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1 Schwarzschild black hole

Here we consider a metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \quad (1.1)$$

and see this metric satisfies the vacuum Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \quad (1.2)$$

Since the Ricci tensor of this metric is

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (1.3)$$

this metric trivially satisfies the equation (1.2).

(Incidentally, the Riemann tensor of this metric is

$$R_{00\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.4)$$

$$R_{01\mu}{}^{\nu} = \begin{bmatrix} 0 & \frac{2G}{x_1^4}M\left(2GM - x_1\right) & 0 & 0 \\ \frac{2GM}{x_1^2(2GM - x_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.5)$$

$$R_{02\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & \frac{GM}{x_1^4}\left(-2GM + x_1\right) & 0 \\ 0 & 0 & 0 & 0 \\ \frac{GM}{x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.6)$$

$$R_{03\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{GM}{x_1^4}\left(-2GM + x_1\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{GM}{x_1}\sin^2\left(x_2\right) & 0 & 0 & 0 \end{bmatrix}, \quad (1.7)$$

$$R_{10\mu}{}^{\nu} = \begin{bmatrix} 0 & \frac{2G}{x_1^4} M \left(-2GM + x_1\right) & 0 & 0 \\ \frac{2GM}{x_1^2 \left(-2GM + x_1\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.8}$$

$$R_{11\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.9}$$

$$R_{12\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{GM}{x_1^2 \left(2GM - x_1\right)} & 0 \\ 0 & \frac{GM}{x_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.10}$$

$$R_{13\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{GM}{x_1^2 \left(2GM - x_1\right)} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{GM}{x_1} \sin^2\left(x_2\right) & 0 & 0 \end{bmatrix}, \tag{1.11}$$

$$R_{20\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & \frac{GM}{x_1^4} \left(2GM - x_1\right) & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{GM}{x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.12}$$

$$R_{21\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{GM}{x_1^2 \left(-2GM + x_1\right)} & 0 \\ 0 & -\frac{GM}{x_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.13}$$

$$R_{22\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.14}$$

$$R_{23\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2G}{x_1} M \\ 0 & 0 & -\frac{2G}{x_1} M \sin^2\left(x_2\right) & 0 \end{bmatrix}, \tag{1.15}$$

$$R_{30\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{GM}{x_1^4}\left(2GM - x_1\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{GM}{x_1}\sin^2\left(x_2\right) & 0 & 0 & 0 \end{bmatrix}, \tag{1.16}$$

$$R_{31\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{GM}{x_1^2\left(-2GM + x_1\right)} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{GM}{x_1}\sin^2\left(x_2\right) & 0 & 0 \end{bmatrix}, \tag{1.17}$$

$$R_{32\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2G}{x_1}M \\ 0 & 0 & \frac{2G}{x_1}M\sin^2\left(x_2\right) & 0 \end{bmatrix}, \tag{1.18}$$

$$R_{33\mu}{}^{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.19}$$

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2 Regge-Wheeler coordinates

For $r \geq 2GM$, we define the Regge-Wheeler radius r^* by

$$r^* \equiv r + 2GM \ln\left(\frac{r}{2GM} - 1\right). \tag{2.1}$$

Since the derivative of this radius is

$$\begin{aligned} \mathrm{d}r^* &= \left(1 + \frac{1}{r/\left(2GM\right) - 1}\right)\mathrm{d}r \\ &= \frac{1}{1 - 2GM/r}\mathrm{d}r, \end{aligned} \tag{2.2}$$

with this radius the metric of the Schwarzschild solution (1.1) becomes

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \\ &= \left(1 - \frac{2GM}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}{r^*}^2\right) + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \end{aligned} \tag{2.3}$$

Besides, with this coordinates the Laplacian becomes

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\left(\frac{f}{r}\right)\right) \\ &= -\frac{1}{r}\frac{1}{1 - 2GM/r}\left(\frac{\partial^2}{\partial t^2}f - \frac{\partial^2}{\partial {r^*}^2}f + \left(1 - \frac{2GM}{r}\right)\left(-\frac{1}{r^2\sin\left(\theta\right)}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}f - \frac{1}{r^2\sin^2\left(\theta\right)}\frac{\partial^2}{\partial\phi^2}f + \frac{2GM}{r^3}f\right)\right) \end{aligned} \tag{2.4}$$

(beware that r and r^* are different).

References

[1] 福間 将文, 酒谷雄峰, 『重力とエントロピー 〜 重力の熱力学的性質を理解するために 〜』(SGC ライブラリ 90), サイエンス社 (2014).