Multiway Cut

Sean Lowen

2-2/k Approx

Understan

LP

Randomization

Analysis

Modifying LP

Multiway Cut

Sean Lowen

Franklin W. Olin College of Engineering

November 22, 2011

Problem

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Randomization

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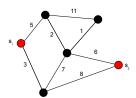
Input

An undirected graph G = (V, E) with

- ▶ Non-negative weights w_e , $\forall e \in E$
- ▶ Set of terminals $S = \{s_1, s_2, \cdots, s_k\} \subseteq V$

Output

Subset of edges at min cost whose removal disconnects all terminals from each other



Problem

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2-2/k Approx

LP

Randomizatio

Analysis

Modifying LP

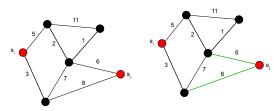
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$$Cost = 14$$

Problem

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Understandir

LP

Randomization

Analysi

Modifying LP

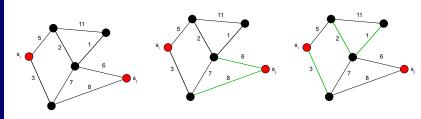
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Outline

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understandin

Pandomizatio

Analysis

Modifying LP

Simple 2-2/k Approximation

LP Relaxation

Understanding the LP Solution

Randomized Rounding Algorithm

Analysis

Modifying LP Solution

Algorithm

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understandin

Randomization

Analysis

Modifying LP

Multiway Cut Algorithm

- 1. $S = \{s_1, \dots, s_k\}$
- 2. **for** $i \leftarrow 1$ to k **do**:

$$t_i \leftarrow \text{contraction of } S \setminus \{s_i\}$$

$$Z_i \leftarrow \min s_i - t_i \text{ cut}$$

3. let
$$w(Z_1) \leq w(Z_2) \leq \cdots \leq w(Z_k)$$

4. **return**
$$\bigcup_{i=1}^{k-1} Z_i$$

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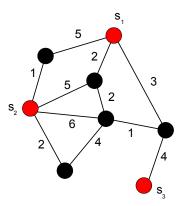
2-2/k Approx

Simplex

Jnderstandin D

Randomization

Analysis



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2-2/k Approx

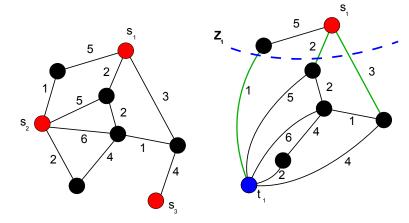
Simple

Understandin

Randomization

Analysi

Modifying LP Solution



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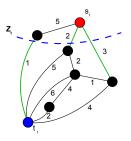
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Simplex

Jnderstandin

Randomization

Analysis



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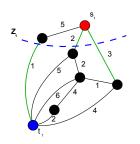
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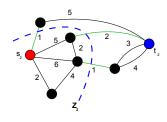
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Jnderstandin

Randomization

Analysis





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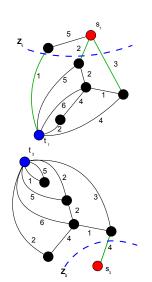
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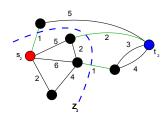
Simple

Understandin

Randomization

Analysi





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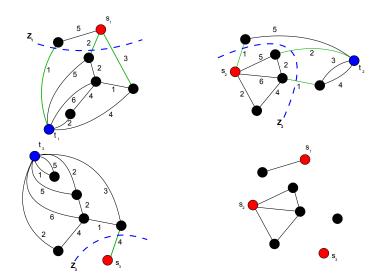
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Simple

Understandin

Randomization

Analysis



Cost = 11

Analysis

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Simple>

Understandin

Randomizatio

Analysis

Modifying LP

► A*: optimal solution

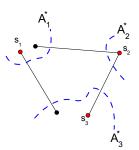
▶ A_i^* : edges in A^* with exactly one endpoint in the partition containing s_i

$$w(Z_1) \leq w(A_1^*)$$

$$w(Z_2) \leq w(A_2^*)$$

$$\vdots$$

$$w(Z_k) \leq w(A_k^*)$$



$$w(\bigcup_{i} Z_{i}) \leq \sum_{i=1}^{k} w(A_{i}^{*}) = 2 \cdot OPT$$

$$w(\bigcup_{i=1}^{k-1} Z_{i}) \leq (1 - \frac{1}{k})w(\bigcup_{i} Z_{i}) \leq 2(1 - \frac{1}{k}) \cdot OPT$$

Analysis

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2-2/k Approx

Simple

Understandi LP

Randomizatio

Analysis

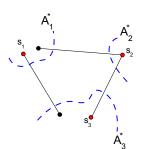
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$$w(\bigcup_{i} Z_{i}) \leq \sum_{i=1}^{k} w(A_{i}^{*}) = 2 \cdot OPT$$

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2-2/k Approx

Simple

Inderstanding

Randomization

Analysis

Modifying LP Solution

Theorem

The above gives a $(2-\frac{2}{k})$ -approximation for multiway cut.

Outline

Multiway Cut

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2-2/k Approx

Simplex Understandi

Randomizatio

Analysis

Modifying LP

Simple 2-2/k Approximation

LP Relaxation

Understanding the LP Solution

Randomized Rounding Algorithm

Analysis

Modifying LP Solution

Multiway Cut

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2-2/k Approx

Simplex

Understandin LP

Randomization

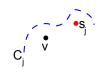
Analysis

Modifying LP

- \triangleright $s_i \in C_i$
- ▶ $\delta(C_i)$: set of edges leaving C_i

- ▶ Obj: min $\bigcup_i w(\delta(C_i))$
- In other words: $\min \frac{1}{2} \sum_{e \in F} w_e \sum_{i=1}^k z_e^i$





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2-2/k Approx

Simplex

Understanding LP

Randomization

Analysis

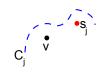
Modifying LP

- $ightharpoonup s_i \in C_i$
- ▶ $\delta(C_i)$: set of edges leaving C_i

$$z_e^i = \begin{cases} 1, & \text{if } e \in \delta(C_i) \\ 0, & \text{otherwise} \end{cases}$$

- ▶ Obj: min $\bigcup_i w(\delta(C_i))$
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2-2/k Approx

Simplex

Understanding

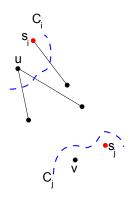
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Multiway Cut

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2-2/k Approx

Simplex

Understanding LP

Randomization

Analysis

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- In other words: $\min \frac{1}{2} \sum_{e \in F} w_e \sum_{i=1}^{k} z_e^i$

$$C_{i}$$

$$S_{i} \quad u_{i} = 1$$

$$U_{i} \quad u_{j} = 0$$

$$\begin{aligned} \mathbf{v}_i &= 0 \\ \mathbf{v}_j &= 1, & \\ \mathbf{C}_j^I & & \\ \end{aligned}$$

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Simplex

Understandin LP

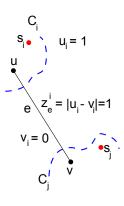
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2-2/k Approx

Simplex

Understanding

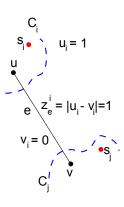
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2-2/k Approx

Simplex

Understandin

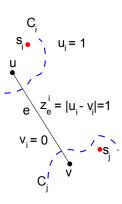
Randomization

Analysis

Modifying LP Solution

- ▶ W.l.o.g., any feasible solution will partition the vertices into C_1, C_2, \dots, C_k
- $ightharpoonup s_i \in C_i$
- ▶ $\delta(C_i)$: set of edges leaving C_i

- ▶ Obj: min $\bigcup_i w(\delta(C_i))$
- In other words: $\min \frac{1}{2} \sum_{e \in F} w_e \sum_{i=1}^k z_e^i$



Constructing the LP

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2-2/k Approx

Simplex

Understa

Randomization

Analysis

Modifying LI

Integer Program

minimize
$$\frac{1}{2} \sum_{e \in E} w_e \sum_{i=1}^k z_e^i$$
 (1)

s.t.
$$\sum_{i=1}^{k} v_{i} = 1, \qquad \forall v \in V$$

$$z_{e}^{i} = |u_{i} - v_{i}|, \quad \forall e = (u, v) \in E, \quad 1 \leq i \leq k$$

$$s_{i} = 1, \quad \forall s \in S$$

$$v_{i} \in \{0, 1\}, \quad \forall v \in V, \quad 1 \leq i \leq k$$
(2)
$$(2)$$

$$(3)$$

$$(4)$$

$$(5)$$

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Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understan

Randomization

Analysis

Modifying LP

LP Relaxation

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$$z_e^i = |u_i - v_i|, \quad \forall e = (u, v) \in E, \quad 1 \le i \le k$$

$$s_{i} = 1, \quad \forall s \in S$$

$$v_i > 0, \quad \forall v \in V, \quad 1 < i < k$$

$$(5)$$

Interpreting the LP

Multiway Cut

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2-2/k Appro

Simplex Understan

LP

Randomizatio

Analysis

- $ightharpoonup x_v = (x_v^1, x_v^2, ..., x_v^k)$, a point in \mathbb{R}^k
- $\rightarrow x_v^i \leftarrow v_i$
- ▶ all coordinates of x_v sum to 1
- x_v belongs to the (k-1)-simplex
- \triangleright x_{s_i} : unit vector with i^{th} coordinate equal to 1

▶ Let
$$d_e = \frac{1}{2} \sum_{i=1}^k z_e^i = \frac{1}{2} \sum_{i=1}^k |x_u^i - x_v^i|$$

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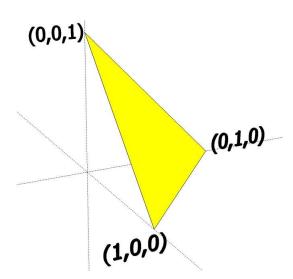
2-2/k Approx

Simplex

Understandin

Randomization

Analysis



Interpreting the LP

Multiway Cut

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2-2/k Appro

Simplex Understa

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- ► Let $d_e = \frac{1}{2} \sum_{i=1}^k z_e^i = \frac{1}{2} \sum_{i=1}^k |x_u^i x_v^i|$

Constructing the LP

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understanding LP

Randomization

Analysis

Modifying LP

LP Relaxation

$$\text{minimize} \qquad \frac{1}{2} \sum_{e \in E} w_e \sum_{i=1}^k z_e^i$$

s.t.
$$\sum_{i=1}^{k} v_i = 1, \quad \forall v \in V$$
$$z_e^i = |u_i - v_i|, \quad \forall e = (u, v) \in E, \quad 1 \le i \le k$$

$$egin{aligned} s_{i_i} &= 1, & \forall s \in S \ v_i &\geq 0, & \forall v \in V, & 1 \leq i \leq k \end{aligned}$$

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Simplex

LP

s.t.

minimize
$$\sum_{z=z}$$

$$\sum_{e \in E} w_e d_e$$

$$\sum_{i=1}^{K} x_{v}^{i} = 1,$$

$$\forall v \in V$$

$$\sum_{i=1}^{k} x_{v}^{i} = 1,$$
 $d_{e} = \frac{1}{2} \sum_{i=1}^{k} |x_{u}^{i} - x_{v}^{i}|$

$$\forall e = (u, v) \in E$$

$$x_{s_i}^i = 1$$

$$\forall s \in S$$

$$\forall v \in V$$
, $1 \leq i$

$$\forall v \in V, \ 1 \leq i \leq k$$

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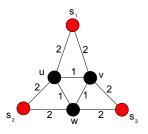
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Randomization

Analysis

Modifying LP



Multiway Cut

Sean Lowen

2-2/k Appro

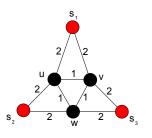
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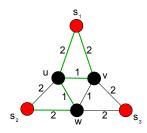
Understandi

Randomization

Analysis

Modifying LP





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2-2/k Appro

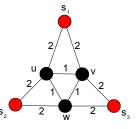
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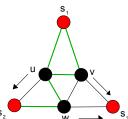
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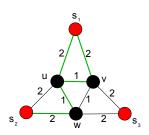
Randomization

Analysis

Modifying LP







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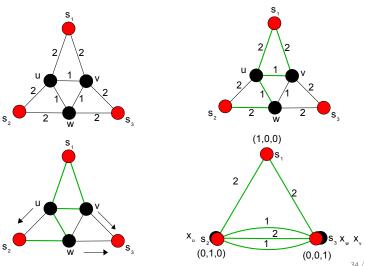
Simplex

Understand

Daniel and and a

Analysis

Modifying LP



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Simplex

LP

minimize

$$\sum_{e \in E} w_e d_e$$

s.t.

$$\sum_{i=1}^k x_{v}^i = 1,$$

$$\sum_{i=1}^{N} x_{v}^{i} = 1,$$

$$\sum_{i=1}^{k} x_{v}^{i} = 1,$$
 $d_{e} = \frac{1}{2} \sum_{i=1}^{k} |x_{u}^{i} - x_{v}^{i}|$

$$\sum_{i=1}^{K} |x_u^i - x_v^i|$$

 $\forall v \in V$

 $\forall e = (u, v) \in E$

$$x_{s_i}=1,$$
 $x_v^i\geq 0,$

$$\forall s \in S$$

$$\forall v \in V, \ 1 \leq i \leq k$$

Outline

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understanding LP

Randomizatio

Analysis

Modifying LP

Simple 2-2/k Approximation

LP Relaxation

Understanding the LP Solution

Randomized Rounding Algorithm

Analysis

Modifying LP Solution

Multiway Cut

Sean Lowen

2-2/k Approx

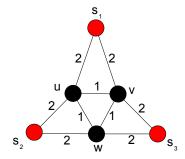
Understanding

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Randomizatio

Analysi

LP



$$OPT = 8$$

Multiway Cut

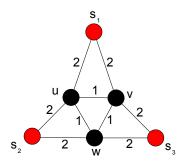
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2-2/k Approx

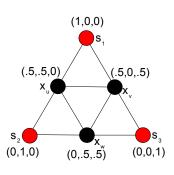
Understanding

Pandamization

Analysis



$$OPT = 8$$



IP-OPT = 7.5

Multiway Cut

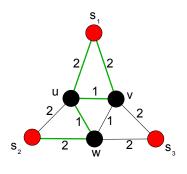
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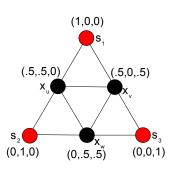
Understanding

Pandamization

Analysis



$$OPT = 8$$



LP-OPT = 7.5

Multiway Cut

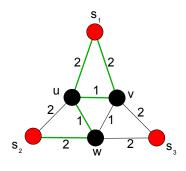
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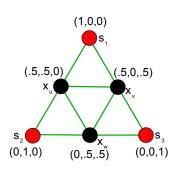
Understanding

Pandomization

Analysis



$$OPT = 8$$



$$LP-OPT = 7.5$$

Outline

Multiway Cut

Sean Lowen

2-2/k Approx

Undanatana

Understand LP

Randomization

Analysis

Modifying LP

Simple 2-2/k Approximation

LP Relaxation

Understanding the LP Solution

Randomized Rounding Algorithm

Analysis

Notation

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Jnderstanding

Randomization

Analysis

Modifying LP

► Assume that the LP gives a solution in which, for any edge (u,v), its endpoints differ in at most two coordinates.

 \triangleright E_i : edges whose endpoints differ in coordinate i.

 $ightharpoonup W_i$: $\sum_{e \in E_i} w_e d_e$

 $B(s_i, \rho): \{v \in V | x_v^i \ge \rho\}.$

 $ightharpoonup C_i$: partition containing s_i

Algorithm

Multiway Cut

Sean Lowen

2-2/k Appro

Simplex

Understanding

Randomization

Analysi

Modifying LP

Algorithm

- 1. Compute optimum LP solution.
- 2. Renumber the terminals so that $W_1 \leq \cdots \leq W_k$
- 3. Pick uniformly at random $\rho \in (0,1)$ and $\sigma \in \{(1,2,\ldots,k-1,k),(k-1,k-2,\ldots,1,k)\}.$
- 4. **for** i=1 to k-1: $C_{\sigma_{(i)}} \leftarrow B(s_i, \rho) \bigcup_{j < i} C_{\sigma_{(j)}}$
- 5. $C_k \leftarrow$ all remaining vertices
- 6. C: set of edges that run between sets C_1, \ldots, C_k
- 7. return C

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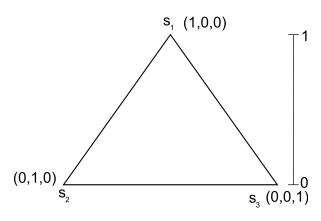
2-2/k Approx

Simplex

Understandin

Randomization

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$$\rho < \frac{1}{2}, \ \sigma = (1, 2, 3)$$

Multiway Cut

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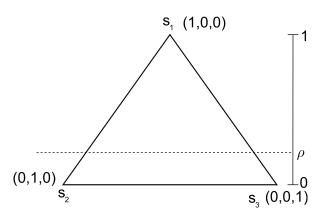
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Simplex

Understandin

Randomization

Analysis



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Multiway Cut

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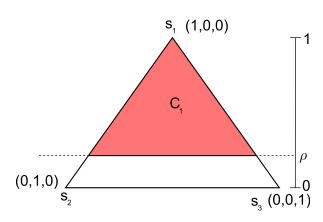
2-2/k Approx

Simplex

Understandin

Randomization

Analysis



$$\rho < \frac{1}{2}, \ \sigma = (1, 2, 3)$$

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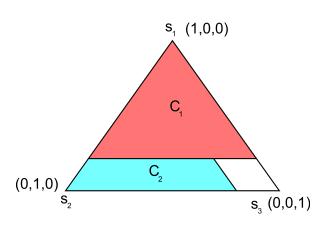
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Understandir

Randomization

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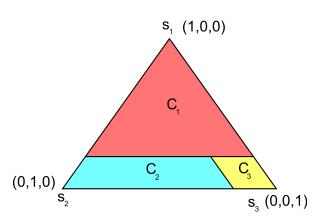
2-2/k Approx

Simplex

Understandir

Randomization

Analysis



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Outline

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understand

Randomizatio

Analysis

Modifying LP Solution Simple 2-2/k Approximation

LP Relaxation

Understanding the LP Solution

Randomized Rounding Algorithm

Analysis

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understanding LP

Randomization

Analysis

Modifying LP

Lemma

- ► Consider edge e = (u, v), where u and v differ in coordinates i and j
- ▶ By definition, $x_u^i x_v^i = x_v^j x_u^j = d_e$
- ▶ Edge *e* will only be cut if x_u and x_v end up in different partitions, i.e., ρ falls in either the (x_u^i, x_v^i) or (x_v^j, x_u^j) intervals
- ▶ Let's call those intervals α and β , respectively

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

LP LP

Randomization

Analysis

Modifying LP

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2-2/k Approx

Jimplex

LP

Randomizatio

Analysis

Modifying LP

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Multiway Cut

Sean Lowen

2-2/k Approx

Understandin

Randomization

Analysis

Modifying LP

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 $\mathsf{Multiway}\ \mathsf{Cut}$

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2-2/k Approx

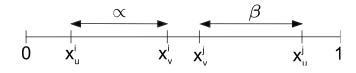
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Understandi

Randomization

Analysis





$$\Pr[e \in C] = \Pr[\rho \in (\alpha \cup \beta)] = |\alpha| + |\beta| \le 2d_e$$

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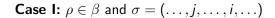
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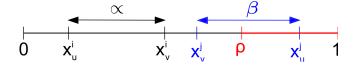
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Understanding

Randomization

Analysis





Case I: $\rho \in \beta$ and $\sigma = (\ldots, j, \ldots, i, \ldots)$

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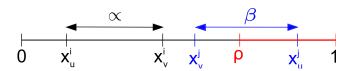
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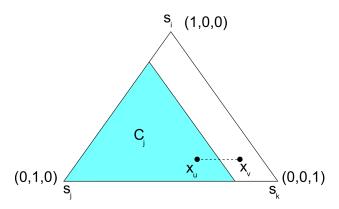
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Understandin

Randomization

Analysis





Multiway Cut

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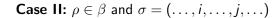
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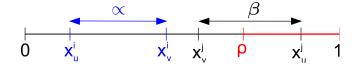
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Understanding

Randomization

Analysis





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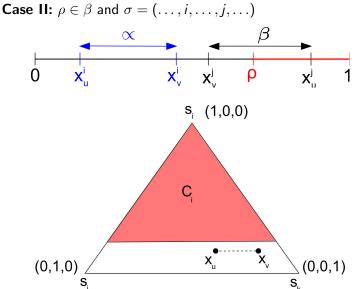
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Understandin

Randomization

Analysis



Multiway Cut

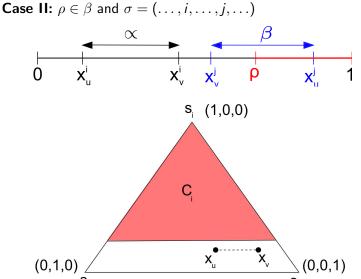
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2-2/k Approx

Understandin

Randomization

Analysis



Multiway Cut

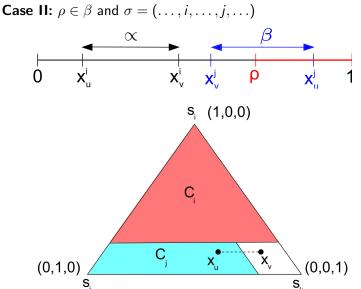
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2-2/k Approx

Understandin

Randomization

Analysis



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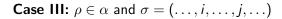
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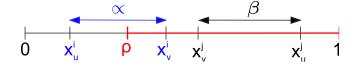
Simplex

Understanding

Randomization

Analysis





Multiway Cut

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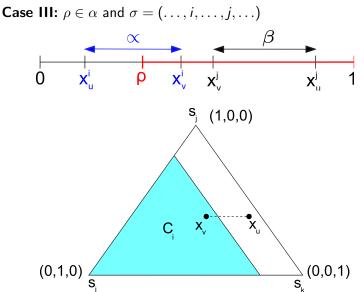
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Understandin

Randomization

Analysis



Multiway Cut

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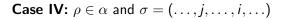
2-2/k Approx

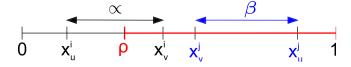
Simplex

Understanding

Randomization

Analysis





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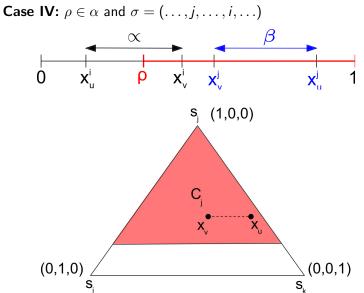
2-2/k Approx

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Understandin

Randomization

Analysis



Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understand

Randomization

Analysis

Modifying LP

$$\Pr[e \in C] = \Pr[\rho \in \beta] + \Pr[(\rho \in \alpha) \land (\sigma = (\dots, i, \dots, j, \dots))]$$
$$= |\beta| + \frac{|\alpha|}{2} \le 1.5d_e$$

Lemma

If $e \in E - E_k$, $\Pr[e \in C] \le 1.5d_e$

Analysis: $e \in E_k$

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

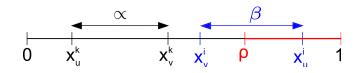
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Randomization

Analysis

Modifying LF

An edge in E_k can only be cut if $\rho \in \beta$.



- $ightharpoonup \sigma$ always = (\ldots, i, \ldots, k)
- ▶ k gets all remaining vertices
- ▶ $\Pr[e \in C] = |\beta| = d_e$

Analysis: $e \in E_k$

Multiway Cut

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2-2/k Approx

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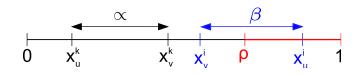
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Randomization

Analysis

Modifying LP

An edge in E_k can only be cut if $\rho \in \beta$.



- $ightharpoonup \sigma$ always = (\ldots, i, \ldots, k)
- ▶ k gets all remaining vertices
- ▶ $\Pr[e \in C] = |\beta| = d_e$

Lemma

If $e \in E_k$, $\Pr[e \in C] \leq d_e$

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understandin

Randomization

Analysis

Modifying LP

$$OPT_f = \sum_{e \in E} w_e d_e$$

Note that every edge belongs to exactly 2 of the E_i sets.

$$\sum_{i=1}^{k} W_i = 2 \cdot OPT_f$$

Since W_k is the largest of the sets

$$W_k = \sum_{e \in E_k} w_e d_e \ge \frac{2}{k} \cdot OPT_f$$

Multiway Cut

Sean Lowen

2-2/k Approx

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LP

Randomization

Analysis

Modifying LP

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Multiway Cut

Sean Lowen

2-2/k Approx

Understandi

Randomization

Analysis

Modifying LP

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Multiway Cut

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2-2/k Approx

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Understandi

Randomization

Analysis

$$\begin{aligned} \mathbf{E}[w(C)] &= \sum_{e \in E} w_e \mathbf{Pr}[e \in C] \\ &= \sum_{e \in E - E_k} w_e \mathbf{Pr}[e \in C] + \sum_{e \in E_k} w_e \mathbf{Pr}[e \in E_k] \\ &\leq 1.5 \sum_{e \in E - E_k} w_e d_e + \sum_{e \in E_k} w_e d_e \\ &= 1.5 \sum_{e \in E} w_e d_e - 0.5 \sum_{e \in E_k} w_e d_e \\ &= 1.5 \sum_{e \in E} w_e d_e - 0.5 W_k \\ &\leq (1.5 - 1/k) \cdot OPT_f \end{aligned}$$

Conclusion

Multiway Cut

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2-2/k Approx

Simplex

nderstanding

Randomization

Analysis

Modifying LP

There is a $(1.5 - \frac{1}{k})$ -approximation for multiway cut.

Outline

Multiway Cut

Sean Lowen

2-2/k Approx

The Least cond

LP

Randomizatio

Analysis

Modifying LP Solution Simple 2-2/k Approximation

LP Relaxation

Understanding the LP Solution

Randomized Rounding Algorithm

Analysis

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

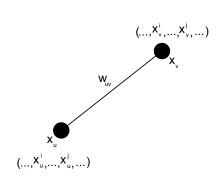
Understandin

Randomization

Analysis

Modifying LP Solution

- \triangleright $x_u \& x_v$ differ in more than two coordinates.
- i: coordinate in which x_u and x_v differ the least. Let xⁱ_v > xⁱ_u.
- $\blacktriangleright \ \text{Let} \ \alpha = x_v^i x_u^i$
- ► There is a coordinate j s.t. $x_u^j x_v^j \ge \alpha$
- $\forall k \neq i, j, x_w^k \leftarrow x_v^k$
- $\triangleright W_{UW}, W_{WV} \leftarrow W_{UV}$



Multiway Cut

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2-2/k Approx

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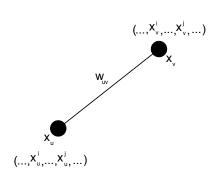
Understanding

Randomization

Analysis

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2-2/k Approx

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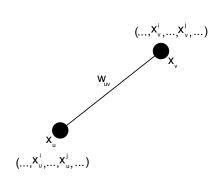
Understandinį

Randomization

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2-2/k Approx

Simple

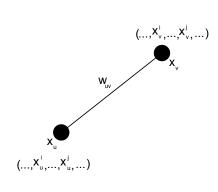
Understandin

Randomization

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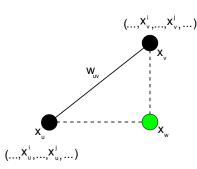


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Modifying LP Solution

- $\triangleright x_{\mu} \& x_{\nu}$ differ in more than two coordinates.
- \triangleright i: coordinate in which x_{ii} and x_{ν} differ the least. Let $x_{i}^{i} > x_{i}^{i}$.
- ightharpoonup Let $\alpha = x_{v}^{i} x_{u}^{i}$
- ► There is a coordinate *i* s.t. $x_{ii}^{j} - x_{ii}^{j} > \alpha$
- $\triangleright \forall k \neq i, j, x_{ii}^k \leftarrow x_{ii}^k$



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2-2/k Approx

Simple

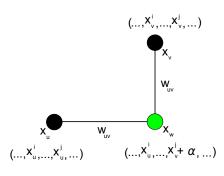
Understandin LP

Randomization

Analysis

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- $\triangleright W_{IJW}, W_{WV} \leftarrow W_{IJV}$



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2-2/k Approx

Simplex

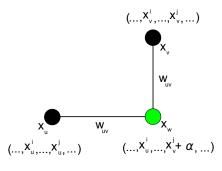
Understandin LP

Randomization

Analysis

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Multiway Cut

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2-2/k Approx

Simplex

Understand

Randomization

Analysis

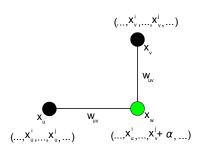
$$w_{uv} d_{uv} \stackrel{?}{=} w_{uv} d_{uw} + w_{uv} d_{wv}$$
$$d_{uv} \stackrel{?}{=} d_{uw} + d_{wv}$$

$$d_{uw} = \sigma - 2\alpha$$

$$d_{uw} = 2\alpha$$

$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$d_{uw} + d_{uv} = \sigma = d_{uv}$$



Multiway Cut

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2-2/k Approx

Understand

Randomization

Analysis

$$w_{uv} d_{uv} \stackrel{?}{=} w_{uv} d_{uw} + w_{uv} d_{wv}$$
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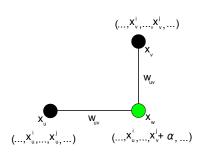
Let
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Multiway Cut

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2-2/k Approx

Simpley

Understand

Randomization

Analysis

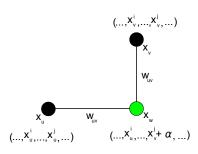
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Multiway Cut

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2-2/k Approx

C'........

Understand

Randomization

Analysis

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$$d_{uv} \stackrel{?}{=} d_{uw} + d_{wv}$$

$$\text{Let } d_{uv} = \sigma$$

$$d_{uw} = \sigma - 2\alpha$$

$$d_{wv} = 2\alpha$$

$$x_{uv} \stackrel{?}{=} w_{uv} d_{uv} + w_{uv} d_{wv}$$

$$x_{v} \stackrel{(..., x_{v}^{i}, ..., x_{v}^{i}, ...)}{}$$

Multiway Cut

Sean Lowen

2-2/k Approx

Simplex

Understand

Randomization

Analysis

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$$\text{Let } d_{uv} = \sigma$$

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$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$(..., x_{u}^{i}, ..., x_{u}^{i}, ...)$$

 $d_{\mu\nu} + d_{\mu\nu} = \sigma = d_{\mu\nu}$

Multiway Cut

Sean Lowen

2-2/k Approx

Simpley

Understand

Randomization

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$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$(..., x_u^i, ..., x_u^i, ...)$$

Multiway Cut

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2-2/k Approx

C'........

Understandi

Randomization

Analysis

Modifying LP Solution

$$w_{uv} d_{uv} \stackrel{?}{=} w_{uv} d_{uw} + w_{uv} d_{wv}$$

$$d_{uv} \stackrel{?}{=} d_{uw} + d_{wv}$$

$$(..., x_{v}^{i}, ..., x_{v}^{i}, ...)$$

$$\downarrow x_{v}$$

$$d_{uw} = \sigma - 2\alpha$$

$$d_{uw} = 2\alpha$$

$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$d_{uw} + d_{wv} = \sigma = d_{uv}$$

$$(..., x_{u}^{i}, ..., x_{u}^{i}, ...)$$

$$(..., x_{u}^{i}, ..., x_{v}^{i}, + \alpha, ...)$$

$$d_{uw} + d_{wv} = \sigma = d_{uv}$$

Lemma

There is a way to modify the LP s.t. for any edge e = (u, v), x_u and x_v differ in 0 or 2 coordinates, at no additional cost.

References

Multiway Cut

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2-2/k Appro

Simplex

Inderstandin P

Randomization

Analysis

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2-2/k Approx

Understandin

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Anaiysis

Modifying LP Solution Thank you!