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Francisco Marcellán
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Orthogonal Polynomials and Linear Functionals

An Algebraic Approach and Applications

PRESS

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Authors:

Juan Carlos García-Ardila
Departamento de Matemática Aplicada a la Ingeniería Industrial
Universidad Politécnica de Madrid
Calle José Gutiérrez Abascal 2
28006 Madrid, Spain
E-mail: juancarlos.garciaa@upm.es

Francisco Marcellán
Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés, Spain
E-mail: pacomarc@ing.uc3m.es

Misael E. Marriaga
Departamento de Matemática Aplicada
Ciencia e Ingeniería de Materiales y Tecnología Electrónica
Universidad Rey Juan Carlos
Calle Tulipán s/n
28933 Móstoles, Spain
E-mail: misael.marriaga@urjc.es

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Preface

This manuscript was originally conceived as the lecture notes for an introductory graduate-level course taught by Francisco Marcellán during the First Orthonet Summer School in Seville, Spain, 2016. The purpose of the course (and by transitivity, the purpose of these notes as well) was to introduce young researchers to the basic theory of orthogonal polynomials using linear functionals as the main tool for treating several notions of the theory.

Since these lecture notes were first written, we have gradually added several results about orthogonal polynomials obtained by working with linear functionals that appear scattered throughout the literature and that we thought would nicely fit in with the contents of the original notes. Our intention was to prepare a document that students and interested researchers can consult as an introductory text to this branch of study. Moreover, most of these results appear in regular scientific journals and their proofs are tailored for a more mature audience. Thus, in the spirit of making this manuscript a point of first approach to the field, we have filled in some of the details in the proofs of a few results whenever we thought appropriate to do so.

We must say that these notes are far from being an exhaustive account of the development of the general theory of orthogonal polynomials. We adhere to describing results concerning standard orthogonality with respect to linear functionals. Nevertheless, other types of orthogonality are widely studied as well. For instance, orthogonal polynomials with respect to the so-called Sobolev bilinear forms have been of great interest in recent years. This type of orthogonality is quite different from that associated with linear functionals, and the properties of the corresponding orthogonal polynomials deviate greatly from those presented in these notes. Hoping to make up for this lack of exhaustiveness, we have added a list of references that an interested reader can consult.

Contents

Chapter 1
Introduction

Orthogonal polynomials were introduced in the 18th century when Adrien M. Legendre studied the problem of gravitational attraction between a body and a sphere in his paper entitled "Sur l'attraction des sphèroïdes." Legendre proved the following statement: if the force of attraction exerted by a solid of revolution is known on an exterior point along its axis of revolution, then the force of attraction is also known for every point on the exterior of the solid. Here Legendre introduced a family of orthogonal polynomials $(P_n(x))_{n\geq 0}$ and he showed that the zeros of $P_n(x)$ are all real, simple, and located in the closed interval $[-1, 1]$ (see [2]). These polynomials can be represented by the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n,$$

given by Olinde Rodrigues.

The Hermite polynomials $(H_n(x))_{n\geq 0}$ made their appearance between 1799 and 1825. Even though they are named in honor of Charles Hermite (1822–1901), it seems like the first person to consider them was Pierre-Simon Laplace who used them for the first time in his celebrated "Traité de mécanique céleste" to treat problems of the theory of probabilities. These polynomials were also studied by P. L. Chebyshev and finally by Hermite, who studied them extensively. The Hermite polynomials satisfy the following orthogonality relation:

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2}\, dx = 2^n n! \sqrt{\pi}\, \delta_{n,m}.$$

Another well-known family of orthogonal polynomials, named after Edmond Nicolas Laguerre, are the Laguerre polynomials $(L_n^{(\alpha)}(x))_{n\geq 0}$ which satisfy the following orthogonality condition:

$$\int_0^{\infty} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^{\alpha} e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}, \quad \alpha > -1.$$

These polynomials were first studied by Niels Henrik Abel and Joseph-Louis Lagrange, but it was Chebyshev who first dealt with them in more detail. In 1879, Laguerre used the particular case $\alpha = 0$ to study the integral $\int_x^{\infty} e^{-t} t^{-1} dt$ and found that these polynomials are solutions of the differential equation

$$xy'' + (x+1)y' = ny, \quad n \geq 0.$$

In some texts, the polynomials $(L_n^{(\alpha)}(x))_{n\geq 0}$ are also known as the Laguerre–Sonin polynomials, after Nikolai Yakovlevich Sonin who continued Sojotkin's work for $\alpha > -1$, discovering properties for those polynomials.

The German mathematician Karl Jacobi was the first to introduce a family of orthogonal polynomials without trying to solve a specific physical or mathematical problem. Jacobi introduced these polynomials in terms of hypergeometric functions, which had already been studied by a famous mathematician Carl F. Gauss. Jacobi defined them as

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1, \frac{1-x}{2}\right),$$

with $\alpha > -1$ and $\beta > -1$. These polynomials are orthogonal with respect to the measure $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\chi_{[-1,1]}(x)dx$ where $\chi_{[-1,1]}$ is the characteristic function defined on the interval $[-1, 1]$.

The Laguerre, Hermite, and Jacobi polynomials are known as the families of classical orthogonal polynomials (see Chapter 9).

From the emergence of the classical orthogonal polynomials until today, the theory of orthogonal polynomials has grown exponentially mainly due to its numerous applications in physics, approximation theory, differential and difference equations, mechanics, and statistics (among others).

The above paragraphs have only been a small overview of the whole history behind orthogonal polynomials, so we invite the reader interested in deepening his/her knowledge on this subject to see, for example, [2, 4, 30, 37, 41, 52, 55] and the references therein.

Chapter 2

Moment functionals on $\mathbb{P}$ and orthogonal polynomials

The first part of this chapter is dedicated to collecting some preliminary concepts and fixing the notation that will be used throughout these lecture notes. Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the usual sets of natural, real, and complex numbers, respectively. Let $\mathbb{P}$ be the linear space of polynomials of a real variable and complex coefficients.

The algebraic dual of $\mathbb{P}$, denoted by $\mathbb{P}^*$, is defined as the set of all linear mappings from $\mathbb{P}$ into $\mathbb{C}$, that is,

$$\mathbb{P}^* = \{\mathbf{u} : \mathbb{P} \to \mathbb{C} : \mathbf{u} \text{ is linear}\}.$$

Note that $\mathbb{P}^*$ is a vector space over $\mathbb{C}$. If $\mathbf{u} \in \mathbb{P}^*$, then the image of a polynomial p under $\mathbf{u}$ will be expressed using duality brackets as $\langle \mathbf{u}, p \rangle$. The elements of $\mathbb{P}^*$ are usually called *linear functionals*.

Recall that any sequence of polynomials $(p_n(x))_{n\geq 0}$ satisfying $\deg p_n(x) = n$ constitutes a basis for $\mathbb{P}$. In particular, the basis $(x^n)_{n\geq 0}$ is called the monomial or standard basis of $\mathbb{P}$. Any linear functional $\mathbf{u}$ is completely defined by the values $\mu_n := \langle \mathbf{u}, x^n \rangle$, $n \geq 0$ (and extended by linearity to all polynomials), where μ_n is called the nth *moment* of the linear functional $\mathbf{u}$.[1] Hence, we refer to $\mathbf{u}$ as a moment functional.

Example 2.1. We introduce an important family of linear functionals. For each $a \in \mathbb{R}$, we define the functionals $\boldsymbol{\delta}_a^{(k)}$, $k \geq 0$, as follows:

$$\begin{aligned} \boldsymbol{\delta}_a^{(k)} &: \mathbb{P} \to \mathbb{C}, \\ p(x) &\mapsto \left\langle \boldsymbol{\delta}_a^{(k)}, p(x) \right\rangle = (-1)^k p^{(k)}(a). \end{aligned}$$

When $a = 0$, we only write $\boldsymbol{\delta}^{(k)}$. Taking the monomial basis, we check that the moments are

$$\left\langle \boldsymbol{\delta}^{(k)}, x^n \right\rangle = \begin{cases} 0, & k \neq n, \\ (-1)^k k!, & n = k, \end{cases} \qquad n \geq 0.$$

Example 2.2. Another important example is the linear functional $\mathbf{u}$ defined by

$$\langle \mathbf{u}, p \rangle = \int_I p(x)\, d\mu(x), \qquad p(x) \in \mathbb{P},$$

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[1] More generally, given any basis $(p_n(x))_{n\geq 0}$ of $\mathbb{P}$, a linear functional $\mathbf{u} \in \mathbb{P}^*$ can be defined by assigning values $\tilde{\mu}_n := \langle \mathbf{u}, p_n \rangle$, $n \geq 0$, where $\tilde{\mu}_n \in \mathbb{C}$ are called generalized moments. In this way, $\mathbf{u}$ is defined on all of $\mathbb{P}$ by linearity.

where I is a subset of $\mathbb{R}$ and $d\mu$ is a probability measure defined on I. In particular, when $d\mu = dx$ (Lebesgue measure) and $I = [0, 1]$, the moments of $\mathbf{u}$ are

$$\mu_n = \langle \mathbf{u}, x^n \rangle = \int_0^1 x^n \, dx = \frac{1}{n+1}, \quad n \geq 0.$$

The following discussion is quite technical, but is included here for completeness. We will show that linear functionals on $\mathbb{P}$ are continuous functions in the sense that we describe below. Let $\mathfrak{S}$ be the linear space of formal[2] power series with coefficients in $\mathbb{C}$, that is,

$$\mathfrak{S} = \left\{ f(t) = \sum_{k=0}^{\infty} f_k t^k : f_k \in \mathbb{C} \right\}.$$

Define the bilinear form $(\cdot, \cdot) : \mathfrak{S} \times \mathbb{P} \to \mathbb{C}$ by

$$\big(f(t), x^n\big) = f_n, \tag{2.1}$$

for every $f(t) = \sum_{k=0}^{\infty} f_k t^k \in \mathfrak{S}$. In particular, $(t^k, x^n) = \delta_{k,n}$. Hereafter, $\delta_{n,k}$ denotes the Kronecker delta defined as

$$\delta_{n,k} = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases} \tag{2.2}$$

Recall that a bilinear form is linear in *both* entries and, thus, (2.1) is extended by linearity to all of $\mathbb{P}$. Moreover, since every polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ is a *finite* linear combination of monomials,

$$\big(f(t), p(x)\big) = \sum_{k=0}^{n} f_k a_k$$

is a finite sum.

Every linear functional can be identified with a unique formal series, and vice versa. Indeed, for $\mathbf{u} \in \mathbb{P}^*$, define the formal power series

$$f_{\mathbf{u}}(t) = \sum_{k=0}^{\infty} \langle \mathbf{u}, x^k \rangle t^k.$$

Observe that $(f_{\mathbf{u}}(t), x^n) = \langle \mathbf{u}, x^n \rangle$ for $n \geq 0$. On the other hand, since each moment functional is completely defined by its moments, for $f(t) = \sum_{k=0}^{\infty} f_k t^k \in \mathfrak{S}$, let us define the linear functional $\mathbf{u}_f$ as

$$\langle \mathbf{u}_f, x^n \rangle = f_n = \big(f(t), x^n\big), \quad n \geq 0.$$

[2] The word "formal" means that we are not concerned with convergence.

Then, we can identify $\mathfrak{S}$ with $\mathbb{P}^*$ (that is, these two sets are isomorphic). The pair $(\mathfrak{S}, \mathbb{P})$, together with the bilinear form (2.1), is known in the literature as a dual pair [49].

Now, we equip $\mathbb{P}$ with a topology (more precisely, the weak topology [49, 56]) induced by a family of seminorms $(\|\cdot\|_f)_{f\in\mathfrak{S}}$ where

$$\|p(x)\|_f = \left|\left(f(t), p(x)\right)\right|, \quad f \in \mathfrak{S}.$$

Once the topology is established, it makes sense to talk about open sets and continuous functions. In particular, we define the set of continuous linear functionals $\mathbb{P}' \subseteq \mathbb{P}^*$ as the topological dual of $\mathbb{P}$. Under this topology and taking into account (2.1), it can be proved that we can identify $\mathfrak{S}$ with $\mathbb{P}'$. Together with the previous identification, we get the following important equality (see [44, 56]):

$$\mathbb{P}^* = \mathbb{P}'.$$

That is, every linear functional defined on $\mathbb{P}$ is continuous.

It is not hard to find a spanning set for $\mathbb{P}'$. Indeed, if we define the space of linear functionals $\mathcal{F} := \mathrm{Span}(\boldsymbol{\delta}^{(k)})_{k\geq 0}$ (see Example 2.1), then $\mathcal{F} \subseteq \mathbb{P}'$. The following proposition shows that $\mathcal{F} = \mathbb{P}'$ (see [44, 53]) and thus, $(\boldsymbol{\delta}^{(k)})_{k\geq 0}$ constitutes a spanning set for $\mathbb{P}'$.

Proposition 2.3. *Given* $\mathbf{u} \in \mathbb{P}'$, *there exist complex numbers* $(a_n)_{n\geq 0}$ *such that*

$$\mathbf{u} = \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n!} \boldsymbol{\delta}^{(n)}.$$

Proof. Notice that for each $n \geq 0$,

$$\mu_n = \langle \mathbf{u}, x^n \rangle = (-1)^n \frac{a_n}{n!} \langle \boldsymbol{\delta}^{(n)}, x^n \rangle = a_n.$$

Thus, by linearity of $\mathbf{u}$, the result follows. ∎

Intuitively, $\langle \mathbf{u}, p(x) \rangle$ is well defined for every polynomial $p(x)$ since

$$\langle \mathbf{u}, p(x) \rangle = \sum_{n=0}^{\deg p(x)} a_n \frac{p^{(n)}(0)}{n!} < \infty$$

is a finite sum, as every polynomial is a finite linear combination of monomials. For a more rigorous treatment of infinite series of linear functionals, see [53].

The functional $\boldsymbol{\delta}^{(0)}$ (which we will denote by $\boldsymbol{\delta}(x)$) is known in the literature as the Dirac delta functional supported on $x = 0$.

Definition 2.4. Given a moment functional $\mathbf{u} \in \mathbb{P}'$, we define the following operations:

i) **Left multiplication by a polynomial.** Given $q(x) \in \mathbb{P}$, we define the left multiplication of $\mathbf{u}$ by $q(x)$, as the functional $q(x)\mathbf{u}$ such that

$$\langle q(x)\mathbf{u}, p(x)\rangle = \langle \mathbf{u}, q(x)p(x)\rangle, \quad p(x) \in \mathbb{P}.$$

ii) **Division by a polynomial of degree 1**. Given the monic polynomial $(x-a)$, we define the functional $(x-a)^{-1}\mathbf{u}$ as

$$\langle (x-a)^{-1}\mathbf{u}, p(x)\rangle = \left\langle \mathbf{u}, \frac{p(x)-p(a)}{x-a}\right\rangle.$$

iii) **Derivative of a functional.** The distributional derivative $D\mathbf{u}$ is defined as

$$\langle D\mathbf{u}, p(x)\rangle = -\langle \mathbf{u}, p'(x)\rangle,$$

where D is the derivative operator.

iv) Given the linear application $\Delta : \mathbb{P} \to \mathbb{P}$ defined by $\Delta p(x) = p(x+1) - p(x)$, we define the functional $\Delta\mathbf{u}$ as

$$\langle \Delta\mathbf{u}, p(x)\rangle = \langle \mathbf{u}, \Delta p(x)\rangle.$$

v) Given the linear application $\tau_b : \mathbb{P} \to \mathbb{P}$ defined by $\tau_b p(x) = p(x-b)$, we define the shift functional $\tau_b\mathbf{u}$ as

$$\langle \tau_b\mathbf{u}, p(x)\rangle = \langle \mathbf{u}, \tau_{-b} p(x)\rangle.$$

vi) Given the linear application $h_a : \mathbb{P} \to \mathbb{P}$ defined by $h_a p(x) = p(ax)$, $a \neq 0$, we define the dilation functional $h_a\mathbf{u}$ as

$$\langle h_a\mathbf{u}, p(x)\rangle = \langle \mathbf{u}, h_a p(x)\rangle.$$

vii) Given the linear application $\sigma : \mathbb{P} \to \mathbb{P}$ defined by $\sigma p(x) = p(x^2)$, we define the symmetrization functional $\sigma\mathbf{u}$ as

$$\langle \sigma\mathbf{u}, p(x)\rangle = \langle \mathbf{u}, \sigma p(x)\rangle.$$

Exercise 2.1. Determine the moments of the linear functionals introduced in Definition 2.4 in terms of the moments of $\mathbf{u}$.

Example 2.5. Consider the moment functional $\mathbf{u}$ defined by $\langle \mathbf{u}, p(x)\rangle = p(a) + p(b)$ with $a, b \in \mathbb{R}$, $a \neq b$. In fact, $\mathbf{u} = \boldsymbol{\delta}_a + \boldsymbol{\delta}_b$. Moreover, the moments of $\mathbf{u}$ are $\mu_n = a^n + b^n$, $n \geq 0$.

- For $q(x) \in \mathbb{P}$,
$$\langle q(x)\mathbf{u}, p(x)\rangle = q(a)p(a) + q(b)p(b).$$

- Let $c \in \mathbb{R}$. If $c \neq a$ and $c \neq b$, then

$$\left\langle (x-c)^{-1}\mathbf{u}, p(x) \right\rangle = \frac{p(a)-p(c)}{a-c} + \frac{p(b)-p(c)}{b-c}.$$

If $a = c$, then

$$\left\langle (x-a)^{-1}\mathbf{u}, p(x) \right\rangle = p'(a) + \frac{p(b)-p(a)}{b-a}.$$

Example 2.6. Let $w(x)$ be a nonnegative function and integrable on an closed interval $I \subseteq \mathbb{R}$. Assume that $\int_I w(x)dx > 0$ and let us define the moment functional $\mathbf{u}$ as follows:

$$\left\langle \mathbf{u}, p(x) \right\rangle = \int_I p(x)w(x)dx.$$

- The derivative of $\mathbf{u}$ is given by

$$\left\langle D\mathbf{u}, p(x) \right\rangle = -\int_I p'(x)w(x)dx.$$

- The functional $\sigma\mathbf{u}$ is given by

$$\left\langle \sigma\mathbf{u}, p(x) \right\rangle = \int_I p(x^2)w(x)dx.$$

Note that all the moments of $\sigma\mathbf{u}$ are positive real numbers.

- Finally, consider the functional $(x-c)^{-1}\mathbf{u}$ with $c \notin I$. Then

$$\begin{aligned}\left\langle (x-c)^{-1}\mathbf{u}, p(x) \right\rangle &= \int_I \frac{p(x)-p(c)}{x-c} w(x)dx \\ &= \int_I p(x)\frac{w(x)}{x-c}dx - p(c)\int_I \frac{w(x)}{x-c}dx.\end{aligned}$$

From the above, the functional $(x-c)^{-1}\mathbf{u}$ is usually expressed as $(x-c)^{-1}\mathbf{u} = \hat{\mathbf{u}} + \hat{\mu}_0\boldsymbol{\delta}_c$, where

$$\left\langle \hat{\mathbf{u}}, p(x) \right\rangle = \int_I p(x)\frac{w(x)}{x-c}dx,$$

and $\hat{\mu}_0$ is the first moment of $\hat{\mathbf{u}}$.

Definition 2.7. Let $\mathbf{u}, \mathbf{v} \in \mathbb{P}'$ be two functionals with moments $\langle \mathbf{u}, x^n \rangle = \mu_n$ and $\langle \mathbf{v}, x^n \rangle = \nu_n$. We define the convolution product of $\mathbf{u}$ and $\mathbf{v}$ as the functional whose nth moment is given by

$$\left\langle \mathbf{u} * \mathbf{v}, x^n \right\rangle = \sum_{k=0}^{n} \mu_{n-k}\nu_k.$$

In particular, if $\mathbf{v} = \boldsymbol{\delta}$, then $\langle \mathbf{u} * \boldsymbol{\delta}, x^n \rangle = \mu_n$, $n \geq 0$, i.e., the functional $\boldsymbol{\delta}$ is a neutral (identity) element for the convolution product.

Let $\mathbf{w}$ be a moment functional such that $\mathbf{w} = \mathbf{u} * \mathbf{v}$. If $\mu_0 \neq 0$, then the moments of $\mathbf{v}$ have the following representation:

$$\begin{aligned} \nu_0 &= \frac{\omega_0}{\mu_0}, \\ \nu_n &= \frac{\omega_n - \sum_{k=0}^{n-1} \mu_{n-k} \nu_k}{\mu_0}, \quad n \geq 1, \end{aligned} \tag{2.3}$$

where $\omega_n = \langle \mathbf{w}, x^n \rangle$. Consequently, we obtain the following result.

Proposition 2.8. *If* $\mathbf{u}$ *is a functional such that* $\mu_0 \neq 0$*, then there exists a unique functional* $\mathbf{u}^{-1}$ *such that*

$$\mathbf{u} * \mathbf{u}^{-1} = \boldsymbol{\delta}.$$

The functional $\mathbf{u}^{-1}$ *is said to be the inverse functional of* $\mathbf{u}$. *Observe that the above is equivalent to*

$$\left\langle \mathbf{u} * \mathbf{u}^{-1}, 1 \right\rangle = 1, \qquad \left\langle \mathbf{u} * \mathbf{u}^{-1}, x^n \right\rangle = 0, \quad n \geq 1.$$

Exercise 2.2. Given the functional $\mathbf{u}$, with $\mu_0 \neq 0$, find the relation between the moments of $\mathbf{u}$ and $\mathbf{u}^{-1}$. *Hint.* Use equation (2.3).

The following definition will be useful in the sequel.

Definition 2.9. Given a linear functional $\mathbf{u}$ and a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, we define the polynomial $(\mathbf{u} * p)(x)$ as

$$\begin{aligned} (\mathbf{u} * p)(x) := \left\langle \mathbf{u}_y, \frac{xp(x) - yp(y)}{x - y} \right\rangle &= \sum_{k=0}^{n} \left(\sum_{m=k}^{n} a_m \mu_{m-k} \right) x^k \\ &= \left(1, x, \ldots, x^n\right) \begin{pmatrix} \mu_0 & \cdots & \cdots & \mu_n \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_0 \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}. \end{aligned}$$

The notation $\mathbf{u}_y$ indicates that the functional $\mathbf{u}$ acts on the variable y.

Definition 2.10. Let $(P_n(x))_{n\geq 0}$ be a monic basis of $\mathbb{P}$. The sequence of linear functionals $(\mathbf{u}_n)_{n\geq 0}$ defined by

$$\left\langle \mathbf{u}_m, P_n(x) \right\rangle = \delta_{m,n},$$

where $\delta_{m,n}$ is defined in (2.2), is called the dual basis of $(P_n(x))_{n\geq 0}$.

A relation between differentiation in $\mathbb{P}$ and $\mathbb{P}'$ is established in the following proposition.

Proposition 2.11. *Let $(P_n(x))_{n\geq 0}$ be a monic basis of $\mathbb{P}$ and $(\mathbf{u}_n)_{n\geq 0}$ its corresponding dual basis. Consider the sequence of polynomials $(Q_n(x))_{n\geq 0}$ defined by $Q_n(x) = \frac{P'_{n+1}(x)}{n+1}$. Then $(Q_n(x))_{n\geq 0}$ is a basis of $\mathbb{P}$ and its dual basis $(\mathbf{v}_n)_{n\geq 0}$ satisfies $D\mathbf{v}_n = -(n+1)\mathbf{u}_{n+1}$ for every $n \geq 0$.*

Proof. It is clear that for each $n \geq 0$, $Q_n(x) = \frac{P'_{n+1}(x)}{n+1}$ is a monic polynomial with $\deg Q_n(x) = n$. Therefore, the sequence of polynomials $(Q_n(x))_{n\geq 0}$ is a basis of $\mathbb{P}$. On the other hand, since $(\mathbf{u}_n)_{n\geq 0}$ is a basis of $\mathbb{P}'$, then for each $n \geq 0$, we have

$$D\mathbf{v}_n = \sum_{k=0}^{\infty} a_{n,k}\mathbf{u}_k,$$

where

$$a_{n,k} = \langle D\mathbf{v}_n, P_k(x)\rangle = -\langle \mathbf{v}_n, P'_k(x)\rangle = -k\langle \mathbf{v}_n, Q_{k-1}(x)\rangle.$$

Thus, $a_{n,n+1} = -(n+1)$ and $a_{n,k} = 0$ for $k \neq n+1$. ∎

We turn our attention to orthogonality with respect to a moment functional.

Definition 2.12. Let $\mathbf{u} \in \mathbb{P}'$ be a moment functional. A sequence of polynomials $(P_n(x))_{n\geq 0}$ is called an orthogonal polynomial sequence (OPS) with respect to $\mathbf{u}$ if

(1) $\deg P_n(x) = n$,

(2) $\langle \mathbf{u}, P_n(x)P_m(x)\rangle = \delta_{n,m}h_n$, with $h_n \neq 0$.

Here, $\delta_{m,n}$ is the Kronecker delta defined by (2.2).

Observe that an OPS constitutes a basis for $\mathbb{P}$. If for all $n \geq 0$, the leading coefficient of $P_n(x)$ is 1, that is,

$$P_n(x) = x^n + \text{terms of lower degree},$$

then $(P_n(x))_{n\geq 0}$ is said to be a *monic orthogonal polynomial sequence* (MOPS). In a similar way, if for all $n \geq 0$, $\langle \mathbf{u}, P_n^2(x)\rangle = 1$, then $(P_n(x))_{n\geq 0}$ is called an *orthonormal polynomial sequence* (ONPS).

The following result follows directly from the definition of orthogonality.

Theorem 2.13. *Let $(P_n(x))_{n\geq 0}$ be a sequence of polynomials with* $\deg P_n(x) = n$ *and let* $\mathbf{u}$ *be a moment functional. Then the following statements are equivalent:*

(1) *$(P_n(x))_{n\geq 0}$ is an OPS with respect to* $\mathbf{u}$.

(2) $\langle \mathbf{u}, P_n(x)q(x)\rangle = 0$, *if* $\deg q(x) \leq n-1$, *and* $\langle \mathbf{u}, P_n(x)q(x)\rangle \neq 0$, *if* $\deg q(x) = n$.

(3) $\langle \mathbf{u}, x^k P_n(x)\rangle = 0$ *for every* $k \leq n-1$, *and* $\langle \mathbf{u}, x^n P_n(x)\rangle \neq 0$.

A question that arises naturally is whether the family of orthogonal polynomials associated with a linear functional is unique. Clearly, if we have an OPS $(P_n(x))_{n\geq 0}$ and a sequence of nonzero numbers $(\alpha_n)_{n\geq 0}$, then the sequence of polynomials $(\alpha_n P_n(x))_{n\geq 0}$ is also orthogonal. But are there other orthogonal families that are not of this form? The answer is no.

Theorem 2.14. *Two sequences of polynomials $(P_n(x))_{n\geq 0}$, $(Q_n(x))_{n\geq 0}$ are orthogonal with respect to the same moment functional $\mathbf{u}$ if and only if for each $n \geq 0$, there exists a nonzero constant α_n such that $Q_n(x) = \alpha_n P_n(x)$.*

Proof. Suppose that there are two sequences of polynomials $(P_n(x))_{n\geq 0}$, $(Q_n(x))_{n\geq 0}$ that are orthogonal with respect to $\mathbf{u}$. Since $(P_n(x))_{n\geq 0}$ is a basis of $\mathbb{P}$, there exist constants $(\alpha_{n,k})_{k=0}^{n}$ such that

$$Q_n(x) = \sum_{k=0}^{n} \alpha_{n,k} P_k(x),$$

where

$$\alpha_{n,k} = \frac{\langle \mathbf{u}, Q_n(x)P_k(x)\rangle}{\langle \mathbf{u}, P_k^2(x)\rangle} \quad \text{and} \quad \alpha_{n,n} = \frac{\langle \mathbf{u}, Q_n(x)P_n(x)\rangle}{\langle \mathbf{u}, P_n^2(x)\rangle} \neq 0.$$

From orthogonality we have $\langle \mathbf{u}, Q_n(x)P_k(x)\rangle = 0$ for $k \leq n-1$, thus we deduce that $Q_n(x) = \alpha_{n,n} P_n(x)$.

The other implication is a direct consequence of the definition of orthogonality. ∎

2.1 Existence of orthogonal polynomial sequences

Once orthogonality with respect to a moment functional $\mathbf{u} \in \mathbb{P}'$ is defined, the next step is to study conditions for the existence of an OPS associated with $\mathbf{u}$. Thus, suppose that $(P_n(x))_{n\geq 0}$ is a MOPS with respect to $\mathbf{u}$. By Theorem 2.13, if $P_n(x) = x^n + \sum_{k=0}^{n-1} a_{n,k} x^k$, then

$$\begin{aligned} \left\langle \mathbf{u}, x^m P_n(x) \right\rangle = a_{n,0}\mu_m + a_{n,1}\mu_{m+1} + \cdots + a_{n,n-1}\mu_{n+m-1} \\ + \mu_{n+m} = 0, \quad 0 \leq m \leq n-1. \end{aligned} \tag{2.4}$$

If we define the matrix

$$H_n := \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{pmatrix},$$

then the system of linear equations obtained in (2.4) can be written in matrix form as follows:

$$H_n \begin{pmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ a_{n,n-1} \end{pmatrix} = - \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix}.$$

Thus, we have the following result.

Theorem 2.15. *The sequence of monic orthogonal polynomials* $(P_n(x))_{n\geq 0}$ *associated with the functional* **u** *exists if and only if* $\det H_n \neq 0$ *for every* $n \geq 1$*. Moreover, the polynomial* $P_n(x)$ *can be expressed as*

$$P_n(x) = x^n - \begin{pmatrix} 1 & x & \cdots & x^{n-1} \end{pmatrix} H_n^{-1} \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix}$$

$$= \frac{1}{\det H_n} \det \left(\begin{array}{cccc|c} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ \hline 1 & x & \cdots & x^{n-1} & x^n \end{array} \right), \quad n \geq 1.$$

In addition,

$$\langle \mathbf{u}, P_0^2(x) \rangle = \det H_1 = \mu_0,$$
$$\langle \mathbf{u}, P_n^2(x) \rangle = \frac{\det H_{n+1}}{\det H_n}, \quad n \geq 1.$$

The matrix H_n has a Hankel structure and is said to be the leading principal submatrix of moments of size $n \times n$. Observe that it is a consequence of Definition 2.12 that $H_1 = \mu_0 = \langle \mathbf{u}, P_0^2(x) \rangle \neq 0$.

Definition 2.16. A moment functional **u** is said to be quasidefinite if and only if $\det H_n \neq 0$ for every $n \geq 1$, and positive definite if and only if $\det H_n > 0$ for every $n \geq 1$. Clearly, if **u** is positive definite then it is quasidefinite.

In other words, a moment functional **u** is quasidefinite if and only if there is an OPS associated with **u**.

If the functional **u** is positive definite, then for any polynomial $p(x)$ not identically zero ($p(x) \not\equiv 0$), $\langle \mathbf{u}, p^2(x) \rangle > 0$. Therefore, if $(P_n(x))_{n\geq 0}$ is an OPS associated with **u**, then the sequence of polynomials $(Q_n(x))_{n\geq 0}$ defined by $Q_0(x) := \mu_0^{-1/2}$

and

$$Q_n(x) = \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle^{1/2}}, \quad n \geq 1,$$

is the ONPS associated with $\mathbf{u}$ (it is not difficult to show that an ONPS associated with a positive definite moment functional is unique).

Exercise 2.3. Show that an ONPS associated with a positive definite moment functional is unique. The proof is similar to the proof of Theorem 2.14.

Lemma 2.17. *Let $h(x)$ be a polynomial such that $h(x) \geq 0$ for all $x \in \mathbb{R}$. Then there exist polynomials $p(x)$ and $q(x)$ with real coefficients such that*

$$h(x) = p^2(x) + q^2(x).$$

Observe that the above lemma implies that if $\mathbf{u}$ is a positive definite functional and $h(x) \geq 0$ for all $x \in \mathbb{R}$, then $\langle \mathbf{u}, h(x)\rangle \geq 0$.

Theorem 2.18. *If $\mathbf{u}$ is a positive definite functional, then*

(i) $\mathbf{u}$ *has real moments;*

(ii) *the ONPS associated with* $\mathbf{u}$ *consists of polynomials with real coefficients.*

Proof. Let $(\mu_n)_{n\geq 0}$ be the moments of $\mathbf{u}$. From the hypothesis, we have that the even moments $(\mu_{2n})_{n\geq 0}$ are positive real numbers. If we apply $\mathbf{u}$ to the polynomial $(x+1)^2$, we obtain

$$\mu_1 = \frac{1}{2}\left(\left\langle \mathbf{u}, (x+1)^2\right\rangle - \mu_2 - \mu_0\right) \in \mathbb{R}.$$

In general,

$$\mu_{2n-1} = \frac{1}{2n}\left(\left\langle \mathbf{u}, (x+1)^{2n}\right\rangle - \mu_{2n} - \sum_{k=0}^{2n-2}\binom{2n}{k}\mu_{2n-k}\right),$$

and (i) follows by induction.

Let $(Q_n(x))_{n\geq 0}$ be the ONPS associated with $\mathbf{u}$. Observe that $\det H_n > 0$ for $n \geq 1$, since $\mathbf{u}$ is positive definite. Then, from (i) and Theorem 2.15, we have that $Q_n(x)$ has real coefficients for $n \geq 0$. ∎

Example 2.19. Let $\mathbf{u}$ be a functional defined by

$$\langle \mathbf{u}, p\rangle = \int p(x)d\mu(x),$$

where $d\mu$ is a measure supported on an infinite subset of the real line. Recall that every measure can be written as the sum of an absolutely continuous measure with respect to Lebesgue measure and a singular measure. In particular, if μ is absolutely continuous, then there exists a function $w(x)$, positive on the support of μ, such that $d\mu(x) = w(x)dx$ (the moment functional $\mathbf{u}$ is positive definite). The function $w(x)$ is called the weight function associated with $\mathbf{u}$. The most well-known weight functions in the literature, as well as their corresponding sequences of orthogonal polynomials, are (see Chapter 8)

(1) Beta distribution, $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, $x \in [-1, 1]$, $\alpha, \beta > -1$. The associated OPS are Jacobi polynomials.

(2) Gamma distribution, $d\mu(x) = x^{\alpha}e^{-x}dx$, $x \in (0, \infty)$, $\alpha > -1$. The associated OPS are Laguerre polynomials.

(3) Normal distribution, $d\mu(x) = e^{-x^2}dx$, $x \in (-\infty, \infty)$. The associated OPS are Hermite polynomials.

It is interesting to note that if $\mathbf{u}$ is a positive definite moment functional, then the bilinear form $(\cdot, \cdot) : \mathbb{P} \times \mathbb{P} \to \mathbb{C}$ defined by

$$\big(p(x), q(x)\big) = \big\langle \mathbf{u}, p(x)q(x) \big\rangle$$

is an inner product. Then, $\mathbf{u}$ induces a norm $\| \cdot \|$ on $\mathbb{P}$ defined by

$$\|p(x)\| = \sqrt{\big(p(x), p(x)\big)} = \sqrt{\big\langle \mathbf{u}, p^2(x) \big\rangle}.$$

Clearly, an ONPS associated with $\mathbf{u}$ constitutes an orthonormal basis of $\mathbb{P}$ with respect to $(\cdot, \cdot)$.

2.2 Three-term recurrence relation

One of the most important properties of OPS is that they satisfy a three-term recurrence relation. This property will allow us to deduce results about their zeros [13,54].

Proposition 2.20 (Three-term recurrence relation). *Let* $\mathbf{u}$ *be a quasidefinite functional and let* $(P_n(x))_{n\geq 0}$ *be its corresponding sequence of monic orthogonal polynomials. Then there exist two sequences of complex numbers* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 0}$*, with* $a_n \neq 0$ *for* $n \geq 1$*, such that*

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 0, \\ P_{-1}(x) &= 0, \quad P_0(x) = 1. \end{aligned} \tag{2.5}$$

In addition,

$$b_n = \frac{\langle \mathbf{u}, xP_n^2(x) \rangle}{\langle \mathbf{u}, P_n^2(x) \rangle}, \quad n \geq 0, \qquad a_n = \frac{\langle \mathbf{u}, P_n^2(x) \rangle}{\langle \mathbf{u}, P_{n-1}^2(x) \rangle}, \quad n \geq 1. \tag{2.6}$$

If $\mathbf{u}$ *is positive definite, then* b_n *and* a_n *are real numbers with* $a_n > 0$*. Moreover, its sequence of orthonormal polynomials* $(Q_n(x))_{n\geq 0}$ *satisfies the following relation:*

$$\begin{aligned} &xQ_n(x) = c_{n+1}Q_{n+1}(x) + b_nQ_n(x) + c_nQ_{n-1}(x), \quad n \geq 0, \\ &Q_{-1}(x) = 0, \quad Q_0(x) = \mu_0^{-1/2}, \end{aligned} \tag{2.7}$$

where $c_n^2 = a_n$*.*

Proof. Since $(P_n(x))_{n\geq 0}$ is a basis of $\mathbb{P}$ and $xP_n(x)$ is a monic polynomial of degree $n+1$, there exist complex numbers $(\alpha_{n,k})_{k=0}^n$ such that

$$xP_n(x) = P_{n+1}(x) + \sum_{k=0}^{n} \alpha_{n,k}P_k(x).$$

From orthogonality (see Theorem 2.13), we get

$$\alpha_{n,k} = \frac{\langle \mathbf{u}, xP_n(x)P_k(x)\rangle}{\langle \mathbf{u}, P_k^2(x)\rangle} = \begin{cases} 0, & \text{if } k \leq n-2, \\ \dfrac{\langle \mathbf{u}, xP_n(x)P_{n-1}(x)\rangle}{\langle \mathbf{u}, P_{n-1}^2(x)\rangle}, & \text{if } k = n-1, \\ \dfrac{\langle \mathbf{u}, xP_n^2(x)\rangle}{\langle \mathbf{u}, P_n^2(x)\rangle}, & \text{if } k = n. \end{cases}$$

Taking into account that

$$\big\langle \mathbf{u}, xP_n(x)P_{n-1}(x)\big\rangle = \big\langle \mathbf{u}, P_n(x)xP_{n-1}(x)\big\rangle = \big\langle \mathbf{u}, P_n^2(x)\big\rangle,$$

we obtain the result. The second property is a direct consequence of the above. ■

Exercise 2.4. By comparing coefficients on both sides of (2.5), show that if the sequence $(P_n(x))_{n\geq 0}$ satisfies (2.5) and $P_n(x) = x^n + \sum_{k=0}^{n-1} \lambda_{n,k}x^k$, then

$$b_n = \lambda_{n,n-1} - \lambda_{n+1,n} \quad \text{and} \quad a_n = \lambda_{n,n-2} - \lambda_{n+1,n-1} - b_n\lambda_{n,n-1}. \tag{2.8}$$

If we define the semiinfinite matrices

$$J_{\text{mon}} = \begin{pmatrix} b_0 & 1 & & \\ a_1 & b_1 & 1 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} b_0 & c_0 & & \\ c_0 & b_1 & c_1 & \\ & c_1 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

then the three-term recurrence relations (2.5) and (2.7) can be written in matrix form as

$$x\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = J_{\text{mon}}\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix}, \qquad x\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix} = J\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix}, \tag{2.9}$$

respectively. The matrices J_{mon} and J are known in the literature as Jacobi matrices. Taking into account the structure of J (tridiagonal and symmetric), it can be associated with a discrete Sturm–Liouville operator (see Chapter 8).

The following interesting result states that the MOPS and ONPS associated with a positive definite linear functional share zeros, which, in turn, are the eigenvalues of the truncated Jacobi matrices.

Proposition 2.21. *Let* $\mathbf{u}$ *be a positive definite moment functional and let* $(P_n(x))_{n\geq 0}$ *and* $(Q_n(x))_{n\geq 0}$ *be its associated MOPS and ONPS, respectively. For* $n \geq 1$, $P_n(x)$ *and* $Q_n(x)$ *have the same zeros. Moreover, if* $x_{n,k}$, $k = 1, \ldots, n$ *are the zeros of* $Q_n(x)$*, then each* $x_{n,k}$ *is an eigenvalue of the matrix*

$$J_n = \begin{pmatrix} b_0 & c_0 & & & \\ c_0 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-2} \\ & & & c_{n-2} & b_{n-1} \end{pmatrix},$$

where J_n *is the leading principal submatrix of* J *of size* $n \times n$.

Proof. Since $\mathbf{u}$ is a positive definite functional, its sequence of orthonormal polynomials $(Q_n(x))_{n\geq 0}$ exists, and because the sequence of monic orthogonal polynomials satisfies $P_n(x) = \langle \mathbf{u}, P_n^2(x)\rangle^{1/2} Q_n(x)$, polynomials $Q_n(x)$ and $P_n(x)$ have the same zeros. On the other hand, from (2.9),

$$x \begin{pmatrix} Q_0(x) \\ \vdots \\ Q_{n-1}(x) \end{pmatrix} = J_n \begin{pmatrix} Q_0(x) \\ \vdots \\ Q_{n-1}(x) \end{pmatrix} + Q_n(x) \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus, if $Q_n(x_{n,k}) = 0$, then

$$x_{n,k} \begin{pmatrix} Q_0(x_{n,k}) \\ \vdots \\ Q_{n-1}(x_{n,k}) \end{pmatrix} = J_n \begin{pmatrix} Q_0(x_{n,k}) \\ \vdots \\ Q_{n-1}(x_{n,k}) \end{pmatrix},$$

and the result is obtained. ■

Corollary 2.22. *If* $\mathbf{u}$ *is a positive definite moment functional and* $(P_n(x))_{n\geq 0}$ *is an OPS associated with* $\mathbf{u}$*, then the zeros of* $P_n(x)$ *are real numbers.*

We have already shown that the sequence of orthogonal polynomials satisfies a three-term recurrence relation. A natural question to ask is if the converse holds. That is, given a sequence of polynomials $(P_n(x))_{n\geq 0}$ defined recursively as in (2.5), does there exist a quasidefinite functional **u** such that $(P_n(x))_{n\geq 0}$ is an OPS associated with **u**? J. Favard showed in 1935 that the answer to this question is affirmative, although this result had previously been proved independently by J. Shohat and I. Natanson. For a modern alternative proof in the positive define case, see [7].

Theorem 2.23 (Favard's theorem [13, 38, 51]). *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers with $a_n \neq 0$, $n \geq 0$, and let $(P_n(x))_{n\geq 0}$ be the sequence of monic polynomials defined by*

$$\begin{aligned}
&xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 0,\\
&P_{-1}(x) = 0, \quad P_0(x) = 1.
\end{aligned}$$

*Then there exists a quasidefinite moment functional **u** such that $(P_n(x))_{n\geq 0}$ is the MOPS with respect to **u**. Moreover, if b_n, and a_n are real with $a_n > 0$ for every $n \geq 0$, then **u** is positive definite.*

Proof. We will define the moment functional **u** inductively from its moments and the requirement that $\langle \mathbf{u}, P_n(x)\rangle = 0$ for $n \geq 1$.

Define $\langle \mathbf{u}, 1\rangle = a_0 = \mu_0$. By hypothesis, we have $P_1(x) = x - b_0$. Define $\mu_1 := b_0\mu_0$. Then $\langle \mathbf{u}, P_1(x)\rangle = 0$. In a similar way, $P_2(x) = x^2 - (b_1 + b_0)x + (b_1 b_0 - a_1)$, and define $\mu_2 := (b_1 + b_0)\mu_1 - (b_1 b_0 - a_1)\mu_0$. Then $\langle \mathbf{u}, P_2(x)\rangle = 0$.

In general, if we know the moments $\mu_0, \mu_1, \ldots, \mu_{n-1}$, then using the representation of $P_n(x)$ in terms of the monomial basis

$$P_n(x) = x^n + \sum_{k=0}^{n-1} \lambda_{n,k} x^k, \quad n \geq 1,$$

we can define $\mu_n := -\sum_{k=0}^{n-1} \lambda_{n,k}\mu_k$. With this in mind, we see that $\langle \mathbf{u}, P_n(x)\rangle = 0$.

Once the functional **u** has been defined from its moments, we will prove the orthogonality property. From the recurrence relation, we get

$$\begin{aligned}
&\big\langle \mathbf{u}, xP_n(x)\big\rangle = \big\langle \mathbf{u}, P_{n+1}(x)\big\rangle + b_n\big\langle \mathbf{u}, P_n(x)\big\rangle + a_n\big\langle \mathbf{u}, P_{n-1}(x)\big\rangle = 0, \quad n \geq 2,\\
&\big\langle \mathbf{u}, xP_1(x)\big\rangle = a_1 a_0.
\end{aligned}$$

Using the recurrence relation again and the previous equation, we obtain

$$\begin{aligned}
\big\langle \mathbf{u}, x^2 P_n(x)\big\rangle &= \big\langle \mathbf{u}, xP_{n+1}(x)\big\rangle + b_n\big\langle \mathbf{u}, xP_n(x)\big\rangle + a_n\big\langle \mathbf{u}, xP_{n-1}(x)\big\rangle\\
&= \big\langle \mathbf{u}, P_{n+2}(x)\big\rangle + b_{n+1}\big\langle \mathbf{u}, P_{n+1}(x)\big\rangle + a_{n+1}\big\langle \mathbf{u}, P_n(x)\big\rangle\\
&\quad + a_n\big(\big\langle \mathbf{u}, P_n(x)\big\rangle + b_{n-1}\big\langle \mathbf{u}, P_{n-1}(x)\big\rangle + a_{n-1}\big\langle \mathbf{u}, P_{n-2}(x)\big\rangle\big)\\
&= 0, \quad n \geq 3,\\
\big\langle \mathbf{u}, x^2 P_2(x)\big\rangle &= a_2 a_1 a_0.
\end{aligned}$$

Recursively, for $0 \leq k \leq n-1$,

$$\big\langle \mathbf{u}, x^k P_n(x)\big\rangle = \big\langle \mathbf{u}, x^{k-1} P_{n+1}(x)\big\rangle + b_n\big\langle \mathbf{u}, x^{k-1} P_n(x)\big\rangle + a_n\big\langle \mathbf{u}, x^{k-1} P_{n-1}(x)\big\rangle = 0$$

and

$$\langle \mathbf{u}, x^n P_n(x)\rangle = a_n\langle \mathbf{u}, x^{n-1} P_{n-1}(x)\rangle = a_n a_{n-1}\langle \mathbf{u}, x^{n-2} P_{n-2}(x)\rangle = \cdots$$
$$= a_n a_{n-1} \cdots a_1 a_0.$$

Since for all $n \geq 0$, $a_n \neq 0$, the functional $\mathbf{u}$ is quasidefinite. Notice that, in particular, if $a_n > 0$ for all $n \geq 0$, then $\mathbf{u}$ is positive definite. ■

Example 2.24. Let $\mathbf{u}$ be the moment functional defined by

$$\langle \mathbf{u}, p(x)\rangle = \int_{-1}^{1} p(x)\big(1 - x^2\big)^{1/2}\, dx, \quad p(x) \in \mathbb{P}.$$

The polynomials defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{2^n \sin\theta}, \quad x = \cos\theta,\ n = 0, 1 \ldots,$$

known in the literature as the monic Chebyshev polynomials of the second kind, constitute an OPS with respect to $\mathbf{u}$. Using the explicit expressions of $U_n(x)$ and $\mathbf{u}$, it can be proved that

$$\langle \mathbf{u}, U_n(x) U_m(x)\rangle = 2^{-(n+m+1)}\pi\delta_{n,m}$$

and

$$\begin{aligned} xU_n(x) &= U_{n+1}(x) + \frac{1}{4}U_{n-1}(x), \quad n \geq 0, \\ U_{-1}(x) &= 0, \quad U_0(x) = 1. \end{aligned} \tag{2.10}$$

Exercise 2.5. Use the identity $2 \sin a \cos b = \sin(a + b) + \sin(a - b)$ to deduce the three-term recurrence relation satisfied the Chebyshev polynomials of the second kind.

Example 2.25. In this example, we illustrate the important role that generating functions play in the construction of sequences of orthogonal polynomials.

A function $F(x,t)$ is said to be a generating function for the sequence $(g_k(x))_{k\geq 0}$ if there exists a sequence of numbers $(c_k)_{k\geq 0}$ such that

$$F(x,t) = \sum_{k=0}^{\infty} c_k g_k(x) t^k.$$

Here, we consider the function $F(x,t) = e^{-at}(1+t)^x$ with $a \neq 0$. Using power series expansions and the Cauchy product for infinite series, we get that $F(x,t)$ can be expressed as

$$F(x,t) = \left(\sum_{m=0}^{\infty} \frac{(-a)^m}{m!} t^m\right)\left(\sum_{n=0}^{\infty} \binom{x}{n} t^n\right) = \sum_{m=0}^{\infty} P_m^{(a)}(x) t^m,$$

where $\binom{x}{0} = 1$, $\binom{x}{\nu} = \frac{1}{\nu!}x(x-1)\cdots(x-\nu+1)$ for $\nu \geq 1$, and

$$P_m^{(a)}(x) = \sum_{k=0}^{m} \binom{x}{k} \frac{(-a)^{m-k}}{(m-k)!}.$$

Observe that $P_m^{(a)}(x)$ is a polynomial of degree m with the leading coefficient $1/m!$.

An orthogonality relation for these polynomials can be deduced as follows. On the one hand, we have

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{a^k}{k!} F(k,s)F(k,t) &= \sum_{k=0}^{\infty} \frac{e^{-a(s+t)}[a(1+s)(1+t)]^k}{k!} \\ &= e^{-a(s+t)} e^{a(1+s)(1+t)} \\ &= e^a e^{ast} \\ &= \sum_{n=0}^{\infty} \frac{e^a a^n}{n!}(st)^n\end{aligned}$$

and, on the other hand,

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} F(k,s)F(k,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty} P_m(k)P_n(k)\frac{a^k}{k!}\right)s^m t^n.$$

That is,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty} P_m(k)P_n(k)\frac{a^k}{k!}\right)s^m t^n = \sum_{n=0}^{\infty} \frac{e^a a^n}{n!}(st)^n.$$

Then, comparing the coefficients of $s^m t^n$, we get the following orthogonality relation:

$$\sum_{k=0}^{\infty} P_m(k)P_n(k)\frac{a^k}{k!} = \frac{e^a a^n}{n!}\delta_{m,n}.$$

Now, we are ready to express this orthogonality relation in terms of a moment functional. For $a \neq 0$, consider the moment functional $\mathbf{u}$ defined by

$$\mathbf{u} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\boldsymbol{\delta}_k,$$

with moments given by

$$\langle \mathbf{u}, x^n \rangle = \sum_{k=0}^{\infty} \frac{a^k}{k!} k^n.$$

Then, the sequence of monic polynomials $(C_n^{(a)}(x))_{n\geq 0}$, where $C_n^{(a)}(x) = n!P_n^{(a)}(x)$ (known as Charlier polynomials), are orthogonal with respect to $\mathbf{u}$. Moreover, using (2.6), (2.8), and the fact that the coefficient of x^{n-1} in $C_n^{(a)}(x)$ is

$$\lambda_{n,n-1} = -an - \sum_{k=1}^{n-1} k = -\frac{n(n-1+2a)}{2},$$

we get the three-term recurrence relation

$$\begin{aligned} &xC_n^{(a)}(x) = C_{n+1}^{(a)}(x) + (n+a)C_n^{(a)}(x) + anC_{n-1}^{(a)}(x), \quad n \geq 0, \\ &C_{-1}^{(a)}(x) = 0, \quad C_0^{(a)}(x) = 1. \end{aligned}$$

As a last comment, it is important to remark that if $a > 0$, then $\mathbf{u}$ is a positive definite moment functional (see Theorem 2.23).

Exercise 2.6. Use the ratio test (d'Alembert's criterion) for infinite series to show that the moments in the previous example are well defined.

2.3 Christoffel–Darboux kernel polynomials

Let $\mathbf{u}$ be a quasidefinite moment functional and $(P_n(x))_{n\geq 0}$ the MOPS associated with $\mathbf{u}$. For every polynomial $p(x) \in \mathbb{P}$ and $n \geq 0$, the orthogonal projection $\hat{p}(y) \in \mathbb{P}_n$ of $p(x)$ onto $\mathbb{P}_n$ is

$$\hat{p}(y) = \sum_{k=0}^{n} \frac{\langle \mathbf{u}, p(x)P_k(x)\rangle}{\langle \mathbf{u}, P_k^2(x)\rangle} P_k(y) = \left\langle \mathbf{u}, p(x) \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle \mathbf{u}, P_k^2(x)\rangle} \right\rangle.$$

This motivates the following definition.

Definition 2.26. Let $(P_n(x))_{n\geq 0}$ be the MOPS with respect to the quasidefinite moment functional $\mathbf{u}$. We define the nth Christoffel–Darboux (C–D) kernel polynomial as

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle \mathbf{u}, P_k^2(x)\rangle}.$$

Proposition 2.27 (Reproducing kernel property). *Given a moment functional* $\mathbf{u}$ *and* $(P_n(x))_{n\geq 0}$ *its corresponding MOPS, for every polynomial* $q(x) \in \mathbb{P}_n$*, it holds that*

$$\langle \mathbf{u}, K_n(x, y)q(x)\rangle = q(y).$$

From the three-term recurrence relation, we can deduce a closed form for C–D kernel polynomials.

Theorem 2.28 (Christoffel–Darboux formula [13,51]). *Let $(P_n(x))_{n\geq 0}$ be the MOPS associated with a quasidefinite moment functional* $\mathbf{u}$. *Then,*

$$(x-y)K_n(x,y) = \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{\langle \mathbf{u}, P_n^2(x)\rangle}.$$

Proof. Since $(P_n(x))_{n\geq 0}$ is the MOPS associated with $\mathbf{u}$, these polynomials satisfy (2.5). Thus,

$$\begin{aligned} xP_k(x) &= P_{k+1}(x) + b_k P_k(x) + a_k P_{k-1}(x), \quad k \geq 0, \\ yP_k(y) &= P_{k+1}(y) + b_k P_k(y) + a_k P_{k-1}(y), \quad k \geq 0, \end{aligned}$$

where a_k is given in (2.6). Multiplying the first equation by $P_k(y)$ and the second by $P_k(x)$, and then subtracting the corresponding expressions, we get

$$\begin{aligned} \frac{(x-y)P_k(x)P_k(y)}{\langle \mathbf{u}, P_k^2(x)\rangle} &= \frac{P_{k+1}(x)P_k(y) - P_k(x)P_{k+1}(y)}{\langle \mathbf{u}, P_k^2(x)\rangle} \\ &\quad - \frac{P_k(x)P_{k-1}(y) - P_{k-1}(x)P_k(y)}{\langle \mathbf{u}, P_{k-1}^2(x)\rangle}. \end{aligned}$$

Summing the above equation from 0 to n, we obtain

$$\begin{aligned} (x-y)K_n(x,y) &= (x-y)\sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle \mathbf{u}, P_k^2(x)\rangle} \\ &= \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{\langle \mathbf{u}, P_n^2(x)\rangle}. \end{aligned}$$

■

Using the Christoffel–Darboux formula, it is easy to show that

$$K_n(x,x) = \lim_{y\to x} K_n(x,y).$$

Corollary 2.29. *We have*

$$K_n(x,x) = \sum_{k=0}^{n} \frac{P_k^2(x)}{\langle \mathbf{u}, P_k^2(x)\rangle} = \frac{P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}. \tag{2.11}$$

Using C–D kernel polynomials, we can deduce the following important result about OPS associated with positive definite moment functionals.

Corollary 2.30. *If* $\mathbf{u}$ *is positive definite, then, for $n \geq 1$, $P_n(x)$ has real and simple zeros.*

Proof. From Corollary 2.22, we know that $P_n(x)$ has real zeros. Now, from (2.11) we get

$$0 < \sum_{k=0}^{n} \frac{P_k^2(x)}{\langle \mathbf{u}, P_k^2(x)\rangle} = K_n(x,x).$$

If $P_n(x)$ has a nonsimple zero α, then $P_n(\alpha) = P_n'(\alpha) = 0$ and $K_n(\alpha, \alpha) = 0$, which is a contradiction. ■

Given a positive definite moment functional $\mathbf{u}$ and its corresponding ONPS $(Q_n(x))_{n\geq 0}$, we are interested in studying the following extremal problem:

$$\inf_{\substack{p(x)\in\mathbb{P}_n\\ p(a)=1}} \langle \mathbf{u}, |p(x)|^2 \rangle.$$

Let $p(x)$ be a polynomial in $\mathbb{P}_n$ such that $p(a) = 1$. Then there exists a sequence of complex numbers $(\lambda_{n,k})_{k=0}^n$ such that $p(x) = \sum_{k=0}^n \lambda_{n,k} Q_k(x)$. Due to orthogonality, we have

$$\langle \mathbf{u}, |p(x)|^2 \rangle = \left\langle \mathbf{u}, \left| \sum_{k=0}^n \lambda_{n,k} Q_k(x) \right|^2 \right\rangle = \sum_{k=0}^n |\lambda_{n,k}|^2 \langle \mathbf{u}, |Q_k(x)|^2 \rangle = \sum_{k=0}^n |\lambda_{n,k}|^2.$$

On the other hand, using the Cauchy–Schwarz inequality, we get

$$1 = |p(a)|^2 = \left| \sum_{k=0}^n \lambda_{n,k} Q_k(a) \right|^2 \leq \left(\sum_{k=0}^n |\lambda_{n,k}|^2 \right) \left(\sum_{k=0}^n |Q_k(a)|^2 \right).$$

Thus,

$$\langle \mathbf{u}, |p(x)|^2 \rangle = \sum_{k=0}^n |\lambda_{n,k}|^2 \geq \frac{1}{\sum_{k=0}^n |Q_k(a)|^2} = \frac{1}{K_n(a,a)}.$$

Observe that $K_n(x,a)$ is a polynomial of degree at most n. From Proposition 2.27, it follows that

$$\left\langle \mathbf{u}, \left| \frac{K_n(x,a)}{K_n(a,a)} \right|^2 \right\rangle = \frac{1}{K_n(a,a)},$$

that is, the polynomial $\frac{K_n(x,a)}{K_n(a,a)}$ is the infimum we seek.

2.4 Polynomials of the first kind and the Stieltjes function

If $(P_n(x))_{n\geq 0}$ is a sequence of monic polynomials satisfying (2.5), we define the polynomial of the first kind $(P_n^{(1)}(x))_{n\geq 0}$ as the sequence of polynomials satisfying the following recurrence relation:

$$\begin{aligned} xP_n^{(1)}(x) &= P_{n+1}^{(1)}(x) + b_{n+1} P_n^{(1)}(x) + a_{n+1} P_{n-1}^{(1)}(x), \quad n \geq 0, \\ P_0^{(1)}(x) &= 1, \quad P_{-1}^{(1)}(x) = 0. \end{aligned} \tag{2.12}$$

A way to represent the polynomials $(P_n^{(1)}(x))_{n\geq 0}$ is given in the following proposition.

Proposition 2.31. *If* $\mathbf{u}$ *is a quasidefinite moment functional and* $(P_n(x))_{n\geq 0}$ *is its corresponding MOPS, then the sequence of monic polynomials* $(P_n^{(1)}(x))_{n\geq 0}$ *defined by*

$$P_{n-1}^{(1)}(x) = \frac{1}{\mu_0}\left\langle \mathbf{u}_y, \frac{P_n(x) - P_n(y)}{x-y}\right\rangle, \quad n \geq 1,$$

satisfies (2.12).[3]

Proof. Since $(P_n(x))_{n\geq 0}$ satisfies the recurrence relation (2.5), we have

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 0, \\ yP_n(y) &= P_{n+1}(y) + b_n P_n(y) + a_n P_{n-1}(y), \quad n \geq 0. \end{aligned}$$

If we subtract one equation from the other and divide by $\mu_0(x-y)$, we get that

$$\begin{aligned} \frac{xP_n(x) - yP_n(y)}{\mu_0(x-y)} = &\frac{P_{n+1}(x) - P_{n+1}(y)}{\mu_0(x-y)} \\ &+ b_n \frac{P_n(x) - P_n(y)}{\mu_0(x-y)} \\ &+ a_n \frac{P_{n-1}(x) - P_{n-1}(y)}{\mu_0(x-y)}, \quad n \geq 1. \end{aligned}$$

By applying the moment functional to both sides of the above equation and taking into account that

$$\begin{aligned} \left\langle \mathbf{u}_y, \frac{xP_n(x) - yP_n(y)}{\mu_0(x-y)}\right\rangle &= \left\langle \mathbf{u}_y, \frac{xP_n(x) - xP_n(y) + xP_n(y) - yP_n(y)}{\mu_0(x-y)}\right\rangle \\ &= xP_{n-1}^{(1)}(x) + \frac{1}{\mu_0}\langle \mathbf{u}_y, P_n(y)\rangle \\ &= xP_{n-1}^{(1)}(x), \quad n \geq 1, \end{aligned}$$

we obtain the result. ■

Notice that the families of polynomials $(P_n(x))_{n\geq 0}$ and $(P_{n-1}^{(1)}(x))_{n\geq 0}$ are linearly independent solutions of the recurrence relation (2.5). Thus, any other polynomial solution can be written as a linear combination of them.

[3]The expression $\mathbf{u}_y$ indicates that the functional $\mathbf{u}$ acts on the variable y.

Observation 2.32. From the representation given in Theorem 2.15 and the fact that $\frac{x^k-y^k}{x-y} = \sum_{j=0}^{k-1} x^j y^{k-j-1}$, we obtain that the polynomial $P_{n-1}^{(1)}(x)$ can be written as

$$P_{n-1}^{(1)}(x) = \frac{1}{\mu_0}\begin{pmatrix}1 & x & \cdots & x^{n-1}\end{pmatrix}$$
$$\times \left(\begin{pmatrix} \mu_{n-1} \\ \mu_{n-2} \\ \vdots \\ \mu_0 \end{pmatrix} - \begin{pmatrix} 0 & \mu_0 & \mu_1 & \cdots & \mu_{n-2} \\ 0 & 0 & \mu_0 & \cdots & \mu_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & \mu_0 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} H_n^{-1} \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix} \right).$$

Definition 2.33. Let $\mathbf{u}$ be a functional with moments $(\mu_n)_{n\geq 0}$ and let $(P_m(x))_{m\geq 0}$ be its corresponding MOPS. If $P_m(x) = \sum_{k=0}^{m} a_{m,k} x^k$ with $a_{m,m} = 1$, we define $\mathcal{S}_m(z)$ as the formal series

$$\mathcal{S}_m(z) = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{m} a_{m,k}\mu_{n+k}}{z^{n+1}}.$$

In particular, $\mathcal{S}_0(z) := \mathcal{S}(z)$ is called the Stieltjes function associated with the functional $\mathbf{u}$ (see [64]).

A way to approximate the values of the Stieltjes function is by using the (n,n)-Padé approximant. That is, we construct a pair of polynomials $T_n(z)$ and $R_n(z)$ of degree at most n such that

$$T_n(z)\mathcal{S}(z) - R_n(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right). \tag{2.13}$$

If

$$T_n(z) = z^n + d_{n-1}z^{n-1} + \cdots + d_1 z + d_0,$$
$$R_n(z) = r_n z^n + r_{n-1}z^{n-1} + \cdots + r_1 z + r_0,$$

then (2.13) holds if the coefficients of the monomials

$$z^{-n}, z^{-n+1}, \ldots, z^{n-1}, z^n$$

are zero. The coefficients of z^{-k}, $1 \leq k \leq n$, vanish if

$$d_0\mu_{k-1} + d_1\mu_k + \cdots + d_{n-1}\mu_{n+k-2} = -\mu_{n+k-1}, \quad 1 \leq k \leq n.$$

In matrix form, these equations read

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix} = - \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix}.$$

From Theorem 2.15, we see that the elements $(d_k)_{k=0}^{n-1}$ are precisely the coefficients of monic orthogonal polynomial $P_n(z)$ associated with $\mathbf{u}$. Therefore, $T_n(z) = P_n(z)$.

Now, the coefficients of z^k, $0 \le k \le n$, vanish if

$$d_k\mu_0 + d_{k+1}\mu_1 + \cdots + d_{n-1}\mu_{n-k-1} + \mu_{n-k} = -r_{k-1}, \quad 1 \le k \le n, \tag{2.14}$$

as well as $r_n = 0$. This last condition implies that the polynomial $R_n(z)$ is of degree at most $n-1$. Writing (2.14) in matrix form, we obtain

$$\begin{aligned} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{pmatrix} &= \begin{pmatrix} \mu_{n-1} \\ \mu_{n-2} \\ \vdots \\ \mu_0 \end{pmatrix} + \begin{pmatrix} 0 & \mu_0 & \mu_1 & \cdots & \mu_{n-2} \\ 0 & 0 & \mu_0 & \cdots & \mu_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & \mu_0 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \mu_{n-1} \\ \mu_{n-2} \\ \vdots \\ \mu_0 \end{pmatrix} - \begin{pmatrix} 0 & \mu_0 & \mu_1 & \cdots & \mu_{n-2} \\ 0 & 0 & \mu_0 & \cdots & \mu_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & \mu_0 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} H_n^{-1} \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix}. \end{aligned}$$

Thus, from Remark 2.32, it follows that $R_n(z) = \mu_0 P_{n-1}^{(1)}(z)$. Therefore

$$P_n(z)\mathcal{S}(z) - \mu_0 P_{n-1}^{(1)}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right). \tag{2.15}$$

In this way, we can associate $\mathcal{S}(z)$ with a MOPS $(P_n(x))_{n\ge0}$. Moreover, since $(P_n^{(1)}(x))_{n\ge0}$ satisfies (2.7), by Favard's theorem, there exists a quasidefinite linear functional $\mathbf{u}^{(1)}$ such that $(P_n^{(1)}(x))_{n\ge0}$ is the corresponding MOPS. Therefore, the Stieltjes function associated with $\mathbf{u}^{(1)}$, or, equivalently, with $(P_n^{(1)}(x))_{n\ge0}$, namely

$$\mathcal{S}^{(1)} = \sum_{n=0}^{\infty} \frac{\mu_n^{(1)}}{z^{n+1}},$$

where $(\mu_n^{(1)})_{n\geq 0}$ are the moments of $\mathbf{u}^{(1)}$, satisfies

$$P_n^{(1)}(z)\mathcal{S}^{(1)}(z) - \mu_0^{(1)} P_{n-1}^{(2)}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right),$$

with

$$P_{n-1}^{(2)}(x) = \frac{1}{\mu_0^{(1)}}\left\langle \mathbf{u}_y^{(1)}, \frac{P_n^{(1)}(x) - P_n^{(1)}(y)}{x-y}\right\rangle, \quad n \geq 1.$$

These polynomials are known as *polynomials of the second kind*.

New Stieltjes functions can be constructed from old ones as follows.

Definition 2.34. A rational spectral transformation of the Stieltjes function $\mathcal{S}(z)$ is a transformation of the form

$$\tilde{\mathcal{S}}(z) := \frac{A(z)\mathcal{S}(z) + B(z)}{C(z)\mathcal{S}(z) + D(z)}, \quad A(z)D(z) - B(z)C(z) \neq 0,$$

where $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are polynomials such that $\tilde{\mathcal{S}}(z)$ has the same asymptotic behavior as $\mathcal{S}(z)$ at infinity, i.e., $\tilde{\mathcal{S}}(z)$ can be written as a formal series as follows:

$$\tilde{\mathcal{S}}(z) = \sum_{k=0}^{\infty} \frac{\tilde{\mu}_k}{z^{k+1}},$$

where $(\tilde{\mu}_k)_{k\geq 0}$ are called the transformed moments. If $C(z) = 0$, we say that the spectral transformation is linear.

Chapter 3

Continued fractions

Continued fractions have played a central role in approximation theory. For example, they were used as a tool to prove that π and e are transcendental numbers. Continued fractions have also been used to solve Diophantine equations, as well as the classical moment problem. The latter gives us a connection between moment functionals and orthogonal polynomials.

Recall that given two sequences of real numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ with $a_n \neq 0$, $n \geq 1$, the expression

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}}$$

is said to be a continued fraction. Let $C_0 = b_0$ and

$$C_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}}, \quad n \geq 1.$$

Then C_n is called the nth approximant of the continued fraction. It is clear that the sequence of numbers $(C_n)_{n\geq 0}$ converges to the value of the continued fraction. Thus, our goal is to study properties of this sequence.

Proposition 3.1 (Wallis 1655 [61]). *Let $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ be two sequences of real numbers defined by the recurrence relations*

$$\begin{aligned} &A_n = b_n A_{n-1} + a_n A_{n-2}, \quad n \geq 1, && B_n = b_n B_{n-1} + a_n B_{n-2}, \quad n \geq 1, \\ &A_{-1} = 1, \quad A_0 = b_0, && B_{-1} = 0, \quad B_0 = 1. \end{aligned}$$

For $n \geq 0$, we have

$$C_n = \frac{A_n}{B_n}. \tag{3.1}$$

Proof. The proof is by induction. For $n = 1$, it is clear that

$$C_1 = \frac{b_1 b_0 + a_1}{b_1} = \frac{A_1}{B_1}.$$

Suppose that $C_n = \frac{A_n}{B_n}$. Now, we will verify that $C_{n+1} = \frac{A_{n+1}}{B_{n+1}}$. Note that C_{n+1} can be obtained from C_n by substituting b_n by $b_n + \frac{a_{n+1}}{b_{n+1}}$. With this in mind, we write

$$C_{n+1} = \frac{A_n^*}{B_n^*},$$

where

$$\begin{aligned} A_n^* &= \left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) A_{n-1} + a_n A_{n-2}, \\ B_n^* &= \left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) B_{n-1} + a_n B_{n-2}. \end{aligned}$$

From the recurrent definition of A_n, it follows that

$$\begin{aligned} A_n^* &= \left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) A_{n-1} + a_n A_{n-2} \\ &= \frac{1}{b_{n+1}} \big(b_{n+1}(b_n A_{n-1} + a_n A_{n-2}) + a_{n+1} A_{n-1}\big) \\ &= \frac{1}{b_{n+1}} A_{n+1}. \end{aligned}$$

In a similar way, we find

$$B_n^* = \frac{1}{b_{n+1}} B_{n+1}.$$

Thus, we can conclude that equation (3.1) holds for every $n \geq 0$. ■

3.1 Continued fractions and orthogonal polynomials

Since $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ in Proposition 3.1 are defined by a three-term recurrence relation, we are immediately reminded of orthogonal polynomials. Let **u** be a moment functional and let $(P_n(x))_{n\geq 0}$ be its corresponding MOPS. They satisfy a three-term recurrence relation as in (2.5). In particular, if we consider the continued fraction

$$\cfrac{\mu_0}{(x-b_0) - \cfrac{a_1}{(x-b_1) - \cfrac{a_2}{(x-b_2) - \cfrac{a_3}{(x-b_3) - \ddots}}}},$$

then from Proposition 3.1, we conclude that the elements of the sequence $(C_n(x))_{n\geq 0}$ are rational functions of the form $C_n(x) = \frac{A_n(x)}{B_n(x)}$, where

$$A_n(x) = (x - b_{n-1})A_{n-1}(x) - a_{n-1}A_{n-2}(x), \quad n \geq 1,$$
$$A_{-1}(x) = 1, \quad A_0(x) = 0, \quad A_1(x) = \mu_0,$$

and

$$B_n(x) = (x - b_{n-1})B_{n-1}(x) - a_{n-1}B_{n-2}(x), \quad n \geq 1,$$
$$B_{-1}(x) = 0, \quad B_0(x) = 1.$$

Observe that $(B_n(x))_{n\geq 0}$ satisfies the same recurrence relation as $(P_n(x))_{n\geq 0}$, hence $B_n(x) = P_n(x)$. In the same way, $A_n(x) = \mu_0 P_{n-1}^{(1)}(x)$. The above, together with (2.15), implies that

$$\mathcal{S}(x) = \cfrac{\mu_0}{(x - b_0) - \cfrac{a_1}{(x - b_1) - \cfrac{a_2}{(x - b_2) - \cfrac{a_3}{(x - b_3) - \ddots}}}}$$
$$= \frac{\mu_0}{(x - b_0) - \frac{a_1}{\mu_0^{(1)}}\mathcal{S}^{(1)}(x)},$$

where $\mathcal{S}^{(1)}(x)$ denotes the Stieltjes function associated with the polynomials of the first kind.

Chapter 4

Zeros of orthogonal polynomials

The zeros of orthogonal polynomials have many applications in numerical analysis, mathematical physics, and approximation theory. Later we will see two specific applications, quadrature formulas (Chapter 5) and potential theory (Chapter 10).

Let $\mathbf{u}$ be a linear functional. We say that $\mathbf{u}$ is supported on $E \subseteq \mathbb{R}$ if and only if $\langle \mathbf{u}, p(x) \rangle \neq 0$ for every polynomial $p(x)$ such that $p(x) \not\equiv 0$ on E. If $\mathbf{u}$ is a positive definite moment functional, then $\mathbf{u}$ is supported on E if and only if $\langle \mathbf{u}, p(x) \rangle > 0$ for every nonzero polynomial $p(x)$ such that $p(x) \geq 0$ on E.

Example 4.1. Let $E = \{x_1, x_2, \ldots, x_n\}$ be a finite set of real numbers with n elements, and let $\mathbf{u} = \sum_{k=1}^{n} M_k \boldsymbol{\delta}_{x_k}$ with $M_k > 0$. Then $\mathbf{u}$ is not positive definite because the polynomial

$$p(x) = \prod_{k=1}^{n} (x - x_k)^2$$

is nonnegative on E but $\langle \mathbf{u}, p(x) \rangle = 0$.

The following theorem gives us information about the zeros of the orthogonal polynomials associated with a positive definite moment functional.

Theorem 4.2 ([13]). *Let* $\mathbf{u}$ *be a positive definite functional supported on an interval* I *and let* $(P_n(x))_{n\geq 0}$ *be its corresponding MOPS. The zeros of* $P_n(x)$ *are all real, simple, and located in the interior of* I.

Proof. Since $\langle \mathbf{u}, P_n(x) \rangle = 0$, the polynomial $P_n(x)$ changes sign in I (otherwise $\mathbf{u}$ would not be supported on I). Thus, $P_n(x)$ has at least one zero of odd multiplicity in I. Let $\{x_1, x_2, \ldots, x_k\}$ be the distinct zeros of $P_n(x)$ of odd multiplicity located in the interior of I and define the polynomial

$$h(x) = (x - x_1)(x - x_2) \cdots (x - x_k) P_n(x).$$

Since $h(x)$ does not have zeros of odd multiplicity in I, the inequality $h(x) > 0$ holds for every $x \in I$. Therefore,

$$\big\langle \mathbf{u}, h(x) \big\rangle = \big\langle \mathbf{u}, (x - x_1)(x - x_2) \cdots (x - x_k) P_n(x) \big\rangle > 0,$$

but, from orthogonality, it follows that $k = n$. This means that

$$P_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n),$$

and the result follows. ∎

Notice that this result completes Corollary 2.30.

4.1 The interlacing property of zeros

In the sequel, we denote by $(x_{n,k})_{k=1}^n$ the zeros of the monic polynomial $P_n(x)$ in increasing order, that is,

$$x_{n,1} < x_{n,2} < \cdots < x_{n,n}.$$

Taking into account that the leading coefficient of $P_n(x)$ is positive, for $x > x_{n,n}$, we have sgn $P_n(x) = 1$ and, for $x < x_{n,1}$, sgn $P_n(x) = (-1)^n$, where

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

In the same way, we have that sgn $P_n'(x_{n,n}) = 1$ and sgn $P_n'(x_{n,n-1}) = -1$. In general,

$$\operatorname{sgn} P_n'(x_{n,k}) = (-1)^{n-k}. \tag{4.1}$$

The following theorem states the so-called interlacing property of zeros of orthogonal polynomials (see Figure 4.1).

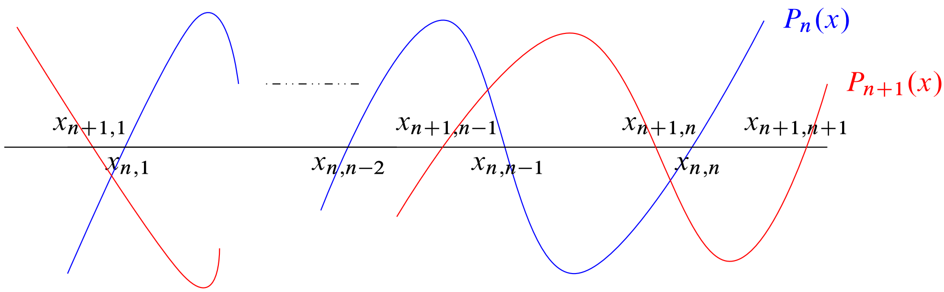

Figure 4.1. Interlacing of zeros.

Theorem 4.3 ([13]). *Let* **u** *be a positive definite functional and let* $(P_n(x))_{n\geq 0}$ *be its corresponding MOPS. The zeros of the polynomials* $P_n(x)$ *and* $P_{n+1}(x)$ *interlace, that is,*

$$x_{n+1,k} < x_{n,k} < x_{n+1,k+1}, \quad 1 \leq k \leq n.$$

Proof. From (2.11), we have

$$P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x) > 0.$$

In particular,

$$P_{n+1}'(x_{n+1,k})P_n(x_{n+1,k}) > 0, \quad P_{n+1}'(x_{n+1,k+1})P_n(x_{n+1,k+1}) > 0, \quad 1 \leq k \leq n.$$

This, together with (4.1), implies that $\operatorname{sgn} P_n(x_{n+1,k}) = -\operatorname{sgn} P_n(x_{n+1,k+1})$. It follows that $P_n(x)$ has at least one zero in each of the n intervals $I_k = [x_{n+1,k}, x_{n+1,k+1}]$, $1 \le k \le n$. But $\deg P_n(x) = n$, meaning that there is exactly one zero of $P_n(x)$ in each I_k. ■

Example 4.4. Figure 4.2 illustrates the interlacing of the zeros of four consecutive Chebyshev polynomials of the second kind.

				Zeros				
$U_1(x)$				0				
$U_2(x)$			−0.5		0.5			
$U_3(x)$		−0.7071		0		0.7071		
$U_4(x)$	−0.8090		−0.3090		0.3090			0.8090

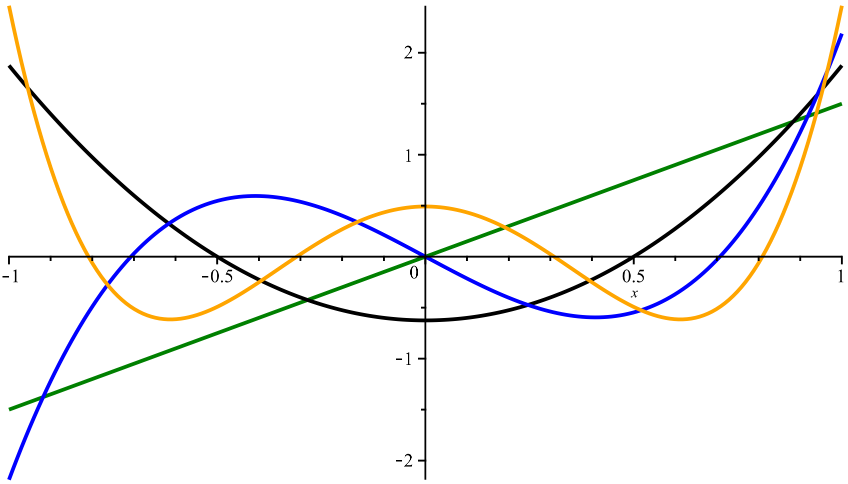

Figure 4.2. Chebyshev polynomials of the second kind.

In the previous theorem we saw that orthogonality with respect to a positive definite moment functional implies the interlacing property of zeros. In the next result, we see that the interlacing property of zeros is a fundamental property of orthogonality in the sense that if the zeros of any two polynomials of consecutive degrees interlace, then these polynomials are elements of an OPS associated with a positive definite moment functional.

Theorem 4.5 ([62]). *Let $q_n(x)$ and $q_{n-1}(x)$ be two monic polynomials with simple real zeros $(x_{n,k})_{k=1}^{n}$ and $(x_{n-1,k})_{k=1}^{n-1}$, respectively. If*

$$x_{n,1} < x_{n-1,1} < \cdots < x_{n,k} < x_{n-1,k} < x_{n,k+1} < \cdots < x_{n,n-1} < x_{n,n},$$

then $q_n(x)$ and $q_{n-1}(x)$ are orthogonal polynomials with respect to some positive definite moment functional.

Proof. Given that $q_n(x)$ and $q_{n-1}(x)$ are polynomials of degree n and $n-1$, respectively, then using the Euclidean division algorithm, there exist a polynomial $r_{n-2}(x)$ of degree at most $n-2$ and a polynomial $(x - b_{n-1})$ of degree 1 such that

$$q_n(x) = (x - b_{n-1})q_{n-1}(x) - r_{n-2}(x).$$

It is clear that $r_{n-2}(x)$ can be written as $a_{n-1}q_{n-2}(x)$, where $q_{n-2}(x)$ is a monic polynomial. Thus

$$q_n(x) = (x - b_{n-1})q_{n-1}(x) - a_{n-1}q_{n-2}(x).$$

We will prove that $q_{n-2}(x)$ is a polynomial of degree exactly $n-2$. Note that from the hypothesis

$$x_{n-1,k} < x_{n,k+1} < x_{n-1,k+1}, \quad 1 \le k \le n-2,$$

it follows that

$$\begin{aligned} q_n(x_{n-1,k}) &= -a_{n-1}q_{n-2}(x_{n-1,k}), \\ q_n(x_{n-1,k+1}) &= -a_{n-1}q_{n-2}(x_{n-1,k+1}). \end{aligned}$$

Since the zeros of $q_n(x)$ and $q_{n-1}(x)$ interlace, we know that

$$\operatorname{sgn} q_n(x_{n-1,k}) = -\operatorname{sgn} q_n(x_{n-1,k+1}),$$

and, consequently, $\operatorname{sgn} q_{n-2}(x_{n-1,k}) = -\operatorname{sgn} q_{n-2}(x_{n-1,k+1})$. Then, $q_{n-2}(x)$ has at least one zero in each of the $n-2$ intervals $I_k = [x_{n-1,k}, x_{n-1,k+1}]$, $1 \le k \le n-1$. But $\deg q_{n-2}(x) \le n-2$. Hence, there is exactly one zero in each I_k and thus $\deg(q_{n-2}(x)) = n-2$. Moreover, its zeros are interlaced with the zeros of $q_{n-1}(x)$.

Note that

$$a_{n-1} = -\frac{q_n(x_{n-1,n-1})}{q_{n-2}(x_{n-1,n-1})},$$

and, since $\operatorname{sgn} q_n(x_{n-1,n-1}) = -1$ and $\operatorname{sgn} q_{n-2}(x_{n-1,n-1}) = 1$, a_{n-1} is a positive real number. If we iterate the above process, we can find a set of polynomials $(q_k(x))_{k=0}^{n}$ and two sequences of numbers $(a_k)_{k=1}^{n-1}$, $a_k > 0$, and $(b_n)_{n=0}^{n-1}$ such that, for $0 \le k \le n$, $q_k(x)$ satisfies the three-term recurrence relation

$$\begin{aligned} &q_{k+1}(x) = (x - b_k)q_k(x) - a_k q_{k-1}(x), \quad 0 \le k \le n-1, \\ &q_0(x) = 1, \quad q_{-1}(x) = 0. \end{aligned}$$

Let $(a_k)_{k=1}^{n-1} \cup (a_k)_{k\ge n}$ and $(b_k)_{k=0}^{n-1} \cup (b_k)_{k\ge n}$, where, for $k \ge n$, the numbers a_k and b_k are arbitrary and $a_k > 0$. By Favard's theorem, there exists a positive definite moment functional $\mathbf{u}$ such that the polynomials $(q_k(x))_{k=0}^{n}$ are the first $n+1$ orthogonal polynomials with respect to $\mathbf{u}$. ■

The above result can also be deduced using the following theorem about interlacing eigenvalues of Hermitian matrices found in [29, p. 185].

Theorem 4.6. *Let n be a given positive integer, and let $(x_{n,k})_{k=1}^{n}$ and $(x_{n-1,k})_{k=1}^{n-1}$ be two given sequences of numbers such that*

$$x_{n,1} < x_{n-1,1} < \cdots < x_{n,k} < x_{n-1,k} < x_{n,k+1} < \cdots < x_{n,n-1} < x_{n,n}.$$

Let $\Lambda = \operatorname{diag}(x_{n-1,1}, x_{n-1,2}, \ldots, x_{n-1,n-1})$. There exist a real number b and a real vector $y = (y_1, \ldots, y_{n-1})^t \in \mathbb{R}^{n-1}$ such that $(x_{n,k})_{k=1}^{n}$ is the set of eigenvalues of the real symmetric matrix

$$B = \left(\begin{array}{c|c} \Lambda & y \\ \hline y^t & b \end{array}\right).$$

Indeed, for Λ and B as in Theorem 4.6, let $q_{n-1}(x) = \det(xI - \Lambda)$ and $q_n(x) = \det(xI - B)$. These are two monic polynomials with simple real zeros $(x_{n-1,k})_{k=1}^{n-1}$ and $(x_{n,k})_{k=1}^{n}$, respectively. Using Cauchy's expansion for determinants, we get the following relation:

$$\begin{aligned} q_n(x) &= (x-b)\det(xI - \Lambda) - y^t \operatorname{adj}(xI - \Lambda)y \\ &= (x-b)q_{n-1}(x) - r_{n-2}(x), \end{aligned}$$

where

$$r_{n-2}(x) = \sum_{i=1}^{n-1} y_i^2 \frac{q_{n-1}(x)}{x - x_{n-1,i}}.$$

Clearly, $r_{n-2}(x)$ can be written as $a_{n-1}q_{n-2}(x)$, where $a_{n-1} = \sum_{k=1}^{n-1} y_k^2$ and $q_{n-2}(x)$ is a monic polynomial of degree $n-2$. As in the proof of Theorem 4.5, it can be shown that the interlacing of $(x_{n-1,k})_{k=1}^{n-1}$ and $(x_{n,k})_{k=1}^{n}$ implies and that the zeros of $q_{n-2}(x)$ interlace with the zeros of $q_{n-1}(x)$ and that $a_{n-1} > 0$. Iterating the above argument and following the proof of Theorem 4.5, we can find polynomials $(q_k(x))_{k=0}^{n}$ and a positive definite moment functional $\mathbf{u}$ such that $(q_k(x))_{k=0}^{n}$ constitutes the first $n+1$ elements of an OPS with respect to $\mathbf{u}$.

Theorem 4.7 ([50]). *Consider two sequences of real numbers $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ such that $(x_n)_{n\geq 1}$ is a decreasing sequence, $(y_n)_{n\geq 1}$ is a increasing sequence, and $x_1 = y_1$ (see Figure 4.3). There exists a unique sequence of monic orthogonal polynomials $(P_n(x))_{n\geq 0}$ such that*

$$P_n(x_n) = P_n(y_n) = 0, \quad n \geq 1.$$

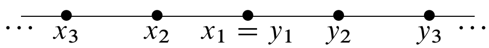

Figure 4.3. Sequences of numbers $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$.

Proof. The idea of the proof is to construct the sequence of polynomials $(P_n(x))_{n\geq 0}$ in a recursive way.

It is clear that $P_0(x) = 1$. Let

$$P_1(x) = (x - x_1), \qquad P_2(x) = (x - x_2)(x - y_2).$$

Since we want the polynomials to be orthogonal with respect to some moment functional, we must have

$$P_2(x) = (x - b_1)P_1(x) - a_1 P_0(x).$$

Comparing each coefficient, we find that

$$b_1 = x_2 + y_2 - x_1 \quad \text{and} \quad a_1 = (x_1 - y_2)(x_2 - x_1) > 0.$$

We can construct $P_3(x)$ as

$$P_3(x) = (x - b_2)P_2(x) - a_2 P_1(x)$$

so that

$$\begin{aligned} 0 &= P_3(x_3) = (x_3 - b_2)P_2(x_3) - a_2 P_1(x_3), \\ 0 &= P_3(y_3) = (y_3 - b_2)P_2(y_3) - a_2 P_1(y_3). \end{aligned}$$

The above equations are equivalent to the system

$$\begin{pmatrix} P_2(x_3) & P_1(x_3) \\ P_2(y_3) & P_1(y_3) \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_3 P_2(x_3) \\ y_3 P_2(y_3) \end{pmatrix},$$

and it has a unique solution if and only if

$$P_2(x_3)P_1(y_3) - P_2(y_3)P_1(x_3) \neq 0.$$

But, from the hypothesis we have $P_2(x_3) > 0$, $P_2(y_3) > 0$, $P_1(x_3) < 0$, and $P_1(y_3) > 0$. Thus,

$$P_2(x_3)P_1(y_3) - P_2(y_3)P_1(x_3) > 0.$$

Moreover, using Cramer's rule, we obtain

$$a_2 = \frac{(y_3 - x_3)P_2(x_3)P_2(y_3)}{P_2(x_3)P_1(y_3) - P_2(y_3)P_1(x_3)} > 0.$$

Iterating this procedure, we can construct a sequence of monic polynomials $(P_n(x))_{n\geq 0}$ satisfying a three-term recurrence relation as in (2.5) and such that $a_n > 0$ for all $n \geq 0$. But this implies that there exists a positive definite moment functional such that $(P_n(x))_{n\geq 0}$ is its associated MOPS. ■

Chapter 5

Gauss quadrature rules

Given a continuous function $f(x)$, a problem of great interest is to approximate the integral

$$\int f(x)\,d\mu(x)$$

by means of numerical integration (quadrature rule)

$$\int f(x)\,d\mu(x) \approx f(x_1)w_1 + \cdots + f(x_n)w_n.$$

Here, the integral is approximated by a sum of n terms that involves the weights w_k and the n nodes x_k that must be properly chosen.

Definition 5.1. We say that a quadrature rule has accuracy of order k if for every polynomial $q(x) \in \mathbb{P}_k$,

$$\int q(x)\,d\mu(x) = q(x_1)w_1 + \cdots + q(x_n)w_n, \tag{5.1}$$

and there exists a polynomial of degree $k+1$ such that (5.1) does not hold.

A simple way to construct quadrature rules consists in using interpolation formulas. Recall that given a set of n points $\mathcal{N} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, where $x_i \neq x_j$, if $i \neq j$, the *Lagrange interpolating polynomial* of degree at most $n-1$ corresponding to $\mathcal{N}$ is defined as

$$L_n(x) := \sum_{k=1}^{n} y_k \ell_k(x), \quad \text{where } \ell_k(x) := \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{(x-x_j)}{(x_k-x_j)}.$$

Hence, when $n=1$, we take $L_1(x) \equiv y_1$. The numbers x_j, $1 \le j \le n$, are called *nodes* and the polynomials $(\ell_k(x))_{k=1}^n$ are known in the literature as the *Lagrange polynomial basis*. The polynomial $L_n(x)$ has the property $L_n(x_j) = y_j$, $1 \le j \le n$. Indeed, since $\ell_k(x_j) = \delta_{k,j}$, we have

$$L_n(x_j) = \sum_{k=1}^{n} y_k \delta_{k,j} = y_j, \quad 1 \le j \le n.$$

Observe that if $F(x) = \prod_{j=1}^{n}(x - x_j)$, then

$$\lim_{x\to x_j} \frac{F(x)}{x - x_j} = F'(x_j) \neq 0.$$

It follows that

$$\ell_k(x) = \frac{F(x)}{(x - x_k)F'(x_k)}.$$

The Lagrange polynomial $L_n(x)$ provides a unique solution to the problem of finding a polynomial of degree at most $n-1$ whose graph contains the points (x_j, y_j), $1 \leq j \leq n$.

Given a continuous function $f(x)$ and n nodes $(x_j, f(x_j))$, $1 \leq j \leq n$, we can write

$$f(x) = L_n(x) + R_n(x),$$

where $R_n(x)$ is the interpolation error. Observe that if $f(x)$ is a polynomial of degree at most $n-1$, then by the uniqueness of Lagrange polynomial, $R_n(x) = 0$.

Let us now return to our problem concerning integration. Notice that for the function $f(x)$,

$$\begin{aligned}\int f(x)\, d\mu(x) &= \int L_n(x)\, d\mu(x) + \int R_n(x)\, d\mu(x) \\ &= \sum_{k=1}^{n} f(x_k) \int \ell_k(x)\, d\mu(x) + \int R_n(x)\, d\mu(x).\end{aligned}$$

With this in mind, we have the following quadrature rule:

$$\int f(x)\, d\mu(x) \approx \sum_{k=1}^{n} f(x_k) \int \ell_k(x)\, d\mu(x).$$

A question that arises naturally is the following. How should we choose the nodes x_j so that this quadrature formula has the maximum accuracy possible? The following theorem gives us the answer.

Theorem 5.2 (Gauss quadrature rule). *Let* $\mathbf{u}$ *be a positive definite functional and let* $(P_n(x))_{n\geq 0}$ *be its corresponding MOPS. There exist positive real numbers* $A_{n,1}, \ldots,$ $A_{n,n}$ *such that for every polynomial* $q(x)$ *of degree at most* $2n-1$*, we have*

$$\langle \mathbf{u}, q(x) \rangle = \sum_{k=1}^{n} A_{n,k} q(x_{n,k}), \tag{5.2}$$

where $(x_{n,k})_{k=1}^{n}$ *are the zeros of de* $P_n(x)$.

Proof. Let $q(x)$ be a polynomial of degree at most $2n-1$. From the Euclidean division algorithm we know that there exist polynomials $C(x)$ and $R(x)$, both of degree at most $n-1$, such that

$$q(x) = P_n(x)C(x) + R(x).$$

Note that $q(x_{n,k}) = R(x_{n,k})$ for $1 \leq k \leq n$.

Since $R(x)$ has degree less than n, it can be written uniquely as

$$R(x) = \sum_{k=1}^{n} R(x_{n,k})\ell_k(x),$$

where

$$\ell_k(x) = \frac{P_n(x)}{(x - x_{n,k})P'(x_{n,k})}.$$

This, together with orthogonality, gives us

$$\begin{aligned}\langle \mathbf{u}, q(x)\rangle &= \langle \mathbf{u}, P_n(x)C(x) + R(x)\rangle = \langle \mathbf{u}, R(x)\rangle \\ &= \sum_{k=1}^{n} R(x_{n,k})\langle \mathbf{u}, \ell_k(x)\rangle = \sum_{k=1}^{n} q(x_{n,k})\langle \mathbf{u}, \ell_k(x)\rangle.\end{aligned}$$

Thus, taking $A_{n,k} = \langle \mathbf{u}, \ell_k(x)\rangle$, we obtain (5.2).

Notice that $\ell_k^2(x)$ is a polynomial of degree at most $2n-2$. From the quadrature rule, we have

$$0 < \langle \mathbf{u}, \ell_k^2(x)\rangle = \sum_{j=1}^{n} \ell_k(x_{n,j})\langle \mathbf{u}, \ell_j(x)\rangle = \langle \mathbf{u}, \ell_k(x)\rangle = A_{n,k}.$$

Hence, the $A_{n,k}$ are positive real numbers. ∎

The constants $A_{n,k}$ involved in the Gauss quadrature rule are known in the literature as *Christoffel numbers*.

Corollary 5.3. *The Christoffel numbers $A_{n,k}$, $1 \le k \le n$, have the following representation:*

$$A_{n,k} = \frac{1}{K_{n-1}(x_{n,k}, x_{n,k})},$$

where $K_n(x, y)$ is the nth C–D kernel polynomial as in Definition 2.26.

Proof. From the quadrature rule, it is easy to see that

$$\langle \mathbf{u}, \ell_k(x)\ell_m(x)\rangle = A_{n,k}\delta_{k,m}. \tag{5.3}$$

On the other hand, since $\frac{K_{n-1}(x,x_{n,k})}{K_{n-1}(x_{n,k},x_{n,k})}$ is a polynomial of degree $n-1$, it can be written in a unique way as

$$\frac{K_{n-1}(x, x_{n,k})}{K_{n-1}(x_{n,k}, x_{n,k})} = \ell_k(x) + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{K_{n-1}(x_{n,j}, x_{n,k})}{K_{n-1}(x_{n,k}, x_{n,k})}\ell_j(x).$$

Thus, using Corollary 2.27 and (5.3), we obtain

$$\frac{1}{K_{n-1}(x_{n,k}, x_{n,k})} = \left\langle \mathbf{u}, \ell_k(x)\frac{K_{n-1}(x, x_{n,k})}{K_{n-1}(x_{n,k}, x_{n,k})}\right\rangle = \langle \mathbf{u}, \ell_k^2(x)\rangle = A_{n,k}.$$ ∎

Observation 5.4. Notice that the functional $\mathbf{u}$ restricted to $\mathbb{P}_{2n-1}$ has the following representation:

$$\mathbf{u}|_{\mathbb{P}_{2n-1}} = \sum_{k=1}^{n} \frac{1}{K_{n-1}(x_{n,k}, x_{n,k})} \delta_{x_{n,k}}.$$

Sometimes we may be interested in the Gauss quadrature rules with prefixed nodes outside of the support of the moment functional. Although the following theorem states the result in the case of one node outside of the support, it is not difficult to extend it to more nodes (see [21]).

Theorem 5.5 (Gauss–Radau quadrature rule). *Let* $\mathbf{u}$ *be a positive definite moment functional supported on an interval* $E \subseteq \mathbb{R}$ *and let* $a \in \mathbb{R}$ *be such that for every* $x \in E$, $a < x$. *Let* $(\tilde{P}_n(x))_{n\geq 0}$ *be the MOPS associated with the positive definite moment functional* $\tilde{\mathbf{u}} := (x-a)\mathbf{u}$.

There exist positive constants $\alpha_1, A_{n,1}, \ldots, A_{n,n}$, *such that, for every polynomial* $q(x) \in \mathbb{P}_{2n}$,

$$\langle \mathbf{u}, q(x) \rangle = \alpha_1 q(a) + \sum_{k=1}^{n} A_{n,k} q(\tilde{x}_{n,k}), \tag{5.4}$$

where $(\tilde{x}_{n,k})_{k=1}^{n}$ *are the zeros of the orthogonal polynomial* $\tilde{P}_n(x)$.

Proof. Note first that since $x > a$ for all $x \in E$, $\tilde{\mathbf{u}}$ is positive definite. If $q(x)$ is a polynomial of degree at most $2n$, then from the Euclidean division algorithm we know that there exist polynomials $C(x)$ and $R(x)$ of degree at most $n-1$ and n, respectively, such that

$$q(x) = \tilde{P}_n(x)(x-a)C(x) + R(x).$$

If we define the polynomials $F(x) = (x-a)\tilde{P}_n(x)$ and

$$\ell_k(x) = \frac{F(x)}{(x-\tilde{x}_{n,k})F'(\tilde{x}_{n,k})}, \qquad \ell_a(x) = \frac{F(x)}{(x-a)F'(a)},$$

then the set $(\ell_k(x))_{k=1}^{n} \cup \{\ell_a(x)\}$ is a Lagrange basis for $\mathbb{P}_n$. Therefore,

$$R(x) = R(a)\ell_a(x) + \sum_{k=1}^{n} R(\tilde{x}_{n,k})\ell_k(x).$$

This, together with orthogonality, gives us

$$\begin{aligned}
\langle \mathbf{u}, q(x) \rangle &= \langle \mathbf{u}, \tilde{P}_n(x)(x-a)C(x) + R(x) \rangle = \langle \tilde{\mathbf{u}}, \tilde{P}_n(x)C(x) \rangle + \langle \mathbf{u}, R(x) \rangle \\
&= R(a)\langle \mathbf{u}, \ell_a(x) \rangle + \sum_{k=1}^{n} R(\tilde{x}_{n,k})\langle \mathbf{u}, \ell_k(x) \rangle \\
&= q(a)\langle \mathbf{u}, \ell_a(x) \rangle + \sum_{k=1}^{n} q(\tilde{x}_{n,k})\langle \mathbf{u}, \ell_k(x) \rangle,
\end{aligned}$$

and (5.4) follows. In a similar way as in Theorem 5.2, the constants $\alpha_1, A_{n,1}, \dots, A_{n,n}$ are positive. ■

Remark 5.6. As an example of an extension of Theorem 5.5 to more nodes, we mention the well-known Gauss–Lobbato quadrature rule. In this case, two nodes are fixed outside of the support $E = (a, b)$ of the positive definite moment functional $\mathbf{u}$. Generally, these two nodes are fixed at the endpoints of the interval E. In this way, the extended result of Theorem 5.5 states that there exist positive constants $\alpha_1, \alpha_2, A_{n,1}, \dots, A_{n,n}$ and nodes $\tilde{x}_{n,1}, \dots, \tilde{x}_{n,n}$ such that, for every polynomial $q(x) \in \mathbb{P}_{2n+1}$,

$$\langle \mathbf{u}, q(x) \rangle = \alpha_1 q(a) + \alpha_2 q(b) + \sum_{k=1}^{n} A_{n,k} q(\tilde{x}_{n,k}).$$

As in the case of the Gauss–Radau quadrature rule, it is not difficult to check that the nodes $\tilde{x}_{n,1}, \dots, \tilde{x}_{n,n}$ turn out to be the zeros of the orthogonal polynomial of degree n associated with the modified positive definite moment functional $(x-a)(b-x)\mathbf{u}$. A special case of study is when $\mathbf{u}$ is associated with the Jacobi weight $(1-x)^{\alpha}(x+1)^{\beta}$ supported on $[-1, 1]$. In this case, the Gauss–Radau quadrature formula uses the nodes $a = -1$, $b = 1$, and the zeros of the orthogonal polynomials associated with $\tilde{\mathbf{u}} = (1-x)(x+1)\mathbf{u}$, which is again a Jacobi weight but now with parameters $\alpha + 1$ and $\beta + 1$ (see [22]).

Chapter 6

Symmetric functionals

A quasidefinite moment functional $\mathbf{u}$ is said to be *symmetric* if, for all $k \geq 0$,

$$\langle \mathbf{u}, x^{2k+1} \rangle = 0,$$

that is, all its odd moments are equal to zero. A simple example of a symmetric moment functional is

$$\langle \mathbf{u}, p(x) \rangle = \int_{-\infty}^{\infty} p(x) e^{-x^2} dx, \quad p(x) \in \mathbb{P}.$$

The following result characterizes symmetric quasidefinite moment functionals in terms of the recurrence relation satisfied by its MOPS.

Theorem 6.1 ([13]). *Let $(T_n(x))_{n \geq 0}$ be a MOPS associated with a quasidefinite moment functional* $\mathbf{u}$. *Then the following statements are equivalent:*

(i) $\mathbf{u}$ *is symmetric.*

(ii) *$T_n(x)$ has the same parity as n, that is, $T_n(x)$ is an even (resp. odd) function when n is even (resp. odd).*

(iii) *$(T_n(x))_{n \geq 0}$ satisfies* (2.5) *with $b_n = 0$ for $n \geq 0$.*

Proof. First we show (i) $\Leftrightarrow$ (ii).

If we have a symmetric moment functional $\mathbf{u}$ and two polynomials $\pi(x) = \sum_{i=0}^{m} a_i x^i$ and $\varpi(x) = \sum_{j=0}^{n} b_j x^j$, then

$$\langle \mathbf{u}, \pi(-x)\varpi(-x) \rangle = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} a_i b_j \langle \mathbf{u}, x^{i+j} \rangle = \langle \mathbf{u}, \pi(x)\varpi(x) \rangle.$$

In particular,

$$\langle \mathbf{u}, T_n(-x) T_m(-x) \rangle = \langle \mathbf{u}, T_n(x) T_m(x) \rangle.$$

Moreover, since the sequence of orthogonal polynomials is unique up to constant multiplication, there exists a sequence of numbers $(a_n)_{n \geq 0}$ such that $T_n(x) = a_n T_n(-x)$. Thus, comparing the leading coefficients, we deduce that

$$T_n(x) = (-1)^n T_n(-x).$$

This means that $T_n(x)$ has the same parity of n. Conversely, if $T_n(x)$ has the same parity of n, then $T_n(x)$ is a linear combination of monomials with odd powers when n is odd. Then, $\mu_1 = \langle \mathbf{u}, T_1(x) \rangle = 0$, $\mu_3 = \langle \mathbf{u}, T_3(x) \rangle = 0$, and $\mu_{2k+1} = \langle \mathbf{u}, T_{2k+1}(x) \rangle = 0$, $k \geq 0$, follows by induction.

Now, we show (ii) $\Leftrightarrow$ (iii). If $T_n(x)$ has the same parity of n, then $b_n = 0$, $n \geq 1$, in (2.5) follows from (2.8). Conversely, suppose that $(T_n(x))_{n\geq 0}$ satisfies the recurrence relation

$$T_{n+1}(x) = xT_n(x) - a_n T_{n-1}(x), \quad n \geq 0,$$
$$T_{-1}(x) = 0, \quad T_0(x) = 1.$$

We show by induction that $T_n(x) = (-1)^n T_n(-x)$, $n \geq 0$. Clearly, $T_0(x) = T_0(-x)$ and $T_1(x) = -T_1(-x)$. Suppose that $T_n(x) = (-1)^n T_n(-x)$ and $T_{n-1}(x) = (-1)^{n-1} T_{n-1}(-x)$. Then,

$$\begin{aligned} T_{n+1}(x) &= (-1)^n x T_n(-x) - (-1)^{n-1} a_n T_{n-1}(-x) \\ &= (-1)^{n+1} \big[-xT_n(-x) - a_n T_{n-1}(-x) \big] \end{aligned}$$

and, thus, $T_{n+1}(x) = (-1)^{n+1} T_{n+1}(-x)$. ∎

If $(T_n(x))_{n\geq 0}$ is the MOPS associated with a symmetric moment functional, then $T_n(x)$ only contains powers of x with the same parity of n. Therefore, there are two monic polynomial sequences $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ such that

$$T_{2n}(x) = P_n(x^2) \quad \text{and} \quad T_{2n+1}(x) = xQ_n(x^2). \tag{6.1}$$

The following result states that $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ are also sequences of orthogonal polynomials.

Proposition 6.2 (Chihara [13]). *Let* $\mathbf{u}$ *be a quasidefinite symmetric moment functional. Then the sequence of polynomials* $(P_n(x))_{n\geq 0}$ *is orthogonal with respect to the functional* $\sigma\mathbf{u}$ *(see Definition 2.4) and* $(Q_n(x))_{n\geq 0}$ *is orthogonal with respect to the functional* $x\sigma\mathbf{u}$.

Proof. For $n \geq 0$, let $h_n = \langle \mathbf{u}, T_n^2(x) \rangle$. Note that for $n, m \geq 0$,

$$\begin{aligned} h_{2n}\delta_{n,m} &= \langle \mathbf{u}, T_{2n}(x)T_{2m}(x) \rangle = \langle \mathbf{u}, P_n(x^2)P_m(x^2) \rangle = \langle \sigma\mathbf{u}, P_n(x)P_m(x) \rangle, \\ h_{2n+1}\delta_{n,m} &= \langle \mathbf{u}, T_{2n+1}(x)T_{2m+1}(x) \rangle = \langle \mathbf{u}, x^2 Q_n(x^2)Q_m(x^2) \rangle \\ &= \langle x\sigma\mathbf{u}, Q_n(x)Q_m(x) \rangle, \end{aligned}$$

and the result follows. ∎

Corollary 6.3. *If a symmetric functional* $\mathbf{u}$ *is quasidefinite, then the functionals* $\sigma\mathbf{u}$ *and* $x\sigma\mathbf{u}$ *are also quasidefinite.*

Since $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ are sequences of monic orthogonal polynomials associated with the functionals $\sigma\mathbf{u}$ and $x\sigma\mathbf{u}$, respectively, they satisfy the three-term recurrence relations

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geq 0, \\ P_0(x) &= 1, \quad P_{-1}(x) = 0, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} &xQ_n(x) = Q_{n+1}(x) + d_n Q_n(x) + c_n Q_{n-1}(x), \quad n \geq 0, \\ &Q_0(x) = 1, \quad Q_{-1}(x) = 0, \end{aligned} \tag{6.3}$$

where $a_n \neq 0$, $c_n \neq 0$, b_n, and d_n are real numbers.

6.1 LU factorization

Recall that if $\mathbf{u}$ is a symmetric moment functional, then its associated MOPS satisfies the following recurrence relation:

$$xT_n(x) = T_{n+1}(x) + \gamma_n T_{n-1}(x), \quad n \geq 0.$$

Thus, taking into account (6.1), the relation

$$xT_{2n}(x) = T_{2n+1}(x) + \gamma_{2n} T_{2n-1}(x), \quad n \geq 0,$$

leads to

$$P_n(x) = Q_n(x) + \gamma_{2n} Q_{n-1}(x), \quad n \geq 0,$$

or, equivalently,

$$\mathbf{P} := \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \gamma_2 & 1 & & \\ & \gamma_4 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix} := L\mathbf{Q},$$

where the elements γ_{2k}, $k \geq 0$, are nonzero.

On the other hand, the relation

$$xT_{2n+1}(x) = T_{2n+2}(x) + \gamma_{2n+1} T_{2n}(x), \quad n \geq 0,$$

leads to

$$xQ_n(x) = P_{n+1}(x) + \gamma_{2n+1} P_n(x), \quad n \geq 0,$$

or, equivalently,

$$x\mathbf{Q} = \begin{pmatrix} \gamma_1 & 1 & & & \\ & \gamma_3 & 1 & & \\ & & \gamma_5 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \mathbf{P} := U\mathbf{P},$$

where again the elements γ_{2k+1}, $k \geq 0$, are nonzero.

The above relations allow us to state the following result.

Proposition 6.4. *Let $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ be the MOPS associated with the linear functionals $\sigma\mathbf{u}$ and $x\sigma\mathbf{u}$, respectively. Then,*

$$x\mathbf{P} = (LU)\mathbf{P},$$
$$x\mathbf{Q} = (UL)\mathbf{Q},$$

where U and L are upper and lower triangular matrices, respectively, given by

$$L = \begin{pmatrix} 1 & & & \\ \gamma_2 & 1 & & \\ & \gamma_4 & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \gamma_1 & 1 & & \\ & \gamma_3 & 1 & \\ & & \gamma_5 & 1 \\ & & & \ddots & \ddots \end{pmatrix},$$

and $(\gamma_n)_{n\geq 0}$ are the coefficients of the recurrence relation satisfied by $(T_n(x))_{n\geq 0}$.

Note that $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ satisfy (6.2) and (6.3), respectively, with

$$\begin{aligned} b_n &= (\gamma_{2n+1} + \gamma_{2n}), & n &\geq 0, & a_n &= \gamma_{2n}\gamma_{2n-1}, & n &\geq 1, \\ d_n &= (\gamma_{2n+2} + \gamma_{2n+1}), & n &\geq 0, & c_n &= \gamma_{2n+1}\gamma_{2n}, & n &\geq 1, \end{aligned} \tag{6.4}$$

where $\gamma_0 = 0$. It is easy to prove by induction that

$$\gamma_{2n+1} = -\frac{P_{n+1}(0)}{P_n(0)}.$$

Indeed, taking $n = 0$ in (6.2), we get $\gamma_1 = b_0 = -\frac{P_1(0)}{P_0(0)}$. Next, suppose that $\gamma_{2k-1} = -\frac{P_k(0)}{P_{k-1}(0)}$ for $k \leq n$. Since

$$-P_{n+1}(0) = b_n P_n(0) + a_n P_{n-1}(0),$$

we get

$$\begin{aligned} -\frac{P_{n+1}(0)}{P_n(0)} &= b_n + a_n \frac{P_{n-1}(0)}{P_n(0)} \\ &= \gamma_{2n+1} + \gamma_{2n} - \gamma_{2n}\gamma_{2n-1}(\gamma_{2n-1})^{-1} \\ &= \gamma_{2n+1} + \gamma_{2n} - \gamma_{2n} \\ &= \gamma_{2n+1}, \end{aligned}$$

and the result follows.

Example 6.5. Recall the Chebyshev polynomials of the second kind introduced in Example 2.24. Taking into account (2.10), from Theorem 6.1 we know that the quasidefinite moment functional $\mathbf{u}$ associated with these polynomials is symmetric.

Then, from Proposition 6.2, we have that the two families of polynomials $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ such that

$$U_{2n}(x) = P_n(x^2) \quad \text{and} \quad U_{2n+1}(x) = xQ_n(x^2)$$

are orthogonal with respect to $\mathbf{v} = \sigma\mathbf{u}$ and $\mathbf{w} = x\sigma\mathbf{u}$, respectively. Moreover, from the integral representation of $\mathbf{u}$, we deduce that

$$\langle \mathbf{v}, p(x) \rangle = \int_0^1 p(x) x^{-1/2} \sqrt{1-x}\, dx$$

and

$$\langle \mathbf{w}, p(x) \rangle = \int_0^1 p(x) x^{1/2} \sqrt{1-x}\, dx.$$

Using (6.4), we obtain the three-term recurrence relations for $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$. From (2.10), we see that $\gamma_n = 1/4$, $n \geq 1$ ($\gamma_0 = 0$). Therefore, $b_0 = 1/4$, $b_n = 1/2$, $a_n = 1/16$, $n \geq 1$, in (6.2):

$$\begin{aligned} &xP_n(x) = P_{n+1}(x) + \frac{1}{2}P_n(x) + \frac{1}{16}P_{n-1}(x), \quad n \geq 1, \\ &P_0(x) = 1, \quad P_1(x) = x - \frac{1}{4}, \end{aligned}$$

and $d_n = 1/2$, $c_n = 1/16$, $n \geq 0$, in (6.3):

$$\begin{aligned} &xQ_n(x) = Q_{n+1}(x) + \frac{1}{2}Q_n(x) + \frac{1}{16}Q_{n-1}(x), \quad n \geq 0, \\ &Q_0(x) = 1, \quad Q_{-1}(x) = 0. \end{aligned}$$

Chapter 7

Transformations of moment functionals

The analysis of transformations (or perturbations) of moment functionals is an interesting topic in the theory of orthogonal polynomials on the real line (see [11, 12, 18, 19, 64] and the references therein). In particular, when dealing with a positive definite case, i.e., the moment functional has an integral representation in terms of a probability measure supported in an infinite subset of the real line; such perturbations provide useful information in the study of Gauss quadrature rules for the transformed linear functional, taking into account that the perturbation yields to new nodes and Christoffel numbers (see Remark 5.6) [13, 21]. An example of their application is the Gauss–Radau quadrature rules (Chapter 5). Recall that in Theorem 5.5, the polynomials $(\tilde{P}_n(x))_{n\geq 0}$ were orthogonal to the positive definite functional $(x-a)\mathbf{u}$ that is just a transformation of the moment functional $\mathbf{u}$. In this chapter, we deal with the following so-called *linear spectral transformations* of $\mathbf{u}$:

(1) canonical Christoffel transformation $\tilde{\mathbf{u}} = (x-a)\mathbf{u}$,

(2) canonical Geronimus transformation $\hat{\mathbf{u}} = (x-a)^{-1}\mathbf{u} + M\boldsymbol{\delta}_a$,

(3) Uvarov transformation $\check{\mathbf{u}} = \mathbf{u} + M\boldsymbol{\delta}_a$,

and study the relation between their corresponding MOPS and the MOPS associated with the original moment functional $\mathbf{u}$. We will also show that any linear spectral transformation of $\mathbf{u}$ can be considered as a composition of canonical Christoffel and Geronimus transformations.

7.1 Canonical Christoffel transformation

Let $\mathbf{u}$ be a quasidefinite moment functional and let $(P_n(x))_{n\geq 0}$ be its corresponding MOPS. If a is a real number, then we define its *canonical Christoffel transformation* (see [14]) as the functional

$$\tilde{\mathbf{u}} = (x-a)\mathbf{u}. \tag{7.1}$$

Suppose that $\tilde{\mathbf{u}}$ is quasidefinite and let $(\tilde{P}_n(x))_{n\geq 0}$ be its associated MOPS. Since $(P_k(x))_{k=0}^{n+1}$ is a basis of the vector subspace $\mathbb{P}_{n+1}$, there exist real numbers $(\lambda_{n,k})_{k=0}^{n}$ such that

$$(x-a)\tilde{P}_n(x) = P_{n+1}(x) + \sum_{k=0}^{n} \lambda_{n,k} P_k(x).$$

Taking into account that $\langle \mathbf{u}, (x-a)\tilde{P}_n(x)P_k(x)\rangle = \langle \tilde{\mathbf{u}}, \tilde{P}_n(x)P_k(x)\rangle$, we obtain

$$\lambda_{n,k} = \begin{cases} 0, & \text{if } k \leq n-1, \\ \dfrac{\langle \tilde{\mathbf{u}}, \tilde{P}_n^2(x)\rangle}{\langle \mathbf{u}, P_n^2(x)\rangle}, & \text{if } k = n. \end{cases}$$

Therefore,

$$(x-a)\tilde{P}_n(x) = P_{n+1}(x) + \lambda_n P_n(x). \tag{7.2}$$

If we evaluate the above expression at $x = a$, we deduce that $\lambda_n = -\frac{P_{n+1}(a)}{P_n(a)}$.

On the other hand, following a similar reasoning, we get

$$P_n(x) = \tilde{P}_n(x) + \nu_n \tilde{P}_{n-1}(x), \quad \nu_n = -\frac{P_{n-1}(a)\langle \mathbf{u}, P_n^2(x)\rangle}{P_n(a)\langle \mathbf{u}, P_{n-1}^2(x)\rangle}. \tag{7.3}$$

Proposition 7.1. *The moment functional* $\tilde{\mathbf{u}} = (x-a)\mathbf{u}$ *is quasidefinite if and only if* $\mu_1 - a\mu_0 \neq 0$ *and, for* $n \geq 1$, $P_n(a) \neq 0$.

Proof. Suppose that $\tilde{\mathbf{u}}$ is quasidefinite. From (7.3), the polynomials $(\tilde{P}_n(x))_{n\geq 0}$ are given recursively by $\tilde{P}_0(x) = 1$ and

$$\tilde{P}_n(x) = P_n(x) - \nu_n \tilde{P}_{n-1}(x), \quad n \geq 1.$$

Clearly, if $P_k(a) = 0$ for some $k \geq 1$, then $\tilde{P}_k(x)$ is not well-defined, which contradicts the quasidefinite character of $\tilde{\mathbf{u}}$. Thus, $P_n(a) \neq 0$ for $n \geq 1$. Moreover,

$$0 \neq \tilde{\mu}_0 = \langle \tilde{\mathbf{u}}, 1\rangle = \langle \mathbf{u}, x-a\rangle = \mu_1 - a\mu_0.$$

Conversely, suppose that $\mu_1 - a\mu_0 \neq 0$ and, for $n \geq 1$, $P_n(a) \neq 0$. Define the sequence of monic polynomials $(\tilde{P}_n(x))_{n\geq 0}$, where $\tilde{P}_0(x) = 1$ and, for $n \geq 1$, $\tilde{P}_n(x)$ is given by (7.2). Then

$$\left\langle \tilde{\mathbf{u}}, \tilde{P}_0^2(x)\right\rangle = \langle \mathbf{u}, x-a\rangle = \mu_1 - a\mu_0 \neq 0.$$

For $n \geq 1$, if $0 \leq k \leq n-1$, then

$$\begin{aligned}\left\langle \tilde{\mathbf{u}}, \tilde{P}_n(x)P_k(x)\right\rangle &= \left\langle \mathbf{u}, (x-a)\tilde{P}_n(x)P_k(x)\right\rangle \\ &= \left\langle \mathbf{u}, \big(P_{n+1}(x) + \lambda_n P_n(x)\big)P_k(x)\right\rangle = 0\end{aligned}$$

and

$$\left\langle \tilde{\mathbf{u}}, \tilde{P}_n(x)P_n(x)\right\rangle = \left\langle \mathbf{u}, \big(P_{n+1}(x) + \lambda_n P_n(x)\big)P_n(x)\right\rangle = \lambda_n\left\langle \mathbf{u}, P_n^2(x)\right\rangle \neq 0.$$

Therefore, $(\tilde{P}_n(x))_{n\geq 0}$ is a MOPS associated with $\tilde{\mathbf{u}}$, hence, $\tilde{\mathbf{u}}$ is quasidefinite. ∎

Exercise 7.1. Let $\mathbf{u}$ be a quasidefinite symmetric moment functional. Is $x\mathbf{u}$ a quasidefinite moment functional?

Observation 7.2. Note that equations (7.2) and (7.3) can be written in matrix form as follows:

$$(x-a)\begin{pmatrix}\tilde{P}_0(x)\\ \tilde{P}_1(x)\\ \tilde{P}_2(x)\\ \vdots\end{pmatrix} = \begin{pmatrix}\lambda_0 & 1 & & & \\ & \lambda_1 & 1 & & \\ & & \lambda_2 & \ddots & \\ & & & \ddots & \ddots\end{pmatrix}\begin{pmatrix}P_0(x)\\ P_1(x)\\ P_2(x)\\ \vdots\end{pmatrix}$$

and

$$\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \nu_1 & 1 & & \\ & \nu_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{P}_0(x) \\ \tilde{P}_1(x) \\ \tilde{P}_2(x) \\ \vdots \end{pmatrix}.$$

Returning to the functional $\tilde{\mathbf{u}} = (x-a)\mathbf{u}$, the connection formulas between the polynomials $P_n(x)$ and $\tilde{P}_n(x)$ given in Observation 7.2 allow us to find relations between the Jacobi matrices associated with the functionals $\mathbf{u}$ and $\tilde{\mathbf{u}}$.

Theorem 7.3 ([11, 63]). *Let J_{mon} and $\tilde{J}_{\text{mon}}$ be the Jacobi matrices associated with $\mathbf{u}$ and $\tilde{\mathbf{u}} = (x-a)\mathbf{u}$, respectively. There exist a lower triangular matrix L with 1's in its main diagonal and an upper triangular matrix U*

$$L = \begin{pmatrix} 1 & & & \\ \nu_1 & 1 & & \\ & \nu_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \lambda_0 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

such that

$$J_{\text{mon}} - aI = LU \quad \textit{and} \quad \tilde{J}_{\text{mon}} - aI = UL. \tag{7.4}$$

The above relation is known in the literature as Darboux transformation *without* parameter.

Proof. From Observation 7.2, we deduce that

$$(x-a)\tilde{\mathbf{P}} = U\mathbf{P} \quad \text{and} \quad \mathbf{P} = L\tilde{\mathbf{P}},$$

where $\mathbf{P} = (P_0(x), P_1(x), \dots)^t$, $\tilde{\mathbf{P}} = (\tilde{P}_0(x), \tilde{P}_1(x), \dots)^t$, L is a lower triangular matrix with 1's in its main diagonal, and U is an upper triangular matrix.

Notice that $(x-a)\mathbf{P} = (J_{\text{mon}} - aI)\mathbf{P}$. Therefore,

$$(J_{\text{mon}} - aI)\mathbf{P} = (x-a)(L\tilde{\mathbf{P}}) = L\big((x-a)\tilde{\mathbf{P}}\big) = LU\mathbf{P},$$

and, taking into account that $(P_n(x))_{n\geq 0}$ is a basis of the space of polynomials, we deduce

$$J_{\text{mon}} - aI = LU.$$

Similarly, $(x-a)\tilde{\mathbf{P}} = (\tilde{J}_{\text{mon}} - aI)\tilde{\mathbf{P}}$ and, thus,

$$(\tilde{J}_{\text{mon}} - aI)\tilde{\mathbf{P}} = U\mathbf{P} = UL\tilde{\mathbf{P}}.$$

Since $(\tilde{P}_n(x))_{n\geq 0}$ is a basis of polynomials, we obtain

$$(\tilde{J}_{\text{mon}} - aI) = UL. \qquad \blacksquare$$

Example 7.4. The classical Jacobi polynomials are well-known families of polynomials depending on two parameters. Let $(P_n^{(\alpha,\beta)}(x))_{n\geq 0}$ be the monic Jacobi polynomials of parameters (α,β). For $\alpha,\beta > -1$, these polynomials are orthogonal with respect to the positive definite moment functional $\mathbf{u}_{\alpha,\beta}$ defined by

$$\langle \mathbf{u}_{\alpha,\beta}, p(x)\rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx, \quad p(x)\in\mathbb{P}.$$

The explicit expression for these polynomials is

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{S_n(\alpha,\beta)}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}, \quad n\geq 0,$$

where

$$S_n(\alpha,\beta) = \binom{2n+\alpha+\beta}{n}.$$

Here $\binom{r}{k} = \frac{\Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k)}$ and $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function. The explicit orthogonality relation reads

$$\begin{aligned}&\langle \mathbf{u}_{\alpha,\beta}, P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\rangle\\ &\quad = 2^{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)n!}{(2n+\alpha+\beta+1)(\Gamma(2n+\alpha+\beta+1))^2}\delta_{n,m}.\end{aligned}$$

The Jacobi polynomials satisfy the three-term recurrence relation

$$\begin{aligned}&xP_n^{(\alpha,\beta)}(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_nP_n^{(\alpha,\beta)}(x) + a_nP_{n-1}^{(\alpha,\beta)}(x), \quad n\geq 0,\\ &P_0^{(\alpha,\beta)}(x) = 1, \quad P_{-1}^{(\alpha,\beta)}(x) = 0,\end{aligned}$$

where

$$\begin{aligned}b_n &= \frac{\beta^2-\alpha^2}{(2n+2+\alpha+\beta)(2n+\alpha+\beta)},\\ a_n &= \frac{4(n+\beta)(n+\alpha+\beta)(n+\alpha)n}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\end{aligned}$$

except that when $\alpha=-\beta$, $b_0=\beta$ and $b_n=0$, $n\geq 1$.

Jacobi polynomials with parameters $(\alpha+1,\beta)$ and $(\alpha,\beta+1)$ are associated with the moment functionals $\mathbf{u}_{\alpha+1,\beta} = (1-x)\mathbf{u}_{\alpha,\beta}$ and $\mathbf{u}_{\alpha,\beta+1} = (x+1)\mathbf{u}_{\alpha,\beta}$, respectively. Since $\mathbf{u}_{\alpha+1,\beta}$ and $\mathbf{u}_{\alpha,\beta+1}$ are Christoffel transformations of $\mathbf{u}_{\alpha,\beta}$, we can use the results discussed above to study the relations between Jacobi polynomials with adjacent parameters.

Here, we study the MOPS associated with the moment functional $\mathbf{u}_{\alpha,\beta+1}$ given by

$$\begin{aligned}\langle \mathbf{u}_{\alpha,\beta+1}, p(x)\rangle &= \langle \mathbf{u}_{\alpha,\beta}, (x+1)p(x)\rangle\\ &= \int_{-1}^{1} p(x)(1-x)^{\alpha}(1+x)^{\beta+1}\,dx, \quad p(x)\in\mathbb{P}.\end{aligned}$$

If $J_{\text{mon}}^{\alpha,\beta}$ and $J_{\text{mon}}^{\alpha,\beta+1}$ are the monic Jacobi matrices associated with $\mathbf{u}_{\alpha,\beta}$ and $\mathbf{u}_{\alpha,\beta+1}$, respectively, then $J_{\text{mon}}^{\alpha,\beta}+I$ has an LU factorization and $J_{\text{mon}}^{\alpha,\beta+1}+I$ has an UL factorization as in (7.4) with $a=-1$. Taking into account that

$$P_n^{(\alpha,\beta)}(-1)=\frac{(-1)^n\Gamma(n+\beta+1)2^n\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)},$$

we see that $(P_n^{(\alpha,\beta)}(x))_{n\geq 0}$ and $(P_n^{(\alpha,\beta+1)}(x))_{n\geq 0}$ satisfy (7.2) and (7.3) with

$$\lambda_n=\frac{2(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)},\quad n\geq 0,$$
$$\nu_n=\frac{2(n+\alpha)n}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},\quad n\geq 1.$$

Explicitly, we obtain the so-called connection formulas:

$$\begin{aligned}(x+1)P_n^{(\alpha,\beta+1)}(x)&=P_{n+1}^{(\alpha,\beta)}(x)\\&\quad+\frac{2(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)}P_n^{(\alpha,\beta)}(x),\\P_n^{(\alpha,\beta)}(x)&=P_n^{(\alpha,\beta+1)}(x)\\&\quad+\frac{2(n+\alpha)n}{(2n+\alpha+\beta)(2n+a+\beta+1)}P_{n-1}^{(\alpha,\beta+1)}(x).\end{aligned}\tag{7.5}$$

Exercise 7.2. Consider Example 6.5.

1. Show that $(2^nP_n(\frac{x+1}{2}))_{n\geq 0}$ and $(2^nQ_n(\frac{x+1}{2}))_{n\geq 0}$ are sequences of monic Jacobi polynomials of adjacent parameters.
   Hint: Under the change of variable $t=\frac{x+1}{2}$, for which parameters (α,β) is $\langle\mathbf{u}_{\alpha,\beta},P_n(\frac{x+1}{2})P_m(\frac{x+1}{2})\rangle=\langle\mathbf{v},P_n(t)P_m(t)\rangle$?
2. It is clear that the moment functional $\mathbf{w}=x\mathbf{v}$ is a Christoffel transformation of $\mathbf{v}$, and, therefore, $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$ satisfy (7.2) and (7.3) for some λ_n and ν_n. Use the previous item and (7.5) to deduce λ_n and ν_n.
   Hint: Use the change of variable $t=\frac{x+1}{2}$ to turn (7.5) into connection formulas for $(P_n(x))_{n\geq 0}$ and $(Q_n(x))_{n\geq 0}$.

Let $\mathcal{S}(z)$ and $\tilde{\mathcal{S}}(z)$ be the Stieltjes functions associated with $\mathbf{u}$ and $\tilde{\mathbf{u}}$, respectively. Then $\tilde{\mathcal{S}}(z)$ is a linear rational transformation of $\mathcal{S}(z)$. Indeed, the moments of $\tilde{\mathbf{u}}$ and $\mathbf{u}$ satisfy the following relation:

$$\tilde{\mu}_n=\langle\tilde{\mathbf{u}},x^n\rangle=\langle\mathbf{u},(x-a)x^n\rangle=\mu_{n+1}-a\mu_n.$$

Then,

$$\tilde{\mathcal{S}}(z)=\sum_{n=0}^{\infty}\frac{\tilde{\mu}_n}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{\mu_{n+1}}{z^{n+1}}-a\sum_{n=0}^{\infty}\frac{\mu_n}{z^{n+1}}$$

$$= z\left(\mathcal{S}(z) - \frac{\mu_0}{z}\right) - a\mathcal{S}(z)$$
$$= (z-a)\mathcal{S}(z) - \mu_0.$$

Transformations of this kind are often called *canonical Christoffel rational transformations*.

As a consequence, any linear spectral transformation of the form

$$\mathcal{S}^{(N)}(z) = (z - x_1)\cdots(z - x_N)\mathcal{S}(z) + B(z),$$

where $B(z)$ is a polynomial of degree at most $N-1$ defined uniquely by $\mathcal{S}(z)$ (see [64]), can be seen as N successive canonical Christoffel rational transformations.

7.2 Canonical Geronimus transformation

Let $\mathbf{u}$ be a quasidefinite moment functional and $(P_n(x))_{n\geq 0}$ its corresponding MOPS. Given a real number a, we define the *canonical Geronimus transformation* of $\mathbf{u}$ (see [23]) as the functional

$$\hat{\mathbf{u}} = (x-a)^{-1}\mathbf{u} + M\boldsymbol{\delta}_a. \tag{7.6}$$

In contrast with the canonical Christoffel transformation, a canonical Geronimus transformation is not unique because it depends on the choice of the parameter M. Moreover, $M = \hat{\mu}_0$, that is, M is the first moment of the functional $\hat{\mathbf{u}}$ since

$$\hat{\mu}_0 = \langle \hat{\mathbf{u}}, 1\rangle = \left\langle (x-a)^{-1}\mathbf{u}, 1\right\rangle + M = M.$$

Therefore, a necessary condition for $\hat{\mathbf{u}}$ to be quasidefinite is that $M \neq 0$.

Suppose that $\hat{\mathbf{u}}$ is a quasidefinite functional and let $(\hat{P}_n(x))_{n\geq 0}$ be its associated MOPS. Since $(P_n(x))_{n\geq 0}$ is a basis of $\mathbb{P}$, there exist constants $(\varsigma_{n,k})_{k=0}^{n-1}$ such that

$$\hat{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} \varsigma_{n,k} P_k(x), \quad n \geq 1.$$

Notice that $\mathbf{u} = (x-a)\hat{\mathbf{u}}$. Then,

$$\left\langle \mathbf{u}, \hat{P}_n(x)P_k(x)\right\rangle = \left\langle (x-a)\hat{\mathbf{u}}, \hat{P}_n(x)P_k(x)\right\rangle = \left\langle \hat{\mathbf{u}}, \hat{P}_n(x)(x-a)P_k(x)\right\rangle,$$

and, consequently,

$$\varsigma_{n,k} = \begin{cases} 0, & \text{if } k \leq n-2, \\ \dfrac{\langle \hat{\mathbf{u}}, \hat{P}_n^2(x)\rangle}{\langle \mathbf{u}, P_{n-1}^2(x)\rangle}, & \text{if } k = n-1. \end{cases}$$

Therefore,

$$\hat{P}_n(x) = P_n(x) + \varsigma_n P_{n-1}(x), \quad n \geq 1. \tag{7.7}$$

In order to express ς_n in terms of known values, note that from (7.7) we get

$$\frac{\hat{P}_n(x) - \hat{P}_n(a)}{\mu_0(x-a)} = \frac{P_n(x) - P_n(a)}{\mu_0(x-a)} + \varsigma_n \frac{P_{n-1}(x) - P_{n-1}(a)}{\mu_0(x-a)}, \quad n \geq 1.$$

Applying the functional $\mathbf{u}$ to both sides of this equation, we obtain

$$\begin{aligned} P_{n-1}^{(1)}(a) + \varsigma_n P_{n-2}^{(1)}(a) &= \frac{1}{\mu_0}\langle \hat{\mathbf{u}}, \hat{P}_n(x)\rangle - \frac{M}{\mu_0}\hat{P}_n(a) \\ &= -\frac{M}{\mu_0}\hat{P}_n(a), \end{aligned}$$

where $(P_n^{(1)}(x))_{n\geq 0}$ are the polynomials of the first kind defined in (2.12). Thus, multiplying (7.7) by $M\mu_0^{-1}$ and evaluating at $x = a$, we deduce

$$\varsigma_n\left(\mu_0 P_{n-2}^{(1)}(a) + MP_{n-1}(a)\right) = -\mu_0 P_{n-1}^{(1)}(a) - MP_n(a),$$

or, equivalently,

$$\varsigma_n = -\frac{\mu_0 P_{n-1}^{(1)}(a) + MP_n(a)}{\mu_0 P_{n-2}^{(1)}(a) + MP_{n-1}(a)}.$$

Following a similar reasoning as above, we obtain

$$(x-a)P_n(x) = \hat{P}_{n+1}(x) + \rho_n \hat{P}_n(x), \tag{7.8}$$

where

$$\begin{aligned} \rho_n &= -\frac{(\mu_0 P_{n-2}^{(1)}(a) + MP_{n-1}(a))\langle \mathbf{u}, P_n^2(x)\rangle}{(\mu_0 P_{n-1}^{(1)}(a) + MP_n(a))\langle \mathbf{u}, P_{n-1}^2(x)\rangle}, \quad n \geq 1, \\ \rho_0 &= \frac{\mu_0}{\hat{\mu}_0}. \end{aligned}$$

Proposition 7.5. *The moment functional $\hat{\mathbf{u}} = (x-a)^{-1}\mathbf{u} + M\delta_a$ is quasidefinite if and only if $M \neq 0$ and, for $n \geq 0$,*

$$\mu_0 P_{n-1}^{(1)}(a) + MP_n(a) \neq 0.$$

Proof. Suppose that $\hat{\mathbf{u}}$ is quasidefinite and that $P_{k-1}^{(1)}(a) + MP_k(a) = 0$ for some $k \geq 0$. By (7.8), $(\hat{P}_n(x))_{n\geq 0}$ is given recursively by $\hat{P}_0(x) = 1$ and, for $n \geq 0$,

$$\hat{P}_{n+1}(x) = (x-a)P_n(x) - \rho_n \hat{P}_n(x).$$

Then, $\hat{P}_{k+1}(x)$ is not well-defined, which contradicts the quasidefinite character of $\hat{\mathbf{u}}$. Hence, $\mu_0 P_{n-1}^{(1)}(a) + MP_n(a) \neq 0$ for $n \geq 0$. Moreover, it was shown above that $M = \hat{\mu}_0 \neq 0$.

Conversely, suppose that $M \neq 0$ and $\mu_0 P_{n-1}^{(1)}(a) + MP_n(a) \neq 0$ for $n \geq 0$. Let $(\hat{P}_n(x))_{n\geq 0}$ be the sequence of monic polynomials defined recursively by $\hat{P}_0(x) = 1$ and, for $n \geq 1$,

$$\hat{P}_n(x) = P_n(x) + \varsigma_n P_{n-1}(x),$$

where

$$\varsigma_n = -\frac{\mu_0 P_{n-1}^{(1)}(a) + MP_n(a)}{\mu_0 P_{n-2}^{(1)}(a) + MP_{n-1}(a)}.$$

Then, $\langle \hat{\mathbf{u}}, \hat{P}_0^2(x)\rangle = M \neq 0$. For $n \geq 0$, if $0 \leq k \leq n$, then

$$\begin{aligned}
\big\langle \hat{\mathbf{u}}, \hat{P}_n(x)P_k(x)\big\rangle &= \big\langle \hat{\mathbf{u}}, P_n(x)P_k(x)\big\rangle + \varsigma_n\big\langle \hat{\mathbf{u}}, P_{n-1}(x)P_k(x)\big\rangle \\
&= \left\langle \mathbf{u}, \frac{P_n(x)P_k(x) - P_n(a)P_k(a)}{x-a}\right\rangle + MP_n(a)P_k(a) \\
&\quad + \varsigma_n\left(\left\langle \mathbf{u}, \frac{P_{n-1}(x)P_k(x) - P_{n-1}(a)P_k(a)}{x-a}\right\rangle + MP_{n-1}(a)P_k(a)\right).
\end{aligned}$$

Taking into account that $\frac{P_k(x)-P_k(a)}{x-a}$ is a polynomial of degree at most $k-1$ and

$$\frac{P_m(x)P_k(x) - P_m(a)P_k(a)}{x-a} = P_m(x)\frac{P_k(x) - P_k(a)}{x-a} + \frac{P_m(x) - P_m(a)}{x-a}P_k(a),$$

we obtain

$$\begin{aligned}
\big\langle \hat{\mathbf{u}}, \hat{P}_n(x)P_k(x)\big\rangle &= \big(\mu_0 P_{n-1}^{(1)}(x) + MP_n(a) + \varsigma_n\big(\mu_0 P_{n-2}^{(1)}(x) + MP_{n-1}(a)\big)\big)P_k(a) \\
&\quad + \varsigma_n\left\langle \mathbf{u}, P_{n-1}(x)\frac{P_k(x) - P_k(a)}{x-a}\right\rangle \\
&= \varsigma_n\left\langle \mathbf{u}, P_{n-1}(x)\frac{P_k(x) - P_k(a)}{x-a}\right\rangle.
\end{aligned}$$

Hence, for $0 \leq k \leq n-1$, $\langle \hat{\mathbf{u}}, \hat{P}_n(x)P_k(x)\rangle = 0$ and

$$\big\langle \hat{\mathbf{u}}, \hat{P}_n(x)P_n(x)\big\rangle = \varsigma_n\big\langle \mathbf{u}, P_{n-1}^2(x)\big\rangle \neq 0.$$

Therefore, $(\hat{P}_n(x))_{n\geq 0}$ is a MOPS associated with $\hat{\mathbf{u}}$ and, thus, $\hat{\mathbf{u}}$ is quasidefinite. ■

Observation 7.6. Relations (7.7) and (7.8) can be written in a matrix form as follows:

$$\begin{pmatrix} \hat{P}_0(x) \\ \hat{P}_1(x) \\ \hat{P}_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \varsigma_1 & 1 & & \\ & \varsigma_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix}$$

and

$$(x-a)\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \rho_0 & 1 & & & \\ & \rho_1 & 1 & & \\ & & \rho_2 & \ddots & \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \hat{P}_0(x) \\ \hat{P}_1(x) \\ \hat{P}_2(x) \\ \vdots \end{pmatrix}.$$

Similarly to canonical Christoffel transformations, we have the following relations between the Jacobi matrices associated with the functionals $\mathbf{u}$ and $\hat{\mathbf{u}}$. Nevertheless, this relation is not unique because it depends on the choice of the free parameter $\hat{\mu}_0$, which cannot be zero (otherwise the quasidefinite character of $\hat{\mathbf{u}}$ is contradicted). In the proof, we keep $\hat{\mathbf{u}}$ fixed.

Theorem 7.7 ([11, 63]). *Let J_{mon} and $\hat{J}_{\text{mon}}$ be the Jacobi matrices associated with $\mathbf{u}$ and $\hat{\mathbf{u}}$, respectively. There exist a lower triangular matrix L with 1's in its main diagonal and an upper triangular matrix U,*

$$L = \begin{pmatrix} 1 & & & \\ \varsigma_1 & 1 & & \\ & \varsigma_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \rho_0 & 1 & & & \\ & \rho_1 & 1 & & \\ & & \rho_2 & \ddots & \\ & & & \ddots & \ddots \end{pmatrix},$$

such that

$$J_{\text{mon}} - aI = UL \quad \textit{and} \quad \hat{J}_{\text{mon}} - aI = LU.$$

The above relation is known in the literature as Darboux transformation *with* parameter.

Proof. From Observation 7.6, we have

$$(x-a)\mathbf{P} = U\hat{\mathbf{P}} \quad \text{and} \quad \hat{\mathbf{P}} = L\mathbf{P},$$

where $\mathbf{P} = (P_0(x), P_1(x), \dots)^t$, $\hat{\mathbf{P}} = (\hat{P}_0(x), \hat{P}_1(x), \dots)^t$, L is a lower triangular matrix with 1's in the main diagonal, and U is an upper triangular matrix.

Taking into account that $(J_{\text{mon}} - aI)\mathbf{P} = (x-a)\mathbf{P}$, we have

$$(J_{\text{mon}} - aI)\mathbf{P} = U\hat{\mathbf{P}} = UL\mathbf{P}.$$

But $(P_n(x))_{n\geq 0}$ is a basis of $\mathbb{P}$, therefore we obtain

$$J_{\text{mon}} - aI = UL.$$

Similarly, $(\hat{J}_{\text{mon}} - aI)\hat{\mathbf{P}} = (x-a)\hat{\mathbf{P}}$ and, thus,

$$(\hat{J}_{\text{mon}} - aI)\hat{\mathbf{P}} = (x-a)(L\mathbf{P}) = L\big((x-a)\mathbf{P}\big) = LU\hat{\mathbf{P}}.$$

But $(\hat{P}_n(x))_{n\geq 0}$ is a basis of polynomials. Therefore,

$$\hat{J}_{\text{mon}} - aI = LU. \qquad \blacksquare$$

Example 7.8. For $\alpha > -1$, let $(L_n^{(\alpha+1)}(x))_{n\geq 0}$ be the monic Laguerre polynomials (of parameter $\alpha+1$). These polynomials are explicitly given by

$$L_n^{(\alpha+1)}(x) = (-1)^n(\alpha+2)_n \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+2)_k} \frac{x^k}{k!}, \tag{7.9}$$

where $(\nu)_k := \nu(\nu+1)\cdots(\nu+k-1)$, $(\nu)_0 = 1$, denotes the Pochhammer symbol. They are orthogonal with respect to the positive definite moment functional $\mathbf{v}_{\alpha+1}$ defined by

$$\langle \mathbf{v}_{\alpha+1}, p(x)\rangle = \int_0^\infty p(x)x^{\alpha+1}e^{-x}\,dx, \quad p(x) \in \mathbb{P}.$$

Using the explicit expressions for $L^{(\alpha+1)}(x)$ and $\mathbf{v}_{\alpha+1}$, we obtain the following orthogonality relation:

$$\langle \mathbf{v}_{\alpha+1}, L_n^{(\alpha+1)}(x)L_m^{(\alpha+1)}(x)\rangle = n!\Gamma(n+\alpha+2)\delta_{n,m}, \tag{7.10}$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function.

They satisfy the following three-term recurrence relation:

$$\begin{aligned} xL_n^{(\alpha+1)}(x) &= L_{n+1}^{(\alpha+1)}(x) + (2n+\alpha+2)L_n^{(\alpha+1)}(x) \\ &\quad + n(n+\alpha+1)L_{n-1}^{(\alpha+1)}(x), \quad n \geq 0, \\ L_0^{(\alpha+1)}(x) &= 1, \quad L_{-1}^{(\alpha+1)}(x) = 0. \end{aligned}$$

Now, let $\hat{\mathbf{v}}$ be the linear functional defined by the Geronimus transformation (7.6) with $a = 0$, that is, $\hat{\mathbf{v}} = x^{-1}\mathbf{v}_{\alpha+1} + M\boldsymbol{\delta}$. Choosing $M := \Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}dx$, we get

$$\begin{aligned} \langle \hat{\mathbf{v}}, p(x)\rangle &= \langle x^{-1}\mathbf{v}_{\alpha+1} + M\boldsymbol{\delta}, p(x)\rangle \\ &= \int_0^\infty \frac{p(x)-p(0)}{x} x^{\alpha+1}e^{-x}\,dx + p(0)\int_0^\infty x^\alpha e^{-x}\,dx \\ &= \int_0^\infty p(x)x^\alpha e^{-x}\,dx, \quad p(x) \in \mathbb{P}. \end{aligned}$$

That is, $\hat{\mathbf{v}} = x^{-1}\mathbf{v}_{\alpha+1} + \Gamma(\alpha+1)\boldsymbol{\delta} = \mathbf{v}_\alpha$. In other words, the moment functional $\mathbf{v}_\alpha$ associated with the Laguerre polynomials of parameter α is a Geronimus transformation of $\mathbf{v}_{\alpha+1}$.

Let $J_{\text{mon}}^{\alpha+1}$ and J_{mon}^{α} be the monic Jacobi matrices associated with $\mathbf{v}_{\alpha+1}$ and $\mathbf{v}_\alpha$, respectively. The LU factorization of $J_{\text{mon}}^{\alpha+1}$ and the UL factorization of J_{mon}^{α} in Theorem 7.7 can be obtained with $\rho_n = \alpha+n+1$, $n \geq 0$, and $\varsigma_n = n$, $n \geq 1$. From this, together with (7.7) and (7.8), we get the well-known connection formulas:

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x),$$
$$xL_n^{(\alpha+1)}(x) = L_{n+1}^{(\alpha)}(x) + (\alpha+n+1)L_n^{(\alpha)}(x).$$

If $\mathcal{S}(z)$ and $\hat{\mathcal{S}}(z)$ are the Stieltjes functions associated with $\mathbf{u}$ and $\hat{\mathbf{u}}$, respectively, then $\hat{\mathcal{S}}(z)$ is a linear spectral transformation of $\mathcal{S}(z)$. Indeed, taking into account that the moments of $\hat{\mathbf{u}}$ and $\mathbf{u}$ satisfy

$$\mu_n = \langle \mathbf{u}, x^n \rangle = \langle \hat{\mathbf{u}}, (x-a)x^n \rangle = \hat{\mu}_{n+1} - a\hat{\mu}_n,$$

we obtain

$$\begin{aligned} \mathcal{S}(z) &= \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{\hat{\mu}_{n+1}}{z^{n+1}} - a\sum_{n=0}^{\infty} \frac{\hat{\mu}_n}{z^{n+1}} \\ &= z\left(\hat{\mathcal{S}}(z) - \frac{\hat{\mu}_0}{z}\right) - a\hat{\mathcal{S}}(z) \\ &= (z-a)\hat{\mathcal{S}}(z) - \hat{\mu}_0. \end{aligned}$$

It follows that

$$\hat{\mathcal{S}}(z) = \frac{\mathcal{S}(z) + \hat{\mu}_0}{(z-a)}.$$

Transformations of this kind are often called *canonical Geronimus spectral transformations*. As in the canonical Christoffel case, it can be proved by induction that any linear spectral transformation of the form

$$\mathcal{S}^{(N)}(z) = \frac{\mathcal{S}(z) + B(z)}{(z-x_1)\cdots(z-x_N)},$$

where the polynomial $B(z)$ is of degree at most $N-1$ completely defined by $S(z)$ (see [64]), can be seen as N successive canonical Geronimus transformations.

7.3 Uvarov transformation

Let $\mathbf{u}$ be a quasidefinite moment functional and $(P_n(x))_{n\geq 0}$ its corresponding MOPS. Given a real number a, we define the *Uvarov transformation* of $\mathbf{u}$ as the functional

$$\check{\mathbf{u}} = \mathbf{u} + M\boldsymbol{\delta}_a, \tag{7.11}$$

with $M > 0$ (see [58]). Note that, as for canonical Geronimus transformations, Uvarov transformations depend on the parameter M.

Suppose that $\check{\mathbf{u}}$ is a quasidefinite functional and let $(\check{P}_n(x))_{n\geq 0}$ be its corresponding MOPS. There exist real numbers $(\beta_{n,k})_{k=0}^{n-1}$ such that

$$\check{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} \beta_{n,k} P_k(x), \quad n \geq 1. \tag{7.12}$$

For $0 \leq k \leq n-1$,

$$\beta_{n,k} = \frac{\langle \mathbf{u}, \check{P}_n(x) P_k(x)\rangle}{\langle \mathbf{u}, P_n^2(x)\rangle} = \frac{\langle \check{\mathbf{u}} - M\delta_a, \check{P}_n(x) P_k(x)\rangle}{\langle \mathbf{u}, P_n^2(x)\rangle} = -\frac{M\check{P}_n(a)P_k(a)}{\langle \mathbf{u}, P_n^2(x)\rangle}.$$

Therefore, (7.12) can be rewritten as

$$\check{P}_n(x) = P_n(x) - M\check{P}_n(a)K_{n-1}(x,a).$$

Moreover, evaluating the above expression at $x = a$, we obtain

$$\check{P}_n(a) = \frac{P_n(a)}{1 + MK_{n-1}(a,a)}$$

and, thus,

$$\check{P}_n(x) = P_n(x) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)} K_{n-1}(x,a). \tag{7.13}$$

Observe that the sequence of polynomials $(\check{P}_n(x))_{n\geq 0}$ exists if

$$M \neq -\frac{1}{K_{n-1}(a,a)}, \quad n \geq 1.$$

Proposition 7.9. *The moment functional $\check{\mathbf{u}} = \mathbf{u} + M\delta_a$ is quasidefinite if and only if for $n \geq 0$, $1 + MK_{n-1}(a,a) \neq 0$ and*

$$\left\langle \mathbf{u}, P_n^2(x)\right\rangle + \frac{MP_n^2(a)}{1 + MK_{n-1}(a,a)} \neq 0.$$

Here $K_{-1}(a,a) = 0$.

Proof. Suppose that $\check{\mathbf{u}}$ is quasidefinite. Then, for $n \geq 0$,

$$\begin{aligned}
0 \neq \left\langle \check{\mathbf{u}}, \check{P}_n(x)P_n(x)\right\rangle &= \left\langle \check{\mathbf{u}}, \left(P_n(x) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)} K_{n-1}(x,a)\right) P_n(x)\right\rangle \\
&= \left\langle \mathbf{u}, P_n^2(x)\right\rangle \\
&\quad + M\left(P_n(a) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)} K_{n-1}(a,a)\right) P_n(a) \\
&= \left\langle \mathbf{u}, P_n^2(x)\right\rangle + \frac{MP_n^2(a)}{1 + MK_{n-1}(a,a)}.
\end{aligned}$$

Clearly, if $1 + MK_{n-1}(a,a) = 0$ for some $n \geq 0$, then the quasidefinite character of $\check{\mathbf{u}}$ is contradicted.

Conversely, suppose that, for $n \geq 0$, $1 + MK_{n-1}(a,a) \neq 0$ and

$$\langle \mathbf{u}, P_n^2(x)\rangle + \frac{MP_n^2(a)}{1 + MK_{n-1}(a,a)} \neq 0.$$

Let $(\check{P}_n(x))_{n\geq 0}$ be the sequence of monic polynomials defined by (7.13) for $n \geq 1$, and $\check{P}_0(x) = 1$. Then, for $n \geq 0$,

$$\langle \check{\mathbf{u}}, \check{P}_n(x)P_n(x)\rangle = \langle \mathbf{u}, P_n^2(x)\rangle + \frac{MP_n^2(a)}{1 + MK_{n-1}(a,a)} \neq 0.$$

For $n \geq 1$ and $0 \leq k \leq n-1$, we use the reproducing C–D kernel property to compute

$$\begin{aligned}
\langle \check{\mathbf{u}}, \check{P}_n(x)P_k(x)\rangle &= \left\langle \check{\mathbf{u}}, \left(P_n(x) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)}K_{n-1}(x,a)\right)P_k(x)\right\rangle \\
&= -\frac{MP_n(a)}{1 + MK_{n-1}(a,a)}\langle \mathbf{u}, K_{n-1}(x,a)P_k(x)\rangle \\
&\quad + M\left(P_n(a) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)}K_{n-1}(a,a)\right)P_k(a) \\
&= MP_n(a)P_k(a) - \frac{MP_n(a)P_k(a)}{1 + MK_{n-1}(a,a)}\big(1 + MK_{n-1}(a,a)\big) \\
&= 0.
\end{aligned}$$

Therefore, $(\check{P}_n(x))_{n\geq 0}$ is the MOPS associated with $\check{\mathbf{u}}$ and, hence, $\check{\mathbf{u}}$ is quasidefinite. ∎

Example 7.10. For $\alpha > 1$, let $(L_n^{(\alpha)}(x))_{n\geq 0}$ be the sequence of monic Laguerre polynomials of parameter α, orthogonal with respect the positive definite moment functional $\mathbf{v}_\alpha$ (see Example 7.8).

Let $\check{\mathbf{v}}$ be an Uvarov transformation of $\mathbf{v}_\alpha$ given by (7.11) with $a = 0$, that is, $\check{\mathbf{v}} = \mathbf{v}_\alpha + M\boldsymbol{\delta}$, and let $K_n^{(\alpha)}(x, y)$ be the C–D kernel polynomial of degree n associated with the Laguerre polynomials of parameter α. We leave it as an exercise to show that

$$K_n^{(\alpha)}(x,0) = \frac{L_n^{(\alpha)}(0)}{\langle \mathbf{u}, (L_n^{(\alpha)}(x))^2\rangle}L_n^{(\alpha+1)}(x). \tag{7.14}$$

By (7.9), we have $L_n^{(\alpha)}(0) = (-1)^n\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}$, and, thus, $K_n^{(\alpha)}(0,0) \neq 0$. Choose M such that $\check{\mathbf{v}}$ is quasidefinite (see Proposition 7.9). In this case, let $(\check{L}_n^{(\alpha)}(x))_{n\geq 0}$ be the associated sequence of monic orthogonal polynomials. From (7.10) and (7.13), we obtain an expression for $\check{L}_n^{(\alpha)}(x)$ in terms of Laguerre polynomials:

$$\check{L}_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) - \frac{ML_n^{(\alpha)}(0)}{1 + MK_{n-1}^{(\alpha)}(0,0)}\frac{L_{n-1}^{(\alpha)}(0)}{\langle \mathbf{u}, (L_{n-1}^{\alpha}(x))^2\rangle}L_{n-1}^{(\alpha+1)}(x)$$

$$= L_n^{(\alpha)}(x) + \frac{M(a+1)^2\Gamma(n+a+1)}{M(a+1)\Gamma(n+a+1) + (\Gamma(a+2))^2(n-1)!} L_{n-1}^{(\alpha+1)}(x).$$

Exercise 7.3. Observe that $\mathbf{v}_{\alpha+1}$ is a Christoffel transformation of $\mathbf{v}_\alpha$ since $\mathbf{v}_{\alpha+1} = x\mathbf{v}_\alpha$. Therefore, $(L_n^{(\alpha)}(x))_{n\geq 0}$ and $(L_n^{(\alpha+1)}(x))_{n\geq 0}$ satisfy a relation of the form (7.2) with $a = 0$:

$$xL_n^{(\alpha+1)}(x) = L_{n+1}^{(\alpha)}(x) - \frac{L_{n+1}^{(\alpha)}(0)}{L_n^{(\alpha)}(0)} L_n^{(\alpha)}(x).$$

Use this relation and the Christoffel–Darboux formula in Theorem 2.28 to deduce (7.14).

Let $\mathcal{S}(z)$ and $\check{\mathcal{S}}(z)$ be the Stieltjes functions associated with $\mathbf{u}$ and $\check{\mathbf{u}}$, respectively. As in the Christoffel and Geronimus cases, we deduce that $\check{\mathcal{S}}(z)$ is a linear spectral transformation of $\mathcal{S}(z)$ of the form

$$\check{\mathcal{S}}(z) = \mathcal{S}(z) + \frac{1}{z-a}.$$

Notice that canonical Christoffel and Geronimus transformations are not inverse transformation to each other. If we apply a canonical Geronimus transformation followed by a canonical Christoffel transformation to a moment functional, we recover the original functional. However, if we apply a canonical Christoffel transformation followed by a canonical Geronimus transformation, we get the original functional plus a constant multiplied by a Dirac delta, that is, an Uvarov transformation.

We will conclude this section with the following theorem.

Theorem 7.11 (Zhedanov [64]). *Every linear spectral transformation is a superposition of canonical Christoffel and Geronimus transformations.*

Proof. Let

$$\mathcal{S}^{(1)}(z) = \frac{A(z)\mathcal{S}(z) + B(z)}{D(z)}$$

be a Stieltjes function obtained by applying a linear spectral transformation to $\mathcal{S}(z)$. We can apply a chain of consecutive canonical Christoffel transformations to $\mathcal{S}^{(1)}(z)$ in such a way that

$$\mathcal{S}^{(2)}(z) = \tilde{A}(z)\mathcal{S}^{(1)}(z) + \tilde{B}(z),$$

where $\tilde{A}(z)$ and $\tilde{B}(z)$ are polynomials such that the zeros of $\tilde{A}(z)$ can be chosen arbitrarily. If, in particular, we take $\tilde{A}(z) = D(z)$, then

$$\mathcal{S}^{(2)}(z) = A(z)\mathcal{S}(z) + B(z) + \tilde{B}(z).$$

Therefore, $\mathcal{S}^{(2)}(z)$ is obtained from $\mathcal{S}(z)$ by applying a chain of consecutive canonical Christoffel transformations. On the other hand, by construction, $\mathcal{S}^{(1)}(z)$ is obtained from $\mathcal{S}^{(2)}(z)$ by applying a chain of consecutive canonical Geronimus transformations. Hence, $\mathcal{S}^{(1)}(z)$ is obtained from $\mathcal{S}(z)$ by applying canonical Christoffel and Geronimus transformations. ∎

Chapter 8

Classical orthogonal polynomials

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, Bessel) are the most extensively studied families of orthogonal polynomials in the literature.

The following expressions for the Hermite, Laguerre, and Jacobi polynomials appear in [1, p. 775], and the expression for the Bessel polynomials are found in [36, p. 108]:

$$\text{(Hermite)} \quad H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!(n-2k)!} (2x)^{n-2k},$$

$$\text{(Laguerre)} \quad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{1}{k!} x^k, \quad \alpha \neq -1, -2, \ldots,$$

$$\text{(Jacobi)} \quad P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k} (x+1)^k,$$

$$\alpha, \beta \neq -1, -2, \ldots,$$

$$\text{(Bessel)} \quad B_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n+a-1)_k \left(\frac{x}{2}\right)^k, \quad a \neq -1, -2, \ldots.$$

It is well known that Hermite polynomials are associated with a positive definite moment functional, as well as the Laguerre polynomials for $\alpha > -1$, and the Jacobi polynomials for $\alpha, \beta > -1$. In contrast, the Bessel case corresponds to a quasidefinite linear functional that is not positive definite.

Although there are several ways to introduce the classical orthogonal polynomials, we focus our attention on the characterization given in 1929 by S. Bochner [9], where he studied the families of polynomials $(P_n(x))_{n\geq 0}$ that are eigenfunctions of the linear differential operator

$$L = \phi(x) \frac{d^2}{dx^2} + \psi(x) \frac{d}{dx} + \mu(x),$$

where $\phi(x)$, $\psi(x)$, and $\mu(x)$ are fixed polynomials. That is,

$$L\big[P_n(x)\big] = \lambda_n P_n(x), \quad \lambda_n \in \mathbb{R}, \tag{8.1}$$

for every $n \in \mathbb{N}$ and $\lambda_n \neq \lambda_m$ when $n \neq m$.

8.1 The linear differential operator and its solutions

Let us study the linear operator L and the polynomial solutions of (8.1). Let $y_0(x) \not\equiv 0$ be a polynomial solution with $\deg y_0(x) = 0$, that is,

$$L[y_0] = \phi(x)y_0'' + \psi(x)y_0' + \mu(x)y_0 = \lambda_0 y_0.$$

It can be easily deduced that $\mu(x) = \lambda_0$ is a constant that does not depend on the degree of the polynomial solutions of (8.1). Similarly, let

$$\begin{aligned} y_1(x) &= y_{1,1}x + y_{1,0}, & y_{1,1} &\neq 0, \\ y_2(x) &= y_{2,2}x^2 + y_{2,1}x + y_{2,0}, & y_{2,2} &\neq 0, \end{aligned}$$

be polynomial solutions of (8.1) with $\deg y_1(x) = 1$ and $\deg y_2(x) = 2$. Substituting $y_1(x)$ into (8.1), we obtain

$$\begin{aligned} L\big[y_1(x)\big] &= \phi(x)y_1''(x) + \psi(x)y_1'(x) + \mu(x)y_1(x) = \psi(x)y_{1,1} + \lambda_0 y_1(x) \\ &= \lambda_1 y_1(x). \end{aligned}$$

If $\lambda_1 \neq \lambda_0$, then

$$\psi(x) = \frac{\lambda_1 - \lambda_0}{y_{1,1}} y_1(x)$$

is a polynomial with $\deg \psi(x) = 1$. Finally,

$$\begin{aligned} L\big[y_2(x)\big] &= \phi(x)y_2''(x) + \psi(x)y_2'(x) + \lambda_0 y_2(x) \\ &= \phi(x)(2y_{2,2}) + \psi(x)(2y_{2,2}x + y_{2,1}) + \lambda_0 y_2(x) = \lambda_2 y_2(x) \end{aligned}$$

implies

$$\phi(x) = \frac{(\lambda_2 - \lambda_0)y_2(x) - \psi(x)(2y_{2,2}x + y_{2,1})}{2y_{2,2}}$$

and, therefore, $\deg \phi(x) \leq 2$.

For $n \geq 1$, if

$$y_n(x) = x^n + \text{terms of lower degree},$$

is a monic solution of (8.1), then, by comparing coefficients in both sides of $L[y_n] = \lambda_n y_n$, we get

$$\lambda_n - \lambda_0 = \frac{n(n-1)}{2}\phi'' + n\psi' \neq 0, \quad n \geq 1.$$

For convenience, we set $\lambda_0 = 0$. Observe that

$$\frac{1}{2}n\phi'' + \psi' \neq 0, \quad n \geq 0,$$

must hold.

Summarizing, if for $n \geq 0$, (8.1) has a polynomial solution $y_n(x)$ with $\deg y_n(x) = n$, then L has the form

$$L = \phi(x)\frac{d^2}{dx^2} + \psi(x)\frac{d}{dx},$$

with $\deg \phi(x) \leq 2$, $\deg \psi(x) = 1$. Moreover, $\frac{1}{2}n\phi'' + \psi' \neq 0$, $n \geq 0$.

The classical orthogonal polynomials are classified, up to affine transformations of the independent variable x, according to the canonical forms of the polynomial $\phi(x)$,

$$\phi(x) = \begin{cases} 1, & \text{Hermite,} \\ x, & \text{Laguerre,} \\ 1 - x^2, & \text{Jacobi,} \\ x^2, & \text{Bessel.} \end{cases}$$

We remark that when $\phi(x) = 0$, the polynomial solutions of (8.1) are the elements of the sequence $(x^n)_{n \geq 0}$, which cannot be associated with a quasidefinite moment functional.

8.2 Weight function and inner product

We will continue to study the differential operator L from the point of view of the Sturm–Liouville theory. Thus, we seek to write the operator in a convenient form for the study of the orthogonality of its eigenfunctions. Consider two cases, $\phi'(x) = \psi(x)$ and $\phi'(x) \neq \psi(x)$.

The simplest case is when $\phi'(x) = \psi(x)$. Here, we have that

$$2\phi'(x) - \psi(x) = \psi(x) \quad \text{and} \quad \phi''(x) - \psi'(x) = 0.$$

Consequently, the operator L can be written as

$$L = \frac{d}{dx}\left(\phi(x)\frac{d}{dx}\right).$$

The following result is easily verified using integration by parts twice.

Proposition 8.1. *Suppose that the polynomial $\phi(x)$ has two distinct real zeros a and b with $a < b$, and that $\phi'(x) = \psi(x)$. Then, the operator L is symmetric with respect to the scalar product $(\cdot,\cdot)$ defined as*

$$(p, q) = \int_a^b p(x)q(x)\,dx, \quad \forall\, p(x), q(x) \in \mathbb{P},$$

that is, $(L[p], q) = (p, L[q])$ for all $p(x), q(x) \in \mathbb{P}$.

Observe that under the hypotheses of Proposition 8.1, the condition $\lambda_n \neq \lambda_m$ when $n \neq m$ is of great importance to guarantee the orthogonality of the eigenfunctions of L. Indeed, let $y_n(x)$ and $y_m(x)$ be two distinct eigenfunctions of L. We know that $(L[y_n], y_m) = (y_n, L[y_m])$, therefore

$$(\lambda_n - \lambda_m)(y_n, y_m) = (\lambda_n y_n, y_m) - (y_n, \lambda_m y_m) = \big(L[y_n], y_m\big) - \big(y_n, L[y_m]\big) = 0.$$

Since $\lambda_n - \lambda_m \neq 0$, it follows that $(y_n, y_m) = 0$, that is, $y_n(x)$ and $y_m(x)$ are orthogonal.

For the case when $\phi'(x) \neq \psi(x)$, we have the following result.

Proposition 8.2. *Suppose that there exists a function $w(x)$ of class $\mathcal{C}^2$ and an interval $(a,b) \subseteq \mathbb{R}$ such that*

$$\frac{d}{dx}\big(\phi(x)w(x)\big) = \psi(x)w(x), \tag{8.2}$$

with boundary conditions

$$\lim_{x\to a} \phi(x)r(x)w(x) = \lim_{x\to b} \phi(x)r(x)w(x), \quad \forall r(x) \in \mathbb{P}. \tag{8.3}$$

Then, the differential operator L is symmetric with respect to the inner product $(\cdot,\cdot)$ defined by

$$(p,q) = \int_a^b p(x)q(x)w(x)\,dx, \quad \forall p(x), q(x) \in \mathbb{P},$$

that is, $(L[p], q) = (p, L[q])$.

Proof. If $w(x)$ satisfies (8.2), then $w(x)L$ can be written as

$$w(x)L = \frac{d}{dx}\left(w(x)\phi(x)\frac{d}{dx}\right).$$

Given $p(x), q(x) \in \mathbb{P}$,

$$\big(L[p], q\big) = \int_a^b L\big[p(x)\big]q(x)w(x)\,dx = \int_a^b \frac{d}{dx}\left(w(x)\phi(x)\frac{d}{dx}p(x)\right)q(x)\,dx.$$

Integrating by parts twice, we get

$$\begin{aligned}\big(L[p], q\big) &= \phi(x)w(x)\big(p'(x)q(x) - p(x)q'(x)\big)\Big|_a^b + \int_a^b p(x)L\big[q(x)\big]w(x)\,dx \\ &= \phi(x)w(x)\big(p'(x)q(x) - p(x)q'(x)\big)\Big|_a^b + \big(p, L[q]\big).\end{aligned}$$

From the boundary conditions (8.3), it follows that $(L[p], q) = (p, L[q])$, that is, L is a symmetric operator with respect to $(\cdot,\cdot)$. ∎

Clearly, the orthogonality of the eigenfunctions of L in this case is also a necessary consequence of the condition $\lambda_n \neq \lambda_m$ when $n \neq m$.

The function $w(x)$ in Proposition 8.2 is called a *symmetry factor* of the operator L, and the first-order differential equation (8.2) is called the *Pearson equation* for $w(x)$. Moreover, when $w(x)$ is positive and integrable on the interval (a, b), and

$$\int_a^b x^n w(x)\, dx < \infty, \quad n \geq 0,$$

$w(x)$ is called a *weight function*.

The general solution of the Pearson equation (8.2) is

$$w(x) = \frac{C}{\phi(x)} \exp\left(\int_a^x \frac{\psi(t)}{\phi(t)}\, dt\right), \quad C \in \mathbb{R} \setminus \{0\}.$$

Notice that when $\phi'(x) = \psi(x)$, the solution of (8.2) reduces to $w(x) = C$, that is, the Lebesgue measure on $[a, b]$ up to multiplication by a constant.

Table 8.1 shows the symmetry factors of each family of classical orthogonal polynomials, as well as the polynomials $\phi(x)$ and $\psi(x)$ and the corresponding intervals of orthogonality.

Family	$\phi(x)$	$\psi(x)$	$w(x)$	(a, b)
Hermite	1	$-2x$	e^{-x^2}	$\mathbb{R}$
Laguerre	x	$\alpha + 1 - x$	$x^\alpha e^{-x}$	$(0, +\infty)$
Jacobi	$1 - x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$	$(1 - x)^\alpha (1 + x)^\beta$	$(-1, 1)$

Table 8.1. Pearson equations and classical weight functions.

It is important to note that $w(x)$ is a weight function for the Hermite case, for the Laguerre case when $\alpha > -1$, and for the Jacobi case when $\alpha, \beta > -1$.

Chapter 9

Classical functionals

Alternatively, classical orthogonal polynomials can be introduced as follows.

Definition 9.1. Let $\mathbf{u}$ be a quasidefinite moment functional, and let $(P_n(x))_{n\geq 0}$ be a sequence of orthogonal polynomials associated with $\mathbf{u}$. Then $\mathbf{u}$ is classical if there exist nonzero polynomials $\phi(x)$ and $\psi(x)$ with $\deg \phi(x) \leq 2$, $\deg \psi(x) = 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}. \tag{9.1}$$

The sequence $(P_n(x))_{n\geq 0}$ is called a sequence of classical orthogonal polynomials.

We remark that (9.1) must be understood in the distributional sense. That is, for every $p(x) \in \mathbb{P}$, the following must hold:

$$\big\langle D\big(\phi(x)\mathbf{u}\big), p(x)\big\rangle = \big\langle \psi(x)\mathbf{u}, p(x)\big\rangle,$$

or, equivalently,

$$\big\langle \mathbf{u}, \phi(x)p'(x) + \psi(x)p(x)\big\rangle = 0.$$

The definition of classical moment functionals in terms of the distributional Pearson equation not only encompasses positive definite moment functionals (Jacobi, Laguerre, Hermite) associated with weight functions, but includes the nonpositive definite case as well. Considering the nonpositive definite case gives rise to the Bessel classical moment functional satisfying the distributional Pearson equation (9.1) with $\phi(x) = x^2$ and $\psi(x) = ax + 2$. The Bessel functional is quasidefinite when $a \neq -1, -2, \ldots$. Moreover, it has the following integral representation:

$$\big\langle \mathbf{u}, p(x)\big\rangle = \int_c p(z)w(z)\,dz, \quad p(x) \in \mathbb{P},$$

where $w(z) = (2\pi i)^{-1} z^{a-2} e^{-2/z}$, and c is the unit circle oriented counterclockwise.

The following result is a direct consequence of (9.1) and the quasidefinite character of $\mathbf{u}$.

Lemma 9.2. *The functional* $\mathbf{u}$ *satisfies* (9.1) *if and only if its sequence of moments* $(\mu_n)_{n\geq 0}$ *satisfies the three-term relation*

$$d_n\mu_{n+1} + e_n\mu_n + n\phi(0)\mu_{n-1} = 0, \quad n \geq 0, \tag{9.2}$$

where

$$d_n = \frac{1}{2}n\phi'' + \psi', \quad e_n = n\phi'(0) + \psi(0).$$

If $\mathbf{u}$ *is quasidefinite, then* $d_n \neq 0$ *for* $n \geq 0$ *and, thus,* (9.2) *is a second-order recurrence relation.*

Proof. If $\mathbf{u}$ satisfies (9.1), then, for $n \geq 0$,

$$\big\langle D\big(\phi(x)\mathbf{u}\big), x^n\big\rangle = \big\langle \psi(x)\mathbf{u}, x^n\big\rangle,$$

equivalently,

$$\big\langle \mathbf{u}, n\phi x^{n-1} + \psi x^n\big\rangle = 0.$$

Writing

$$\phi(x) = \frac{1}{2}\phi''(0)x^2 + \phi'(0)x + \phi(0), \quad \psi(x) = \psi'(0)x + \psi(0)$$

and substituting in the previous equation, we obtain

$$\left\langle \mathbf{u}, \frac{1}{2}n\phi''(0)x^{n+1} + n\phi'(0)x^n + n\phi(0)x^{n-1} + \psi'(0)x^{n+1} + \psi(0)x^n \right\rangle = 0,$$

and so (9.2) follows. It is easy to verify that the implication in the opposite direction holds by inverting each of the previous steps.

Additionally, suppose that $\mathbf{u}$ is quasidefinite and let $(P_n(x))_{n\geq 0}$ be an orthogonal polynomial sequence associated with $\mathbf{u}$. If $\deg \phi(x) < 2$, then $d_n = \psi'(0) \neq 0$, $n \geq 0$, since $\deg \psi(x) = 1$. On the other hand, if $\deg \phi(x) = 2$, we have that $\phi(x)P_m''(x) + \psi(x)P_m'(x)$ is a polynomial of degree at most m. From (9.1) we obtain

$$\begin{aligned}
\big\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\big\rangle &= \big\langle \mathbf{u}, \phi(x)\big(P_n(x)P_k'(x)\big)'\big\rangle - \big\langle \mathbf{u}, \phi(x)P_n(x)P_k''(x)\big\rangle \\
&= -\big\langle D\big(\phi(x)\mathbf{u}\big), P_n(x)P_k'(x)\big\rangle - \big\langle \mathbf{u}, \phi(x)P_n(x)P_k''(x)\big\rangle \\
&= -\big\langle \psi(x)\mathbf{u}, P_n(x)P_k'(x)\big\rangle - \big\langle \mathbf{u}, \phi(x)P_n(x)P_k''(x)\big\rangle \\
&= -\big\langle \mathbf{u}, P_n(x)\big(\phi(x)P_k''(x) + \psi(x)P_k'(x)\big)\big\rangle.
\end{aligned}$$

Therefore, $\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\rangle = 0$ when $n > k$. Similarly,

$$\big\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\big\rangle = -\big\langle \mathbf{u}, \big(\phi(x)P_n''(x) + \psi(x)P_n'(x)\big)P_k(x)\big\rangle,$$

and, thus, $\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\rangle = 0$ when $n < k$.

Then,

$$\big\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\big\rangle = \begin{cases} 0, & k \neq n, \\ -nd_{n-1}\langle \mathbf{u}, P_n^2(x)\rangle, & k = n, \end{cases} \quad n, k \geq 0.$$

Since $(P_{n+1}'(x))_{n\geq 0}$ constitutes a basis for $\mathbb{P}$, we can write

$$P_{n+2}(x) = \sum_{k=0}^{n+2} a_{n,k}P_{k+1}'(x), \quad a_{n,n+2} \neq 0, \; n \geq 0.$$

Multiplying both sides of this expansion by $\phi(x)P_{n+1}'(x)$ and applying $\mathbf{u}$, we get

$$\begin{aligned}
\big\langle \mathbf{u}, \phi(x)P_{n+1}'(x)P_{n+2}(x)\big\rangle &= a_{n,n}\big\langle \mathbf{u}, \phi(x)P_{n+1}'(x)P_{n+1}'(x)\big\rangle \\
&= -a_{n,n}nd_n\big\langle \mathbf{u}, P_{n+1}^2(x)\big\rangle, \quad n \geq 0.
\end{aligned}$$

But $(P_n(x))_{n\geq 0}$ is a sequence of orthogonal polynomials and $\deg\phi(x) = 2$, then the left-hand side is not equal to zero, hence, $d_n \neq 0$ for $n \geq 0$. ■

Exercise 9.1. Let $P_n(x) = k_n x^n +$ "terms of lower degree" be an orthogonal polynomial of degree n associated with the quasidefinite moment functional $\mathbf{u}$. Observe that $\langle \mathbf{u}, P_n^2(x)\rangle = k_n\langle \mathbf{u}, x^n P_n(x)\rangle$. Show that if $\mathbf{u}$ satisfies (9.1), then

$$\big\langle \mathbf{u}, \phi(x)P_n'(x)P_n'(x)\big\rangle = -n d_{n-1}\big\langle \mathbf{u}, P_n^2(x)\big\rangle,$$

where $d_n = \frac{1}{2}n\phi'' + \psi'$.

Now, we present the characterizations of classical orthogonal polynomials obtained from Definition 9.1. The polynomials $\phi(x)$ and $\psi(x)$ that appear in each of the characterizations are the polynomial coefficients in (9.1). First, we need the following result.

Lemma 9.3. *Let $\mathbf{u}$ be a quasidefinite functional and let $(P_n(x))_{n\geq 0}$ be its associated MOPS. If there exist real numbers $(\lambda_n)_{n\geq 0}$ such that*

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) = \lambda_n P_n(x), \quad n \geq 0,$$

then,

$$D^2\big(\phi(x)\mathbf{u}\big) - D\big(\psi(x)\mathbf{u}\big) = 0.$$

Proof. For $n \geq 0$,

$$\begin{aligned}\big\langle D^2\big(\phi(x)\mathbf{u}\big) - D\big(\psi(x)\mathbf{u}\big), P_n(x)\big\rangle &= \big\langle \mathbf{u}, \phi(x)P_n''(x) + \psi(x)P_n'(x)\big\rangle \\ &= \big\langle \mathbf{u}, \lambda_n P_n(x)\big\rangle = 0.\end{aligned}$$

Hence, $D^2(\phi(x)\mathbf{u}) - D(\psi(x)\mathbf{u})$ is the null functional over the linear space of polynomials $\mathbb{P}$. ■

Theorem 9.4 (Bochner [9]). *Let $\mathbf{u}$ be a quasidefinite functional and $(P_n(x))_{n\geq 0}$ its associated MOPS. Then $\mathbf{u}$ is classical if and only if there exist nonzero polynomials $\phi(x)$ and $\psi(x)$, with $\deg\phi(x) \leq 2$ and $\deg\psi(x) = 1$, such that, for $n \geq 0$, $P_n(x)$ satisfies*

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) = \lambda_n P_n(x), \tag{9.3}$$

where $\lambda_n = n(\frac{n-1}{2}\phi'' + \psi')$.

Proof. Suppose that $\mathbf{u}$ is a classical functional. On the one hand, from the Pearson equation (8.2), we get

$$\big\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\big\rangle = \begin{cases} 0, & k \neq n, \\ -n d_{n-1}\langle \mathbf{u}, P_n^2(x)\rangle, & n = k, \end{cases} \quad n \geq 0,$$

where $d_n = \frac{1}{2}n\phi'' + \psi'$. On the other hand,

$$\begin{aligned}\left\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\right\rangle &= \left\langle \mathbf{u}, \phi(x)\left(\left[P_n'(x)P_k(x)\right]' - P_n''(x)\right)P_k(x)\right\rangle \\ &= -\left\langle \mathbf{u}, \left(\phi(x)P_n''(x) + \psi(x)P_n'(x)\right)P_k(x)\right\rangle, \\ &\qquad 0 \le k \le n,\ n \ge 0,\end{aligned}$$

and, since $d_n \neq 0,\ n \geq 0$ (see Lemma 9.2), we have that

$$\deg\left(\phi(x)P_n''(x) + \psi(x)P_n'(x)\right) = n, \quad n \ge 1.$$

We deduce that $\phi(x)P_n''(x) + \psi(x)P_n'(x)$ is an orthogonal polynomial with respect to $\mathbf{u}$,

$$\left\langle \mathbf{u}, \left(\phi(x)P_n''(x) + \psi(x)P_n'(x)\right)P_k(x)\right\rangle = \begin{cases} 0, & k \neq n, \\ n d_{n-1}\langle \mathbf{u}, P_n^2(x)\rangle, & n = k, \end{cases} \quad n \geq 0,$$

and, therefore, is proportional to $P_n(x)$. That is, for $n \geq 1$, there exists a real number $\lambda_n \neq 0$ such that

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) = \lambda_n P_n(x).$$

The explicit expression of λ_n in terms of ϕ'' and ψ' is obtained by comparing coefficients on both sides of (9.3).

Now suppose that $(P_n(x))_{n\geq 0}$ satisfies (9.3). Then, by Lemma 9.3, we have $D^2(\phi(x)\mathbf{u}) - D(\psi(x)\mathbf{u}) = 0$. For $n \geq 2$ and $n = 0$,

$$\begin{aligned} 0 &= \left\langle D^2\left(\phi(x)\mathbf{u}\right) - D\left(\psi(x)\mathbf{u}\right), xP_n(x)\right\rangle \\ &= \left\langle \mathbf{u}, \phi(x)\left(xP_n(x)\right)'' + \psi(x)\left(xP_n(x)\right)'\right\rangle \\ &= \left\langle \mathbf{u}, x\left(\phi(x)P_n''(x) + \psi(x)P_n'(x)\right)\right\rangle + 2\left\langle \mathbf{u}, \phi(x)P_n'(x)\right\rangle + \left\langle \mathbf{u}, \psi(x)P_n(x)\right\rangle \\ &= \lambda_n\left\langle \mathbf{u}, xP_n(x)\right\rangle + 2\left\langle \mathbf{u}, \phi(x)P_n'(x)\right\rangle + \left\langle \mathbf{u}, \psi(x)P_n(x)\right\rangle \\ &= 2\left\langle \mathbf{u}, \phi(x)P_n'(x)\right\rangle + \left\langle \mathbf{u}, \psi(x)P_n(x)\right\rangle. \end{aligned}$$

Moreover,

$$\begin{aligned} 0 &= \left\langle D^2\left(\phi(x)\mathbf{u}\right) - D\left(\psi(x)\mathbf{u}\right), P_1(x)\right\rangle = \left\langle \mathbf{u}, \phi(x)P_1''(x) + \psi(x)P_1'(x)\right\rangle \\ &= \left\langle \mathbf{u}, \psi(x)\right\rangle, \end{aligned}$$

that is, $\psi(x)$ is orthogonal to $P_0(x) = 1$. Therefore, for $n \geq 2$ and $n = 0$,

$$\left\langle \psi(x)\mathbf{u}, P_n(x)\right\rangle = \left\langle \mathbf{u}, \psi(x)P_n(x)\right\rangle = 0$$

and

$$\left\langle D\left(\phi(x)\mathbf{u}\right), P_n(x)\right\rangle = -\left\langle \mathbf{u}, \phi(x)P_n'(x)\right\rangle = \frac{1}{2}\left\langle \mathbf{u}, \psi(x)P_n(x)\right\rangle = 0.$$

For $n = 1$, we must show that $\langle D(\phi(x)\mathbf{u}), P_1(x)\rangle = \langle \psi(x)\mathbf{u}, P_1(x)\rangle$, or, equivalently,

$$\big\langle \mathbf{u}, \phi(x)P_1'(x) + \psi(x)P_1(x)\big\rangle = \big\langle \mathbf{u}, \phi(x) + x\psi(x)\big\rangle = 0.$$

Note that

$$0 = \big\langle D^2\big(\phi(x)\mathbf{u}\big) - D\big(\psi(x)\mathbf{u}\big), x^2\big\rangle = 2\big\langle \mathbf{u}, \phi(x) + x\psi(x)\big\rangle.$$

It follows that $D(\phi(x)\mathbf{u})$ and $\psi(x)\mathbf{u}$ coincide over the basis $(P_n(x))_{n\geq 0}$. ∎

In fact, the Jacobi, Laguerre, Hermite, and Bessel polynomials satisfy the following second-order linear differential equations:

$$\begin{aligned}
&\text{(Jacobi)} && (1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]y'(x) \\
& && \quad = -n(n+\alpha+\beta+1)y(x),\\
&\text{(Bessel)} && x^2y''(x) + (ax+2)y'(x) = n(n+a-1)y(x),\\
&\text{(Laguerre)} && xy''(x) + (\alpha+1-x)y'(x) = -ny(x),\\
&\text{(Hermite)} && y''(x) - 2xy'(x) = -2ny(x).
\end{aligned}$$

Note that from the proof of Lemma 9.2 we obtain an orthogonality relation for $(P_{n+1}'(x))_{n\geq 0}$. That is, if $\mathbf{u}$ is a classical functional and $(P_n(x))_{n\geq 0}$ is its associated sequence of monic orthogonal polynomials, then

$$\big\langle \mathbf{u}, \phi(x)P_n'(x)P_k'(x)\big\rangle = \begin{cases} 0, & n \neq k,\\ -nd_{n-1}\langle \mathbf{u}, P_n^2(x)\rangle, & n = k, \end{cases} \qquad n,k \geq 1. \tag{9.4}$$

In 1935, W. Hahn [25] characterized the classical orthogonal polynomials as those sequences of orthogonal polynomials whose derivatives are again orthogonal. We state this characterization in the following theorem.

Theorem 9.5 (Hahn [25]). *Let $(P_n(x))_{n\geq 0}$ be the sequence of monic orthogonal polynomials associated with the functional $\mathbf{u}$. Then $\mathbf{u}$ is classical if and only if there exists a nonzero polynomial $\phi(x)$ with $\deg\phi(x) \leq 2$ such that $\big(\frac{P_{n+1}'(x)}{n+1}\big)_{n\geq 0}$ is the sequence of monic orthogonal polynomials associated with the functional $\mathbf{v} = \phi(x)\mathbf{u}$.*

Proof. Let $\mathbf{u}$ be a classical functional and $(P_n(x))_{n\geq 0}$ its associated sequence of monic orthogonal polynomials. Moreover, define $\mathbf{v} = \phi(x)\mathbf{u}$ with $\phi(x)$ satisfying (9.1). For $n,k \geq 1$,

$$\begin{aligned}
\big\langle \mathbf{v}, P_n'(x)P_k'(x)\big\rangle &= \big\langle \phi(x)\mathbf{u}, P_n'(x)P_k'(x)\big\rangle = \big\langle \mathbf{u}, \phi P_n'(x)P_k'(x)\big\rangle\\
&= \begin{cases} 0, & n \neq k,\\ nd_{n-1}\langle \mathbf{u}, P_n^2(x)\rangle, & n = k. \end{cases}
\end{aligned}$$

Then, $\big(\frac{P_{n+1}'(x)}{n+1}\big)_{n\geq 0}$ is the sequence of monic orthogonal polynomials associated with $\mathbf{v}$.

Now, suppose that $\left(\frac{P'_{n+1}(x)}{n+1}\right)_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to $\mathbf{v}$. Since

$$\left(\frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u}\right)_{n\geq 0}$$

constitutes a basis for the dual space of $\mathbb{P}$, we can write

$$D\big(\phi(x)\mathbf{u}\big) = \sum_{n=0}^{\infty} a_n \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u},$$

where

$$a_n = \big\langle D\big(\phi(x)\mathbf{u}\big), P_n(x)\big\rangle = -\big\langle \phi(x)\mathbf{u}, P'_n(x)\big\rangle, \quad n \geq 0.$$

Clearly, $a_0 = 0$. From the orthogonality of the derivatives, we get

$$a_1 = -\big\langle \phi(x)\mathbf{u}, P'_1(x)\big\rangle = -\big\langle \mathbf{v}, P'_1(x)^2\big\rangle \neq 0$$

and $a_n = 0$ for $n \geq 2$. Hence,

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u},$$

with

$$\psi(x) = -\frac{\langle \mathbf{v}, P'_1(x)^2\rangle}{\langle \mathbf{u}, P_1^2(x)\rangle} P_1(x). \qquad \blacksquare$$

We must remark that under the hypotheses of the previous theorem, the quasi-definite functional $\mathbf{v} = \phi(x)\mathbf{u}$ is classical. Indeed, for every $q(x) \in \mathbb{P}$,

$$\begin{aligned}
\big\langle D\big(\phi(x)\mathbf{v}\big), q(x)\big\rangle &= -\big\langle \mathbf{v}, \phi(x)q'(x)\big\rangle \\
&= -\big\langle \phi(x)\mathbf{u}, \big(\phi(x)q(x)\big)'\big\rangle - \big\langle \mathbf{v}, \phi'(x)q(x)\big\rangle \\
&= \big\langle \phi(x)D\big(\phi(x)\mathbf{u}\big), q(x)\big\rangle - \big\langle \phi'(x)\mathbf{v}, q(x)\big\rangle \\
&= \big\langle \big(\psi(x) - \phi'(x)\big)\mathbf{v}, q(x)\big\rangle.
\end{aligned}$$

Since $\deg \phi(x) \leq 2$ and $\deg(\psi(x) - \phi'(x)) = 1$, the functional $\mathbf{v}$ satisfies the Pearson equation

$$D\big(\phi(x)\mathbf{v}\big) = \big(\psi(x) + \phi'(x)\big)\mathbf{v}.$$

Iterating this idea, we obtain the following results.

Corollary 9.6 ([25, 33, 34]). *Let* $\mathbf{u}$ *be a classical functional satisfying* (9.1) *and* $(P_n(x))_{n\geq 0}$ *its associated sequence of monic orthogonal polynomials. For each* $k \geq 1$, *let* $\mathbf{v}_k$ *be the functional* $\mathbf{v}_k = \phi^k(x)\mathbf{u}$ *and* $(Q_{n,k}(x))_{n\geq 0}$ *be the sequence of monic polynomials given by*

$$Q_{n,k}(x) = \frac{P^{(k)}_{n+k}(x)}{(n+1)_k}, \quad n \geq 0,$$

where $P_n^{(k)}(x)$ *denotes the* k*th derivative of* $P_n(x)$. *Then, for each* $k \geq 1$, $(Q_{n,k}(x))_{n\geq 0}$ *is the sequence of monic orthogonal polynomials associated with the functional* $\mathbf{v}_k$, *satisfying*

$$D\big(\phi(x)\mathbf{v}_k\big) = \big(\psi(x) + k\phi'(x)\big)\mathbf{v}_k.$$

Hence, $\mathbf{v}_k$ *is a classical functional.*

Observe that if $\mathbf{u}$ be a classical functional and $(P_n(x))_{n\geq 0}$ its associated MOPS, then

$$\big\langle \mathbf{u}, \phi(x)\big\rangle = \big\langle \mathbf{v}, P_1'(x)^2\big\rangle \neq 0.$$

Furthermore, the following result implies that $\langle \mathbf{u}, \phi^n(x)\rangle \neq 0$ for $n \geq 0$.

Corollary 9.7 ([42]). *Let* $\mathbf{u}$ *be a classical functional and* $(P_n(x))_{n\geq 0}$ *its associated sequence of monic orthogonal polynomials. Then,*

$$\big\langle \mathbf{u}, \phi^n(x)\big\rangle = \frac{(-1)^n}{n!}\prod_{j=0}^{n-1} d_{n+j-1}\big\langle \mathbf{u}, P_n^2(x)\big\rangle, \quad n \geq 0, \tag{9.5}$$

where $d_k = \dfrac{1}{2}k\phi'' + \psi'$.

Proof. The result is trivial for $n = 0$. Suppose that, for every classical functional $\mathbf{u}$, (9.5) holds for $n \geq 1$ fixed. We will prove the result by induction on n. By Corollary 9.6, the functional $\mathbf{v} = \phi(x)\mathbf{u}$ is classical and satisfies $D(\phi(x)\mathbf{v}) = \tilde{\psi}(x)\mathbf{v}$ with

$$\tilde{\psi}(x) = \psi(x) + \phi'(x) = \big(\psi' + \phi''\big)x + \big(\psi(0) + \phi'(0)\big).$$

Thus, $\mathbf{v}$ is classical of the same type as $\mathbf{u}$, and

$$Q_n(x) = \frac{P_{n+1}'(x)}{n+1}, \quad n \geq 0,$$

is its associated sequence of monic orthogonal polynomials. By the induction hypothesis, we can write

$$\big\langle \mathbf{v}, \phi^n(x)\big\rangle = \frac{(-1)^n}{n!}\prod_{j=0}^{n-1}\left[(n-1+j)\frac{1}{2}\phi'' + \tilde{\psi}'\right]\big\langle \mathbf{v}, Q_n^2(x)\big\rangle.$$

Using (9.4), we obtain

$$\begin{aligned}\big\langle \mathbf{u}, \phi^{n+1}(x)\big\rangle = \big\langle \mathbf{v}, \phi^n(x)\big\rangle &= \frac{(-1)^n}{n!}\prod_{j=0}^{n-1} d_{n+j+1}\big[-(n+1)d_n\big]\big\langle \mathbf{u}, P_{n+1}^2(x)\big\rangle \\ &= \frac{(-1)^{n+1}}{(n+1)!}\prod_{j=0}^{n} d_{n+j}\big\langle \mathbf{u}, P_{n+1}^2(x)\big\rangle.\end{aligned}$$

■

Example 9.8. For $\alpha, \beta > -1$, consider the classical moment functional $\mathbf{u}_{\alpha,\beta}$ associated with the (monic) Jacobi polynomials $(P_n^{(\alpha,\beta)}(x))_{n\geq 0}$. Then, the functional $\mathbf{u}_{\alpha,\beta}$ is defined by

$$\langle \mathbf{u}_{\alpha,\beta}, p(x) \rangle = \int_{-1}^{1} p(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx, \quad p(x) \in \mathbb{P}.$$

By Theorem 9.5, we have that the sequence of monic polynomials

$$\left(\frac{(P_{n+1}^{(\alpha,\beta)}(x))'}{n+1} \right)_{n\geq 0}$$

constitutes a MOPS associated with the moment functional $\mathbf{u}_{\alpha+1,\beta+1} = (1-x^2)\mathbf{u}_{\alpha,\beta}$. Since $\mathbf{u}_{\alpha+1,\beta+1}$ is the moment functional associated with the Jacobi polynomials of parameters $(\alpha+1, \beta+1)$, we have that, for $n \geq 0$,

$$P_n^{(\alpha+1,\beta+1)}(x) = \frac{(P_{n+1}^{(\alpha,\beta)}(x))'}{n+1}.$$

Furthermore, from Corollary 9.6, we have that for $k \geq 1$,

$$P_n^{(\alpha+k,\beta+k)}(x) = \frac{(P_{n+k}^{(\alpha,\beta)}(x))^{(k)}}{(n+1)_k}, \quad n \geq 0.$$

Similarly, let $(B_n^{(a)}(x))_{n\geq 0}$, $(L_n^{(\alpha)}(x))_{n\geq 0}$, and $(H_n(x))_{n\geq 0}$ be the Bessel, Laguerre, and Hermite monic polynomials, respectively. Then, for $k \geq 1$ and $n \geq 0$,

$$\begin{aligned} B_n^{(a+k)}(x) &= \frac{(B_{n+k}^{(a)}(x))^{(k)}}{(n+1)_k}, \\ L_n^{(\alpha+k)}(x) &= \frac{(L_{n+k}^{(\alpha)}(x))^{(k)}}{(n+1)_k}, \\ H_n(x) &= \frac{H_{n+k}^{(k)}(x)}{(n+1)_k}. \end{aligned}$$

As a consequence of the orthogonality of the higher-order derivatives of classical orthogonal polynomials, we obtain the following characterization in terms of a family of bilinear forms involving derivatives of increasing order.[1] We say that $(P_n(x))_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to a bilinear form

$$(\cdot,\cdot) : \mathbb{P} \times \mathbb{P} \to \mathbb{R},$$

[1] Orthogonality with respect to a bilinear form involving derivatives is known in the literature as *Sobolev orthogonality*. A survey of the main ideas and recent developments in Sobolev orthogonal polynomials can be found in [43] (see also [20]).

if

$$\big(P_n(x), P_k(x)\big) = \begin{cases} 0, & n \neq k, \\ h_n \neq 0, & n = k. \end{cases}$$

Proposition 9.9. *Let* $\mathbf{u}$ *be a quasidefinite functional and* $(P_n(x))_{n\geq 0}$ *its corresponding sequence of monic orthogonal polynomials. Then* $\mathbf{u}$ *is classical if and only if* $(P_n(x))_{n\geq 0}$ *is a sequence of orthogonal polynomials with respect to every bilinear form* $(\cdot\,,\cdot)_N$ *defined on* $\mathbb{P}$ *as*

$$\big(p(x), q(x)\big)_N = \left\langle \mathbf{u}, p(x)q(x) + \sum_{k=1}^{N} \lambda_k \phi^k(x) p^{(k)}(x) q^{(k)}(x) \right\rangle, \quad N \geq 1,$$

where $\lambda_k > 0$ *are real numbers and* $p^{(n)}(x)$ *denotes the derivative of order* n*.*

Furthermore, for every $N \geq 1$*, there exists a linear differential operator* L_N *of order* $2N$ *such that the following statements hold for every* $p(x), q(x) \in \mathbb{P}$*:*

(1) $(p(x), q(x))_N = \langle \mathbf{u}, p(x) L_N[q(x)] \rangle$,

(2) $(L_N[p(x)], q(x))_N = (p(x), L_N[q(x)])_N$, *and*

(3) $\deg L_N[q(x)] \leq \deg q(x)$.

That is, L_N *is symmetric with respect to* $(\cdot\,,\cdot)_N$ *and does not increase the degree of any polynomial.*

Proof. If $\mathbf{u}$ is classical, then the orthogonality of $(P_n(x))_{n\geq 0}$ with respect to $(\cdot\,,\cdot)_N$ for $N \geq 1$ follows from Corollary 9.6.

Suppose that $(P_n(x))_{n\geq 0}$ is orthogonal with respect to $(\cdot\,,\cdot)_N$ for every $N \geq 1$. Since

$$\left(\frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x) \rangle} \mathbf{u} \right)_{n\geq 0}$$

constitutes a basis for the dual space of $\mathbb{P}$, we can write

$$D\big(\phi(x)\mathbf{u}\big) = \sum_{n=0}^{\infty} a_n \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x) \rangle} \mathbf{u},$$

where

$$a_n = \big\langle D\big(\phi(x)\mathbf{u}\big), P_n(x) \big\rangle = -\big\langle \mathbf{u}, \phi(x) P_n'(x) \big\rangle, \quad n \geq 0.$$

Clearly, $a_0 = 0$. If $N \geq 2$, then for $n \geq 2$,

$$\begin{aligned} a_n &= -\big\langle \mathbf{u}, \phi(x) P_n'(x) \big\rangle \\ &= -\lambda_1^{-1} \Bigg[\big(P_n(x), P_1(x)\big)_N - \big\langle \mathbf{u}, P_n(x) P_1(x) \big\rangle \\ &\qquad - \sum_{k=2}^{N} \lambda_k \big\langle \mathbf{u}, \phi^k(x) P_n^{(k)}(x) P_1^{(k)}(x) \big\rangle \Bigg] \\ &= 0, \end{aligned}$$

and if $N = 1$, then

$$a_n = -\lambda_1^{-1}\left[\left(P_n(x), P_1(x)\right)_N - \left\langle \mathbf{u}, P_n(x)P_1(x)\right\rangle\right] = 0, \quad n \geq 2.$$

That is, $a_n = 0$, $n \geq 2$, independently of N.

Finally, $a_1 = -\langle \mathbf{u}, \phi(x)P_1'(x)\rangle = -\langle \mathbf{u}, \phi(x)\rangle = -d_0\langle \mathbf{u}, P_0^2(x)\rangle \neq 0$ (see Corollary 9.7). Hence,

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u},$$

with

$$\psi(x) = -\frac{\langle \mathbf{u}, \phi(x)\rangle}{\langle \mathbf{u}, P_1^2(x)\rangle} P_1(x).$$

Now, we prove the existence of the linear operator L_N using induction. In fact, we will show that

$$L_N\big[q(x)\big] = q(x) + \sum_{k=1}^{N}\sum_{j=0}^{k}(-1)^k\lambda_k\binom{k}{j}\psi_{k,k-j}(x)\phi^j(x)q^{(k+j)}(x), \quad \forall q(x) \in \mathbb{P},$$

where

$$\begin{aligned}
\psi_{k,0}(x) &= 1,\\
\psi_{k,m}(x) &= \phi\psi_{k,m-1}'(x) + \psi_{k,m-1}(x)\big[\psi(x) + (k-m)\phi'(x)\big], \quad 0 \leq m \leq k,
\end{aligned}$$

are polynomials with $\deg \psi_{k,m}(x) = m$.

Observe that $\deg L_N[q(x)] \leq \deg q(x)$. Furthermore,

$$D^m\big(\phi^k(x)\mathbf{u}\big) = \psi_{k,m}(x)\phi^{k-m}(x)\mathbf{u}, \quad 0 \leq m \leq k,\ k \geq 0,$$

and

$$\begin{aligned}
L_1\big[q(x)\big] &= q(x) - \lambda_1\psi(x)q'(x) - \lambda_1\phi(x)q''(x),\\
L_N\big[q(x)\big] &= L_{N-1}\big[q(x)\big]\\
&\qquad + (-1)^N\lambda_N\sum_{k=0}^{N}\binom{N}{k}\psi_{N,N-k}(x)\phi^k(x)q^{(N+k)}(x),\ N \geq 2.
\end{aligned}$$

For $N = 1$ and every $p(x), q(x) \in \mathbb{P}$, using the Pearson equation for $\mathbf{u}$, we obtain

$$\begin{aligned}
\big(p(x), q(x)\big)_1 &= \big\langle \mathbf{u}, p(x)q(x)\big\rangle + \lambda_1\big\langle \mathbf{u}, \phi(x)p'(x)q'(x)\big\rangle\\
&= \big\langle \mathbf{u}, p(x)\big[q(x) - \lambda_1\psi(x)q'(x) - \lambda_1\phi(x)q''(x)\big]\big\rangle\\
&= \big\langle \mathbf{u}, p(x)L_1\big[q(x)\big]\big\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\big(p(x), q(x)\big)_1 &= \big\langle \mathbf{u}, \big[p(x) - \lambda_1\psi(x)p'(x) - \lambda_1\phi(x)p''(x)\big]q(x)\big\rangle\\
&= \big\langle \mathbf{u}, L_1\big[p(x)\big]q(x)\big\rangle.
\end{aligned}$$

Therefore, (1), (2), and (3) hold with L_1.

Now, assume that (1), (2), and (3) hold with L_{N-1} for some $N \geq 2$. Then,

$$\big(p(x), q(x)\big)_N = \big\langle \mathbf{u}, p(x) L_{N-1}\big[q(x)\big]\big\rangle + \lambda_N \big\langle \mathbf{u}, \phi^N(x) p^{(N)}(x) q^{(N)}(x)\big\rangle,$$

for every $p(x), q(x) \in \mathbb{P}$. Using the Leibniz product rule, it is not hard to verify that

$$p^{(N)}(x) q^{(N)}(x) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \big(p(x) q^{(N+k)}(x)\big)^{(N-k)}$$

and, thus,

$$\begin{aligned}
&\big\langle \mathbf{u}, \phi^N(x) p^{(N)}(x) q^{(N)}(x)\big\rangle \\
&\quad = \left\langle (-1)^N \sum_{k=0}^{N} \binom{N}{k} q^{(N+k)}(x) D^{N-k}\big(\phi^N(x)\mathbf{u}\big), p(x)\right\rangle \\
&\quad = \left\langle \mathbf{u}, p(x)\left[(-1)^N \lambda_N \sum_{k=0}^{N} \binom{N}{k} \psi_{N,N-k}(x) \phi^k(x) q^{(N+k)}(x)\right]\right\rangle.
\end{aligned}$$

Hence, $(p(x), q(x))_N = \langle \mathbf{u}, p(x) L_N[q(x)]\rangle$. Similarly,

$$(p(x), q(x))_N = \langle \mathbf{u}, L_N[p(x)] q(x)\rangle,$$

and the result follows. ■

The following two results are called *structure relations* which are satisfied by classical orthogonal polynomials and their derivatives.

Theorem 9.10 ([3]). *Let $(P_n(x))_{n\geq 0}$ be the sequence of monic orthogonal polynomials associated with the quasidefinite functional* $\mathbf{u}$. *Then* $\mathbf{u}$ *is classical if and only if there exist a nonzero polynomial $\phi(x)$ with* $\deg \phi(x) \leq 2$ *and real numbers a_n, b_n, c_n, $n \geq 1$, with $c_n \neq 0$, such that*

$$\phi(x) P_n'(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 1. \tag{9.6}$$

Proof. Suppose that $\mathbf{u}$ is classical and let $\phi(x)$ and $\psi(x)$ be the polynomials satisfying (9.1). Since $\phi(x) P_n'(x)$ is a polynomial of degree at most $n+1$, we can write

$$\phi(x) P_n'(x) = \sum_{k=0}^{n+1} a_{n,k} P_k(x),$$

where

$$\begin{aligned}
a_{n,k}\big\langle \mathbf{u}, P_k^2(x)\big\rangle &= \big\langle \mathbf{u}, \phi P_n'(x) P_k(x)\big\rangle \\
&= -\big\langle \mathbf{u}, P_n(x)\big(\phi(x) P_k'(x) + \psi(x) P_k(x)\big)\big\rangle, \quad 0 \leq k \leq n+1.
\end{aligned}$$

But as $\deg\phi(x) \le 2$ and $\deg\psi(x) = 1$, $\phi(x)P_k'(x) + \psi(x)P_k(x)$ is a polynomial of degree $k+1$ (since $d_n \ne 0,\ n \ge 0$, see Lemma 9.2) and, therefore, $a_{n,k} = 0$ for $k \le n-2$. Thus,

$$\phi(x)P_n'(x) = \sum_{k=n-1}^{n+1} a_{n,k}P_k(x) = a_{n,n+1}P_{n+1}(x) + a_{n,n}P_n(x) + a_{n,n-1}P_{n-1}(x).$$

Moreover, since $\phi(x)P_{n-1}'(x) + \psi(x)P_{n-1}(x)$ is a polynomial of degree n, we get $a_{n,n-1} \ne 0$.

Now, suppose that $(P_n(x))_{n\ge 0}$ satisfies (9.6) with $\phi(x)$ some nonzero polynomial of $\deg\phi(x) \le\ 2$. Since

$$\left(\frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u}\right)_{n\ge 0}$$

constitutes a basis for the dual space of $\mathbb{P}$, we can write

$$D\big(\phi(x)\mathbf{u}\big) = \sum_{k=0}^{\infty} \alpha_k \frac{P_k(x)}{\langle \mathbf{u}, P_k^2(x)\rangle}\mathbf{u},$$

where

$$\alpha_k = \big\langle D\big(\phi(x)\mathbf{u}\big), P_k(x)\big\rangle = -\big\langle \phi(x)\mathbf{u}, P_k'(x)\big\rangle = -\big\langle \mathbf{u}, \phi(x)P_k'(x)\big\rangle, \quad k \ge 0.$$

Therefore,

$$\begin{aligned} \alpha_k &= -a_k\big\langle \mathbf{u}, P_{k+1}(x)\big\rangle - b_k\big\langle \mathbf{u}, P_k(x)\big\rangle - c_k\big\langle \mathbf{u}, P_{k-1}(x)\big\rangle = 0, \quad k \ge 2,\\ \alpha_1 &= -c_1\big\langle \mathbf{u}, P_0(x)\big\rangle \ne 0,\\ \alpha_0 &= -b_0\big\langle \mathbf{u}, P_0(x)\big\rangle. \end{aligned}$$

It follows that $\mathbf{u}$ satisfies

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u},$$

with

$$\psi(x) = -\left(\frac{c_1}{\langle \mathbf{u}, P_1^2(x)\rangle}P_1(x) + \frac{b_0}{\langle \mathbf{u}, P_0^2(x)\rangle}P_0(x)\right)\big\langle \mathbf{u}, P_0(x)\big\rangle.$$

Moreover, $c_1 \ne 0$ implies that $\deg\psi(x) = 1$. ∎

Theorem 9.11 ([24,39]). *Let $(P_n(x))_{n\ge 0}$ be the sequence of monic orthogonal polynomials associated with the quasidefinite functional $\mathbf{u}$. Then, $\mathbf{u}$ is classical if and only if there exist real numbers r_n and s_n, $n \ge 2$, such that*

$$P_n(x) = \frac{P_{n+1}'(x)}{n+1} + r_n\frac{P_n'(x)}{n} + s_n\frac{P_{n-1}'(x)}{n-1}, \quad n \ge 2. \tag{9.7}$$

Proof. Suppose that $\mathbf{u}$ is classical. Since $(\frac{P'_{n+1}(x)}{n+1})_{n\geq 0}$ is a basis for $\mathbb{P}$, we can write

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} \frac{P'_{k+1}(x)}{k+1}, \quad n \geq 2.$$

Multiplying both sides by $\phi(x)\frac{P'_{k+1}(x)}{k+1}$ and applying $\mathbf{u}$, we get

$$a_{n,k} = \frac{\langle \mathbf{u}, \phi(x) P'_{k+1}(x) P_n(x)\rangle/(k+1)}{\langle \mathbf{u}, \phi(x)(P'_{k+1}(x))^2\rangle/(k+1)^2} = \frac{\langle \mathbf{u}, \phi(x) P'_{k+1}(x) P_n(x)\rangle}{-d_k \langle \mathbf{u}, P^2_{k+1}(x)\rangle},$$

where $d_n = \frac{1}{2}n\phi'' + \psi'$. Since $\deg(\phi(x)P'_{k+1}(x)) \leq k+2$, $a_{n,k} = 0$ for $k \leq n-3$. Therefore,

$$\begin{aligned} P_n(x) &= \sum_{k=n-2}^{n} a_{n,k} \frac{P'_{k+1}(x)}{k+1} \\ &= a_{n,n}\frac{P'_{n+1}(x)}{n+1} + a_{n,n-1}\frac{P'_n(x)}{n} + a_{n,n-2}\frac{P'_{n-1}(x)}{n-1}. \end{aligned}$$

Conversely, suppose that $(P_n(x))_{n\geq 0}$ satisfies (9.7). We must show that there exist nonzero polynomials $\phi(x)$ and $\psi(x)$, with $\deg \phi(x) \leq 2$ and $\deg \psi(x) = 1$, such that $\mathbf{u}$ satisfies (9.1). Let $(\mathbf{v}_n)_{n\geq 0}$ be a basis of the dual space of $\mathbb{P}$ associated with

$$Q_n(x) = \frac{P'_{n+1}(x)}{n+1}, \quad n \geq 0,$$

that is,

$$\langle \mathbf{v}_n, Q_k(x)\rangle = \delta_{n,k}.$$

We can write

$$\mathbf{v}_n = \sum_{k=0}^{\infty} a_{n,k} \frac{P_k(x)}{\langle \mathbf{u}, P_k^2(x)\rangle}\mathbf{u}, \quad n \geq 0,$$

where

$$\begin{aligned} a_{n,k} &= \langle \mathbf{v}_n, P_k(x)\rangle = \langle \mathbf{v}_n, Q_k(x)\rangle + r_k\langle \mathbf{v}_n, Q_{k-1}(x)\rangle + s_k\langle \mathbf{v}_n, Q_{k-2}(x)\rangle \\ &= 0, \quad k \neq n, n+1, n+2, \\ a_{n,n} &= 1, \quad a_{n,n+1} = r_{n+1}, \quad a_{n+2} = s_{n+2}. \end{aligned}$$

Therefore, for every $n \geq 0$,

$$\mathbf{v}_n = \left(s_{n+2}\frac{P_{n+2}(x)}{\langle \mathbf{u}, P^2_{n+2}(x)\rangle} + r_{n+1}\frac{P_{n+1}(x)}{\langle \mathbf{u}, P^2_{n+1}(x)\rangle} + \frac{P_n(x)}{\langle \mathbf{u}, P^2_n(x)\rangle}\right)\mathbf{u}.$$

On the other hand, for $n \geq 0$, $D\mathbf{v}_n = -(n+1)P_{n+1}(x)\mathbf{u}/\langle \mathbf{u}, P_{n+1}^2(x)\rangle$. Taking $n = 0$, we get that $\mathbf{u}$ satisfies (9.1) with

$$\begin{aligned}
\phi(x) &= \frac{s_2}{\langle \mathbf{u}, P_2^2(x)\rangle} P_2(x) + \frac{r_1}{\langle \mathbf{u}, P_1^2(x)\rangle} P_1(x) + \frac{1}{\langle \mathbf{u}, P_0^2(x)\rangle} P_0(x), \\
\psi(x) &= -\frac{P_1(x)}{\langle \mathbf{u}, P_1^2(x)\rangle}.
\end{aligned}$$

■

Example 9.12. Recall the connection formulas for monic Laguerre polynomials presented in Example 7.8:

$$\begin{aligned}
L_n^{(\alpha)}(x) &= L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x), \\
xL_n^{(\alpha+1)}(x) &= L_{n+1}^{(\alpha)}(x) + (\alpha+n+1)L_n^{(\alpha)}(x).
\end{aligned}$$

Since $L_n^{(\alpha+1)}(x) = \frac{(L_{n+1}^{(\alpha)}(x))'}{n+1}$, $n \geq 0$ (see Example 9.8), we deduce that the monic Laguerre polynomials satisfy (9.6) and (9.7):

$$\begin{aligned}
L_n^{(\alpha)}(x) &= \frac{1}{n+1}\big(L_{n+1}^{(\alpha)}(x)\big)' + \big(L_n^{(\alpha)}(x)\big)', \quad n \geq 0, \\
x\big(L_n^{(\alpha)}(x)\big)' &= nL_n^{(\alpha)}(x) + n(\alpha+n)L_{n-1}^{(\alpha)}(x), \quad n \geq 1.
\end{aligned}$$

The following characterization is known as the *distributional Rodrigues formula* for classical orthogonal polynomials.

Theorem 9.13 (Tricomi [57]). *Let $(P_n(x))_{n\geq 0}$ be the sequence of monic orthogonal polynomials associated with the quasidefinite functional $\mathbf{u}$. Then $\mathbf{u}$ is classical if and only if, for every $n \geq 0$, there exist a nonzero polynomial $\phi(x)$ with $\deg \phi(x) \leq 2$ and a real number $k_n \neq 0$ such that*

$$D^n\big(\phi^n(x)\mathbf{u}\big) = k_n P_n(x)\mathbf{u}. \tag{9.8}$$

Proof. Suppose that (9.8) holds for $n \geq 0$. The classical character of $\mathbf{u}$ follows immediately by taking $n = 1$.

Now, suppose that $\mathbf{u}$ is classical. Using the basis of the dual space of $\mathbb{P}$ associated with $(P_n(x))_{n\geq 0}$, we can write

$$D^n\big(\phi^n(x)\mathbf{u}\big) = \sum_{k=0}^{\infty} a_{n,k} \frac{P_k(x)}{\langle \mathbf{u}, P_k^2(x)\rangle}\mathbf{u}, \quad n \geq 1,$$

where

$$a_{n,k} = \big\langle D^n\big(\phi^n(x)\mathbf{u}\big), P_k(x)\big\rangle.$$

For $k < n$, we have

$$a_{n,k} = \big\langle D^n\big(\phi^n(x)\mathbf{u}\big), P_k(x)\big\rangle = (-1)^n\big\langle \phi^n(x)\mathbf{u}, P_k^{(n)}(x)\big\rangle = 0.$$

Moreover, by Corollary 9.6, we have $a_{n,k} = 0$ for $k > n$. Hence, (9.8) holds with $k_0 = 1$ and

$$k_n = (-1)^n n! \langle \mathbf{u}, \phi^n(x) \rangle.$$

From Corollary 9.7, $k_n \neq 0$. ■

Example 9.14. Let $\mathbf{u}$ be the classical moment functional associated with the Hermite polynomials defined by

$$\langle \mathbf{u}, p(x) \rangle = \int_{-\infty}^{\infty} p(x) e^{-x^2}\, dx, \quad p(x) \in \mathbb{P}.$$

Then $\mathbf{u}$ satisfies (9.8) with $\phi(x) = 1$ and $k_n = (-2)^n$, $n \geq 0$, that is, $D^n(\mathbf{u}) = (-2)^n H_n(x)\mathbf{u}$, where $H_n(x)$ is the monic Hermite polynomial of degree n.

Many properties of classical orthogonal polynomials can be deduced from (9.8). For instance, the three-term recurrence relation for Hermite polynomials can be deduced using $D^n(\mathbf{u}) = (-2)^n H_n(x)\mathbf{u}$. Indeed, for $p(x) \in \mathbb{P}$,

$$\begin{aligned}
\langle (-2)^n H_n(x)\mathbf{u}, p(x) \rangle &= \langle D^n(\mathbf{u}), p(x) \rangle \\
&= -\langle D^{n-1}(\mathbf{u}), p'(x) \rangle \\
&= -\langle (-2)^{n-1} H_{n-1}(x)\mathbf{u}, p'(x) \rangle \\
&= -(-2)^{n-1} \langle \mathbf{u}, H_{n-1}(x) p'(x) \rangle \\
&= -(-2)^{n-1} \int_{-\infty}^{\infty} H_{n-1}(x) p'(x) e^{-x^2}\, dx.
\end{aligned}$$

Now, integrating by parts, we obtain

$$\begin{aligned}
\langle (-2)^n H_n(x)\mathbf{u}, p(x) \rangle &= -(-2)^{n-1} H_{n-1}(x) p(x) e^{-x^2} \Big|_{-\infty}^{\infty} \\
&\quad + (-2)^{n-1} \int_{-\infty}^{\infty} p(x) \big(H_{n-1}(x) e^{-x^2}\big)'\, dx \\
&= (-2)^{n-1} \int_{-\infty}^{\infty} p(x) \big(H'_{n-1}(x) - 2x H_{n-1}(x)\big) e^{-x^2}\, dx \\
&= \langle (-2)^{n-1} \big(H'_{n-1}(x) - 2x H_{n-1}(x)\big)\mathbf{u}, p(x) \rangle.
\end{aligned}$$

That is, $(-2H_n(x) - H'_{n-1}(x) + 2x H_{n-1}(x))\mathbf{u} = \mathbf{0}$. Since $(n-1)H_{n-2}(x) = H'_{n-1}(x)$ and $\mathbf{u}$ is a quasidefinite moment functional, we obtain the three-term recurrence relation

$$H_n(x) = x H_{n-1}(x) - \frac{n-1}{2} H_{n-2}(x).$$

Exercise 9.2. In Example 9.14, we used the quasidefinite character of $\mathbf{u}$ to deduce the three-term recurrence relation for the Hermite polynomials. Show that if $\mathbf{u}$ is a quasidefinite moment functional and if $q(x)\mathbf{u} = \mathbf{0}$ for some $q(x) \in \mathbb{P}$, then $q(x) \equiv 0$. *Hint:* Let $(P_n(x))_{n\geq 0}$ be an OPS associated with $\mathbf{u}$. Then $q(x)$ can be written as

$q(x) = \sum_{k=0}^{\deg q(x)} a_k P_k(x)$ for some real numbers a_k. Show that $a_k = 0$, $0 \leq k \leq \deg q(x)$.

Now, we state an important result for classical functionals.

Theorem 9.15 (Marcellán, Branquinho, and Petronilho [39]; Maroni [48]). *Let* **u** *be a classical functional and* $(P_n(x))_{n\geq 0}$ *the sequence of monic orthogonal polynomials associated with* **u**. *Then, the Stieltjes function associated with* **u**,

$$\mathcal{S}(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}},$$

is a formal solution of the first-order linear differential equation

$$\phi(z)\mathcal{S}'(z) + \big[\phi'(z) - \psi(z)\big]\mathcal{S}(z) + \left(\frac{1}{2}\phi'' - \psi'\right)\mu_0 = 0.$$

Proof. Multiplying both sides of (9.2) by z^{-n-1}, and then summing over $n \geq 0$, we obtain

$$\sum_{n=0}^{\infty}\left(\frac{n}{2}\phi'' + \psi'\right)\frac{\mu_{n+1}}{z^{n+1}} + \sum_{n=0}^{\infty}\big(n\phi'(0) + \psi(0)\big)\frac{\mu_n}{z^{n+1}} + \sum_{n=0}^{\infty} n\phi(0)\frac{\mu_{n-1}}{z^{n+1}} = 0.$$

Since

$$\mathcal{S}'(z) = -\sum_{n=0}^{\infty}(n+1)\frac{\mu_n}{z^{n+2}},$$

we have that

$$\phi(z)\mathcal{S}'(z) + \big[(\phi'' - \psi')z - \phi'(0) - \psi(0)\big]\mathcal{S}(z) + \left(\frac{1}{2}\phi'' - \psi'\right)\mu_0 = 0,$$

and the result follows. ∎

Chapter 10

Electrostatic interpretation for the zeros of classical orthogonal polynomials

In this chapter, we discuss an interesting interpretation for the zeros of classical orthogonal polynomials. We will see here that for a certain electric potential $V(x)$, the zeros of a classical orthogonal polynomial of degree n determine the positions of electrostatic equilibrium for a system of unitary charged particles under the influence of $V(x)$ (see [54]).

Before starting our discussion, let us establish some facts. We consider a system of charged particles in a straight line under the interaction of a logarithmic potential. In this model, for each point x, the logarithmic potential corresponding to a positive charged particle q located at a point c is given by

$$E(x) = q \ln \frac{1}{|x - c|}.$$

Let us consider a system of n unitary charged particles in the presence of an external potential $V(x)$. The particles are constrained to the interval $[a, b]$. Let x_1, $x_2, \dots, x_n$ be the positions of the particles. The total energy of the system is

$$E(x_1, \dots, x_n) = \sum_{k=1}^{n} V(x_k) + 2 \sum_{1 \le i < j \le n} \ln \frac{1}{|x_i - x_j|}.$$

10.1 Equilibrium points on a bounded interval with charged end points

The following electrostatic equilibrium problem was studied by T. J. Stieltjes.

Let us consider a system of n unitary charged particles constrained to be located in the interval $(-1, 1)$, and two external particles with charges $q > 0$ and $p > 0$, located at $x = -1$ and $x = 1$, respectively. In this case, the external potential is given by

$$V(x) = p \ln \frac{1}{|1 - x|} + q \ln \frac{1}{|1 + x|}$$

and the total energy of the system is given by

$$E(x_1, \dots, x_n) = p \sum_{k=1}^{n} \ln \frac{1}{(1 - x_k)} + q \sum_{k=1}^{n} \ln \frac{1}{(1 + x_k)} + 2 \sum_{1 \le \ell < k \le n} \ln \frac{1}{|x_\ell - x_k|},$$

where $x_i \in (-1, 1)$, $i = 1, \dots, n$, are the positions of the n charged particles.

The equilibrium points correspond to a local minimum of the total energy of the system. Therefore, we must find the critical points of the energy by solving the following system of equations:

$$\frac{\partial E}{\partial x_i} = \frac{p}{1-x_i} - \frac{q}{1+x_i} + \sum_{\substack{k=1\\k\neq i}}^{n} \frac{2}{x_k - x_i} = 0, \quad 1 \le i \le n. \tag{10.1}$$

Let

$$f(x) = \prod_{k=1}^{n} (x - x_k).$$

We have

$$\begin{aligned}\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{x_k - x_i} &= \lim_{x\to x_i} \left(\frac{1}{x - x_i} - \frac{f'(x)}{f(x)} \right)\\ &= \lim_{x\to x_i} \left(\frac{f(x) - (x-x_i) f'(x)}{(x-x_i) f(x)} \right).\end{aligned}$$

Using L'Hôpital's rule, we obtain

$$\begin{aligned}\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{x_k - x_i} &= \lim_{x\to x_i} \frac{-(x-x_i) f''(x)}{(x-x_i) f'(x) + f(x)}\\ &= -\lim_{x\to x_i} \frac{f''(x) + (x-x_i) f'''(x)}{2f'(x) + (x-x_i) f''(x)}\\ &= -\frac{f''(x_i)}{2f'(x_i)},\end{aligned}$$

that is,

$$\sum_{\substack{k=1\\k\neq i}}^{n} \frac{2}{x_k - x_i} = -\frac{f''(x_i)}{f'(x_i)}. \tag{10.2}$$

From (10.1) and (10.2), we deduce that the polynomial of degree n,[1]

$$(1-x^2) f''(x) - \big[(1+x)p - (1-x)q\big] f'(x),$$

vanishes at the x_i's, and, therefore,

$$(1-x^2) f''(x) - \big[(1+x)p - (1-x)q\big] f'(x) = \lambda f(x), \quad \lambda \in \mathbb{R} \setminus \{0\}, \tag{10.3}$$

that is, $f(x)$ is a ${}_2F_1$ hypergeometric polynomial.

[1] The leading coefficient is

$$-n(n-1) - (p+q)n = -n(n-1+p+q) \neq 0.$$

Recall that the Jacobi polynomials $(P_n^{(\alpha,\beta)}(x))_{n\geq 0}$ satisfy (10.3) with $p = \alpha + 1$ and $q = \beta + 1$ (see Table 8.1). This means that the zeros of $P_n^{(\alpha,\beta)}(x)$ are located at the equilibrium points of the n unitary charged particles under the influence of the potential $V(x)$.

To establish that the energy has a local minimum at the x_i's, we must study the Hessian matrix $H = (h_{i,j})_{i,j=1}^n$, where

$$h_{i,j} = \frac{\partial^2 E}{\partial x_i \partial x_j} = \begin{cases} \dfrac{-2}{(x_j - x_i)^2}, & i \neq j, \\ \displaystyle\sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{2}{(x_k - x_i)^2} + \frac{p}{(1 - x_i)^2} + \frac{q}{(1 + x_i)^2}, & i = j. \end{cases}$$

This implies that H is a real symmetric matrix such that

$$h_{j,j} > \sum_{\substack{k=1 \\ k\neq j}}^{n} |h_{j,k}|.$$

This means that H is a diagonally dominant matrix. Therefore, H is positive definite (see [29, Theorem 6.1.10]) and E has a minimum at the zeros of the polynomial solution of (10.3).

10.2 Equilibrium points on the complex plane: The Bessel case

Next we will study the energy of a system formed by a dipole with charge of $(a + 1)$ at the origin and a charge of $(c - a)$ at the point $\frac{1}{a}$, with $a \to \infty$ (see [59]).

Let us consider n unit positive charges located at the points $\{z_k\}_{k=1}^n$ on the complex plane. Then, the electrostatic energy of the system is

$$\begin{aligned} E(z_1, \ldots, z_n) &= \lim_{a\to\infty} \Bigg[(a+1) \sum_{k=1}^{n} \ln \frac{1}{|z_k|} + (c - a) \sum_{k=1}^{n} \ln \frac{1}{|z_k - \frac{1}{a}|} \\ &\quad + 2 \sum_{1\leq \ell<k\leq n} \ln \frac{1}{|z_\ell - z_k|} \Bigg] \\ &= \lim_{a\to\infty} \left[a \sum_{k=1}^{n} \ln \left| \frac{z_k - \frac{1}{a}}{z_k} \right| + c \sum_{k=1}^{n} \ln \frac{1}{|z_k - \frac{1}{a}|} \right] \\ &\quad + \sum_{k=1}^{n} \ln \frac{1}{|z_k|} + 2 \sum_{1\leq \ell<k\leq n} \ln \frac{1}{|z_\ell - z_k|} \end{aligned}$$

$$= \lim_{a\to\infty} a \sum_{k=1}^{n} \ln\left|1 - \frac{1}{a z_k}\right| + (c+1)2 \sum_{k=1}^{n} \ln \frac{1}{|z_k|}$$
$$+ \sum_{1\le \ell<k\le n} \ln \frac{2}{|z_\ell - z_k|}.$$

Now, we write

$$a \ln\left|1 - \frac{1}{a z_k}\right| = \frac{1}{2} a \ln\left(1 - \frac{1}{a z_k}\right)\left(1 - \frac{1}{a \bar{z}_k}\right)$$
$$= -\frac{1}{2} z_k^{-1} \frac{\ln(1 - \frac{1}{a z_k})}{-\frac{1}{a z_k}} - \frac{1}{2} \bar{z}_k^{-1} \frac{\ln(1 - \frac{1}{a \bar{z}_k})}{-\frac{1}{a \bar{z}_k}}.$$

Therefore,

$$\lim_{a\to\infty} a \ln\left|1 - \frac{1}{a z_k}\right| = -\frac{1}{2}\left(z_k^{-1} + \bar{z}_k^{-1}\right) = -\frac{\operatorname{Re} z_k}{|z_k|^2}$$

and

$$E(z_1, \ldots, z_n) = (c+1) \sum_{k=1}^{n} \ln \frac{1}{|z_k|} - \sum_{k=1}^{n} \frac{\operatorname{Re} z_k}{|z_k|^2} + \sum_{1\le \ell<k\le n} \ln \frac{2}{|z_\ell - z_k|}$$
$$= \sum_{k=1}^{n} \ln \frac{1}{|z_k|^{c+1} e^{-|z_k|^{-2} \operatorname{Re} z_k}} + 2 \sum_{1\le \ell<k\le n} \ln \frac{1}{|z_\ell - z_k|}.$$

In this case, the external potential is

$$V(z) = \ln \frac{1}{|z|^{c+1} e^{-|z|^{-2} \operatorname{Re} z}}.$$

Writing the complex numbers as $z_k = x_k + i y_k$, $1 \le k \le n$, we can find the critical points of $E(z_1, \ldots, z_n)$ by setting the partial derivatives $\partial E/\partial x_k$ and $\partial E/\partial y_k$, $1 \le k \le n$, to zero. Hendriksen and van Rossum proved in [28] that for the solution points $z_1, z_2, \ldots, z_n$, the polynomial

$$f(z) = \prod_{k=1}^{n} (z - z_k)$$

and the Bessel polynomial of degree n satisfy the same second-order differential equation. The Hessian matrix is not positive definite (see [59]), therefore the critical points of $E(z_1, \ldots, z_n)$ are saddle points. This means that the electrostatic (unstable) equilibrium points coincide with the zeros of the Bessel polynomial of degree n.

When the degree of the Bessel polynomials increases, the zeros approach a fixed curve on the complex plane (see [17]).

10.3 Classical orthogonal polynomials and the inverse problem

We can pose the inverse problem as follows:

- Let $(P_n(x))_{n\geq 0}$ be a sequence of classical orthogonal polynomials. For each $n \geq 1$, let $\{x_{n,i}\}_{i=1}^n$ be the zeros of the polynomial $P_n(x)$. If the electrostatic equilibrium points of n unitary charged particles are $\{x_{n,i}\}_{i=1}^n$, what is the external potential and the total energy of the system?

Let us consider the Laguerre and Hermite families. In both cases, the charged particles are constrained to be located in an unbounded interval.

Laguerre case. We will deduce the external potential of the system from the differential equation satisfied by the Laguerre polynomials.

The Laguerre orthogonal polynomials satisfy the differential equation

$$xy_n'' + (\alpha + 1 - x)y_n' = \lambda y_n, \quad \alpha > -1.$$

For $n \geq 1$, if $\{x_{n,i}\}_{i=1}^n \subset (0, \infty)$ are the zeros of the polynomial of degree n, then

$$x_{n,i}\, y_n''(x_{n,i}) + (\alpha + 1 - x_{n,i})y_n'(x_{n,i}) = 0, \quad 1 \leq i \leq n,$$

or, equivalently,

$$\frac{y_n''(x_{n,i})}{y_n'(x_{n,i})} + \frac{\alpha+1}{x_{n,i}} - 1 = 0, \quad 1 \leq i \leq n.$$

If we write

$$y_n(x) = \prod_{k=1}^{n} (x - x_{n,k}),$$

then we have

$$\sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{2}{x_{n,k} - x_{n,i}} = -\frac{y_n''(x_{n,i})}{y_n'(x_{n,i})}, \tag{10.4}$$

and, therefore,

$$\sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{2}{x_{n,k} - x_{n,i}} + 1 - \frac{\alpha+1}{x_{n,i}} = 0, \quad 1 \leq i \leq n.$$

It is not difficult to see that the above system of equations can be written as

$$\frac{\partial E}{\partial x_{n,i}} = 0, \quad 0 \leq i \leq n,$$

where

$$
\begin{aligned}
E(x_{n,1},\dots,x_{n,n}) &= 2\sum_{1\le \ell<k\le n} \ln\frac{1}{|x_{n,\ell}-x_{n,k}|} - \sum_{k=1}^{n}\big[(\alpha+1)\ln x_{n,k} - x_{n,k}\big] \\
&= 2\sum_{1\le \ell<k\le n} \ln\frac{1}{|x_{n,\ell}-x_{n,k}|} + \sum_{k=1}^{n}\ln\frac{1}{x_{n,k}} + \sum_{k=1}^{n}\ln\frac{1}{x_{n,k}^{\alpha}e^{-x_{n,k}}}.
\end{aligned}
$$

If E is interpreted as the total energy of the system of n unitary charged particles in electrostatic equilibrium, then we can deduce that the external potential $V(x)$ of the system is

$$
V(x) = \ln\frac{1}{x} + \ln\frac{1}{x^{\alpha}e^{-x}}, \qquad x>0,
$$

which corresponds to the combination of a potential generated by a unitary charged particle at the origin and another potential whose influence on $(0,\infty)$ is modeled by the function $w(x)=x^{\alpha}e^{-x}$ (see [30]).

Hermite case. Similarly, for $n\ge 1$, let $\{x_{n,i}\}_{i=1}^{n}\subset\mathbb{R}$ be the zeros of the Hermite polynomial of degree n. Since the Hermite orthogonal polynomials satisfy the differential equation

$$
y_n'' - 2xy_n' = \lambda y_n,
$$

we have

$$
\frac{y_n''(x_{n,i})}{y_n'(x_{n,i})} - 2x_{n,i} = 0, \qquad 1\le i\le n.
$$

Hence, if we write $y_n(x)=\prod_{k=1}^{n}(x-x_{n,k})$, then (10.4) holds and we obtain

$$
2\sum_{\substack{k=1\\k\ne i}}^{n}\frac{1}{|x_{n,k}-x_{n,i}|} + 2x_{n,k} = 0, \qquad 1\le i\le n.
$$

We deduce that the total energy of the system is

$$
E(x_1,\dots,x_n) = 2\sum_{1\le \ell<k\le n}\ln\frac{1}{|x_{n,\ell}-x_{n,k}|} + \sum_{k=1}^{n}\ln\frac{1}{e^{-x_{n,k}^2}},
$$

and the external potential $V(x)$ is

$$
V(x) = \ln\frac{1}{e^{-x^2}}.
$$

Remark 10.1. The weight function $w(x)=x^{\alpha}e^{-x}$ associated with the Laguerre polynomials appears in the expression for the external potential of the first system. Similarly, the external potential of the second system involves the weight function $w(x)=e^{-x^2}$ associated with the Hermite polynomials (see [30]).

Chapter 11

Semiclassical functionals

The semiclassical orthogonal polynomials were studied by P. Maroni in [45] as those sequences of orthogonal polynomials whose derivatives are quasiorthogonal polynomials (see also [20, 46]) as defined in Definition 11.7 below. Recently, the notion of semiclassical orthogonal polynomials has be extended to other types of orthogonality [5].

In this chapter, we introduce the semiclassical functionals as a natural generalization of the classical functionals (see [27]).

Definition 11.1. A quasidefinite functional $\mathbf{u}$ is semiclassical if there exist nonzero polynomials $\phi(x)$ and $\psi(x)$, with $\deg\phi(x) := r \geq 0$ and $\deg\psi(x) := t \geq 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}. \tag{11.1}$$

A sequence of orthogonal polynomials associated with $\mathbf{u}$ is called a semiclassical sequence of orthogonal polynomials.

Notice that $\deg\psi(x) \geq 1$, since if $\psi(x) = \psi_0 \neq 0$ is a nonzero constant, then

$$0 = -\langle\phi(x)\mathbf{u}, 0\rangle = \langle D\big(\phi(x)\mathbf{u}\big), 1\rangle = \langle\psi(x)\mathbf{u}, 1\rangle = \langle\mathbf{u}, \psi(x)\rangle = \psi_0\mu_0,$$

where $\mu_0 = \langle\mathbf{u}, 1\rangle$ is the first moment of the functional. But this implies that $\mu_0 = 0$, contradicting the quasidefinite character of $\mathbf{u}$. Additionally, we must require the polynomial $\phi(x)$ to be nonzero to ensure that $\mathbf{u}$ is quasidefinite.

In order to avoid any incompatibility with the quasidefinite character on the semiclassical functional $\mathbf{u}$, it will be required from now on that if

$$\phi(x) = a_r x^r + \cdots \quad \text{and} \quad \psi(x) = b_t x^t + \cdots,$$

then, for any $n \geq 0$, if $r = t + 1$, then

$$n a_r + b_{r-1} \neq 0, \tag{11.2}$$

so that every moment is well defined. To justify this, from Definition 11.1, we have

$$\langle D\big(\phi(x)\mathbf{u}\big), x^n\rangle = \langle\psi(x)\mathbf{u}, x^n\rangle, \quad n \geq 0,$$

then,

$$\langle\mathbf{u}, n\phi(x)x^{n-1} + \psi(x)x^n\rangle = 0, \quad n \geq 0.$$

Writing

$$\phi(x) = \sum_{k=0}^{r} a_k x^k, \qquad \psi(x) = \sum_{k=0}^{t} b_k x^k,$$

the Pearson equation is equivalent to the following recurrence relation for the moments:

$$n\sum_{k=0}^{r} a_k\mu_{n+k-1} + \sum_{k=0}^{t} b_k\mu_{n+k} = 0, \quad n \geq 0.$$

If $r = t + 1$, we have

$$(na_r + b_t)\mu_{n+t} = -\sum_{k=0}^{t-1}(na_{k+1} + b_k)\mu_{n+k} - na_0\mu_{n-1}, \quad n \geq 0. \tag{11.3}$$

Therefore, if there exists an n_0 such that $n_0a_r + b_t = 0$, then the corresponding moment μ_{n_0+t} cannot be determined by (11.3), a situation that can lead to some incompatibility with the quasidefinite character of $\mathbf{u}$.

If $\mathbf{u}$ is semiclassical, then (11.1) may not be minimal in the sense that the polynomials $\phi(x)$ and $\psi(x)$ are not unique. Indeed, for any nonzero polynomial $q(x)$, $\mathbf{u}$ satisfies

$$D\big(\tilde{\phi}(x)\mathbf{u}\big) = \tilde{\psi}(x)\mathbf{u},$$

where $\tilde{\phi}(x) = q(x)\phi(x)$ and $\tilde{\psi}(x) = q'(x)\phi(x) + q(x)\psi(x)$. Furthermore,

$$\begin{aligned}
&\deg q(x)\phi(x) = \deg q(x) + \deg \phi(x),\\
&\deg\big(q'(x)\phi(x) + q(x)\psi(x)\big) = \deg q(x) + \max\big\{\deg \phi(x) - 1, \deg \psi(x)\big\}.
\end{aligned}$$

The nonuniqueness of the Pearson equation satisfied by $\mathbf{u}$ motivates the following definition.

Definition 11.2. The class of a semiclassical functional $\mathbf{u}$ is defined as

$$\mathfrak{s}(\mathbf{u}) := \min \max\big\{\deg \phi(x) - 2, \deg \psi(x) - 1\big\}, \tag{11.4}$$

where the minimum is taken among all pairs of polynomials $\phi(x)$ and $\psi(x)$ such that $\mathbf{u}$ satisfies (11.1).

Hence, a quasidefinite functional $\mathbf{u}$ is classical if it is semiclassical of class $\mathfrak{s}(\mathbf{u}) = 0$.

For a semiclassical functional $\mathbf{u}$, it can be proved that the polynomials $\phi(x)$ and $\psi(x)$ such that the minimum of (11.4) is attained are unique up to a multiplicative constant.

Lemma 11.3. *Let* $\mathbf{u}$ *be a semiclassical functional such that*

$$D\big(\phi_1(x)\mathbf{u}\big) = \psi_1(x)\mathbf{u}, \quad s_1 := \max\big\{\deg \phi_1(x) - 2, \deg \psi_1(x) - 1\big\}, \tag{11.5}$$

$$D\big(\phi_2(x)\mathbf{u}\big) = \psi_2(x)\mathbf{u}, \quad s_2 := \max\big\{\deg \phi_2(x) - 2, \deg \psi(x)_2 - 1\big\}, \tag{11.6}$$

where $\phi_i(x)$ *and* $\psi_i(x)$, $i = 1, 2$, *are nonzero polynomials with* $\deg \phi_i(x) \geq 0$ *and* $\deg \psi_i(x) \geq 1$. *Let* $\phi(x)$ *be the greatest common divisor of* $\phi_1(x)$ *and* $\phi_2(x)$.

Then there exists a polynomial $\psi(x)$ *such that*

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}, \quad s := \max\big\{\deg\phi(x) - 2, \deg\psi(x) - 1\big\}.$$

Moreover, $s - \deg\phi(x) = s_1 - \deg\phi_1(x) = s_2 - \deg\phi_2(x)$.

Proof. By the hypothesis, there exist polynomials $\tilde{\phi}_1(x)$ and $\tilde{\phi}_2(x)$ such that $\phi_1(x) = \phi(x)\tilde{\phi}_1(x)$ and $\phi_2(x) = \phi(x)\tilde{\phi}_2(x)$. If $\phi_1(x)$ and $\phi_2(x)$ are coprime, then set $\phi(x) = 1$. From (11.5) and (11.6), we obtain

$$\tilde{\phi}_2 D(\phi_1\mathbf{u}) - \tilde{\phi}_1 D(\phi_2\mathbf{u}) = (\tilde{\phi}_2\psi_1 - \tilde{\phi}_1\psi_2)\mathbf{u}. \tag{11.7}$$

Observe that, for any polynomial $p(x) \in \mathbb{P}$,

$$\begin{aligned}
&\big\langle\tilde{\phi}_2(x) D\big(\phi_1(x)\mathbf{u}\big) - \tilde{\phi}_1(x) D\big(\phi_2(x)\mathbf{u}\big), p(x)\big\rangle \\
&\quad = \big\langle\mathbf{u}, -\phi_1(x)\big(\tilde{\phi}_2(x)p(x)\big)' + \phi_2(x)\big(\tilde{\phi}_1(x)p(x)\big)'\big\rangle \\
&\quad = \big\langle\mathbf{u}, \big(\phi_2(x)\tilde{\phi}_1'(x) - \phi_1(x)\tilde{\phi}_2'(x)\big)p(x)\big\rangle \\
&\qquad + \big\langle\mathbf{u}, \phi(x)\big(\tilde{\phi}_1(x)\tilde{\phi}_2(x) - \tilde{\phi}_2(x)\tilde{\phi}_1(x)\big)p'(x)\big\rangle \\
&\quad = \big\langle\mathbf{u}, \big(\phi_2(x)\tilde{\phi}_1'(x) - \phi_1(x)\tilde{\phi}_2'(x)\big)p(x)\big\rangle \\
&\quad = \big\langle\big(\phi_2(x)\tilde{\phi}_1'(x) - \phi_1(x)\tilde{\phi}_2'(x)\big)\mathbf{u}, p(x)\big\rangle.
\end{aligned}$$

Therefore, (11.7) becomes

$$\big(\phi_2(x)\tilde{\phi}_1'(x) - \phi_1(x)\tilde{\phi}_2'(x) - \tilde{\phi}_2(x)\psi_1(x) + \tilde{\phi}_1(x)\psi_2(x)\big)\mathbf{u} = 0.$$

Since $\mathbf{u}$ is quasidefinite, we get

$$\phi_2(x)\tilde{\phi}_1'(x) - \phi_1(x)\tilde{\phi}_2'(x) - \tilde{\phi}_2(x)\psi_1(x) + \tilde{\phi}_1(x)\psi_2(x) = 0,$$

or, equivalently, $(\tilde{\phi}_1'(x)\phi(x) - \psi_1(x))\tilde{\phi}_2(x) = (\tilde{\phi}_2'(x)\phi(x) - \psi_2(x))\tilde{\phi}_1(x)$.

But $\tilde{\phi}_1(x)$ and $\tilde{\phi}_2(x)$ are coprime polynomials. Hence, there exists a polynomial $\psi(x)$ such that

$$\tilde{\phi}_1'(x)\phi(x) - \psi_1(x) = -\psi(x)\tilde{\phi}_1(x), \quad \tilde{\phi}_2'(x)\phi(x) - \psi_2(x) = -\psi(x)\tilde{\phi}_2(x). \tag{11.8}$$

Since $\phi_1(x) = \phi(x)\tilde{\phi}_1(x)$ and $\phi_2(x) = \phi(x)\tilde{\phi}_2(x)$, (11.5) and (11.6) can be written as

$$\begin{aligned}
\tilde{\phi}_1(x) D\big(\phi(x)\mathbf{u}\big) + \big(\tilde{\phi}_1'(x)\phi(x) - \psi_1(x)\big)\mathbf{u} = 0, \\
\tilde{\phi}_2(x) D\big(\phi(x)\mathbf{u}\big) + \big(\tilde{\phi}_2(x)\phi(x) - \psi_2(x)\big)\mathbf{u} = 0.
\end{aligned}$$

Using (11.8), we write

$$\tilde{\phi}_1(x)\big(D\big(\phi(x)\mathbf{u}\big) - \psi(x)\mathbf{u}\big) = 0, \quad \tilde{\phi}_2(x)\big(D\big(\phi(x)\mathbf{u}\big) - \psi(x)\mathbf{u}\big) = 0.$$

From Bézout identity for coprime polynomials, we have that there exist polynomials $a(x)$ and $b(x)$ such that $a(x)\tilde{\phi}_1(x) + b(x)\tilde{\phi}_2(x) = 1$. With this in mind,

$$D\big(\phi(x)\mathbf{u}\big) = \phi(x)\mathbf{u}.$$

Finally, observe that from (11.8)

$$\begin{aligned}\deg\psi_1 - 1 &= \deg\tilde{\phi}_1 + \max\big\{\deg\phi - 2, \deg\psi - 1\big\}\\ &= \deg\phi_1 + \max\big\{\deg\phi - 2, \deg\psi - 1\big\} - \deg\phi\\ &= \deg\phi_1 + s - \deg\phi.\end{aligned}$$

Besides as $s - \deg\phi > \deg\phi_1 - 2 - \deg\phi_1$, then also $s - \deg\phi = s_1 - \deg\phi_1$. The other equality is obtained in a similar way. ∎

Theorem 11.4 ([48]). *For any semiclassical linear functional* $\mathbf{u}$*, the polynomials* $\phi(x)$ *and* $\psi(x)$ *in* (11.1) *such that*

$$\mathfrak{s}(\mathbf{u}) = \max\big\{\deg\phi(x) - 2, \deg\psi(x) - 1\big\}$$

are unique up to a constant factor.

Proof. Suppose that $\mathbf{u}$ satisfies (11.1) with $\phi_i(x)$ and $\psi_i(x)$, $i = 1, 2$, and suppose that $\mathfrak{s}(\mathbf{u}) = \max\{\deg\phi_i(x) - 2, \deg\psi_i(x) - 1\}$, $i = 1, 2$. If in Lemma 11.3 we take $s_1 = s_2$, then $s = s_1 = s_2$. But this implies that $\deg\phi(x) = \deg\phi_1(x) = \deg\phi_2(x)$, or, equivalently, $\phi(x) = \phi_1(x) = \phi_2(x)$. Notice also that $\psi(x)$ is unique up to a constant factor. ∎

The polynomials $\phi(x)$ and $\psi(x)$ such that

$$\mathfrak{s}(\mathbf{u}) = \max\big\{\deg\phi(x) - 2, \deg\psi(x) - 1\big\}$$

are characterized in the following result.

Proposition 11.5 ([45]). *Let* $\mathbf{u}$ *be a semiclassical linear functional and let* $\phi(x)$ *and* $\psi(x)$ *be nonzero polynomials, with* $\deg\phi(x) := r$ *and* $\deg\psi(x) := t$*, such that* (11.1) *holds. Let* $s := \max\{r-2, t-1\}$*. Then* $s = \mathfrak{s}(\mathbf{u})$ *if and only if*

$$\prod_{c:\phi(c)=0}\Big(\big|\psi(c) - \phi'(c)\big| + \big|\big\langle\mathbf{u}, \theta_c\psi(x) - \theta_c^2\phi(x)\big\rangle\big|\Big) > 0. \tag{11.9}$$

Here, $\theta_c f(x) = \frac{f(x)-f(c)}{x-c}$.

Proof. Let c be a zero of $\phi(x)$, then there exists a polynomial $\phi_c(x)$ of degree $r-1$ such that $\phi(x) = (x-c)\phi_c(x)$. On the other hand, since

$$\theta_c^2\phi(x) = \frac{\phi(x) - \phi(c)}{(x-c)^2} - \frac{\phi'(c)}{x-c},$$

we get

$$\psi(x) - \phi_c(x) = (x - c)\psi_c(x) + r_c,$$

where

$$\psi_c(x) = \theta_c\psi(x) - \theta_c^2\phi(x), \quad r_c = \psi(c) - \phi'(c).$$

With this in mind, (11.1) can be written as $(x - c)D(\phi_c(x)\mathbf{u}) = (x - c)\psi_c(x)\mathbf{u} + r_c\mathbf{u}$. From here, we obtain

$$\begin{aligned} D\big(\phi_c(x)\mathbf{u}\big) &= \psi_c(x)\mathbf{u} + \frac{r_c}{(x-c)}\mathbf{u} + \big\langle \mathbf{u}, \psi_c(x)\big\rangle\boldsymbol{\delta}_c \\ &= \psi_c(x)\mathbf{u} + \frac{\psi(c) - \phi'(c)}{(x-c)}\mathbf{u} + \big\langle \mathbf{u}, \theta_c\psi(x) - \theta_c^2\phi(x)\big\rangle\boldsymbol{\delta}_c. \end{aligned}$$

We proceed to the proof of the proposition.

Suppose that $\mathfrak{s}(\mathbf{u}) = s$, $r_c = 0$, and $\langle \mathbf{u}, \psi_c(x)\rangle = 0$ for some c such that $\phi(c) = 0$. Then $D(\phi_c(x)\mathbf{u}) = \psi_c(x)\mathbf{u}$. But $\deg \phi_c(x) = r - 1$ and $\deg \psi_c(x) = t - 1$. This means that $\mathfrak{s}(\mathbf{u}) = s - 1$, which is a contradiction.

Now, suppose that (11.9) holds and that $\mathbf{u}$ is of class $\tilde{s} \le s$, with $D(\tilde{\phi}(x)\mathbf{u}) = \tilde{\psi}(x)\mathbf{u}$. From Lemma 11.3, there exists a polynomial $\rho(x)$ such that

$$\phi(x) = \rho(x)\tilde{\phi}(x), \quad \psi(x) = \rho(x)\tilde{\psi}(x) + \rho'(x)\tilde{\phi}(x).$$

If $\tilde{s} < s$, then necessarily $\deg \rho(x) \ge 1$. Let c be a zero of $\rho(x)$ and let $\rho_c(x)$ be the polynomial such that $\rho(x) = (x - c)\rho_c(x)$. Then,

$$\psi(x) - \phi_c(x) = (x - c)\big(\rho_c(x)\tilde{\psi}(x) + \rho_c'(x)\tilde{\phi}(x)\big).$$

It follows that

$$r_c = 0, \quad \psi_c(x) = \rho_c(x)\tilde{\psi}(x) + \rho_c'(x)\tilde{\phi}(x).$$

Hence,

$$\big\langle \mathbf{u}, \psi_c(x)\big\rangle = \big\langle \mathbf{u}, \tilde{\psi}(x)\rho_c(x)\big\rangle + \big\langle \mathbf{u}, \rho_c'(x)\tilde{\phi}(x)\big\rangle = \big\langle D\big(\tilde{\phi}(x)\mathbf{u}\big) - \tilde{\psi}(x)\mathbf{u}, \rho_c(x)\big\rangle = 0.$$

But this means that $\phi(c) = 0$ and

$$\big|\psi(c) - \phi'(c)\big| + \big|\big\langle \mathbf{u}, \theta_c\psi - \theta_c^2\phi\big\rangle\big| = 0,$$

which contradicts (11.9). Thus, $s = \tilde{s}$ and, by Theorem 11.4, $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ are multiples of $\phi(x)$ and $\psi(x)$, respectively, up to a constant factor. ∎

Proposition 11.6 ([26, 47]). *Let* $\mathbf{u}$ *be a quasidefinite functional. The following statements are equivalent:*

(1) $\mathbf{u}$ *is semiclassical.*

(2) *There exist two nonzero polynomials* $\phi(z)$ *and* $\psi(z)$*, with* $\deg\phi(z) := r \geq 0$ *and* $\deg\psi(z) =: t \geq 1$*, such that the Stieltjes function associated with* $\mathbf{u}$*, namely*

$$\mathcal{S}(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}},$$

satisfies

$$\phi(z)\mathcal{S}'(z) + \big(\phi'(z) - \psi(z)\big)\mathcal{S}(z) = C(z), \tag{11.10}$$

where

$$C(z) = \big(\mathbf{u} * \theta_0(\phi' - \psi)\big)(z) + (D\mathbf{u} * \theta_0\phi)(z).$$

The operator θ_0 *and the polynomial* $(\mathbf{u} * p)(x)$ *are defined in Definitions 2.4 and 2.9, respectively.*

Proof. (1) $\Rightarrow$ (2) Let $\mathbf{u}$ be a semiclassical functional of class s satisfying (8.2). For

$$\phi(z) = \sum_{k=0}^{r} \frac{\phi^{(k)}(0)}{k!} z^k, \quad \psi(z) = \sum_{m=0}^{t} \frac{\psi^{(m)}(0)}{m!} z^m,$$

we have

$$\begin{aligned} 0 = \big\langle D\big(\phi(x)\mathbf{u}\big) - \psi(x)\mathbf{u}, x^n\big\rangle &= -\big\langle \mathbf{u}, nx^{n-1}\phi(x) + x^n\psi(x)\big\rangle \\ &= -n\sum_{k=0}^{r} \frac{\phi^{(k)}(0)}{k!}\mu_{n+k-1} - \sum_{m=0}^{t} \frac{\psi^{(m)}(0)}{m!}\mu_{n+m}. \end{aligned}$$

Multiplying the above relation by $1/z^{n+1}$ and taking the infinite sum over n, we obtain

$$0 = -\sum_{n=0}^{\infty} n \sum_{k=0}^{r} \frac{\phi^{(k)}(0)}{k!} \frac{\mu_{n+k-1}}{z^{n+1}} - \sum_{n=0}^{\infty}\sum_{m=0}^{t} \frac{\psi^{(m)}(0)}{m!} \frac{\mu_{n+m}}{z^{n+1}}. \tag{11.11}$$

It is straightforward to verify that

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{m=0}^{t} \frac{\psi^{(m)}(0)}{m!} \frac{\mu_{n+m}}{z^{n+1}} &= \psi(z)\mathcal{S}(z) - \sum_{m=1}^{t}\sum_{n=0}^{m-1} \frac{\psi^{(m)}(0)}{m!} \mu_n z^{m-1-n} \\ &= \psi(z)\mathcal{S}(z) - (\mathbf{u} * \theta_0\psi)(z). \end{aligned}$$

On the other hand,

$$\mathcal{S}'(z) = -\sum_{n=0}^{\infty}(n+1)\frac{\mu_n}{z^{n+2}}.$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{r} n\frac{\phi^{(k)}(0)}{k!}\frac{\mu_{n+k-1}}{z^{n+1}} &= -\phi(z)\mathcal{S}'(z)-\phi'(z)\mathcal{S}(z)+\sum_{k=2}^{r}\sum_{n=0}^{k-2}\frac{\phi^{(k)}(0)}{(k-1)!}\frac{\mu_n}{z^{n-k+2}}\\ &\quad -\sum_{k=2}^{r}\sum_{n=0}^{k-2}(n+1)\frac{\phi^{(k)}(0)}{k!}\frac{\mu_n}{z^{n-k+2}}\\ &= -\phi(z)\mathcal{S}'(z)-\phi'(z)\mathcal{S}(z)+\big(\mathbf{u}*\theta_0\phi'\big)(z)\\ &\quad +(D\mathbf{u}*\theta_0\phi)(z).\end{aligned}$$

Hence, (11.10) follows from (11.11).

(2) ⇒ (1) Suppose that (11.10) holds for some nonzero polynomials $\phi(z)$ and $\psi(z)$. Since each step above is also true in the reverse direction, (11.10) is equivalent to (11.11). But this implies that, for every $n \geq 0$,

$$0=-\big\langle \mathbf{u}, nx^{n-1}\phi(x)+x^n\psi(x)\big\rangle=\big\langle D\big(\phi(x)\mathbf{u}\big)-\psi(x)\mathbf{u}, x^n\big\rangle.$$

Therefore, $\mathbf{u}$ is semiclassical. ■

We present several characterizations of semiclassical functionals. Notice that some of them are natural extensions of the characterizations of classical functionals.

Definition 11.7. Let $\mathbf{u}$ be a linear functional and let $(P_n(x))_{n\geq 0}$ be a sequence of polynomials with $\deg P_n(x)=n$. We say that $(P_n(x))_{n\geq 0}$ is quasiorthogonal of order m with respect to $\mathbf{u}$ if

$$\begin{aligned}\langle \mathbf{u}, P_k P_n(x)\rangle &= 0, && k+1\leq |n-m|,\\ \langle \mathbf{u}, P_{n-m}P_n(x)\rangle &\neq 0, && \text{for some } n\geq m.\end{aligned}$$

Proposition 11.8 ([48]). *Let $\mathbf{u}$ be a quasidefinite functional, and let $(P_n(x))_{n\geq 0}$ be its sequence of monic orthogonal polynomials. The following statements are equivalent:*

(1) *The functional $\mathbf{u}$ is semiclassical of class s.*

(2) *For $n\geq 0$, let $Q_n(x)=\frac{P'_{n+1}(x)}{n+1}$. There exists a nonzero polynomial $\phi(x)$ with $\deg\phi(x)=r$ such that the sequence of monic polynomials $(Q_n(x))_{n\geq 0}$ is quasiorthogonal of order s with respect to the functional $\phi(x)\mathbf{u}$.*

Proof. (1) ⇒ (2) Let $\phi(x)$ and $\psi(x)$ be nonzero polynomials with $\deg\phi(x):=r\geq 0$ and $\deg\psi(x):=t\geq 1$ such that $\mathbf{u}$ satisfies $D(\phi(x)\mathbf{u})=\psi(x)\mathbf{u}$ and $s:=\max\{r-2, t-1\}$ is the class of $\mathbf{u}$. Note that

$$\begin{aligned}\big\langle \phi(x)\mathbf{u}, P_m(x)P'_{n+1}(x)\big\rangle &= \big\langle \phi(x)\mathbf{u}, \big(P_m(x)P_{n+1}(x)\big)'\big\rangle-\big\langle \phi(x)\mathbf{u}, P'_m(x)P_{n+1}(x)\big\rangle\\ &= -\big\langle \mathbf{u}, \big(\psi(x)P_m(x)+\phi(x)P'_m(x)\big)P_{n+1}(x)\big\rangle.\end{aligned}$$

The above implies that $\langle \phi(x)\mathbf{u}, P_m(x)P'_{n+1}(x)\rangle = 0$ for $0 \le m \le n-s-1$. Moreover, from (11.2), $\psi(x)P_m(x) + \phi(x)P'_m(x)$ has degree $s+m+1$ and, thus, $\langle \phi(x)\mathbf{u}, P_{n-s}(x)P'_{n+1}(x)\rangle \neq 0$. Hence, $Q_n(x)$ is quasiorthogonal of order s.

(2) ⇒ (1) Suppose that there exists a nonzero polynomial $\phi(x)$ with $\deg\phi(x) := r \ge 0$ such that the sequence of polynomials $(Q_n(x))_{n\ge 0}$ is quasiorthogonal of order s with respect to the functional $\phi(x)\mathbf{u}$. Since $(\frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u})_{n\ge 0}$ is a basis of the dual space of $\mathbb{P}$,

$$D\big(\phi(x)\mathbf{u}\big) = \sum_{n=0}^{\infty} \alpha_n \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u},$$

where $\alpha_n = \langle D(\phi(x)\mathbf{u}), P_n(x)\rangle = -\langle \phi\mathbf{u}, P'_n(x)\rangle = -n\langle \mathbf{u}, Q_{n-1}(x)\rangle$, $n \ge 1$ with $\alpha_0 = 0$.

From the quasiorthogonality of $(Q_n(x))_{n\ge 0}$, $\alpha_n = 0$ when $s+2 \le n$. Thus,

$$D\big(\phi(x)\mathbf{u}\big) = \psi(x)\mathbf{u}, \quad \text{where } \psi(x) = \sum_{n=1}^{s+1} \alpha_n \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}. \qquad \blacksquare$$

Corollary 11.9. *A quasidefinite functional* $\mathbf{u}$ *with associated sequence of monic orthogonal polynomials* $(P_n(x))_{n\ge 0}$ *is semiclassical of class* s *if and only if there exists a nonzero polynomial* $\phi(x)$ *such that sequence of monic polynomials* $(Q_{n,m}(x))_{n\ge 0}$, *where* $Q_{n,m}(x) = \frac{P^{(m)}_{n+m}(x)}{(n+1)_m}$, *is quasiorthogonal of order* s *with respect to the linear functional* $\phi^m(x)\mathbf{u}$.

Exercise 11.1. State and prove the analogous result of Proposition 9.9 for semiclassical functionals.

Proposition 11.10 ([48]). *Let* $\mathbf{u}$ *be a quasidefinite functional and* $(P_n(x))_{n\ge 0}$ *its sequence of monic orthogonal polynomials. The following statements are equivalent:*

(1) $\mathbf{u}$ *is semiclassical of class* s.

(2) *There exist a nonnegative integer number* s *and a monic polynomial* $\phi(x)$ *of degree* r *with* $0 \le r \le s+2$ *such that*

$$\phi(x)P'_{n+1}(x) = \sum_{k=n-s}^{n+r} \lambda_{n,k}P_k(x), \quad n \ge s,\ \lambda_{n,n-s} \neq 0. \tag{11.12}$$

If $s \ge 1$, $r \ge 1$ *and* $\lambda_{s,0} \neq 0$, *then* s *is the class of* $\mathbf{u}$.

Proof. (1) ⇒ (2) Suppose that $\mathbf{u}$ is of class s satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\deg\phi(x) = r$. Since $(P_n(x))_{n\ge 0}$ is a basis of $\mathbb{P}$, for each $n \ge 0$ there exists a set of real numbers $(\lambda_{n,k})_{k=0}^{n+r}$ such that

$$\phi(x)P'_{n+1}(x) = \sum_{k=0}^{n+r} \lambda_{n,k}P_k(x).$$

Using orthogonality,

$$\lambda_{n,k} = \frac{\langle \phi(x)\mathbf{u}, P'_{n+1}(x)P_k(x)\rangle}{\langle \mathbf{u}, P_k^2(x)\rangle} = \frac{(n+1)\langle \phi(x)\mathbf{u}, Q_n(x)P_k(x)\rangle}{\langle \mathbf{u}, P_k^2(x)\rangle},$$

where, for each $n \geq 0$, $Q_n(x) = \frac{P'_{n+1}(x)}{n+1}$. But $\mathbf{u}$ is semiclassical of class s, so by Proposition 11.8, $Q_n(x)$ is quasiorthogonal of order s with respect to $\phi(x)\mathbf{u}$. Therefore, $\lambda_{n,k} = 0$, when $s+1 \leq n-k$, and $\lambda_{n,n-s} \neq 0$.

(2) ⇒ (1) Assume that $(P_n(x))_{n\geq 0}$ satisfies (11.12). Since $(\frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u})_{n\geq 0}$ is a basis of the dual space of $\mathbb{P}$,

$$D\big(\phi(x)\mathbf{u}\big) = \sum_{n=0}^{\infty} \alpha_n \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle}\mathbf{u}.$$

Using (11.12),

$$\alpha_n = -\big\langle \mathbf{u}, \phi(x)P'_n(x)\big\rangle = -\sum_{k=n-s}^{n+r} \lambda_{n,k}\big\langle \mathbf{u}, P_k(x)\big\rangle = \begin{cases} 0, & n > s, \\ -\lambda_{n,0}\langle \mathbf{u}, P_0(x)\rangle, & n \leq s. \end{cases}$$

Therefore, $\mathbf{u}$ satisfies $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with

$$\psi(x) = \sum_{n=0}^{s+1} \alpha_n \frac{P_n(x)}{\langle \mathbf{u}, P_n^2(x)\rangle},$$

hence, $\mathbf{u}$ is semiclassical. Observe that if, in particular, $\lambda_{s,0} \neq 0$, then $\mathbf{u}$ is of class s. ∎

Using the three term recurrence relation (2.5), (11.12) can be written in a compact form as shown in the following result.

Theorem 11.11 ([48]). *Let* $\mathbf{u}$ *be a semiclassical functional of class* s *and* $(P_n(x))_{n\geq 0}$ *its associated sequence of monic orthogonal polynomials. Then,*

$$\phi(x)P'_{n+1}(x) = \frac{C_{n+1}(x) - C_0(x)}{2}P_{n+1}(x) - D_{n+1}(x)P_n(x), \quad n \geq 0, \tag{11.13}$$

where $(C_n(x))_{n\geq 0}$ *and* $(D_n(x))_{n\geq 0}$ *satisfy*

$$\begin{aligned} C_{n+1}(x) &= -C_n(x) + \frac{2D_n(x)}{a_n}(x-b_n), \quad n \geq 0, \\ C_0(x) &= -\psi(x) + \phi'(x), \end{aligned} \tag{11.14}$$

and

$$\begin{aligned} D_{n+1}(x) = -\phi(x) &+ \frac{a_n}{a_{n-1}}D_{n-1}(x) + \frac{D_n(x)}{a_n}(x-b_n)^2 \\ &- C_n(x)(x-b_n), \quad n \geq 0, \\ D_0(x) = (\mathbf{u} * \theta_0\phi)'(x) &- (\mathbf{u} * \theta_0\psi)(x), \quad D_{-1}(x) = 0. \end{aligned}$$

The above expression leads to the so-called ladder operators associated with the semiclassical functional **u**. Using (11.14) and the three-term recurrence relation (2.5), we can deduce from (11.13) that, for $n \geq 0$,

$$\phi(x)P'_{n+1}(x) = -\left(\frac{C_{n+2}(x) + C_0(x)}{2}\right)P_{n+1}(x) + \frac{D_{n+1}(x)}{a_{n+1}}P_{n+2}(x). \quad (11.15)$$

Relations (11.13) and (11.15) are essential to deduce a second-order differential equation satisfied by the polynomials $(P_n(x))_{n\geq 0}$ (see [26, 30, 48]), which reads

$$J(x,n)P''_{n+1}(x) + K(x,n)P'_{n+1}(x) + L(x,n)P_{n+1}(x) = 0, \quad n \geq 0,$$

where, for $n \geq 0$,

$$J(x,n) = \phi(x)D_{n+1}(x),$$
$$K(x;n) = \big(\phi'(x) + C_0(x)\big)D'_{n+1}(x) - \phi(x)D'_{n+1}(x),$$

and

$$L(x,n) = \left(\frac{C_{n+1}(x) - C_0(x)}{2}\right)D'_{n+1}(x) - \left(\frac{C'_{n+1}(x) - C'_0(x)}{2}\right)D_{n+1}(x) - D_{n+1}(x)\sum_{k=0}^{n}\frac{D_k(x)}{a_k}.$$

Notice that the degrees of the polynomials J, K, L are at most $2s+2$, $2s+1$, and $2s$, respectively.

Theorem 11.12 ([8]). *Let* **u** *be a quasidefinite linear functional and* $(P_n(x))_{n\geq 0}$ *the sequence of monic orthogonal polynomials associated with* **u**. *The following statements are equivalent:*

(1) **u** *is semiclassical.*

(2) $(P_n(x))_{n\geq 0}$ *satisfies the following nonlinear differential equation:*

$$\phi(x)\big[P_{n+1}(x)P_n(x)\big]' = \frac{D_n(x)}{a_n}P^2_{n+1}(x) - C_0(x)P_{n+1}(x)P_n(x) - D_{n+1}(x)P^2_n(x), \quad (11.16)$$

where $D_n(x)$, $C_0(x)$, *and* a_n *are the same as in* (11.13).

Proof. (1) $\Rightarrow$ (2) Suppose that **u** is semiclassical. From (11.13), we have

$$\phi(x)\big[P_{n+1}(x)P_n(x)\big]' = P_n(x)\left(\frac{C_{n+1}(x) - C_0(x)}{2}P_{n+1}(x) - D_{n+1}(x)P_n(x)\right) + P_{n+1}(x)\left(\frac{C_n(x) - C_0(x)}{2}P_n(x) - D_n(x)P_{n-1}(x)\right)$$

$$
\begin{aligned}
&= -D_{n+1}(x)P_n^2(x) \\
&\quad + \left(\frac{C_{n+1}(x) + C_n(x) - 2C_0(x)}{2}\right) P_{n+1}(x)P_n(x) \\
&\quad - D_n(x)P_{n+1}(x)P_{n-1}(x).
\end{aligned}
$$

Now, taking into account that $P_{n-1}(x) = \frac{(x-b_n)}{a_n} P_n(x) - \frac{1}{a_n} P_{n+1}(x)$, the above relation becomes

$$
\begin{aligned}
\phi(x)\big[P_{n+1}(x)P_n(x)\big]' = {} & -D_{n+1}P_n^2(x) + \frac{D_n(x)}{a_n} P_{n+1}^2(x) \\
& + \left(\frac{C_{n+1}(x) + C_n(x) - 2C_0(x)}{2} - \frac{(x-b_n)}{a_n} D_n(x)\right) \\
& \cdot P_{n+1}(x)P_n(x).
\end{aligned}
$$

Using the relation (11.14), we get the result.

(2) $\Rightarrow$ (1) Let **u** be a quasi-definite linear functional, and let $(P_n(x))_{n\geq 0}$ be the sequence of monic orthogonal polynomials associated with **u**.

Suppose that $(P_n(x))_{n\geq 0}$ satisfies (11.16). Using the three term recurrence relation $P_{n+1}(x) = (x - b_n)P_n(x) - a_n P_{n-1}(x)$ and (11.14), we can write (11.16) as

$$
\begin{aligned}
\phi(x)P_{n+1}'(x)P_n(x) = {} & \left(\frac{C_{n+1}(x) + C_n(x) - 2C_0(x)}{2}\right) P_{n+1}(x)P_n(x) \\
& - D_{n+1}(x)P_n^2(x) - D_n(x)P_{n+1}(x)P_{n-1}(x) \\
& - \phi(x)P_{n+1}(x)P_n'(x).
\end{aligned}
\tag{11.17}
$$

Multiplying the above relation by $P_{n-1}(x)$ and replacing $\phi(x)P_n'(x)P_{n-1}(x)$ with (11.17) for $n-1$, we obtain

$$
\begin{aligned}
\phi(x)P_{n+1}'(x)P_{n-1}(x) = {} & \left(\frac{C_{n+1}(x) - C_{n-1}(x)}{2}\right) P_{n+1}(x)P_{n-1}(x) \\
& - D_{n+1}(x)P_n(x)P_{n-1}(x) \\
& + P_{n+1}(x)\big(D_{n-1}(x)P_{n-2}(x) + \phi(x)P_{n-1}'(x)\big).
\end{aligned}
$$

Similarly, multiplying the above relation by $P_{n-2}(x)$ and then replacing $\phi(x)P_{n-1}'(x)P_{n-2}(x)$ by (11.17) for $n-2$, we get

$$
\begin{aligned}
\phi(x)P_{n+1}'(x)P_{n-2}(x) = {} & \left(\frac{C_{n+1}(x) + C_{n-2}(x) - 2C_0(x)}{2}\right) P_{n+1}(x)P_{n-2}(x) \\
& - D_{n+1}(x)P_n(x)P_{n-2}(x) \\
& - P_{n+1}(x)\big(D_{n-2}(x)P_{n-3}(x) + \phi(x)P_{n-2}'(x)\big).
\end{aligned}
$$

Iterating this process, we obtain that, for odd $k \leq n$,

$$\begin{aligned}\phi(x)P'_{n+1}(x)P_{n-k}(x) = &\left(\frac{C_{n+1}(x) - C_{n-k}(x)}{2}\right)P_{n+1}(x)P_{n-k}(x)\\ &- D_{n+1}(x)P_n(x)P_{n-k}(x)\\ &+ P_{n+1}(x)\big(D_{n-k}(x)P_{n-(k+1)}(x) + \phi(x)P'_{n-k}(x)\big)\end{aligned}$$

and, for even $k \leq n$,

$$\begin{aligned}\phi(x)P'_{n+1}(x)P_{n-k}(x) = &\left(\frac{C_{n+1}(x) + C_{n-k}(x) - 2C_0(x)}{2}\right)P_{n+1}(x)P_{n-k}(x)\\ &- D_{n+1}(x)P_n(x)P_{n-k}(x)\\ &- P_{n+1}(x)\big(D_{n-k}(x)P_{n-(k+1)}(x) + \phi(x)P'_{n-k}(x)\big).\end{aligned}$$

If n is either odd or even, when $k = n$, we obtain (11.12), but this implies that **u** is semiclassical. ■

Before dealing with the next result, we fix some notation. Let $(Q_n(x))_{n\geq 0}$ be a basis of $\mathbb{P}$. We define the vectors $\mathbf{Q} := (Q_0(x), Q_1(x), Q_2(x), \ldots)^t$ and $\chi(x) = (1, x, x^2, x^3, \ldots)^t$. Let N be the semiinfinite matrix such that $\chi'(x) = N\chi(x)$. Therefore,

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We denote by $\tilde{N}$ the semiinfinite matrix such that $\mathbf{Q}' = \tilde{N}\mathbf{Q}$. Observe that if S is a matrix of change of basis from the monomials $\chi(x)$ to $\mathbf{Q}$, that is, $\mathbf{Q} = S\chi(x)$, then $\tilde{N} = SNS^{-1}$.

If $\mathbf{Q}$ is semiclassical, we write (11.12) in matrix form as $\phi(x)\mathbf{Q}' = F\mathbf{Q}$, where F is a semiinfinite band matrix. Finally, for square matrices A and B of size n, we define its commutator as $[A, B] = AB - BA$.

Proposition 11.13. *Let* **u** *be a positive definite semiclassical functional satisfying the Pearson equation $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$, and let $(Q_n(x))_{n\geq 0}$ be the sequence of orthonormal polynomials associated with* **u**. *Then,*

(i) $[J, F] = \phi(J)$,

(ii) $\tilde{N}\phi(J)^t + \phi(J)\tilde{N}^t = -\psi(J)$,

(iii) $F + F^t = -\psi(J)$,

where J is the Jacobi matrix associated with $(Q_n(x))_{n\geq 0}$.

Remark 11.14. This is the matrix representation of the Laguerre–Freud equations satisfied by the parameters of the three-term recurrence relation of semiclassical orthonormal polynomials. As a direct consequence, one can deduce nonlinear difference equations that the coefficients of the three-term recurrence relation satisfy. They are related to discrete Painlevé equations. Some illustrative examples appear in [60].

Proof. (i) Differentiating $x\mathbf{Q} = J\mathbf{Q}$ and then multiplying by $\phi(x)$, we get

$$J\phi(x)\mathbf{Q}' = \phi(x)\mathbf{Q} + x\phi(x)\mathbf{Q}'.$$

But $\phi(x)\mathbf{Q}' = F\mathbf{Q}$ and $\phi(x)\mathbf{Q} = \phi(J)\mathbf{Q}$. Hence,

$$JF\mathbf{Q} = \phi(J)\mathbf{Q} + xF\mathbf{Q} = \big(\phi(J) + FJ\big)\mathbf{Q},$$

and, since $\mathbf{Q}$ is a basis, the result follows.

(ii) From the Pearson equation,

$$\begin{aligned} 0 &= \big\langle D\big(\phi(x)\mathbf{u}\big), \mathbf{Q}\mathbf{Q}^t\big\rangle - \big\langle \psi(x)\mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle \\ &= -\big\langle \phi(x)\mathbf{u}, \mathbf{Q}'\mathbf{Q}^t + \mathbf{Q}\big(\mathbf{Q}'\big)^t\big\rangle - \big\langle \psi(x)\mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle \\ &= -\tilde{N}\big\langle \mathbf{u}, \phi(x)\mathbf{Q}\mathbf{Q}^t\big\rangle - \big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\phi(x)\big\rangle\tilde{N}^t - \big\langle \mathbf{u}, \psi(x)\mathbf{Q}\mathbf{Q}^t\big\rangle \\ &= -\tilde{N}\phi(J)\big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle - \big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle\phi(J)^t\tilde{N}^t - \psi(J)\big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle. \end{aligned}$$

But $\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\rangle$ is equal to the identity matrix since $(Q_n(x))_{n\geq 0}$ are orthonormal, and the result follows.

(iii) Similarly, from the Pearson equation,

$$\begin{aligned} 0 &= \big\langle D\big(\phi(x)\mathbf{u}\big), \mathbf{Q}\mathbf{Q}^t\big\rangle - \big\langle \psi(x)\mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle \\ &= -\big\langle \phi(x)\mathbf{u}, \mathbf{Q}'\mathbf{Q}^t + \mathbf{Q}\big(\mathbf{Q}'\big)^t\big\rangle - \big\langle \psi(x)\mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle \\ &= -\big\langle \mathbf{u}, \phi(x)\mathbf{Q}'\mathbf{Q}^t\big\rangle - \big\langle \mathbf{u}, \mathbf{Q}\big(\mathbf{Q}'\big)^t\phi(x)\big\rangle - \big\langle \mathbf{u}, \psi(x)\mathbf{Q}\mathbf{Q}^t\big\rangle \\ &= -F\big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle - \big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle F^t - \psi(J)\big\langle \mathbf{u}, \mathbf{Q}\mathbf{Q}^t\big\rangle, \end{aligned}$$

and the result follows. ∎

Chapter 12

Examples of semiclassical orthogonal polynomials

It is well known that the semiclassical functionals of class $s = 0$ are the classical linear functionals (Hermite, Laguerre, Jacobi, and Bessel).

If $\mathbf{u}$ is a semiclassical functional of class $s = 1$, we can distinguish two cases:

$$\text{(A)} \quad \deg\psi(x) = 2, \quad 0 \le \deg\phi(x) \le 3; \qquad \text{(B)} \quad \deg\psi(x) = 1, \quad \deg\phi(x) = 3.$$

S. Belmehdi [8] obtained the following canonical forms of the functionals of the class 1, up to linear changes of the variable, according to the degree of $\phi(x)$ and the multiplicity of its zeros:

(A) $\deg\psi(x) = 2$	
$\deg\phi(x) = 0$	1
$\deg\phi(x) = 1$	x
$\deg\phi(x) = 2$	x^2 $x^2 - 1$
$\deg\phi(x) = 3$	x^3 $x^2(x-1)$ $\left(x^2-1\right)(x-c)$

(B) $\deg\psi(x) = 1$	
$\deg\phi(x) = 3$	x^3 $x^2(x-1)$ $\left(x^2-1\right)(x-c)$

Example 12.1. Let $\mathbf{u}$ be the quasidefinite functional defined by

$$\langle \mathbf{u}, p(x) \rangle = \int_0^\infty p(x)x^\alpha e^{-x}\,dx + Mp(0), \quad \forall p(x) \in \mathbb{P},$$

with $\alpha > -1$ and $M > 0$. Then $\mathbf{u}$ is a semiclassical functional of class $s = 1$ satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\phi(x) = x^2$ and $\psi(x) = x(\alpha + 2 - x)$.

The sequence of polynomials orthogonal with respect to the above functional is known in the literature as Laguerre-type orthogonal polynomials (see [31,40], among others).

Example 12.2. Let $\mathbf{u}$ be the quasidefinite functional defined by (see [8])

$$\langle \mathbf{u}, p(x) \rangle = \int_{-1}^1 p(x)(x-1)^{(a+b-2)/2}(x+1)^{(b-a-2)/2}e^{ax}\,dx, \quad \forall p(x) \in \mathbb{P},$$

with $b > a$. Then $\mathbf{u}$ satisfies $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\phi(x) = x^2 - 1$ and $\psi(x) = ax^2 + bx$, hence, it is semiclassical of class $s = 1$.

Example 12.3. Let $\mathbf{u}$ be the quasidefinite functional defined by (see [10, 60])

$$\langle \mathbf{u}, p(x) \rangle = \int_0^{\infty} p(x) x^{\alpha} e^{-x^2+tx}\, dx, \quad \forall p(x) \in \mathbb{P},$$

with $\alpha > -1$ and $t \in \mathbb{R}$. In [10], it is shown that $\mathbf{u}$ is a semiclassical functional of class $s = 1$ satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\phi(x) = x$ and $\psi(x) = -2x^2 + tx + \alpha + 1$.

Example 12.4. Let $\mathbf{u}$ be the quasidefinite functional defined by (see [8])

$$\langle \mathbf{u}, p(x) \rangle = \int_0^{N} p(x) x^{\alpha} e^{-x}\, dx, \quad \forall p(x) \in \mathbb{P},$$

with $\alpha > -1$ and $N > 0$. The functional $\mathbf{u}$ is semiclassical of class $s = 1$ satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\phi(x) = (x - N)x$ and $\psi(x) = x(\alpha + N - x) - (\alpha + 1)N$.

This functional is known in the literature as the truncated Gamma functional and the corresponding sequences of orthogonal polynomials are called truncated Laguerre orthogonal polynomials.

Semiclassical functionals can be constructed via discrete Darboux transformations. First, we need to prove the following theorem.

Theorem 12.5. *Let $\mathbf{u}$ and $\mathbf{v}$ be two linear functionals related by*

$$A(x)\mathbf{u} = B(x)\mathbf{v},$$

where $A(x)$ and $B(x)$ are nonzero polynomials. Then $\mathbf{u}$ is semiclassical if and only if $\mathbf{v}$ is semiclassical.

Proof. Suppose that $\mathbf{u}$ is semiclassical satisfying $D(\phi_0(x)\mathbf{u}) = \psi_0(x)\mathbf{u}$. Let

$$\phi_1(x) = A(x)B(x)\phi_0(x).$$

Then, for every polynomial $p(x) \in \mathbb{P}$,

$$\begin{aligned}
\langle D(\phi_1(x)\mathbf{v}), p(x) \rangle &= \langle D(A(x)B(x)\phi_0(x)\mathbf{v}), p(x) \rangle \\
&= \langle D(A^2(x)\phi_0(x)\mathbf{u}), p(x) \rangle \\
&= -\langle \phi_0(x)\mathbf{u}, A^2(x)p'(x) \rangle \\
&= -\langle \phi_0(x)\mathbf{u}, (A^2(x)p(x))' \rangle + \langle \phi_0(x)\mathbf{u}, (A^2(x))' p(x) \rangle \\
&= \langle A^2(x)\psi_0(x)\mathbf{u}, p(x) \rangle + \langle 2\phi_0(x)A'(x)A(x)\mathbf{u}, p(x) \rangle \\
&= \langle (A(x)\psi_0(x) + 2A'(x)\phi_0(x))B(x)\mathbf{v}, p(x) \rangle.
\end{aligned}$$

Therefore, $\mathbf{v}$ is semiclassical with $\psi_1(x) = (A(x)\psi_0(x) + 2\phi_0(x)A'(x))B(x)$.

Similarly, if $\mathbf{v}$ is semiclassical, by interchanging the role of the functionals above, it follows that $\mathbf{u}$ is semiclassical. ∎

Corollary 12.6. *Any linear spectral transformation of a semiclassical functional is also a semiclassical functional.*

Exercise 12.1. Use Theorem 12.5 to prove Corollary 12.6. *Hint:* Consider each transformation (Christoffel, Geronimus, and Uvarov) separately. If $\mathbf{u}$ is a semiclassical functional satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$, find the polynomials $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ such that the modified functional $\mathbf{v}$ satisfies $D(\tilde{\phi}(x)\mathbf{v}) = \tilde{\psi}(x)\mathbf{v}$.

Remark 12.7.

- For canonical Christoffel (7.1) and Geronimus (7.6) transformations, the class of the new functional depends on the location of the point a in terms of the zeros of $\phi(x)$.
- Uvarov transformations (7.11) of classical orthogonal polynomials yield semiclassical linear functionals. The so called Krall-type linear functionals appear when a Dirac measure, or mass point, is located at a zero of $\phi(x)$. The corresponding sequences of orthogonal polynomials satisfy, for some choices of the parameters (in the Laguerre case, for α a nonnegative integer number) higher order linear differential equations with order depending on α. It is an open problem to describe the sequences of orthogonal polynomials which are eigenfunctions of higher order differential operators. For order two (S. Bochner [9]) and four (H. L. Krall [35]), the problem has been completely solved.

Example 12.8. The linear functional obtained from an Uvarov transformation of the Laguerre functional will be of class 1 if the mass point is located at $a = 0$. It will be of class 2 if the mass point is located at $a \neq 0$. The details are left as an exercise.

Other examples of semiclassical functionals of class 2 are also known.

Example 12.9. Let $\mathbf{u}$ be the functional defined by

$$\langle \mathbf{u}, p(x)\rangle = \int_{\mathbb{R}} p(x)e^{-\frac{x^4}{4}-tx^2}\,dx, \quad \forall p(x) \in \mathbb{P},$$

where $t \in \mathbb{R}$. In this case, $\mathbf{u}$ is a semiclassical functional of class $s = 2$ satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\phi(x) = 1$ and $\psi(x) = -2tx - x^3$.

This is a particular case of the so-called generalized Freud linear functionals (see [15, 16]).

Example 12.10. Let $\mathbf{u}$ be a quasidefinite functional defined by

$$\langle \mathbf{u}, p(x)\rangle = \int_{-N}^{N} p(x)e^{-x^2}\,dx, \quad \forall p(x) \in \mathbb{P},$$

where $N > 0$. The functional $\mathbf{u}$ is semiclassical of class $s = 2$ satisfying $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$ with $\phi(x) = x^2 - N^2$ and $\psi(x) = 2x(1 + N^2 - x^2)$.

New semiclassical functionals can also be constructed through symmetrized functionals. Indeed, given a functional with Stieltjes function $\mathcal{S}(z)$, the Stieltjes function $\tilde{\mathcal{S}}(z)$ of its symmetrized functional satisfies $\tilde{\mathcal{S}}(z) = z\mathcal{S}(z^2)$.

Theorem 12.11 ([6]). *Let* $\mathbf{u}$ *be a semiclassical functional satisfying* $D(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$, *and let* $\mathcal{S}(z)$ *be its Stieltjes function, which satisfies* (11.10)

$$\phi(z)\mathcal{S}'(z) + \big(\phi'(z) - \psi(z)\big)\mathcal{S}(z) = C(z).$$

The Stieltjes function $\tilde{\mathcal{S}}(z)$ *associated with the symmetrized linear functional* $\mathbf{v}$ *satisfies*

$$z\phi(z^2)\tilde{\mathcal{S}}'(z) + \big[2z^2\big(\phi'(z^2) - \psi(z^2)\big) - \phi(z^2)\big]\tilde{\mathcal{S}}(z) = 2z^3C(z^2).$$

Thus, the symmetrized functional of a semiclassical linear functional is semiclassical. The class of $\mathbf{v}$ *is either* $2s$, $2s+1$, *or* $2s+3$, *according to the coprimality of the polynomial coefficients in the ordinary linear differential equation satisfied by* $\tilde{\mathcal{S}}(z)$.

Chapter 13

The Askey scheme

The Askey scheme (see, for instance, [32]) is a way to organize orthogonal polynomials of hypergeometric type into a hierarchy. This scheme includes families of orthogonal polynomials of a discrete variable (Hahn, Meixner, Krawtchouk, Charlier) and the classical orthogonal polynomials (Hermite, Laguerre, Bessel, and Jacobi) as limiting cases of the former. We include this chapter at the end of the lecture notes as reference for the interested reader and as an illustration of the relations between the included families of orthogonal polynomials.

First, we will summarize the hypergeometric representations and orthogonality relations of the polynomials. In this chapter, we introduce part of the Askey scheme and describe the relations between the families of polynomials that appear in the figure below.

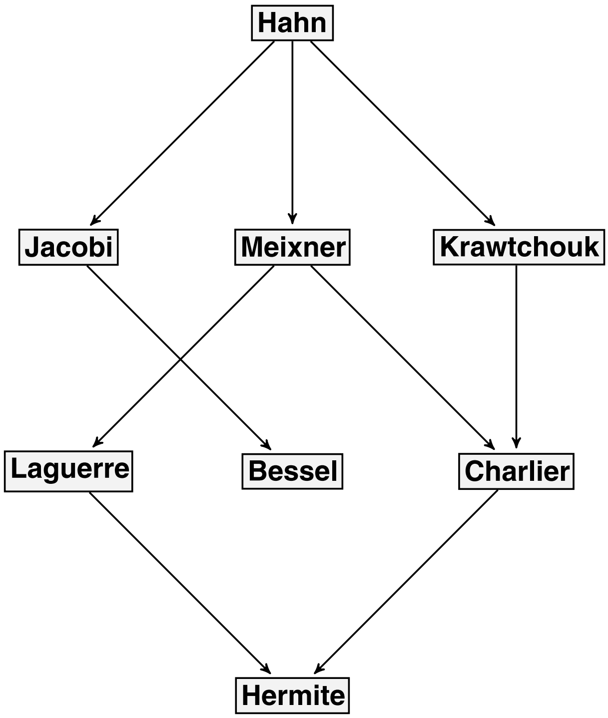

Figure 13.1. The Askey scheme

13.1 Hahn polynomials

Hypergeometric representation

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x \\ \alpha+1, -N\end{matrix}; 1\right), \quad 0 \le n \le N. \tag{13.1}$$

Orthogonality. For $\alpha, \beta > -1$, or for $\alpha, \beta < -N$,

$$\sum_{x=0}^{N} \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x} Q_m(x;\alpha,\beta,N) Q_n(x;\alpha,\beta,N)$$
$$= \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!} \delta_{mn}.$$

13.2 Jacobi polynomials

Hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1 \\ \alpha+1\end{matrix}, \frac{1-x}{2}\right), \quad n \ge 0.$$

Orthogonality. For $\alpha, \beta > -1$,

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta \, dx$$
$$= \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1) n!} \delta_{mn}.$$

13.3 Meixner polynomials

Hypergeometric representation

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-n, -x \\ \beta\end{matrix}, 1-\frac{1}{c}\right), \quad n \ge 0.$$

Orthogonality

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_m(x;\beta,c) M_n(x;\beta,c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^\beta} \delta_{mn}, \quad \beta > 0,\ 0 < c < 1.$$

13.4 Krawtchouk polynomials

Hypergeometric representation

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix}-n, -x \\ -N\end{matrix}; \frac{1}{p}\right), \quad 0 \le n \le N.$$

Orthogonality

$$\sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} K_m(x; p, N) K_n(x; p, N)$$
$$= \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p}\right)^n \delta_{mn}, \quad 0 < p < 1.$$

13.5 Laguerre polynomials

Hypergeometric representation

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix}-n \\ \alpha+1\end{matrix}; x\right), \quad n \ge 0.$$

Orthogonality

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{mn}, \quad \alpha > -1,\ n \ge 0.$$

13.6 Bessel polynomials

Hypergeometric representation

$$B_n^{(a)}(x) = {}_2F_0\left(\begin{matrix}-n, n+a+1 \\ -\end{matrix}; -\frac{x}{2}\right), \quad n \ge 0.$$

Orthogonality

$$(2\pi i)^{-1} \int_C B_m^{(a)}(z) B_n^{(a)}(z) z^a e^{-2/z}\, dz$$
$$= \frac{2(-1)^{n+1} n!}{(2n+a+1)(a+2)_{n-1}} \delta_{mn}, \quad a \ne 0, -1, -2, \ldots,$$

where C is the unit circle in the complex plane oriented counterclockwise.

13.7 Charlier polynomials

Hypergeometric representation

$$C_n(x;a) = {}_2F_0\left(\begin{matrix}-n, -x \\ -\end{matrix}; -\frac{1}{a}\right), \quad n \geq 0.$$

Orthogonality

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} C_m(x;a) C_n(x;a) = a^{-n} e^a n! \delta_{nm}, \quad a > 0.$$

13.8 Hermite polynomials

Hypergeometric representation

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix}-n/2, -(n-1)/2 \\ -\end{matrix}; -\frac{1}{x^2}\right), \quad n \geq 0.$$

Orthogonality

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2}\, dx = 2^n n! \delta_{mn}.$$

13.9 Limit relations

Hahn → Jacobi. Jacobi polynomials are obtained from Hahn polynomials of a discrete variable by taking $x \to Nx$ and letting $N \to \infty$. In fact, we have

$$\lim_{N\to\infty} Q_n(Nx; \alpha, \beta, N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha\beta)}(1)}.$$

Jacobi → Laguerre. Laguerre polynomials are obtained from Jacobi polynomials by taking $x \to 1 - 2\beta^{-1}x$ and letting $\beta \to \infty$:

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - 2\beta^{-1}x\right) = L_n^{(\alpha)}(x).$$

Jacobi → Bessel. Bessel polynomials are obtained from Jacobi polynomials by taking $\beta = a - \alpha$ and letting $\alpha \to -\infty$:

$$\lim_{\alpha\to-\infty} \frac{P_n^{(\alpha,a-\alpha)}(1+\alpha x)}{P_n^{(\alpha,\alpha-a)}(1)} = B_n^{(a)}(x).$$

Jacobi → Hermite. Hermite polynomials are obtained from Jacobi polynomials by taking $\beta \to \alpha$ and letting $\alpha \to \infty$ as follows:

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} P_n^{(\alpha,\alpha)}\big(\alpha^{-\frac{1}{2}}x\big) = \frac{H_n(x)}{2^n n!}.$$

Hahn → Meixner. If we take $\alpha = b-1$, $\beta = N(1-c)c^{-1}$ in (13.1) and let $N \to \infty$, then Meixner polynomials are obtained from Hahn polynomials as follows:

$$\lim_{N\to\infty} Q_n\big(x; b-1, N(1-c)c^{-1}, N\big) = M_n(x; b, c).$$

Meixner → Laguerre. Laguerre polynomials are obtained from Meixner polynomials by taking $\beta = \alpha + 1$ and $x \to (1-c)^{-1}x$, and letting $c \to 1$:

$$\lim_{c\to 1} M_n\big((1-c)^{-1}x; \alpha+1, c\big) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(1)}.$$

Meixner → Charlier. Charlier polynomials are obtained from Meixner polynomials by taking $c = (\alpha+\beta)^{-1}a$ and letting $\beta \to \infty$:

$$\lim_{\beta\to\infty} M_n\big(x; \beta, (\alpha+\beta)^{-1}a\big) = C_n(x; a).$$

Hahn → Krawtchouk. If we take $\alpha = pt$ and $\beta = (1-p)t$ in (13.1) and let $t \to \infty$, then we can obtain Krawtchouk polynomials from Hahn polynomials as follows:

$$\lim_{t\to\infty} Q_n\big(x; pt, (1-p)t, N\big) = K_n(x; p, N).$$

Krawtchouk → Charlier. Charlier polynomials are obtained from Krawtchouk polynomials by taking $p = N^{-1}a$ and letting $N \to \infty$:

$$\lim_{N\to\infty} K_n\big(x; N^{-1}a, N\big) = C_n(x; a).$$

Krawtchouk → Hermite. Hermite polynomials are obtained from Krawtchouk polynomials by taking $x \to pN + x\sqrt{2p(1-p)N}$ and letting $N \to \infty$:

$$\lim_{N\to\infty} \sqrt{\binom{N}{n}}\, K_n\big(pN + x\sqrt{2p(1-p)N}; p, N\big) = \frac{(-1)^n H_n(x)}{\sqrt{2^n n! (\frac{p}{1-p})^n}}.$$

Laguerre → Hermite. Hermite polynomials are obtained from Laguerre polynomials by letting $\alpha \to \infty$ as follows:

$$\lim_{\alpha\to\infty} \left(\frac{2}{\alpha}\right)^{\frac{1}{2}n} L_n^{(\alpha)}\big((2\alpha)^{\frac{1}{2}}x + \alpha\big) = \frac{(-1)^n}{n!} H_n(x).$$

Charlier → Hermite. Hermite polynomials are obtained from Charlier polynomials by taking $x \to (2a)^{1/2}x + a$ and letting $a \to \infty$:

$$\lim_{a\to\infty} (2a)^{\frac{1}{2}n} C_n\big((2a)^{1/2}x + a; a\big) = (-1)^n H_n(x).$$

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