An Improved Approximation Algorithm for Multiway Cut
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Interim report for CS 6150

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1 TODO

- What is Multiway cut
- Why is multiway cut MAXSNP hard ?
- what is MAXSNP/SNP ?
- Related work - Dalhaus - the below one . check if we should add some more ?
- Ford fulkerson for max flow ????
- Chopra and Rao [6] and Cunningham [7] develop a polyhedral approach to Multiway Cut ?
- Bertisimas non-linear formulation
- The case k=2 is not the only polynomially solvable instance of the multiway cut problem. Lovdsz and Cherkasskij show that when c(e) = 1 and G is Eulerian, then the multiwaycut problem is polynomially solvable.
- write about simplex, polytope

2 Understanding

Isolation Heuristic - In isolation heuristic, if there are K terminals, in each iteration we attach one terminal to the source and all the other terminals to the sink and then run a max-flow and and find the min-cut. Let the edge set after each iteration be E_i. Now the lowest K-1 cuts are taken and the union of them gives the multi-way cut. This algorithm gives an approximation of $2(1 - \frac{1}{k})$. Proof:

- Run the isolation heuristic and get the set of edges. Let them be A.
- Lets say E^* is the optimal edge set for the multiway cut with K terminals. Then it means if E^* is removed then there will be K disjoint connected graphs$(V_1, V_2, \cdots, V_k)$, each having one terminal respectively
- Lets say $E^* = \sum i = 1^k E_i^*$ and each E_i^* represents the edges removed to disconnect V_i from the rest.
- Lets say $\delta(V_i)$ gives the set of all outgoing edges from V_i.
- Now we can say that, $w(E_i) \leq w(\delta(V_i))$ because, both of isolates the terminal i from the rest and we know E_i is the mincut. So $w(E_i)$ cannot be greater than $w(\delta(V_i))$
- Lets says there is m edges between V_i and V_{i+1}. Then both $\delta(V_i)$ and $\delta(V_{i+1})$ will include those m edges.
- $2w(E^*) = \sum_{i=1} k\delta(V_i)$. Its 2 times since each edges is double counted.
- We know $w(A) \leq (1 - \frac{1}{k})(\sum_{i=1}(E_i^*))$. This is because, A is the union of first K-1 smallest set. In this expression, we are adding up all the K values. Since only $k - 1$ values are taken, we are multiplying it with $1 - \frac{1}{k}$ and since it the union of all these set, it will be equal to or less than the summation of indiviual sets.

$$
\begin{aligned}
w(A) &\leq (1 - \frac{1}{k})(\sum_{i=1}(E_i^*)) \\
&\leq (1 - \frac{1}{k})(\sum_{i=1} \delta(V_i) \\
&\leq 2(1 - \frac{1}{k})w(E^*) \\
&\leq 2(1 - \frac{1}{k})OPT
\end{aligned}
$$

How did the authors comeup with the Linear program which they say is an SLP.

- As discussed earlier, any valid solution will split the Graphi $G(V, E)$ into K graphs$(C_1, C_2, \cdots, C_k)$, with each C_i having a terminal S_i
- Let $\delta(C_i)$ represent the set of edges which goes out of C_i
- Each vertex $v \in V$ is represented as x_i^j and this value is set to 1 if the i^{th} vertex is part of the componenet C_i, else its set to 0. Since a vertex can be a part of only one component, we can say that $\sum_{j=1}^k x^j = 1, \forall x \in V$,

- We define another notation z_e^i which operates on all the edges in E and is set to 1 if the edge e is in $\delta(C_i)$ else its set to 0

- If e is in the $\delta(C_i)$, then it connect (u, v) where $u \in C_i, v \notin C_i$. Thus we can write z_e^i as $z_e^i = x_u^i - x_v^i$, since x_u^i will be 1 and x_v^i will be 0, as per the definition of x. If e is not in the $\delta(C_i)$, then both u and v belong to a same component, then both x_u^i and x_v^i will be 1 if both belong to component C_i else both will be 0.

- Now for the objective function is $\frac{1}{2} \sum\limits_{e \in E} c_e * \sum\limits_{i=1}^{k} z_e^i$, where c_e is the cost/weight of the edge. We are multiplying it with $\frac{1}{2}$ because of the double counting which we discussed in the previous explanation.

- Thus the LP is

$$
\begin{aligned}
\min \quad & \frac{1}{2} \sum_{e \in E} c_e * \sum_{i=1}^{k} z_e^i \\
\text{such that} \quad & z_e^i = |x_u^i - x_v^i|, \quad \forall e \in E \\
& \sum_{i=1}^{k} x_v^i = 1, \quad \forall v \in V \\
& x_{s_i}^i = 1, \quad \forall s_i \in T (\text{Set of Terminals}) \\
& x_v^j \geq 0, \quad \forall v \in V \quad (\text{Got by adding LP relaxation to } x_v^j \in (0,1), \quad \forall v \in V)
\end{aligned}
$$