^2_{2k-2}(d\hat{\hat{\alpha}},a) \,.$$

Thus

$$p^2_{2k}(d\hat{\hat{\alpha}},a) \ge \frac{e^{8k-2}}{k}\, \lambda^{-1}_{2k}(d\hat{\hat{\alpha}},a) \,.$$

Now let $k \to \infty$.

Let us remark that both $\hat{\alpha}$ and $\hat{\hat{\alpha}}$ are continuous at a .

<u>Definition 8.</u> For $z \in \mathbb{C}\backslash[-1,1]$ we define $\rho(z)$ by

$$\rho(z) = z + \sqrt{z^2 - 1}$$

where we take that branch of $\sqrt{z^2 - 1}$ for which $|\rho(z)| > 1$ whenever $z \in \mathbb{C}\backslash[-1,1]$. We have

$$\lim_{z\to\infty} \rho(z) = \infty, \quad \lim_{z\to\infty} |z/\rho(z)| = \frac{1}{2}.$$

Lemma 9. Let $|z| > 1$. If

$$\lim_{k\to\infty} a_k z^{-k} = a$$

then

$$\lim_{n\to\infty} \frac{z-1}{z^{n+1}} \sum_{k=0}^{n} a_k = a.$$

Proof. The matrix $\mu = [\mu_{kn}]$ where $\mu_{kn} = (z - 1)z^{k-n-1}$ satisfies the conditions of Toeplitz-Silverman's theorem.

Lemma 10. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and let v_τ denote the Chebyshev weight corresponding to $\tau = [a-b, a+b]$. Then for every $z \in \mathbb{C}\backslash[a-b, a+b]$

$$\lim_{n\to\infty} \lambda_n(v_\tau, t)\ |p_n^2(v_\tau, t)| = |\rho(\frac{z-a}{b})|^2 - 1$$

and

$$\lim_{n\to\infty} \lambda_n^*(v_\tau, z)\ p_n^2(v_\tau, z) = \rho(\frac{z-a}{b})^2 - 1.$$

Proof. Use Lemma 9 and the formula

$$p_n(v_\tau, z) = \frac{1}{\sqrt{2\pi}} [\rho(\frac{z-a}{b})^n + \rho(\frac{z-a}{b})^{-n}]$$

for $n = 1, 2, \ldots$.

Theorem 11. Let $\alpha \in M(a,b)$ with $b > 0$. Then

(i) for every $z \notin \mathrm{supp}(d\alpha)$

$$\lim_{n\to\infty} \lambda_n(d\alpha, z)\ |p_n^2(d\alpha, z)| = |\rho(\frac{z-a}{b})|^2 - 1$$

and

$$\lim_{n\to\infty} \lambda_n^*(d\alpha, z)\ p_n^2(d\alpha, z) = \rho(\frac{z-a}{b})^2 - 1,$$

(ii) for every $x \in \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) = 0 ,$$

(iii) the convergence in (ii) is uniform inside $(a-b,a+b)$.

Proof. (i) If $z \notin \Delta(d\alpha)$ then both $|z-t|^{-2}$ and $(z-t)^{-2}$ are continuous functions of $t \in \Delta(d\alpha)$. Thus by (4), (5) and Theorem 3.2.3

$$\lim_{n\to\infty} \lambda_n(d\alpha,z)\, |p_n^2(d\alpha,z)| = [\frac{1}{2\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{|z-t|^2}\, dt]^{-1} \tag{8}$$

and

$$\lim_{n\to\infty} \lambda_n^*(d\alpha,z)\, p_n^2(d\alpha,z) = [\frac{1}{2\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{(z-t)^2}\, dt]^{-1} . \tag{9}$$

If $z \in \Delta(d\alpha)$ but $z \in \operatorname{supp}(d\alpha)$ then we take ϵ from Theorem 3.3.8 and put

$$f_1(t) = \begin{cases} |z-t|^{-2} & \text{for } |z-t| > \epsilon \\ 0 & \text{for } |z-t| \le \epsilon , \end{cases}$$

$$f_2(t) = \begin{cases} (z-t)^{-2} & \text{for } |z-t| > \epsilon \\ 0 & \text{for } |z-t| \le \epsilon . \end{cases}$$

Both f_1 and f_2 satisfy the conditions of Theorem 3.2.3. By Theorem 3.3.8 neither (4) nor (5) will change of we replance $|z-t|^{-2}$ and $(z-t)^2$ by $f_1(t)$ and $f_2(t)$ respectively for $n \ge N$. Thus (8) and (9) hold for every $z \notin \operatorname{supp}(d\alpha)$. To calculate the integrals on the right sides of (8) and (9) let us remark that it is the same for every $\alpha \in M(a,b)$, in particular, for the Chebyshev weight corresponding to $[a-b,a+b]$. Now we use Lemma 10.

(ii) If $x \in [a-b,a+b]$ then use Theorem 3. If $x \in \operatorname{supp}(d\alpha)\backslash[a-b,a+b]$ then by Theorem 3.3.7 α has a jump at x which implies (ii) again.

(iii) See Theorem 3.

Theorem 12. Let $\operatorname{supp}(d\alpha)$ be compact and let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$. If there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \in \mathbb{C}$, $\lim_{k\to\infty} z_k = \infty$ and

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)} = \rho(\frac{z_k - a}{b})^{-1}$$

for $k = 1,2,\ldots$ then $\alpha \in M(a,b)$.

Proof. Suppose without loss of generality that $z_k \notin \Delta(d\alpha)$ for every k . We have

$$(10) \qquad \frac{zp_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} [1 + \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, x_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{z - x_{kn}}]$$

which can easily be checked. Let $d(z) = \mathrm{dist}(z, \Delta(d\alpha))$. Then we get with $C = C(\mathrm{supp}(d\alpha))$

$$\left|\frac{z_k\, p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)}\right| \leq \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} [1 + Cd(z_k)^{-1}] .$$

Letting first $n \to \infty$ and then $k \to \infty$ we obtain

$$\frac{b}{2} \leq \liminf_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} .$$

On the other hand we have by the recurrence formula and Lemma 3.3.1

$$\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \leq [|z_k| + C_1]\left|\frac{p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)}\right| + C_2\left|\frac{p_{n-2}(d\alpha, z_k)}{p_n(d\alpha, z_k)}\right|$$

where C_1 and C_2 depend on $\mathrm{supp}(d\alpha)$. First let $n \to \infty$ and then $k \to \infty$. We get

$$\limsup_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \leq \frac{b}{2} .$$

Using again the recurrence formula we obtain

$$\alpha_n(d\alpha) = z_1 - \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} \frac{p_{n+1}(d\alpha, z_1)}{p_n(d\alpha, z_1)} - \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha, z_1)}{p_n(d\alpha, z_1)} .$$

Thus $\alpha_n(d\alpha)$ is convergent and letting $n \to \infty$ we see that

$$\lim_{n\to\infty} \alpha_n(d\alpha) = z_1 - \frac{b}{2}\rho(\frac{z_1 - a}{b}) - \frac{b}{2}\rho(\frac{z_1 - a}{b})^{-1} = a .$$

Theorem 13. Let $\alpha \in M(a,b)$ and let $z \in \mathbb{C} \setminus \mathrm{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} = \begin{cases} 0 & \text{for } b = 0 \\ \rho(\frac{z-a}{b})^{-1} & \text{for } b > 0 . \end{cases}$$

Proof. If $b = 0$ then the theorem follows immediately from (10) and Theorem 3.3.8. If $b > 0$ then by (10) and Theorems 3.2.3 and 3.3.8

$$t^{2n} - L_n(d\alpha, y^{2n}, t) = p_n(d\alpha, t)[\Gamma\, p_n(d\alpha, x) + \pi_{n-1}(x)] .$$

Comparing the leading coefficients we see that $\Gamma = \gamma_n(d\alpha)^{-2}$. Consequently

$$(1) \qquad \sum_{k=1}^{n} (1 - x_k^2)\, p_{n-1}^2(d\beta, x_k)\, \lambda_k = 1 + \frac{\gamma_{n-1}^2(d\beta)}{\gamma_n^2(d\alpha)} .$$

Expand $(1 - x^2)\, p_{n-1}(d\beta, x)$ in a Fourier series in $p_k(d\alpha, x)$.*) It is easy to see that

$$(1 - x^2)\, p_{n-1}(d\beta, x) = \sum_{k=n-1}^{n+1} a_k\, p_k(d\alpha, x)$$

with $a_{n-1} = \gamma_{n-1}(d\alpha)/\gamma_{n-1}(d\beta)$ and $a_{n+1} = -\gamma_{n-1}(d\beta)/\gamma_{n+1}(d\alpha)$. Thus

$$(1 - x_k^2)\, p_{n-1}(d\beta, x_k) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)}\, p_{n-1}(d\alpha, x_k) - \frac{\gamma_{n-1}(d\beta)}{\gamma_{n+1}(d\alpha)}\, p_{n+1}(d\alpha, x_k) .$$

We obtain from the recursion formula that

$$p_{n+1}(d\alpha, x_k) = -\frac{\gamma_{n-1}(d\alpha)\ \gamma_{n+1}(d\alpha)}{\gamma_n^2(d\alpha)}\, p_{n-1}(d\alpha, x_k) .$$

Hence

$$(1 - x_k^2)\, p_{n-1}(d\beta, x_k) = \left[\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} + \frac{\gamma_{n-1}(d\beta)\ \gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)}\right] \cdot p_{n-1}(d\alpha, x_k) .$$

Putting this into (1) we obtain

$$\sum_{k=1}^{n} \lambda_k \frac{p_{n-1}^2(d\alpha, x_k)}{1 - x_k^2} = \left[1 + \frac{\gamma_{n-1}^2(d\beta)}{\gamma_n^2(d\alpha)}\right] \cdot \left[\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} + \frac{\gamma_{n-1}(d\beta)\ \gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)}\right]^{-2} .$$

From $\alpha \in S$ follows $\beta \in S$ and we can use Lemma 2 to show that the limit of the right hand side is 2 . By Theorem 3.2.3, if $\epsilon \in (0,1)$ then

$$\lim_{n\to\infty} \sum_{|x_{kn}| \le 1-\epsilon} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{1 - x_{kn}^2} = \frac{2}{\pi} \int_{|t| \le 1-\epsilon} v(t)\, dt .$$

Since by the previous calculation

*) This argument is due to Christoffel and is given in Szegö, Chapter 3. In the following, this argument will be used several times.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} = 2 = \frac{2}{\pi}\int_{-1}^{1} v(t)\,dt$$

we get

$$\lim_{n\to\infty} \sum_{|x_{kn}|>1-\epsilon} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} = \frac{4}{\pi}\int_{1-\epsilon}^{1} v(t)\,dt$$

for $0 < \epsilon < 1$. Now let f be an arbitrary Riemann integrable function on $[-1,1]$. We have

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} =$$

$$= \sum_{|x_{kn}|\le 1-\epsilon} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} + \sum_{|x_{kn}|>1-\epsilon} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2}$$

for $0 < \epsilon < 1$. Since $fv^2 1_{[-1+\epsilon,1-\epsilon]}$ is Riemann integrable we obtain from Theorem 3.2.3 that

$$\lim_{n\to\infty} \sum_{|x_{kn}|\le 1-\epsilon} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} = \frac{2}{\pi}\int_{-1+\epsilon}^{1-\epsilon} f(t) \frac{dt}{\sqrt{1-t^2}} .$$

Therefore

$$\limsup_{n\to\infty} \left| \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} - \frac{2}{\pi}\int_{-1+\epsilon}^{1-\epsilon} f(t)\frac{dt}{\sqrt{1-t^2}} \right| \le$$

$$\le \sup_{-1\le t\le 1} |f(t)| \lim_{n\to\infty} \sum_{|x_{kn}|>1-\epsilon} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{1-x_{kn}^2} = \sup_{-1\le t\le 1} |f(t)| \frac{4}{\pi}\int_{1-\epsilon}^{1} v(t)\,dt.$$

Letting $\epsilon \to 0$ the theorem follows.

Theorem 3 will be used to investigate some interpolation processes.

Definition 4. Let $\alpha \in S$. Then

$$\Gamma(\theta) = -\frac{1}{2\pi}\int_{-1}^{1} \frac{\log W(t) - \log W(x)}{t-x} \frac{\sqrt{1-x^2}}{\sqrt{1-t^2}}\,dt , \qquad x = \cos\theta ,$$

where $W(x) = \alpha'(x)\sqrt{1-x^2}$, $-1 \le x \le 1$, $0 \le \theta \le \pi$.

Lemma 5. If $\alpha \in S$ then

$$\lim_{n\to\infty} \int_0^{\pi} \left| p_n(d\alpha,\cos\theta)\sqrt{\alpha'(\cos\theta)\sin\theta} - \sqrt{\frac{2}{\pi}}\cos[n\theta - \Gamma(\theta)] \right|^2 d\theta = 0 .$$

$$\lim \frac{zp_{n-1}(d\alpha,z)}{p_n(d\alpha,z)}$$

exists for $z \notin \operatorname{supp}(d\alpha)$ and equals to

$$\frac{z}{\pi b}\int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{z-t}\,dt\ .$$

It can directly be calculated that the latter expression equals $z\rho(\frac{z-a}{b})^{-1}$ but it is easier if we remark that $zp_{n-1}(v_\tau,z)/p_n(v_\tau,z)$ converges to the same limit when v_τ is the Chebyshev weight corresponding to $\tau = [a-b,a+b]$.

Let us remark that Theorem 13 could have been deduced from H. Poincare's famous result on linear difference equations [17].

Using Theorems 7, 11 and 13 we can prove limit relations for $\lambda_{n+1}(d\alpha,z)\,|p_n^2(d\alpha,z)|$ and $\lambda_{n+1}^*(d\alpha,z)\,p_n^2(d\alpha,z)$ as well. We will not go into details, we only formulate one result which we will use later.

Theorem 14. Let $\alpha \in M(a,b)$ with $b > 0$ and let $z \in C \setminus \operatorname{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \lambda_{n+1}^*(d\alpha,z)\,p_n^2(d\alpha,z) = 1 - \rho(\frac{z-a}{b})^{-2}\ .$$

Lemma 15. Let $\alpha \in M(a,b)$ and $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \frac{p_n'(d\alpha,z)}{n\,p_n(d\alpha,z)} = \begin{cases} \dfrac{1}{z-a} & \text{for } b = 0 \\[2ex] \dfrac{1}{\sqrt{(z-a)^2-b^2}} & \text{for } b > 0\ . \end{cases}$$

where $\sqrt{z^2-1} > 0$ for $z > 1$.

Proof. We have for $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$

$$\frac{p_n'(d\alpha,z)}{p_n(d\alpha,z)} = \sum_{k=1}^{n} \frac{1}{z - x_{kn}(d\alpha)}\ .$$

Let $\varepsilon > 0$ and

$$f(t) = \begin{cases} 0 & \text{for } |z - t| < \varepsilon \\ \dfrac{1}{z - t} & \text{for } |z - t| \geq \varepsilon . \end{cases}$$

By Theorem 3.3.8 if $\varepsilon = \varepsilon(z)$ is small enough and $n \geq N(z)$ then

$$\frac{p_n'(d\alpha, z)}{n\, p_n(d\alpha, z)} = \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) .$$

Using Theorems 5.2 and 5.3 we obtain

$$\lim_{n\to\infty} \frac{p_n'(d\alpha, z)}{n\, p_n(d\alpha, z)} = \begin{cases} f(a) & \text{for } b = 0 \\ \dfrac{1}{\pi} \displaystyle\int_{a-b}^{a+b} \frac{f(t)}{\sqrt{b^2 - (t-a)^2}}\, dt & \text{for } b > 0 . \end{cases}$$

and $f(a) = (z - a)^{-1}$ for $b = 0$ and $f(t) = (z - t)^{-1}$ for $b > 0$, $t \in [a - b, a + b]$. The calculation of the above integral is simple: put α = Chebyshev weight corresponding to $[a - b, a + b]$.

Theorem 16. Let $\alpha \in M(a, b)$ and $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} n\left[\frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} - \frac{p_{n-1}'(d\alpha, z)}{p_n'(d\alpha, z)}\right] = \begin{cases} 0 & \text{for } b = 0 \\ \rho\left(\dfrac{z-a}{b}\right) & \text{for } b > 0 . \end{cases}$$

Proof. From

$$p_{n-1}(d\alpha, z) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_n(d\alpha, z) \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \frac{p_{n-1}^2(d\alpha, x_{kn})}{z - x_{kn}}$$

follows

$$n\left[\frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} - \frac{p_{n-1}'(d\alpha, z)}{p_n'(d\alpha, z)}\right] = n\, \frac{p_n(d\alpha, z)}{p_n'(d\alpha, z)} \cdot \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \frac{p_{n-1}^2(d\alpha, x_{kn})}{(z - x_{kn})^2} .$$

If $b = 0$ then use Theorem 3.3.8 and Lemma 15. If $b > 0$ then use Theorems 3.2.3, 3.3.8 and Lemma 15. For $b > 0$ we get

$$\lim_{n\to\infty} n\left[\frac{p_{n-1}(d\alpha,z)}{p_n(d\alpha,z)} - \frac{p'_{n-1}(d\alpha,z)}{p'_n(d\alpha,z)}\right] = \sqrt{\left(\frac{z-a}{b}\right)^2 - 1} \cdot \frac{1}{\pi}\int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{(z-t)^2}\,dt$$

and this integral has been calculated in the course of proof of Theorem 11.

From Theorems 13 and 16 we obtain

Theorem 17. Let $\alpha \in M(a,b)$ and $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \frac{p'_{n-1}(d\alpha,z)}{p'_n(d\alpha,z)} = \begin{cases} 0 & \text{for } b = 0 \\ \rho\left(\frac{z-a}{b}\right)^{-1} & \text{for } b > 0 . \end{cases}$$

The following result is rather surprising if we compare it with Theorem 13.

Theorem 18. Let $\alpha \in M(a,b)$ with $b > 0$. Then for every $x \in \operatorname{supp}(d\alpha) \setminus [a-b, a+b]$

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha,x)}{p_n(d\alpha,x)} = \rho\left(\frac{x-a}{b}\right) .$$

Proof. We have

$$p_{n-1}(d\alpha,x) = \int_{-\infty}^{\infty} p_{n-1}(d\alpha,t)\, K_n(d\alpha,x,t)\, d\alpha(t) .$$

If $x \in \operatorname{supp}(d\alpha) \setminus [a-b,a+b]$ then by Theorem 3.3.7, x is an isolated point of $\operatorname{supp}(d\alpha)$. Hence, we can find $\epsilon > 0$ such that

$$p_{n-1}(d\alpha,x) = \int_{|x-t|>\epsilon} p_{n-1}(d\alpha,t)\, K_n(d\alpha,x,t)\, d\alpha(t) + \frac{\alpha(x+0)-\alpha(x-0)}{\lambda_n(d\alpha,x)}\, p_{n-1}(d\alpha,x).$$

Using

$$K_n(d\alpha,x,t) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha,t)\, p_n(d\alpha,x) - p_n(d\alpha,t)\, p_{n-1}(d\alpha,x)}{x-t}$$

we obtain

$$p_{n-1}(d\alpha,x)\left[1 - \frac{\alpha(x+0)-\alpha(x-0)}{\lambda_n(d\alpha,x)} + \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \cdot \int_{|x-t|>\epsilon} \frac{p_{n-1}(d\alpha,t)p_n(d\alpha,t)}{x-t}\, d\alpha(t)\right] =$$

$$= p_n(d\alpha,x)\, \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \int_{|x-t|>\epsilon} \frac{p^2_{n-1}(d\alpha,t)}{x-t}\, d\alpha(t) .$$

We have

$$\lim_{n\to\infty} \frac{\alpha(x+0) - \alpha(x-0)}{\lambda_n(d\alpha,x)} = 1 .$$

(See Freud, Section II.2, $\operatorname{supp}(d\alpha)$ is compact!). Thus by Theorem 4.2.13 , $p_n(d\alpha,x) \neq 0$ for n large and

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha,x)}{p_n(d\alpha,x)} = \frac{\frac{1}{\pi}\int_{a-b}^{a+b} (x-t)^{-1}\, [b^2 - (t-a)^2]^{-1/2}\, dt}{\frac{1}{\pi}\int_{a-b}^{a+b} t(x-t)^{-1}\, [b^2 - (t-a)^2]^{-1/2}\, dt}$$

which equals $\rho(\frac{x-a}{b})$.

Theorem 19. Let $\alpha \in M(0,1)$ and ℓ be a fixed nonnegative integer. Then

$$\lim_{n\to\infty} \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, p_{k+\ell}(d\alpha,x) = T_\ell(x) \tag{11}$$

for each $x \in [-1,1]$ provided that α is continuous at x ; in particular, (11) holsd for almost every $x \in \operatorname{supp}(d\alpha)$. If α is continuous on $\tau \subset (-1,1)$ then (11) is satisfied uniformly for $x \in \tau$.

Proof. Recall that

$$\lim_{n\to\infty} \lambda_n(d\alpha,x) = 0$$

at every x where α is continuous and the convergence is uniform on every interval of continuity of α since $\operatorname{supp}(d\alpha)$ is compact. (See Freud, Section II.3.) If $\ell = 1$, then the theorem follows from Theorem 11 and from the formula

$$x - \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, p_{k+1}(d\alpha,x) = \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} \alpha_k(d\alpha)\, p_k^2(d\alpha,x) +$$

$$+ \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} \left[2\, \frac{\gamma_n(d\alpha)}{\gamma_{k+1}(d\alpha)} - 1\right] p_k(d\alpha,x)\, p_{k+1}(d\alpha,x) -$$

$$- \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_{n-1}(d\alpha,x)\, p_n(d\alpha,x)\, \lambda_n(d\alpha,x)$$

which is a direct consequence of the recurrence formula. Now let $\ell > 1$. Then by Theorem 3.1.1

$$\lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, p_{k+\ell}(d\alpha,x) = U_{\ell-1}(x)\, \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, p_{k+1}(d\alpha,x) -$$

$$- U_{\ell-2}(x) + \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, R_{k+\ell,k+1}(d\alpha,x) .$$

Since $U_{\ell-1}(x)x - U_{\ell-2}(x) = T_\ell(x)$ we obtain that (11) holds at those points x where it holds with $\ell = 1$ and where

$$\lim_{n\to\infty} \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, R_{k+\ell,k+1}(d\alpha,x) = 0$$

is also satisfied. To finish the proof, we apply 3.1.(3). We have

$$|R_{k+\ell,k+1}(d\alpha,x)| \leq \epsilon_k(x)\ [\,|p_k(d\alpha,x)| + |p_{k+1}(d\alpha,x)|\,]$$

for ℓ fixed where

$$\lim_{k\to\infty} \epsilon_k(x) = 0$$

uniformly for $x \in [-1,1]$. Thus

$$\left|\sum_{k=0}^{n-1} p_k(d\alpha,x)\, R_{k+\ell,k+1}(d\alpha,x)\right| \leq \sum_{k=0}^{n-1} \epsilon_k^*(x)\, p_k^2(d\alpha,x)$$

with

$$\lim_{n\to\infty} \epsilon_k^*(x) = 0$$

uniformly for $x \in [-1,1]$. Consequently,

$$\lim_{n\to\infty} \lambda_n(d\alpha,x) \sum_{k=0}^{n-1} p_k(d\alpha,x)\, R_{k+\ell,k+1}(d\alpha,x) = 0$$

holds for every $x \in [-1,1]$ for which $\lim_{n\to\infty} \lambda_n(d\alpha,x) = 0$ and the congergence is uniform for $x \in \Delta \subset (-1,1)$ whenever $\lim_{n\to\infty} \lambda_n(d\alpha,x) = 0$ is true uniformly for $x \in \Delta$.

4.2. Weak Limits

Definition 1. We write $\alpha \in S$ if $\operatorname{supp}(d\alpha) = [-1,1]$ and $v \log \alpha' \in L^1(-1,1)$.

Lemma 2. If $\alpha \in S$ then

$$\lim_{n\to\infty} \gamma_n(d\alpha)2^{-n} = \frac{1}{\sqrt{\pi}} \exp\{-\frac{1}{2\pi}\int_{-1}^{1} v(t) \log \alpha'(t)\, dt\}.$$

Proof. See e.g., Freud, §V.6.

Theorem 3. Let $\alpha \in S$ and f be Riemann integrable on $[-1,1]$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, \frac{p_{n-1}^2(d\alpha,x_{kn})}{1 - x_{kn}^2} = \frac{2}{\pi}\int_{-1}^{1} f(t)\, \frac{dt}{\sqrt{1-t^2}} \;.$$

Proof. Let β be defined by $d\beta(x) = (1 - x^2)\, d\alpha(x)$. Then $(1 - x^2) \cdot$
$\cdot\, [p_{n-1}^2(d\beta,x) - \gamma_{n-1}^2(d\beta)x^{2n-2}]$ is a polynomial of degree $2n-1$ and we have by the Gauss-Jacobi mechanical quadrature formula

$$\sum_{k=1}^{n} (1 - x_k^2)[p_{n-1}^2(d\beta,x_k) - \gamma_{n-1}^2(d\beta)x_k^{2n-2}]\lambda_k = 1 + \gamma_{n-1}^2(d\beta)\int_{-1}^{1}(t^2 - 1)\, t^{2n-2}\, d\alpha(t)$$

(here $x_k = x_{kn}(d\alpha)$ and $\lambda_k = \lambda_{kn}(d\alpha)$. Thus

$$\sum_{k=1}^{n} (1 - x_k^2)\, p_{n-1}^2(d\beta,x_k)\, \lambda_k = 1 + \gamma_{n-1}^2(d\beta)\, [\int_{-1}^{1} t^{2n}\, d\alpha(t) - \sum_{k=1}^{n} x_k^{2n}\, \lambda_k] \,.$$

Further

$$\sum_{k=1}^{n} x_k^{2n}\, \lambda_k = \sum_{k=1}^{n} L_n(d\alpha,y^{2n},x_k)\, \lambda_k = \int_{-1}^{1} L_n(d\alpha,y^{2n},t)\, d\alpha(t) \,.$$

Hence we obtain

$$\sum_{k=1}^{n} (1 - x_k^2)\, p_{n-1}^2(d\beta,x_k)\, \lambda_k = 1 + \gamma_{n-1}^2(d\beta)\int_{-1}^{1} [t^{2n} - L_n(d\alpha,y^{2n},t)]\, d\alpha(t) \,.$$

Since $t^{2n} - L_n(d\alpha,y^{2n},t)$ is a polynomial of degree $2n$ which vanishes at the zeros of $p_n(d\alpha,x)$ we have

Proof. See Geronimus, Chapter IX.

In the following, we shall apply Lemma 5 several times, here we give three applications of it.

Theorem 6. If $\alpha \in S$ then

$$\lim_{n\to\infty} \int_{-1}^{1} p_n^2(d\alpha,x)\ d[\alpha_s(x) + \alpha_j(x)] = 0 .$$

Proof. By the Riemann-Lebesgue lemma and Lemma 5 $\int_{-1}^{1} p_n^2(d\alpha,x)\alpha'_{ac}(x)dx \to 1$ when $n \to \infty$.

Theorem 7. Let $\alpha \in S$ and $f \in L^{\infty}_{d\alpha}$. Then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\ p_n^2(d\alpha,x)\ d\alpha(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\ dx .$$

Proof. Use Lemma 5, Theorem 6 and the Riemann-Lebesgue lemma.

Theorem 8. Let $\alpha \in S$, $0 < p < \infty$, $g(\geq 0) \in L^1_{d\alpha}$. If

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(d\alpha,t)|^p\ g(t)\ d\alpha(t) = 0$$

then $g(t) = 0$ for almost every $t \in [-1,1]$.

Proof. Let first $2 \leq p < \infty$. Put for $M > 0$

$$g_M(t) = \min\{g(t),M\} .$$

Then $g_M \in L^{\infty}_{d\alpha}$. Further

$$\int_{-1}^{1} p_n^2(d\alpha,t)\ g_M(t)^{2/p}\ d\alpha(t) \leq [\int_{-1}^{1} |p_n(d\alpha,t)|^p\ g(t)d\alpha(t)]^{2/p} \cdot [\alpha(1) - \alpha(-1)]^{\frac{p-2}{p}}.$$

By the hypothesis and Theorem 7

$$\frac{1}{\pi}\int_{-1}^{1} g_M(t)^{2/p}\ \frac{dt}{\sqrt{1-t^2}} = 0$$

for every $M > 0$. Hence $g = 0$. Let now $1 \leq p < 2$. Let $g^*(t) = g(\cos t)$, $\varepsilon > 0$. Then

$$\int_{-1}^{1} |p_n(d\alpha,t)|^p \, g(t) \, d\alpha(t) \geq \int_{-1}^{1} |p_n(d\alpha,t)|^p \, g(t) \, \alpha'(t) \, dt =$$

$$= \int_0^{\pi} |p_n(d\alpha,\cos t) \sqrt{\alpha'(\cos t)\sin t}\,|^p \, [\alpha'(\cos t)\sin t]^{1-\frac{p}{2}} \, g^*(t) \, dt \, .$$

Let $g_1(t) = [\alpha'(\cos t)\sin t]^{1-\frac{p}{2}} g^*(t)$. Then

$$[\int_{-1}^{1} |p_n(d\alpha,t)|^p \, g(t) d\alpha(t)]^{1/p} \geq [\epsilon \int_{g_1 \geq \epsilon} |p_n(d\alpha,\cos t) \sqrt{\alpha'(\cos t)\sin t}\,|^p \, dt]^{1/p} \geq$$

$$\geq \epsilon^{1/p} [\int_{g_1 \geq \epsilon} |\sqrt{\tfrac{2}{\pi}} \cos(nt - \Gamma(t))|^p]^{1/p} -$$

$$- \epsilon^{1/p} [\int_0^{\pi} |p_n(d\alpha,\cos t) \sqrt{\alpha'(\cos t)\sin t} - \sqrt{\tfrac{2}{\pi}} \cos(nt - \Gamma(t))|^p \, dt]^{1/p} \, .$$

Since $p < 2$, $|\cos|^p \geq |\cos|^2$ and the second integral here converges to 0 by Lemma 5. Thus by the hypothesis

$$\liminf_{n\to\infty} \int_{g_1 \geq \epsilon} \cos^2(nt - \Gamma(t)) dt = 0 \, ,$$

that is $\text{meas}(g_1 \geq \epsilon) = 0$. Thus $g_1 = 0$ and consequently $g = 0$. If $0 < p < 1$ we can repeat the previous arguments, the only difference is that we consider $\int |\,|^p$ instead of $[\int |\,|^p]^{1/p}$.

<u>Lemma 9</u>. Let $\alpha \in M(a,0)$. Let $\{n_k\}$ and $\{m_k\}$ be two sequences of natural integers such that at least one of them converges to ∞ when $k \to \infty$. If f is continuous on $\Delta(d\alpha)$ then

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(t) \, p_{n_k}(d\alpha,t) \, p_{m_k}(d\alpha,t) \, d\alpha(t) = \begin{cases} f(a) & \text{if } \lim\limits_{k\to\infty} (n_k - m_k) = 0 \\ 0 & \text{if } \liminf\limits_{k\to\infty} |n_k - m_k| > 0. \end{cases}$$

<u>Proof</u>. In the first case we can suppose without loss of generality that $n_k = m_k = k$ for every k . Because of continuity we can also suppose that f is a polynomial. If f is constant, the Lemma is certainly true. Otherwise

$$f(x) = f(a) + \sum_{j=1}^{\deg f} \frac{f^{(j)}(a)}{j!} (x - a)^j .$$

We shall show that for every $j \geq 1$

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} (x - a)^j \, p_k^2(d\alpha,x) \, d\alpha(x) = 0 . \tag{2}$$

If $j = 1$, then (2) means that $\lim_{k\to\infty} \alpha_k(d\alpha) = a$. If $j = 2$ then

$$(x - a)^2 \, p_k^2(d\alpha,x) = (x - a) \, p_k(d\alpha,x) \, [\frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)} p_{k+1}(d\alpha,x) +$$
$$+ (\alpha_k(d\alpha) - a) \, p_k(d\alpha,x) + \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} p_{k-1}(d\alpha,x)] .$$

Hence,

$$\int_{-\infty}^{\infty} (x - a)^2 \, p_k^2(d\alpha,x) \, d\alpha(x) = \frac{\gamma_k^2}{\gamma_{k+1}^2} + (\alpha_k - a)^2 + \frac{\gamma_{k-1}^2}{\gamma_k^2} \xrightarrow[k\to\infty]{} 0 .$$

Since $\operatorname{supp}(d\alpha)$ is compact, (2) holds also for $j > 2$ if it holds for $j = 2$. The second case can be obtained from the first one as follows. If k is large, then $m_k \neq n_k$. Thus

$$\int_{-\infty}^{\infty} f(t) \, p_{n_k}(d\alpha,t) \, p_{m_k}(d\alpha,t) \, d\alpha(t) = \int_{-\infty}^{\infty} [f(t) - f(a)] \, p_{m_k}(d\alpha,t) \, p_{n_k}(d\alpha,t) d\alpha(t),$$

that is the absolute value of the left side is not greater than

$$\{\int_{-\infty}^{\infty} |f(t) - f(a)| \, p_{n_k}^2(d\alpha,t) \, d\alpha(t) \cdot \int_{-\infty}^{\infty} |f(t) - f(a)| \, p_{m_k}^2(d\alpha,t) \, d\alpha(t)\}^{1/2} .$$

Here both factors are bounded and at least one of them tends to 0 when $k\to\infty$.

__Theorem 10.__ Lemma 9 remains true if f, instead of being continuous on $\Delta(d\alpha)$, is merely bounded on $\operatorname{supp}(d\alpha)$, continuous at a and it is $d\alpha$ measurable.

__Proof.__ Let $\epsilon > 0$. Then

$$\int_{-\infty}^{\infty} g(t) \, p_k^2(d\alpha,t) \, d\alpha(t) \leq \int_{a-\epsilon}^{a+\epsilon} p_k^2(d\alpha,t) \, d\alpha(t) \leq 1$$

where g is continuous function vanishing outside $[a - \epsilon, a + \epsilon]$ with $g(a) = 1$ and $0 \leq g(t) \leq 1$ for $|a - t| \leq \epsilon$. Thus by Lemma 9

$$\lim_{k\to\infty} \int_{a-\epsilon}^{a+\epsilon} p_k^2(d\alpha,t)\, d\alpha(t) = 1 .$$

We have

$$\left|\int_{-\infty}^{\infty} f(t)\, p_k^2(d\alpha,t)\, d\alpha(t) - f(a)\right| \le \sup_{|t-a|\le\epsilon} |f(t) - f(a)| \cdot$$

$$\cdot \int_{a-\epsilon}^{a+\epsilon} p_k^2(d\alpha,t)\, d\alpha(t) + 2 \sup_{t\in \operatorname{supp}(d\alpha)} |f(t)|\, [1 - \int_{a-\epsilon}^{a+\epsilon} p_k^2(d\alpha,t)\, d\alpha(t)] .$$

Letting first $k\to\infty$ and then $\epsilon\to 0$ we obtain

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(t)\, p_k^2(d\alpha,t)\, d\alpha(t) = f(a) .$$

the case when $\liminf_{k\to\infty} |m_k - n_k| > 0$ follows from the case when $\lim_{k\to\infty} (m_k - n_k) = 0$.

<u>Definition 11</u>. Let us define the numbers $\alpha_{nk}(d\alpha)$ for $n = 1,2,\ldots$ and $k = n-1,n,n+1$ as follows

$$\alpha_{nk}(d\alpha) = \begin{cases} \gamma_{n-1}(d\alpha)/\gamma_n(d\alpha) & \text{for } k = n-1 \\ \alpha_n(d\alpha) & \text{for } k = n \\ \gamma_n(d\alpha)/\gamma_{n+1}(d\alpha) & \text{for } k = n+1 . \end{cases}$$

<u>Lemma 12</u>. Let m be a nonnegative integer and $n > m$. Then

$$x^m p_n(d\alpha,x) = \sum_{\substack{-1\le k_i\le 1\\ i=1,2,\ldots,m}} \alpha_{n,n+k_1}(d\alpha)\, \alpha_{n+k_1,n+k_1+k_2}(d\alpha)$$

$$\cdot \ldots \alpha_{n+k_1+\ldots+k_{m-1},n+k_1+\ldots+k_m}(d\alpha)\, p_{n+k_1+\ldots+k_m}(d\alpha,x) .$$

<u>Proof</u>. Apply the recursion formula repeatedly.

<u>Theorem 13</u>. Let $\alpha \in M(a,b)$ with $b > 0$. Let $\{m_k\}$ and $\{n_k\}$ be two sequences of natural integers such that at least one of them converges to ∞ when $k\to\infty$ and the finite or infinite $\lim_{k\to\infty} (m_k - n_k)$ exists. Let f be $d\alpha$ measurable, bounded on $\operatorname{supp}(d\alpha)$ and continuous on $[a-b,a+b]$. Then

$$(3)\quad \lim_{k\to\infty} \int_{-\infty}^{\infty} f(t)\, p_{m_k}(d\alpha,t)\, p_{n_k}(d\alpha,t)\, d\alpha(t) = \lim_{k\to\infty} \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{T_{|m_k-n_k|}\left(\frac{t-a}{b}\right)}{\sqrt{b^2 - (t-a)^2}} dt.$$

Proof. Let, without loss of generality, $\alpha \in M(0,1)$. First we shall prove (3) when f is continuous on $\Delta(d\alpha)$. If $\lim_{k\to\infty} (m_k - n_k) = \infty$ then the right side in (3) equals 0 . If f is a polynomial then the integral on the left side of (3) equals 0 if k is big. Thus for every continuous function f the left side in (3) also equals 0 . Now let $\lim_{k\to\infty} (m_k - n_k) < \infty$. Then we may assume that for every k, $n_k = k$, $m_k = k+\ell$ where ℓ is a fixed nonnegative integer. Because of linearity and continuity arguments; that is, because of Banach-Steinhaus' theorem, we can suppose that f is of the form $f(t) = t^m$ where m is a fixed nonnegative integer. Thus we have to show that for $m = 0,1,2,\ldots$

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} t^m\, p_k(d\alpha,t)\, p_{k+\ell}(d\alpha,t)\, d\alpha(t) = \frac{1}{\pi}\int_{-1}^{1} t^m \frac{T_\ell(t)}{\sqrt{1-t^2}}\, dt\ . \tag{4}$$

Let us remark that (4) is true if α is the Chebyshev weight. For if $k \geq 1$

$$\int_{-\infty}^{\infty} t^m\, p_k(d\alpha,t)\, p_{k+\ell}(d\alpha,t)\, d\alpha(t) = \frac{2}{\pi}\int_0^{\pi} \cos^m\theta \cdot \frac{1}{2}[\cos \ell\theta + \cos(2k+\ell)\theta]d\theta$$

which equals to

$$\frac{1}{\pi}\int_{-1}^{1} t^m \frac{T_\ell(t)}{\sqrt{1-t^2}}\, dt \tag{5}$$

if $2k + \ell > m$. If $\alpha \in M(0,1)$ then by Lemma 12 we have for $k > m$

$$\int_{-\infty}^{\infty} t^m\, p_k(d\alpha,t)\, p_{k+\ell}(d\alpha,t)\, d\alpha(t) =$$

$$= \sum_{\substack{-1\leq k_i \leq 1\\ i=1,2,\ldots,m\\ \sum_{i=1}^m k_i = \ell}} \alpha_{k,k+k_1}(d\alpha)\, \alpha_{k+k_1,k+k_1+k_2}(d\alpha) \cdots \alpha_{k+k_1+\ldots+k_{m-1},k+\ell}(d\alpha)\ .$$

The right side here is convergent when $k \to \infty$ since $\alpha \in M(0,1)$ and ℓ is fixed, its limit depends only on $\lim_{j\to\infty} \alpha_j(d\alpha)$ and $\lim_{j\to\infty} \gamma_{j-1}(d\alpha)/\gamma_j(d\alpha)$, that is in our case this limit equals (5). Hence (4) holds if f is continuous on $\Delta(d\alpha)$. If f is continuous only on $\operatorname{supp}(d\alpha)$ which is closed then f can be extended to a function which is continuous on $\Delta(d\alpha)$. If f is a function satisfying the conditions of the theorem then we can write $f = f_1 + f_2$ where f_1 is continuous on $\Delta(d\alpha)$, f_2 is bounded and $d\alpha$ measurable, further f_2

vanishes on $[a-b, a+b]$. Hence, if we can show that

$$(6) \qquad \lim_{k\to\infty} \int_{a\pm b}^{\pm\infty} p_k^2(d\alpha, t)\, d\alpha(t) = 0$$

then we finish the proof of the theorem. Let g be continuous function on $\Delta(d\alpha)$ such that $g(t) \geq 0$ for $t \in \Delta(d\alpha)$, $g(t) = 1$ for $t \in \Delta(d\alpha)\setminus[a-b, a+b]$ and

$$\frac{1}{\pi}\int_{a-b}^{a+b} g(t) \frac{1}{\sqrt{b^2 - (t-a)^2}}\, dt < \epsilon$$

where $\epsilon > 0$ is given. Then

$$\int_{-\infty}^{a-b} + \int_{a+b}^{\infty} p_k^2(d\alpha, t)\, d\alpha(t) \leq \int_{-\infty}^{\infty} g(t)\, p_k^2(d\alpha, t)\, d\alpha(t)$$

and using the fact that the theorem has already been proved for continuous functions we obtain (6) by first letting $k \to \infty$ and then $\epsilon \to 0$.

Using one-sided approximation machinery we obtain immediately from Theorem 13 the following

Theorem 14. Let $\alpha \in M(a,b)$ with $b > 0$. If f is $d\alpha$ measurable, bounded on $\mathrm{supp}(d\alpha)$ and Riemann integrable on $[a-b, a+b]$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, p_n^2(d\alpha, t)\, d\alpha(t) = \frac{1}{\pi}\int_{a-b}^{a+b} f(t) \frac{dt}{\sqrt{b^2 - (t-a)^2}} .$$

Compare this theorem with Theorem 7.

Corollary 15. Let $\alpha \in M(a,b)$, $b > 0$, $x \in (a-b, a+b)$ and let α be continuous in a neighborhood of x . Then

$$\lim_{h\to 0}\lim_{n\to\infty} \frac{1}{2h}\int_{x-h}^{x+h} p_n^2(d\alpha, t)\, d\alpha(t) = \frac{1}{\pi} \frac{1}{\sqrt{b^2 - (x-a)^2}} .$$

Proof. The function $1_{(x-\epsilon, x+\epsilon)}$ is $d\alpha$ measurable for $\epsilon > 0$ small.

Theorem 16. Let $\alpha \in M(a,b)$, $b > 0$. Let f be bounded on $\Delta(d\alpha)$ and Riemann integrable on $[a-b, a+b]$. Then for every fixed integer ℓ

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p_{n-1}(d\alpha, x_{kn})\, p_{n+\ell}(\alpha, x_{kn}) =$$

$$= -\operatorname{sign} \ell \, \frac{2}{\pi b^2} \int_{a-b}^{a+b} f(t)\, U_{|\ell|-1}\left(\frac{t-a}{b}\right) \sqrt{b^2 - (t-a)^2}\, dt \, .$$

Proof. We obtain from Theorems 3.1.3 ($\ell > 0$) and 3.1.13 ($\ell < 0$) and from the recurrence formula that

$$p_{n+\ell}(d\alpha, x_{kn}) = -\operatorname{sign} \ell \; U_{|\ell|-1}\left(\frac{x_{kn} - a}{b}\right) p_{n-1}(d\alpha, x_{kn}) +$$

$$+ \; o(1) \; [\,|p_{n-1}(d\alpha, x_{kn})| + |p_{n-2}(d\alpha, x_{kn})|\,]$$

where $\lim_{n\to\infty} o(1) = 0$ uniformly for $1 \le k \le n$ if ℓ is fixed. Thus the theorem follows from Theorem 3.2.3 .

Theorem 17. Let $\alpha \in S$, f be Riemann integrable on $[-1,1]$ and ℓ be a fixed integer. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p_{n-1}(d\alpha, x_{kn}) \frac{p_{n+\ell}(d\alpha, x_{kn})}{1 - x_{kn}^2} =$$

$$= -\operatorname{sign} \ell \, \frac{1}{\pi} \int_{-1}^{1} f(t)\, U_{|\ell|-1}(t) \, \frac{dt}{\sqrt{1 - t^2}} \, .$$

Proof. Repeat the proof of Theorem 16 and use Theorem 3 instead of Theorem 3.2.3.

5. Eigenvalues of Toepliz Matrices

In Grenander-Szegö the proof of Theorem 7.7 and the example 8.1(f) are not correct. In the first the Gauss-Jacobi mechanical quadrature formula is used for polynomials of degree more than $2n-1$, in the second, an orthonormal system is constructed, but it is, in fact, only normed but not orthogonal. In this section, we will obtain results which are a little bit more general than those of Grenander-Szegö, we will use some methods of the above book in a simplified form.

<u>Lemma 1</u>. Let $\operatorname{supp}(d\alpha)$ be compact and let f be continuous on $\Delta(d\alpha)$ with the modulus of continuity ω. Then

$$\left|\sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \sum_{k=0}^{n-1}\int_{-\infty}^{\infty} f(t)\, p_k^2(d\alpha,t)\, d\alpha(t)\right| \le n\, \omega(n^{-1/3})\, [1+\tfrac{1}{2}|\Delta(d\alpha)|^3]$$

for $n > |\Delta(d\alpha)|^{-3}$.

<u>Proof</u>. Since $\sum_{k=0}^{n-1} p_k^2(t) = \sum_{k=1}^{n} \frac{\ell_k^2(t)}{\lambda_k}$ and $\lambda_k = \int_{-\infty}^{\infty} \ell_k^2(t)\, d\alpha(t)$ we have

$$\sum_{k=1}^{n} f(x_{kn}) - \sum_{k=0}^{n-1}\int_{-\infty}^{\infty} f(t)\, p_k^2(t)\, d\alpha(t) = \sum_{k=1}^{n}\int_{\Delta(d\alpha)} [f(x_{kn})-f(t)]\, \frac{\ell_{kn}^2(d\alpha,t)}{\lambda_{kn}(d\alpha)}\, d\alpha(t) \equiv A.$$

Let $\epsilon > 0$. Then

$$|A| \le n\,\omega(\epsilon) + \sum_{k=1}^{n}\int_{|t-x_k|\ge\epsilon} |f(x_k)-f(t)|\, \frac{\ell_k^2(t)}{\lambda_k}\, d\alpha(t) \le$$

$$\le n\,\omega(\epsilon) + \frac{\gamma_{n-1}^2}{\gamma_n^2}\, \frac{\omega(|\Delta(d\alpha)|)}{\epsilon^2} \sum_{k=1}^{n} \lambda_k\, p_{n-1}^2(x_k) \int_{-\infty}^{\infty} p_n^2(t)\, d\alpha(t) =$$

$$= n\,[\omega(\epsilon) + \frac{\gamma_{n-1}^2}{\gamma_n^2}\, \frac{\omega(|\Delta(d\alpha)|)}{n\,\epsilon^2}]\,.$$

By easy calculation, $\gamma_{n-1}/\gamma_n \le |\Delta(d\alpha)|/2$. Thus

$$|A| \le n\,[\omega(\epsilon) + \tfrac{1}{4}\,|\Delta(d\alpha)|^2\, \frac{\omega(|\Delta(d\alpha)|)}{n\,\epsilon^2}].$$

Now let $\epsilon = n^{-1/3}$ and use the inequality

$$\frac{\omega(a)}{a} \leq 2\,\frac{\omega(b)}{b} \qquad (0 < b \leq a)$$

which holds for every modulus of continuity ω.

Theorem 2. Let $\alpha \in M(a,0)$. Let f be bounded on $\Delta(d\alpha)$ and continuous at a. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) = f(a) .$$

Proof. If f is continuous on $\Delta(d\alpha)$ then the theorem follows immediately from Lemma 1 and Lemma 4.2.9. If f is continuous only at a and it is bounded on $\Delta(d\alpha)$ then we fix $\epsilon > 0$ and we construct two continuous functions f_1 and f_2 on $\Delta(d\alpha)$ so that

$$f_1(x) \leq f(x) \leq f_2(x)$$

for $x \in \Delta(d\alpha)$ and $f_2(a) - f_1(a) \leq \epsilon$ and we use the fact that for f_1 and f_2 the theorem has already been proved.

Theorem 3. Let $\alpha \in M(a,b)$ with $b > 0$. Let f be bounded on $\Delta(d\alpha)$ and Riemann integrable on $[a-b,a+b]$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) = \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{1}{\sqrt{b^2 - (t-a)^2}}\, dt ,$$

in particular, for every segment $\Delta \subset \Delta(d\alpha)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x_{kn}(d\alpha) \in \Delta} 1 = \frac{1}{\pi} \int_{\Delta\cap[a-b,a+b]} \frac{1}{\sqrt{b^2 - (t-a)^2}}\, dt .$$

Proof. If f is continuous on $\Delta(d\alpha)$ then use Lemma 1 and Theorem 4.2.13, otherwise apply the one-sided approximation machinery.

Now we will translate the previous results into a different languate. Let $\operatorname{supp}(d\alpha)$ be compact, $f \in L^1_{d\alpha}$ be real valued and let us consider the Toeplitz matrix $A(f,d\alpha)$ defined as

$$A(f,d\alpha) = [[\int_{-\infty}^{\infty} f(t)\, p_i(d\alpha,t)\, p_j(d\alpha,t)\, d\alpha(t)]]_{i,j=0}^{\infty} .$$

Let, further, $A_n(f,d\alpha)$ be the truncated matrix consisting of n^2 elements. The characteristic polynomial $h_n(f,d\alpha,x)$ is $\det[A_n(f,d\alpha)-xE]$, the zeros of $h_n(f,d\alpha,x)$, which we denote by $x_{kn}(f,d\alpha)$, $(k = 1,2,\dots,n)$, are called the eignevalues of $A_n(f,d\alpha)$. Since $A_n^* = A_n$ all x_{kn} are real.

If $f(t) \equiv 1$ then $A(f,d\alpha) = E$ and $h_n(f,d\alpha,x) = (1-x)^n$, that is $x_{kn}(f,d\alpha) = 1$ for $k = 1,2,\dots,n$.

<u>Lemma 4</u>. Let $f(t) \equiv t$. Then for $n = 1,2,\dots$

$$h_n(f,d\alpha,x) = (-1)^n \gamma_n^{-1}(d\alpha)\, p_n(d\alpha,x) .$$

<u>Proof</u>. $(-1)^n h_n(f,d\alpha,x)$ satisfy the same recurrence formula as $\gamma_n^{-1}(d\alpha)\, p_n(d\alpha,x)$, and for $n = 1,2$ the lemma can easily be checked.

<u>Definition 5</u>. $\{a_{kn}\}_{k=1}^{n}$ and $\{b_{kn}\}_{k=1}^{n}$, $(n = 1,2,\dots;\ a_{kn} \in \mathbb{R};\ b_{kn} \in \mathbb{R})$ are equally distributed if there exists an interval Δ such that $a_{kn} \in \Delta$ and $b_{kn} \in \Delta$ for $n = 1,2,\dots$ and $k = 1,2,\dots,n$, further for every continuous function f on Δ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [f(a_{kn}) - f(b_{kn})] = 0 .$$

We obtain from Theorems 2 and 3 the following

<u>Theorem 6</u>. Let $f(t) \equiv t$, $a \in \mathbb{R}$, $b \geq 0$. Then for every pair of weights α_1 and α_2 from $M(a,b)$ the eigenvalues of $A_n(f,d\alpha_1)$ and $A_n(f,d\alpha_2)$ are equally distributed.

<u>Definition 7</u>. Let $A = [[a_{ik}]]_{i,k=1}^{n}$ be real $n \times n$ matrix. Then

$$\mathrm{Tr}A = \sum_{k=1}^{n} a_{kk} ,$$

$$\|A\| = \left|\frac{1}{n} T_r A\right| ,$$

$$((A))^2 = \sup_u \frac{(Au,Au)}{(u,u)} ,$$

here $u = (u_1,\dots,u_n)$, $(u,v) = \sum_{k=1}^{n} u_k v_k$, further

$$\sqrt{A} = [[\sqrt{a_{ik}}]]_{i,k=1}^{n} .$$

<u>Properties 8.</u> $\operatorname{Tr} AB = \operatorname{Tr} BA$, $\operatorname{Tr} A = \Sigma$ (eigenvalues of A) ,

$$((A))^2 \geq \max_{k=1,2,\ldots,n} \sum_{j=1}^{n} a_{jk}^2 , \quad ((AB)) \leq ((A))((B)) .$$

If $A^* = A$ then $((A)) = \max |\text{eigenvalues of } A|$.

<u>Lemma 9.</u> For every $n \in \mathbb{N}^+$

$$((A_n(f,d\alpha))) \leq \sup_{t \in \operatorname{supp}(d\alpha)} |f(t)| .$$

<u>Proof.</u> Let λ be an eigenvalue of $A_n(f,d\alpha)$ for which $((A_n f,d\alpha))) = |\lambda|$ and let u be the corresponding eigenvector with $(u,u) = 1$. Then

$$((A_n(f,d\alpha))) = |\lambda|\ (u,u) = |(\lambda u,u)| = |(A_n(f,d\alpha)u,u)| =$$

$$= \left|\int_{-\infty}^{\infty} f(t) \left(\sum_{k=0}^{n-1} p_k(d\alpha,t)u_k\right)^2 d\alpha(t)\right| \leq \sup_{t \in \operatorname{supp}(d\alpha)} |f(t)|\ (u,u)$$

where $u = (u_0,u_1,\ldots,u_{n-1})$.

<u>Lemma 10.</u> Let $m \in \mathbb{N}$ be fixed and let f_i $(i = 1,2,\ldots,m)$ be given. Then for every $j \in \{1,2,\ldots,m\}$

$$\limsup_{n\to\infty} \left\|\prod_{i=1}^{m} A_n(f_i,d\alpha)\right\| \leq \limsup_{n\to\infty} \|\sqrt{A_n(f_j^2,d\alpha)}\| \prod_{\substack{i=1\\ i\neq j}}^{m} \sup_{t \in \operatorname{supp}(d\alpha)} |f_i(t)| .$$

<u>Proof.</u> For $m = 1$ the lemma is certainly true. Let $m \geq 2$. By Properties 8 we can suppose that $j = 1$. Let

$$B = \prod_{i=2}^{m} A_n(f_i,d\alpha) .$$

Then

$$\|A_n(f_1,d\alpha)\ B\| = \left|\frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj}(f_1,d\alpha)\ b_{jk}\right| \leq$$

$$\leq \left|\frac{1}{n} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj}^2(f_1,d\alpha) \sum_{j=1}^{n} b_{jk}^2\right)^{1/2}\right| =$$

$$\leq \max_{k=1,2,\ldots,n} \left(\sum_{j=1}^{n} b_{jk}^2\right)^{1/2} \left|\frac{1}{n} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj}^2(f_1,d\alpha)\right)^{1/2}\right| .$$

By Bessel's inequality

$$\sum_{j=1}^{n} a_{kj}^2(f_1,d\alpha) = \sum_{j=0}^{n-1} [\int_{-\infty}^{\infty} p_{k-1}(d\alpha,t)\; p_j(d\alpha,t)\; f_1(t)\; d\alpha(t)]^2 \le$$

$$\le \int_{-\infty}^{\infty} p_{k-1}^2(d\alpha,t)\; f_1^2(t)\; d\alpha(t) = a_{kk}(f_1^2,d\alpha)\ .$$

Thus by Properties 8

$$\|A_n(f_1,d\alpha)\; B\| \le \|\sqrt{A_n(f_1^2,d\alpha)}\|\; ((B)) \le \|\sqrt{A_n(f_1,d\alpha)}\| \prod_{i=2}^{m} ((A_n(f_i,d\alpha)))\ .$$

Now use Lemma 9 .

Lemma 11. Let $\mathrm{supp}(d\alpha)$ be compact, $s \in \mathbb{N}$ and π be a polynomial. Then

$$\lim_{n\to\infty} \|A_n^s(\pi,d\alpha) - A_n(\pi^s,d\alpha)\| = 0\ .$$

Proof. See Grenander-Szegö, §8.1 .

Let us remark that in the previous lemma it is sufficient to suppose that for $d\alpha$ the moment problem is well defined, that is for $x^k f$, $x^k g \in L^2 d\alpha$ $(k = 0,1,\ldots)$, $A(fg,d\alpha) = A(f,d\alpha)\; A(g,d\alpha)$.

Lemma 12. Let $\alpha \in M(a,b)$, f be $d\alpha$ measurable and bounded on $\mathrm{supp}(d\alpha)$. If $b = 0$ and f is continuous at a then

$$\lim_{n\to\infty} \|\sqrt{A_n(f^2,d\alpha)}\| = |f(a)|\ .$$

If $b > 0$ and f is Riemann integrable on $[a-b,a+b]$ then

$$\lim_{n\to\infty} \|\sqrt{A_n(f^2,d\alpha)}\| = [\frac{1}{\pi}\int_{a-b}^{a+b} f^2(t)\; \frac{dt}{\sqrt{b^2-(ta)^2}}]^{1/2}\ .$$

Proof. See Theorems 4.2.10 and 4.2.14 .

Let us recall that we consider Toeplitz matrices $A(f,d\alpha)$ for real valued functions f .

Theorem 13. Let $\alpha \in M(a,b)$, $s \in \mathbb{N}$, f be $d\alpha$ measurable and bounded on $\mathrm{supp}(d\alpha)$. Let for $b = 0$, f be continuous at a and for $b > 0$, f be Riemann integrable on $[a-b,a+b]$. Then

$$\lim_{n\to\infty} \|A_n^s(f,d\alpha) - A_n(f^s,d\alpha)\| = 0 .$$

Proof. Let π be a polynomial. Since $\operatorname{Tr} AB = \operatorname{Tr} BA$, we have

$$\|A_n^s(f,d\alpha) - A_n(f^s,d\alpha)\| = \|A_n^s(f-\pi+\pi,d\alpha) - A_n((f-\pi+\pi)^s,d\alpha)\| =$$

$$= \|[A_n^s(\pi,d\alpha) - A_n(\pi^s,d\alpha)] + \sum_{j=1}^{s}\binom{s}{j}A_n^j(f-\pi,d\alpha)A_n^{s-j}(\pi,d\alpha) + A_n(-\sum_{j=1}^{s}\binom{s}{j}(f-\pi)^j\pi^{s-j},d\alpha)\| =$$

$$= \|A_I + A_{II} + A_{III}\| .$$

Let, for simplicity, $b > 0$. If $b = 0$, then we shall see from the proof that we can put $\pi(t) \equiv f(a)$. By Lemma 11

$$\lim_{n\to\infty} \|A_I\| = 0 .$$

By Theorem 4.2.14

$$(1)\quad \lim_{n\to\infty}\|A_{III}\| = \left|\frac{1}{\pi}\int_{a-b}^{a+b}\left[\sum_{j=1}^{s}\binom{s}{j}(f(t)-\pi(t))^j\cdot\pi(t)^{s-j}\right]\frac{dt}{\sqrt{b^2-(t-a)^2}}\right| .$$

We have, further, by Lemma 10

$$\limsup_{n\to\infty}\|A_{II}\| \le \sum_{j=1}^{s}\binom{s}{j}\limsup_{n\to\infty}\|\sqrt{A_n((f-\pi)^2,d\alpha)}\|\cdot\sup_{t\in\operatorname{supp}(d\alpha)}|f(t)-\pi(t)|^{j-1}\sup_{t\in\operatorname{supp}(d\alpha)}|\pi(t)|^{s-j} .$$

Hence, by Lemma 12

$$\limsup_{n\to\infty}\|A_{II}\| \le \left[\frac{1}{\pi}\int_{a-b}^{a+b}\frac{[f(t)-\pi(t)]^2}{\sqrt{b^2-(t-a)^2}}dt\right]^{\frac{1}{2}}\cdot 2^s \sup_{t\in\operatorname{supp}(d\alpha)}(|f(t)|+|\pi(t)|)^{s-1} \equiv R(f,\pi) ,$$

and from (1) we get the same estimate for $\lim_{n\to\infty}\|A_{III}\|$:

$$\lim_{n\to\infty}\|A_{III}\| \le R(f,\pi) .$$

The theorem will be proved if we show that for every $\epsilon > 0$ one can find a polynomial π such that $R(f,\pi) < \epsilon$. This latter can be shown easily. Let f_1 be a function on $\Delta(d\alpha)$ such that $f_1(t) = f(t)$ for $t \in [a-b,a+b]$, $|f(t)| \le |f_1(t)|$ for $t \in \Delta(d\alpha)$ and $f_1 \in L^\infty(\Delta(d\alpha))$. Let us send $\Delta(d\alpha)$ to $[-1,1]$ by a linear transformation and then to $[0,\pi]$ by $x = \cos\theta$, $(-1\le x\le 1,$ $0\le\theta\le\pi)$. Set $g(\theta) = f_1(t)$, $(t \in \Delta(d\alpha),\ \theta \in [0,\pi])$. Let g^* denote the even extension of g to $[-\pi,\pi]$. Consider the Fejer sums of g^*. They are cosine polynomials, they are bounded in maximum norm: $\|\sigma_n(g^*)\| \leq \sup_{t \in \Delta(d\alpha)} |f_1(t)|$, they converge to g^* in e.g., $L^{10}[-\pi,\pi]$. Let us return now to $\Lambda(d\alpha)$ and remark that

$$\int_{a-b}^{a+b} \frac{|f(t) - \pi(t)|^2}{\sqrt{b^2 - (t-a)^2}} dt \leq C[\int_{\Delta(d\alpha)} |f(t) - \pi(t)|^{10} v_{\Delta(d\alpha)}(t)\, dt]^{1/5} .$$

<u>Theorem 14</u>. Let $\alpha \in M(a,0)$, f be $d\alpha$ measurable, bounded on $\text{supp}(d\alpha)$ and continuous at a. Let $\mathfrak{F}$ be bounded on $\Lambda \subset f(\text{supp}(d\alpha))$ and continuous at $f(a)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathfrak{F}(x_{kn}(f,d\alpha)) = \mathfrak{F}(f(a)).$$

<u>Proof.</u> Observe that if $f(\text{supp}(d\alpha)) \subset \Lambda$ then $x_{kn}(f,d\alpha) \in \Lambda$ for $n = 1,2,\ldots$ and $k = 1,2,\ldots,n$. This can be shown by exactly the same argument as in Lemma 9. Let first $\mathfrak{F}(u) = u^s$, $(s = 0,1,2,\ldots)$. For $s = 0$ the theorem is true. For $s = 1$

$$\sum_{k=1}^{n} x_{kn}(f,d\alpha) = \text{Tr } A_n(f,d\alpha)$$

and we apply Theorem 4.2.10. If $s \geq 2$ then

$$\sum_{k=1}^{n} x_{kn}^s(f,d\alpha) = \text{Tr } A_n^s(f,d\alpha)$$

as is well known and we use Theorems 13 and 4.2.10. Hence the theorem is true if $\mathfrak{F}$ is a polynomial, and consequently it is true if $\mathfrak{F}$ is continuous on Δ. Otherwise we use one-sided approximations.

<u>Theorem 15</u>. Let $\alpha \in M(a,b)$ with $b > 0$. Let f be $d\alpha$ measurable, bounded on $\text{supp}(d\alpha)$ and Riemann integrable on $[a-b,a+b]$. Let $\mathfrak{F}$ be continuous on $\Delta \supset f(\text{supp}(d\alpha))$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathfrak{F}(x_{kn}(f,d\alpha)) = \frac{1}{\pi} \int_{a-b}^{a+b} \frac{\mathfrak{F}(f(t))}{\sqrt{b^2 - (t-a)^2}} dt .$$

<u>Proof.</u> The same as that of Theorem 14.

Theorems 14 and 15 give us the following

Theorem 16. Let $a \in \mathbb{R}$, $b \geq 0$ and f be continuous on $\mathbb{R}$. Then for each pair of weights α_1 and α_2 belonging to $M(a,b)$ the eigenvalues of $A_n(f, d\alpha_1)$ and $A_n(f, d\alpha_2)$ are equally distributed.

6. Christoffel Functions

6.1 An Interpolation Process

The Hermite-Fejer interpolation polynomial $H_n(d\alpha,f,x)$ is the unique polynomial of degree at most $2n-1$ which satisfies the conditions

$$H_n(d\alpha,f,x_{kn}) = f(x_{kn}) , \qquad H_n'(d\alpha,f,x_{kn}) = 0$$

for $k = 1,2,\ldots,n$. Here $x_{kn} = x_{kn}(d\alpha)$. Hence

$$H_n(d\alpha,f,x) = \sum_{k=1}^{n} f(x_{kn}) \, [1 - 2\ell_{kn}'(d\alpha,x_{kn}) \, (x - x_{kn})] \, \ell_{kn}^2(d\alpha,x) .$$

Let us compute $\ell_k'(x_k)$. We have

$$\lambda_n^{-1}(x) = \sum_{k=1}^{n} \frac{\ell_k^2(x)}{\lambda_{kn}} ,$$

that is

$$-\lambda_n'(x) \, \lambda_n^{-2}(x) = \sum_{k=1}^{n} \frac{2\ell_k'(x) \, \ell_k(x)}{\lambda_{kn}} .$$

Putting here $x = x_{kn}$ we obtain

$$(1) \qquad -2\ell_k'(x_k) = \lambda_n'(x_k) \, \lambda_{kn}^{-1} .$$

Thus

$$H_n(d\alpha,f,x) = \sum_{k=1}^{n} f(x_{kn}) \, [\lambda_{kn}(d\alpha) + \lambda_n'(d\alpha,x_{kn}) \, (x - x_{kn})] \cdot \frac{\ell_{kn}^2(d\alpha,x)}{\lambda_{kn}(d\alpha)} .$$

This is Freud's representation for $H_n(d\alpha,f,x)$. (See [6]). In the brackets here we find an approximate expression for $\lambda_n(d\alpha,x)$:

$$\lambda_n(x) = \lambda_{kn} + \lambda_n'(x_{kn}) \, (x - x_{kn}) + \frac{\lambda_n''(\theta)}{2} \, (x - x_{kn})^2$$

where θ is between x and x_{kn} . Let us replace the expression in the brackets by $\lambda_n(d\alpha,x)$. Denote the resulting expression by $F_n(d\alpha,f,x)$:

$$F_n(d\alpha,f,x) = \lambda_n(d\alpha,x) \sum_{k=1}^{n} f(x_{kn}) \frac{\ell_{kn}^2(d\alpha,x)}{\lambda_{kn}(d\alpha)} .$$

For $z \in \mathbb{C}$ put

$$F_n(d\alpha,f,z) = \lambda_n^*(d\alpha,z) \sum_{k=1}^{n} f(x_{kn}) \frac{\ell_{kn}^2(d\alpha,z)}{\lambda_{kn}(d\alpha)}$$

and

$$\tilde{F}_n(d\alpha,f,z) = \lambda_n(d\alpha,z) \sum_{k=1}^{n} f(x_{kn}) \frac{|\ell_{kn}^2(d\alpha,z)|}{\lambda_{kn}(d\alpha)} .$$

(See 4.1 .)

Properties 1. (i) If $f(x) \equiv 1$ then $F_n(f,x) \equiv 1$. (ii) If $f(x) \geq 0$ for $x \in \Delta(d\alpha)$ then $F_n(f,x) \geq 0$ for $x \in \mathbb{R}$. (iii) $F_n(d\alpha,f,x_{kn}) = f(x_{kn})$ for $k = 1,2,\ldots,n$. (iv) $F_n'(d\alpha,f,x_{kn}) = 0$ for $k = 1,2,\ldots,n$ (use (1)) . (v) F_n is a rational function at degree $(2n-2,2n-2)$, only the numerator depends on f .

Because of (ii) we can expect that for many weights α, $F_n(d\alpha,f)$ converges to f whenever f is continuous. The surprising result is that the above class of weights α is very large. We shall consider convergence of $F_n(d\alpha,f)$ for $\alpha \in M(a,b)$ with $b > 0$ since for our purposes the case when $\alpha \in M(a,0)$ is less interesting. In order to avoid complicated formulas we shall assume, without loss of generality, that $\alpha \in M(0,1)$. Concerning $\rho(z)$ see Definition 4.1.8 .

Theorem 2. Let $\alpha \in M(0,1)$. Let f be bounded on $\Delta(d\alpha)$. If f is continuous at some $x \in \mathrm{supp}(d\alpha)$ then

(2) $$\lim_{n\to\infty} F_n(d\alpha,f,x) = f(x) .$$

If f is continuous on the segment $\Delta \subset (-1,1)$ then (2) is satisfied uniformly for $x \in \Delta$. If f is Riemann integrable on $[-1,1]$ and bounded on $\Delta(d\alpha)$ then for every $z \in \mathbb{C} \setminus \mathrm{supp}(d\alpha)$

$$\lim_{n\to\infty} F_n(d\alpha,f,z) = \frac{\rho^2(z) - 1}{2\pi} \int_{-1}^{1} f(t) \frac{\sqrt{1-t^2}}{(z-t)^2} dt$$

and

$$\lim_{n\to\infty} \tilde{F}_n(d\alpha,f,z) = \frac{|\rho(z)|^2 - 1}{2\pi} \int_{-1}^{1} f(t) \frac{\sqrt{1-t^2}}{|z-t|^2}\, dt .$$

Proof. The theorem follows from Theorems 3.2.3, 3.3.8, 4.1.11 and Properties 1. Let us prove e.g., the first part of the theorem. Let $\varepsilon > 0$. Then

$$|F_n(f,x) - f(x)| \le \sup_{|x-t|\le\varepsilon} |f(t) - f(x)| + 2\frac{\gamma_{n-1}^2}{\gamma_n^2} \varepsilon^{-2} \lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) \sup_{t\in\Delta(d\alpha)} |f(t)|.$$

First let $n \to \infty$ and then $\varepsilon \to 0$. By Theorem 4.1.11 (2) follows.

Definition 3. Let $g(\ge 0) \in L^1_{d\alpha}$. Then α_g is defined by

$$\alpha_g(t) = \int_{-\infty}^{t} g(u)\, d\alpha(u) .$$

Let us remark that α_g may not be a weight, it can happen that either α_g has only a finite number of points of increase or not each moment of α_g is finite. If g is a polynomial then α_g certainly is a weight. If $\operatorname{supp}(d\alpha)$ is compact and $g^{-1} \in L^1_{d\alpha}$ then also α_g is a weight.

Lemma 4. Let g be a linear function, nonnegative on $\operatorname{supp}(d\alpha)$ $(g(t) = c_1 t + c_2,$ $c_1 \ne 0)$. Then

$$\lambda_n^{-1}(d\alpha_g,x) = \sum_{k=1}^{n} \frac{\ell_{kn}^2(d\alpha,x)}{g(x_{kn}(d\alpha))\, \lambda_{kn}(d\alpha)} . \tag{3}$$

Proof. (By Freud [7]). Let us denote the right hand side of (3) by A. We have to show that for every π_{n-1}

$$\pi_{n-1}^2(x) \le A \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, d\alpha_g(t) \tag{4}$$

and for every $x \in \mathbb{R}$ there exists a π_{n-1}^* which turns (4) into equality. We have $\pi_{n-1} \equiv L_n(d\alpha,\pi_{n-1})$. Hence

$$\pi_{n-1}^2(x) \le \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \pi_{n-1}^2(x_{kn})\, g(x_{kn})\, A .$$

Since $\deg \pi_{n-1}^2 g \le 2n-1$ we can use the Gauss-Jacobi mechanical quadrature to obtain

$$\pi_{n-1}^2(x) \le A \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, g(t)\, d\alpha(t) = A \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, d\alpha_g(t) .$$

On the other hand we can define π_{n-1}^* by

$$\pi_{n-1}^*(t) = \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha,t)\, \ell_{kn}(d\alpha,x)}{g(x_{kn}(d\alpha))\, \lambda_{kn}(d\alpha)} .$$

<u>Lemma 5</u>. Let $\operatorname{supp}(d\alpha)$ be compact and let c be one of the endpoints of $\Delta(d\alpha)$. Then

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{|c - x_{kn}|} \le \frac{\gamma_n^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)} |c - \alpha_n(d\alpha)| .$$

<u>Proof</u>. (By Freud [8]). Since $p_n(d\alpha,c) \ne 0$

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha,x_{kn})}{c - x_{kn}} = \frac{\gamma_n}{\gamma_{n-1}} \frac{p_{n-1}(d\alpha,c)}{p_n(d\alpha,c)} .$$

Further $\operatorname{sign} p_{n+1}(d\alpha,c) = \operatorname{sign} p_{n-1}(d\alpha,c)$. Thus by the recurrence formula

$$|c - \alpha_n|\, |p_n(d\alpha,c)| \ge \frac{\gamma_{n-1}}{\gamma_n} |p_{n-1}(d\alpha,c)| .$$

The lemma follows from the above two formulas.

<u>Theorem 6</u>. Let $\alpha \in M(0,1)$. Let $g(t) = c_1 t + c_2$, $(c_1 \ne 0)$, be nonnegative on $\operatorname{supp}(d\alpha)$. Then for every $x \in \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)} = g(x)$$

and the convergence is uniform for $x \in \Delta \subset (-1,1)$.

<u>Proof</u>. We have by Lemma 4

$$\frac{\lambda_n(d\alpha,x)}{\lambda_n(d\alpha_g,x)} = F_n(d\alpha,g^{-1},x) .$$

If g is positive on $\Delta(d\alpha)$ we can directly apply Theorem 2. Next let g vanish at one of the endpoints of $\Delta(d\alpha)$ which we denote by c. First we show that

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,c)}{\lambda_n(d\alpha,c)} = 0 \qquad (= g(c)) .$$

Let $\epsilon > 0$. Then $g + \epsilon$ is positive on $\Delta(d\alpha)$. Hence

$$0 \leq \frac{\lambda_n(d\alpha_g, c)}{\lambda_n(d\alpha, c)} \leq \frac{\lambda_n(d\alpha_{g+\varepsilon}, c)}{\lambda_n(d\alpha, c)} \underset{n\to\infty}{\to} g(c) + \varepsilon = \varepsilon .$$

Now let $\varepsilon \to 0$. If $x \in \operatorname{supp}(d\alpha) \setminus c$ then for small $\delta > 0$, g^{-1} is bounded in $[x-\delta, x+\delta]$ (and g^{-1} is uniformly bounded in $\Delta \subset (-1,1)$). Writing $g(t) = A(t - c)$ we have

$$|F_n(d\alpha, g^{-1}, x) - g^{-1}(x)| = \Big| \sum_{|x-x_k|\leq\delta} + \sum_{|x-x_k|>\delta} \Big| \leq$$

$$\leq \frac{A\delta}{g(x)} \max_{|t-x|\leq\delta} g^{-1}(t) + \delta^{-2} \frac{\gamma_{n-1}^2}{\gamma_n^2} \lambda_n(x)\, p_n^2(x) \cdot$$

$$\cdot \sum_{|x-x_k|>\delta} |g^{-1}(x) - g^{-1}(x_k)|\, \lambda_k\, p_{n-1}^2(x_k) \leq$$

$$\leq \frac{A\delta}{g(x)} \max_{|t-x|\leq\delta} g^{-1}(t) + \delta^{-2} \frac{\gamma_{n-1}^2}{\gamma_n^2} \lambda_n(x)\, p_n^2(x)\, g^{-1}(x) +$$

$$+ A^{-1} \delta^{-2} \frac{\gamma_{n-1}^2}{\gamma_n^2} \lambda_n(x)\, p_n^2(x) \sum_{k=1}^{n} \lambda_k \frac{p_{n-1}^2(x_k)}{|x_k - c|} .$$

By Lemma 5 we obtain

$$|F_n(d\alpha, g^{-1}, x) - g^{-1}(x)| \leq A\,\delta\, g^{-1}(x) \max_{|x-t|\leq\delta} g^{-1}(t)$$

$$+ \delta^{-2} \frac{\gamma_{n-1}^2}{\gamma_n^2} \lambda_n(x)\, p_n^2(x)\; g^{-1}(x) + A^{-1} \delta^{-2} \lambda_n(x)\, p_n^2(x)\, |c - \alpha_n| .$$

This estimate and Theorem 4.1.11 shows that

$$\lim_{n\to\infty} F_n(d\alpha, g^{-1}, x) = g^{-1}(x)$$

for $x \in \operatorname{supp}(d\alpha)\setminus c$ and the convergence is uniform for $x \in \Delta \subset (-1,1)$.

<u>Lemma 7</u>. Let $u \in \mathbb{C}\setminus(-1,1)$ and $z \in \mathbb{C}\setminus[-1,1]$. Then

$$\frac{\rho^2(z) - 1}{2\pi} \int_{-1}^{1} \frac{\sqrt{1 - t^2}}{(t-u)(t-z)^2}\, dt = \frac{1}{z - u} - \frac{\sqrt{z^2 - 1}\, \rho(z)}{(z - u)^2} [\rho^{-1}(u) - \rho^{-1}(z)]$$

where $\sqrt{z^2 - 1} > 0$ for $z > 1$.

Proof. Because of continuity arguments we can suppose that $z \neq u$ and $u \in \mathbb{C}\setminus[-1,1]$. Since

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{z-t}\,dt = \frac{1}{\rho(z)}$$

(See the proof of Theorem 4.1.13) we have

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{(u-t)(z-t)}\,dt = \frac{1}{(z-u)}\,[\rho^{-1}(u) - \rho^{-1}(z)]\,.$$

Differentiating this identity with respect to z we obtain the lemma.

Theorem 8. Let $\alpha \in M(0,1)$. Let $g(t) = A(t-B)$ be positive on $\Delta(d\alpha)$. Then for every $z \in \mathbb{C}\setminus \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty}\frac{\lambda_n^*(d\alpha,z)}{\lambda_n^*(d\alpha_g,z)} = g^{-1}(z) + \sqrt{z^2-1}\;\rho(z)\,\frac{d}{dz}\,g^{-1}(z)\cdot[\rho^{-1}(B) - \rho^{-1}(z)]\,,$$

in particular

$$\lim_{n\to\infty}\frac{\lambda_n(d\alpha,B)}{\lambda_n(d\alpha_g,B)} = \frac{\rho(b)}{2(B^2-1)\,A}\,.$$

Proof. By Lemma 4 we have

$$\frac{\lambda_n^*(d\alpha,z)}{\lambda_n^*(d\alpha_g,z)} = F_n(d\alpha,g^{-1},z)\,.$$

Therefore Theorem 2 yields

$$\lim_{n\to\infty}\frac{\lambda_n^*(d\alpha,z)}{\lambda_n^*(d\alpha_g,z)} = \frac{\rho^2(z)-1}{2\pi}\int\frac{\sqrt{1-t^2}}{g(t)(z-t)^2}\,dt\,.$$

Thus the theorem follows from Lemma 7.

Lemma 9. Let $g(t) = A(t-B)$, $(A \neq 0)$ be nonnegative on $\operatorname{supp}(d\alpha)$. Then

$$\frac{\gamma_{n-1}^2(d\alpha_g)}{\gamma_{n-1}^2(d\alpha)} = -\frac{1}{A}\,\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)}\,\frac{p_{n-1}(d\alpha,B)}{p_n(d\alpha,B)}\,.$$

Proof. We have

$$\frac{\gamma_{n-1}^2(d\alpha_g)}{\gamma_{n-1}^2(d\alpha)} = \lim_{x\to\infty}\frac{\lambda_n(d\alpha,x)}{\lambda_n(d\alpha_g,x)}$$

which equals by Lemma 4

$$\lim_{x\to\infty}\frac{\sum_{k=1}^{n}\frac{\ell_k^2(d\alpha,x)\,x^2}{\lambda_k(d\alpha)\,g(x_k)}}{\sum_{k=1}^{n}\frac{\ell_k^2(d\alpha,x)\,x^2}{\lambda_k(d\alpha)}}=\frac{\sum_{k=1}^{n}\lambda_k(d\alpha)\,p_{n-1}^2(d\alpha,x_k)\,g^{-1}(x_k)}{\sum_{k=1}^{n}\lambda_k(d\alpha)\,p_{n-1}^2(d\alpha,x_k)}=$$

$$=-\frac{1}{A}\sum_{k=1}^{n}\lambda_k(d\alpha)\,p_{n-1}^2(d\alpha,x_k)\,\frac{1}{B-x_k}=$$

$$=-\frac{1}{A}\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)}\,L_n(d\alpha,p_{n-1}(d\alpha),B)\cdot\frac{1}{p_n(d\alpha,B)}=$$

$$=-\frac{1}{A}\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)}\,\frac{p_{n-1}(d\alpha,B)}{p_n(d\alpha,B)}\,.$$

Lemma 10. Let $\alpha\in M(0,1)$. Let $g(t)=A(t-B)$ be positive on $\Delta(d\alpha)$. Then $\alpha_g\in M(0,1)$ and

(5) $$\lim_{n\to\infty}\frac{\gamma_{n-1}(d\alpha_g)}{\gamma_{n-1}(d\alpha)}=\left|\frac{2}{A\ \rho(B)}\right|^{1/2}=\exp\{-\frac{1}{2\pi}\int_{-1}^{1}\log g(t)\,\frac{dt}{\sqrt{1-t^2}}\}\,.$$

Proof. If g is positive on $\Delta(d\alpha)$ then B is outside $\Delta(d\alpha)$. Hence by Theorem 4.1.13

(6) $$\lim_{n\to\infty}\frac{p_{n-1}(d\alpha,B)}{p_n(d\alpha,B)}=\rho^{-1}(B)<\infty\,.$$

Applying Lemma 9 we see that the equality on the left side of (5) holds and consequently

$$\lim_{n\to\infty}\frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)}=\frac{1}{2}\,.$$

Putting α = Chebyshev weight we have $\alpha\in S$ and $\alpha_g\in S$ (Let us recall that $[-1,1]\subset\Delta(d\beta)$ for $\beta\in M(0,1)$ and hence g is positive on $[-1,1]$). Using Lemma 4.2.2 we obtain the right side equality in (5). Now we have to show that

$$\lim_{n\to\infty}\alpha_n(d\alpha_g)=0\,.$$

Let us develop $g\,p_n(d\alpha_g)$ into a Fourier series in $p_k(d\alpha)$. It is easy to see that

$$g(x)\,p_n(d\alpha_g,x)=\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)}\,p_n(d\alpha,x)+A\,\frac{\gamma_n(d\alpha_g)}{\gamma_{n+1}(d\alpha)}\,p_{n+1}(d\alpha,x)\,.$$

Hence
$$\int_{-\infty}^{\infty} g^2(x)\, p_n^2(d\alpha_g,x)\, d\alpha(x) = \frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} + A^2 \frac{\gamma_n^2(d\alpha_g)}{\gamma_{n+1}^2(d\alpha)} .$$

The left side equals
$$\int_{-\infty}^{\infty} A(x - B)\, p_n^2(d\alpha_g,x)\, d\alpha_g(x) = A\,\alpha_n(d\alpha_g) - AB .$$

Thus by Lemma 9
$$\alpha_n(d\alpha_g) = B - \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} \left\{ \frac{p_{n+1}(d\alpha,B)}{p_n(d\alpha,B)} + \frac{p_n(d\alpha,B)}{p_{n+1}(d\alpha,B)} \right\} .$$

By (6) $\lim_{n\to\infty} \alpha_n(d\alpha_g)$ exists and equals $B - \frac{1}{2}[\rho(B) + \rho^{-1}(B)] = 0$. Consequently $\alpha_g \in M(0,1)$.

<u>Remark 11</u>. Lemma 9 and the proof of Lemma 10 show that if $g(t) = A(t - B)$ is positive on $\Delta(d\alpha)$ and $\alpha \in M(0,1)$ then
$$\lim_{n\to\infty} \frac{g(z)\, p_n(d\alpha_g,z)}{p_n(d\alpha,z)} = \frac{A}{\sqrt{2}} \left| \frac{1}{A\,\rho(B)} \right|^{1/2} [\rho(z) - \rho(B)]$$
for $z \in [\mathbb{C}\setminus \mathrm{supp}(d\alpha)]\setminus\{B\} = [\mathbb{C}\setminus \mathrm{supp}(d\alpha_g)] \setminus \{B\}$.

This remark and Theorem 4.1.11 give a new proof of Theorem 8.

<u>Lemma 12</u>. Let $\alpha \in M(0,1)$. Let $g(x) = (x - A)^2 + B^2$ with $A \in \mathbb{R}$, $B^2 > 0$. Then $\alpha_g \in M(0,1)$ and

$$(7) \qquad \lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = 2 \left| \frac{1}{\rho(A + iB)\ \rho(A - iB)} \right|^{1/2} = \exp\left\{ -\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1 - t^2}} \right\}.$$

<u>Proof</u>. Let us develop $g\,p_n(d\alpha_g)$ in a Fourier series in $p_k(d\alpha)$. We have

$$(8) \quad g(x)\, p_n(d\alpha_g,x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p_n(d\alpha,x) + d_{n+1}\, p_{n+1}(d\alpha,x) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} p_{n+2}(d\alpha,x) .$$

Unfortunately, we cannot directly calculate d_{n+1} . Let us note that $g(A \pm iB) = 0$. Hence
$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p_n(d\alpha, A \pm iB) + d_{n+1}\, p_{n+1}(d\alpha, A \pm iB) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} p_{n+2}(d\alpha, A \pm iB) = 0 .$$

Consequently

(9) $$-d_{n+1} = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)}\frac{p_n(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)} + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)}\frac{p_{n+2}(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)}$$

and

$$\frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} = \frac{\gamma_n(d\alpha)}{\gamma_{n+2}(d\alpha)}\left[\frac{p_{n+2}(d\alpha,A+iB)}{p_{n+1}(d\alpha,A+iB)} - \frac{p_{n+2}(d\alpha,A-iB)}{p_{n+1}(d\alpha,A-iB)}\right] \cdot \left[\frac{p_n(d\alpha,A-iB)}{p_{n+1}(d\alpha,A-iB)} - \frac{p_n(d\alpha,A+iB)}{p_{n+1}(d\alpha,A+iB)}\right]^{-1}.$$

Letting $n \to \infty$ and using Theorem 4.1.13 we obtain

$$\lim_{n\to\infty} \frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} = \frac{1}{4}[\rho(A+iB) - \rho(A-iB)] \cdot [\rho^{-1}(A-iB) - \rho^{-1}(A+iB)]^{-1} = \frac{1}{4}\ \rho(A+iB)\ \rho(A-iB)$$

which proves the left side equality in (7). The right side equality in (7) follows from Lemma 4.2.2. Now we shall show that for every $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha) = \mathbb{C} \setminus \operatorname{supp}(d\alpha_g)$, $(z \neq A \pm iB)$

(10) $$\lim_{n\to\infty} \frac{p_n(d\alpha_g, z)}{p_{n+1}(d\alpha_g, z)} = \rho(z)^{-1} .$$

If (10) holds then by Theorem 4.1.12 $\alpha_g \in M(0,1)$. We obtain from (8) and (9)

$$\frac{g(z)\ p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)}\ \Big[\frac{\gamma_{n+2}(d\alpha)\ \gamma_n(d\alpha)}{\gamma_n^2(d\alpha_g)} -$$

$$\Big(\frac{\gamma_{n+2}(d\alpha)\gamma_n(d\alpha)}{\gamma_n^2(d\alpha)}\ \frac{p_n(d\alpha,A+iB)}{p_{n+1}(d\alpha,A+iB)} + \frac{p_{n+2}(d\alpha,A+iB)}{p_{n+1}(d\alpha,A+iB)}\Big) \cdot \frac{p_{n+1}(d\alpha,z)}{p_n(d\alpha,z)} + \frac{p_{n+2}(d\alpha,z)}{p_n(d\alpha,z)}\ \Big] .$$

By (7) and Theorem 4.1.13 for $z \in [C \setminus \operatorname{supp}(d\alpha)] \setminus \{A \pm B\}$

(11) $$\lim_{n\to\infty} \frac{g(z)p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2}\left|\frac{1}{\rho(A+iB)\rho(A-iB)}\right|^{1/2} \cdot [\rho(z) - \rho(A+iB)][\rho(z) - \rho(A-iB)].$$

Now (10) follows from Theorem 4.1.13. Hence $\alpha_g \in M(0,1)$.

Let us remark that by Theorem 4.1.13, (10) holds also for $z = A \pm iB$.

<u>Lemma 13</u>. Let $\alpha \in M(0,1)$. Let $g(x) = (x - A)(x - B)$, $(A \neq B)$ be positive on $\operatorname{supp}(d\alpha)$. Then $\alpha_g \in M(0,1)$,

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = 2\ \left|\frac{1}{\rho(A)\ \rho(B)}\right|^{1/2} = \exp\Big(-\frac{1}{2\pi}\int_{-1}^{1} \log g(t)\ \frac{dt}{\sqrt{1-t^2}}\ \Big) ,$$

further for every $z \in [C \setminus \mathrm{supp}(d\alpha)] \setminus \{A, B\}$

$$\lim_{n\to\infty} \frac{g(z)\, p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left|\frac{1}{\rho(A)\ \rho(B)}\right|^{1/2} \cdot [\rho(z) - \rho(A)]\, [\rho(z) - \rho(B)] \;.$$

Proof. The proof of Lemma 12 can be repeated. Note that $A, B \notin [-1,1]$ but may belong to $\Delta(d\alpha)$.

Lemma 14. Let $\alpha \in M(0,1)$ and let $g(x) = (x-A)^2$ where $A \in \mathbb{R} \setminus \mathrm{supp}(d\alpha)$, that is A may belong to $\Delta(d\alpha)$. Then $\alpha_g \in M(0,1)$ and

$$(12) \qquad \lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = 2\left|\frac{1}{\rho(A)}\right| = \exp\Big(-\frac{1}{2\pi} \int_{-1}^{1} \log g(t)\, \frac{dt}{\sqrt{1-t^2}}\Big) \;.$$

Proof. The proof of Lemma 12 has to be modified. We have (8) and $g(A) = g'(A) = 0$. Hence

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)}\, p_n(d\alpha, A) + d_{n+1}\, p_{n+1}(d\alpha, A) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)}\, p_{n+2}(d\alpha, A) = 0$$

and

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)}\, p_n'(d\alpha, A) + d_{n+1}\, p_{n+1}'(d\alpha, A) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)}\, p_{n+2}'(d\alpha, A) = 0 \;.$$

From here

$$(13) \qquad -d_{n+1} = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)}\, \frac{p_n(d\alpha, A)}{p_{n+1}(d\alpha, A)} + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)}\, \frac{p_{n+2}(d\alpha, A)}{p_{n+1}(d\alpha, A)}$$

and

$$\frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} = \frac{\gamma_n(d\alpha)}{\gamma_{n+2}(d\alpha)} \left[\frac{p_{n+2}'(d\alpha, A)}{p_{n+1}'(d\alpha, A)} - \frac{p_{n+2}(d\alpha, A)}{p_{n+1}(d\alpha, A)} \right] \cdot \left[\frac{p_n(d\alpha, A)}{p_{n+1}(d\alpha, A)} - \frac{p_n'(d\alpha, A)}{p_{n+1}'(d\alpha, A)} \right]^{-1}.$$

Now (12) follows from Theorems 4.1.13, 4.1.16 and 4.1.17, further from Lemma 4.2.2. Using (8), (12), (13) and Theorem 4.1.13 we obtain by the same way as we did in the proof of Lemma 12 that

$$(14) \qquad \lim_{n\to\infty} \frac{g(z)\, p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left| \frac{1}{\rho(A)} \right| [\rho(z) - \rho(A)]^2$$

for $z \in \mathbb{C} \setminus \mathrm{supp}(d\alpha) \setminus \{A\}$ which together with Theorems 4.1.12 and 4.1.13

shows that $\alpha_g \in M(0,1)$.

Theorem 15. Let $\alpha \in M(0,1)$. Let

$$g(x) = A \prod_{k=1}^{N} (x - B_k)$$

be positive on $\operatorname{supp}(d\alpha)$. Then $\alpha_g \in M(0,1)$,

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = \exp\{-\frac{1}{2\pi}\int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\}$$

and for every $z \in [\mathbb{C} \setminus \operatorname{supp}(d\alpha)] \setminus \{B_k\}$

$$\lim_{n\to\infty} \frac{p_n(d\alpha_g,z)}{p_n(d\alpha,z)} = \frac{1}{2^N} \exp\{-\frac{1}{2\pi}\int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\} \cdot \prod_{k=1}^{N} \frac{\rho(z) - \rho(B_k)}{z - B_k} .$$

Proof. Repeated application Lemmas 10, 12, 13, 14, of Remark 11 and of formulas (11) and (14).

Definition 16. Let $\beta \in S$. Then the Szegö function $D(d\beta,z)$ is defined by

$$D(d\beta,z) = \exp\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log \beta'(\cos t) \cdot \frac{1 + ze^{-it}}{1 - ze^{-it}} dt\}$$

for $|z| < 1$.

Properties 17. $D \in H_2(|z| < 1)$, for almost every $t \in [-\pi,\pi]$

$$\lim_{r\to 1-0} D(d\beta, re^{it}) = D(d\beta, e^{it})$$

exists and $|D(d\beta,e^{it})|^2 = \beta'(\cos t)$ for almost every $t \in [-\pi,\pi]$, $D(d\beta,z) \neq 0$ for $|z| < 1$, $D(d\beta,0) > 0$. (See e.g. Freud, Chapter V.)

Recall that v denotes the Chebyshev weight.

Lemma 18. Let $\beta \in S$, $z \in \mathbb{C} \setminus [-1,1]$. Then

$$\lim_{n\to\infty} p_n(d\beta,z)\, \rho(z)^{-n} = \frac{1}{\sqrt{2\pi}} D(v^{-1}d\beta, \rho(z)^{-1})^{-1}$$

and the convergence is uniform for $|\rho(z)| \geq R > 1$.

Proof. See e.g. Freud, §V.5.

Lemma 19. Let $g(x) = A \prod_{k=1}^{N} (x - B_k)$ be positive on $[-1,1]$. Then for $z \in \mathbb{C} \setminus [-1,1]$

$$(15) \quad D(g, \rho(z)^{-1}) = 2^N \exp\{\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1 - t^2}}\} \cdot \prod_{k=1}^{N} \frac{z - B_k}{\rho(z) - \rho(B_k)} .$$

Proof. Put in Theorem 15 α = Chebyshev weight and use Lemma 18. Then for $z \in [\mathbb{C} \setminus [-1,1]] \setminus \{B_k\}$, (15) holds and consequently it holds for every $z \in \mathbb{C} \setminus [-1,1]$.

Let us note that - because of continuity arguments - (15) holds if $g(\pm 1) = 0$, further (15) holds for $z = \infty$ if g is only nonnegative on $[-1,1]$.

Theorem 20. Let $\alpha \in M(0,1)$ and let g be a polynomial which is positive on $\operatorname{supp}(d\alpha)$. Then $\alpha_g \in M(0,1)$ and

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} = D(g,0) ,$$

$$\lim_{n\to\infty} \frac{p_n(d\alpha, z)}{p_n(d\alpha_g, z)} = D(g, \rho(z)^{-1})$$

for $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$.

Proof. By Theorem 15 and Lemma 19, the only thing which we have to show is that if $g(B) = 0$ then

$$(16) \quad \lim_{n\to\infty} \frac{p_n(d\alpha, B)}{p_n(d\alpha_g, B)} = D(g, \rho(z)^{-1}) .$$

Let $\delta > 0$ be small enough. Then for $|z - B| = \delta$

$$\lim_{n\to\infty} \frac{p_n(d\alpha, z)}{p_n(d\alpha_g, z)} = D(g, \rho(z)^{-1}) ,$$

that is by Theorem 4.1.13

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha_g, z)} = \rho(z)^{-1} D(g, \rho(z)^{-1})$$

for $|z - B| = \delta$. Since $p_{n-1}(d\alpha, z) = L_n(d\alpha_g, p_{n-1}(d\alpha), z)$ we have

$$\left|\frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha_g, z)}\right| \leq \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} \frac{1}{\epsilon} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) \, |p_{n-1}(d\alpha, x_{kn})| \cdot |p_{n-1}(d\alpha_g, x_{kn})| \leq$$

$$\leq \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} \frac{1}{\epsilon} \{\int_{-\infty}^{\infty} p_{n-1}^2(d\alpha,t)\ d\alpha_g(t)\}^{1/2} \leq \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} \frac{1}{\epsilon} \max_{t\in \operatorname{supp}(d\alpha)} [g(t)]^{1/2} .$$

for $n \geq N$ where ϵ and N are defined by Theorem 3.3.8. Since both $p_{n-1}(d\alpha,z)/p_n(d\alpha_g,z)$ and $\rho(z)^{-1}\ D(g,\rho(z)^{-1})$ are analytic in $|z - B| \leq \delta$ if $\delta \leq \epsilon$ and $n \geq N$ we can apply Cauchy's integral formula and Lebesgue's theorem about $\lim \int f_n = \int \lim f_n$ and we obtain

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha,B)}{p_n(d\alpha_g,B)} = \rho(B)^{-1}\ D(g,\rho(B)^{-1}) .$$

Thus by Theorem 4.1.13, (16) holds.

Now we can easily generalize Theorem 8.

Theorem 21. Let $\alpha \in M(0,1)$. Let g be a polynomial which is positive on $\operatorname{supp}(d\alpha)$. Then for every $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{\lambda_n^*(d\alpha_g,z)}{\lambda_n^*(d\alpha,z)} = D(g,\rho(z)^{-1})^2$$

and

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,z)}{\lambda_n(d\alpha,z)} = |D(g,\rho(z)^{-1})|^2 .$$

Proof. Apply Theorems 20 and 4.1.11.

Remark 22. Let us put $\rho(z)^{-1} = re^{i\theta}$ $(0 < r < 1)$ in (17). Then $z = \frac{1}{2}(re^{i\theta} + r^{-1}e^{-i\theta}) \underset{r\to 1-0}{\to} \cos\theta$. By Properties 17 for almost every $\theta \in [-\pi,\pi]$

$$\lim_{r\to 1-0}\ \lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,z)}{\lambda_n(d\alpha,z)} = g(x) \qquad (x = \cos\theta)$$

which suggests

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)} = g(x) \qquad (-1 \leq x \leq 1) .$$

Property 23. Let $z = re^{i\theta}$, $0 \leq r < 1$. Then

$$|D(d\beta,z)|^2 = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \beta'(\cos t)\ \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\ dt\} .$$

(See e.g. Freud, Chapter V.)

Lemma 24. Let f be Riemann integrable on $\Delta \supset [-1,1]$ and let $f(x) \geq C > 0$ for $x \in \Delta$. Let $0 < R < 1$ be fixed. Then for every $\epsilon > 0$ there exist two polynomials π_1 and π_2 such that

$$\frac{C}{2} \leq \pi_1(x) \leq f(x) \leq \pi_2(x)$$

for $x \in \Delta$ and

$$|D(\pi_2,z)|^2 (1 - \epsilon) \leq |D(f,z)|^2 \leq |D(\pi_1,z)|^2 (1 + \epsilon)$$

for $|z| \leq R$.

Proof. Let $\epsilon > 0$. We construct a polynomial π_2 such that

$$f(x) \leq \pi_2(x) \qquad (x \in \Delta)$$

and

$$\int_{-\pi}^{\pi} [\pi_2(\cos t) - f(\cos t)]dt < \epsilon .$$

(See Szegő, 1.5.) Then by Property 23

$$|D(\pi_2,z)|^2 \leq |D(\pi_2 f^{-1},z)|^2 \; |D(f,z)|^2$$

and by Jensen's inequality

$$|D(\pi_2 f^{-1},z)|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi_2(\cos t)}{f(\cos t)} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\, dt$$

$(z = re^{i\theta})$. Hence

$$|D(\pi_2 f^{-1},z)|^2 \leq 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi_2(\cos t) - f(\cos t)}{f(\cos t)} \cdot \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\, dt \leq 1 + \text{const} \cdot \epsilon$$

for $|z| \leq R$. The second part of the lemma can be proved in the same way. Since $f(x) \geq C > 0$ for $x \in \Delta$ we can choose π_1 so that $\pi_1(x) \geq \frac{C}{2}$ for $x \in \Delta$. (See Szegő, 1.5.)

Theorem 25. Let $\alpha \in M(0,1)$. Let $g(\geq 0)$ be $d\alpha$ measurable, $g^{\pm 1}$ be bounded on $\operatorname{supp}(d\alpha)$ and g be Riemann integrable on $[-1,1]$. Then for every $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$

$$\lim_{n \to \infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} = |D(g, \rho(z)^{-1})|^2 .$$

Proof. By the assumptions α_g is a weight, since $g^{-1} \in L^1_{d\alpha}$. Recall that

$$\lambda_n(d\alpha, z) = \min_{\pi_{n-2}} \int_{-\infty}^{\infty} |(1 + (z - t)\, \pi_{n-2}(t))|^2 \, d\alpha(t) .$$

Then from $d\alpha \leq d\beta$ it follows that $\lambda_n(d\alpha, z) \leq \lambda_n(d\beta, z)$ for every $z \in \mathbb{C}$. Let us construct two functions f_1 and f_2 such that both f_1 and f_2 are Riemann integrable on $\Delta(d\alpha)$, $f_1(x) = f_2(x) = g(x)$ for $x \in [-1,1]$ and

$$0 < c_1 \leq f_1(x) \leq g(x) \leq f_2(x) \leq c_2 < \infty$$

for $x \in \Delta(d\alpha)$. We can do this by Theorem 3.3.7. Let $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$. Then $|\rho(z)^{-1}| < 1$. Let $\epsilon > 0$. Then by Lemma 24 we can find two polynomials π_1 and π_2 such that $\pi_1(x) \geq c_1/2$, $\pi_2(x) \geq c_1/2$ for $x \in \Delta(d\alpha)$,

$$\lambda_n(d\alpha_{\pi_1}, z) \leq \lambda_n(d\alpha_g, z) \leq \lambda_n(d\alpha_{\pi_2}, z)$$

and

$$(1 - \epsilon)|D(\pi_2, \rho(z)^{-1})|^2 \leq |D(g, \rho(z)^{-1})|^2 \leq (1 + \epsilon)|D(\pi_1, \rho(z)^{-1})|^2 .$$

Thus by Theorem 21

$$|D(g, \rho(z)^{-1})|^2 \frac{1}{1+\epsilon} \leq \liminf_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} \leq \limsup_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} \leq \frac{1}{1-\epsilon} |D(g, \rho(z)^{-1})|^2 .$$

Now let $\epsilon \to 0$.

Theorem 26. Let $\alpha \in M(0,1)$ and g be as in Theorem 25. Then

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} = D(g, 0) ,$$

and so

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} = \frac{1}{2} .$$

Proof. Recall that from $d\alpha \leq d\beta$ follows $\gamma_n(d\beta) \leq \gamma_n(d\alpha)$. Now we can repeat the proof of Theorem 25, the only difference is that this time we apply Theorem 20 instead of Theorem 21.

Theorem 27. Let $\alpha \in M(0,1)$ and g be as in Theorem 25. Then $\alpha_g \in M(0,1)$.

Proof. If we could directly calculate $\alpha_n(d\alpha_g)$ the proof would probably be nice. Unfortunately we cannot do this. Let $x \notin \mathrm{supp}(d\alpha)$. Then by Theorem 25

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,x)}{\lambda_{n+1}(d\alpha_g,x)} \cdot \frac{\lambda_{n+1}(d\alpha,x)}{\lambda_n(d\alpha,x)} = 1 ,$$

that is

$$\lim_{n\to\infty} \frac{1 + p_n^2(d\alpha_g,x)\ \lambda_n(d\alpha_g,x)}{1 + p_n^2(d\alpha,x)\ \lambda_n(d\alpha,x)} = 1 .$$

Thus by Theorem 4.1.11

$$\lim_{n\to\infty} p_n^2(d\alpha_g,x)\ \lambda_n(d\alpha_g,x) = \rho(x)^2 - 1 .$$

Using Theorem 26 we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) \frac{p_{n-1}^2(d\alpha_g,x_{kn})}{(x - x_{kn})^2} = \frac{4}{\rho(x)^2 - 1} = -2 \frac{d}{dx} \rho(x)^{-1}$$

for $x \notin \mathrm{supp}(d\alpha)$. By Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) \frac{p_{n-1}^2(d\alpha_g,x_{kn})}{x - x_{kn}} = \frac{2}{\rho(x)} = \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1 - t^2}}{x - t}\, dt$$

for $x \notin \Delta(d\alpha)$. If f is continuous on $\Delta(d\alpha)$ then for every $\epsilon > 0$ we can find a function F of the form

$$F(t) = \sum_{j=1}^{N} a_j \frac{1}{x_j - t}$$

where $a_j \in \mathbb{C}$ and $x_j \in \mathbb{R} \setminus \Delta(d\alpha)$ such that

$$\max_{t \in \Delta(d\alpha)} |F(t) - f(t)| \le \epsilon .$$

(See e.g. Ahiezer, section of problems.) Hence if f is continuous on $\Delta(d\alpha)$ then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g)\ f(x_{kn})\ p_{n-1}^2(d\alpha_g,x_{kn}) = \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{1 - t^2}\, dt .$$

Consequently, by Theorem 3.2.4, $\alpha_g \in M(0,1)$.

Remark 2.8. Later we shall show (with the aid of the Pollaczek polynomials), that if w is defined by

$$w(x) = \exp\{-(1 - x^2)^{-1/2}\}$$

for $-1 \le x \le 1$ and $\mathrm{supp}(w) = [-1,1]$ then $w \in M(0,1)$. Consequently by the previous theorem $gw \in M(0,1)$ if $g > 0$ is continuous on $[-1,1]$. Let us remark that the above w is the "nicest" weight which does not belong to S.

Theorem 29. Let $\alpha \in M(0,1)$ and let g satisfy the conditions of Theorem 25. Let $K \subset [\mathbb{C} \cup \{\infty\}] \setminus \operatorname{supp}(d\alpha)$ be an arbitrary closed set. Then

$$\lim_{n\to\infty} \frac{p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = D(g, \rho(z)^{-1})^{-1} \tag{18}$$

uniformly for $z \in K$.

Proof. If $z \in \mathbb{R} \setminus \operatorname{supp}(d\alpha)$ then (18) follows immediately from Theorem 25, 27, 3.3.8 and 4.1.11. Let K^* be a region in $C \cup \{\infty\}$ such that $K \subset K^*$, $\overline{K}^* \cap \operatorname{supp}(d\alpha) = \emptyset$ and $K^* \cap \mathbb{R} \neq \emptyset$. By Theorem 3.3.8 the functions $p_n(d\alpha_g, z)\, p_n(d\alpha, z)^{-1}$ are analytic in K^*. If we can show that

$$\frac{|p_n(d\alpha_g, z)|}{|p_n(d\alpha, z)|} \le \text{const} \tag{19}$$

for $z \in \overline{K}^*$ and $n = N, N+1, \ldots$ where $N = N(\overline{K}^*)$ then the theorem will follow from Vitali's theorem. Let d_N be defined by $d_N = \operatorname{dist}(\overline{K}^*, \{x_{kn}(d\alpha)\}_{n=N,k=1}^{\infty\ \ n})$. By Theorem 3.3.8, $d_N > 0$ for some $N \in \mathbb{N}$. Let $n \ge N$ and $z \in \overline{K}^*$. Then

$$|p_n(d\alpha_g, z)|^2 \le \lambda_{n+1}(d\alpha, z)^{-1} \int_{-\infty}^{\infty} p_n^2(d\alpha_g, t)\, d\alpha(t) \le$$

$$\le C\, \lambda_{n+1}(d\alpha, z)^{-1} \int_{-\infty}^{\infty} p_n^2(d\alpha_g, t)\, d\alpha_g(t)$$

where $C^{-1} = \inf\limits_{t \in \operatorname{supp}(d\alpha)} g(t)$. Hence

$$|p_n(d\alpha_g, z)|^2 \le C\, |p_n(d\alpha, z)|^2 + C\, \lambda_n(d\alpha, z)^{-1}.$$

Further we have

$$\lambda_n(d\alpha, z)^{-1} = \sum_{k=1}^{n} \frac{|\ell_{kn}(d\alpha, z)|^2}{\lambda_{kn}(d\alpha)} \le \frac{\gamma_{n-1}(d\alpha)^2}{\gamma_n(d\alpha)^2}\, |p_n(d\alpha, z)|^2\, d_N^{-2}.$$

Consequently (19) is satisfied with $\text{const} = [C(1 + d_N^{-2}\, |\Delta(d\alpha)|^2 \cdot 0.25)]^{1/2}$.

6.2. A Sequence of Positive Operators

Using the well known formula

$$\ell_{kn}(d\alpha,x) = \lambda_{kn}(d\alpha)\ K_n(d\alpha,x,x_{kn})$$

we obtain

$$F_n(d\alpha,f,x) = \lambda_n(d\alpha,x) \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\ f(x_{kn})\ K_n^2(d\alpha,x,x_{kn})$$

which is the Riemann-Stieltjes sum for

$$G_n(d\alpha,f,x) = \lambda_n(d\alpha,z) \int_{-\infty}^{\infty} f(t)\ K_n^2(d\alpha,x,t)\ d\alpha(t)\ .$$

For $z \in \mathbb{C}$ we put

$$G_n(d\alpha,f,z) = \lambda_n^*(d\alpha,z) \int_{-\infty}^{\infty} f(t)\ k_n^2(d\alpha,z,t)\ d\alpha(t)\ .$$

(See 4.1.)

Properties 1. (i) If $f(x) \equiv 1$ then $G_n(f,x) \equiv 1$. (ii) If $f(x) \geq 0$ for $x \in \operatorname{supp}(d\alpha)$ then $G_n(f,x) \geq 0$ for $x \in \mathbb{R}$. (iii) G_n is a rational function of degree $(2n-2,2n-2)$ where the denominator does not depend on f .

Theorem 2. Let $\alpha \in M(0,1)$. Let f be $d\alpha$ measurable and bounded on $\operatorname{supp}(d\alpha)$. Then for each $x \in \operatorname{supp}(d\alpha) \setminus [-1,1]$

$$\lim_{n\to\infty} G_n(d\alpha,f,x) = f(x)\ . \tag{1}$$

If $x \in [-1,1]$ and f is continuous at x then (1) holds. If f is continuous on $\Delta \subset (-1,1)$ then (1) holds uniformly for $x \in \Delta$. If f is continuous on $\operatorname{supp}(d\alpha)$ and $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$ then

$$\lim_{n\to\infty} G_n(d\alpha,f,z) = \frac{\sqrt{z^2-1}}{\pi} \int_{-1}^{1} \frac{f(t)}{(z-t)\sqrt{1-t^2}}\, dt\ . \tag{2}$$

Here $\sqrt{z^2-1} > 0$ for $z > 1$.

Proof. (i) Let $x \in \operatorname{supp}(d\alpha) \setminus [-1,1]$. Then by Theorem 3.3.7, x is an isolated point of $\operatorname{supp}(d\alpha)$. Hence there exists $\epsilon > 0$ such that

$$G_n(f,x) = f(x)\,\frac{\alpha(x+0)-\alpha(x-0)}{\lambda_n(x)} + \lambda_n(x) \int_{|x-t|\geq\varepsilon} f(t)\,K_n^2(x,t)\,d\alpha(t)\ .$$

Here the first term converges to $f(x)$ when $n \to \infty$. (See Freud, §II.2, $\operatorname{supp}(d\alpha)$ is compact!) Remembering that

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n}\,\frac{p_{n-1}(t)\,p_n(x) - p_n(t)\,p_{n-1}(x)}{x-t}$$

and using Theorem 4.1.11 we see that

$$\lim_{n\to\infty} \lambda_n(x) \int_{|x-t|>\varepsilon} f(t)\,K_n^2(x,t)\,d\alpha(t) = 0\ . \tag{3}$$

(ii) Let $x \in [-1,1]$. Then by Theorem 4.1.11 for every $\varepsilon > 0$, (3) is satisfied and the convergence id uniform for $x \in \Delta \subset (-1,1)$. Thus by Properties 1 the usual machinery of positive operators can be applied. We do not go into details. (iii) Let $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$. By Tietze's theorem we can suppose that f is continuous on $\Delta(d\alpha)$. The function $(z-t)^{-2}$ restricted to $\operatorname{supp}(d\alpha)$ is continuous and we can extend it to a function g which is continuous on $\Delta(d\alpha)$. We have

$$G_n(f,z) = \lambda_n^*(z)\,\frac{\gamma_{n-1}^2}{\gamma_n^2} \int_{-\infty}^{\infty} f(t)\,g(t)\,[p_{n-1}(t)\,p_n(z) - p_n(t)\,p_{n-1}(z)]^2\,d\alpha(t) =$$

$$= \frac{\gamma_{n-1}^2}{\gamma_n^2}\,[\lambda_n^*(z)\,p_n^2(z) \int_{-\infty}^{\infty} f(t)\,g(t)\,p_{n-1}^2(t)\,d\alpha(t) +$$

$$+ \lambda_n^*(z)\,p_n^2(z)\,\frac{p_{n-1}^2(z)}{p_n^2(z)} \int_{-\infty}^{\infty} f(t)\,g(t)\,p_n^2(t)\,d\alpha(t) -$$

$$- 2\lambda_n^*(z)\,p_n^2(z)\,\frac{p_{n-1}(z)}{p_n(z)} \int_{-\infty}^{\infty} f(t)\,g(t)\,p_{n-1}(t)\,p_n(t)\,d\alpha(t)]\ .$$

Now we apply Theorems 4.1.11, 4.1.13 and 4.2.13. We obtain

$$\lim_{n\to\infty} G_n(f,z) = \frac{1}{4\pi}\,[\rho^2(z) - 1][1 + \rho^{-2}(z)] \int_{-1}^{1} \frac{f(t)}{(z-t)^2\sqrt{1-t^2}}\,dt -$$

$$- \frac{1}{2\pi}\,[\rho^2(z) - 1]\,\rho^{-1}(z) \int_{-1}^{1} \frac{t\,f(t)}{(z-t)^2\sqrt{1-t^2}}\,dt\ .$$

But $[\rho^2(z) - 1][1 + \rho^{-2}(z)] = 4z\sqrt{z^2 - 1}$ and $[\rho^2(z) - 1]\ \rho^{-1}(z) = 2\sqrt{z^2 - 1}$. Hence

$$\lim_{n\to\infty} G_n(f,z) = \frac{\sqrt{z^2 - 1}}{\pi} \int_{-1}^{1} \frac{z - t}{(z - t)^2} \frac{f(t)}{\sqrt{1 - t^2}}\, dt\ .$$

Let us note that once (2) holds for continuous functions then it also holds for Riemann integrable functions if $x \in \mathbb{R} \setminus \mathrm{supp}(d\alpha)$. We shall not go into details since in the following we shall concentrate on convergence of $G_n(d\alpha,f,x)$ for $x \in \mathrm{supp}(d\alpha)$. The following theorem explains why we introduced the opertors $G_n(d\alpha,f)$ and why we should investigate them for as many weights α as possible.

<u>Theorem 3</u>. Let $g(\geq 0) \in L^1_{d\alpha}$. If α_g is a weight then

(4) $$\frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)} \leq G_n(d\alpha,g,x)$$

for $x \in \mathbb{R}$ and if $g^{-1} \in L^1_{d\alpha}$ then

(5) $$G_n^{-1}(d\alpha,g^{-1},x) \leq \frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)}$$

for $x \in \mathbb{R}$.

Before the proof let us remark that if $\mathrm{supp}(d\alpha)$ is compact and $g^{-1} \in L^1_{d\alpha}$ then α_g is a weight.

<u>Proof</u>. From

$$\lambda_n(d\alpha,x) = \min_{\pi_{n-1}} \pi_{n-1}^{-2}(x) \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\ d\alpha(t)$$

follows that

$$\lambda_n(d\alpha_g,x) \leq K_n^{-2}(d\alpha,x,x) \int_{-\infty}^{\infty} K_n^2(d\alpha,x,t)\ d\alpha_g(t) =$$

$$= \lambda_n^2(d\alpha,x) \int_{-\infty}^{\infty} K_n^2(d\alpha,x,t)\ g(t)\ d\alpha(t) = \lambda_n(d\alpha,x)\ G_n(d\alpha,g,x)\ .$$

On the other hand

$$\pi_{n-1}(x) = \int_{-\infty}^{\infty} K_n(d\alpha,x,t)\ \pi_{n-1}(t)\ d\alpha(t)\ .$$

Thus

$$\pi_{n-1}^2(x) \le \int_{-\infty}^{\infty} K_n^2(d\alpha,x,t)\, g^{-1}(t)\, d\alpha(t) \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, g(t)\, d\alpha(t) =$$

$$= \lambda_n^{-1}(d\alpha,x)\, G_n(d\alpha,g^{-1},x) \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, d\alpha_g(t) ,$$

that is

$$\lambda_n^{-1}(d\alpha_g, x) \le \lambda_n^{-1}(d\alpha,x)\, G_n(d\alpha,g^{-1},x) .$$

From Theorems 2 and 3 we could immediately obtain limit relations for

$$\frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)}$$

when both g and g^{-1} are bounded on $\operatorname{supp}(d\alpha)$. This condition however may be weakened by using the following two results.

Lemma 4. Let $\alpha \in M(0,1)$. Let $\{k_n\}$ be a sequence of natural integers which is bounded: $k_n \le k$ for every n . Then

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha,x)}{\lambda_{n+k_n}(d\alpha,x)} = 1$$

for every $x \in \operatorname{supp}(d\alpha)$ and the convergence is uniform for $x \in \Delta \subset (-1,1)$.

Proof. Since

$$1 \le \frac{\lambda_n(x)}{\lambda_{n+k_n}(x)} \le \frac{\lambda_n(x)}{\lambda_{n+k}(x)} = \prod_{j=n}^{n+k-1} \frac{\lambda_j(x)}{\lambda_{j+1}(x)}$$

we have to consider only $\lambda_n(x)/\lambda_{n+1}(x)$ which equals

$$1 + \lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) .$$

Now we apply Theorem 4.1.11.

Theorem 5. Let $g(\ge 0) \in L^1_{d\alpha}$. Let α_g be a weight. Then for every polynomial P_1 of degree m_1

$$\text{(6)} \qquad \frac{P_1^2(x)\, \lambda_n(d\alpha_g,x)}{\lambda_{n-m_1}(d\alpha,x)} \le G_{n-m_1}(d\alpha,gP_1^2,x) \qquad (n > m_1) .$$

If P_2 is a polynomial of degree m_2 such that $P_2^2\, g^{-1} \in L^1_{d\alpha}$ then

$$\text{(7)} \qquad P_2^2(x)\, G_{n+m_2}^{-1}(d\alpha,g^{-1}P_2^2,x) \le \frac{\lambda_n(d\alpha_g,x)}{\lambda_{n+m_2}(d\alpha,x)} .$$

Let us note that if $\operatorname{supp}(d\alpha)$ is compact and $P_2^2 g^{-1} \in L^1_{d\alpha}$ for some polynomial P_2 then α_g is a weight.

<u>Proof.</u> Inequality (6) follows from

$$\lambda_n(d\alpha_g,x) \leq P_1^{-2}(x)\ K_{n-m_1}^{-2}(d\alpha,x,x) \int_{-\infty}^{\infty} P_1^2(t)\ K_{n-m_1}^2(d\alpha,x,t)\ d\alpha_g(t)$$

whenever $n > m_1$. Further for every π_{n-1}

$$\pi_{n-1}(x)\ P_2(x) = \int_{-\infty}^{\infty} \pi_{n-1}(t)\ P_2(t)\ K_{n+m_2}(d\alpha,x,t)\ d\alpha(t)\ ,$$

that is

$$\pi_{n-1}^2(x)\ P_2^2(x) \leq \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\ g(t)\ d\alpha(t) \int_{-\infty}^{\infty} P_2^2(t)\ g^{-1}(t)\ K_{n+m_2}^2(d\alpha,x,t)\ d\alpha(t)$$

which implies (7).

<u>Theorem 6.</u> Let $\alpha \in M(0,1)$. Let $g(\geq 0) \in L^1_{d\alpha}$ and suppose that there exist two polynomials P_1 and P_2 such that gP_1^2 and $g^{-1}P_2^2$ are bounded on $\operatorname{supp}(d\alpha)$. Then

(i) for every $x \in \operatorname{supp}(d\alpha) \setminus [-1,1]$

(8)
$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)} = g(x)\ .$$

(ii) if $x \in [-1,1]$ and g is continuous at x then (8) holds.

(iii) if g is continuous on $\Delta \subset (-1,1)$ and $g(t) > 0$ for $t \in \Delta$ then (8) is satisfied uniformly for $x \in \Delta$.

<u>Proof.</u> Let first $x \in \operatorname{supp}(d\alpha) \setminus [-1,1]$. Then by Theorem 3.3.7, x is an isolated point of $\operatorname{supp}(d\alpha)$. Hence g must be finite at x and then we can suppose that P_1 does not vanish at x. We obtain from Theorems 2,5 and Lemma 4 that

(9)
$$\limsup_{n\to\infty} \frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)} \leq g(x)$$

which implies (8) if $g(x) = 0$. If $g(x) > 0$ then we can assume that P_2 does not vanish at x. Then by the same argument

(10) $$\liminf_{n\to\infty} \frac{\lambda_n(d\alpha_g,x)}{\lambda_n(d\alpha,x)} \geq g(x) .$$

Now let $x \in [-1,1]$. If g is continuous at x then $g(x) < \infty$ and thus we can suppose that $P_1(x) > 0$. Hence (9) holds again. If $g(x) = 0$ then (8) follows from (9). If $g(x) > 0$ then we can suppose $P_2(x) > 0$ which implies (10). If g is continuous on $\Delta \subset (-1,1)$ then the above argument can be used if only g is positive on Δ.

In order to illustrate the strength of this theorem we give a few examples.

Definition 7. u denotes the Jacobi weight, that is $\text{supp}(u) = [-1,1]$ and

$$u(x) \equiv u^{(a,b)}(x) = (1-x)^a(1+x)^b$$

for $-1 \leq x \leq 1$ where $a,b > -1$. Hence $u^{(-1/2,-1/2)} = v$.

In the following it will always be clear if a,b are related with $u^{(a,b)}$ or $M(a,b)$.

Example 8. Let α be the Chebyshev weight $(d\alpha(x) = v(x)dx)$ and let $w(x)\sqrt{1-x^2} = g(x)$ be positive and continuous on $[-1,1]$. Then $w = v_g$. Further, by easy calculation,

$$\lambda_n^{-1}(v,x) = \frac{1}{\pi}\left[n - \frac{1}{2} + \frac{1}{2}U_{2n-2}(x)\right]$$

where U_n is the Chebyshev polynomial of second kind. Hence for every $x \in [-1,1]$

$$\lim_{n\to\infty} \left[n - \frac{1}{2} + \frac{1}{2}U_{2n-2}(x)\right] \lambda_n(w,x) = \pi w(x)\sqrt{1-x^2} = \pi g(x)$$

(later we shall show that the convergence is uniform for $x \in [-1,1]$), in particular

$$\lim_{n\to\infty} n\, \lambda_n(w,\pm 1) = \frac{\pi}{2}\, g(\pm 1) .$$

Example 9. Let $w = \varphi u$ where $\varphi > 0$ is continuous on $[-1,1]$. Then

$$\lim_{n\to\infty} n\, \lambda_n(w,x) = \pi\sqrt{1-x^2}\, w(x)$$

uniformly for $x \in \Delta \subset (-1,1)$. This is, of course, not new. (See e.g. Freud.)

Example 10. Let $b > 0$, $\operatorname{supp}(w) = [-b,b]$ and $w > 0$ be continuous on $[-b,b]$. Then

$$w^b(x) = w(bx)$$

is a weight on $[-1,1]$. From the definition of Christoffel function we obtain

$$\lambda_n(w,x) = b\lambda_n(w^b, xb^{-1}) .$$

Hence

$$\lim_{n\to\infty} n\lambda_n(w,x) = \pi\sqrt{b^2 - x^2}\, w(x)$$

uniformly for $x \in \Delta \subset (-b,b)$.

Example 11. Let w be continuous on $[-1,1]$ and $w(x) > 0$ for $x \in (-1,1)$. Let $\epsilon > 0$, $\delta > 0$ and $\Delta = [-1+\delta, 1-\delta]$. Then

$$1_\Delta(x)\, w(x) \le w(x) \le w(x) + \epsilon$$

for $-1 \le x \le 1$. Hence

$$n\,\lambda_n(1_\Delta w,x) \le n\,\lambda_n(w,x) \le n\,\lambda_n(w+\epsilon,x) ,$$

where $\operatorname{supp}(w+\epsilon) = \operatorname{supp}(w)$ is assumed. Thus by the previous examples we have

$$w(x)\,\pi\sqrt{(1-\delta)^2 - x^2} \le \begin{matrix}\liminf\limits_{n\to\infty} \\ \limsup\limits_{n\to\infty}\end{matrix} n\lambda_n(w,x) \le \pi\sqrt{1-x^2}\,[w(x)+\epsilon]$$

uniformly for $x \in \Delta_1 \subset \Delta^0$. Since $\epsilon > 0$ and $\delta > 0$ are arbitrary we obtain

$$\lim_{n\to\infty} n\,\lambda_n(w,x) = \pi\sqrt{1-x^2}\, w(x)$$

uniformly for $x \in \Delta_1 \subset (-1,1)$. Recall that $w(\pm 1)$ may vanish.

Definition 12. Let $a,b \in \mathbb{R}$ with $a > |b|$. The Pollaczek weight $w^{(a,b)}$ is defined by $\operatorname{supp}(w^{(a,b)}) = [-1,1]$ and

$$w^{(a,b)}(\cos\theta) = 2\exp\{\frac{\theta}{\sin\theta}(a\cos\theta + b)\}[1 + \exp[\frac{\pi}{\sin\theta}(a\cos\theta + b)]]^{-1} ,$$

$\theta \in [0,\pi]$, $x = \cos\theta$.

Properties 13.

(i) We have

$$\alpha_n(w^{(a,b)}) = -\frac{b}{2n+1+a} = -\frac{b}{2n} + O(\frac{1}{n^2})$$

for $n = 0,1,2,\ldots$ and

$$\frac{\gamma_{n-1}(w^{(a,b)})}{\gamma_n(w^{(a,b)})} = \frac{n}{2\sqrt{(n+\frac{a}{2})^2 - \frac{1}{4}}} = \frac{1}{2} - \frac{a}{4n} + O(\frac{1}{n^2})$$

for $n = 1,2,\ldots$ (See Szegö, Appendix)*.

(ii) $w^{(a,b)} \in M(0,1)$ but $w^{(a,b)} \notin S$.

(iii) $\{p_n^2(w^{a,b}),x)\}$ is uniformly bounded for $x \in \Delta \subset (-1,1)$. To prove this use Theorem 3.1.11 and Example 11.

(iv) Let

$$\varphi(x) = w^{(a,b)}(x) \exp\{\frac{(a+b)\pi}{\sqrt{2}\sqrt{1-x}} + \frac{(a-b)\pi}{\sqrt{2}\sqrt{1+x}}\} .$$

Then φ is continuous and positive on $[-1,1]$.

(v) Let $w^{(a)} = w^{(a,0)}$. Then $w^{(a)}$ is even and

$$\lim_{n\to\infty} \frac{p_n^2(w^{(a)},1)}{\sqrt{n}\exp[4\sqrt{a}\sqrt{n}\ ]} = \frac{1}{4\pi} e^{-a} \frac{1}{\sqrt{a}} .$$

(See Szegö, Appendix.)

(vi)

$$\lim_{n\to\infty} \frac{\gamma_n(w^{(a,b)})}{2^n n^{a/2}} = \Gamma(\frac{a+1}{2})^{-1}$$

where Γ denotes the Γ function of Euler. (See Szegö, Appendix.)

(vii)

$$\lim_{n\to\infty} \lambda_n(w^{(a)},x)n = \pi\sqrt{1-x^2}\, w^{(a)}(x)$$

uniformly for $x \in \Delta \subset (-1,1)$ and

$$\lim_{n\to\infty} \lambda_n(w^{(a)},\pm 1)n \exp(4\sqrt{a}\sqrt{n}) = 2\pi e^a .$$

The first limit relation follows from Example 11, the second one from (v) by a somewhat tedious calculation.

(viii) Let $z \in \mathbb{C}\setminus[-1,1]$. Then

*Let us note that the formula (1.7) in the Appendix of Szegö's book is not quite correct, it should be written

$$nP_n(x;a,b) = [(2n-1+a)x+b]P_{n-1}(x;a,b) - (n-1)P_{n-2}(x;a,b),\ n=2,3,4,\ldots .$$

$$\lim_{n\to\infty} p_n(w^{(a,b)},z)\ \rho^{-n}(z)\, n^{-\frac{az+b}{2\sqrt{z^2-1}}} = \Gamma\left(\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}\right)^{-1}\left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}}$$

(See Szegö, Appendix.)

(ix) If $z \in \mathbb{C}\setminus[-1,1]$ then

$$\lim_{n\to\infty} \lambda_n^*(w^{(a,b)},z)^{-1}\ \rho^{-2n}(z)\, n^{-\frac{az+b}{\sqrt{z^2-1}}} =$$

$$= \Gamma\left(\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}\right)^{-2}\frac{1}{4(z^2-1)}\left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{\frac{az+b}{\sqrt{z^2-1}}}$$

and

$$\lim_{n\to\infty} \lambda_n(w^{(a,b)},z)^{-1}\ |\rho(z)|^{-2n}\ \left|n^{\frac{az+b}{2\sqrt{z^2-1}}}\right|^{-2} =$$

$$= \frac{1}{|\rho(z)|^2-1}\left|\Gamma\left(\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}\right)\right|^{-2}\left|\left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}}\right|^2.$$

These follow from (ii), (viii) and Theorem 4.1.11.

Example 14. Let w be defined by $\operatorname{supp}(w) = [-1,1]$ and

(11) $$w(x) = \exp\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

for $-1 \le x \le 1$. By Property 13(iv)

$$g(x) = w(x)\ w^{(\frac{1}{\pi})}(x)^{-1}$$

is positive and continuous on $[-1,1]$. We have $g(\pm 1) = \frac{1}{2}\exp(-\frac{1}{\pi})$.
Hence by Properties (ii), (vii) and Theorem 6

$$\lim_{n\to\infty} \lambda_n(w,x)\, n = \pi\sqrt{1-x^2}\ w(x)$$

uniformly for $x \in \Delta \subset (-1,1)$ and

$$\lim_{n\to\infty} \lambda_n(w,\pm 1)\, n \exp\left\{4\sqrt{\frac{n}{\pi}}\right\} = \pi .$$

By Theorem 6.1.27, $w \in M(0,1)$ since $w^{(\frac{1}{\pi})} \in M(0,1)$. By Property 13(vi) and Theorem 6.1.26

$$\lim_{n\to\infty} \frac{\gamma_n(w)}{2^n n^{\frac{1}{2\pi}}} = \Gamma\left(\frac{\pi+1}{2\pi}\right) D\left(\frac{w}{w^{(\frac{1}{\pi})}}, 0\right)^{-1}$$

and by Property 13(ix) and Theorem 6.1.25, for every $z \in \mathbb{C} \setminus [-1,1]$

$$\lim_{n\to\infty} \lambda_n(w,z) \, |\rho(z)|^{2n} \, \left|\frac{z}{n^{2\pi\sqrt{z^2-1}}}\right|^2 =$$

$$= (|\rho(z)|^2 - 1) \, \left|\Gamma\left(\frac{1}{2} + \frac{z}{2\pi\sqrt{z^2-1}}\right)\right|^2 \, \left|\left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{1-\frac{z}{\pi\sqrt{z^2-1}}}\right| \, \left|D\left(\frac{w}{w^{(\frac{1}{\pi})}}, \rho(z)^{-1}\right)\right|^2 .$$

Further by Theorem 6.1.29 and Property 13(viii) for every $z \in \mathbb{C} \setminus [-1,1]$

$$\lim_{n\to\infty} p_n(w,z) \, \rho(z)^{-n} \, n^{\frac{-zi}{2\pi\sqrt{1-z^2}}} =$$

$$= \Gamma\left(\frac{1}{2} + \frac{z}{2\pi\sqrt{z^2-1}}\right)^{-1} \left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}+\frac{z}{2\pi\sqrt{z^2-1}}} D\left(\frac{w}{w^{(\frac{1}{\pi})}}, \rho(z)^{-1}\right)^{-1}$$

Using Example 14 and Theorems 6.1.25-27 we immediately obtain

<u>Theorem 15</u>. Let w be defined by (11). Let $g(\geq 0) \in L^1_w$ be equivalent to a strictly positive and Riemann integrable function. Then every result in Example 14 remains true if we replace w by $w_g = gw$, in particular, $w_g \in M(0,1)$.

Now we shall investigate $G_n(u,f)$ where $u = u^{(a,b)}$ is a Jacobi weight.

<u>Lemma 16</u>. There exists a constant $C = C(u)$ such that

$$p_n^2(u,x) \leq C\left[\sqrt{1-x} + \frac{1}{n}\right]^{-2a-1} \left[\sqrt{1+x} + \frac{1}{n}\right]^{-2b-1}$$

for $|x| \leq 1$, $n = 1,2,\ldots$.

<u>Proof</u>. See Szegő, §7.32.

<u>Lemma 17</u>. For $m = n-1, n$ and $n = 1,2,\ldots$

$$\max_{|x|\leq 1} \lambda_n(u,x) \, p_m^2(u,x) = O\left(\frac{1}{n}\right) .$$

Proof. Apply Lemmas 16 and 6.3.5.

Let us note that Theorem 3.1.11 and Lemma 4 give

$$\max_{x \in \Delta \subset (-1,1)} \lambda_n(u,x)\, p_m^2(u,x) = O(\tfrac{1}{n}) \qquad (m = n-1, n)\ .$$

Lemma 18. Let $g \in L^1 \cap L_u^1$. Then

$$\lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} |g(t)|\, p_n^2(u,t)\, u(t)\, dt = 0\ .$$

Proof. Let us consider $\int_0^1$. If $a < 0$ then use Lemma 16. Let $a \geq 0$. Fix $\epsilon > 0$. Then

$$\frac{1}{n}\int_0^1 |g(t)| p_n^2(u,t)\, u(t) dt \leq \frac{1}{n} \max_{0\leq t\leq 1-\epsilon} p_n^2(u,t) \int_{-1}^{1} |g(t)|\, u(t) dt + C\int_{1-\epsilon}^{\epsilon} |g(t)| dt$$

again by Lemma 16. First let $n\to\infty$ and then $\epsilon\to 0$.

Theorem 19. Let $g \in L^1 \cap L_u^1$. If g is continuous on a closed set $\mathfrak{B} \subset [-1,1]$ then

$$\lim_{n\to\infty} G_n(u,g,x) = g(x)$$

uniformly for $x \in \mathfrak{B}$.

Proof. We have to show that

$$\lim_{n\to\infty} \max_{|x|\leq 1} \{\lambda_n(u,x) \int_{-1}^{1} (x - t)^2\, K_n^2(u,x,t)\, |g(t)|\, u(t) dt\} = 0$$

and then the machinery of positive operators can be applied. But this is so by Lemmas 17 and 18.

Theorem 20. Let $g \in L^1 \cap L_u^1$. Let $x \in (-1,1)$ be a Lebesgue point of g. Then

$$\lim_{n\to\infty} G_n(u,g,x) = g(x)\ .$$

Proof. Let $x \in (-1,1)$ and $\epsilon > 0$ be fixed. If ϵ is small enough then

$|G_n(u,g,x) - g(x)| \leq$

$$\leq \lambda_n(u,x) \int\limits_{|x-t|<\frac{1}{n}} + \int\limits_{\frac{1}{n}\leq|x-t|\leq\epsilon} + \int\limits_{|x-t|>\epsilon} |g(t)-g(x)|\, K_n^2(u,x,t)\, u(t)dt \; .$$

By Lemmas 16-18

$$(12) \qquad \lambda_n(u,x) \int\limits_{|x-t|<\frac{1}{n}} < Cn \int\limits_{|x-t|<\frac{1}{n}} |g(t) - g(x)|dt \; ,$$

$$(13) \qquad \lambda_n(u,x) \int\limits_{\frac{1}{n}\leq|x-t|\leq\epsilon} \leq \frac{c}{n} \int\limits_{\frac{1}{n}\leq|x-t|\leq\epsilon} \frac{|g(t) - g(x)|}{(x-t)^2}\, dt$$

and the third term converges to 0 when $n\to\infty$. We have

$$\lim_{n\to\infty} (\text{right side of } (12)) = 0$$

because x is a Lebesgue point of g. To estimate the right side of (13) we integrate by parts and remember that $\epsilon > 0$ is arbitrary.

Lemma 21. If $a,b > -1$ then

$$\lim_{n\to\infty} n^{2a+2}\, \lambda_n(u,1) = (a+1)\, 2^{a+b+1}\, \Gamma(a+1)^2$$

and

$$\lim_{n\to\infty} n^{2b+2}\, \lambda_n(u,-1) = (b+1)\, 2^{a+b+1}\, \Gamma(b+1)^2 \; .$$

Proof. Szegö, §4.5 and some calculation.

Theorem 22. Let $\operatorname{supp}(w) = [-1,1]$ and suppose that there exists a polynomial P such that $w^{-1} P^2 \in L^1(-1,1)$. Then for almost every $x \in [-1,1]$

$$(14) \qquad \lim_{n\to\infty} n\lambda_n(w,x) = \pi w(x)\sqrt{1-x^2} \; .$$

If w is positive and continuous on a closed set $\mathfrak{M} \subset (-1,1)$ then (14) holds uniformly for $x \in \mathfrak{M}$. If w is continuous at $x \in (-1,1)$ then (14) holds. If there exists a Jacobi weight u such that w/u is positive and continuous on $\Delta \subset [-1,1]$ then

$$\lim_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(u,x)} = \frac{w(x)}{u(x)}$$

uniformly for $x \in \Delta$. If w/u is positive and continuous at 1 then

$$\lim_{n\to\infty} n^{2a+2} \lambda_n(w,1) = \frac{w(1)}{u(1)} (a+1)\, 2^{a+b+1}\, \Gamma(a+1)^2 .$$

If w/u is positive and continuous at -1 then

$$\lim_{n\to\infty} n^{2b+2} \lambda_n(w,-1) = \frac{w(-1)}{u(-1)} (b+1)\, 2^{a+b+1}\, \Gamma(b+1)^2 .$$

Proof. Put $g = w$ and α = Legendre weight. Then $w(x)dx = d\alpha_g(x)$. Therefore by Theorem 3

$$\frac{\lambda_n(w,x)}{\lambda_n(d\alpha,x)} \le G_n(d\alpha,g,x) .$$

Applying Theorem 20 we obtain

$$\limsup_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(d\alpha,x)} \le g(x) = w(x)$$

for almost every $x \in (-1,1)$, that is by Example 9 ($\varphi = 1$, u = Legendre weight)

$$\limsup_{n\to\infty} n\, \lambda_n(w,x) \le \pi\, w(x) \sqrt{1-x^2}$$

for almost every $x \in (-1,1)$. By the conditions of the theorem $w^{-1}P^2 \in L^1$ with some fixed polynomial P . Let $m = \deg(P)$. Then by Theorem 5

$$P^2(x)\, G^{-1}_{n+m}(d\alpha, g^{-1}P^2, x) \le \frac{\lambda_n(w,x)}{\lambda_{n+m}(d\alpha,x)}$$

where $g = w$ and α = Legendre weight. Using again Theorem 20 we get

$$w(x) = g(x) \le \liminf_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_{n+m}(d\alpha,x)}$$

for almost every $x \in (-1,1)$. Thus by Lemma 4

$$w(x) \le \liminf_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(d\alpha,x)}$$

for almost every $x \in (-1,1)$. Applying Example 9 we see that for almost every $x \in (-1,1)$ the inequality

$$\pi\, w(x) \sqrt{1-x^2} \le \liminf_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(d\alpha,x)}$$

holds. Therefore (14) is satisfied whenever $w^{-1}P^2 \in L^1$. The statement about points of continuity can be proved in the same way. The only difference is that Theorem 19 should be used instead of Theorem 20. Now we will show that if $w^{-1}P^2 \in L^1$ with a suitable polynomial P and there exist a $\Delta \subset [-1,1]$ and a Jacobi weight u such that w/u is positive and continuous on Δ then

$$\lim_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(u,x)} = \frac{w(x)}{u(x)}$$

holds uniformly for $x \in \Delta$. If $\Delta \subset (-1,1)$ then this follows immediately from Example 9 and the first part of the part of the theorem. Otherwise we can assume without loss of generality that $\Delta = [\delta,1]$, $(0 < \delta < 1)$. We can also assume that u is of the form $u^{(a,0)}$, that is $u(x) = (1-x)^a$. Let $g = w/u$. Then by the conditions $g \in L_u^1 \cap L^1$ and $P_1^2 g^{-1} \in L_u^1 \cap L^1$ with some polynomial P_1. Actually, we can put $P_1(x) = (1-x^2)^2 P(x)$. By Theorem 19

$$\lim_{n\to\infty} G_n(u,g,x) = g(x)$$

and

$$\lim_{n\to\infty} G_n(u,P_1^2 g^{-1},x) = P_1^2(x) g^{-1}(x)$$

uniformly for $x \in [\delta,1]$. Consequently by Theorem 3 and 5

$$\lim_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(u,x)} = \frac{w(x)}{u(x)}$$

holds uniformly for $x \in [\delta,1]$. Here we also had to use that fact that for fixed m

$$\lim_{n\to\infty} \frac{\lambda_{n+m}(u,x)}{\lambda_n(u,x)} = 1$$

uniformly for $-1 \le x \le 1$. This follows from Lemma 17. Finally, using Lemma 21, the last part of the theorem can be proved in a similar way.

<u>Corollary 23</u>. Let $\operatorname{supp}(w) \subset [-1,1]$. Then for almost every $x \in (-1,1)$

$$\limsup_{n\to\infty} n\,\lambda_n(w,x) \le \pi\sqrt{1-x^2}\, w(x) .$$

<u>Proof</u>. Let $\epsilon > 0$. Then $\lambda_n(w,x) \le \lambda_n(w + \epsilon 1_{[-1,1]},x)$ and $(w+\epsilon 1_{[-1,1]})^{-1} \in L^1$.

<u>Corollary 24</u>. If $\frac{1}{\alpha'} \in L^1(\Delta)$ then

$$\limsup_{n\to\infty} \frac{1}{n\,\lambda_n(d\alpha,x)} < \infty$$

for almost every $x \in \Delta$.

Proof. $\lambda_n(d\alpha,x) \geq \lambda_n(\alpha',x) \geq \lambda_n(\alpha' 1_\Delta,x)$ and transform Δ to $[-1,1]$.

In the following, we shall improve both corollaries. Corollary 24 is a very strong result. To see this, compare Corollary 24 with Freud's result (See Freud, §IV.6.)

Theorem. Let $\operatorname{supp}(w) \subset [-1,1]$ and

$$\int_0^\pi \frac{|w(\cos(\theta+h))\,\sin(\theta+h) - w(\cos\theta)\,\sin\theta\,|}{w(\cos\theta)\,\sin\theta}\,d\theta = O(\log^{-\epsilon}\frac{1}{|h|})$$

for h small with $\epsilon > 1$. Then for almost every $x \in [-1,1]$

$$\limsup_{n\to\infty} \frac{1}{n\,\lambda_n(w,x)} < \infty .$$

Let us mention two applications of Corollary 24:

Theorem 25. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$ and $\frac{1}{\alpha'} \in L^1(\Delta)$ where $\Delta \subset [a-b,a+b]$. If

$$\limsup_{n\to\infty} \sum_{j=n}^{2n} j\, C_j^{a,b}(d\alpha)^2 < \infty$$

(See Definition 3.1.4.) then the sequence $\{p_n^2(d\alpha,x)\}$ is bounded for almost every $x \in \Delta$.

Proof. Theorem 3.1.11 and Corollary 24.

Later we will see that in both Theorems 25 and 26 the condition $\frac{1}{\alpha'} \in L^1(\Delta)$ may be weakened to $[\alpha']^{-\epsilon} \in L^1(\Delta)$ for some $\epsilon > 0$.

Now we will consider $G_n(d\alpha,f)$ for weights α which are less nice than the Jacobi weights. In the following, τ, τ_1 etc. will denote closed intervals. Recall that τ^0 denotes the interior of τ.

Theorem 27. Let $\alpha \in M(0,1)$, $\tau \subset (-1,1)$. Let $\alpha'(t) \geq c > 0$ for almost every $t \in \tau$. Let the sequence $\{p_n^2(d\alpha,t)\}$ be uniformly bounded on every $\tau_1 \subset \tau^0$. Let $f \in L^1_{d\alpha}$ and π be a polynomial vanishing at the endpoints of τ. Let $|f(t)\ \pi(t)| \le M < \infty$ for $d\alpha$ almost every $t \in \mathrm{supp}(d\alpha) \setminus \tau$. Then for almost every $x \in \tau$

$$\lim_{n\to\infty} G_n(d\alpha, f\pi, x) = f(x)\ \pi(x) . \tag{15}$$

<u>Proof.</u> Let $x \in \tau^0$ be a $d\alpha$ Lebesgue point of $f\pi$, that is let

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t)\ \pi(t) - f(x)\ \pi(x)|\ d\alpha(t) = 0 .$$

It is well known that almost every $x \in \tau^0$ is a $d\alpha$ Lebesgue point of $f\pi$. (See Freud, §IV.2.) First we will show that

$$\exists\ \alpha'(x) < \infty \Rightarrow \limsup_{n\to\infty} n\ \lambda_n(d\alpha,x) < \infty . \tag{16}$$

Let v_Δ be the Chebyshev weight corresponding to $\Delta(d\alpha)$. Then

$$\lambda_n(d\alpha,x) \le K_n^{-2}(v_\Delta,x,x) \int_{-\infty}^{\infty} K_n^2(v_\Delta,x,t)\ d\alpha(t) .$$

Since $x \in \tau^0 \subset \tau \subset \Delta(d\alpha)^0$ we have by Example 8

$$n\ \lambda_n(d\alpha,x) \le c\ n[\alpha(x + \tfrac{1}{n}) - \alpha(x - \tfrac{1}{n})] + \frac{c}{n} \int_{|x-t|\ge\frac{1}{n}} \frac{d\alpha(t)}{(x-t)^2}$$

and integrating the integral by parts we obtain (16). Since $\alpha' \in L^1(-1,1)$, $\alpha'(x) < \infty$ for almost every $x \in \tau^0$. Let now $x \in \tau^0$ be $d\alpha$ Lebesgue point of $f\pi$ and let $\alpha'(x) < \infty$. We will prove (15) for such points x. Let $\varepsilon > 0$ be so small that $x \pm \varepsilon \in \tau^0$. If n is large then

$$G_n(d\alpha,f\pi,x) - f(x)\ \pi(x) = \lambda_n(d\alpha,x) \left\{ \int_{|x-t|\le\frac{1}{n}} + \int_{\frac{1}{n}<|x-t|\le\varepsilon} + \int_{\substack{|x-t|>\varepsilon \\ t\in\tau}} + \int_{t\notin\tau} \right\}$$

$$[f(t)\ \pi(t) - f(x)\ \pi(x)]\ K_n^2(d\alpha,x,t)\ d\alpha(t) .$$

By (16)

$$\lim_{n\to\infty} \lambda_n(d\alpha,x) \int_{|x-t|\le\frac{1}{n}} = 0 .$$

We have further by (16)

$$\lambda_n(d\alpha,x)\ \Big|\int_{\frac{1}{n}\le|x-t|\le\epsilon}\Big| \le \frac{c}{n}\int_{\frac{1}{n}\le|x-t|\le\epsilon}|f(t)\,\pi(t)-f(x)\,\pi(x)|\,\frac{d\alpha(t)}{(x-t)^2}\ .$$

Integrating by parts we obtain

$$\limsup_{n\to\infty}\lambda_n(d\alpha,x)\ \Big|\int_{\frac{1}{n}\le|x-t|\le\epsilon}\Big| \le C\sup_{|h|\le\epsilon}\frac{1}{h}\int_x^{x+h}|f(t)\,\pi(t)-f(x)\,\pi(x)|\,d\alpha(t)\ .$$

Now consider $\displaystyle\int_{\substack{|x-t|>\epsilon\\ t\in\tau}}$. We have

$$\lambda_n(d\alpha,x)\Big|\int_{\substack{|x-t|>\epsilon\\ t\in\tau}}\Big| \le \frac{c}{n\epsilon^2}\int_{t\in\tau}[p_n^2(d\alpha,t)+p_{n-1}^2(d\alpha,t)][|f(t)\pi(t)|+|f(x)\pi(x)|]d\alpha(t) \le$$

$$\le \frac{c}{n\epsilon^2}|f(x)\pi(x)| + \sum_{k=n-1}^{n}\frac{c}{n\epsilon^2}\int_{t\in\tau}p_k^2(d\alpha,t)|f(t)\pi(t)|d\alpha(t)\ .$$

Here we cannot use simple estimates since the sequence $\{p_k^2(d\alpha,t)\}$ is uniformly bounded only for $x\in\tau_1\subset\tau^0$ but not for $x\in\tau$. By Theorem 4.1.11

$$\lim_{k\to\infty}\max_{t\in\tau}\ \lambda_k(d\alpha,t)\ p_k^2(d\alpha,t) = 0\ .$$

Hence for $k = n-1,n$

$$\frac{1}{n}\int_{t\in\tau}p_k^2(d\alpha,t)\ |f(t)\,\pi(t)|\ d\alpha(t) = o(1)\int_{t\in\tau}\frac{1}{k\,\lambda_k(d\alpha,t)}\ |f(t)\,\pi(t)|\ d\alpha(t)\ .$$

Let v_τ denote the Chebyshev weight corresponding to τ . Then from $\alpha'(t)\ge c>0$ for almost every $t\in\tau$ follows that

$$\lambda_n(d\alpha,t)\ge\frac{c}{n}\,v_\tau(t)^{-1}$$

for $t\in\tau$ (See Freud, §III.3.) Hence, for $k = n-1,n$

$$\frac{1}{n}\int_{t\in\tau}p_k^2(d\alpha,t)\ |f(t)\,\pi(t)|\ d\alpha(t) = o(1)\int_{t\in\tau}v_\tau(t)\ |f(t)\,\pi(t)|\ d\alpha(t)\ .$$

Let us recall that π vanishes at the endpoints of τ . Thus

$$\lim_{n\to\infty}\lambda_n(d\alpha,x)\int_{\substack{|x-t|>\epsilon\\ t\in\tau}} = 0\ .$$

Finally, by the conditions and (16)

$$\lambda_n(d\alpha,x)\int_{t\notin\tau}\leq\frac{c}{n}\,.$$

Consequently (15) holds for almost every $x\in\tau$.

Theorem 28. Let $\alpha\in M(0,1)$, $\tau\subset(-1,1)$, $\alpha'(t)\geq c>0$ for almost every $t\in\tau$, $\{p_n^2(d\alpha,t)\}$ be uniformly bounded on each $\tau_1\subset\tau^0$. Let $f\in L^1_{d\alpha}$, π be a polynomial vanishing at the endpoints of τ and let $|f(t)\,\pi(t)|\leq M<\infty$ for $d\alpha$ almost every $t\in\operatorname{supp}(d\alpha)\setminus\tau$. If f is continuous at $x\in\tau^0$ and $|\alpha(x)\;\alpha(t)|\leq K|x-t|$ for $|x-t|$ small then (15) holds. If f is continuous on $\tau_1\subset\tau^0$ and $\alpha\in\operatorname{Lip}1$ on $\tau_2(\tau_1\subset\tau_2^0)$ then (15) is satisfied uniformly for $x\in\tau_1$.

Proof. Repeat the proof of Theorem 27 with some obvious modifications.

Lemma 29. Let $\alpha\in S$. Let $\tau\subset(-1,1)$ and α be absolutely continuous on τ with $\alpha'(t)\equiv1$ for $t\in\tau$. Then the sequence $\{p_n^2(d\alpha,x)\}$ is uniformly bounded on each $\tau_1\subset\tau^0$ and

$$\lim_{n\to\infty}n\,\lambda_n(d\alpha,x)=\pi\sqrt{1-x^2}$$

uniformly for $x\in\tau_1\subset\tau^0$. Moreover, $\alpha\in M(0,1)$.

Proof. See Geronimus, §5.4.

Lemma 30. Let $\beta\in S$. Let $\tau\subset(-1,1)$ and β be absolutely continuous on τ with $1/\beta'\in L^1(\tau)$. Then for almost every $x\in\tau$

$$\lim_{n\to\infty}n\,\lambda_n(d\beta,x)=\pi\,\beta'(x)\sqrt{1-x^2}\,.$$

Proof. Let us define α and g by

$$d\alpha(x)=\begin{cases}d\beta(x) & \text{for}\quad x\in[-1,1]\setminus\tau\\ dx & \text{for}\quad x\in\tau\end{cases}$$

with $\operatorname{supp}(d\alpha)=[-1,1]$ and

$$g(x) = \begin{cases} 1 & \text{for } x \in [-1,1] \setminus \tau \\ \beta'(x) & \text{for } x \in \tau \end{cases}$$

Then $\alpha \in S$, $g(x)^{\pm 1} = 1 < \infty$ for $x \in [-1,1] \setminus \tau$ and $\beta = \alpha_g$. Further α satisfies the conditions of Lemma 29 and consequently α satisfies also the conditions of Theorem 27. Let us put $P_1 = v_\tau^{-2}$ where v_τ is the Chebyshev weight corresponding to τ. Then by Theorem 5

$$\frac{v_\tau^{-4}(x)\ \lambda_n(d\beta,x)}{\lambda_{n-2}(d\alpha,x)} \le G_{n-2}(d\alpha, gv_\tau^{-4}, x) .$$

Since v_τ^{-4} vanishes at the endpoints of τ we obtain from Lemmas 4, 29 and Theorem 27

$$\limsup_{n\to\infty} n\ \lambda_n(d\beta,x) \le \pi\sqrt{1 - x^2}\ \beta'(x)$$

for almost every $x \in \tau$.

On the other hand, putting $P_2 = v_\tau^{-2}$ and using Theorem 5

$$v_\tau^{-4}(x)\ G_{n+2}^{-1}(d\alpha, g^{-1}v_\tau^{-4}) \le \frac{\lambda_n(d\beta,x)}{\lambda_{n+2}(d\alpha,x)} .$$

Thus by the same arguments $(g^{-1} \in L^1_{d\alpha})$.

$$\pi\sqrt{1 - x^2}\ \beta'(x) \le \liminf_{n\to\infty} \frac{\lambda_n(d\beta,x)}{\lambda_n(d\alpha,x)}$$

Lemma 31. Let α be an arbitrary weight. Then for almost every $x \in [-1,1]$

$$\lim_{n\to\infty} \lambda_n(v,x) \int_{-1}^{1} K_n^2(v,x,t)\ d[\alpha_s(t) + \alpha_j(t)] = 0 .$$

Proof. We have for almost every $x \in [-1,1]$

$$\lim_{h\to 0} \frac{1}{h} \int_{x-h}^{x+h} |d[\alpha_s(t) + \alpha_j(t)]| = \lim_{h\to 0} \frac{1}{h} \int_{x-h}^{x+h} d[\alpha_s(t) + \alpha_j(t)] = 0$$

and we can use standard argument. In this case the standard argument is the following. We replace

$$\lambda_n(v,x)\ K_n^2(v,x,t)$$

by

$$\text{const}\ \min\left(n, \frac{1}{n(x - t)^2}\right) .$$

Then we fix $\epsilon > 0$ and write

$$\int_{-1}^{1} = \int_{|x-t| \le \frac{1}{n}} + \int_{\frac{1}{n} \le |x-t| \le \epsilon} + \int_{|x-t| \ge \epsilon} .$$

Here the first and third integrals on the right side obviously converge to 0 when $n \to \infty$ for almost every $x \in [-1,1]$. To extimate the second integral we use integration by parts. Finally we let $\epsilon \to 0$.

<u>Lemma 32</u>. Let α be an arbitrary weight. Then for almost every $x \in [-1,1]$

$$\lim_{n\to\infty} \lambda_n(v,x) \int_{-1}^{1} K_n^2(v,x,t)\, d\alpha(t) = \sqrt{1-x^2}\, \alpha'(x) .$$

<u>Proof</u>. Since $d\alpha(t) = \alpha_a'(t)dt + d[\alpha_s(t) + \alpha_j(t)] = g(t)v(t)dt + d[\alpha_s(t) + \alpha_j(t)]$ where $g = \alpha_a v^{-1}$ the lemma follows from Theorem 20 and Lemma 31.

<u>Theorem 33</u>. If $\operatorname{supp}(d\alpha) \subset [-1,1]$ then

$$\limsup_{n\to\infty} n\, \lambda_n(d\alpha,x) \le \pi\, \alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in [-1,1]$.

<u>Proof</u>. Use Lemma 32 and the inequality

$$\frac{\lambda_n(d\alpha,x)}{\lambda_n(v,x)} \le \lambda_n(v,x) \int_{-1}^{1} K_n^2(v,x,t)\, d\alpha(t)$$

which follows from $\operatorname{supp}(d\alpha) \subset [-1,1]$.

Now we can prove the following

<u>Theorem 34</u>. Let $\alpha \in S$, $\tau \subset [-1,1]$. If $\frac{1}{\alpha'} \in L^1(\tau)$ then

$$\lim_{n\to\infty} n\, \lambda_n(d\alpha,x) = \pi\, \alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in \tau$.

<u>Proof</u>. By Theorem 33 we have to show that

(17) $$\liminf_{n\to\infty} n\, \lambda_n(d\alpha,x) \ge \pi\, \alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in \tau$. We can assume $\tau \subset (-1,1)$. Since

$$\frac{\lambda_n(d\beta,x)}{\lambda_n(d\beta_\tau,x)} = \beta'(x) + O(1)\cdot \begin{cases} \frac{1}{n}\int_{-1}^{1} \frac{\omega(t)}{t^2}\,dt & (\beta' \in A^{\omega}_{\tau_1}) \\ \frac{1}{n}\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\,dt & (\beta' \in B^{\omega}_{\tau_1}) \end{cases}$$

uniformly for $x \in \tau_1$.

Proof. If τ is small then $d(\beta_\tau)_g = d\beta$, $g^{\pm 1}$ is bounded on $[-1,1]$, $g^{\pm 1} \in A^{\omega}_x(B^{\omega}_x)$ or $g^{\pm 1} \in A^{\omega}_{\tau_1}(B^{\omega}_{\tau_1})$ respectively by Remark 41. Further β_τ satisfies the conditions of Lemma 29 and consequently β_τ satisfies the conditions of Theorems 38 and 40. Finally, apply Theorem 3.

Lemma 44. If $\tau \subset [-1,1]$ then

$$\lambda_n(v,x)\int_{-1}^{1} K_n^2(v,x,t)\,d\beta_\tau(t) \le \sqrt{1-x^2} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$ and consequently if $\operatorname{supp}(d\beta) \subset [-1,1]$ then

$$n\,\lambda_n(d\beta_\tau,x) \le \pi\sqrt{1-x^2} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$.

Proof. See Freud, §V.6.

Lemma 45. Let $\operatorname{supp}(d\beta) \subset [-1,1]$, $\tau \subset (-1,1)$. Let exist a polynomial π such that $\pi^2/\beta'_\tau \in L^1(-1,1)$. Then

$$\frac{1}{n\,\lambda_n(d\beta_\tau,x)} \le \frac{1}{\pi\sqrt{1-x^2}} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$.

Proof. Let us consider (7) in Theorem 5. We put there α = Chebyshev weight, $g = \beta'_\tau/v$ so that $d\alpha_g(x) = \beta'_\tau(x)dx$. Let $P_2 = v^{-2}\pi$. We obtain

$$\frac{v^{-4}(x)\,\pi^2(x)\,\lambda_{n+m}(v,x)}{\lambda_n(\beta'_\tau,x)} \le G_{n+m}(v,\frac{v^{-3}\pi^2}{\beta'_\tau},x)$$

where $m = \deg\pi + 2$ and $v^{-3}\pi^2/\beta'_\tau \in L^1_v$. Since $\beta'_\tau(t) = 1$ for $t \in \tau$ we can suppose that π has no zeros in τ^0 . Hence for $x \in \tau^0$

$$\lambda_n^{-1}(d\beta_\tau,x) \le \pi^{-2}(x)\ v^4(x)\ \lambda_{n+m}^{-1}(v,x)\ G_{n+m}(v,\frac{v^{-3}\pi^2}{\beta_\tau'},x)\ .$$

Now we should apply Theorem 40 with α = Chebyshev weight, $f = v^{-3}\pi^2/\beta_\tau' \in B_{\tau_1}^{\omega}$ if $\tau_1 \subset \tau^0$ and $\omega(t) \equiv t$, but we cannot do this directly since in our case f is not bounded on $\operatorname{supp}(v)$. This small problem can be avoided by remarking that the Chebyshev polynomials are uniformly bounded on $[-1,1]$ and thus

$$\int_{|x-t|>\epsilon} |f(t)|\ K_n^2(v,x,t)\ v(t)dt \le \frac{c}{\epsilon^2}\int_{-1}^{1} |f(t)|\ v(t)dt$$

which in our case is finite. Hence

$$G_{n+m}(v,\frac{v^{-3}\pi^2}{\beta_\tau'},x) = \frac{v^{-3}(x)\ \pi^2(x)}{\beta_\tau'(x)} + O(\frac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$ which proves the lemma.

Lemmas 44 and 45 give us

Theorem 46. Let $\operatorname{supp}(d\beta) \subset [-1,1]$, $\tau \subset (-1,1)$ and let exists a polynomial π such that $\pi^2/\beta_\tau' \in L^1(-1,1)$. Then

$$n\ \lambda_n(d\beta_\tau,x) = \pi\sqrt{1-x^2} + O(\frac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$.

We obtain immediately from Theorems 43 and 46 the following

Theorem 47. Let $\operatorname{supp}(d\alpha) = [-1,1]$ and suppose that there exists a polynomial π such that $\pi^2/\alpha' \in L^1(-1,1)$. If $x \in (-1,1)$, α is absolutely continuous near x, $\alpha' \in A_x^{\omega}(B_x^{\omega})$ and $\alpha'(x) > 0$ then

$$n\ \lambda_n(d\alpha,x) = \pi\alpha'(x)\sqrt{1-x^2} + O(1)\begin{cases} \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\,dt & (\alpha' \in A_x^{\omega}) \\[2ex] \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\,dt & (\alpha' \in B_x^{\omega})\ . \end{cases}$$

If $\tau \subset (-1,1)$, α is absolutely continuous in a neighborhood of τ, $\alpha' \in A_\tau^{\omega}(B_\tau^{\omega})$ and $\alpha'(t) > 0$ for $t \in \tau$ then

$$n\,\lambda_n(d\alpha,x) = \pi\alpha'(x)\sqrt{1-x^2} + O(1)\begin{cases} \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1}\frac{\omega(t)}{t^2}\,dt & (\alpha' \in A_\tau^{\omega}) \\[2ex] \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1}\frac{\omega(t)}{t}\,dt & (\alpha' \in B_\tau^{\omega}) \end{cases}$$

uniformly for $x \in \tau_1 \subset \tau^0$.

The reader should compare Theorem 47 with Freud's results where $\pi^2/\alpha' \in L^\infty$ has to be assumed. (See Freud, §V.6.)

In Theorem 40, we have shown that $G_n(d\alpha,f)$ will converge to f with rate $\frac{1}{n}$ if f is good. On the other hand for $f \in \mathrm{Lip}\ 1$, we have only obtained $\log n/n$ as convergence rate for $G_n(d\alpha,f)$. (See Theorem 38.) We may ask two questions, namely, whether $\log n/n$ occurs because of our weak techniques and how to improve convergence.

<u>Theorem</u> 48. Let $f(x) \equiv |x|$. Then

$$G_n(v,f,0) \geq C\,\frac{\log n}{n}$$

for $n \geq 3$.

<u>Proof</u>. Since

$$G_n(v,f,0) = 2\,\lambda_n(v,0)\int_0^1 t\,K_n^2(v,0,t)\,v(t)dt$$

we have only to show that for k odd

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 kt}{\cos t}\,dt \geq C\,\log k \qquad (k \geq 3)\,.$$

The left side here equals

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 kt}{\sin t}\,dt \geq \int_0^{\frac{\pi}{2}} \frac{\sin^2 kt}{t}\,dt = \int_0^{\frac{k\pi}{2}} \frac{\sin^2 u}{u}\,du \geq C\,\log k\,.$$

By Theorem 48 if we want to improve the convergence properties of $G_n(d\alpha,f)$ then we have to modify these operators. Let us put for $n \leq m$

$$G_{n,m}(d\alpha,f,x) = \lambda_n(d\alpha,x)\int_{-\infty}^{\infty} f(t)\,K_n(d\alpha,x,t)\,K_m(d\alpha,x,t)\,d\alpha(t)\,.$$

For $z \in \mathbb{C}$, $G_{n,m}(d\alpha,f,z)$ can be defined by

$$G_{n,m}(d\alpha,f,z) = \lambda_n^*(d\alpha,z) \int_{-\infty}^{\infty} f(t)\, k_n(d\alpha,z,t)\, k_m(d\alpha,z,t)\, d\alpha(t) .$$

(See 4.1.)

Properties 49. (i) $G_{n,m}(d\alpha,\pi_{m-n}) \equiv \pi_{m-n}$. (ii) $G_{n,m}$ is a rational function of degree $(n+m-2,2n-2)$. (iii) The Lebesgue function $G^*_{n,m}(d\alpha,x)$ of the operator $G_{n,m}(d\alpha)$ is not greater than $[\lambda_n(d\alpha,x)\, \lambda_m^{-1}(d\alpha,x)]^{1/2}$.

Consequently if f is good globally and α is nice locally (near $x \in \operatorname{supp}(d\alpha)$) then for e.g., $m = 2n$, $G_n(d\alpha,f,x)$ may converge to $f(x)$ very rapidly. On the other hand, if α is nice near x, then the kernel function of $G_{n,2n}(d\alpha)$ has the same majorant only as that of $G_n(d\alpha) \equiv G_{n,n}(d\alpha)$, namely

$$\frac{Cn}{1 + n^2(x-t)^2}, \tag{21}$$

which - as is well known- is too weak to assume good convergence properties for $G_{n,2n}(d\alpha,f,x)$ if f is nice only at x. For this reason, we introduce another operator $G_N(d\alpha)$, $(N = (n_1,n_2,\ldots,n_k))$.

Let $k \geq 2$ be fixed and let $n_1 \leq n_2$, $n_1-1 + n_2-1 \leq n_3-1$ and, in general $\sum_{j=1}^{i-1} (n_j - 1) \leq n_i - 1$ for $i = 2,\ldots,k$. We put

$$G_N(d\alpha,f,x) = \prod_{i=1}^{k-1} \lambda_{n_i}(d\alpha,x) \int_{-\infty}^{\infty} f(t) \prod_{i=1}^{k} K_{n_i}(d\alpha,x,t)\, d\alpha(t)$$

and for $z \in \mathbb{C}$ we define $G_N(d\alpha,f,z)$ in exactly the same way as we did when $k = 2$.

Let us note that if e.g., α = Chebyshev weight then the kernel function of $G_N(d\alpha)$ may be majorized by

$$\frac{Cn}{1 + n^k|x-t|^k} \tag{22}$$

($x,t \in [-1,1]$ and all n_i are of order n), which differes very much from (21)! (22) implies that for $k \geq 3$, $G_N(v,f,x)$ converges to $f(x)$ with rate $\frac{1}{n}$ if all n_i are of order n and $f \in A_x^{\omega}$ with $\omega(t) \equiv t$.

It should be possible to improve most of the results of this section by using $G_N(d\alpha)$ instead of $G_n(d\alpha)$. At the present time we cannot do this.

Let us mention a simple result which we shall need later.

Lemma 50. Let $x \in \mathbb{R}$. Then for every polynomial π_n the inequality

$$|\pi_n(x)| \leq \frac{1}{2} \lambda_n(d\alpha,x) \int_{-\infty}^{\infty} |\pi_n(t)|[K_n^2(d\alpha,x,t) + K_{2n}^2(d\alpha,x,t)]\, d\alpha(t)$$

holds.

Proof. Use Property 49(i).

Theorem 51. Let $\alpha \in M(0,1)$. Then

$$\limsup_{n\to\infty} n\, \lambda_n(d\alpha,x) \leq \pi\, \alpha'(x)\sqrt{1-x^2} \tag{23}$$

for almost every $x \in \operatorname{supp}(d\alpha)$.

Proof. By Theorem 3.3.7 it is enough to show that (23) holds for almost every $x \in [-1,1]$. If $\operatorname{supp}(d\alpha) = [-1,1]$ then (23) follows from Theorem 33. Let now $\Delta = \operatorname{supp}(d\alpha) \setminus [-1,1]$ be not empty. Then there exists $\epsilon_1 > 0$ such that for every $\epsilon \in (0,\epsilon_1)$, $\Delta_\epsilon = \operatorname{supp}(d\alpha) \setminus [-1-\epsilon, 1+\epsilon]$ is not empty. By Theorem 3.3.7, Δ_ϵ contains finitely many points. Let $\Delta_\epsilon = \{a_k\}_{k=1}^m$ where $m = m(\epsilon)$. Let π be defined by

$$\pi(x) = \prod_{k=1}^{m} (x - a_k)\,.$$

Then for $n > m$

$$\lambda_n(d\alpha,x) \leq \int_{-\infty}^{\infty} \frac{K_{n-m}^2(v_\epsilon,x,t)\,\pi^2(t)}{K_{n-m}^2(v_\epsilon,x,x)\,\pi^2(x)}\, d\alpha(t)$$

for $x \in [-1,1]$ where v_ϵ denotes the Chebyshev weight corresponding to $[-1-\epsilon,1+\epsilon]$. Since π vanishes on Δ_ϵ we obtain

$$\frac{\lambda_n(d\alpha,x)}{\lambda_n(v_\epsilon,x)} \leq \frac{\lambda_{n-m}(v_\epsilon,x)}{\lambda_n(v_\epsilon,x)}\, \pi(x)^{-2}\, \lambda_{n-m}(v_\epsilon,x) \int_{-1-\epsilon}^{1+\epsilon} K_{n-m}^2(v_\epsilon,x,t)\, \pi^2(t)\, d\alpha(t)\,.$$

Transforming $[-1-\epsilon,1+\epsilon]$ into $[-1,1]$ we get

$$\frac{\lambda_n(d\alpha,x)}{\lambda_n(v,\frac{x}{1+\epsilon})} \leq \frac{\lambda_{n-m}(v,\frac{x}{1+\epsilon})}{\lambda_n(v,\frac{x}{1+\epsilon})}\, \pi(x)^{-2} \,\cdot$$

$$\cdot\, \lambda_{n-m}(v,\tfrac{x}{1+\epsilon}) \int_{-1}^{1} K_{n-m}^2(v,\tfrac{x}{1+\epsilon},t)\, \pi^2((1+\epsilon)t)\, d\alpha((1+\epsilon)t)\,.$$

Let β be defined by $d\beta(t) = \pi^2((1+\epsilon)t)d\alpha((1+\epsilon)t)$. Then β is a weight on $[-1,1]$. Hence by Lemmas 4 and 32

$$\limsup_{n\to\infty} \frac{\lambda_n(d\alpha,x)}{\lambda_n(v,\frac{x}{1+\epsilon})} \leq \sqrt{1-\frac{x^2}{(1+\epsilon)^2}}\,\pi(x)^{-2}\,\beta'(\tfrac{x}{1+\epsilon}) = \sqrt{(1+\epsilon)^2 - x^2}\,\alpha'(x)$$

for almost every $x \in [-1,1]$, that is by Example 9

$$\limsup_{n\to\infty} n\,\lambda_n(d\alpha,x) \leq \pi\,\alpha'(x)\sqrt{(1+\epsilon)^2 - x^2}$$

for almost every $x \in [-1,1]$. Now let $\epsilon\to 0$.

Now we can prove the following theorem which is one of our main results.

<u>Theorem 52</u>. Let $\alpha \in M(0,1)$, $\tau \subset (-1,1)$, $1/\alpha' \in L^1(\tau)$. Assume that there exists a sequence $\{\epsilon_k(\geq 0)\}$ with $\lim_{k\to\infty} \epsilon_k = 0$ such that $\log \alpha'(t)/\sqrt{(1-\epsilon_k)^2 - x^2} \in L^1(-1+\epsilon_k, 1-\epsilon_k)$ for every fixed k . Then

(24)
$$\lim_{n\to\infty} n\,\lambda_n(d\alpha,x) = \pi\,\alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in \tau$.

<u>Proof</u>. Because of Theorem 51 we only have to show that

$$\liminf_{n\to\infty} n\,\lambda_n(d\alpha,x) \geq \pi\,\alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in \tau$. We have $\lambda_n(d\alpha,x) \geq \lambda_n(\alpha',x)$. Let k be fixed and w be defined by $w(t) = \alpha'((1-\epsilon_k)t)$ for $-1 \leq t \leq 1$ with $\mathrm{supp}(w) = [-1,1]$. By the conditions, $w \in S$ and $w^{-1} \in L^1(\tau_1)$, where $\tau_1 = [(1-\epsilon_k)^{-1}c_1, (1-\epsilon_k)^{-1}c_2]$ if $\tau = [c_1, c_2]$. By Theorem 34

$$\lim_{n\to\infty} n\,\lambda_n(w,x) = \pi\sqrt{1-x^2}\,w(x)$$

for almost every $x \in \tau_1$. We have by Example 10

$$\lambda_n(w,x) = (1-\epsilon_k)^{-1}\,\lambda_n(1_\Delta\alpha',(1-\epsilon_k)x) \leq (1-\epsilon_k)^{-1}\,\lambda_n(\alpha',(1-\epsilon_k)x)$$

where $\Delta = [-1+\epsilon_k, 1-\epsilon_k]$. Hence

$$\liminf_{n\to\infty} n\,\lambda_n(\alpha',(1-\epsilon_k)x) \geq \pi\sqrt{(1-\epsilon_k)^2 - (1-\epsilon_k)^2x^2}\,\alpha'((1-\epsilon_k)x)$$

for almost every $x \in \tau_1$, that is

$$\liminf_{n\to\infty} n\,\lambda_n(\alpha',x) \geq \pi\sqrt{(1-\epsilon_k)^2 - x^2}\,\alpha'(x)$$

for almost every $x \in \tau$. Now let $k\to\infty$.

Later we shall show that if e.g.,

$$\sum_{j=0}^{\infty} \{|\alpha_j(d\alpha)| + |\frac{\gamma_j(d\alpha)}{\gamma_{j+1}(d\alpha)} - \frac{1}{2}|\} < \infty$$

then for each $\tau \subset (-1,1)$ all the conditions of Theorem 52 are satisfied and thus (24) holds for almost every $x \in [-1,1]$. Let us note that Theorem 34 follows from Theorem 52.

<u>Corollary 53</u>. Let α and τ satisfy the conditions of Theorem 52 and let ℓ be a fixed nonnegative integer. Then for almost every $x \in \tau$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} p_k(d\alpha,x)\,p_{k+\ell}(d\alpha,x) = \frac{T_\ell(x)}{\pi\,\alpha'(x)\sqrt{1-x^2}}$$

where T_ℓ denotes the ℓ-th Chebyshev polynomial.

<u>Proof</u>. Use Theorems 52 and 4.1.19.

<u>Theorem 54</u>. Let $\alpha \in M(0,1)$. Then for almost every $x \in \operatorname{supp}(d\alpha)$

$$\limsup_{n\to\infty} n\,\lambda_n(d\alpha,x) = \pi\,\alpha'(x)\sqrt{1-x^2}\,. \tag{25}$$

<u>Proof</u>. Let $\mathfrak{B}$ be defined by $\mathfrak{B} = \{x:\ \alpha'(x) > 0\}$. By Theorem 33 we have to show that (25) holds for almost every $x \in \mathfrak{B}$.

Since α is almost everywhere continuous, for every $x \in (-1,1)$ we can find a sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$, $\lim_{m\to 0} \epsilon_m = 0$, $[x-\epsilon_m, x+\epsilon_m] \subset (-1,1)$ and α is continuous at $x-\epsilon_m$ and $x+\epsilon_m$. By Theorem 4.2.14

$$\lim_{n\to\infty} \int_{x-\epsilon_m}^{x+\epsilon_m} \frac{1}{n\,\lambda_n(d\alpha,t)}\,d\alpha(t) = \frac{1}{\pi}\int_{x-\epsilon_m}^{x+\epsilon_m} \frac{dt}{\sqrt{1-t^2}}$$

for $m = 1,2,\ldots$. Thus by Fatou's lemma

$$\frac{1}{2\epsilon_m}\int_{x-\epsilon_m}^{x+\epsilon_m} \liminf_{n\to\infty} \frac{1}{n\lambda_n(d\alpha,t)}\,\alpha'(t)dt \le \frac{1}{2\pi\epsilon_m}\int_{x-\epsilon_m}^{x+\epsilon_m} \frac{dt}{\sqrt{1-t^2}} .$$

Letting $m\to\infty$ and using Lebesgue's theorem we obtain

$$\liminf_{n\to\infty} \frac{1}{n\lambda_n(d\alpha,x)}\,\alpha'(x) \le \frac{1}{\pi\sqrt{1-x^2}}$$

for almost every $x \in [-1,1]$. By Theorem 3.3.7, $\mathfrak{M} \subset [-1,1]$. Hence for almost every $x \in \mathfrak{M}$

$$\limsup_{n\to\infty} n\,\lambda_n(d\alpha,x) \ge \pi\,\alpha'(x)\sqrt{1-x^2} .$$

The converse inequality has been proved in Theorem 33.

<u>Theorem 55</u>. Let $\operatorname{supp}(d\alpha) = [-1,1]$ and $\alpha'(x) > 0$ for almost every $x \in [-1,1]$. Then (25) holds for almost every $x \in [-1,1]$.

<u>Proof</u>. If f is continuous on $[-1,1]$ then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_{kn}(d\alpha)) = \frac{1}{\pi}\int_{-1}^{1} f(t)\,\frac{dt}{\sqrt{1-t^2}} .$$

(See Freud, §III.9.) Hence by Lemma 5.1

$$\lim_{n\to\infty}\int_{-1}^{1} f(t)\,\frac{1}{n\,\lambda_n(d\alpha,t)}\,d\alpha(t) = \frac{1}{\pi}\int_{-1}^{1} f(t)\,\frac{dt}{\sqrt{1-t^2}} \tag{26}$$

if f is continuous on $[-1,1]$. Using one-sided approximation we obtain that (26) remains valid if f is the characteristic function of a $d\alpha$ measurable interval $\Delta \subset (-1,1)$. Now we can repeat the proof of Theorem 54.

6.3. Generalized Christoffel Functions

Definition 1. Let $0 < p < \infty$. Then the generalized Christoffel function $\lambda_n(d\alpha,p,x)$ is defined by

$$\lambda_n(d\alpha,p,x) = \inf_{\pi_{n-1}} \frac{1}{|\pi_{n-1}(x)|^p} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p \, d\alpha(t) .$$

Later we will see why we do not introduce a normalization: $\lambda_n(d\alpha,p,x) = \inf[\quad]^{1/p}$.

Properties 2. (i) If $d\alpha \le d\beta$ then $\lambda_n(d\alpha,p,x) \le \lambda_n(d\beta,p,x)$. (ii) $\lambda_n(d\alpha,2,x) \equiv \lambda_n(d\alpha,x)$. (iii) If $\operatorname{supp}(d\alpha)$ is compact then

$$\lambda_n(d\alpha,p,x) = \min_{\pi_{n-1}} \frac{1}{|\pi_{n-1}(x)|^p} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p \, d\alpha(t) .$$

Proof. Let us fix α,n,p and $\Delta \supset \Delta(d\alpha)$. Let us show that $\lambda_n(d\alpha,p,y) \ge \lambda > 0$ for $y \in \Delta$. Let m be an integer such that $m \ge p$. Let $y \in \Delta \supset \Delta(d\alpha)$. Then

$$\pi_{n-1}^m(y) = \int_\Delta \pi_{n-1}^m(t) \; K_{mn}(d\alpha,y,t) \, d\alpha(t)$$

for $y \in \Delta$. Hence

$$\max_{y\in\Delta} |\pi_{n-1}(y)|^m \le c \int_\Delta |\pi_{n-1}(t)|^m \, d\alpha(t)$$

where $c = c(n,m,d\alpha,\Delta)$ does not depend on π_{n-1}. Writing $|\pi_{n-1}(t)|^m =$ $= |\pi_{n-1}(t)|^p \, |\pi_{n-1}(t)|^{m-p}$ $(m-p \ge 0)$ we obtain

$$\max_{y\in\Delta} |\pi_{n-1}(y)|^p \le c \int_\Delta |\pi_{n-1}(t)|^p \, d\alpha(t) ,$$

in particular $\lambda_n(d\alpha,p,y) \ge \lambda > 0$ for $y \in \Delta$. Thus

$$\int_{-\infty}^{\infty} |\pi_{n-1}(y)| \, d\alpha(y) \le \lambda^{-\frac{1}{p}} [\int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p \, d\alpha(t)]^{\frac{1}{p}} .$$

If we write $\pi_{n-1}(x) = \sum_{k=0}^{n-1} a_k p_k(d\alpha,x)$ then we obtain from the previous inequality that

$$(0) \qquad a_k \le c_1 \, [\int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p \, d\alpha(t)]^{\frac{1}{p}}$$

where $C_1 = C_1(n,p,d\alpha)$ does not depend on π_{n-1}. Now (iii) follows from Bolozano-Weierstrass' theorem by the following argument. We fix n and x. Then we can find a sequence of polynomials $\pi_{n-1,m} \in \mathbb{P}_{n-1}$, $(m = 1,2,\dots)$ such that $\pi_{n-1,m}(x) = 1$ for every m and

$$(0') \qquad \lambda_n(d\alpha,p,x) \le \int_{-\infty}^{\infty} |\pi_{n-1,m}(t)|^p \, d\alpha(t) \le \lambda_n(d\alpha,p,x) + \frac{1}{m}$$

for $m = 1,2,\dots$. Let

$$\pi_{n-1,m} = \sum_{k=0}^{n-1} a_{km} \, p_k(d\alpha) .$$

Then by (0)

$$|a_{km}| \le C_1 \, [\lambda_n(d\alpha,p,x) + 1]$$

for $k = 0,1,\dots,n-1$ and $m = 1,2,\dots$. By Bolzano-Weierstrass' theorem we can choose a subsequence m_j such that

$$\lim_{j\to\infty} a_{km_j} = a_k$$

for $k = 0,1,\dots,n-1$. Consequently, on every compact set the sequence π_{n-1,m_j} uniformly converges to some π_{n-1} when $j\to\infty$. Since $\pi_{n-1,m_j}(x) = 1$ for every j, $\pi_{n-1}(x) = 1$ also holds. Finally, it follows from (0') that

$$\lambda_n(d\alpha,p,x) = \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p \, d\alpha(t) .$$

First of all we will investigate the simplest case, that is when α is a Jacobi weight. Let us recall that the Jacobi weight is denoted by $u = u^{(a,b)}$.. We will find the exact order of $\lambda_n(u,p,x)$ on $[-1,1]$ when $n\to\infty$.

Definition 3. We write $\varphi_n(x) \sim \psi_n(x)$ if for every n and for every x in consideration (usually for $-1 \le x \le 1$)

$$0 < c_1 \le \varphi_n(x)/\psi_n(x) \le c_2 < \infty .$$

$\varphi(x) \sim \psi(x)$, $n \sim m$ etc. have similar meanings.

Definition 4. Let $a,b \in \mathbb{R}$. Then u_n is defined by

$$u_n(x) \equiv u_n^{(a,b)}(x) = [\sqrt{1-x} + \tfrac{1}{n}]^{2a+1} \, [\sqrt{1+x} + \tfrac{1}{n}]^{2b+1}$$

for $x \in [-1,1]$.

Let us remark that if $m \sim n$ then $u_n(x) \sim u_m(x)$.

Lemma 5. We have

$$\lambda_n(u^{(a,b)},x) \sim \frac{1}{n} u_n^{(a,b)}(x)$$

for $-1 \leq x \leq 1$.

Proof. See [11].

Lemma 6. Let $\cos \theta_{kn} = x_{kn}(u)$ for $k = 0,1,\ldots,n+1$ with $x_{0n} = 1$ and $x_{n+1,n} = -1$. Then

$$\theta_{kn} - \theta_{k-1,n} \sim \frac{1}{n}$$

for $k = 1,2,\ldots,n+1$.

Proof. See e.g., [12].

Corollary 7. Let $x \in [x_{kn}(u), x_{k-1,n}(u)]$, $(k = 1,2,\ldots,n+1)$. Then

$$u_n^{(a_1,b_1)}(x) \sim u_n^{(a_1,b_1)}(x_k) \sim u_n^{(a_1,b_1)}(x_{k-1})$$

for $a_1, b_1 \in \mathbb{R}$.

Let v be - as before - the Chebyshev weight. Then

$$|K_n(v,x,t)| \leq C \min\{n, \frac{1}{|x - t|}\}$$

for $x,t \in [-1,1]$. This extimate is good inside $(-1,1)$ but not for x and t close to the endpoints of $[-1,1]$. We will need a better estimate which we formulate as

Lemma 8. Let $x,t \in [-1,1]$. Then

$$|K_n(v,x,t)| \leq C \min\{n, \frac{\sqrt{1 - x^2} + \sqrt{1 - t^2}}{|x - t|}\}.$$

Proof. The idea comes from Pollard probably:

$$T_n(x)T_{n-1}(t) - T_{n-1}(x)T_n(t) = [T_n(x) - T_{n-1}(x)]T_{n-1}(t) + T_{n-1}(x)[T_{n-1}(t) - T_n(t)].$$

Lemma 9. Let $0 < \delta < 1$ and $a,b,c \in \mathbb{R}$. Then we have uniformly in n and m such that $1 \le m \le \delta n$

$$\sum_{\substack{k=1\\k\neq m}}^{n} k^a \,|k+m|^b\, |k-m|^c \sim$$

$$\sim m^{b+c}\begin{Bmatrix} 1 & \text{if } a<-1\\ \log\,(m+2) & \text{if } a=-1\\ m^{a+1} & \text{if } a>-1\end{Bmatrix} +$$

$$+\, m^{a+b}\begin{Bmatrix} 1 & \text{if } c<-1\\ \log\,(m+2) & \text{if } c=-1\\ m^{c+1} & \text{if } c>-1\end{Bmatrix} +$$

$$+\begin{Bmatrix} m^{a+b+c\;+1} & \text{if } a+b+c<-1\\ \log(\frac{n}{m}+2) & \text{if } a+b+c=-1\\ n^{a+b+c+1} & \text{if } a+b+c>-1\end{Bmatrix} .$$

Proof. We write

$$\sum_{\substack{k=1\\k\neq m}}^{n} = \sum_{k=1}^{[\frac{m}{2}]} + \sum_{k=[\frac{m}{2}]+1}^{m-1} + \sum_{k=m+1}^{[\frac{m}{2}(\delta^{-1}+1)]} + \sum_{k=[\frac{m}{2}(\delta^{-1}+1)]+1}^{n}$$

and each sum can easily be estimated.

Now we can compute the following important

Theorem 10. Let $u = u^{(a,b)}$ be given. Then there exists a natural integer $N_1 = N_1(a,b)$ such that for every fixed $N \ge N_1$

(1) $$\int_{-1}^{1} \left|\frac{K_n(v,x,t)}{K_n(v,x,x)}\right|^N u(t)dt \le c\,\frac{1}{n}\, u_n^{(a,b)}(x)$$

for $-1 \le x \le 1$ and $n = 1,2,\ldots$.

Proof. Let us note that if (1) holds for $N_1 = N$ then it holds also for every fixed $N > N_1$. Let N be an even integer. Then $n \sim nN \equiv M$ and by Definition 4, $u_n(x) \sim u_M(x)$ for $|x| \le 1$. Let us compute the integral on the left side of (1) by the Gauss-Jacobi mechanical quadrature formula. We can do this since N is even. The above integral equals to

$$(2) \qquad \sum_{k=1}^{M} \lambda_{kM}(u) \left[\frac{K_n(v,x,x_{kM}(u))}{K_n(v,x,x)}\right]^N$$

and we will estimate this sum. We may suppose without loss of generality that $0 \leq x \leq 1$. Take a suitable value for N such that (1) holds. If $-1 \leq x < 0$ then by similar arguments, we obtain another value for N such that (1) is satisfied. Taking the maximum of these two values of N we get a new N which is good for every $x \in [-1,1]$. We have $K_n(v,x,x)^{-N} \sim M^{-N}$ for $-1 \leq x \leq 1$ by Example 6.2.8. Let m be defined by

$$|x - x_{kM}| \geq |x - x_{mM}| \qquad \text{for } k = 1,2,\ldots,M .$$

Then

$$(2) \sim M^{-N} \Big(\sum_{x_k \leq -\frac{1}{2}} + \sum_{\substack{-\frac{1}{2} < x_k < 1 \\ k \neq m}} + \sum_{k=m} \Big) \lambda_{kM}(u) K_n^N(v,x,x_{kM}) .$$

By Lemma 8 the first sum here is $O(1)$. By Lemma 5 and Corollary 7 the last sum is of order $M^{N-1} u_M(x)$. Hence

$$(2) \leq C \left[M^{-N} + \frac{1}{M} u_M(x)\right] + M^{-N} \sum_{\substack{-\frac{1}{2} \leq x_k < 1 \\ k \neq m}} \lambda_{kM}(u) K_n^N(v,x,x_{kM}) .$$

By Lemma 6 for $k \neq m$, $|x - x_{kM}| > C \frac{1}{M}\sqrt{1 - x^2}$ and $|x - x_{kM}| \geq C\frac{1}{M^2}|k-m||k+m|$. Consequently by Lemmas 5,6 and 8

$$M^{-N} \sum_{\substack{-\frac{1}{2} \leq x_k < 1 \\ k \neq m}} \lambda_{kM}(u) K_n^N(v,x,x_{kM}) \leq$$

$$\leq CM^{-N-1} \sum_{\substack{-\frac{1}{2} \leq x_k < 1 \\ k \neq m}} (1 - x_{kM})^{a+\frac{1}{2}} \left(\frac{\sqrt{1-x^2} + \sqrt{1-x_{kM}^2}}{x - x_{kM}}\right)^N \leq$$

$$\leq CM^{-N-1}(1 - x^2)^{\frac{N}{2}} \sum_{\substack{k=1 \\ k \neq m}}^{c_1 M} \left(\frac{k}{M}\right)^{2a+1} \left[\frac{(k-m)(k+m)}{M^2}\right]^{-N} \equiv A + B$$

where $0 < c_1 < 1$. To estimate A and B we use Lemma 9. We obtain that if

$$N > 1 \quad \text{and} \quad 2a + 1 - 2N < -1$$

then

$$A \sim (1-x^2)^{\frac{N}{2}} M^{N-2-2a}\, m^{2a+1-N} \sim (1-x^2)^{\frac{N}{2}} \frac{1}{M} [\sqrt{1-x} + \frac{1}{M}]^{2a+1-N} \le C \frac{1}{M} u_M^{(a,b)}(x) ,$$

and if

$$N > 1 \quad \text{and} \quad 2a + 1 - N < -1$$

then

$$B \sim M^{-2-2a}\, m^{2a+1} \sim \frac{1}{M} u_M^{(a,b)}(x) .$$

Thus for $N > \max\{1, 2a+2\}$

$$(2) \le C\, [M^{-N} + \frac{1}{M} u_M^{(a,b)}(x)] \le C \frac{1}{M} u_M^{(a,b)}(x) \le \frac{c}{n} u_n^{(a,b)}(x) .$$

To finish the proof of the theorem we choose $N > 0$ that $N > \max(1, 2a+2, 2b+2)$.

<u>Lemma 11</u>. Let $a \ge -\frac{1}{2}$, $0 < p < \infty$. Then

$$(3) \qquad |\pi_{n-1}(x)|^p \le C\, n^{2(a+1)} \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt$$

for $\frac{1}{2} \le x \le 1$ and there exists a number $c_1 = c_1(a,p) > 0$ such that

$$(4) \qquad \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt \le 2 \int_0^{1-c_1 n^{-2}} |\pi_{n-1}(t)|^p (1-t)^a dt .$$

<u>Proof</u>. Let k be a natural integer such that $2k \ge p$. Then $\deg \pi_{n-1}^k \sim n$ and by Lemma 5

$$\pi_{n-1}^{2k}(x) \le C\, n^{2(a+1)} \int_{-1}^1 \pi_{n-1}^{2k}(t)\, (1-t^2)^a dt$$

for $-1 \le x \le 1$, that is

$$\max_{|x|\le 1} |\pi_{n-1}(x)|^{2k} \le \max_{|x|\le 1} |\pi_{n-1}(x)|^{2k-p}\, C\, n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^p (1-t^2)^a dt .$$

Therefore

$$|\pi_{n-1}(x)|^p \le C\, n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^p (1-t^2)^a dt$$

for $-1 \le x \le 1$. Let us put here $x^{2M} \pi_{n-1}(x^2)$ instead of $\pi_{n-1}(x)$. Then for M fixed

$$|x^{2M} \pi_{n-1}(x^2)|^p \le C\, n^{2(a+1)} \int_0^1 |\pi_{n-1}(t)|^p\, t^{Mp-\frac{1}{2}} (1-t)^a\, dt$$

for $0 \le x \le 1$. Let now M be so large that $Mp - \frac{1}{2} \ge 0$. Hence (3) holds. We obtain from (3) that for $c_1 > 0$

$$\int_{1-\frac{c_1}{n^2}}^{1} |\pi_{n-1}(x)|^p (1-x)^a dx \le Cn^{2(a+1)} \int_{1-\frac{c_1}{n^2}}^{1} (1-x)^a dx \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt$$

and if $c_1 > 0$ is small then

$$C\, n^{2(a+1)} \int_{1-\frac{c_1}{n^2}}^{1} (1-x)^a\, dx \le \frac{1}{2}, \tag{5}$$

which implies (4).

<u>Lemma 12</u>. Let $\min(a,b) \ge -\frac{1}{2}$, $0 < p < \infty$. Then

$$\lambda_n(u,x) \le C\, \lambda_n(u,p,x) \tag{6}$$

for $n = 1,2,\ldots$ and $-1 \le x \le 1$.

<u>Proof</u>. Let k be an integer such that $2k \ge p$. Put $a_1 = (a + \frac{1}{2})\frac{2k}{p} - \frac{1}{2}$ and $b_1 = (b + \frac{1}{2})\frac{2k}{p} - \frac{1}{2}$. Then $a_1, b_1 \ge -\frac{1}{2}$ and by Lemma 5 and 11 there exists $\varepsilon > 0$ such that

$$u^{(a_1,b_1)}(x)\sqrt{1-x^2}\, \pi_{n-1}(x)^{2k} \le C\, n \int_{-1+\frac{\varepsilon}{n^2}}^{1-\frac{\varepsilon}{n^2}} \pi_{n-1}(t)^{2k}\, u^{(a_1,b_1)}(t)\sqrt{1-t^2}\, \frac{1}{\sqrt{1-t^2}}\, dt$$

for $|x| \le 1 - \frac{\varepsilon}{n^2}$. Hence

$$\left|[u^{(a_1,b_1)}(x)\sqrt{1-x^2}]^{\frac{1}{2k}}\, \pi_{n-1}(x)\right|^p \le$$

$$\le C\, n \int_{-1+\frac{\varepsilon}{n^2}}^{1-\frac{\varepsilon}{n^2}} \left|[u^{(a_1,b_1)}(t)\sqrt{1-t^2}]^{\frac{1}{2k}}\, \pi_{n-1}(t)\right|^p \frac{1}{\sqrt{1-t^2}}\, dt,$$

that is

$$|\pi_{n-1}(x)|^p \le n\, u_n^{(a,b)}(x)^{-1} \int_{-1}^{1} |\pi_{n-1}(t)|^p\, u(t) dt$$

for $|x| \le 1 - \frac{\varepsilon}{n^2}$. By Lemma 11 the latter inequality also holds for $1 - \frac{\varepsilon}{n^2} \le |x| \le 1$. Now (6) follows from Lemma 5.

Theorem 13. Let u be a Jacobi weight and $0 < p < \infty$. Then

$$\lambda_n(u,p,x) \sim \lambda_n(u,x)$$

for $-1 \le x \le 1$.

Proof. Let N be such that (1) holds. Let $m \in \mathbb{N}$ be so big that $mp \ge N$. If we put $n_1 = [\frac{n}{m}]$ then

$$\lambda_n(u,p,x) \le \int_{-1}^{1} \left|\frac{K_{n_1}(v,x,t)}{K_{n_1}(v,x,x)}\right|^{mp} u(t)dt \le c\frac{1}{n_1}u_{n_1}(x) \sim \frac{1}{n}u_n(x) \sim \lambda_n(u,x) \quad (|x|\le 1)$$

by Lemma 5 and Theorem 10. Now we shall show that

$$\lambda_n(u,x) \le c\, \lambda_n(u,p,x) \tag{7}$$

for $|x| \le 1$. Let $u = u^{(a,b)}$, $u^* = u^{(\max\{a,-\frac{1}{2}\},\max\{b,-\frac{1}{2}\})}$ and $\hat{u} = u^{\left(\max\{-\frac{(a+\frac{1}{2})}{p}-\frac{1}{2},-\frac{1}{2}\},\max\{-\frac{(b+\frac{1}{2})}{p}-\frac{1}{2},-\frac{1}{2}\}\right)}$. We have by Lemma 5 and 12

$$|\pi_{n-1}(x)\, K_n(\hat{u},x,x)|^p \le c\, n[u_n^*(x)]^{-1} \int_{-1}^{1} |\pi_{n-1}(t)\, K_n(\hat{u},t,t)|^p\, u^*(t)dt \qquad (|x|\le 1),$$

$$|K_n(\hat{u},t,t)|^p\, u^*(t) \le c\, n^p\, u(t) \qquad (|x|\le 1)$$

and

$$u_n^*(x)\ |K_n(\hat{u},x,x)|^p \sim n^p\, u_n(x)$$

for $|x| \le 1$. Hence

$$|\pi_{n-1}(x)|^p \le n\, u_n(x)^{-1} \int_{-1}^{1} |\pi_{n-1}(t)|^p\, u(t)dt .$$

From this inequality and Lemma 5 we obtain (7).

There is a very important consequence of Theorem 13 which we formulate as

Theorem 14. Let u be a Jacobi weight and $0 < p < \infty$. Then there exists a number $c_1 = c_1(u,p) > 0$ such that

$$\int_{-1}^{1} |\pi_n(t)|^p\, u(t)dt \le 2 \int_{-1+\frac{c_1}{n^2}}^{1-\frac{c_1}{n^2}} |\pi_n(t)|^p\, u(t)dt$$

for every π_n.

Proof. We have

$$|\pi_n(x)|^p \leq \lambda_n^{-1}(u,p,x) \int_{-1}^{1} |\pi_n(t)|^p\, u(t)dt \qquad (-1\leq x\leq 1).$$

Thus by Theorem 13

$$|\pi_n(x)|^p \leq C\, \lambda_n^{-1}(u,x) \int_{-1}^{1} |\pi_n(t)|^p\, u(t)dt \qquad (-1\leq x\leq 1).$$

By Lemma 5

$$\lambda_n^{-1}(u,x) \leq \begin{cases} C\, n^{2(a+1)} & \text{if} \quad 1-x \sim \frac{1}{n^2} \\ C\, n^{2(b+1)} & \text{if} \quad 1+x \sim \frac{1}{n^2} \end{cases} .$$

Consequently, if $\epsilon > 0$ is fixed then

$$\left(\int_{-1}^{-1+\frac{\epsilon}{n^2}} + \int_{1-\frac{\epsilon}{n^2}}^{1}\right) |\pi_n(x)|^p\, u(x)dx \leq$$

$$\leq C\,[n^{2(b+1)} \int_{-1}^{-1+\frac{\epsilon}{n^2}} (1+x)^b dx + n^{2(a+1)} \int_{1-\frac{\epsilon}{n^2}}^{1} (1-x)^a dx] \int_{-1}^{1} |\pi_n(t)|^p\, u(t)dt .$$

Now if we choose $\epsilon > 0$ small enough we obtain

$$\left(\int_{-1}^{-1+\frac{\epsilon}{n^2}} + \int_{1-\frac{\epsilon}{n^2}}^{1}\right) |\pi_n(x)|^p\, u(x)dx \leq \frac{1}{2} \int_{-1}^{1} |\pi_n(t)|^p\, u(t)dt .$$

Hence the theorem follows.

<u>Corollary 15</u>. Let u be a Jacobi weight, $0 < p < \infty$, $\epsilon > 0$. Then for every π_n

$$\int_{-1}^{1} |\pi_n(t)|^p\, u(t)dt \leq C\, n^{2\epsilon} \int_{-1}^{1} |\pi_n(t)|^p\, u(t)(1-t^2)^{\epsilon} dt$$

with $C = C(p,u,\epsilon)$.

<u>Corollary 16</u>. Let $0 < q < p < \infty$. Let u^p be a Jacobi weight. Then

$$\left(\int_{-1}^{1} |\pi_n(t)\, u(t)|^p dt\right)^{\frac{1}{p}} \leq C\, n^{2(\frac{1}{q}-\frac{1}{p})} \left(\int_{-1}^{1} |\pi_n(t)\, u(t)|^q dt\right)^{\frac{1}{q}}$$

for every π_n where $C = C(p,q,u)$.

Proof. By Lemma 5 and Theorems 13, 14

$$\int_{-1}^{1} |\pi_n(t)\ u(t)|^p dt \le 2 \int_{-1+\frac{c_1}{n^2}}^{1-\frac{c_1}{n^2}} |\pi_n(t)\ u(t)|^{p-q+q} dt \le$$

$$\le c\ n^{2\frac{p-q}{p}} \left(\int_{-1}^{1} |\pi_n(t)\ u(t)|^p dt\right)^{\frac{p-q}{p}} \int_{-1+\frac{c_1}{n^2}}^{1-\frac{c_1}{n^2}} |\pi_n(t)\ u(t)|^q dt\ .$$

Let us note that Corollaries 15 and 16 for $1 \le p < \infty$ and $1 \le q < p < \infty$ are not new. (See Khalilova [9].)

Before we begin to investigate the generalized Christoffel functions for weights different from the Jacobi ones we will need a few lemmas.

Lemma 17. Let $\alpha(x) + \alpha(-x) \equiv \text{const.}$ Let α_1 and α_2 be defined by

$$\alpha_1(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \alpha(\sqrt{x}) - \alpha(0) & \text{for} \quad x \ge 0 \end{cases}$$

and

$$\alpha_2(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \int_0^x t\ d\alpha(\sqrt{t}) & \text{for} \quad x \ge 0\ . \end{cases}$$

Then

$$\sum_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\ 2)}}^{n} p_k^2(d\alpha, x) = \begin{cases} \frac{1}{2}\ \lambda^{-1}_{\frac{n}{2}+1}(d\alpha_1, x^2) & \text{for} \quad n \text{ even} \\ \frac{x^2}{2}\ \lambda^{-1}_{\frac{n+1}{2}}(d\alpha_2, x^2) & \text{for} \quad n \text{ odd} \end{cases}$$

and

$$\sum_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\ 2)}}^{n} p_k^2(d\alpha, x) = \max_{\pi_n \equiv n (\mathrm{mod}\ 2)} \pi_n^2(x) \Big/ \int_{-\infty}^{\infty} \pi_n^2(t)\ d\alpha(t)\,,$$

where $\pi_n \equiv n(\mathrm{mod}\ 2)$ means that π_n is even if n is even and π_n is odd if n is odd.

Proof. Easy calculation.

Having in mind later applications we shall prove the following

Lemma 18. Let $\alpha \in M(0,1)$ and let $\alpha(x) + \alpha(-x) \equiv \text{const.}$ Let α_1 and α_2 be defined as in Lemma 17. Then $\alpha_1, \alpha_2 \in M(\frac{1}{2}, \frac{1}{2})$.

Proof. The lemma follows from the relations

$$\alpha_n(d\alpha_1) = \frac{\gamma_{2n}^2(d\alpha)}{\gamma_{2n+1}^2(d\alpha)} + \frac{\gamma_{2n-1}^2(d\alpha)}{\gamma_{2n}^2(d\alpha)} ,$$

$$\frac{\gamma_{n-1}(d\alpha_1)}{\gamma_n(d\alpha_1)} = \frac{\gamma_{2n-2}(d\alpha)}{\gamma_{2n}(d\alpha)}$$

and

$$\alpha_n(d\alpha_2) = \frac{\gamma_{2n+1}^2(d\alpha)}{\gamma_{2n+2}^2(d\alpha)} + \frac{\gamma_{2n}^2(d\alpha)}{\gamma_{2n+1}^2(d\alpha)}$$

$$\frac{\gamma_{n-1}(d\alpha_2)}{\gamma_n(d\alpha_2)} = \frac{\gamma_{2n-1}(d\alpha)}{\gamma_{2n+1}(d\alpha)}$$

which can easily be checked. Recall that $\gamma_{-1} = 0$.

Lemma 19. Let $w(t) = |t|^{\Gamma}(1 - t^2)^a$ for $-1 \leq t \leq 1$ with $\Gamma, a > -1$ and $\operatorname{supp}(w) = [-1,1]$. Then

$$\lambda_n(w,x) \sim \frac{1}{n}(|x| + \frac{1}{n})^{\Gamma} (\sqrt{1-x} + \frac{1}{n})^{2a+1} (\sqrt{1+x} + \frac{1}{n})^{2a+1}$$

for $-1 \leq x \leq 1$.

Proof. Apply Lemmas 5 and 17.

Theorem 20. Let $\operatorname{supp}(d\alpha)$ be compact, $\Delta \subset \operatorname{supp}(d\alpha)$, $t^* \in \Delta^0$, $\Gamma > -1$. Let α be absolutely continuous in Δ and let

$$\alpha'(t) \sim |t - t^*|^{\Gamma} \qquad (t \in \Delta) .$$

Then

$$\lambda_n(d\alpha,x) \sim \frac{1}{n}(|x - t^*| + \frac{1}{n})^{\Gamma}$$

for $x \in \Delta_1 \subset \Delta^0$.

Proof. Recall that

$$\lambda_n(d\beta,x) \leq \lambda_n(d\alpha,x)$$

whenever $d\beta \leq d\alpha$. Therefore by Lemma 19 we have to show that

(8)
$$\lambda_n(d\alpha,x) \leq c\frac{1}{n}(|x - t^*| + \frac{1}{n})^\Gamma$$

for $x \in \Delta_1 \subset \Delta^0$. Let $\hat{v}$ denote the Chebyshev weight corresponding to $\Delta(d\alpha)$ and m be a natural integer. Then

$$\lambda_n(d\alpha,x) \leq \int_{-\infty}^{\infty} \left[\frac{K_{[\frac{n}{m}]}(\hat{v},x,t)}{K_{[\frac{n}{m}]}(\hat{v},x,x)}\right]^{2m} d\alpha(t)$$

and hence

$$\lambda_n(d\alpha,x) \leq c\, n^{-2m} \int_{t \notin \Delta} d\alpha(t) + c \int_{t \in \Delta} |t - t^*|^\Gamma \frac{1}{1 + n^{2m}(x - t)^{2m}}\, dt$$

uniformly for $x \in \Delta_1 \subset \Delta^0$ if m is fixed. The second integral on the right side of the latter inequality can be estimated by standard methods. Finally, we obtain that (8) is satisfied if we choose m large enough.

Let us note that the calculation in Theorem 20 was simple because we needed estimates only for $x \in \Delta_1 \subset \Delta^0$ and not for $x \in \Delta$.

<u>Lemma 21</u>. Let $0 < p < \infty$, $\Gamma \geq 0$, $0 \in \Delta_1^0$, $\Delta_1 \subset \Delta^0$. Then for every π_{n-1}

$$|\pi_{n-1}(x)|^p \leq c\, n^{\Gamma+1} \int_\Delta |\pi_{n-1}(t)|^p\, |t|^\Gamma\, dt$$

uniformly for $x \in \Delta_1$.

<u>Proof</u>. Let $\epsilon > 0$ be such that $[-2\epsilon, 2\epsilon] \subset \Delta_1^0$. If $x \in \Delta_1 \setminus [-\epsilon,\epsilon]$ then by Lemma 5 and Theorem 13

$$|\pi_{n-1}(x)|^p \leq c\, n \int_{\Delta\setminus[-\frac{\epsilon}{2},\frac{\epsilon}{2}]} |\pi_{n-1}(t)|^p\, dt \leq c_1\, n^{\Gamma+1} \int_\Delta |\pi_{n-1}(t)|^p\, |t|^\Gamma\, dt$$

since $\Gamma \geq 0$. Let now $x \in [-\epsilon,\epsilon]$. Let v_ϵ denote the Chebyshev weight corresponding to $[-2\epsilon,2\epsilon]$, M be an even natural integer and m be an integer such that $2m \geq p$. Then by Lemma 19

$$[v_\epsilon(x)^{-M}\, \pi_{n-1}(x)]^{2m}\, v_\epsilon(x)^{-1} \leq c\, n^{\Gamma+1} \int_{-2\epsilon}^{2\epsilon} [v_\epsilon(t)^{-M}\, \pi_{n-1}(t)]^{2m}\, |t|^\Gamma\, dt$$

for $x \in [-2\epsilon,2\epsilon]$. Hence

$$|v_\epsilon(x)^{-M-\frac{1}{2m}}\pi_{n-1}(x)|^p \le c\, n^{\Gamma+1}\int_{-2\epsilon}^{2\epsilon}|\pi_{n-1}(t)|^p\, v_\epsilon(t)^{-Mp-\frac{p}{2m}+1}|t|^\Gamma\, dt$$

for $x \in [-2\epsilon, 2\epsilon]$. Now let M be so large that $-Mp-\frac{p}{2m}+1 \le 0$.

From Lemma 21, we obtain the following

Lemma 22. Let $0<p<\infty$, $\Gamma \ge 0$, $0 \in \Delta^0$. Then there exists $c_1 = c_1(p,\Gamma,\Delta)$ such that

$$\int_\Delta |\pi_{n-1}(t)|^p\,|t|^\Gamma\,dt \le 2\int_{\substack{t\in\Delta\\ |t|\ge\frac{c_1}{n}}}|\pi_{n-1}(t)|^p\,|t|^\Gamma\,dt\ .$$

Lemma 23. Let $0<p<\infty$, $\Gamma\ge 0$, $0\in\Delta_1^0$, $\Delta_1\subset\Delta^0$. Let $w(t) = |t|^\Gamma$ for $t\in\Delta$ with $\operatorname{supp}(w) = \Delta$. Then

$$\frac{1}{n}\left(|x|+\frac{1}{n}\right)^\Gamma \le c\,\lambda_n(w,p,x) \qquad (n=1,2,\dots) \tag{9}$$

uniformly for $x\in\Delta_1$.

Proof. Let $\epsilon>0$ be such that $[-2\epsilon,2\epsilon]\subset\Delta_1^0$. If $x\in\Delta_1\setminus[-\epsilon,\epsilon]$ then (9) follows from Lemma 5 and Theorem 13. If $x\in[-\epsilon,\epsilon]$ then we find an integer m such that $2m\ge p$. We put $w^*(t) = |t|^{2m\Gamma/p}(4\epsilon^2-t^2)^{(m/p)-(1/2)}$ for $|t|\le 2\epsilon$ with $\operatorname{supp}(w^*) = [-2\epsilon,2\epsilon]$. Then by Lemmas 19 and 22

$$\sqrt{4\epsilon^2-x^2}\,w^*(x)\,|\pi_{n-1}(x)|^{2m} \le c\,n\int_{\frac{c_1}{n}\le|t|\le 2\epsilon}|\pi_{n-1}(t)|^{2m}\,w^*(t)\,dt$$

for $\frac{c_1}{n}\le|x|\le 2\epsilon$. Hence

$$\sqrt{4\epsilon^2-x^2}\,w(x)\,|\pi_{n-1}(x)|^p \le cn\int_{\frac{c_1}{n}\le|t|\le 2\epsilon}|\pi_{n-1}(t)|^p\,w(t)dt \le cn\int_\Delta|\pi_{n-1}(t)|^p w(t)dt$$

for $\frac{c_1}{n}\le|x|\le 2\epsilon$. Hence (9) holds also for $\frac{c_1}{n}\le|x|\le\epsilon$. If $|x|\le\frac{c_1}{n}$ we apply Lemma 21.

Lemma 24. Lemma 23 remains true if $-1<\Gamma<0$ instead of $\Gamma\ge 0$.

Proof. Let $\epsilon>0$ be such that $\Delta\subset(-\epsilon,\epsilon)$. Let $w^*(t) = |t|^{-\frac{\Gamma}{p}}$ for

$|t| \le \epsilon$ with $\mathrm{supp}(w^*) = [-\epsilon,\epsilon]$. Then by Lemma 19

$$\lambda_n(w^*,x) \sim \frac{1}{n}(|x| + \frac{1}{n})^{-\frac{\Gamma}{p}}$$

for $x \in \Delta$. By Lemma 23

$$|\pi_{n-1}(x)\ \lambda_n^{-1}(w^*,x)|^p \le c\ n \int_\Delta |\pi_{n-1}(t)\ \lambda_n^{-1}(w^*,t)|^p\ dt$$

for $x \in \Delta_1$. Hence

$$|\pi_{n-1}(x)|^p\ (|x| + \frac{1}{n})^\Gamma \le c\ n \int_\Delta |\pi_{n-1}(t)|^p\ |t|^\Gamma\ dt$$

for $x \in \Delta_1$.

Theorem 25. Let $\mathrm{supp}(d\alpha)$ be compact, $\Delta \subset \mathrm{supp}(d\alpha)$, $t^* \in \Delta^0$, $\Gamma > -1$, $0 < p < \infty$. Let α be absolutely continuous in Δ and let

$$\alpha'(t) \sim |t - t^*|^\Gamma \qquad (t \in \Delta) .$$

Then

$$\lambda_n(d\alpha,p,x) \sim \lambda_n(d\alpha,x) \sim \frac{1}{n}(|x - t^*| + \frac{1}{n})^\Gamma$$

for $x \in \Delta_1 \subset \Delta^0$.

Proof. The inequality

$$\lambda_n(d\alpha,p,x) \le \frac{c}{n}(|x - t^*| + \frac{1}{n})^\Gamma \qquad (x \in \Delta_1 \subset \Delta^0)$$

can be proved exactly by the same way as in Theorem 20 for $p = 2$. For the estimate from below we can suppose that $t^* \in \Delta_1^0$ and then we apply Lemmas 23 and 24.

Corollary 26. Lemma 22 remains valid for $-1 < \Gamma < 0$ and consequently if $\Gamma > -1$, $\epsilon > 0$, $0 \in \Delta^0$ and $0 < p < \infty$ then for every π_n

$$\int_\Delta |\pi_n(t)|^p\ |t|^\Gamma\ dt \le c\ n^\epsilon \int_\Delta |\pi_n(t)|^p\ |t|^{\Gamma+\epsilon}\ dt$$

where $c = c(p,\Gamma,\epsilon,\Delta)$.

Theorem 27. Let $\mathrm{supp}(d\alpha)$ be compact, $0 < p < \infty$, $a > -1$. Let $\Delta(d\alpha) = [c_1,c_2]$, $\delta > 0$ and let α be absolutely continuous in $[c_2 - \delta, c_2]$. Let

$$\alpha'(t) \sim (c_2 - t)^a \qquad (t \in [c_2-\delta, c_2]) .$$

Then

$$\lambda_n(d\alpha,p,x) \sim \frac{1}{n}(\sqrt{c_2 - x} + \frac{1}{n})^{2a+1}$$

for $x \in [c_2 - \frac{\delta}{2}, c_2]$.

Proof. We have by Theorem 10 and standard arguments

$$\lambda_n(d\alpha,p,x) \leq \frac{1}{n}(\sqrt{c_2 - x} + \frac{1}{n})^{2a+1}$$

for $x \in [c_2 - \frac{\delta}{2}, c_2]$. The converse inequality follows from Lemma 5 and Theorem 13.

From Theorems 25 and 27 we obtain

Theorem 28. Let $1 = t_1 > t_2 > \ldots > t_N = -1$, $\Gamma_k > -1$ for $k = 1,2,\ldots,N$ and let w be defined by

$$w(t) = \prod_{k=1}^{N} |t - t_k|^{\Gamma_k} \qquad (-1 \leq t \leq 1)$$

with $\mathrm{supp}(w) = [-1,1]$. Let

$$\bar{w}_n(t) = (\sqrt{1 - t} + \frac{1}{n})^{2\Gamma_1+1} \prod_{k=2}^{N-1} (|t - t_k| + \frac{1}{n})^{\Gamma_k} (\sqrt{1 + t} + \frac{1}{n})^{2\Gamma_N+1}$$

for $-1 \leq t \leq 1$. Then for every $0 < p < \infty$

$$\lambda_n(w,p,x) \sim \lambda_n(w,x) \sim \frac{1}{n} \bar{w}_n(x)$$

for $|x| \leq 1$.

Remark 29. We can establish inequalities similar to those in Theorem 14 and Corollaries 15 and 16. The exact formulation of those results is left to the reader.

Recall that v denotes the Chebyshev weight.

Lemma 30. Let $p > 1$. Then

$$\int_{-1}^{1} |K_n(v,x,t)|^p \, v(t)dt \sim n^{p-1} \sim \lambda_n(v,x)^{1-p}$$

for $-1 \leq x \leq 1$.

Proof. The estimate

$$n^{p-1} \le C \int_{-1}^{1} |K_n(v,x,t)|^p \, v(t)dt$$

follows immediately from Theorem 13. The converse estimate is obvious when $p \ge 2$ and can be obtained by a simple calculation from Lemma 8 when $1<p<2$.

<u>Lemma 31</u>. Let α be an arbitrary weight, $p > 1$. Then for almost every $x \in [-1,1]$

$$\lim_{n\to\infty} \frac{\int_{-1}^{1} |K_n(v,x,t)|^p \, d\alpha(t)}{\int_{-1}^{1} |K_n(v,x,t)|^p \, v(t)dt} = \sqrt{1-x^2}\,\alpha'(x).$$

<u>Proof</u>. Using Lemma 30 the lemma can be proved in almost the same way as Lemma 6.2.32. We will not go into details.

<u>Theorem 32</u>. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$ and $0 < p < \infty$. Then

$$\limsup_{n\to\infty} n\,\lambda_n(d\alpha,p,x) \le C\,\alpha'(x)\sqrt{1-x^2} \tag{10}$$

for almost every $x \in [-1,1]$ where $C = C(p)$.

<u>Proof</u>. Let m be a natural integer such that $mp > 1$. Then

$$\lambda_n(d\alpha,p,x) \le \int_{-1}^{1} \left| \frac{K_{[\frac{n}{m}]}(v,x,t)}{K_{[\frac{n}{m}]}(v,x,x)} \right|^{mp} d\alpha(t) .$$

Now we apply Lemmas 30 and 31.

<u>Theorem 33</u>. Let $\alpha \in M(0,1)$ and $0 < p < \infty$. Then for almost every $x \in \operatorname{supp}(d\alpha)$, (10) holds with $C = C(p)$.

<u>Proof</u>. Combine the arguments used in the proof of Theorems 32 and 6.2.51.

<u>Theorem 34</u>. Let α be an arbitrary weight. Let Δ and $\epsilon > 0$ be given and let v_Δ denote the Chebyshev weight corresponding to Δ. Let $[\alpha']^{-\epsilon} \in L^1(\Delta)$. Then for each $p \in (0,\infty)$

$$\liminf_{n\to\infty} n\,\lambda_n(d\alpha,p,x) \ge C\,\alpha'(x)\; v_\Delta(x)^{-1}$$

for almost every $x \in \Delta$ where $C = C(\epsilon, \Delta, p)$.

Proof. Let $q = \epsilon p(1+\epsilon)^{-1}$, m and M be natural integers such that $mp\epsilon > 1$ and $2\epsilon pM > 1+\epsilon$. Let $N = [\frac{n}{m}]$. We can suppose without loss of generality that $\Delta = [-1,1]$. Then by Theorem 13

$$|\lambda_N^{-m}(v,x)\ v(x)^{-2M}\ \pi_{n-1}(x)|^q \le C\ n \int_{-1}^{1} |K_N^m(v,x,t)\ v(t)^{-2M}\ \pi_{n-1}(t)|^q\ v(t)dt\ .$$

Hence by Hölder's inequality

$$|\pi_{n-1}(x)|^p \le C\ n^{\frac{p}{q}}\ \lambda_N(v,x)^{pm}\ v(x)^{2Mp} \int_{-1}^{1} |\pi_{n-1}(t)|^p\ \alpha'(t)dt\ \cdot$$

$$\cdot \left(\int_{-1}^{1} |K_N(v,x,t)|^{\frac{mpq}{p-q}}\ v(t)^{\frac{p}{p-q}-\frac{2Mpq}{p-q}}\ \alpha'(t)^{-\frac{q}{p-q}}dt\right)^{\frac{p-q}{q}}\ .$$

Using Lemma 30 we obtain

$$|\pi_{n-1}(x)|^p \le C\ n\ v(x)^{2Mp} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p\ d\alpha(t)\ \cdot$$

$$\cdot \left(\frac{\int_{-1}^{1} |K_N(v,x,t)|^{m\epsilon p}\ v(t)^{1+\epsilon-2M\epsilon p}\ \alpha'(t)^{-\epsilon}\ dt}{\int_{-1}^{1} |K_N(v,x,t)|^{m\epsilon p}\ v(t)\ dt}\right)^{1/\epsilon}\ .$$

Consequently

$$\frac{1}{n\ \lambda_n(d\alpha,p,x)} \le C\,n\,v(x)^{2Mp} \left(\frac{\int_{-1}^{1} |K_N(v,x,t)|^{\epsilon mp}\ v(t)^{1+\epsilon-2M\epsilon p}\ \alpha'(t)^{-\epsilon}dt}{\int_{-1}^{1} |K_N(v,x,t)|^{m\epsilon p}\ v(t)dt}\right)^{1/\epsilon}\ .$$

By the conditions $v^{1+\epsilon-2M\epsilon p}\ (\alpha')^{-\epsilon}$ is a weight. Thus the theorem follows from Lemma 31.

In Theorem 34 the most important case is when $p = 2$. Let us formulate it separately as

Theorem 35. Let $\epsilon > 0$. If $(\alpha')^{-\epsilon} \in L^1(\Delta)$ then

$$\limsup_{n\to\infty} \frac{\alpha'(x)\ v_\Delta(x)^{-1}}{n\ \lambda_n(d\alpha,x)} \in L^\infty(\Delta)\ .$$

Let us note that Corollary 6.2.24 is contained in Theorem 35. Hence Theorems 6.2.25 and 6.2.26 remain valid if $[\alpha']^{-\epsilon} \in L^1(\Delta)$ instead of $1/\alpha' \in L^1(\Delta)$.

7. The Coefficients in the Recurrence Formula

Theorem 1. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$ and

$$\sum_{j=1}^{\infty} \left|\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} - \frac{1}{2}\right| < \infty .$$

Then $\alpha \in S$.

Proof. Let $0 \leq k \leq n$. Let us divide both sides in 3.1(2) by x^n and let $x \to \infty$. We obtain

$$2^{-n} \gamma_n(d\alpha) = 2^{-k} \gamma_k(d\alpha) + \sum_{j=k+1}^{n} 2^{-j} \gamma_j(d\alpha) \left[1 - 2 \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}\right] .$$

Let us fix k so that

$$\sum_{j=k+1}^{\infty} \left|1 - 2 \frac{\gamma_{j-1}}{\gamma_j}\right| < \frac{1}{2} .$$

Then for every $n \geq k + 1$

$$2^{-n} \gamma_n \leq 2^{-k} \gamma_k + \frac{1}{2} \max_{k+1 \leq j \leq n} 2^{-j} \gamma_j$$

and thus for every $m \geq k + 1$

$$\max_{k+1 \leq n \leq m} 2^{-n} \gamma_n \leq 2^{-k} \gamma_k + \frac{1}{2} \max_{k+1 \leq j \leq m} 2^{-j} \gamma_j ,$$

in particular for $m \geq k + 1$

$$2^{-m} \gamma_m \leq 2^{1-k} \gamma_k .$$

Let us fix $\epsilon > 0$ and define β by

$$d\beta(x) = \begin{cases} d\alpha(x) + \epsilon \, dx & -1 \leq x \leq 1 \\ 0 & |x| > 1 . \end{cases}$$

Since $\operatorname{supp}(d\alpha) \subset [-1,1]$, the measure $d\beta$ is greater than $d\alpha$. Therefore

$$\gamma_n(d\beta) \leq \gamma_n(d\alpha)$$

for $n = 1,2,\ldots$. Hence

$$2^{-m} \gamma_m(d\beta) \leq 2^{1-k} \gamma_k(d\alpha)$$

for $m \geq k + 1$. Consequently

$$\limsup_{m\to\infty} 2^{-m} \gamma_m(d\beta) \le 2^{1-k} \gamma_k(d\alpha) .$$

Since $\beta \in S$ we obtain from Lemma 4.2.2 that

$$\lim_{m\to\infty} 2^{-m} \gamma_m(d\beta) = \frac{1}{\sqrt{\pi}} \exp\{-\frac{1}{2\pi}\int_{-1}^{1} v(t)\log\beta'(t)\, dt\} ,$$

that is

$$\lim_{m\to\infty} 2^{-m} \gamma_m(d\beta) = \frac{1}{\sqrt{\pi}} \exp\{-\frac{1}{2\pi}\int_{-1}^{1} v(t)\log[\alpha'(t)+\epsilon]\, dt\} .$$

Hence for every $\epsilon > 0$

$$\frac{1}{\sqrt{\pi}} \exp\{-\frac{1}{2\pi}\int_{-1}^{1} v(t)\log[\alpha'(t)+\epsilon]dt\} \le 2^{1-k}\gamma_k .$$

Letting $\epsilon \to 0$ we see that

$$\int_{-1}^{1} v(t)\log\alpha'(t)dt > -\infty .$$

Hence $\alpha \in S$.

<u>Lemma 2</u>. Let w be of the form $w(x) = \varphi(x)\,|x|^{\epsilon}$, $(\epsilon > -1,\ -1 \le x \le 1)$ where $\varphi(>0)$ is an even function of bounded variation with $\varphi(1) > 0$. Then for $n = 1,2,\ldots$

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \frac{1}{2}\,\frac{n+\frac{\epsilon}{2}[1+(-1)^{n+1}]}{[n+\frac{1}{2}(a_n+\epsilon+1)]^{1/2}\,[n+\frac{1}{2}(a_{n-1}+\epsilon-1)]^{1/2} - \frac{1}{2}b_n}$$

where

$$a_n = \int_{-1}^{1} p_n^2(w,x)\, x\,|x|^{\epsilon}\, d\varphi(x)$$

and

$$b_n = \int_{-1}^{1} p_{n-1}(w,x)\, p_n(w,x)\,|x|^{\epsilon}\, d\varphi(x) .$$

<u>Proof</u>. Since $\gamma_{n-1}(w)/\gamma_n(w) = \gamma_{n-1}(cw)/\gamma_n(cw)$ for every $c > 0$ we can suppose that $\varphi(\pm 1) = 1$. We have

$$\int_{-1}^{1} x\,[p_n^2(w,x)]'\, w(x)dx = 2\int_{-1}^{1} p_n(w,x)\,[n\gamma_n(w)x^n + \ldots]\, w(x)dx = 2n .$$

On the other hand since w is even

$$\int_{-1}^{1} x[p_n^2(w,x)]'\, w(x)dx = 2p_n^2(w,1) - \int_{-1}^{1} p_n^2(w,x)dx\, w(x) =$$

$$= 2p_n^2(w,1) - 1 - \int_{-1}^{1} p_n^2(w,x)x \; dw(x) .$$

Thus

$$2n + 1 = 2p_n^2(w,1) - \int_{-1}^{1} p_n^2(w,x)x \; dw(x) .$$

Because $w(x) = \varphi(x) \; |x|^\varepsilon$ we obtain

$$\int_{-1}^{1} p_n^2(w,x)x \; dw(x) = \int_{-1}^{1} p_n^2(w,x)x \; |x|^\varepsilon \; d\varphi(x) +$$

$$+ \varepsilon \int_{-1}^{1} p_n^2(w,x)x \; \varphi(x) \; |x|^{\varepsilon-1} \operatorname{sign} x \, dx = \int_{-1}^{1} p_n^2(w,x)x \; |x|^\varepsilon \; d\varphi(x) + \varepsilon .$$

Consequently

$$p_n^2(w,1) = n + \frac{1+\varepsilon}{2} + \frac{1}{2}\int_{-1}^{1} p_n^2(w,x)x \; |x|^\varepsilon \; d\varphi(x) \tag{1}$$

for $n = 0,1,\ldots$. Now we shall consider another integral:

$$\int_{-1}^{1} [p_n(w,x) \; p_{n-1}(w,x)]' \; w(x)dx = \int_{-1}^{1} p_n'(w,x) \; p_{n-1}(w,x) \; w(x)dx =$$

$$= \int_{-1}^{1} p_{n-1}(w,x) \; [n\gamma_n(w)x^{n-1} + \ldots] \; w(x)dx = n \frac{\gamma_n(w)}{\gamma_{n-1}(w)} .$$

But

$$\int_{-1}^{1} [p_n(w,x)p_{n-1}(w,x)]' \; w(x)dx = 2p_n(w,1)p_{n-1}(w,1) - \int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)dw(x)$$

and

$$\int_{-1}^{1} p_n(w,x) \; p_{n-1}(w,x) \; dw(x) =$$

$$= \int_{-1}^{1} p_n(w,x) \; p_{n-1}(w,x) \; |x|^\varepsilon \; d\varphi(x) + \varepsilon \int_{-1}^{1} p_n(w,x) \; p_{n-1}(w,x) \; \frac{w(x)}{x} \, dx .$$

If n is even then $p_{n-1}(w,x)x^{-1}$ is a polynomial of degree $n-2$ and consequently the latter integral equals 0 . If n is odd then $p_n(w,x)x^{-1}$ is a polynomial of degree $n-1$. Thus

$$\int_{-1}^{1} p_{n-1}(w,x) \; p_n(w,x) \; \frac{w(x)}{x} \, dx = \int_{-1}^{1} p_{n-1}(w,x) \; [\gamma_n x^{n-1} + \ldots] \; w(x)dx = \frac{\gamma_n(w)}{\gamma_{n-1}(w)} .$$

Hence for $n = 1,2,\ldots$

$$\int_{-1}^{1} p_n(w,x) \; p_{n-1}(w,x) \; \frac{w(x)}{x} \, dx = \frac{1}{2} \frac{\gamma_n(w)}{\gamma_{n-1}(w)} [1 + (-1)^{n+1}] .$$

Thus we obtain

$$(2)\quad \{n+\tfrac{\varepsilon}{2}[1+(-1)^{n+1}]\}\frac{\gamma_n(w)}{\gamma_{n-1}(w)} = 2p_n(w,1)p_{n-1}(w,1) - \int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)\,|x|^{\varepsilon}d\varphi(x).$$

Putting (1) into (2) we finish the proof.

Theorem 3. Let $\operatorname{supp}(w) \subset [-1,1]$, $w(>0)$ be even and of bounded variation with $w(1) > 0$. Let

$$(3)\qquad \sup_{n\geq 1} \int_{-1}^{1} p_n^2(w,x)\ |dw(x)| < \infty .$$

If $w^{-\varepsilon} \in L^1(\tau)$, $(\varepsilon > 0,\ \ \tau \subset (-1,1))$ then the sequence $\{|p_n(w,t)|\}$ is bounded for almost every $t \in \tau$.

Proof. We apply Lemma 2 with $\varepsilon = 0$. Both $\{a_n\}$ and $\{b_n\}$ are bounded by (3). Thus

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \frac{1}{2} + O(\frac{1}{n}) .$$

Since w is even we obtain $C_j^{0,1}(w) = O(\frac{1}{n})$. Now we use Theorem 6.2.26.

Theorem 4. Let $w(x) = \varphi(x)\ |x|^{\varepsilon}$, $(\varepsilon > -1,\ \ -1 \leq x \leq 1)$. Let φ be even, continuous and positive and let φ' be also continuous. Then

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \frac{1}{2} + (-1)^{n+1}\ \frac{\varepsilon}{4n} + o(\frac{1}{n})$$

for $n = 1,2,\ldots$. If φ is constant then $o(\frac{1}{n})$ can be replaced by $O(\frac{1}{n^2})$.

Proof. Let us use Lemma 2. By the conditions, $a_n = O(1)$ and $b_n = O(1)$. Thus $w \in M(0,1)$. Further

$$\{n + \tfrac{\varepsilon}{2}[1 + (-1)^{n+1}]\}\ \frac{\gamma_n}{\gamma_{n-1}} =$$

$$= 2n[1 + \frac{1}{2n}(a_n + \varepsilon + 1)]^{1/2}\ [1 + \frac{1}{2n}(a_{n-1} + \varepsilon - 1)]^{1/2} - b_n =$$

$$= 2n[1 + \frac{1}{4n}(a_n + \varepsilon + 1) + O(\frac{1}{n^2})]\ [1 + \frac{1}{4n}(a_{n-1} + \varepsilon - 1) + O(\frac{1}{n^2})] - b_n =$$

$$= 2n + \varepsilon + \frac{1}{2}(a_n + a_{n-1} - 2b_n) + O(\frac{1}{n}) .$$

Thus

$$\frac{\gamma_n}{\gamma_{n-1}} = 2 + \frac{\epsilon(-1)^n}{n + \frac{\epsilon}{2}[1 + (-1)^{n+1}]} + O(\frac{|a_n + a_{n-1} - 2b_n|}{n}) + O(\frac{1}{n^2}) .$$

Finally we obtain

$$\frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2} + (-1)^{n+1} \frac{\epsilon}{4n} + O(\frac{|a_n + a_{n-1} - 2b_n|}{n}) + O(\frac{1}{n^2}) .$$

If φ is constant then $a_n = a_{n-1} = b_n = 0$. If φ is not constant then we have to show that

$$\lim_{n\to\infty} (a_n + a_{n-1} - 2b_n) = 0 . \tag{4}$$

By the recurrence formula

$$a_n = \frac{\gamma_n}{\gamma_{n+1}} b_{n+1} + \frac{\gamma_{n+1}}{\gamma_n} b_n .$$

Hence

$$a_n + a_{n-1} - 2b_n = \frac{\gamma_n}{\gamma_{n+1}} b_{n+1} + 2(\frac{\gamma_{n+1}}{\gamma_n} - 1)b_n + \frac{\gamma_{n+2}}{\gamma_{n-1}} b_{n-1} .$$

Since $w \in M(0,1)$, if $\lim_{n\to\infty} b_n$ exists and it is finite then (4) holds. But

$$b_n = \int_{-1}^{1} p_n(w,x)\, p_{n-1}(w,x)\, \frac{\varphi'(x)}{\varphi(x)}\, w(x)dx .$$

By the conditions, φ'/φ is continuous on $[-1,1]$. Using Theorem 4.2.13 we obtain

$$\lim_{n\to\infty} b_n = \frac{1}{\pi} \int_{-1}^{1} \frac{t\,\varphi'(t)}{\varphi(t)\sqrt{1 - t^2}}\, dt < \infty .$$

Consequently (4) is satisfied.

<u>Theorem 5</u>. Let α be such that either $\mathrm{supp}(d\alpha) \subset [-1,1]$ or $\alpha \in M(0,1)$. Let $\tau \subset [-1,1]$ and φ be defined by

$$\varphi(x) = \sup_{n\geq 0} p_n^2(d\alpha,x) \qquad (x \in \tau) .$$

Then for almost every $x \in \tau$

$$\alpha'(x)\sqrt{1 - x^2} \geq \frac{1}{\pi\,\varphi(x)} ,$$

in particular, if $\varphi(x)$ is finite for almost every $x \in \tau$ then $\alpha'(x) > 0$ for almost every $x \in \tau$ and if $\varphi(x) \leq K < \infty$ for almost every $x \in \mathfrak{M} \subset \tau$ then

$$\alpha'(x)\sqrt{1-x^2} \geq \frac{1}{k\pi} \tag{5}$$

for almost every $x \in \mathfrak{M}$.

Proof. By the definition of φ, $n\lambda_n(d\alpha,x) \geq \varphi(x)^{-1}$ for $x \in \tau$ and we apply Theorems 6.2.33 and 6.2.51.

Let us note that putting α = Chebyshev weight we see that the constant K in (5) is not exact.

Definition 6. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$. Let $\mu = \mu_\alpha$ be the weight on the unit circumference associated with α in the usual way:

$$\mu(\theta) = \begin{cases} \alpha(1) - \alpha(\cos\theta) & \text{for } 0 \leq \theta \leq \pi \\ \alpha(\cos\theta) - \alpha(1) & \text{for } -\pi \leq \theta \leq 0 . \end{cases}$$

Let $\theta_n(d\mu,z) = z^n + \ldots$ $(n = 0,1,\ldots)$ denote the corresponding system of orthogonal polynomials. It is known that the coefficients of $\theta_n(d\mu,z)$ are real. (See e.g., Freud, §V.1.) We put

$$a_n = -\theta_{n+1}(d\mu,0) .$$

Lemma 7. $\alpha \in S$ iff $\sum_{n=0}^{\infty} a_n^2 < \infty$.

Proof. See Geronimus, §8.2.

Lemma 8. We have

$$\frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} = \frac{1}{2}\,[(1 - a_{2n-1})(1 - a_{2n}^2)(1 + a_{2n+1})]^{1/2}$$

and

$$\alpha_n(d\alpha) = \frac{1}{2}\,[a_{2n-2}(1 + a_{2n-1}) - a_{2n}(1 - a_{2n-1})]$$

for $n = 1,2,\ldots$.

Proof. Calculation. For the first relation see e.g. Geronimus, §9.1.

From Lemmas 7 and 8 we obtain

Theorem 9. Let $\alpha \in S$. Then

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha)^2 < \infty .$$

Remark 10. The converse of Theorem 9 is not true. Example: the Pollaczek weight.

Let us note that Geronimus has proved that if $\sum_{n=0}^{\infty} |a_n| < \infty$ then μ is absolutely continuous, μ' is continuous and positive and $|\varphi_n(d\mu,z)| \leq C < \infty$ for $n = 1,2,\ldots$, $-\pi \leq \theta \leq \pi$, $z = e^{i\theta}$. Here $\varphi_n(d\mu,z)$ denotes the corresponding orthonormal polynomial. It is obvious that neither $\alpha'(x) > 0$ for $-1 \leq x \leq 1$ nor $|p_n(d\alpha,x)| \leq C < \infty$ for $n = 1,2,\ldots$, $-1 \leq x \leq 1$ follows from $\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty$. To see this, consider e.g. a Jacobi weight. Later we will show that $\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty$ neither implies that $\mathrm{supp}(d\alpha) = [-1,1]$ nor that α is absolutely continuous but $\mathrm{supp}(\alpha') = [-1,1]$ follows from $\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty$.

Theorem 11. Let $\alpha \in S$. Then the series

$$\sum_{k=1}^{\infty} (1 - x^2)\, \lambda_{2^k+1}(d\alpha,x)\, p_{2^k}^2(d\alpha,x)$$

converges uniformly for $x \in [-1,1]$.

Proof. By Theorem 3.1.8

$$(1 - x^2)\lambda_{2^k+1}(d\alpha,x)\, p_{2^k}^2(d\alpha,x) \leq C\Big[2^{-k} + \sum_{j=2^{k-1}+1}^{2^k} C_j^{0,1}(d\alpha)^2\Big]$$

and we apply Theorem 9.

Theorem 12. Let either $\alpha \in M(0,1)$ or $\mathrm{supp}(d\alpha) \subset [-1,1]$. Let $n_1 < n_2 < \ldots$ be such that $\sum_{k=1}^{\infty} n_k^{-1} < \infty$. Then the series

$$\sum_{k=1}^{\infty} \lambda_{n_k}(d\alpha,x)\, p_{n_k}^2(d\alpha,x)$$

converges for almost every $x \in \mathrm{supp}(d\alpha)$.

Proof. By Theorem 3.3.7, we can assume that $x \in \mathrm{supp}(d\alpha) \cap [-1,1]$. Using Beppo Levi's theorem we obtain that the series

$$\sum_{k=1}^{\infty} \frac{1}{n_k}\, \alpha'(x)\, p_{n_k}^2(d\alpha,x)$$

converges for almost every $x \in \mathrm{supp}(d\alpha) \cap [-1,1]$. Now apply Theorems 6.2.33 and 6.2.51.

Lemma 13. Let $p_n(d\alpha,x) = \gamma_n(d\alpha)x^n + \mu_n(d\alpha)x^{n-1} + \dots$. Then

$$\sum_{j=0}^{n-1} \alpha_j(d\alpha) = -\frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)}$$

for $n = 1,2,\dots,$ in particular,

$$\alpha_n(d\alpha) = \frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)} - \frac{\mu_{n+1}(d\alpha)}{\gamma_{n+1}(d\alpha)} .$$

Proof. We have

$$\sum_{k=0}^{n-1} \alpha_k(d\alpha) = \int_{-\infty}^{\infty} x\, \lambda_n^{-1}(d\alpha,x)d\alpha(x) = \sum_{k=1}^{n} \lambda_{kn}^{-1}(d\alpha) \int_{-\infty}^{\infty} x\, \ell_{kn}^2(d\alpha,x)d\alpha(x) =$$

$$= \sum_{k=1}^{n} x_{kn}(d\alpha) = -\frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)} .$$

Definition 14. Let $t \in \mathbb{R}$. Then δ_t denotes the unit mass concentrated at t, that is

$$\delta_t(x) = \begin{cases} 0 & \text{for } x < t \\ 1 & \text{for } x \geq t . \end{cases}$$

Lemma 15. Let $\epsilon > 0$, $t \in \mathbb{R}$, $\beta = \alpha + \epsilon\delta_t$. Then

$$p_n(d\beta,x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} [p_n(d\alpha,x) - \frac{\epsilon p_n(d\alpha,t)\, K_{n+1}(d\alpha,t,x)}{1 + \epsilon K_{n+1}(d\alpha,t,t)}] \tag{6}$$

where

$$\frac{\gamma_n^2(d\beta)}{\gamma_n^2(d\alpha)} = 1 - \frac{\epsilon p_n^2(d\alpha,t)}{1 + \epsilon K_{n+1}(d\alpha,t,t)} , \tag{7}$$

further

$$\alpha_n(d\beta) = \alpha_n(d\alpha) + \epsilon\frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} \frac{p_n(d\alpha,t)p_{n+1}(d\alpha,t)}{1+\epsilon K_{n+1}(d\alpha,t,t)} - \epsilon\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha,t)p_n(d\alpha,t)}{1+\epsilon K_n(d\alpha,t,t)}. \tag{8}$$

If $\alpha(x) + \alpha(-x) = \text{const.}$ $t > 0$, $\epsilon > 0$ and $\beta = \alpha + \epsilon\delta_t + \epsilon\delta_{-t}$ then

$$p_n(d\beta,x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} [p_n(d\alpha,x) - \epsilon p_n(d\alpha,t) \frac{K_{n+1}(d\alpha,t,x) + (-1)^n K_{n+1}(d\alpha,-t,x)}{1+\epsilon[K_{n+1}(d\alpha,t,t) + (-1)^n K_{n+1}(d\alpha,-t,t)]} .$$

and

$$\frac{\gamma_n^2(d\beta)}{\gamma_n^2(d\alpha)} = 1 - \frac{\epsilon p_n^2(d\alpha,t)}{\frac{1}{2} + \epsilon \sum\limits_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\, 2)}}^{n} p_k^2(d\alpha,t)} .$$

Proof. We will prove only the first part of the Lemma, the second one can be shown exactly in the same way. Let us note that both (7) and (8) follow from (6). If we multiply both sides of (6) by $p_n(d\alpha,x)d\alpha(x)$ and we integrate over $\mathbb{R}$ then we get (7). Using Lemma 13 and comparing coefficients in (6) we obtain (8). Now let us prove (6). Develop $p_n(d\beta,x)$ into a Fourier series in $p_k(d\alpha,x)$. Then

$$p_n(d\beta,x) = \int_{-\infty}^{\infty} p_n(d\beta,u)K_{n+1}(d\alpha,x,u)d\alpha(u) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)}p_n(d\alpha,x) - \epsilon p_n(d\beta,t)K_{n+1}(d\alpha,x,t).$$

Putting here $x = t$ we obtain

$$p_n(d\beta,t) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)}\, p_n(d\alpha,t)\ [1 + \epsilon K_{n+1}(d\alpha,t,t)]^{-1}\ . \tag{9}$$

(6) follows from the above two formulas.

Lemma 16. Let $\alpha \in M(0,1)$, $\epsilon > 0$, $t \in \mathbb{R}$, $\beta_t = \alpha + \epsilon\delta_t$. Then

$$\lim_{n\to\infty} \frac{\gamma_n(d\beta_t)}{\gamma_n(d\alpha)} = \begin{cases} 1 & \text{for } t \in \operatorname{supp}(d\alpha) \\ |\rho(t)|^{-1} & \text{for } t \notin \operatorname{supp}(d\alpha) \end{cases}$$

the convergence is uniform for $t \in \Delta \subset (-1,1)$, further

$$\lim_{n\to\infty} \alpha_n(d\beta_t) = 0$$

for every $t \in \mathbb{R}$ and the convergence is uniform for $t \in \Delta \subset (-1,1)$. If $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha) \setminus \{t\}$ then for $t \notin \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{p_n(d\beta_t,z)}{p_n(d\alpha,z)} = |\rho(t)|\ [1 - \frac{\sqrt{t^2-1}}{\rho(t)}\ \frac{\rho(z)-\rho(t)}{z-t}]$$

and for $t \in \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{p_n(d\beta_t,z)}{p_n(d\alpha,z)} = 1$$

uniformly for $t \in \Delta \subset (-1,1)$. Furthermore

$$\lim_{n\to\infty} p_n(d\beta_t,t)\ p_n(d\alpha,t) = \begin{cases} 0 & \text{for } t \in \operatorname{supp}(d\alpha) \\ \frac{2}{\epsilon}\sqrt{|t|^2-1} & \text{for } t \notin \operatorname{supp}(d\alpha)\ . \end{cases}$$

If $x,t \in \operatorname{supp}(d\alpha)$, $x \neq t$ and the sequence $\{|p_k(d\alpha,x)|\}$ is bounded then

$$(10) \qquad \lim_{n\to\infty} [p_n(d\beta_t,x) - p_n(d\alpha,x)] = 0 .$$

(10) holds uniformly for $x \in \mathfrak{B} = \bar{\mathfrak{B}} \subset \mathrm{supp}(d\alpha)$ if $t \in \mathrm{supp}(d\alpha)\setminus\mathfrak{B}$ and $\{|p_k(d\alpha,x)|\}$ is uniformly bounded for $x \in \mathfrak{B}$. Finally, $\beta_t \in M(0,1)$ for each $t \in \mathbb{R}$.

<u>Proof</u>. The Lemma follows immediately from Lemma 15, Theorems 4.1.11, 4.1.13, 4.1.14 and (9).

Let us note that all the limits in Lemma 16 - except for one - are independent of ϵ.

<u>Lemma 17</u>. Let $\alpha \in M(0,1)$ and

$$\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty .$$

Let for $0 \le k \le n$, $R_{n,k}(d\alpha,x)$ be as in 3.1(2)-(3). Then

$$\lim_{n\ge k\to\infty} R_{n,k}(d\alpha,x) = 0$$

uniformly for $x \in \Delta \subset (-1,1)$.

<u>Proof</u>. By Theorem 3.1.12, the sequence $\{|p_k(d\alpha,x)|\}$ is uniformly bounded for $x \in \Delta \subset (-1,1)$. Now we apply Corollary 3.1.5.

<u>Lemma 18</u>. Let B be a function on $[0,\pi]$. Let

$$\varphi_n(x) = \cos[n\theta + B(\theta)] \qquad (x = \cos\theta)$$

for $n = 0,1,\ldots$. Then for $1 \le k \le n$

$$\varphi_n(x) = \varphi_k(x)\, U_{n-k}(x) - \varphi_{k-1}(x)\, U_{n-k-1}(x)$$

$(-1 \le x \le 1)$.

<u>Proof</u>. Apply Theorem 3.1.1.

<u>Lemma 19</u>. Let $\alpha \in M(0,1)$ and

$$\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty .$$

Suppose that there exist three functions A, B, and C on $[0,\pi]$ and a sequence $n_1 < n_2 < \ldots < n_k \underset{k\to\infty}{\to} \infty$ such that

$$\lim_{k\to\infty} [C(\theta)\, p_{n_k}(d\alpha,x) - \varphi_{n_k}(x)] = 0 \tag{11}$$

and

$$\lim_{k\to\infty} [C(\theta)\, p_{n_k-1}(d\alpha,x) - \varphi_{n_k-1}(x)] = 0 \tag{12}$$

where $x = \cos\theta$, $\varphi_n(x) = A(\theta)\cos[n\theta + B(\theta)]$, $-1 \le x \le 1$, $C(\theta) < \infty$. Then

$$\lim_{n\to\infty} [C(\theta)\, p_n(d\alpha,x) - \varphi_n(x)] = 0\,. \tag{13}$$

If the convergence in (11) and (12) is uniform for $x \in \mathfrak{B} \subset \Delta \subset (-1,1)$ and $C(\theta)$ is uniformly bounded for $x \in \mathfrak{B}$ then (13) holds uniformly for $x \in \mathfrak{B}$.

Proof. By Theorem 3.1.1 and Lemma 18,

$$|C\, p_n - \varphi_n| \le v(|C\, p_k - \varphi_k| + |C\, p_{k-1} - \varphi_{k-1}|) + C|R_{n,k}|$$

for $1 \le k \le n$ where v is the Chebyshev weight. For a fixed n let $k = k(n)$ be defined by $k = \max\{n_j\colon\ n_j \le n\}$. Then $\lim_{n\to\infty} k = \infty$. Now we use Lemma 17.

Recall that the function Γ has been defined in Definition 4.2.4.

Theorem 20. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$ and

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty\,.$$

Then for almost every $x \in [-1,1]$

$$\lim_{n\to\infty} \left[\sqrt{\alpha'(x)\sqrt{1-x^2}}\ p_n(d\alpha,x) - \sqrt{\frac{2}{\pi}}\cos(n\theta - \Gamma(\theta))\right] = 0 \tag{14}$$

where $x = \cos\theta$.

Proof. By Theorem 1, $\alpha \in S$. Thus the Theorem follows from Lemmas 19 and 4.2.5.

Lemma 21. We have

$$\sum_{\substack{k=0\\ k\equiv n \pmod 2}}^{n} p_k^2(v,x) = \frac{n+1}{2\pi} + \frac{1}{4\pi}\frac{U_{2n+1}(x)}{x}$$

and for $t > 1$

$$\frac{p_n^2(v,t)}{\frac{1}{2} + \sum\limits_{\substack{k=0 \\ k\equiv n(\mathrm{mod}\, 2)}}^{n} p_k^2(v,t)} = \frac{4t\sqrt{t^2-1}}{\rho(t)^2} + O(1)\, n\, \rho(t)^{-2n}$$

where $|O(1)| \leq C$ uniformly for $1 + \epsilon \leq t < \infty$.

Proof. Calculation. Recall that v is the Chebyshev weight.

Theorem 22. Let α be the Chebyshev weight, $t > 1$, $\beta = \alpha + \delta_t + \delta_{-t}$. Then

$$\gamma_n(d\beta) = \frac{2^{n-\frac{1}{2}}}{\sqrt{\pi}} [\rho(t)^{-2} + O(1)\, n\, \rho(t)^{-2n}]$$

for $n = 1,2,\ldots$.

Proof. Apply Lemmas 15 and 21.

Theorem 22 is interesting because of the following

Corollary 23. Let $C > 0$ and $\epsilon > 0$. Then there exists a weight $\alpha \in M(0,1)$ such that $[-C,C] \subset \Delta(d\alpha)$ and $c_j^{0,1}(d\alpha) = O(\epsilon^j)$, in particular $\sum\limits_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$.

Proof. For the weight β constructed in Theorem 22, $\alpha_j(d\beta) = 0$ for $j = 0,1,\ldots$.

Lemma 24. Let $\alpha \in M(0,1)$, $\epsilon > 0$, $-1 < t < 1$, $\beta = \alpha + \epsilon\delta_t$. If

$$\limsup_{n\to\infty} \sum_{j=n}^{2n} j\, c_j^{0,1}(d\alpha)^2 < \infty$$

then

$$\alpha_n(d\beta) = \alpha_n(d\alpha) + O(\tfrac{1}{n})$$

and

$$\frac{\gamma_{n-1}(d\beta)}{\gamma_n(d\beta)} = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} + O(\tfrac{1}{n})$$

for $n = 1,2,\ldots$.

Proof. The Lemma follows from Theorem 3.1.8 and Lemma 15. It might be interesting to remark that the estimates do not depend on ϵ.

By Lemma 16 from $\alpha \in M(0,1)$ follows that, $\alpha + \epsilon\delta_t \in M(0,1)$ for every $\epsilon > 0$, $t \in \mathbb{R}$. Hence $\beta = \alpha + \sum_{k=1}^{N} \epsilon_k \delta_{t_k}$ also belongs to $M(0,1)$ and by repeating application of the previous results we obtain asymptotics and estimates for $p_n(d\beta, z)$, $c_j^{0,1}(d\beta)$, $\lambda_n(d\beta, z)$ etc. Let us mention two results.

Theorem 25. Let $\alpha \in M(0,1)$, $t_k \in \mathbb{R}$, $\epsilon_k > 0$ for $k = 1,2,\ldots,N$. Let $\beta = \alpha + \sum_{k=1}^{N} \epsilon_k \delta_{t_k}$. Then $\beta \in M(0,1)$,

$$\lim_{n\to\infty} \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)} = \prod_{t_k \notin \operatorname{supp}(d\alpha)} |\rho(t_k)|^{-1}$$

and for every $z \in \mathbb{C} \setminus \operatorname{supp}(d\beta)$

$$\lim_{n\to\infty} \frac{p_n(d\beta,z)}{p_n(d\alpha,z)} = \prod_{t_k \notin \operatorname{supp}(d\alpha)} \{|\rho(t_k)| \, [1 - \frac{\sqrt{t_k^2 - 1}}{\rho(t_k)} \frac{\rho(z) - \rho(t_k)}{z - t_k}]\} .$$

Theorem 26. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$, $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$, β be defined as in Theorem 25 with $t_k \in (-1,1)$ for $k = 1,2,\ldots,N$. Then for almost every $x \in [-1,1]$ the asymptotic formula (14) holds if we replace there $p_n(d\alpha,x)$ by $p_n(d\beta,x)$.

We suggest the reader combine these results with those of Sections 4.1, 4.2 and 6.1.

Theorem 27. Let

$$\sum_{j=0}^{\infty} j \, c_j^{(0,1)}(d\alpha) < \infty .$$

Then there exists a positive number $K = K(d\alpha)$ such that

$$\sqrt{1 - x^2} \, |p_n(d\alpha,x)| \leq K$$

for $-1 \leq x \leq 1$ and $n = 0,1,\ldots$. Further

$$\alpha'(x) \geq \frac{1}{K^2\pi} \sqrt{1 - x^2}$$

for almost every $x \in [-1,1]$, in particular, $\alpha' \in S$.

Proof. By an inequality of S. Bernstein

$$\max_{|x|\leq 1} |\pi_m(x)| \leq m \max_{|x|\leq 1} |\sqrt{1 - x^2} \, \pi_m(x)|$$

for every π_m if $m \geq 1$. Let $2 \leq k \leq n$. Then we have by Corollaries 3.5.1

$$\max_{|x|\leq 1} |\sqrt{1-x^2}\, p_n(d\alpha,x)| \leq \max_{|x|\leq 1} [|p_k(d\alpha,x)| + |p_{k-1}(d\alpha,x)|] +$$

$$+ 2 \sum_{j=k-1}^{n} j\, c_j^{(0,1)}(d\alpha) \max_{|x|\leq 1} |\sqrt{1-x^2}\, p_j(d\alpha,x)|,$$

or in short

$$A_n \leq C(k) + 2 \sum_{j=k+1}^{n} j\, c_j\, A_j$$

where $c_j = c_j^{0,1}(d\alpha)$, $A_j = \max_{|x|\leq 1} |\sqrt{1-x^2}\, p_j(d\alpha,x)|$ and $C(k)$ is a constant depending on k and $d\alpha$. Let now k be so large that $\sum_{j=k+1}^{\infty} j\, c_j \leq \frac{1}{4}$. Then

$$A_{k+1} \leq 2\, C(k).$$

Suppose that $A_m \leq 2\, C(k)$ for $k < m < n$. If $A_n > 2\, C(k)$ then $A_m \leq 2\, C(k) < A_n$, $m = k+1, \ldots, n-1$, so

$$A_n \leq C(k) + 2 \sum_{j=k+1}^{n} j\, c_j\, A_j < C(k) + A_n\, 2 \sum_{j=k+1}^{n} j\, c_j < C(k) + \frac{1}{2} A_n,$$

that is $A_n < 2C(k)$. This contradiction shows that $A_n \leq 2\, C(k)$ for $n > k$. To finish the proof we apply Theorem 5.

<u>Remark 28</u>. If we put $d\alpha(x) = \sqrt{1-x^2}\, dx$, $\mathrm{supp}(d\alpha) = [-1,1]$ then $c_j^{0,1}(d\alpha) = 0$ fpr every $j = 0,1,\ldots$, $\alpha'(x) = \sqrt{1-x^2}$ and for each $\tau \subset [-1,1]$

$$\max_{x\in\tau} |\sqrt{1-x^2}\, p_n(d\alpha,x)| = \sqrt{\frac{2}{\pi}}$$

if $n \geq n_1(\tau)$, $(n_1([-1,1]) = 0)$. Hence Theorem 27 - except for the constant - cannot be improved.

From Theorems 5 and 3.1.12 we obtain

<u>Theorem 29</u>. Let $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$. Then for each $\tau \subset (-1,1)$ there exists a number $K = K(\tau, d\alpha) > 0$ such that $\alpha'(x) \geq K$ for almost every $x \in \tau$. In particular, if α' is continuous at $t \in (-1,1)$ and $\alpha'(t) = 0$ then

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) = \infty.$$

Corollary 30. If $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ then $\operatorname{supp}(\alpha') = [-1,1]$.

Proof. Theorems 3.3.7 and 29.

Theorem 31. Let α be such that either $\operatorname{supp}(d\alpha) \subset [-1,1]$ or $\alpha \in M(0,1)$. Let $p \geq 2$ and $w(\geq 0) \in L^1(-1,1)$. Then from

$$\sup_{n\geq 0} \int_{-1}^{1} |p_n(d\alpha,x)|^p \, w(x)dx < \infty$$

follows

$$\int_{-1}^{1} [\alpha'(x)\sqrt{1-x^2}]^{-\frac{p}{2}} \, w(x)dx < \infty . \tag{15}$$

Proof. Since $p/2 \geq 1$ we have

$$\|n^{-1}\lambda_n^{-1}(d\alpha)\|_{w,\frac{p}{2}} \leq \frac{1}{n}\sum_{k=0}^{n-1} \|p_n^2(d\alpha)\|_{w,\frac{p}{2}} \leq \sup_{n\geq 0} \|p_n^2(d\alpha)\|_{w,\frac{p}{2}} .$$

Because $\int \liminf |f_n| \leq \limsup \int |f_n|$ the theorem follows from Theorems 6.2.33 and 6.2.51.

Theorem 32. Let $\alpha \in S$, $0 < p < \infty$, $w(\geq 0) \in L^1(-1,1)$. Then

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(d\alpha,x)|^p \, w(x)dx < \infty$$

implies (15).

Proof. First let $0 < p \leq 2$. Then we will use Lemma 4.2.5. Let $N > 0$ and φ_N be defined by

$$\varphi_N(t) = \min\{N, w(\cos t)\sin t[\sin t\alpha'(\cos t)]^{-\frac{p}{2}}\}$$

for $0 \leq t \leq \pi$. Then

$$[\int_0^{\pi} |\sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^2 \, \varphi_N(t)dt]^{\frac{1}{p}} \leq [\int_0^{\pi} |\sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^p \, \varphi_N(t)dt]^{\frac{1}{p}} \leq$$

$$\leq 2^{\frac{1}{p}}[\int_0^{\pi} |p_n(d\alpha,\cos t)\sqrt{\alpha'(\cos t)\sin t} - \sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^p \, \varphi_N(t)dt]^{\frac{1}{p}} +$$

$$+ 2^{\frac{1}{p}} [\int_0^{\pi} |p_n(d\alpha,\cos t)\sqrt{\alpha'(\cos t)\sin t}|^p \, \varphi_N(t)dt]^{\frac{1}{p}} \leq$$

$$\leq 2^{\frac{1}{p}} N^{\frac{1}{p}} \pi^{\frac{2-p}{2}} [\int_0^\pi |p_n(d\alpha, \cos t)\sqrt{\alpha'(\cos t)\sin t} - \sqrt{\frac{2}{\pi}} \cos[nt - \Gamma(t)]|^2 dt]^{\frac{1}{2}} +$$

$$+ 2^{\frac{1}{p}} [\int_{-1}^{1} |p_n(d\alpha, x)|^p w(x)dx]^{\frac{1}{p}} .$$

By Lemma 4.2.5, the first term in the right side converges to 0 when $n \to \infty$. The limit inferior of the second term is finite. By the Riemann-Lebesgue lemma the left side converges to

$$(\frac{1}{\pi} \int_0^\pi \varphi_N(t)dt)^{\frac{1}{p}}$$

when $n \to \infty$. Thus by the above inequalities

$$\frac{1}{\pi} \int_0^\pi \varphi_N(t)dt \leq 2 \liminf_{N\to\infty} \int_{-1}^{1} |p_n(d\alpha, x)|^p w(x)dx .$$

By Beppo Levi's theorem, $\lim\limits_{N\to+\infty} \varphi_N \in L^1(0,\pi)$, that is (15) holds. If $p > 2$ then for $N > 0$

$$\int_{-1}^{1} p_n^2(d\alpha, x) [\alpha'(x)\sqrt{1 - x^2} + N^{-1}]^{-\frac{p}{2}} w_N(x)\, \alpha'(x)\sqrt{1 - x^2} dx \cdot$$

$$\cdot \{\int_{-1}^{1} [\alpha'(x)\sqrt{1 - x^2} + N^{-1}]^{-\frac{p}{2}} w_N(x)dx\}^{\frac{2-p}{p}} \leq \{\int_{-1}^{1} |p_n(d\alpha, x)|^p w(x)dx\}^{\frac{1}{p}} .$$

where $w_N(x) = \min\{N, w(x)\}$. Letting $n\to\infty$ we obtain from Lemma 4.2.5 and from Riemann-Lebesgue's lemma that

$$\pi^{-p}\{\int_{-1}^{1} [\alpha'(x)\sqrt{1 - x^2} + N^{-1}]^{-\frac{p}{2}} w_N(x)dx\}^2 \leq \liminf_{n\to\infty} \int_{-1}^{1} |p_n(d\alpha, x)|^p w(x)dx .$$

Hence again by Beppo Levi's theorem (15) holds.

<u>Theorem 33</u>. Let $w \in M(0,1)$, $\text{supp}(w) = [-1,1]$, w be Riemann integrable on $[-1,1]$. Let $g(\geq 0)$ be almost everywhere continuous on $[-1,1]$ and $p \geq 2$. Then

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(w,x)|^p g(x)dx < \infty$$

implies

$$\int_{-1}^{1} [w(x)\sqrt{1 - x^2}]^{-\frac{p}{2}} g(x)dx < \infty .$$

<u>Proof</u>. In the conditions, the function φ_N defined by

$$\varphi_N(x) = [w(x)\sqrt{1-x^2} + N^{-1}]^{-\frac{p}{2}} \min\{N, g(x)\}\sqrt{1-x^2}$$

$(N > 0)$ is Riemann integrable for each $N > 0$. Now we can repeat the second part of the proof of Theorem 32. Applying Theorem 4.2.14, we obtain the theorem.

<u>Theorem 34</u>. Let $\alpha \in M(0,1)$ and

$$\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty .$$

Then

$$\lim_{k\to\infty} [p_k^2(d\alpha,x) - p_{k-1}(d\alpha,x)\, p_{k+1}(d\alpha,x)] = \frac{2\sqrt{1-x^2}}{\pi\, \alpha'(x)}$$

for almost every $x \in \operatorname{supp}(d\alpha)$.

<u>Proof</u>. Let $0 \le k \le n$ and $\Delta \subset (-1,1)$. Then by Lemma 17

$$p_n(d\alpha,x) = U_{n-k}(x)\, p_k(d\alpha,x) - U_{n-k-1}(x)\, p_{k-1}(d\alpha,x) + \sigma(1)$$

where $\lim\limits_{n\geq k\to\infty} \sigma(1) = 0$ uniformly for $x \in \Delta$. By Theorem 3.1.12, the sequence $\{|p_k(d\alpha,x)|\}$ is uniformly bounded for $x \in \Delta$. Hence

$$p_n^2(d\alpha,x) = U_{n-k}^2(x)\, p_k^2(d\alpha,x) + U_{n-k-1}^2(x)\, p_{k-1}^2(d\alpha,x) -$$

$$- 2p_{k-1}(d\alpha,x)\, p_k(d\alpha,x)\, U_{n-k-1}(x)\, U_{n-k}(x) + \sigma(1).$$

Thus for $m > k$

$$\lambda_{m+1}^{-1}(d\alpha,x) = \sum_{j=0}^{k} p_k^2(d\alpha,x) + \sum_{j=1}^{m-k} U_j^2(x)\, p_k^2(d\alpha,x) + \sum_{j=0}^{m-k-1} U_j^2(x)\, p_{k-1}^2(d\alpha,x) -$$

$$- 2\sum_{j=0}^{m-k-1} U_j(x)\, U_{j+1}(x)\, p_{k-1}(d\alpha,x)\, p_k(d\alpha,x) + m\sigma(1).$$

Let us divide this formula by m and let $m\to\infty$. We obtain from Theorems 29, 6.2.52 and Corollary 6.2.53 that

$$\frac{1}{\pi\alpha'(x)\sqrt{1-x^2}} = \frac{1}{2(1-x^2)}[p_k^2(d\alpha,x) + p_{k-1}^2(d\alpha,x)] - \frac{T_1(x)}{1-x^2}\, p_{k-1}(d\alpha,x)p_k(d\alpha,x) + \sigma(1)$$

for almost every $x \in \Delta$, that is

$$(16) \qquad \frac{2\sqrt{1-x^2}}{\pi\alpha'(x)} = p_k^2(d\alpha,x) + p_{k-1}^2(d\alpha,x) - 2xp_{k-1}(d\alpha,x)p_k(d\alpha,x) + \sigma(1)$$

for almost every $x \in \Delta$. By Theorem 3.1.12, and by the recurrence formula

$$\lim_{k\to\infty} |2xp_{k-1}(d\alpha,x)\, p_k(d\alpha,x) - p_k^2(d\alpha,x) - p_{k-2}(d\alpha,x)\, p_k(d\alpha,x)| = 0$$

uniformly for $x \in \Delta$. Hence

$$(17) \qquad \frac{2\sqrt{1-x^2}}{\pi\,\alpha'(x)} = p_{k-1}^2(d\alpha,x) - p_{k-2}(d\alpha,x)\, p_k(d\alpha,x) + \sigma(1)$$

for almost every $x \in \Delta$ where $\lim_{k\to\infty} \sigma(1) = 0$ uniformly for $x \in \Delta$. Since $\Delta \subset (-1,1)$ is arbitrary, the Theorem follows.

Let us note that the determinant

$$\begin{vmatrix} p_k(d\alpha,x) & p_{k-1}(d\alpha,x) \\ p_{k+1}(d\alpha,x) & p_k(d\alpha,x) \end{vmatrix} = D_k(d\alpha,x)$$

is a rather famous expression, its positivity has been investigated by several authors. (See Szegő, Problems and exercises.) So far $D_k(d\alpha,x)$ has been considered for the classical weights. From Theorem 29 and (17) we obtain the following

<u>Corollary 35</u>. Let $\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty$ and $\Delta \subset (-1,1)$. Then there exists a number $N = N(\alpha,\Delta) > 0$ such that for each $k \geq N$, $D_k(d\alpha,x) > 0$ whenever $x \in \Delta$.

The example of the Chebyshev polynomials shows that Δ cannot be replaced by $[-1,1]$ in Corollary 35.

<u>Corollary 36</u>. If $\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty$ then

$$\limsup_{n\to\infty} \alpha'(x)\sqrt{1-x^2}\, p_n^2(d\alpha,x) = \frac{2}{\pi}$$

for almost every $x \in \operatorname{supp}(d\alpha)$.

<u>Proof.</u> By Theorems 29 and 6.2.51

$$\limsup_{n\to\infty} \alpha'(x)\sqrt{1-x^2}\, p_n^2(d\alpha,x) \geq \frac{2}{\pi}$$

for almost every $x \in \operatorname{supp}(d\alpha)$. On the other hand by (16)

$$p_{k-1}(d\alpha,x) = xp_k(d\alpha,x) \pm [(x^2-1)p_k^2(d\alpha,x) + \frac{2\sqrt{1-x^2}}{\pi\,\alpha'(x)} + \sigma(1)]^{\frac{1}{2}}$$

for almost every $x \in [-1,1]$, that is by Theorem 3.3.7, for almost every $x \in \operatorname{supp}(d\alpha)$. Hence

$$(1 - x^2)p_n^2(d\alpha,x) \le \frac{2\sqrt{1-x^2}}{\pi\,\alpha'(x)} + o(1) .$$

Letting $n\to\infty$ and using Theorem 29 we obtain the Corollary.

Although the proof of the following Theorem is very simple it is one of our strongest results.

<u>Theorem 37</u>. Let $\alpha \in S$. Then

$$\lim_{n\to\infty} \frac{p_n^2(d\alpha,x)}{n} = 0$$

for almost every $x \in [-1,1]$.

<u>Proof</u>. By Corollary 3.1.5,

$$(18)\qquad (1-x^2)p_n^2(d\alpha,x) \le 2\{|p_k(d\alpha,x)| + |p_{k-1}(d\alpha,x)|\}^2 + 8n\sum_{j=k-1}^{\infty} c_j^{0,1}(d\alpha)^2 p_j^2(d\alpha,x).$$

By Theorem 9 and Beppo Levi's theorem

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha)^2\, p_j^2(d\alpha,x) < \infty$$

for almost every $x \in [-1,1]$. Dividing both sides of (18) by n and first letting $n\to\infty$ and then $k\to\infty$ we finish the proof.

<u>Lemma 38</u>. Let $\varphi_{2n}(dw,z)$ be defined as in Theorem 3.1.15. If $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ then

$$(19)\qquad \varphi(d\alpha,z) = \lim_{n\to\infty} \varphi_{2n}(d\alpha,z)$$

$(z = e^{i\theta})$ exists for each $\theta \in (0,2\pi)\setminus\{\pi\}$ and the convergence is uniform for $\theta \in \tau \subset (0,2\pi)\setminus\{\pi\}$, in particular, $\varphi(d\alpha,e^{i\theta})$ is continuous on $(0,2\pi)\setminus\{\pi\}$. If $\sum_{j=0}^{\infty} j\, c_j^{0,1}(d\alpha) < \infty$ then (19) exists for each $|z| \le 1$, the convergence is uniform in the unit disc, $\varphi(d\alpha,z)$ is analytic for $|z| < 1$ and continuous for $|z| \le 1$.

<u>Proof</u>. Apply Theorems 3.1.12, 27 and Bernstein's inequality

$$\max_{|x|\le 1} |\pi_m(x)| \le m \max_{|x|\le 1} |\sqrt{1-x^2}\,\pi_m(x)| \qquad (m \ge 1).$$

Lemma 39. Let $\varphi_{2n}(d\alpha, z)$ be as in Theorem 3.1.15. Let $\sum_{j=0}^{\infty} j\, C_j^{0,1}(d\alpha) < \infty$. Then

$$\lim_{n\to\infty} z^{2n}\, \varphi_{2n}(d\alpha, z^{-1}) = 0$$

uniformly for $|z| \le 1 - \epsilon$, $(\epsilon > 0)$.

Proof. We have

$$z^{2n}\, \varphi_{2n}(d\alpha, z^{-1}) = \sum_{j=0}^{n} a_j\!\left(d\alpha, \frac{z+z^{-1}}{2}\right) z^{2n-j}.$$

By Theorem 27

$$\sqrt{1-x^2}\, |p_j(d\alpha, x)| \le K$$

for $-1 \le x \le 1$, $j = 0,1,\ldots$. Hence

$$\left|(1-z^2)\, z^j\, p_j\!\left(d\alpha, \frac{z+z^{-1}}{2}\right)\right| \le C$$

for $|z| \le 1$, $j = 0,1,\ldots$, Consequently

$$|z^{2n}\, \varphi_{2n}(d\alpha, z^{-1})| \le C(1-|z|^2)^{-1} \cdot$$

$$\cdot \sum_{j=0}^{n} \left\{ \left|1 - 2\frac{\gamma_{j-1}}{\gamma_j}\right| |z|^{2n-2j} + 2|\alpha_{j-1}|\, |z|^{2n-2j+1} + \left|1 - 2\frac{\gamma_{j-2}}{\gamma_{j-1}}\right| |z|^{2n-2j+2} \right\}$$

which converges uniformly to 0 if $|z| \le 1 - \epsilon$.

Theorem 40. Let $\alpha \in M(0,1)$ and $\sum_{j=0}^{\infty} C_j^{0,1}(d\alpha) < \infty$. Then $d\alpha$ can be written in the form

$$d\alpha(t) = \alpha'(t)\, dt + d\alpha_j(t)$$

where α' is continuous and positive in $(-1,1)$, $\operatorname{supp}(\alpha') = [-1,1]$ and $\alpha_j(t)$ is constant for $-1 < t < 1$. Further

$$\lim_{n\to\infty} \left\{\sin\theta\, p_n(d\alpha, \cos\theta) - \left[\frac{2}{\pi}\frac{\sin\theta}{\alpha'(\cos\theta)}\right]^{\frac{1}{2}} \sin[(n+1)\theta - \varphi(\theta)]\right\} = 0 \tag{20}$$

uniformly for $\theta \in \tau \subset (0,\pi)$, where $\varphi(\theta) = \arg\varphi(d\alpha, e^{i\theta})$, (See (19).) is continuous in $(0,\pi)$. α' can be calculated by the formula

$$\frac{2}{\pi}\frac{\sin\theta}{\alpha'(\cos\theta)} = |\varphi(d\alpha, e^{i\theta})|^2 = \lim_{n\to\infty}\{p_n^2(d\alpha, x) - p_{n-1}(d\alpha, x)\, p_{n+1}(d\alpha, x)\}$$

$(x = \cos\theta)$. If $\sum_{j=0}^{\infty} j\, c_j^{0,1}(d\alpha) < \infty$ then

$$\lim_{n\to\infty} \{\sin\theta\, p_n(d\alpha,\cos\theta) - \psi(\theta)\sin[(n+1)\theta - \varphi(\theta)]\} = 0$$

uniformly for $\theta \in [0,\pi]$. Here $\psi(\theta) = |\varphi(d\alpha,e^{i\theta})|$ and $\varphi(\theta) = \arg\varphi(d\alpha,e^{i\theta})$ are continuous functions on $[0,\pi]$.

<u>Proof</u>. Let first $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$. Then by Lemma 38 and Theorem 3.1.15,

$$\lim_{n\to\infty} \{\sin\theta\, p_n(d\alpha,\cos\theta) - |\varphi(d\alpha,e^{i\theta})|\sin[(n+1)\theta - \arg\varphi(d\alpha,e^{i\theta})]\} = 0$$

uniformly for $\theta \in \tau \subset (0,\pi)$. Now let us calculate $|\varphi(d\alpha,e^{i\theta})|$. We have

$$(1-x^2)\frac{1}{n\lambda_n(d\alpha,x)} = |\varphi(d\alpha,e^{i\theta})|^2 \frac{1}{2n}\sum_{k=0}^{n} \{1-\cos[2(k+1)\theta - 2\arg\varphi(d\alpha,e^{i\theta})]\} + o(1)$$

$(x = \cos\theta)$. Here the right side converges to $\frac{1}{2}|\varphi(d\alpha,e^{i\theta})|^2$ when $n\to\infty$ and the convergence is uniform for $\theta \in \tau \subset (0,\pi)$. By Theorem 6.2.54

$$\liminf_{n\to\infty} (1-x^2)\frac{1}{n\,\lambda_n(d\alpha,x)} = \frac{\sqrt{1-x^2}}{\pi\,\alpha'(x)}$$

for almost every $x \in \mathrm{supp}(d\alpha)$. Hence

$$|\varphi(d\alpha,e^{i\theta})|^2 = \frac{2\sqrt{1-x^2}}{\pi\,\alpha'(x)} \tag{21}$$

for almost every $\theta \in \tau \subset (0,\pi)$. Consequently $[\alpha']^{-1}\sqrt{1-x^2}$ is equivalent to a continuous function. By Theorem 29, $\alpha'(x) \geq k > 0$ for almost every $\theta \in \tau \subset (-1,1)$. Thus $\alpha'(\cos\theta)$ and $|\varphi(d\alpha,e^{i\theta})|$ are continuous and positive in $(0,\pi)$ and (21) holds for each $\theta \in (0,\pi)$. Hence (20) holds uniformly for $\theta \in \tau \subset (0,\pi)$. From (20) we obtain

$$\lim_{n\to\infty} \frac{1}{n\,\lambda_n(d\alpha,x)} = \frac{1}{\pi\,\alpha'(x)\sqrt{1-x^2}} \tag{22}$$

uniformly for $x \in \Delta \subset (-1,1)$. Thus

$$\limsup_{n\to\infty} p_n^2(d\alpha,x) > 0$$

for every $x \in (-1,1)$, that is α_j must be constant in $(-1,1)$. Now we will show that α has no singular component. Because α_j is constant in $(-1,1)$ for every $\Delta\subset(-1,1)$, $1_\Delta \alpha' v^{-1}$ is $d\alpha$ measurable. Consequently by Theorem 4.2.14

$$\lim_{n\to\infty} \int_\Delta \alpha'(t)\sqrt{1-t^2}\, p_n^2(d\alpha,t)\, d\alpha(t) = \frac{1}{\pi}\int_\Delta \alpha'(t)dt$$

for every $\Delta \subset (-1,1)$. By (22), we obtain

$$\int_\Delta d\alpha(t) = \int_\Delta \alpha'(t)dt ,$$

that is $\alpha_s(t) \equiv 0$ for $-1 < t < 1$. Finally we apply Theorem 3.3.7. If $\sum_{j=0}^{\infty} j\, C_j^{0,1}(d\alpha) < \infty$ then we use Lemma 38 and Theorem 3.1.15.

<u>Remark 41</u>. In general, the function $\varphi(\theta)$ in (20) does not coincide with $\Gamma(\theta) + \theta - \frac{\pi}{2}$ where Γ is defined in Definition 4.2.4. For instance, if β is the weight introduced in Theorem 22, then $\varphi(\theta) \neq \Gamma(\theta) + \theta - \frac{\pi}{2}$. If we know that $\mathrm{supp}(d\alpha) = [-1,1]$ then by Theorem 1, $\alpha \in S$ and by Theorem 20, $\varphi(\theta) = \Gamma(\theta) + \theta - \frac{\pi}{2}$. Thus by Theorem 41, (14) holds uniformly for $x \in \tau \subset \subset (-1,1)$ if the conditions of Theorem 20 are satisfied.

<u>Theorem 42</u>. Let $\alpha \in M(0,1)$ and $\sum_{j=0}^{\infty} j\, C_j^{0,1}(d\alpha) < \infty$. Then

$$\lim_{n\to\infty} p_n(d\alpha,z)\, \rho(z)^{-n-1} = \frac{1}{2\sqrt{z^2-1}}\, \varphi(d\alpha, \rho(z)^{-1})$$

uniformly for $|\rho(z)| \geq R > 1$ where φ is defined by (19). $\varphi(d\alpha, \rho(z)^{-1})$ is analytic in the domain $|\rho(z)| > 1$ and vanishes for $z \in \mathrm{supp}(d\alpha)\setminus[-1,1]$.

<u>Proof</u>. Use Lemmas 38, 39 and Theorem 3.1.15. If $x \in \mathrm{supp}(d\alpha)\setminus[-1,1]$ then by Theorem 3.1.7, α has a jump at x. Hence $\lim_{n\to\infty} p_n(d\alpha,x) = 0$ that is $\varphi(d\alpha, \rho(x)^{-1}) = 0$.

<u>Remark 43</u>. If $\mathrm{supp}(d\alpha) = [-1,1]$, then $\alpha \in S$ and by Lemma 6.1.18

$$\frac{\rho(z)}{2\sqrt{z^2-1}}\, \varphi(d\alpha, \rho(z)^{-1}) = \frac{1}{\sqrt{2\pi}}\, D(v^{-1}d\alpha, \rho(z)^{-1})^{-1}$$

for $|\rho(z)| > 1$.

<u>Theorem 44</u>. Let $\alpha \in M(0,1)$, $\sum_{j=0}^{\infty} j\, C_j^{0,1}(d\alpha) < \infty$. Let $g(\geq 0)$ be Riemann integrable on $[-1,1]$ and let $g^{\pm 1}$ be bounded on $\mathrm{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} 2^{-n}\, \gamma_n(d\alpha_g) = \varphi(d\alpha,0)\, D(g,0)^{-1}\,, \tag{23}$$

$$\lim_{n\to\infty} p_n(d\alpha_g,z)\, \rho(z)^{-n-1} = \frac{1}{2\sqrt{z^2-1}}\, \varphi(d\alpha,\rho(z)^{-1})\, D(g,\rho(z)^{-1})^{-1} \tag{24}$$

for $|\rho(z)| > 1$ and

$$\lim_{n\to\infty} \lambda_n(d\alpha_g,z)^{-1}\, |\rho(z)|^{-2n-2} = \frac{1}{4(|\rho(z)|^2-1)\,|z^2-1|}\, |\varphi(d\alpha,\rho(z)^{-1}|^2\, |D(g,\rho(z)^{-1})|^{-2}$$

for $|\rho(z)| > 1$.

<u>Proof.</u> Limit relations (23) and (24) follow immediately from Theorems 42, 6.1.25, 6.1.26 and 6.1.29. The last statement of the theorem is a direct consequence of (24) and Theorems 4.1.11 and 6.1.27.

Let us note that some results of Case [4] follows from the previous theorems. For example, Case proved Theorem 40 under the condition $c_j^{0,1}(d\alpha) = O(j^{-2})$.

8. Fourier Series

Recall that for a given weight α the weight α_g was defined in Definition 6.1.3.

Lemma 1. Let $\operatorname{supp}(d\alpha)$ be compact, $g \geq 0$, $g^{\pm 1} \in L^1_{d\alpha}$. Let $f \in L^2_{d\alpha_g}$. Then

(1) $$|S_n(d\alpha_g,f,x) - \lambda_n(d\alpha,x)\,\lambda_n^{-1}(d\alpha_g,x)\,S_n(d\alpha,fg,x)| \leq$$
$$\leq \|f\|_{d\alpha_g,2}\,\{\lambda_n^{-1}(d\alpha_g,x)\,[G_n(d\alpha,g^{-1},x)\,G_n(d\alpha,g,x) - 1]\}^{\frac{1}{2}}$$

for $x \in \mathbb{R}$ and $n = 1,2,\ldots$.

Proof. Let us denote the left side in (1) by $R(x)$. Then

$$R(x) = \int_{-\infty}^{\infty} f(t)\,g(t)\,[K_n(d\alpha_g,x,t) - \frac{\lambda_n(d\alpha,x)}{\lambda_n(\alpha_g,x)}\,K_n(d\alpha,x,t)]\,d\alpha(t)\,.$$

Hence

$$|R(x)|^2 \leq \|f\|^2_{d\alpha_g,2}\,K(x)$$

where

$$K(x) = \int_{-\infty}^{\infty} [K_n(d\alpha_g,x,t) - \frac{\lambda_n(d\alpha,x)}{\lambda_n(d\alpha_g,x)}\,K_n(d\alpha,x,t)]^2\,d\alpha_g(t)\,.$$

Let us calculate $K(x)$. We have

$$K(x) = K_n(d\alpha_g,x,x) - 2\,\frac{\lambda_n(d\alpha,x)}{\lambda_n(d\alpha_g,x)}\int_{-\infty}^{\infty} K_n(d\alpha_g,x,t)\,K_n(d\alpha,x,t)\,d\alpha_g(t) +$$
$$+ \frac{\lambda_n^2(d\alpha,x)}{\lambda_n^2(d\alpha_g,x)}\int_{-\infty}^{\infty} K_n^2(d\alpha,x,t)\,d\alpha_g(t) = \lambda_n^{-1}(d\alpha_g,x)[\frac{\lambda_n(d\alpha,x)}{\lambda_n(d\alpha_g,x)}\,G_n(d\alpha,g,x) - 1]\,.$$

Now use Theorem 6.2.3.

Note that putting $f = p_{n-1}(d\alpha_g)$ in (1) we obtain an inequality which may help us derive asymptotics for $p_{n-1}(d\alpha_g,x)$.

Recall that α_τ, A_x^ω, B_x^ω, g etc. have been defined in 6.2.

Theorem 2. Let $\alpha \in S$, $f \in L^2_{d\alpha}$. Let $x \in (-1,1)$, α be absolutely continuous near x. Let $\alpha' \in B^\omega_x$ with $\omega(t)/t \in L^1$, $\alpha'(x) > 0$. If τ^o is sufficiently small neighborhood of x then

$$\lim_{n\to\infty} [S_n(d\alpha,f,x) - S_n(d\alpha_\tau, f 1_\delta,x)] = 0 \tag{2}$$

where 1_δ denotes the characteristic function of an arbitrary but fixed neighborhood of x. If $\tau_1 \subset (-1,1)$, α is absolutely continuous in $\tau_1(\epsilon)$, $\alpha' \in B^\omega_{\tau_1}$ with $\omega(t)/t \in L^1$, $\alpha'(x) > 0$ for $x \in \tau_1$ and τ^o is a sufficiently small neighborhood of τ_1 then (2) holds uniformly for $x \in \tau_1$ if 1_δ is the characteristic function of a neighborhood of τ_1.

Proof. Since $\alpha = (\alpha_\tau)_g$ we obtain from Theorems 6.2.40, 6.2.43, Remark 6.2.41 and Lemma 1 that

$$\left|S_n(d\alpha,f,x) - \frac{\lambda_n(d\alpha_\tau,x)}{\lambda_n(d\alpha,x)} S_n(d\alpha_\tau,fg,x)\right| \le c\, \|f\|_{d\alpha,2} \tag{3}$$

for $n = 1,2,\ldots$. Note that g is bounded, thus $fg \in L^2_{d\alpha_\tau}$. Let us consider now $S_n(d\alpha_\tau,fg,x)$. We have

$$S_n(d\alpha_\tau,fg,x) - g(x)\, S_n(d\alpha_\tau,f,x) = \int_{-1}^{1} \frac{g(t)-g(x)}{t-x} f(t)(t-x)\, K_n(d\alpha_\tau,x,t)d\alpha_\tau(t). \tag{4}$$

Since the sequence $\{|p_n(d\alpha_\tau,x)|\}$ is uniformly bounded for $x \in \tau^* \subset \tau^o$ (See Lemma 6.2.29.) and

$$\int_{t\in\tau} \left[\frac{g(t)-g(x)}{t-x}\right]^2 |f(t)|^2 dt < \infty$$

we obtain from Bessel's inequality that the right side in (4) tends to 0 when $n\to\infty$. Further, by Theorem 6.2.43

$$\frac{\lambda_n(d\alpha_\tau,x)}{\lambda_n(d\alpha,x)} = \frac{1}{\alpha'(x)} + O(\tfrac{1}{n}) .$$

Hence

$$\frac{\lambda_n(d\alpha_\tau,x)}{\lambda_n(d\alpha,x)} S_n(d\alpha_\tau,fg,x) = S_n(d\alpha_\tau,f,x) + O(\tfrac{1}{n})\, S_n(d\alpha_\tau,f,x) + O(\tfrac{1}{n}) + \sigma(1) .$$

We have further

$$|(S_n(d\alpha_\tau,f,x)| \le \|f\|_{d\alpha_\tau,2}\, \lambda_n^{-1/2}(d\alpha_\tau,x) = O(\sqrt{n})$$

since $d\alpha_\tau(t) = dt$ for $t \in \tau$. Thus

$$\frac{\lambda_n(d\alpha_\tau,x)}{\lambda_n(d\alpha,x)} S_n(d\alpha_\tau,fg,x) = S_n(d\alpha_\tau,f,x) + \sigma(1) .$$

To α_τ we can apply Freud's localization principle (Freud, §IV.5.), by which

$$S_n(d\alpha_\tau, f, x) = S_n(d\alpha_\tau, f\,1_\delta, x) + o(1)\ .$$

Hence by (3)

$$(5)\qquad \limsup_{n\to\infty} \max_{\substack{t=x \\ (\text{or } t\in\tau_1)}} |S_n(d\alpha, f, t) - S_n(d\alpha_\tau, f\,1_\delta, t)| \le C\, \|f\|_{d\alpha,2}$$

where C coes not depend on f. Putting here $f - P$ instead of f where P is a polynomial with $\|f - P\|_{d\alpha,2} < \epsilon$, $(\epsilon > 0)$, and again using Freud's localization principle for α_τ, we obtain that the left side in (5) is not greater than $C\epsilon$. Now let $\epsilon \to 0$.

<u>Theorem 3</u>. Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\tau \subset (-1,1)$, α be absolutely continuous on τ, $\alpha'(t) = 1$ for $t \in \tau$, $\tau_1 \subset \tau^0$. Suppose that there exists a polynomial π such that $\pi^2/\alpha' \in L^1(-1,1)$. Let $f \in L^2_{d\alpha}$ and let 1_δ be the characteristic function of a sufficiently small neighborhood of τ_1. Then

$$\lim_{n\to\infty} [S_n(d\alpha, f, x) - S_n(v, f\,1_\delta, x)] = 0$$

uniformly for $x \in \tau_1$.

<u>Proof</u>. We could repeat Freud's argument (§V.7.), but his proof can be simplified. He requires, moreover, that $\pi^2/\alpha' \in L^\infty$. First of all $f\,1_\delta \in L^2_v$ for δ small. Further, $v^{-1} f\,1_\delta \in L^2_v$ also and it is easy to see that

$$\lim_{n\to\infty} [S_n(v, f\,1_\delta, x) - v(x)\, S_n(v, v^{-1} f\,1_\delta, x)] = 0$$

uniformly for $x \in \tau_1$ since v is nice on τ. By Lemma 6.2.29 and by Freud's localization principle

$$\lim_{n\to\infty} [S_n(d\alpha, f, x) - S_n(d\alpha, f\,1_\delta, x)] = 0$$

uniformly for $x \in \tau_1$. By theorem 6.2.46

$$\frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} = v(x) + O(\tfrac{1}{n}) \qquad (x \in \tau_1).$$

Hence

$$\frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)}\, S_n(v, v^{-1} f\,1_\delta, x) = v(x)\, S_n(v, v^{-1} f\,1_\delta, x) + O(\tfrac{1}{\sqrt{n}})\, \|v^{-1} f\,1_\delta\|_{v,2}$$

$(x \in \tau_1)$. Consequently we have to show that

$$|S_n(d\alpha, f1_\delta, x) - \frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} S_n(v, v^{-1} f1_\delta, x)|$$

converges to 0 uniformly for $x \in \tau_1$ when $n\to\infty$. But this expression equals

$$|\int_{t\in\delta} f(t)\,[K_n(d\alpha,x,t) - \frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} K_n(v,x,t)]dt| \le$$

$$\le \|f1_\delta\|_{d\alpha,2}\,\{\lambda_n^{-1}(d\alpha,x)[\frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)}\,\lambda_n(v,x)\int_{-1}^{1} K_n^2(v,x,t)\,d\alpha(t) - 1]\}^{1/2}.$$

By Theorem 6.2.46 and Lemma 6.2.44, the latter expression is not greater than

$$c\|f1_\delta\|_{d\alpha,2}\sqrt{n}\,\{(v(x) + o(\tfrac{1}{n}))\,(v(x)^{-1} + o(\tfrac{1}{n})) - 1\}^{1/2} = o(1)\,\|f1_\delta\|_{d\alpha,2}$$

and this is enough for our purposes.

From Theorems 2 and 3 we obtain the following equiconvergence

Theorem 4. Let $\pi^2/\alpha' \in L^1(-1,1)$ with a suitable polynomial π. If the conditions of the first part of Theorem 2 are satisfied then

$$\lim_{n\to\infty}[S_n(d\alpha,f,x) - S_n(v, f1_\delta, x)] = 0$$

and under the conditions of the second part of Theorem 2 the convergence is uniform for $x \in \tau_1$. Here $1_\delta = 1_{[x-\delta,x+\delta]}$ (or $= 1_{\tau_1(\delta)}$) and δ is sufficiently small.

Corollary 5. Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\pi^2/\alpha' \in L^1$ with a suitable polynomial π. Let α be absolutely continuous in $\tau \subset (-1,1)$, $\alpha'(t) > c_1 > 0$ for $t \in \tau$, $\alpha' \in C^1(\tau)$, $\omega(\alpha'',t)/t \in L^1$ where ω is the modulus of continuity of α''. Let $f \in L^2_{d\alpha}$. Then

$$\lim_{n\to\infty} S_n(d\alpha,f,x) = f(x)$$

for almost every $x \in \tau$.

Proof. Use Theorem 4 and Carleson [3].

In the following we will investigate the Lebesgue functions

$$K_n(d\alpha,x) = \int_{-\infty}^{\infty} |K_n(d\alpha,x,t)|\,d\alpha(t)$$

$n = 1,2,\ldots$. One trivial thing is sure:

$$K_n^2(d\alpha,x) \leq \lambda_n^{-1}(d\alpha,x)\ [\alpha(\infty) - \alpha(-\infty)]\ .$$

Hence estimating λ_n^{-1} we obtain estimates for K_n. If e.g., $\alpha'(t) \geq c > 0$ for $t \in [x-\varepsilon, x+\varepsilon]$ then

$$K_n^2(d\alpha,x) \leq c\,n\ .$$

It is rather surprising that nobody has ever tried to improve this estimate for weights satisfying weak conditions (e.g., for $\alpha \in S$). We will see that $c\,n$ can be replaced by $\sigma(n)$ in many cases. First we will find conditions for

$$\lim_{n\to\infty} \lambda_n(d\alpha,x)\ K_n^2(d\alpha,x) = 0\ . \tag{6}$$

If $\operatorname{supp}(d\alpha)$ is compact and α has a jump at x then

$$\liminf_{n\to\infty} \lambda_n(d\alpha,x)\ K_n^2(d\alpha,x) \geq \alpha(x+0) - \alpha(x-0)$$

so that (6) cannot hold.

<u>Lemma</u> 6. Let $\varepsilon > 0$, $x \in \mathbb{R}$. Then

$$\begin{aligned}\lambda_n(d\alpha,x)\ K_n^2(d\alpha,x) \leq{}& 2[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] + \\ &+ \frac{2}{\varepsilon^2}\lambda_n(d\alpha,x)\ \{[\frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} + 2(x - \alpha_{n-1}(d\alpha))^2]\ p_{n-1}^2(d\alpha,x) + \\ &+ 2\,\frac{\gamma_{n-2}^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)}\ p_{n-2}^2(d\alpha,x)\}\ [\alpha(\infty) - \alpha(-\infty)]\ .\end{aligned}$$

<u>Proof.</u> We will use the Christoffel-Darboux and the recurrence formulas. We have

$$K_n(d\alpha,x) = \{\int_{|x-t|\leq\varepsilon}\} + \{\int_{|x-t|>\varepsilon}\}\ |K_n(d\alpha,x,t)|\ d\alpha(t)\ .$$

Hence

$$K_n^2(d\alpha,x) \leq 2\{\int_{|x-t|\leq\varepsilon}\}^2 + 2\{\int_{|x-t|>\varepsilon}\}^2\ .$$

Here the first integral in the braces is not greater than

$$[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)]\int_{-\infty}^{\infty} K_n^2(d\alpha,x,t)\ d\alpha(t) = \lambda_n^{-1}(d\alpha,x)\ [\alpha(x+\varepsilon) - \alpha(x-\varepsilon)]\ .$$

Further

$$\{\int_{|x-t|\geq\varepsilon}\}^2 \leq \frac{1}{\varepsilon^2}\,\frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)}\ [p_{n-1}^2(d\alpha,x) + p_n^2(d\alpha,x)]\ [\alpha(\infty) - \alpha(-\infty)]\ .$$

Now the final estimate follows from the recurrence formula.

Corollary 7. Let $\operatorname{supp}(d\alpha)$ be compact, $\varepsilon > 0$ and Δ fixed. Then

$$\lambda_n(d\alpha,x)\ K_n^2(d\alpha,x) \le 2[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] + C\varepsilon^{-2}\lambda_n(d\alpha,x)\ [p_{n-1}^2(d\alpha,x) + p_n^2(d\alpha,x)]$$

for $x \in \Delta$, $n = 1,2,\ldots$ where $C = C(d\alpha,\Delta)$.

Proof. Use Lemmas 6 and 3.3.1.

Theorem 8. Let $\alpha \in M(0,1)$. If α is continuous at $x \in [-1,1]$ then

$$\lim_{n\to\infty} \lambda_n(d\alpha,x)\ K_n^2(d\alpha,x) = 0\ . \tag{7}$$

If α is continuous on the closed set $\mathfrak{B} \subset (-1,1)$ then (7) is satisfied uniformly for $x \in \mathfrak{B}$.

Proof. Use Corollary 7 and Theorem 4.1.11.

Theorem 9. Let $\alpha \in M(0,1)$, $\tau \subset [-1,1]$, $\varepsilon > 0$. If $[\alpha']^{-\varepsilon} \in L^1(\tau)$ then

$$\lim_{n\to\infty} n^{-1/2}\ K_n(d\alpha,x) = 0 \tag{8}$$

for almost every $x \in \tau$. If α is continuous on τ and $\alpha'(t) \ge C > 0$ for almost every $t \in \tau$ then (8) holds uniformly for $x \in \tau_1 \subset \tau^0$.

Proof. Since α is almost everywhere continuous in $[-1,1]$, the first part of the Theorem follows from Theorems 8 and 6.3.35. The second part follows from Theorem 8 and Example 6.2.9.

From Theorem 9 one can easily obtain convergence theorems for $\mathrm{Lip}\frac{1}{2}$. Recall that all known convergence theorems concern the class $\mathrm{lip}\frac{1}{2}$ which is contained in $\mathrm{Lip}\frac{1}{2}$. (See e.g. Freud.) Let us leave the details to the reader.

Let us note that (8) is good for bad weights, for nice weights better estimates can be found.

Theorem 10. Let $\operatorname{supp}(d\alpha)$ be compact, $\tau \subset \operatorname{supp}(d\alpha)$, $\varepsilon > 0$, $[\alpha']^{-\varepsilon} \in L^1(\tau)$.

Then

$$\liminf_{n\to\infty} n^{-1/3}\ K_n(d\alpha,x) < \infty$$

for almost every $x \in \tau$.

Proof. Let us put in Lemma 6 $\epsilon = n^{-1/3}$. Using Lemma 3.3.1, we obtain

$$n^{-\frac{2}{3}} K_n^2(d\alpha,x) \leq$$
$$\leq 2[n\lambda_n(d\alpha,x)]^{-1} n^{\frac{1}{3}} [\alpha(x+n^{-\frac{1}{3}}) - \alpha(x-n^{-\frac{1}{3}})] + Cp_{n-1}^2(d\alpha,x) + Cp_{n-2}^2(d\alpha,x) \ .$$

Summing for $n = 1,2,\ldots,m$ we see that

$$\frac{1}{m} \sum_{n=1}^{m} n^{-\frac{2}{3}} K_n^2(d\alpha,x) \leq$$
$$\leq 2\frac{1}{m} \sum_{n=1}^{m} [n\lambda_n(d\alpha,x)]^{-1} n^{\frac{1}{3}} [\alpha(x+n^{-\frac{1}{3}}) - \alpha(x-n^{-\frac{1}{3}})] + C[m\lambda_m(d\alpha,x)]^{-1} \ .$$

Since α is almost everywhere differentiable we obtain from Theorem 6.3.35 that

$$\limsup_{n\to\infty} \frac{1}{m} \sum_{n=1}^{m} n^{-\frac{2}{3}} K_n^2(d\alpha,x) < \infty$$

for almost every $x \in \tau$. Hence the theorem follows.

Now we will be again in the situation $\alpha,\tau \to g,\alpha_\tau$. (See 6.2.)

Lemma 11. Let $\alpha \in S$, $\tau \subset (-1,1)$. Then

$$K_n(d\alpha_\tau,x) \leq C \log n \qquad (n \geq 3)$$

uniformly for $x \in \tau_1 \subset \tau^o$.

Proof. Use Lemma 6.2.29 and Freud, §IV,4.

Theorem 12. Let $\alpha \in S$. Let $x \in (-1,1)$, α be absolutely continuous near x, $\alpha' \in A_x^\omega$, $\alpha'(x) > 0$. Then

$$K_n(d\alpha,x) \leq C\{\log n + [\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\,dt]^{\frac{1}{2}}\} \qquad (n \geq 3) \tag{9}$$

where C does not depend on n. If $\tau_1 \subset (-1,1)$, α is absolutely continuous near τ_1, $\alpha' \in A_{\tau_1}^\omega$, $\alpha'(t) > 0$ for $t \in \tau_1$ then (9) holds uniformly for $x \in \tau_1$ with C independent of x and n .

Proof. Let us choose τ $(x \in \tau^{o}$ or $\tau_1 \subset \tau^{o})$ so small that the corresponding g is bounded from below and above in $[-1,1]$. We have $\alpha = (\alpha_\tau)_g$. Hence by Lemma 1

$$K_n(d\alpha,x) \leq C\ \lambda_n(d\alpha_\tau x)\ \lambda_n^{-1}(d\alpha,x)\ K_n(d\alpha_\tau,x) +$$
$$+ C\ \{\lambda_n^{-1}(d\alpha,x)\ [G_n(d\alpha_\tau,g^{-1},x)\ G_n(d\alpha_\tau,g,x) - 1]\}^{1/2} .$$

By Theorem 6.2.6

$$\lambda_n(d\alpha_\tau,x)\ \lambda_n^{-1}(d\alpha,x) \leq C .$$

By Example 6.2.9, $\lambda_n^{-1}(d\alpha,x) \leq C\,n$. Now the theorem follows from Lemma 11, Theorem 6.2.38, Remark 6.2.41 and Lemma 6.2.29.

Note that if $\omega(t) = t\ |\log t|$ then

$$\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\,dt \sim [\log n]^2 .$$

A weaker version of Theorem 12 was obtained by Freud, §V.7.

Theorem 13. Let $\alpha \in S$, $u(\geq 0) \in L^1_{d\alpha}$, $w(\geq 0) \in L^1_{d\alpha}$, $\mathrm{meas}(u > 0) > 0$, $\mathrm{meas}(w > 0) > 0$, $1 < q < \infty$, $u^{1/(1-q)} \in L^1_{d\alpha}$ $(q < \infty)$, $u^{-1} \in L^1_{d\alpha}$ $(q = \infty)$, $0 < p < \infty$, $p \leq q$. If $q < \infty$ and for every $f \in L^1_{ud\alpha}$

$$\|S_n(d\alpha,f)\|_{wd\alpha,p} \leq C\ \|f\|_{ud\alpha,q}$$

for $n = 1,2,\ldots$ with C independent of n and f then

$$\int_{-1}^{1} [\alpha'(t)\sqrt{1-t^2}]^{-\frac{p}{2}}\,w(t)\ \alpha'(t)dt < \infty \tag{10}$$

and

$$\int_{-1}^{1} [\alpha'(t)\sqrt{1-t^2}]^{\frac{q}{2(1-q)}}\ u(t)^{\frac{1}{1-q}}\ \alpha'(t)dt < \infty .$$

If $q = \infty$ and for $u\,f \in L^q_{d\alpha}$

$$\|S_n(d\alpha,f)\|_{wd\alpha,p} \leq C\ \|u\,f\|_{d\alpha,q}$$

with $C \neq C(n,f)$ for $n = 1,2,\ldots$ then (10) and

$$\int_{-1}^{1} [\alpha'(t)\sqrt{1-t^2}]^{-\frac{1}{2}}\,u(t)^{-1}\ \alpha'(t)dt < \infty$$

hold.

Proof. For simplicity we will consider the case $1 < q < \infty$. By the conditions

$$\|S_n(d\alpha, f) - S_{n-1}(d\alpha, f)\|_{wd\alpha, p} \leq C \|f\|_{ud\alpha, q} .$$

This means that

$$\|p_n(d\alpha)\|_{wd\alpha, p} \|f p_n(d\alpha)\|_{d\alpha, 1} \leq C \|f\|_{ud\alpha, q} .$$

By Hölder's inequality this is equivalent to

$$\sup_{n \geq 1} \{\|p_n(d\alpha)\|_{wd\alpha, p} \|p_n(d\alpha)u^{-1}\|_{ud\alpha, q'}\} < \infty$$

where $q' = q/(q-1)$. By Theorem 4.2.8 the latter condition is equivalent to

$$\sup_{n \geq 1} \|p_n(d\alpha)\|_{wd\alpha, p} < \infty$$

and

$$\sup_{n \geq 1} \|p_n(d\alpha)u^{-1}\|_{ud\alpha, q'} < \infty .$$

Now the theorem follows from Theorems 7.31 and 7.32.

Let us note that many special cases of Theorem 13 have previously been known. We refer to Badkov [2] and to the literature mentioned there. (In particular, we refer to works of Askey, Muchenhaupt, Newman-Rudin, Pollard, Stein and Wainger.)

Corollary 14. There exists an absolutely continuous $\alpha \in S$ such that from

(11) $$\sup_{n \geq 1} \|S_n(d\alpha)\|_{L^p_{d\alpha} \to L^p_{d\alpha}} < \infty$$

follows $p = 2$.

Proof. Put $\alpha'(x) = \exp\{-(1 - x^2)^{-\frac{1}{4}}\}$. If $1 < p < \infty$, then apply Theorem 13. For $p = 1, \infty$, (11) can never hold.

9. Inequalities

When investigating the Lebesgue functions of Lagrange interpolating processes we will have to be able to estimate the expression

$$\sum_{|x-x_{kn}(d\alpha)|<\epsilon} \lambda_{kn}(d\alpha) \,. \tag{1}$$

The following result is very simple.

<u>Lemma 1.</u> Let $\operatorname{supp}(d\alpha)$ be compact. Then

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \sum_{|x-x_{kn}|<\epsilon} \lambda_{kn}(d\alpha) = \alpha(x+0) - \alpha(x-0)$$

for each $x \in \mathbb{R}$.

<u>Proof</u>. For every $\epsilon > 0$ we can find $y_1 \in (x - 2\epsilon, x - \epsilon)$ and $y_2 \in (x+\epsilon, x+2\epsilon)$ such that α is continuous at y_1 and y_2. Thus by using convergence theorems for mechanical quadrature processes (See e.g. Freud, §III.1.) we obtain

$$\limsup_{n\to\infty} \sum_{|x-x_{kn}|<\epsilon} \lambda_{kn}(d\alpha) \le \int_{x-y_1}^{x+y_2} d\alpha(t) \le \alpha(x+2\epsilon) - \alpha(x-2\epsilon) \,.$$

Now let $\epsilon\to 0$.

Unfortunately, it is not true that (1) is not greater than $\alpha(x+\epsilon) - \alpha(x-\epsilon)$ or $\alpha(x+2\epsilon) - \alpha(x-2\epsilon)$. From the Markov-Stieltjes inequalities we obtain that

$$\sum_{|x-x_{kn}|<\epsilon} \lambda_{kn}(d\alpha) \le \alpha(x^1) - \alpha(x^2)$$

where $x^1 = \min_{x_{kn}\ge x+\epsilon} x_{kn}$ if $\{k: x_n \ge x+\epsilon\}$ is not empty and otherwise $x^1 = +\infty$ and similarly $x^2 = \max_{x_{kn}\le x-\epsilon} x_{kn}$ or $x^2 = -\infty$. Suppose that neither $\{k: x_{kn} < x^1\}$ nor $\{k: x_{kn} > x^2\}$ is empty. Let $x_1^1 = \max_{x_{kn}<x^1} x_{kn}$ and $x_2^2 = \min_{x_{kn}>x^2} x_{kn}$. Then

$$\sum_{|x-x_{kn}|<\epsilon} \lambda_{kn}(d\alpha) \le \alpha(x+\epsilon+x^1-x_1^1) - \alpha(x-\epsilon+x^2-x_2^2) \,. \tag{2}$$

Hence we see that to estimate (1) we have to know the behavior of $x_{kn}(d\alpha) - x_{k+1,n}(d\alpha)$.

<u>Lemma 2</u>. Let $\beta \in S$. Then there exists a number $C = C(d\beta) > 1$ such that

$$p_n^2(d\beta, x) \leq C^{\sqrt{n}}$$

for $x \in [-1,1]$ and $n = 1,2,\ldots$.

<u>Proof</u>. See Geronimus, §8.2.

<u>Lemma 3</u>. Let α be an arbitrary weight, $\Delta \subset \operatorname{supp}(d\alpha)$. Let v_Δ denote the Chebyshev weight corresponding to Δ . If $v_\Delta \log \alpha' \in L^1(\Delta)$ then

$$\max_{x \in \Delta} \pi_n^2(x) \leq C^{\sqrt{n}} \int_{-\infty}^{\infty} \pi_n^2(t)\, d\alpha(t) \qquad (n \geq 1)$$

for each π_n with a suitable $C = C(d\alpha) > 1$.

<u>Proof</u>. Let $\alpha^*(t) = \alpha(t)$ for $t \in \Delta$ and $\alpha^*(t)$ be constant otherwise. Let us transform Δ into $[-1,1]$. We get a weight α^{**} which satisfies the conditions of Lemma 2. Returning to Δ we obtain

$$\max_{x \in \Delta} \pi_n^2(x) \leq \max_{x \in \Delta} \lambda_{n+1}^{-1}(d\alpha^*, x) \int_\Delta \pi_n^2(t) d\alpha^*(t) \leq$$

$$\leq n\, C^{\sqrt{n}} \int_\Delta \pi_n^2(t) d\alpha(t) \leq C_1^{\sqrt{n}} \int_{-\infty}^{\infty} \pi_n^2(t) d\alpha(t) .$$

<u>Theorem 4</u>. Let $\operatorname{supp}(d\alpha)$ be compact, $\Delta \subset \operatorname{supp}(d\alpha)$, $v_\Delta \log \alpha' \in L^1(\Delta)$. Then

$$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \leq C \frac{1}{\sqrt{n}} \qquad (n \geq 1) \tag{3}$$

for $x_k, x_{k+1} \in \Delta$ with C independent of n and k . If $\Delta_1 \subset \Delta^o$ then (3) holds if either x_k or x_{k+1} belongs to Δ_1 .

<u>Proof</u>. Let v^* denote the Chebyshev weight corresponding to $\Delta(d\alpha)$. Let m be a natural integer and $N = [\frac{n}{m}]$. Then

$$\pi_x(t) = K_N^m(v^*, x, t) K_N^{-m}(v^*, x, x)$$

is a π_{n-1} with $\pi_x(x) = 1$. By Lemma 3

$$1 \le C^{\sqrt{n}} \int_{-\infty}^{\infty} \pi_x^2(t)\, d\alpha(t) \tag{4}$$

for $x \in \Delta$. Let $x_k, x_{k+1} \in \Delta$ and $x = \frac{1}{2}(x_k + x_{k+1})$. Then $x \in \Delta$. Further

$$|\pi_x(x_{jn}(d\alpha))| \le [C_1 \tfrac{m}{n}(x_k - x_{k+1})^{-1}]^m$$

for $j = 1,2,\ldots,n$ with $C_1 = C_1(\Delta(d\alpha))$. Calculating the integral on the right side of (4) by the Gauss-Jacobi mechanical quadrature formula we obtain

$$[x_k - x_{k+1}]^{2m} \le C^{\sqrt{n}}\, [C_1 \tfrac{m}{n}]^{2m}\, [\alpha(\infty) - \alpha(-\infty)] ,$$

that is

$$x_k - x_{k+1} \le C_2\, C^{\frac{\sqrt{n}}{2m}}\, \frac{m}{n} .$$

Putting here $m = [\sqrt{n}]$ the first part of the Theorem follows. Using Lemma 3.3.2 we obtain the second part of the Theorem.

Theorem 5. Let $\mathrm{supp}(d\alpha)$ be compact, $\Delta \subset \mathrm{supp}(d\alpha)$, $\epsilon > 0$, $[\alpha']^{-\epsilon} \in L^1(\Delta)$. Then

$$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \le C\, \frac{\log n}{n} \qquad (n \ge 3) \tag{5}$$

for $x_k, x_{k+1} \in \Delta$ where C does not depend on n and k. If $\Delta_1 \subseteq \Delta^o$ then (5) holds for either $x_k \in \Delta_1$ or $x_{k+1} \in \Delta_1$.

Proof. We obtain from Theorem 6.3.13 and from $[\alpha']^{-\epsilon} \in L^1(\Delta)$ that

$$\max_{x \in \Delta} \pi_n^2(x) \le n^A \int_{-\infty}^{\infty} \pi_n^2(t)\, d\alpha(t) \qquad (n \ge 2)$$

for every π_n with a suitable constant $A > 1$. Now we repeat the proof of Theorem 4 and finally we put $m = [\log n]$.

Let us note that the proof of Theorems 4 and 5 is based on an idea of Erdős-Turan but our result is stronger than that of Erdős-Turan. (See Szegő, §6.11.)

Theorem 6. Let $\mathrm{supp}(d\alpha)$ be compact, $\Delta \subset \mathrm{supp}(d\alpha)$, $v_\Delta \log \alpha' \in L^1(\Delta)$, $\epsilon_1 > 0$, $\tau \subset \tau(\epsilon_1) \subset \Delta^o$. Then

$$\sum_{|x - x_{kn}| < \epsilon} \lambda_{kn}(d\alpha) \le \alpha(x + \epsilon + \frac{C}{\sqrt{n}}) - \alpha(x - \epsilon - \frac{C}{\sqrt{n}})$$

uniformly for $n = 1,2,\ldots$, $x \in \tau$, $0 \le \epsilon \le \epsilon_1$ where $C \ne C(n,x,\epsilon)$, $C > 0$.

Proof. Use (2) and Theorem 4.

Lemma 7. Let $\operatorname{supp}(d\alpha)$ be compact, $\epsilon > 0$, $\tau \subset \tau(\epsilon) \subset \Delta^{o} \subset \Delta \subset \operatorname{supp}(d\alpha)$. Then there exists a number $N = N(\epsilon, d\alpha, \Delta)$ such that

$$\sum_{|x-x_{kn}|<\epsilon} \lambda_{kn}(d\alpha) \le \alpha(x+2\epsilon) - \alpha(x-2\epsilon)$$

for $x \in \tau$ and $n \ge N$.

Proof. Apply Lemma 3.2.2, (2) and the Heine-Borel theorem.

In the following we will also need estimates for

$$\sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)\, |p_{n-1}(d\alpha, x_{kn})| \,. \tag{6}$$

It is obvious that (6) is not greater than $[\alpha(\infty) - \alpha(-\infty)]^{1/2}$. The question whether (6) may converge to 0 when $n\to\infty$ seems to be more difficult.

Example 8. Let w be the Hermite weight, that is $w(x) = \exp(-x^2)$ for $x \in \mathbb{R}$. Then

$$|p_{n-1}(w, x_{kn})| \le C n^{-\frac{1}{4}} w(x_{kn})^{-\frac{1}{2}}$$

for $k = 1,2,\ldots,n$. Hence by old theorems about quadrature sums

$$\sum_{k=1}^{n} \lambda_{kn}(w)\, |p_{n-1}(w, x_{kn})| \le C n^{-\frac{1}{4}} \int_{-\infty}^{\infty} w(t)^{\frac{1}{2}} dt \underset{n\to\infty}{\to} 0 \,.$$

We will show that this cannot happen if $\operatorname{supp}(d\alpha)$ is compact and α is nice in a certain sense. For $D(d\alpha, 0)$ see Definition 6.1.16.

Lemma 9. Let $\alpha \in S$. Then

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, |p_{n-1}(d\alpha, x_{kn})| \ge \frac{\sqrt{\pi}}{2} D(d\alpha, 0) \,.$$

Proof. Let $n \ge 1$. Then $T_{n-1}(x) = L_n(d\alpha, T_{n-1}, x)$. Let us divide both sides by x^{n-1} and let $x\to\infty$. We obtain

$$2^{n-2} \le \gamma_{n-1}(d\alpha) \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, |p_{n-1}(d\alpha, x_{kn})| \,.$$

Now apply Lemma 4.2.2.

The following result is a poor but very useful analogue of Theorem 4.2.8.

Theorem 10. Let $\alpha \in S$. Then there exists a number $\delta = \delta(d\alpha) > 0$ such that if $\Omega \subset [-1,1]$ is an arbitrary finite system of disjoint intervals with $|\Omega| \geq 2-\delta$ then

$$\liminf_{n\to\infty} \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)\ |p_{n-1}(d\alpha,x_{kn})| > 0\ . \tag{7}$$

Proof. Let $c\Omega = [-1,1]\backslash\Omega$. Then $1_{c\Omega}$ is Riemann integrable on $[-1,1]$. We have

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\ |p_{n-1}(d\alpha,x_{kn})| =$$

$$= \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)\ |p_{n-1}(d\alpha,x_{kn})| + \sum_{k=1}^{n} 1_{c\Omega}(x_{kn})\ \lambda_{kn}(d\alpha)\,|p_{n-1}(d\alpha,x_{kn})| \leq$$

$$\leq \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)\,|p_{n-1}(d\alpha,x_{kn})| + [(\alpha(1)-\alpha(-1)) \sum_{k=1}^{n} 1_{c\Omega}(x_{kn})\lambda_{kn}(d\alpha)p_{n-1}^2(d\alpha,x_{kn})]^{\frac{1}{2}}.$$

Now let $n\to\infty$. By Lemma 9 and Theorem 3.2.3

$$\frac{\sqrt{\pi}}{2} D(d\alpha,0) \leq$$

$$\leq \liminf_{n\to\infty} \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)\,|p_{n-1}(d\alpha,x_{kn})| + [(\alpha 1)-\alpha(-1))\, \frac{2}{\pi} \int_{c\Omega} \sqrt{1-t^2}\ dt]^{\frac{1}{2}}\ .$$

Hence (7) holds if $|c\Omega|$ is small.

Theorem 11. Let $\alpha \in M(0,1)$, $\tau \subset [-1,1]$. Then

$$\begin{matrix}\limsup\\ \liminf\end{matrix}_{n\to\infty} \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)\ |p_{n-1}(d\alpha,x_{kn})| \geq$$

$$\geq \frac{2}{\pi} \int_{\tau} \sqrt{1-t^2}\ dt\ \{\begin{matrix}\liminf\\ \limsup\end{matrix}_{n\to\infty} \max_{x_{kn}\in\tau} |p_{n-1}(d\alpha,x_{kn})|\}^{-1}\ .$$

Proof. The Theorem follows from Theorem 3.2.3 and the inequality

$$\sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)\, p_{n-1}^2(d\alpha,x_{kn}) \le$$

$$\le \max_{x_{kn}\in\tau} |p_{n-1}(d\alpha,x_{kn})| \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)\, |p_{n-1}(d\alpha,x_{kn})| \;.$$

Theorem 12. Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\alpha'(x) > 0$ for almost every $x \in (-1,1)$, $\Delta \subset [-1,1]$. Then

(8) $$\liminf_{n\to\infty} \{\max_{x\in\Delta} |p_n(d\alpha,x)| \sum_{x_{kn}\in\Delta} \lambda_{kn}(d\alpha)\, |p_{n-1}(d\alpha,x_{kn})|\} \ge \frac{|\Delta|}{2\pi} \int_\Delta v_\Delta(t)^{-1} v(t)\,dt.$$

Proof. By Bernstein's inequality

$$|p_n'(d\alpha,t)| \le \frac{2\pi}{|\Delta|} v_\Delta(t) \max_{x\in\Delta} |p_n(d\alpha,x)|$$

for $t \in \Delta$. Further, $\gamma_{n-1}(d\alpha) \le \gamma_n(d\alpha)$ and

$$\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn}(d\alpha)\, p_{n-1}(d\alpha,x_{kn}) = [p_n'(d\alpha,x_{kn})]^{-1} \;.$$

Hence the left side in (8) is not less than

$$\frac{|\Delta|}{2} \liminf_{n\to\infty} \frac{1}{n} \sum_{x_{kn}\in\Delta} v_\Delta(x_{kn})^{-1} \;.$$

But

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x_{kn}\in\Delta} v_\Delta(x_{kn})^{-1} = \frac{1}{\pi} \int_\Delta v_\Delta(t)^{-1}\, v(t)\, dt \;.$$

(See Freud, §III.9.)

Corollary 13. Let the conditions of Theorem 12 be satisfied. Let the sequence $\{|p_n(d\alpha,x)|\}$ be uniformly bounded for $x \in \Delta$. Then

$$\liminf_{n\to\infty} \sum_{x_{kn}\in\Delta} \lambda_{kn}(d\alpha)\, |p_{n-1}(d\alpha,x_{kn})| > 0 \;.$$

Let us note that the Pollaczek weight satisfies the conditions of Corollary 13.

Now we will deal with weighted Bernstein-Markov inequalities. Our aim will be to generalize the following result of Khalilova [9].

Lemma 14. Let $1 \le p < \infty$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and u be a Jacobi weight. Then

$$\|\pi_n' v^{-1}\|_{u,p} \le c\, n\, \|\pi_n\|_{u,p}$$

and

$$\max_{|x|\le 1} \{|\pi_n'(x)| \, (\sqrt{1-x}+\tfrac{1}{n})^{a+1} \, (\sqrt{1+x}+\tfrac{1}{n})^{b+1}\} \le$$

$$\le C\, n \max_{|x|\le 1} \{|\pi_n(x)| \, (\sqrt{1-x}+\tfrac{1}{n})^{a} \, (\sqrt{1+x}+\tfrac{1}{n})^{b}\}$$

for every π_n where C does not depend on π_n and n.

<u>Lemma 15</u>. Let $a > -1$, $1 \le p \le \infty$. Then

(9) $$\int_{-1}^{1} |\pi_n'(t)|^p \, |t|^a \, dt \le C\, n^p \int_{-2}^{2} |\pi_n(t)|^p \, |t|^a \, dt$$

for every π_n.

<u>Proof.</u> Let first π_n be even, that is let $\pi_n(x) = G_n(x^2)$. Then $\pi_n'(x) = 2\,x\,G_n'(x^2)$ and we have to show that

$$\int_{-1}^{1} |x\, G_n'(x^2)|^p \, |x|^a \, dx \le C\, n^p \int_{-2}^{2} |G_n(x^2)|^p \, |x|^a \, dx$$

or

$$\int_{0}^{1} |G_n'(x)|^p \, |x|^{\frac{p+a-1}{2}} \, dx \le C\, n^p \int_{0}^{4} |G_n(x)|^p \, |x|^{\frac{a-1}{2}} \, dx .$$

But

$$\int_{0}^{1} |G_n'(x)|^p \, |x|^{\frac{p+a-1}{2}} \, dx \le C \int_{0}^{4} |G_n'(x)\, \sqrt{x}\, \sqrt{4-x}|^p \, |x|^{\frac{a-1}{2}} \, dx .$$

Hence (9) follows from Lemma 14 when π_n is even. Let now π_n be odd: $\pi_n(x) = x\,G_n(x^2)$. In this case, $\pi_n'(x) = G_n(x^2) + 2\,x^2\,G_n'(x^2)$ and we will prove that

$$\int_{-1}^{1} |x^2\, G_n'(x^2)|^p \, |x|^a \, dx \le C\, n^p \int_{-2}^{2} |x\, G_n(x^2)|^p \, |x|^a \, dx$$

and

$$\int_{-1}^{1} |G_n(x^2)|^p \, |x|^a \, dx \le C\, n^p \int_{-2}^{2} |x\, G_n(x^2)|^p \, |x|^a \, dx .$$

The first inequality here follows from the first part of the proof by putting there $p+a$ instead of a. The second inequality has been proved in Corollary 6.3.26. Hence (9) holds if π_n is either even or odd. But then it also holds for every π_n with a possibly bigger constant C.

From Lemmas 14 and 15 follows

Theorem 16. Let $1 = t_1 > t_2 > \dots > t_N = -1$, $\Gamma_k > -1$ for $k = 1,2,\dots,N$,

$$w(t) \sim \prod_{k=1}^{N} |t - t_k|^{\Gamma_k} \qquad (-1 \le t \le 1) .$$

Let $1 \le p < \infty$. Then for every π_n

$$\|\pi_n' \, v^{-1}\|_{w,p} \le C\, n \, \|\pi_n\|_{w,p}$$

where C does not depend on n and π_n.

Lemma 17. Let $a \in \mathbb{R}$. Then there exists a number $\epsilon = \epsilon(a) > 0$ such that for every π_n

$$(10) \qquad \max_{|x| \le \frac{\epsilon}{n}} |\pi_n(x)| \le C\, n^a \max_{\frac{\epsilon}{n} \le |x| \le 1} \{ |x|^a \, |\pi_n(x)| \}$$

with $C = C(a)$.

Proof. Let first $a = 0$. Then (10) follows from Lemmas 6.2.50, 6.3.5 and 6.3.22 applied for the Legendre weight. If (10) holds for $a = 0$ then it also holds for $a \ge 0$. If $a < 0$ then we remark that (10) with $a = 0$ implies that

$$(11) \qquad \max_{|x| \le \frac{\epsilon}{n}} |\pi_n(x)| \le C \max_{\frac{\epsilon}{n} \le |x| \le \frac{1}{2}} |\pi_n(x)|$$

with a possibly new $\epsilon > 0$ and C. Let w be defined by $w(t) = |t|^{-a}$ for $-1 \le t \le 1$, $\operatorname{supp}(w) = [-1,1]$. Putting in (11) $K_n(w,x,x)\, \pi_n(x)$ instead of $\pi_n(x)$ we obtain from Lemma 6.3.19

$$\max_{|x| \le \frac{\epsilon}{3n}} |\pi_n(x)| \, n^{1-a} \le C \max_{\frac{\epsilon}{3n} \le |x| \le \frac{1}{2}} \{ |\pi_n(x)| \, |x|^a \} \, n .$$

Hence (1) follows for $a < 0$ also.

Lemma 18. Let $a \in \mathbb{R}$. Then

$$\max_{|x| \le 1} \{ |\pi_n'(x)| \, (|x| + \tfrac{1}{n})^a \} \le C\, n \max_{|x| \le 2} \{ |\pi_n(x)| \, (|x| + \tfrac{1}{n})^a \}$$

for every π_n.

Proof. Repeat the proof of Lemma 15 and use Lemmas 14 and 17.

By Lemmas 14 and 18 we obtain the following

<u>Theorem 19</u>. Let $A_k \in \mathbb{R}$ for $k = 1,2,\dots,N$, $1 > t_2 > \dots > t_{N-1} > -1$,

$$w_n(x) = (\sqrt{1-x} + \tfrac{1}{n})^{2A_1} \prod_{k=2}^{N-1} (|x - t_k| + \tfrac{1}{n})^{A_k} (\sqrt{1+x} + \tfrac{1}{n})^{2A_N}$$

for $|x| \le 1$. Then

$$\max_{|x|\le 1} \{|\pi_n'(x)| \, w_n(x) \, (\sqrt{1-x} + \tfrac{1}{n}) \, (\sqrt{1+x} + \tfrac{1}{n})\} \le$$

$$\le C\, n \max_{|x|\le 1} \{|\pi_n(x)| \, w_n(x)\}$$

for every π_n where C is independent of n and π_n.

Now we return to estimating the distance between two consecutive zeros of orthogonal polynomials.

<u>Theorem 20.</u> Let $\operatorname{supp}(d\alpha)$ be compact, $\Delta \subset \operatorname{supp}(d\alpha)$, $t^* \in \Delta^o$, $\Gamma > -1$. Let α be absolutely continuous in Δ with

$$\alpha'(t) \sim |t - t^*|^\Gamma \qquad (t \in \Delta) .$$

Then

$$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \sim \frac{1}{n}$$

for $x_{kn} \in \Delta_1 \subset \Delta^o$.

<u>Proof</u>. By Lemma 3.3.2 we can suppose that both x_{kn} and $x_{k+1,n}$ belong to Δ_1. First we shall show that

$$x_{kn} - x_{k+1,n} \le C\, n^{-1} .$$

We have the following possibilities.

(12) $\quad t^* \le x_{k+1} < x_k \le t^* + \frac{1}{n}$ or $t^* - \frac{1}{n} \le x_{k+1} < x_k \le t^*$
or $t^* - \frac{1}{n} < x_{k+1} \le t^* < x_k < t^* + \frac{1}{n}$,

(13) $\quad t^* \le x_{k+1} \le t^* + \frac{1}{n} < x_k$ or $x_{k+1} < t^* - \frac{1}{n} \le x_k \le t^*$,

(14) $\quad t^* + \frac{1}{n} \le x_{k+1} < x_k$ or $x_{k+1} < x_k \le t^* - \frac{1}{n}$,

(15) $\quad x_{k+1} \le t^* - \frac{1}{n} < t^* < x_k \le t^* + \frac{1}{n}$ or $t^* - \frac{1}{n} \le x_{k+1} \le t^* < t^* + \frac{1}{n} \le x_k$

and

(16) $\quad x_{k+1} \le t^* - \frac{1}{n} < t^* + \frac{1}{n} \le x_k$.

In all cases (12)-(16) we will use Theorem 6.3.25 and the estimate

$$\int_{x_{k+1,n}}^{x_{kn}} d\alpha(t) \leq \lambda_{kn}(d\alpha) + \lambda_{k+1,n}(d\alpha) \qquad (k = 1,2,\ldots,n-1)$$

which follows from the Markov-Stieltjes inequalities. In the first case of (13) we obtain

$$(x_k - t^*)^{\Gamma+1} \leq c[n^{-\Gamma-1} + \frac{1}{n}(x_k - t^*)^{\Gamma}] .$$

Hence $x_k \leq t^* + c\frac{1}{n}$. In the first case of (14) we have

$$(x_k - t^*)^{\Gamma+1} - (x_{k+1} - t^*)^{\Gamma+1} \leq c\frac{1}{n}[(x_k - t^*)^{\Gamma} + (x_{k+1} - t^*)^{\Gamma}] .$$

Thus

$$x_k - x_{k+1} \leq c\frac{1}{n}\frac{[(x_k - t^*) - (x_{k+1} - t^*)]\,[(x_k - t^*)^{\Gamma} + (x_{k+1} - t^*)^{\Gamma}]}{(x_k - t^*)^{\Gamma+1} - (x_{k+1} - t^*)^{\Gamma+1}}$$

and

$$x_k - x_{k+1} \leq c\frac{1}{n} \sup_{1 \leq x < \infty} \frac{(x - 1)\,(x^{\Gamma} + 1)}{x^{\Gamma+1} - 1} \leq c\frac{1}{n}$$

since $\Gamma + 1 > 0$. The other possibilities may be treated similarly. To estimate $x_k - x_{k+1}$ from below let us remark as Erdős-Turan did that

$$1 = (x_{kn} - x_{k+1,n})\, \frac{d}{dx}\, \ell_{kn}^2(d\alpha, x^*)$$

where $x_{k+1} \leq x^* \leq x_k$. We have

$$\ell_{kn}^2(d\alpha, x) \leq \lambda_{kn}(d\alpha)\, \lambda_n^{-1}(d\alpha, x) .$$

Using Theorems 19 and 6.3.25, we obtain

$$\left|\frac{d}{dx}\, \ell_{kn}^2(d\alpha, x)\right| \leq c\, n$$

uniformly for $x_{kn}, x \in \Delta_1 \subset \Delta^{\circ}$. Hence the estimate from below for $x_k - x_{k+1}$ follows.

Theorem 21. Let $\operatorname{supp}(d\alpha)$ be compact, $\Delta(d\alpha) = [c_1, c_2]$, $a > -1$, $\delta > 0$. Let α be absolutely continuous in $[c_2-\delta, c_2]$ and let

$$\alpha'(t) \sim (c_2 - t)^a$$

for $t \in [c_2-\delta, c_2]$. Let $x_{kn}(d\alpha) = \frac{1}{2}(c_1 + c_2) + \frac{1}{2}(c_2 - c_1)\cos\theta_k$ for $k = 0,1,\ldots,n+1$ where $0 \leq \theta_k \leq \pi$ and $x_{0n} = c_2$, $x_{n+1,n} = c_1$. Then

$$\theta_{k+1} - \theta_k \sim \frac{1}{n}$$

for $x_{kn} \in \Delta \subset (c_2-\delta, c_2]$.

Proof. We can assume without loss of generality that $\Delta(d\alpha) = [-1,1]$ and $\delta \le \frac{1}{2}$. (Concerning the second assumption see e.g. Freud, §III.5.) We obtain immediately from Theorem 6.3.27 and Markov-Stieltjes' inequalities that $\theta_1 = O(\frac{1}{n})$ and $\theta_2^{-1} = O(n)$. Now we will show that $\theta_1^{-1} = O(n)$. Let $m \ge n$ be fixed. Then by the Gauss-Jacobi mechanical quadrature formula

$$(1-x_{1n})\lambda_{1n}(d\alpha) = \int_{-1}^{1} (1-t)\ell_{1n}^2(d\alpha,t)d\alpha(t) = \sum_{k=1}^{m} (1-x_{km})\ell_{1n}^2(d\alpha,x_{km})\,\lambda_{km}(d\alpha) .$$

Hence

$$(1-x_{1n})\lambda_{1n}(d\alpha) \ge (1-x_{2m}) \sum_{k=1}^{m} \ell_{1n}^2(d\alpha,x_{km})\lambda_{km}(d\alpha) - (1-x_{2m})\ell_{1n}(d\alpha,x_{1m})\lambda_{1m}(d\alpha) =$$

$$= (1-x_{2m})\lambda_{1n}(d\alpha)\ [1 - \frac{\ell_{1n}^2(d\alpha,x_{1m})}{\lambda_{1n}(d\alpha)}\,\lambda_{1m}(d\alpha)]$$

and consequently

$$1 - x_{1n} \ge (1 - x_{2m})\ [1 - \frac{\lambda_{1m}(d\alpha)}{\lambda_n(d\alpha,x_{1m})}] .$$

Putting here $m = Nn$ where N is big but fixed we obtain from Theorem 6.3.27 that

$$1 - x_{1n} \ge \frac{1}{2}(1 - x_{2m})$$

if only n is big enough. Hence $\theta_1^{-1} = O(n)$. To prove $\theta_{k+1} - \theta_k = O(n^{-1})$ for $x_k \in \Delta \subset (1-\delta,1]$ we will use the inequalities

$$(17)\quad \sum_{k=i+1}^{n} (1 \pm x_{kn})\lambda_{kn}(d\alpha) \le \int_{-1}^{x_{in}} (1 \pm t)d\alpha(t) \le \sum_{k=i}^{n} (1 \pm x_{kn})\lambda_{kn}(d\alpha) \quad (i=1,\dots,n)$$

which is always true if $\Delta(d\alpha) \subset [-1,1]$. We shall not prove (17), it can be proved in the same way that Freud proves the Markov-Stieltjes inequalities in his book. From (17) and Theorem 6.3.27 we get for $x_k \in \Delta$

$$\theta_{k+1} - \theta_k \le \frac{C}{n} \sup_{0 \le x,y \le \frac{\pi}{2}} \frac{|x-y|\ [\sin^{2a+3}x + \sin^{2a+3}y]}{|\sin^{2a+4}x - \sin^{2a+4}y|}$$

which is of order $\frac{1}{n}$ since $2a+3 > 1$. The estimate $[\theta_{k+1} - \theta_k]^{-1} = O(n)$ for $x_k \in \Delta$ follows from Lemma 14 and Theorem 6.3.27 in the same way as we obtained the estimates from below in Theorem 20.

Theorem 22. Let w be as in Theorem 16, $x_{kn}(w) = \cos\theta_{kn}$ $(x_{0n} = 1,\ x_{n+1,n} = -1,\ 0 \le \theta_{kn} \le \pi)$. Then

$$\theta_{k+1,n} - \theta_{kn} \sim \frac{1}{n}$$

for $k = 0,1,\ldots,n$.

Proof. Use Theorems 20 and 21.

The following inequalities will play a fundamental role in investigations of mean convergence of interpolation processes.

Theorem 23. Let $\alpha, \Delta, t^*, \Gamma$ and Δ_1 be as in Theorem 20. Let $1 \leq p < \infty$ and $m \leq c_1 n$. Then for each π_m and n

$$\sum_{x_{kn} \in \Delta_1} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \leq C \int_\Delta |\pi_m(t)|^p \, d\alpha(t)$$

where $C = C(\alpha, \Delta_1, p, c_1)$.

Proof. Let $w = 1_\Delta \alpha'$ with $\operatorname{supp}(w) = \Delta$. Then by Theorem 6.3.25

$$\lambda_{kn}(d\alpha) \sim \lambda_n(w, x_{kn}(d\alpha)) \sim \lambda_n(w, p, x_{kn}(d\alpha)) \sim \lambda_{n+m}(w, p, x_{kn}(d\alpha))$$

for $x_{kn}(d\alpha) \in \Delta_1 \subset \Delta^o$ since $m \leq c_1 n$. Hence

$$|\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \leq C \int_\Delta |\pi_m(t)|^p \, d\alpha(t)$$

for $x_{kn}(d\alpha) \in \Delta_1$. Further we can suppose that $t^* \in \Delta_1$. Let $j = j(n)$ be defined by $x_{j+1,n} \leq t^* \leq x_{jn}$. Consider

$$\sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) .$$

Observe that

$$|\pi_m(x_{kn})|^p \leq |\pi_m(x)|^p + p \int_{x_{k+1,n}}^{x_{k-1,n}} |\pi_m(t)|^{p-1} |\pi_m'(t)| \, dt$$

for $x_{k+1,n} \leq x \leq x_{k-1,n}$. Thus by the Markov-Stieltjes inequalities we obtain

$$\sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \leq$$

$$\leq 2 \int_\Delta |\pi_m(x)|^p \, d\alpha(x) + p \sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} \lambda_{kn}(d\alpha) \int_{x_{k+1,n}}^{x_{k-1,n}} |\pi_m(t)|^{p-1} |\pi_m'(t)| \, dt .$$

By Theorems 20 and 6.3.25

$$\lambda_{kn}(d\alpha) \sim \frac{1}{n}\,|t - t^*|^\Gamma$$

for $x_{k+1,n} \le t \le x_{k-1,n}$ if $k < j-1$ and $x_{kn} \in \Delta_1$. Consequently

$$\sum_{\substack{x_{kn}\in\Delta_1\\ k<j-1}} \lambda_{kn}(d\alpha) \int_{x_{k+1,n}}^{x_{k-1,n}} |\pi_m(t)|^{p-1}\,|\pi_m'(t)|\,dt \le$$

$$\le c\,\frac{1}{n}\Big\{\int_{\Delta_1(\epsilon)} |\pi_m(t)|^p\,|t - t^*|^\Gamma dt\Big\}^{\frac{p-1}{p}}\Big\{\int_{\Delta_1(\epsilon)} |\pi_m'(t)|^p\,|t - t^*|^\Gamma dt\Big\}^{\frac{1}{p}},$$

where $\epsilon > 0$ is chosen so that $\Delta_1(\epsilon) \subset \Delta^o$. By Lemma 15 we obtain

$$\sum_{\substack{x_{kn}\in\Delta_1\\ k<j-1}} |\pi_m(x_{kn})|^p\,\lambda_{kn}(d\alpha) \le c\,\frac{m+n}{n}\int_\Delta |\pi_m(t)|^p\,d\alpha(t)\,.$$

The sum for $x_{kn} \in \Delta_1$, $k > j+1$ can be estimated similarly.

Theorem 24. Let α, c_2, a, δ and Δ be as in Theorem 21. Let $\Gamma > -1-a$, $1 \le p < \infty$, $m \le c^* n$. Then

$$\sum_{x_{kn}\in\Delta} |\pi_m(x_{kn})|^p\,(c_2 - x_{kn})^\Gamma\,\lambda_{kn}(d\alpha) \le c\int_{c_2-\delta}^{c_2} |\pi_m(t)|^p\,(c_2 - t)^\Gamma\,d\alpha(t)$$

for each π_m where $c = c(\alpha, p, \Delta, \Gamma, c^*)$.

Theorem 25. Let w be as in Theorem 16, $u \in L^1_w$ be a Jacobi weight, $1 \le p < \infty$ and $m \le c^* n$. Then

$$\sum_{k=1}^{n} |\pi_m(x_{kn})|^p\,u(x_{kn})\,\lambda_{kn}(w) \le c\int_{-1}^{1} |\pi_m(t)|^p\,u(t)\,w(t)\,dt$$

for every π_m where $c = c(w,u,p)$.

Theorems 24 and 25 can be proved by the same method as Theorem 23. As an application of the previous results, we will prove two theorems.

Theorem 26. Let $w = w^{(a,b)}$ be the Pollaczek weight defined in Definition 6.2.12. Let $1 \le p \le \infty$, $p \ne 2$. Then the sequence of operators $\{S_n(w)\}$ is not uniformly bounded in L^p_w.

Proof. By Corollary 13 and Theorem 23 ($\Gamma = 0$)

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(w,t)|^q \, w(t)dt > 0$$

for $1 \leq q < \infty$. Let $1 < p < \infty$, $p \neq 2$. Suppose that

$$\sup_{n\geq 1} \|S_n(w)\|_{L_w^p \to L_w^p} < \infty .$$

Then we obtain exactly in the same way as in Theorem 8.13 that

$$\sup_{n\geq 1} \|p_n(w)\|_{w,p} < \infty$$

and

$$\sup_{n\geq 1} \|p_n(w)\|_{w,p'} < \infty$$

where $p' = \frac{p}{p-1}$. Since either p or p' is greater than 2 this cannot happen by Theorem 7.31. For $p = 1$ or $p = \infty$, the Theorem follows from old results. (See e.g. Freud, remarks on Chapter IV.)

Theorem 27. Theorem 26 remains valid if we replace the Pollaczek weight there by the weight defined in Example 6.2.14.

Proof. By Korous' theorem (See Freud, §I.7.) the corresponding system is uniformly bounded on each $\Delta \subset (-1,1)$. Now we can repeat the proof of Theorem 26.

Definition 28. The weight w is called a generalized Jacobi weight ($w \in GJ$) if $\operatorname{supp}(w) = [-1,1]$ and

$$w(t) = \varphi(t)\,(1-t)^{\Gamma_1} \prod_{k=2}^{N-1} |t_k - t|^{\Gamma_k}\,(1+t)^{\Gamma_N}$$

where $\Gamma_k > -1$ ($k = 1,2,\dots,N$), $1 > t_2 > \dots > t_{N-1} > -1$, $\varphi(>0)$ is continuous on $[-1,1]$ and $\omega(\delta)/\delta \in L^1(0,1)$ where ω is the modulus of continuity of φ. If $w^* \sim w$ where $w \in GJ$ then we write $w^* \sim GJ$. Hence if $w \in GJ$ then also $w \sim GJ$. For $w \sim GJ$, w_n is defined by

$$w_n(t) = \left(\sqrt{1-t} + \frac{1}{n}\right)^{2\Gamma_1} \prod_{k=2}^{N-1} \left(|t_k - t| + \frac{1}{n}\right)^{\Gamma_k} \left(\sqrt{1+t} + \frac{1}{n}\right)^{2\Gamma_N}$$

($-1 \leq t \leq 1$; $n = 1,2,\dots$).

Lemma 29. Let $w \in GJ$. Then

$$(\sqrt{1-x}+\frac{1}{n})\ (\sqrt{1+x}+\frac{1}{n})\ w_n(x)\ p_n^2(w,x) \le C$$

for $-1 \le x \le 1$, $n = 1,2,\ldots$ where $C \neq C(n,x)$.

Proof. See Badkov [2].

Lemma 30. Let $\alpha \in S$, $g = v^{-2}$. Then

$$|p_{n-1}(d\alpha,x_k)| \sim (1-x_k^2)\ |p_{n-1}(d\alpha_g,x_k)|$$

where $x_k = x_{kn}(d\alpha)$.

Proof. In course of the proof of Theorem 4.2.3 we have shown that

$$(1-x_k^2)\ p_{n-1}(d\alpha_g,x_k) = p_{n-1}(d\alpha,x_k)\ [\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\alpha_g)} + \frac{\gamma_{n-1}(d\alpha_g)\ \gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)}].$$

The expression in the brackets does not depend on k and by Lemma 4.2.2 it converges to 1 when $n\to\infty$.

Theorem 31. Let $w \in GJ$. Then

$$w_n(x_{kn})\ p_{n-1}^2(w,x_{kn}) \sim \sqrt{1-x_{kn}^2} \sim (\sqrt{1-x_{kn}}+\frac{1}{n})\ (\sqrt{1+x_{kn}}+\frac{1}{n}).$$

Proof. If $w \in GJ$ then $v^{-2}w \in GJ$. Hence by Lemmas 29 and 30

$$w_n(x_{kn})\ p_{n-1}^2(w,x_{kn}) \le C(\sqrt{1-x_{kn}}+\frac{1}{n})\ (\sqrt{1+x_{kn}}+\frac{1}{n}).$$

By Theorem 22 the right side here is $\sim \sqrt{1-x_{kn}^2}$. The converse inequality follows from

$$p_{n-1}(w,x_{kn})^{-1} = \frac{\gamma_{n-1}(w)}{\gamma_n(w)}\ \lambda_{kn}(w)\ p_n'(w,x_{kn})$$

and from Theorems 6.3.28, 19, 22 and Lemma 29. Theorem 6.3.28 gives us bounds for $\lambda_{kn}(w)$. Theorem 19 and Lemma 29 is used to estimate $|p_n'(w,x_{kn})|$. Finally, Theorem 22 shows that

$$\sqrt{1-x_{kn}^2} \sim (\sqrt{1-x_{kn}}+\frac{1}{n})\ (\sqrt{1+x_{kn}}+\frac{1}{n}).$$

Lemma 32. Let α be an arbitrary weight. Let $x_{0n}(d\alpha) = \infty$, $x_{n+1,n}(d\alpha) = -\infty$, $\ell_{0n}(d\alpha,x) \equiv 0$ and $\ell_{n+1,n}(d\alpha,x) \equiv 0$. Let $x_{k+1,n}(d\alpha) \le x \le x_{kn}(d\alpha)$ $(k=0,\ldots,n)$. Then

$$\ell_{kn}(d\alpha,x) + \ell_{k+1,n}(d\alpha,x) \geq 1 .$$

Proof. See Erdős-Turan [5].

Theorem 33. Let $w \in GJ$. Then

$$\lambda_n(w,x)\, p_n^2(w,x) \sim n(x - x_k)^2 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-2} \left(\sqrt{1+x} + \frac{1}{n}\right)^{-2} \tag{18}$$

for $-1 \leq x \leq 1$ where x_k is the zero of $p_n(w,x)$ which is closest to x.

Proof. By Theorems 22 and 6.3.28

$$\ell_{kn}^2(w,x) \leq \lambda_{kn}(w)\, \lambda_n^{-1}(w,x) \leq C$$

for $-1 \leq x \leq 1$ where k is the index of x_k in (18). Further, if $k = 1$ and $x_k \leq x \leq 1$ or $k = n$ and $-1 \leq x \leq x_k$ then by Lemma 32, $\ell_{kn}^2(w,x) \geq 1$. Otherwise x is between either x_{k-1} and x_k $(k > 1)$ or x_k and x_{k+1} $(k < n)$. Let for simplicity, $x_k \leq x \leq x_{k-1}$. Then by Lemma 32

$$\ell_{k-1,n}(w,x) + \ell_{kn}(w,x) \geq 1 .$$

By Theorems 22, 31 and 6.3.28

$$|\lambda_{k-1,n}(w)\, p_{n-1}(w,x_{k-1,n})| \sim |\lambda_{kn}(w)\, p_{n-1}(w,x_{kn})|$$

and obviously $\ell_{k-1,n}(w,x) \geq 0$, $\ell_{kn}(w,x) > 0$ and $\operatorname{sign} p_{n-1}(w,x_{k-1,n}) = -\operatorname{sign} p_{n-1}(w,x_{kn})$. Hence

$$\ell_{k-1,n}(w,x) \leq C\, \ell_{kn}(w,x) .$$

Consequently in all possible cases

$$\ell_{kn}^2(w,x) \sim 1 .$$

The Theorem follows now from Theorems 22, 31 and 6.3.28.

Corollary 34. Let $w \in GJ$. Then

$$p_n(w,1) \sim n^{\Gamma_1 + \frac{1}{2}}$$

and

$$|p_n(w,-1)| \sim n^{\Gamma_N + \frac{1}{2}} .$$

Proof. In this case, either $k = 1$ or $k = n$.

Remark 35. Let α, a and c_2 be as in Theorem 21. Then $p_n(d\alpha,c_2) \geq Cn^a$ for

$n = 1,2,\dots$ where $C \neq C(n)$. This follows immediately from Theorems 21, 6.3.27 and from $\lambda_n^{-1}(d\alpha,x) = \sum_{k=1}^{n} \lambda_{kn}^{-1}(d\alpha)\, \ell_{kn}^2(d\alpha,x)$.

Lemma 36. Let $\operatorname{supp}(d\alpha)$ be compact. Let there exist an interval τ such that the sequence $\{|p_n(d\alpha,x)|\}$ is uniformly bounded for $x \in \tau$. Then

$$\liminf_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} > 0 .$$

Proof. By Theorem 7.5, $\tau \subset \operatorname{supp}(\alpha') \subset \operatorname{supp}(d\alpha)$. Let $\tau = [c_1, c_2]$ and $m_1 = m_1(n)$, $m_2 = m_2(n)$ be defined by $x_{m_1+1,n} < c_1 \le x_{m_1,n}$ and $x_{m_2,n} \le c_2 < x_{m_2-1,n}$ respectively where $x_{n+1,n} = -\infty$, $x_{0n} = \infty$. By Lemma 3.2.2 $\lim_{n\to\infty} x_{m_1,n} = c_1$ and $\lim_{n\to\infty} x_{m_2,n} = c_2$. We can suppose that α is continuous at c_1 and c_2. If not we can replace τ by a smaller interval. Let $m_2 < k \le m_1$. Then by Lemma 32

$$\ell_{k-1,n}\left(d\alpha, \frac{x_{k-1}+x_k}{2}\right) + \ell_{kn}\left(d\alpha, \frac{x_{k-1}+x_k}{2}\right) \ge 1 ,$$

that is

$$x_{k-1} - x_k \le 2\, \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, C^2 [\lambda_{k-1,n}(d\alpha) + \lambda_{kn}(d\alpha)]$$

where $C = \sup_{n\ge 0} \max_{x\in\tau} |p_n(d\alpha,x)|$. Hence

$$x_{m_2} - x_{m_1} \le 4C^2\, \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \sum_{k=m_2}^{m_1} \lambda_{kn}(d\alpha) .$$

Letting $n\to\infty$ we obtain

$$|\tau| \le 4C^2 \liminf_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \int_\tau d\alpha(t) .$$

Theorem 37. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$, $\operatorname{supp}(d\beta) \subset [-1,1]$, $\tau \subset (-1,1)$ and $c\tau = [-1,1]\backslash\tau$. Let $d\alpha(t) = d\beta(t)$ for $t \in \tau$, α be absolutely continuous in $c\tau$ and let exist two polynomials π_1 and π_2 such that $\pi_1\alpha'/\beta' \in L^1_{d\alpha}(c\tau)$ and $|\pi_2(x)\, p_n(d\alpha,x)| \le K$ for $x \in c\tau$ and $n = 1,2,\dots$. Then

$$|p_n(d\beta,x)| \le C[|p_n(d\alpha,x)| + |p_{n-1}(d\alpha,x)|]$$

uniformly for $x \in \tau_1 \subset \tau^{o}$ and $n = 1,2,\dots$.

<u>Proof.</u> We can suppose that neither π_1 nor π_2 has zeros in τ^{o}. Let $\rho = \pi_1\pi_2$, $\deg \rho = m$. Then

$$\rho(x)\, p_n(d\beta,x) = \int_{-1}^{1} \rho(t)\, p_n(d\beta,t)\, K_{n+m+1}(d\alpha,x,t)\, d\alpha(t)\,.$$

By the conditions $\int_{-1}^{1} (\ )d\alpha = \int_{-1}^{1} (\ )d\beta + \int_{c\tau} (\ )d\alpha - \int_{c\tau} (\ )d\beta$. Let $x \in \tau_1 \subset \tau^{o}$. Then

$$\{\int_{c\tau} \rho(t)\, p_n(d\beta,t)\, K_{n+m+1}(d\alpha,x,t)\, d\beta(t)\}^2 \le$$

$$\le C\, [|p_{n+m}(d\alpha,x)| + |p_{n+m+1}(d\alpha,x)|]^2 \int_{-1}^{1} \pi_1^2(t)\, d\beta(t)$$

and

$$\{\int_{c\tau} \rho(t)\, p_n(d\beta,t)\, K_{n+m+1}(d\alpha,x,t)\, d\alpha(t)\}^2 \le$$

$$\le C[|p_{n+m}(d\alpha,x)| + |p_{n+m+1}(d\alpha,x)|]^2 \int_{c\tau} \rho^2(t)\frac{\alpha'(t)^2}{\beta'(t)}dt \int_{c\tau} p_n^2(d\beta,t)\beta'(t)dt.$$

Further for $n > m$

$$\int_{-1}^{1} \rho(t)\, p_n(d\beta,t)\, K_{n+m+1}(d\alpha,x,t)\, d\beta(t) =$$

$$= \int_{-1}^{1} \rho(t)\, p_n(d\beta,t) \sum_{k=n-m}^{n+m} [p_k(d\alpha,x)\, p_k(d\alpha,t)]\, d\beta(t)\,.$$

Consequently

$$|\int_{-1}^{1} \rho(t)\, p_n(d\beta,t)\, K_{n+m+1}(d\alpha,x,t)\, d\beta(t)| \le$$

$$\le C \sum_{k=n-m}^{n+m} |p_k(d\alpha,x)|\, \{\int_{-1}^{1} \rho(t)^2\, p_k^2(d\alpha,t)\, d\beta(t)\}^{\frac{1}{2}}\,.$$

But

$$\int_{-1}^{1} \rho(t)^2\, p_k^2(d\alpha,t)\, d\beta(t) =$$

$$= \int_{\tau} \rho(t)^2\, p_k^2(d\alpha,t)\, d\alpha(t) + \int_{c\tau} \rho(t)^2\, p_k^2(d\alpha,t)\, d\beta(t) \le C\,.$$

Thus we have proved that for $x \in \tau_1 \subset \tau^{o}$

$$|p_n(d\beta,x)| \le C \sum_{k=n-m}^{n+m+1} |p_k(d\alpha,x)|\,.$$

By Lemma 36 the sequence $\{\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\}$ is bounded from below by a positive constant. Thus by Lemma 3.3.1 and by the recurrence formula

$$|p_k(d\alpha,x)| \leq C^*[|p_n(d\alpha,x)| + |p_{n-1}(d\alpha,x)|]$$

whenever $n \geq 1$, $n - k = O(1)$ and $x \in \Delta$ where $C^* = C^*(|n - k|,\alpha,\Delta)$.

Remark 38. Theorem 37 becomes useful if we combine it with Korous' theorem and with results in Freud's and Geronimus' books.

Theorem 39. Theorem 37 remains valid if τ is of the form $[a,1]$ or $[-1,a]$ where $|a| < 1$ and $\tau_1 \subset (a,1]$ or $\tau_1 \subset [-1,a)$ respectively.

Proof. The same as that of Theorem 37.

10. Lagrange Interpolation

First we will consider the Lebesgue function $L_n(d\alpha,x)$ of Lagrange interpolation corresponding to α which is defined by

$$L_n(d\alpha,x) = \sum_{k=1}^{n} |\ell_{kn}(d\alpha,x)| .$$

The estimate

$$L_n^2(d\alpha,x) \leq \lambda_n^{-1}(d\alpha,x)\ [\alpha(\infty) - \alpha(-\infty)] \tag{1}$$

shows that if α has a jump at x then

$$L_n^2(d\alpha,x) \leq \frac{\alpha(\infty) - \alpha(-\infty)}{\alpha(x+0) - \alpha(x-0)}$$

but in general (1) is not a very strong result. Our aim will be to improve (1).

Lemma 1. Let $\operatorname{supp}(d\alpha)$ be compact and $\epsilon > 0$. Then

$$\lambda_n(d\alpha,x)\ L_n^2(d\alpha,x) \leq 2 \sum_{|x-x_{kn}|<\epsilon} \lambda_{kn}(d\alpha) + C\ \epsilon^{-2}\ \lambda_n(d\alpha,x)\ p_n^2(d\alpha,x)$$

for $x \in \Delta$, $n = 1,2,\ldots$ where $C = C(\alpha,\Delta)$.

Proof. Repeat the proof of Lemma 8.6 and apply Lemma 3.3.1.

Theorem 2. Let $\alpha \in M(0,1)$. If α is continuous at $x \in [-1,1]$ then

$$\lim_{n\to\infty} \lambda_n(d\alpha,x)\ L_n^2(d\alpha,x) = 0 . \tag{2}$$

If α is continuous on the closed set $\mathfrak{B} \subset (-1,1)$ then (2) holds uniformly for $x \in \mathfrak{B}$.

Proof. Apply Theorem 4.1.11, Lemmas 9.1, 9.7 and 1.

Theorem 3. Let $\alpha \in M(0,1)$, $\tau \subset [-1,1]$. If $[\alpha']^{-\epsilon} \in L^1(\tau)$ with some $\epsilon > 0$ then

$$\lim_{n\to\infty} n^{-1/2}\ L_n(d\alpha,x) = 0 \tag{3}$$

for almost every $x \in \tau$. If $\alpha'(t) \geq c > 0$ for almost every $t \in \tau$ and α

is continuous on τ then (3) is satisfied uniformly for $x \in \tau_1 \subset \tau^o$.

Proof. Repeat the reasoning in the proof of Theorem 8.9.

Remark 4. Using Theorem 3 we can easily obtain convergence theorems for $\mathrm{Lip}\frac{1}{2}$.

In the following we will investigate $L_n^*(d\alpha,x)$ defined by

$$L_n^*(d\alpha,x)^2 = \frac{1}{n}\sum_{k=1}^{n} L_k^2(d\alpha,x).$$

If we can estimate $L_n^*(d\alpha,x)$ we can also estimate the Lebesgue function of the (C,1) means of Lagrange interpolating polynomials, which we denote by $\hat{L}_n(d\alpha,x)$, since obviously $\hat{L}_n(d\alpha,x) \le L_n^*(d\alpha,x)$. Moreover, we can also estimate the convergence rate of the strong (C,1) means:

$$\sigma_n(d\alpha,f,x) \le \frac{1}{n}\sum_{k=1}^{n} |f(x) - L_k(d\alpha,f,x)|.$$

Let $E_k(f)$ denote the best approximation of f by π_{k-1} in $C(\Delta(d\alpha))$. Then

$$\sigma_n(d\alpha,f,x) \le \frac{1}{n}\sum_{k=1}^{n} [1 + L_k(d\alpha,x)]\, E_k(f).$$

Hence by Jackson's theorem

$$\sigma_n(f,d\alpha,x) \le C\, L_n^*(d\alpha,x)\, \{\frac{1}{n}\int_{\frac{1}{n}}^{1} \frac{\omega_R(f,t)^2}{t^2}\, dt\}^{1/2}$$

where ω_R denotes the R-th modulus of smoothness of f.

Theorem 5. Let $\mathrm{supp}(d\alpha)$ be compact, $\tau \subset \mathrm{supp}(d\alpha)$, $\epsilon > 0$ and $[\alpha']^{-\epsilon} \in$ $\in L^1(\tau)$. Then

$$\limsup_{n\to\infty} n^{-1/3} L_n^*(d\alpha,x) < \infty \tag{4}$$

for almost every $x \in \tau$. If $\alpha \in \mathrm{Lip}_x 1$ and $\alpha'(t) \ge c > 0$ for $|x-t|$ small then (4) holds. If $\alpha \in \mathrm{Lip}\ 1$ on τ and $\alpha'(t) \ge c > 0$ for $t \in \tau$ then (4) is satisfied uniformly for $x \in \tau_1 \subset \tau^o$.

Proof. For simplicity let us prove the first part of the Theorem. Let $x \in \tau_1 \subset \tau^o$ and $\epsilon = n^{-1/3}$. Then by Theorem 9.6 and Lemma 1

$$\lambda_n(d\alpha,x)\, L_n^2(d\alpha,x) \le 2[\alpha(x+cn^{-\frac{1}{3}}) - \alpha(x-cn^{-\frac{1}{3}})] + C_1 n^{\frac{2}{3}} \lambda_n(d\alpha,x)\, p_n^2(d\alpha,x) .$$

Hence

$$L_n^*(d\alpha,x)^2 \le 2n^{-\frac{1}{3}} \sum_{k=1}^{n} \frac{1}{k\lambda_k(d\alpha,x)} k^{\frac{1}{3}}[\alpha x + ck^{-\frac{1}{3}}) - \alpha(x-ck^{-\frac{1}{3}})] + C_1 n^{-\frac{1}{3}} \lambda_{n+1}^{-1}(d\alpha,x).$$

Since α is almost everywhere differentiable we obtain from Theorem 6.3.25 that (3) holds for almost every $x \in \tau_1$. But $\tau_1 \subset \tau^0$ is arbitrary.

Let us note that the second part of Theorem 5 is not new. (See Freud, Some unsolved problems.)

Remark 6. Applying Theorem 5, convergence of $(C,1)$ and strong $(C,1)$ means of Lagrange interpolation polynomials for $f \in \mathrm{lip}\frac{1}{3}$ can be proved.

Corollary 7. If $\mathrm{supp}(d\alpha)$ is compact and $[\alpha']^{-\epsilon} \in L^1(\tau)$ with some $\epsilon > 0$ then

$$\liminf_{n\to\infty} n^{-\frac{1}{3}} L_n(d\alpha,x) < \infty$$

for almost every $x \in \tau$.

Theorem 8. Let $\alpha \in S$ and $x \in [-1,1]$. Then

$$|p_n(d\alpha,x)| \le 4\sqrt{\frac{2}{\pi}}\, D(d\alpha,0)^{-1} L_n(d\alpha,x)$$

for $n = 1,2,\ldots$.

Proof. We obtain from Lemma 6.1.19 and from the inequality between the arithmetic and geometric means that

$$\gamma_{n-1}(d\alpha) \le 2^{n-1}\sqrt{\frac{2}{\pi}}\, D(d\alpha,0)^{-1} .$$

Further

$$\frac{2^{n-2}}{\gamma_{n-1}(d\alpha)} = \int_{-1}^{1} T_{n-1}(t)\, p_{n-1}(d\alpha,t)\, d\alpha(t) \le 2\sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{|p_{n-1}(d\alpha,x_{kn})|}{|x-x_{kn}|}$$

for $-1 \le x \le 1$. Hence the Theorem follows.

Remark 9. In general, Theorem 8 cannot be improved. Let $w \sim GJ$ with $\Gamma_1 > 0$. Then $w \in S$ and

$$\{\sum_{k=1}^{n} \lambda_{kn}(w) \frac{|p_{n-1}(w,x_{kn})|}{1-x_{kn}}\}^2 \le 2\sum_{k=1}^{n} \lambda_{kn}(w) \frac{p_{n-1}^2(w,x_{kn})}{1-x_{kn}^2} \sum_{k=1}^{n} \lambda_{kn}(w)\,(1-x_{kn})^{-1}$$

which is bounded by Theorems 4.2.3, 6.3.28 and 9.22.

Definition 10. Let $x_{0n}(d\alpha) = \infty$, $x_{n+1,n}(d\alpha) = -\infty$, $\ell_{0n}(d\alpha,x) = \ell_{n+1,n}(d\alpha,x) = 0$. Let $m = m(n,x)$ be defined by $x_{m+1,n}(d\alpha) < x \leq x_{mn}(d\alpha)$. Then we put

$$\tilde{L}_n(d\alpha,x) = \sum_{\substack{k=1 \\ k\neq m,m+1}}^{n} |\ell_{kn}(d\alpha,x)| .$$

Lemma 11. Let α be an arbitrary weight. Then

$$L_n(d\alpha,x) - 1 \sim \tilde{L}_n(d\alpha,x)$$

for $x \in \mathbb{R}$ and $n = 1,2,\ldots$.

Proof. By Lemma 9.32

$$\tilde{L}_n(d\alpha,x) + 1 \leq L_n(d\alpha,x) .$$

On the other hand, since $\ell_{mn}(d\alpha,x) \geq 0$ and $\ell_{m+1,n}(d\alpha,x) \geq 0$, we have

$$L_n(d\alpha,x) = \tilde{L}_n(d\alpha,x) + 1 - \sum_{\substack{k=1 \\ k\neq m,m+1}}^{n} \ell_{kn}(d\alpha,x) \leq 1 + 2\tilde{L}_n(d\alpha,x) .$$

Recall that GJ and w_n have been defined in Definition 9.28.

Theorem 12. Let $w \in GJ$. Then

$$(5) \quad L_n(w,x) - 1 \sim \frac{n|x - x_j|}{\sqrt{1-x^2} + \frac{1}{n}} \int\limits_{n|x-t|>\sqrt{1-x^2}+\frac{1}{n}} \left\{\frac{w_n(t)(\sqrt{1-t^2} + \frac{1}{n})}{w_n(x)(\sqrt{1-x^2} + \frac{1}{n})}\right\}^{\frac{1}{2}} \frac{1}{|x-t|}\,dt$$

for $-1 \leq x \leq 1$ where x_j denotes the zero of $p_n(w)$ closest to x .

Proof. The Theorem follows by a long calculation from Theorems 6.3.28, 9.22, 9.31, 9.33 and Lemma 11.

For the case when w is a Jacobi weight Theorem 12 has been proved by Natanson [10]. The integral on the right hand side of (5) is a rather standard one, it can easily be estimated but the final formula is so complicated that we will omit it. We will formulate only one particular case as

Corollary 13. Let $w \in GJ$. Then

$$L_n(w,1) \sim \begin{cases} 1 & \text{for } -1 < \Gamma_1 < -\frac{1}{2} \\ \log n & \text{for } \Gamma_1 = -\frac{1}{2} \\ n^{\Gamma_1+\frac{1}{2}} & \text{for } \Gamma_1 > -\frac{1}{2}. \end{cases}$$

Before finding necessary conditions for mean boundedness of Lagrange interpolation processes, let us make some remarks. If we define $L_n(d\alpha)$ by $L_n(d\alpha)f = L_n(d\alpha, f)$ then the norm of $L_n(d\alpha)$ as a mapping from some L^q $(0 < q < \infty)$ is never bounded, f must always be bounded in $\Delta(d\alpha)^o$. To avoid complication, which we cannot solve at the present time, we will assume that f is bounded on $\Delta(d\alpha)$ and we will write $f \in L^\infty(\Delta(d\alpha))$ where $\|f\|_\infty = \sup_{t\in\Delta(d\alpha)} |f(t)|$. An important difference between Fourier sums and Lagrange interpolation polynomials is that $L_{n+1}(d\alpha,f) - L_n(d\alpha,f)$ is not proportional to $p_n(d\alpha)$. If we write

$$L_n(d\alpha,f,x) = \sum_{k=0}^{n-1} a_k\, p_k(d\alpha,x)$$

and introduce the notation

$$L_{n,k}(d\alpha,f,x) = \sum_{k=0}^{k-1} a_j p_j(d\alpha,x)$$

for $1 \le k \le n-1$ then

$$L_n(d\alpha,f,x) - L_{n,n-1}(d\alpha,f,x) = a_{n-1}\, p_{n-1}(d\alpha,x)$$

where obviously

$$a_{n-1} = \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p_{n-1}(d\alpha,x_{kn}) .$$

Theorem 14. Let either $\alpha \in S$ or α satisfy the conditions of Corollary 9.13. Let β be an arbitrary weight. Let us consider the following three conditions.

(i) $$\sup_{n\ge1} \|L_n(d\alpha)\|_{L^\infty(\Delta(d\alpha))\to L^p_{d\beta}} < \infty ,$$

(ii) $$\sup_{n\ge2} \|L_{n,n-1}(d\alpha)\|_{L^\infty(\Delta(d\alpha))\to L^p_{d\beta}} < \infty$$

and

$$\sup_{n\ge1} \|p_{n-1}(d\alpha)\|_{d\beta,p} < \infty$$

where $p \in (0,\infty)$ is given. Then each pair of (i)-(iii) implies the third one.

Proof. Apply Lemma 9.9 and Corollary 9.13.

The following Theorem is one of our main results.

Theorem 15. Let $\alpha \in S$, $0 < p < \infty$ and $w(\geq 0) \in L^1(-1,1)$. Then from

$$\liminf_{n\to\infty} \|L_n(d\alpha)\|_{L^\infty(-1,1)\to L^p_w} < \infty$$

follows

$$\int_{-1}^{1} [\alpha'(t)\sqrt{1-t^2}]^{-\frac{p}{2}} w(t)dt < \infty . \tag{6}$$

Proof. Let $\delta = \delta(d\alpha) > 0$ be defined by Theorem 9.10. Let $\tau \subset [-1,1]$ with $|\tau| = \frac{\delta}{2}$. Then we can find a system $\Omega = \{\tau_1, \tau_2\}$ with $\tau_1\tau_2 = \emptyset$, $|\Omega| > 2-\delta$ and $\operatorname{dist}(\tau,\Omega) > 0$ such that (9.7) is satisfied. Let f be a function on $[-1,1]$ which satisfies the conditions $\|f\|_\infty = 1$ and

$$f(x_{kn}) = 1_\Omega(x_{kn}) \operatorname{sign}[p_{n-1}(d\alpha,x_{kn})\ (x_{kn} - B)]$$

where B is the center of τ. Of course f depends on n, τ, Ω, and α. We have

$$\|1_\tau\ L_n(d\alpha,f)\|_{w,p} \leq \|L_n(d\alpha)\|_{L^\infty\to L^p_w} .$$

Since $|x - x_{kn}| \leq 2$ for $x \in \tau$, $x_{kn} \in \Omega$ we obtain

$$\|1_\tau\ p_n(d\alpha)\|_{w,p} \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)\ |p_{n-1}(d\alpha,x_{kn})| \leq 2\,\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)}\ \|L_n(d\alpha)\|_{L^\infty\to L^p_w} .$$

Letting $n\to\infty$ we get from Lemma 4.2.2 and Theorem 9.10 that

$$\liminf_{n\to\infty} \|1_\tau\ p_n(d\alpha)\|_{w,p} < \infty .$$

Hence by Theorem 7.32

$$\int_\tau [\alpha'(t)\sqrt{1-t^2}]^{-\frac{p}{2}} w(t)dt < \infty .$$

Since this inequality holds for every $\tau \subset [-1,1]$ with $|\tau| = \frac{\delta}{2}$ and $\delta = \delta(d\alpha) > 0$ it also holds if $\tau = [-1,1]$.

Using the results of sections 7 and 9 we can prove similar theorems when $\alpha \notin S$. We restrict ourselves to the following

Theorem 16. Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\alpha'(x) > 0$ for almost every $x \in [-1,1]$ and let there exist an interval $\tau \subset [-1,1]$ such that the sequence $\{|p_n(d\alpha,x)|\}$ is uniformly bounded for $x \in \tau$. Let $w(\geq 0) \in L^1(-1,1)$. If $0 < p < \infty$ and

(7) $$\limsup_{n\to\infty} \|L_n(d\alpha)\|_{L^\infty(-1,1)\to L^p_w} < \infty$$

then

$$\limsup_{n\to\infty} \|p_n(d\alpha)\|_{w,p} < \infty .$$

If $p \geq 2$ and (7) holds then (6) is satisfied.

Proof. By the conditions $\{\|1_\tau\, p_n(d\alpha)\|_{w,p}\}$ is bounded. Let $c\tau = [-1,1]\setminus\tau$. Let f be defined by $\|f\|_\infty = 1$ and

$$f(x_{kn}) = 1_\tau(x_{kn}) \operatorname{sign} p_{n-1}(d\alpha, x_{kn}) .$$

Then

$$\|1_{c\tau}\, p_n(d\alpha)\|_{w,p} \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)\ |p_{n-1}(d\alpha,x_{kn})| \leq$$

$$\leq 2\,\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)}\ \|L_n(d\alpha)\|_{L^\infty\to L^p_w} .$$

Now the Theorem follows from Theorem 7.31, Corollary 9.13 and Lemma 9.36.

Definition 17. Let us say that α just belongs to S ($\alpha \in JS$) if $\alpha \in S$ but for every $\epsilon > 0$, $[\alpha']^{-\epsilon} v \notin L^1$. Example: $\alpha'(x) = \exp\{-(1-x^2)^{-\delta}\}$ $(0<\delta<\frac{1}{2})$ or $\alpha'(x) = \exp\{-|x|^{-\delta}\}$ $(0 < \delta < 1)$.

Corollary 18. Let either $\alpha \in JS$ or α be a Pollaczek weight or α be defined by

$$\alpha'(t) = \varphi(t) \exp\{-(1-t^2)^{-\frac{1}{2}}\},$$

$\varphi(> 0) \in \operatorname{Lip} 1$, $\operatorname{supp}(d\alpha) = [-1,1]$ and α is absolutely continuous. Then for every $p > 2$, there exists a function $f \in C[-1,1]$ such that

$$\limsup_{n\to\infty} \int_{-1}^{1} |f(t) - L_n(d\alpha,f,t)|^p\, \alpha'(t)dt > 0 .$$

Proof. Use Theorems 15, 16 and Banach-Steinhaus' theorem.

Let us remark that Corollary 18 gives a more or less complete answer to Turan's problem and verifies Askey's conjecture ([1]). Let us recall that Turan asked if there exists a weight α with $\mathrm{supp}(d\alpha) = [-1,1]$ such that the conclusion of Corollary 18 holds and Askey conjectured that the Pollaczek weight solves Turan's problem.

Theorem 19. Let $\mathrm{supp}(d\alpha) \subset [-1,1]$, $0 < p < \infty$, $w(\geq 0) \in L^1(-1,1)$. Let the sequence $n_1 < n_2 < \dots$ be given. If for every $f \in C[-1,1]$

$$\lim_{k\to\infty} \int_{-1}^{1} |L_{n_k}(d\alpha,f,x) - f(x)|^p \, w(x)dx = 0$$

then

$$\limsup_{k\to\infty} \|L_{n_k}(d\alpha)\|_{L^\infty \to L^p_w} < \infty . \tag{8}$$

Proof. If $p \geq 1$, then the Theorem follows from Banach-Steinhaus' theorem. Now let $0 < p < 1$. Let us define the functionals $\varphi_k\colon C[-1,1] \to \mathbb{R}$ by

$$\varphi_k(f) = \int_{-1}^{1} |L_{n_k}(d\alpha,f,x) - f(x)|^p \, w(x)dx .$$

Then $\varphi_k(f+g) \leq \varphi_k(f) + \varphi_k(g)$, $\varphi_k(\lambda f) = |\lambda|^p \varphi_k(f)$, $\varphi_k(f) \geq 0$ and $\lim_{k\to\infty} \varphi_k(f) = 0$ for every $f,g \in C$. Suppose there exists a subsequence $k_1 < k_2 < \dots$ such that

$$c_j = \sup_{\|f\|_C^p \leq 1} \varphi_{k_j}(f) \underset{j\to\infty}{\to} \infty$$

Let us put $j_1 = 1$ and find a function $f_1 \in C$ such that $\|f_1\|_C \leq 1$ and $\varphi_{k_1}(f_1) \geq \frac{1}{2}c_1$. Then there exists a number $j_2 > j_1$ such that for each $j \geq j_2$ $\varphi_{k_j}(f_1) \leq 1$. Now we find a function $f_2 \in C$ such that $\varphi_{k_{j_2}}(f_2) \geq \frac{1}{2}c_{j_2}$ and $\|f_2\|_C \leq 1$. After we choose $j_3 > j_2$ so that $\varphi_{k_j}(f_2) < 1$ for every $j \geq j_3$. Continuing this process we build up two sequences $\{j_\ell\}_{\ell=1}^{\infty}$ and $\{f_\ell\}_{\ell=1}^{\infty}$ so that $f_\ell \in C$, $\|f_\ell\|_C \leq 1$, $\varphi_{k_{j_\ell}}(f_\ell) \geq \frac{1}{2}c_{j_\ell}$ and $\varphi_{k_{j_\ell}}(f_m) \leq 1$ for $m = 1,2,\dots,\ell-1$. Let us choose a subsequence $j_{\ell_1} < j_{\ell_2} < \dots$ such that $\sum_{\nu=1}^{\infty} c_{j_\ell}^{-p} \leq 1$ and

$c_{j_{\ell_m}} \{ \sum_{\nu=m+1}^{\infty} c_{j_{\ell_\nu}}^{-1} \}^p \leq 1$ for $m = 1,2,\ldots$, Let $f = \sum_{\nu=1}^{\infty} c_{j_{\ell_\nu}}^{-1} f_{\ell_\nu}$. Then $f \in C$.

Further for $m \geq 1$

$$\varphi_{k_{j_{\ell_m}}}(f) \geq c_{j_{\ell_m}}^{-p} \varphi_{k_{j_{\ell_m}}}(f_{\ell_m}) - \sum_{\nu=1}^{m-1} c_{j_{\ell_\nu}}^{-p} \varphi_{k_{j_{\ell_m}}}(f_{\ell_\nu}) - c_{j_{\ell_m}} \{ \sum_{\nu=m+1}^{\infty} c_{j_{\ell_\nu}}^{-1} \}^p .$$

Hence

$$\varphi_{k_{j_{\ell_m}}}(f) \geq \frac{1}{2} c_{j_{\ell_m}}^{1-p} - 2 .$$

Letting $m \to \infty$ we obtain

$$\limsup_{k\to\infty} \varphi_k(f) = \infty .$$

The contradiction shows that

$$\limsup_{k\to\infty} \sup_{\|f\|_C^p \leq 1} \varphi_k(f) < \infty$$

which is equivalent to (8).

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