<u>Random variable</u> - Real valued function defined on the sample space

<u>Moments</u> / Mathematical Expectation.

Let X be a discrete random variable with pmf $p_X(x_i)$, $x_i \in X$. We define the expected value of X as

$$\boxed{E(X) = \sum_{x \in X} x_i \, p_X(x_i)}$$

Provided the summation converges

Moments describe the characteristics of the distribution

Symmetric Distribution

Random variable is symmetric about a point α if $P(X \geq \alpha + x) = P(X \leq \alpha - x)$ $\forall x$ or $F(\alpha - x) = 1 - F(\alpha + x) + P(X = \alpha + x)$

$$E(Y) = E(ax + b) = aE(x) + b$$

$$E[g(x)] = \underbrace{\begin{cases} \sum g(x_i)\, p_x(x_i) & \leftarrow \text{Discrete} \\ \int_{-\infty}^{\infty} g(x) f_x(x)\, dx & \leftarrow \text{Cont} \end{cases}}_{\text{Abs Convergence is required}}$$

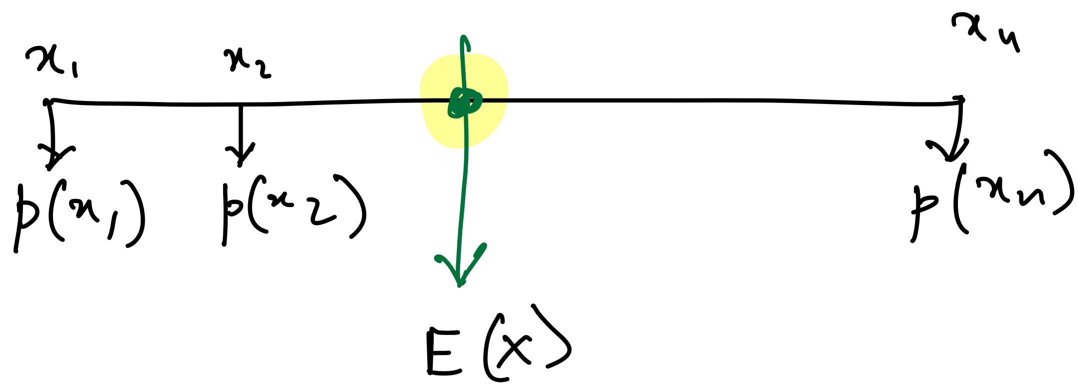
K^{th} Moment

$$\boxed{\mu'_k = E(x^k)}$$

→ k^{th} moment about origin
→ k^{th} non central moment

$$\mu'_1 = E(x)$$

$\mu'_1 \rightarrow$ Expected value



$$\boxed{\mu_k = E\left[(x-\mu)^k\right]}$$

→ k^{th} Central Moment

$\mu_2 \rightarrow$ Variance of X

$$\boxed{\mu_2 = Var(x) = E\left[(x-\mu)^2\right]}$$

$$\mu_1 = 0$$

$$SD = \sqrt{Var(X)}$$

$$\mu_k = E(x-\mu)^k$$

Binomial Expansion

$$= E\left[x^k - \binom{k}{1} x^{k-1}\mu + \binom{k}{2} x^{k-2}\mu^2 - \cdots \right.$$
$$\left. \cdots\cdots + (-1)^k \mu^k \right]$$

$$= \mu_k' - \binom{k}{1}\mu_{k-1}'\mu + \binom{k}{2}\mu_{k-2}'\mu^2 - \cdots\cdots$$
$$\cdots + (-1)^k \mu^k$$

$$\mu_2 = \mu_2' - 2\mu_1'\mu + \mu^2$$

$$\mu_2 = \mu_2' - [\mu_1']^2$$

$$\boxed{Var(X) = \mu_2 = E(x^2) - \{E(x)\}^2}$$

$\mu_2 \geq 0 \Rightarrow E(x^2) \geq \{E(x)\}^2$

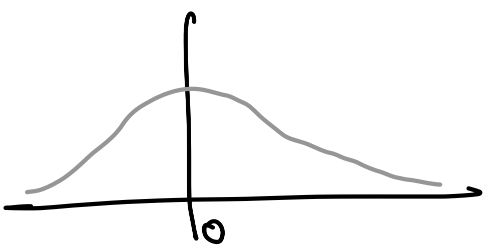
$\beta_k' = E\left[|x|^k\right] \longrightarrow$ k^{th} abs moment about origin

$\beta_k = E\left[|x-\mu|^k\right] \rightarrow$ k^{th} abs central moment

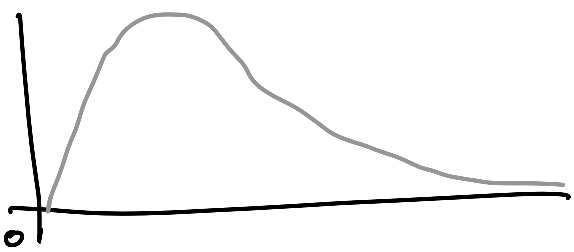
$\alpha_k = E\left[x(x-1)\right] \ldots\ldots \; x-k+1$

↳ k^{th} factorial monet

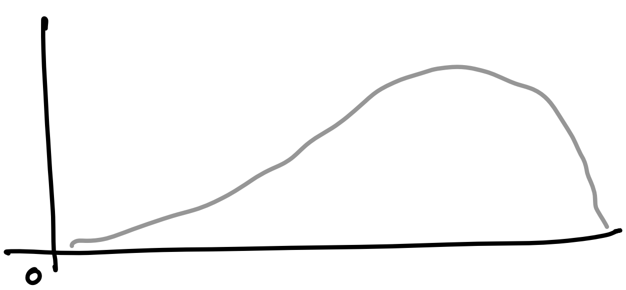
<u>Skewness</u> $\beta_1 = \dfrac{\mu_3}{\mu_2^{3/2}} = \dfrac{\mu_3}{\sigma^3}$



Symmetric $\beta_1 = 0$

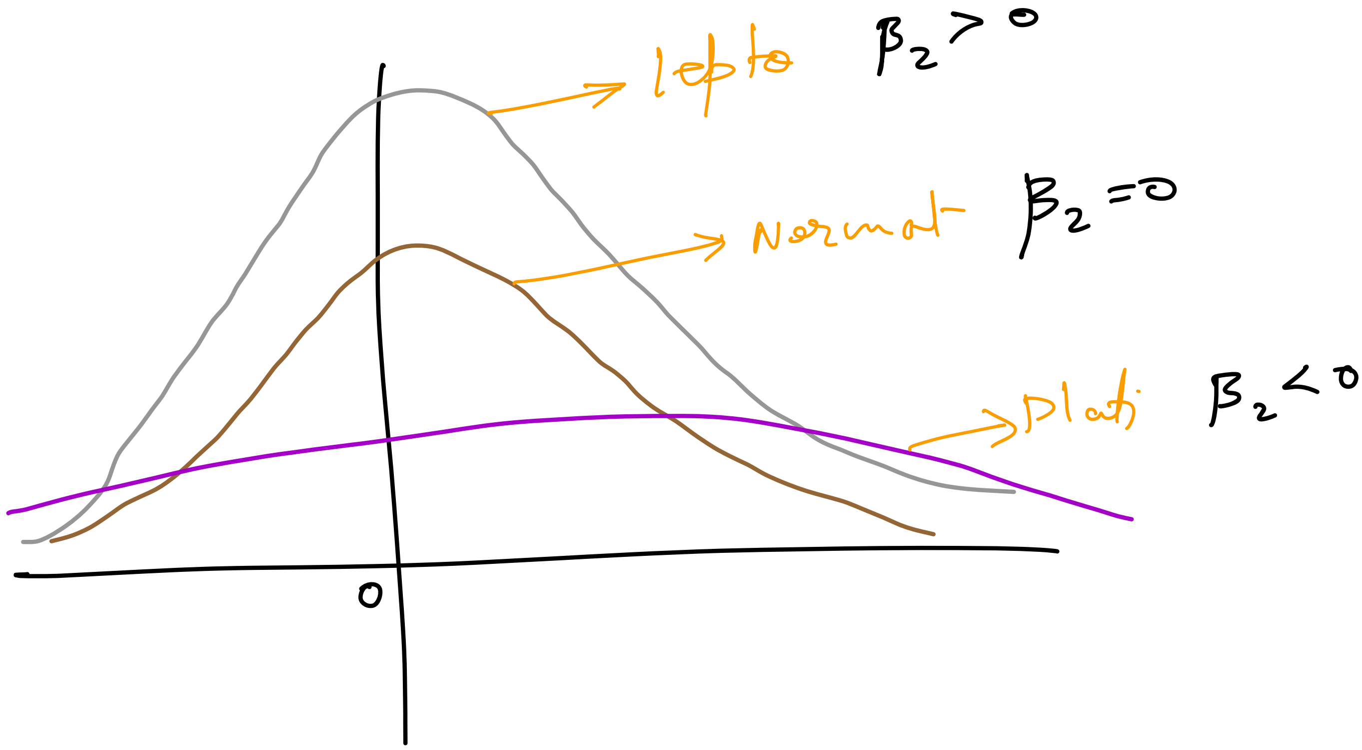


+ve Skew $\beta_1 > 0$



-ve Skew $\beta_1 < 0$

Kurtosis $\quad \beta_2 = \dfrac{\mu_4}{\mu_2^2} - 3$



Theorem: If moment of order t (>0) exists, then the moments of order s $(0 < s < t)$ exists. for a given random variable

Proof

Let X be a continuous random variable with pdf $f_X(x)$

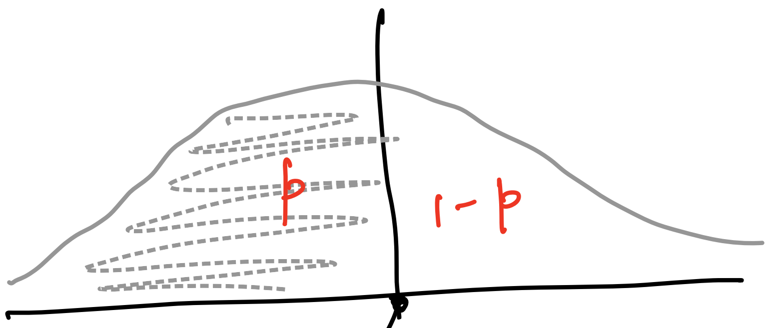
$$E|X|^s = \int_{-\infty}^{\infty} |x|^s f_X(x)\, dx$$

$$= \int_{|x|\le 1} |x|^s f_X(x)\, dx + \int_{|x|>1} |x|^s f_X(x)\, dx$$

$$\le P(|X| \le 1) + \int_{|x|>1} |x|^t f_X(x)\, dx$$

$$\le 1 + E|x|^t \quad < \infty$$

Quartiles



$a \longrightarrow a$ is p^{th} quartile

$Q_{1/2} \longrightarrow$ Median

$Q_{1/4} \quad Q_{3/4} \longrightarrow$ Quartiles

$Q_{1/10} \quad Q_{2/10} \ldots Q_{9/10} \longrightarrow$ Deciles

$Q_{1/100} \quad Q_{2/100} \ldots Q_{99/100} \longrightarrow$ percentiles

In descrete distribution we may not have unique Quantiles

Moment generating function

$$\boxed{M_x(t) = E(e^{tx})}$$ → This may not exist

Theorem: The moment generating function uniquely determines a cdf, and conversely

If the moment generating function exists, it is unique.

Theorem: If the mgf $M_x(t)$ exists for $|t| < t_0$ the derivatives of all orders exists at $t = 0$ and can be evaluated under the integral sign

$$\left.\frac{d^k M_x(t)}{dt^k}\right|_{t=0} = \mu_k' \qquad k = 1, 2 \ldots$$

Remark

$$M_X(t) = E(e^{tX})$$

$$= E\left(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \cdots\right)$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots$$

Coeff of $\frac{t^k}{k!}$ is μ_k' $\quad k = 1, 2 \ldots$

Theorem Let $\{\mu_k'\}$ be the moment sequence of r.v X. If series $\sum \frac{\mu_k'}{k!} t^k$ converges absolutely for some $t > 0$, then $\{\mu_k'\}$ uniquely determines the cdf of the r.v X.

Chebyshev's Inequality

Let X be a r.v. with mean μ and variance σ^2. Then for any $k>0$

$$\underbrace{P\left(|X-\mu| \geq k\right) \leq \frac{\sigma^2}{k^2}}$$

We don't have full information about the random variable except σ and μ.
be we can get the bound for the probability.

Proof

Let X be continuous with pdf $f_X(x)$

$$\sigma^2 = \text{Var}(X) = E(X-\mu)^2$$

we reduced region of integration

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx \geq \underbrace{\int_{|x-\mu| \geq k} (x-\mu)^2 f_X(x)\, dx}$$

$$\int_{|x-\mu|\geq k} (x-\mu)^2 f_X(x)\,dx \geq k^2 \int f_X(x)\,dx$$

from assumption $(x-\mu) \geq k^2$

$$\sigma^2 \geq k^2 P(|X-\mu| \geq k)$$

$$\frac{\sigma^2}{k^2} \geq P(|X-\mu| \geq k)$$

<u>Alternate forms</u>

$$1 - P(|X-\mu| \geq k) \geq 1 - \frac{\sigma^2}{k^2}$$

$$P(|X-\mu| < k) \geq 1 - \frac{\sigma^2}{k^2}$$

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Markov Inequality

Let X be a r.v and g be a non negative even and non decreasing function of $|x|$

$$P(|X| \geq k) \leq \frac{E\, g(X)}{g(k)}$$

Ex

The number of customers who visit a store everyday is a r.v X with $\mu = 18$ and $\sigma = 2.5$ with what prob. can we assert that there will be between 8 to 28 customers

$$P(8 \leq X \leq 28) = P(-10 \leq X - 18 \leq 10)$$

$$= P(|X - 18| \leq 10) \geq 1 - \frac{\sigma^2}{100} = \frac{15}{16}$$

Ex2

Show that for 40 000 flips of a fair coin, the prob. is at least 0.99 that the proportion of heads will be between 0.475 to 0.525

$X \to$ no. of heads $\quad X \sim Bin(40000, \frac{1}{2})$

$n = 40000$

Mean $= np = 20000$

Std $= \sqrt{npq} = 100$

$$P\left(0.475 \le \frac{X}{n} \le 0.525\right)$$

$$= P(19000 \le X \le 21000)$$

$$= P(|X - 20000| \le 1000) \ge 1 - \frac{10000}{(1000)^2}$$

$$= \frac{99}{100}$$

Ex 3

Independent observations are available from a population with mean μ and variance 1. How many observations are needed in order that the prob. is at least 0.9 that the mean of obs. differs from μ but not more than 1

Sol^n

$$E(\bar{x}) = \frac{1}{n} E(x_1 + \dots x_n) = \mu$$

$$V(\bar{x}) = V\left(\frac{x_1 + \dots + x_n}{n}\right) = \frac{1}{n^2}$$

$$\sum v(x_1) = \frac{1}{n}$$

$$P\left(|\bar{x} - \mu| < 1\right) \geq 1 - \frac{1}{n} > 0.9$$

$$\Rightarrow n > 10$$

Discrete Uniform Distribution

$X \rightarrow 1, 2, 3 \ldots N$

$P(X=j) = 1/N \qquad j = 1, 2 \ldots N$

$$\boxed{E(X) = \sum_{j=1}^{N} \frac{j}{N} = \frac{N+1}{2}}$$

$$E(X^2) = \sum_{j=1}^{N} \frac{j^2}{N} = \frac{(N+1)(2N+1)}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\boxed{Var(X) = \frac{N^2 - 1}{12}}$$

All moments exist for uniform distribution

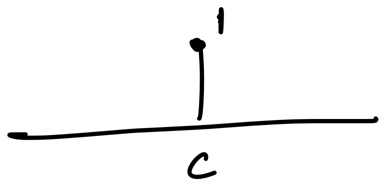
$$M_x(t) = E\left(e^{tx}\right) = \sum_{j=1}^{N} e^{tj} \frac{1}{N}$$

$$\boxed{M_x(t) = \begin{cases} \dfrac{e^t\left(e^{Nt} - 1\right)}{N\left(e^t - 1\right)} & t \neq 0 \\ 1 & t = 0 \end{cases}}$$

Degenerate Distribution

(Sure event)

$$P(X = c) = 1$$



$$E(X) = c$$

$$\mu'_k = c^k$$

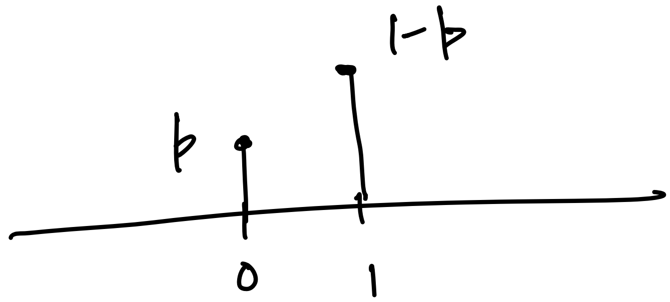
Bernoulli Distribution

$x \rightarrow$ Success or failure

Success $\longrightarrow 1 \longrightarrow p$

Failure $\longrightarrow 0 \longrightarrow 1-p$

$$\begin{cases} p_x(0) = 1-p \\ p_x(1) = p \end{cases} \qquad 0<p<1$$



$$E(x) = 0(1-p) + 1.p$$

$$\boxed{E(x) = p}$$

$$\boxed{\mu'_k = E(x^k) = p} \longrightarrow k=1,2 \dots$$

$$V(x) = p - p^2$$

$$\boxed{V(x) = p(1-p) = pq}$$

$$M_x(t) = E(e^{tx}) = (1-p)e^{t.0} + pe^t$$
$$= 1-p+pe^t$$
$$= q+pe^t$$

Binomial Distribution

(Generalization of Bernoulli Distribution)

n Independent and identical Bernoulli trials

with p of success $= p$

$X \rightarrow$ No of Success in n trials

$\hookrightarrow X = 0, 1, 2 \ldots n-1$

$$\boxed{p_X(j) = P(X = j) = \underbrace{\binom{n}{j}}_{\text{We can choose in } nC_j \text{ ways}} \underbrace{p^j}_{j \text{ success}} \underbrace{(1-p)^{n-j}}_{-j \text{ failure}} \quad j = 0 \ldots n-1}$$

Is it a probability distribution ?

proof

$$\sum_{j=0}^{N-1} p_X(j) = \sum_{j=0}^{N-1} \binom{N}{j} p^j (1-p)^{N-j} = (1-p+p)^N$$

$$= 1^N = 1$$

$$\mu_1' = E(x) = \sum_{j=0}^{n-1} j\, p_x(j)$$

$$= \sum_{j=0}^{n-1} j \binom{n}{j} p^j (1-p)^{-j}$$

$$= \sum_{j=1}^{n-1} j \frac{n!}{j!\,(n-j)!} p^j (1-p)^{n-j}$$

$$= \sum_{j=1}^{n-1} \frac{n!}{(j-1)!\,(n-j)!} p^j (1-p)^{n-j}$$

Set $j-1=i$

$$= np \underbrace{\sum_{i=0}^{n-2} \frac{(n-1)!}{i!\,(n-1-i)!} p^i (1-p)^{n-1-i}}$$

$$\boxed{\mu_1' = E(x) = np}$$

$$\boxed{E[x(x-1)] = n(n-1)p^2}$$

$$E[x^2] = E[x(x-1)] + E[x]$$

$$= n(n-1)p^2 + np$$

$$Var(x) = Ex^2 - (E[x])^2$$

$$\boxed{Var(x) = np(1-p) = npq}$$

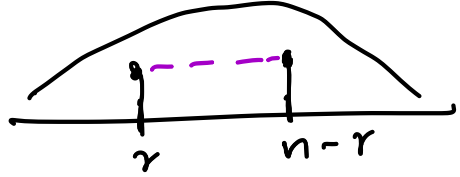
$$\boxed{\mu_3 = \underbrace{np(1-p)}_{\text{always +ve}}\underbrace{(1-2p)}_{\text{This determines the symmetry}}}$$

If $p = 1/2$

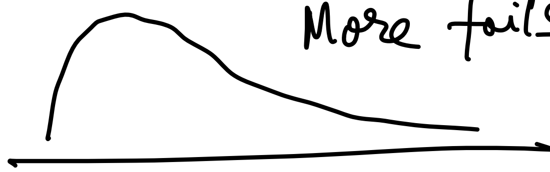
$\mu_3 = 0$

$$\boxed{\beta_1 = \frac{\mu_3}{\sigma^3}} = \frac{npq(1-2p)}{(npq)^{3/2}} = \boxed{\frac{1-2p}{(npq)^{1/2}}}$$

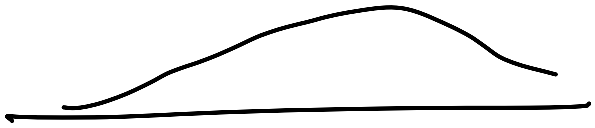
$$\beta_1 = \begin{cases} 0 & \text{if } p = 1/2 \\ >0 & \text{if } p < 1/2 \\ <0 & \text{if } p > 1/2 \end{cases}$$



More fails



More success



$$\mu_4 = 3(npq)^2 + npq(1-6pq)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3$$

$$\beta_2 = \frac{1-6pq}{npq} = \begin{matrix} 0 & \text{if } pq = 1/6 \\ >0 & \text{if } pq < 1/6 \\ <0 & \text{if } pq > 1/6 \end{matrix}$$

Note

$n \to \infty$ the binomial distribution becomes symmetric

$$M_X(t) = E\left(e^{tX}\right)$$

$$= \sum_{j=0}^{n} e^{tj} \binom{n}{j} p^j (1-p)^{n-j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (pe^t)^j (1-p)^{n-j}$$

$$= (1-p+pe^t)^n = (q+pe^t)^n$$

Ex: An airline knows that 5% of the people making reservation do not turn up for the flight. So it sells 52 tickets for a 50 seat flight. What is the prob that every passenger who turns up will get a seat.

p of success $= \dfrac{95}{100}$ $\qquad n = 52$

p of failure $= \dfrac{5}{100}$

$X \rightarrow$ No of people that turn up

$$P(X \leq 50) = \sum_{j=0}^{49} \binom{52}{j} \left(\frac{5}{100}\right)^{j} \left(\frac{95}{100}\right)^{52-j}$$

or

$$= 1 - P(X = 51) - P(X = 52)$$

$$= 0.74$$ ← Not good, they must sell less tickets

Geometric Distribution

(First Success)

Independent Bernoullion trials are conducted till a success is achieved.

Let

$X \to$ no. of trials needed for first success

$X \to 1, 2, 3 \ldots\ldots$

observe that this is a ∞ distribution compared to binomial dist.

$$\boxed{P(X=j) = q^{j-1} p \; ; \; j = 1, 2 \ldots}$$

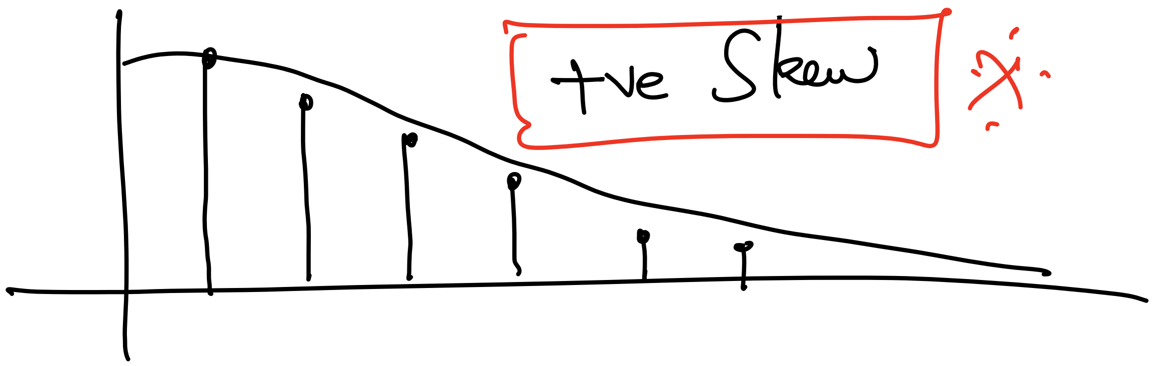
first success at j

$$\boxed{\mu_1' = E(X) = \sum_{j=1}^{\infty} j\, q^{j-1} p = \frac{1}{p}}$$

$$\boxed{\mu_2' = \frac{q+1}{p^2}}$$

$$Var(x) = \mu_2 = \mu_2' - (\mu_1')^2$$

$$\boxed{Var(x) = \frac{q}{p^2}}$$



+ve Skew

$$M_x(t) = \frac{pe^t}{1-qe^t}$$

$$0 < qe^t < 1$$

$$e^t < 1/q$$

$$t < -\log q$$

Ex Suppose independent tests are conducted on monkeys to develop a vaccine. if prob of success is $1/3$ in each trial. what is the prob. that at least 5 trials are needed for first success.

$X \to$ No. of trials need for first success

$$P(X \geq 5) = \sum_{j=5}^{\infty} p q^{j-1}$$

$$= 1 - P(X=1) - P(X=2) - P(X=3) - P(X=4)$$

$$= 1 - pq^0 - pq^1 - pq^2 - pq^3$$

$$= 1 - p - pq - pq^2 - pq^3$$

$$= q - pq - pq^2 - pq^3$$

$$= q^2(1-p) - pq^3$$

$$= q^3 - pq^3$$

$$= q^4 = \left(2/3\right)^4 = \underline{0.19}$$

Q7

$$\sum_{j=5}^{\infty} \left(\frac{2}{3}\right)^{j-1} \frac{1}{3}$$

$$= \left(\frac{2}{3}\right)^4 \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$$

$$= \left(\frac{2}{3}\right)^4 \frac{1}{3} \frac{1}{\left(1 - 2/3\right)} = \left(\frac{2}{3}\right)^4$$

generalizing above problem

$$P(x > m) = \sum_{j=m+1}^{\infty} q^{j-1} p$$

$$= q^m p(1 + q + \dots)$$

$$\boxed{P(x > m) = q^m}$$

$$P(\underbrace{x > m+n}_{A} \mid \underbrace{x > n}_{B}) = \frac{P(A)}{P(B)} = \frac{q^{m+n}}{q^n}$$

Given No Success in n

$$\boxed{P(x > m+n \mid x > n) = P(x > m) = q^m}$$

→ Memoryless property of geometric Distribution (Starting point is immaterial)

Inverse / Negative Binomial Distribution (r^{th} success)

Generalization of geometric

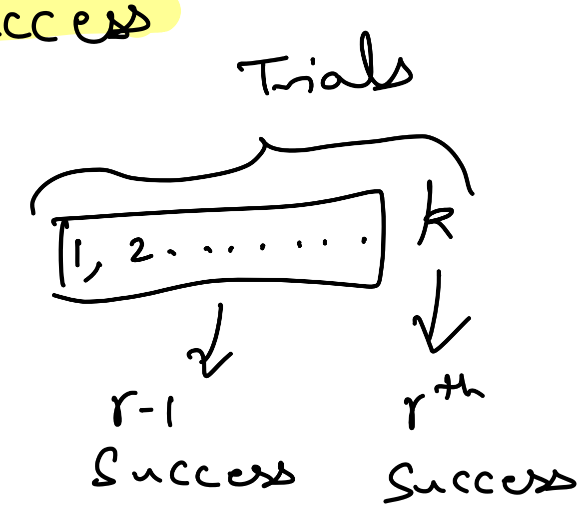
Consider independent Bernoullian trials under identical conditions till $\underline{r^{th}}$ success is achieved.

$X \to$ No of trials for r success

$X \to r, r+1, \ldots\ldots$

$$P(X=k) = \binom{k-1}{r-1} q^{k-r} p^{r}$$

$$k = r, r+1, r+2 \ldots$$



$$\boxed{E(X) = \frac{r}{p}} \qquad \boxed{V(X) = \frac{rq}{p^2}}$$

$$M_X(t) = E(e^{tX}) = \sum e^{tk} \binom{k-1}{r-1} q^{k-r} p^{r}$$

$$M_X(t) = \frac{(pe^t)^r}{(1-qe^t)^r}$$

$qe^t < 1$ or $t < -\log q$

Hypergeometric Distribution

Consists of N items and n items selected at random.

N splits into:

- k (Type 1)
- $N-k$ (Type 2)

→ n items are selected at random without replacement

→ Bernoullian model is applicable if we replace

X → No. of items of type I in the selected sample

$$p_X(x) = P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad x = 0, 1, \ldots, n$$

actual restriction on x

$$\max(0, n-N+k) \le x \le \min(n, k)$$

$$\boxed{\mu_1' = E(x) = \frac{kn}{N}}$$

Proportion of type 1 object

$$E[x(x-1)] = \frac{k(k-1)\, n(n-1)}{N(N-1)}$$

$$E[x^2] = E[x(x-1)] + E[x]$$

$$E[x^2] = \frac{kn(kn-k-n+N)}{N(N-1)}$$

$$\mathrm{Var}(x) = \frac{kn(N-n)(N-k)}{N^2(N-1)}$$

$$\mathrm{Var}(x) = \left(\frac{N-n}{N-1}\right)\frac{kn}{N}\left(1-\frac{k}{N}\right)$$

For large N

$$\frac{k}{N} \rightarrow p$$

$$E(x) = np$$

$$Var(x) = npq$$

Mean and Variance of binomial distribution

If the population size is ∞ly large such that the proportion of type 1 items is p, then if we consider a sample of size n then no. of type 1 must follow a binomial distribution

Theorem: Let $X \sim$ Hypergeometric(k, N, n)

if $k \to \infty$, $N \to \infty$ $\longrightarrow$ $\frac{k}{N} \to p$ then

$$P(X=x) \longrightarrow \binom{n}{x} p^{x} (1-p)^{n-x}$$

Capture - Re Capture Technique

Practical

Ex

How to estimate no. of fish in a lake.

1. Capture k fishes and tag them
2. Release the k fishes
3. Again capture n fishes and see the no. of fishes that are tagged (X)

$$E\left(\frac{X}{n}\right) = \frac{k}{N} \Rightarrow \frac{X}{n} \approx \frac{k}{N}$$

$$\boxed{N \approx \frac{kn}{X}}$$

Poisson Process

observations / occurrences / happenings over a time / area / space.

Assumption that needs to be true

1. The numbers of occurrences/outcomes during disjoint time intervals are independent

2. Probability of a single occurrence is proportional to length of interval.

3. Probability of more than one occurence during a small time interval is negligible

$X \to$ No. of occurrences in a interval of length t

$\hookrightarrow X(t) \to (0, t]$

$$P(X(t) = n) = P_n(t)$$

$0, 1, 2, \ldots$

Under the assumptions 1, 2 & 3

$$\boxed{P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}} \quad n = 0, 1, 2 \ldots$$

What is λ?

From assumption 2

$$P_1(h) = P(X(h) = 1) = \lambda h$$

$h \to$ Small time interval

Similarly from 3

$$P(X(h) \geq 2) = P_2(h) + P_3(h) + \cdots$$

$$= 1 - P_0(h) - P_1(h)$$

$$= o(h)$$ Small order h

$$\frac{o(h)}{h} \to 0$$

Ex Suppose the average no. of telephone calls arriving at the switch board of an operator is 30 calls / hr

i) P(No calls arrive in 3 min period)

ii) P(More than 5 calls in 5 min)

$\lambda = 30$ $\quad t = 1\,hr$

i) $P(X(3)=0) = e^{-\frac{1}{2}\cdot 3} = 0.22$

ii) $$P(X(5) > 5) = \sum_{j=6}^{\infty} \frac{e^{-\frac{1}{2}5}\left(\frac{5}{2}\right)^j}{j!} = 0.42$$

Ex 2

At a certain industrial plant accident take place at an average of 1 every 2 months

$\lambda = 1 \longrightarrow$ 2 months

$\lambda = \frac{1}{2} \longrightarrow$ 1 month

What is the prob of no. accident in a given month

$$P(X(1)=0) = e^{-\frac{1}{2}} = 0.6065$$

Ex 3

A printed page in a book contains 40 lines and each line has 75 positions

Each page has 3000 positions.

A typist makes 1 error per page?

i) What is the prob that a page has no errors?

ii) What is the prob that a 16 page chapter has no errors?

$\lambda = 1 \longrightarrow 6000$ position

$\lambda = 1/2 \longrightarrow 1$ page

i) $$P(X(1) = n) = \frac{e^{-1/2}\left(1/2\right)^{n}}{n!}$$

$$P(X(1) = 0) = e^{-1/2} \approx 0.6065$$

ii) $$P(X(16) = 0) = (0.6065)^{16}$$

Poisson Distribution

$$\boxed{P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \qquad x = 0, 1, 2 \ldots\ldots}$$

$$\boxed{\mu_1' = E(X) = \lambda}$$

$$\boxed{E\left[X(X-1)\right] = \lambda^2}$$

$$\mu_2 = Var(X) = \mu_2' - \mu_1'^{2}$$

$$\boxed{\mu_2 = Var(X) = \lambda}$$

$$\boxed{k^{th} \text{ factorial moment} = \lambda^k}$$

$$\mu_3' = \alpha_3 + 3\alpha_2 + \alpha_1$$

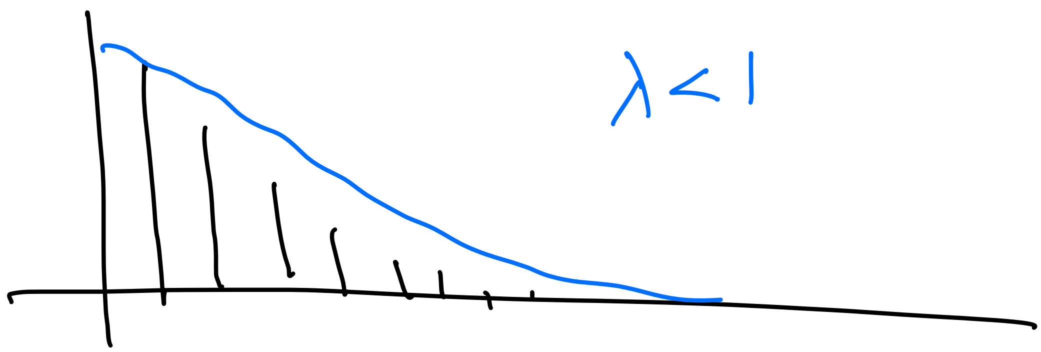
$$\boxed{\mu_3' = \lambda^3 + 3\lambda^2 + \lambda} \qquad \boxed{\mu_3 = \lambda}$$

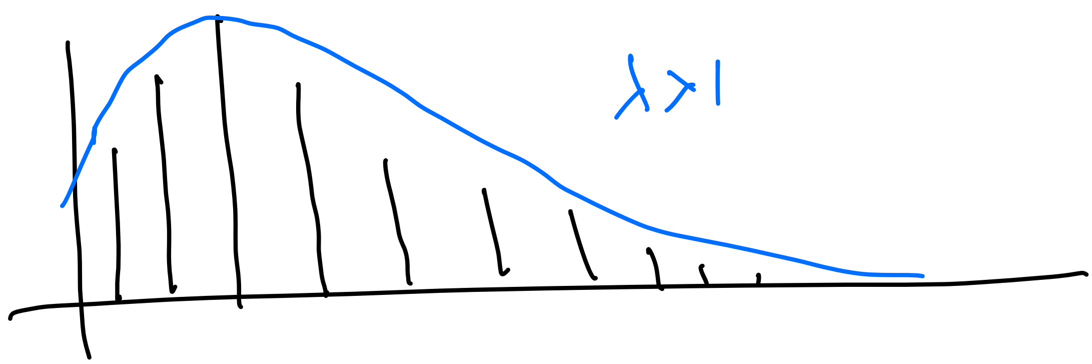
$$\boxed{\mu_4' = \lambda + 3\lambda^2}$$

Skewness

$$\boxed{\beta_2 = \frac{\mu_2}{\sigma^3} = \frac{1}{\sqrt{\lambda}} > 0}$$

Poisson
Always +ve Skew





$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{\lambda + 3\lambda^2}{\lambda^2} - 3$$

Kurtosis

$$\boxed{\beta = \frac{1}{\lambda} > 0}$$ always +ve

$\therefore$ Peak is higher than normal peak.

$\therefore$ $\lambda \to \infty$ poisson tends to normal peak.

$$\boxed{M_x(t) = E(e^{tx}) = e^{\lambda(e^t - 1)}}$$

Theorem

Let $X \sim Bin(n, p)$ binomial

Let $n \to \infty$ $p \to 0$ $\ni$ $np \to \lambda$

then

$$p_X(x) \longrightarrow \frac{e^{-\lambda} \lambda^x}{x!}$$

In a sequence of bernoullian trials if n becomes very large and prob. of occurence is small (and $np \to \lambda$) it converges to poisson

Ex

$X \to$ the no. of survivours from a rare disease.

$$X \sim Bin(1000, 0.05)$$

$$P(X \leq 5) = \sum_{x=0}^{5} \binom{1000}{x} (0.05)^x \underbrace{(0.95)^{1000-x}}$$

very small
can lead to errors

$\therefore$

$$np = \frac{0.05}{100} \times 1000$$

$$np = 50 = \lambda$$

$$\sum_{j=0}^{5} \frac{e^{-50}(50)^j}{j!}$$

Continous

Distributions

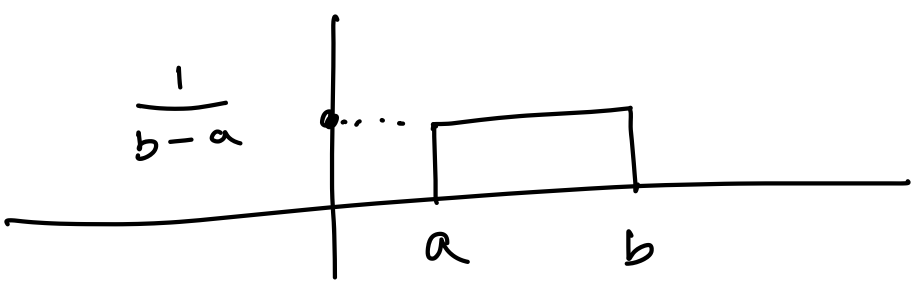
Uniform Distribution

$$f_X(x) = \begin{cases} k & x \in (a, b) \\ = 0 & \text{elsewhere} \end{cases}$$

$$k \int_a^b dx \Rightarrow k(b-a) = 1$$

$$\boxed{k = \frac{1}{b-a}}$$

$$\boxed{f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}}$$



$$\boxed{E(x) = \int_a^b \frac{x}{b-a}\,dx = \frac{a+b}{2}}$$ ← Mid point

$$\boxed{\mu_k' = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}}$$

$$\boxed{Var(x) = \mu_2 = \frac{(b-a)^2}{12}}$$

Variability depends on range

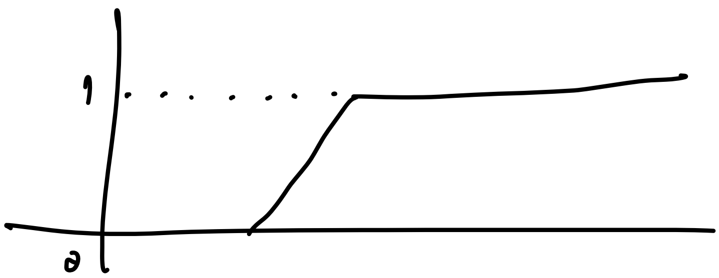
Skewness = 0

Kurtosis → Dependents on range $(b-a)$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

cdf

$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b \end{cases}$$



$$M_X(t) = E\left(e^{tx}\right) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

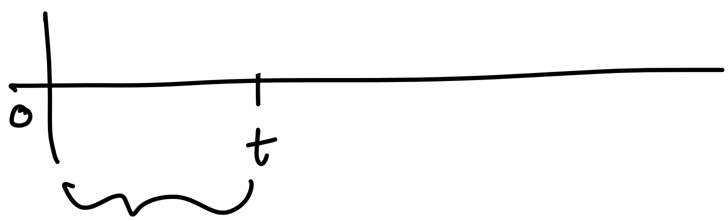
Exponential Distribution

Consider a Poisson process $X(t)$ with rate $(\lambda > 0)$. Let T be the time of the first occurrence.

We want the dist^n of T

Consider

$$P(T > t) = P\big(X(t) = 0\big) = \begin{cases} e^{-\lambda t} & t > 0 \\ 1 & t \le 0 \end{cases}$$



0 occurence because the first occurence happens at T

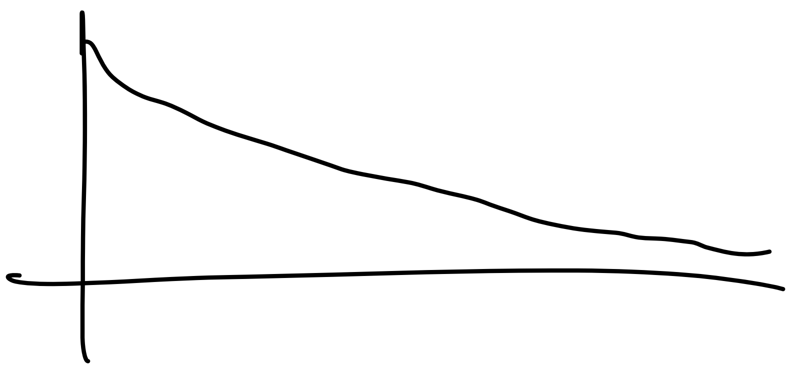
Cdf

$$F_T(t) = 1 - P(T > t) = \begin{cases} 0 & t \le 0 \\ 1 - e^{-\lambda t} & t > 0 \end{cases}$$

pdf

$$f_T(t) = \boxed{\begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t < 0 \end{cases}}$$

Waiting time for first occurence of a poisson process



$$\mu_k' = E(T^k) = \int_0^\infty t^k \lambda e^{-\lambda t}\, dt$$

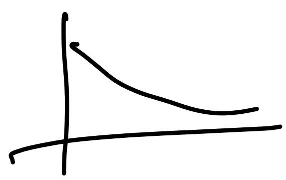
$$= \frac{\lambda\, \Gamma(k+1)}{\lambda^{k+1}} = \frac{\lambda\, k!}{\lambda^{k+1}}$$

$$\boxed{\mu_k' = \frac{k!}{\lambda^k}} \quad k = 1, 2 \ldots$$

$$\boxed{\mu_1' = E(T) = \frac{1}{\lambda}} \longrightarrow \text{Avg wait time for first occurrence}$$

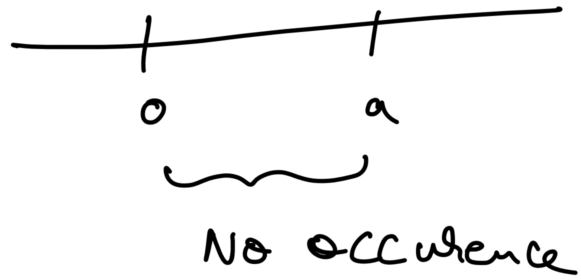
$$\boxed{Var(T) = \frac{1}{\lambda^2}}$$

$\boxed{\beta_1 = 2}$ always +ve



$\boxed{\beta_2 = 6}$ always peak higher than normal

$$P(T>a) = e^{-\lambda a}$$



$$P(\underbrace{T>a+b}_{A} \mid \underbrace{T>b}_{B}) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

$$= \frac{P(T>a+b)}{P(T>b)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}}$$

$$= e^{-\lambda a} = P(T>a)$$

Memoryless property of exponential

Exponential distribution is a continuous form of the geometric distribution

Shifted Exponential

No occurence till a certion point (μ)

$$f_X(x) = \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)} \qquad x > \mu \qquad \sigma > 0$$

$$\mu \in \mathbb{R}$$

$$\boxed{\begin{aligned} E\left[(x-\mu)^k\right] &= k!\,\sigma^k \\ E\left[(X-\mu)\right] &= \sigma \end{aligned}}$$

$$E\left[(x-\mu)^2\right] = 2\sigma^2$$

$$\boxed{Var(X) = \sigma^2} \leftarrow$$ Variance is not affected by shifting.

Theorem: If $X \sim Exp(a\mu + b, a\sigma)$
then $Y = aX + b$, $a \neq 0 \sim Exp(a\mu + b, a\sigma)$

Linearity property of exponential dist.

In particular if $X \sim Exp\left(\mu, \frac{1}{\lambda}\right)$

$X - \mu \sim Exp\left(\frac{1}{\lambda}\right)$

Ex

The time to failure in months X, of the light bulbs produced at two manufacturing plants A & B obeys exponential dist^n with means 5 and 2 respectively. Plant B produces 3 times as many bulbs as A. The bulbs are mixed and sold.

i) What is the prob that a randomly selected bulb will work for atleast 5 months.

$X \rightarrow$ life of bulb

$X/A \longrightarrow \frac{1}{5} e^{-x/5} \qquad x > 0$

$X/B \longrightarrow \frac{1}{2} e^{-x/2} \qquad x > 0$

$$P(X>5) = P(X>5 \mid A)\,P(A) + P(X>5 \mid B)\,P(B)$$

$$= e^{-5\cdot\frac{1}{5}}\,\frac{1}{4} + e^{-5\cdot\frac{1}{2}}\,\frac{3}{4}$$

$$= 0.1534$$

Gamma/ Erlang Distribution

Consider a Poisson process $X(t)$ with rate λ

Let T_r denote the time of r^{th}

occurrence.

$$P(T_r > t) = \begin{cases} P(X(t) \leq r-1) & t > 0 \\ 1 & t \leq 0 \end{cases}$$

$$= \begin{cases} \sum_{j=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^j}{j!} & t > 0 \\ 1 & t \leq 0 \end{cases}$$

<u>cdf</u>

$$F_{T_r}(t) = 1 - P(T_r > t)$$

$$= \begin{cases} 0 & t \leq 0 \\ 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^j}{j!} & t > 0 \end{cases}$$

pdf $\frac{d}{dt} F_{T_r}(t)$

$$\boxed{f_{T_r}(t) = \frac{\lambda^r t^{r-1}}{\Gamma r} e^{-\lambda t} \qquad t>0 \quad \lambda>0 \quad r>0}$$

$$\boxed{\mu_k' = E\left[(T_r)^k\right] = \frac{\Gamma k+r}{\Gamma r} \cdot \frac{1}{\lambda^k} \qquad k=1,2\ldots}$$

$$\boxed{\mu_1' = \frac{\Gamma r+1}{r} \cdot \frac{1}{\lambda} = \frac{r}{\lambda}}$$

If avg waiting time of one occurence is $1/\lambda$ then for r occurence is r/λ

This is due to memoryless property of exponential

$$\mu_2' = \frac{r(r+1)}{2}$$

$$\boxed{\mu_2 = Var(x) = \frac{r}{\lambda^2}}$$

Weibull Distribution

$$f_X(x) = \begin{cases} \alpha\beta\, x^{\beta-1} e^{-\alpha x^{\beta}} & x>0 \\ & \alpha>0 \quad \beta>0 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x^{\beta}} & x>0 \\ 0 & x \le 0 \end{cases}$$

Generalization of exponential dist^n

$$E(X^k) = \frac{\alpha\, \Gamma\left(\frac{k}{\beta}+1\right)}{\alpha^{k/\beta + 1}} = \frac{\Gamma\left(\frac{k+\beta}{\beta}\right)}{\alpha^{k/\beta}}$$

$$\boxed{\mu_1' = \alpha^{-1/\beta}\, \Gamma\left(\frac{\beta+1}{\beta}\right)} \qquad \mu_2' = \alpha^{-2/\beta}\, \Gamma\left(\frac{\beta+2}{\beta}\right)$$

$$\boxed{\mu_2 = \alpha^{-2/\beta}\left[\Gamma\frac{\beta+2}{\beta} - \left(\Gamma\frac{\beta+1}{\beta}\right)^2\right]}$$

Reliability

$T \rightarrow$ life of a system

$$\boxed{P(T>t) = R(t) = 1 - F_T(t)}$$

$\therefore$ System is working at t

$\hookrightarrow$ Reliability of system at time t

Instantaneous failure rate of system at time t

$$\lim_{h \to 0} \frac{1}{h} P\left(t < T \leq t+h \mid T>t\right) = H(t)$$

Hazard rate

$$\boxed{H(t) = \frac{f_T(t)}{1 - F_T(t)}} = -\frac{d}{dt}\log\left(1 - F_T(t)\right)$$

$$\boxed{1 - F_T(t) = \underbrace{k}_{\text{some constant}}\, e^{-\int H(t)\,dt}}$$

$\hookrightarrow$ Some Constant that has to be found from initial condition.

For Weibull

$$\boxed{R(t) = \begin{cases} e^{-\alpha t^{\beta}} & t > 0 \\ 1 & t \le 0 \end{cases}}$$

$$\boxed{H(t) = \frac{f(t)}{R(t)} = \alpha \beta t^{\beta - 1}}$$

If β is fraction it is Increasing hazard

If $\beta > 0$ $\beta \in \mathbb{Z}$ it is polynomial function

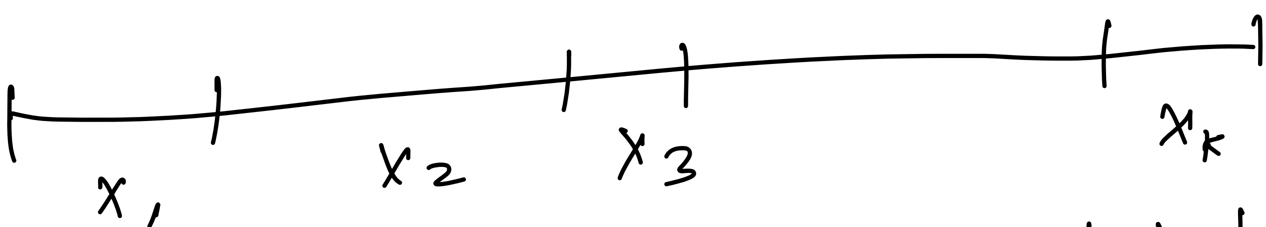
For exponential distribution

$$H(t) = \frac{\lambda \cancel{e^{-\lambda x}}}{\cancel{e^{-\lambda x}}}$$

$$\boxed{H(t) = \lambda}$$ ← constant failure rate (Memoryless)
Does not depend on time

Reliability of a Series System

The entire system will work only if all the component works



Let $x_1, x_2 \ldots\ldots x_k$ denote the lives of k independent components of a system with life X

The reliability of the entire system.

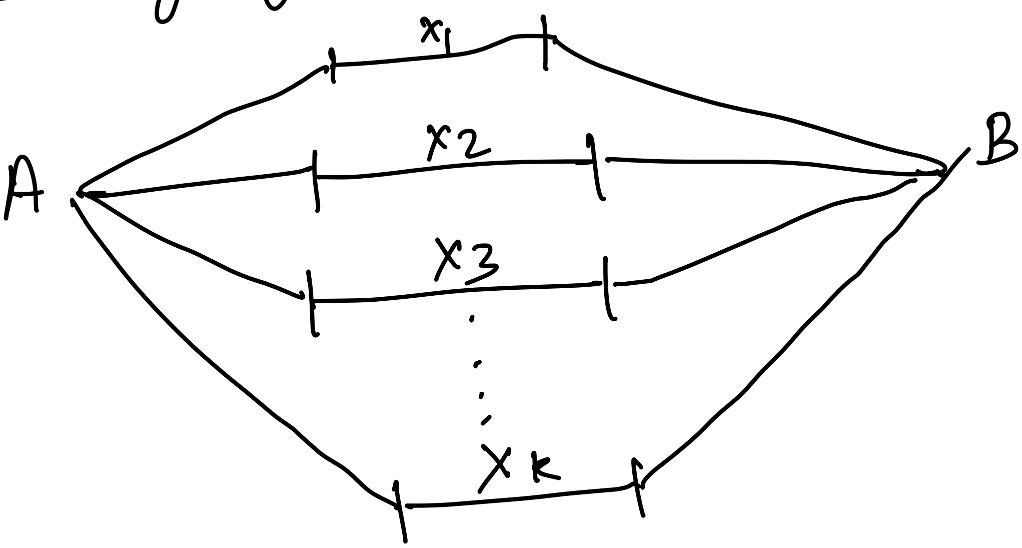
$$R_X(t) = P(X > t) \quad \text{component lives are independent}$$

$$= P(X_1 > t,\ X_2 > t,\ X_3 > t \dots X_k > t)$$

$$= \prod_{i=1}^{k} P(X_i > t)$$

$$\boxed{R_X(t) = \prod_{i=1}^{k} R_{X_i}(t)}$$

Reliability of a Parallel System



$$R_X(t) = P(X > t) = 1 - \underbrace{P(X \le t)}_{\text{System fails before } t}$$

System fails before time t, if each of the component fails

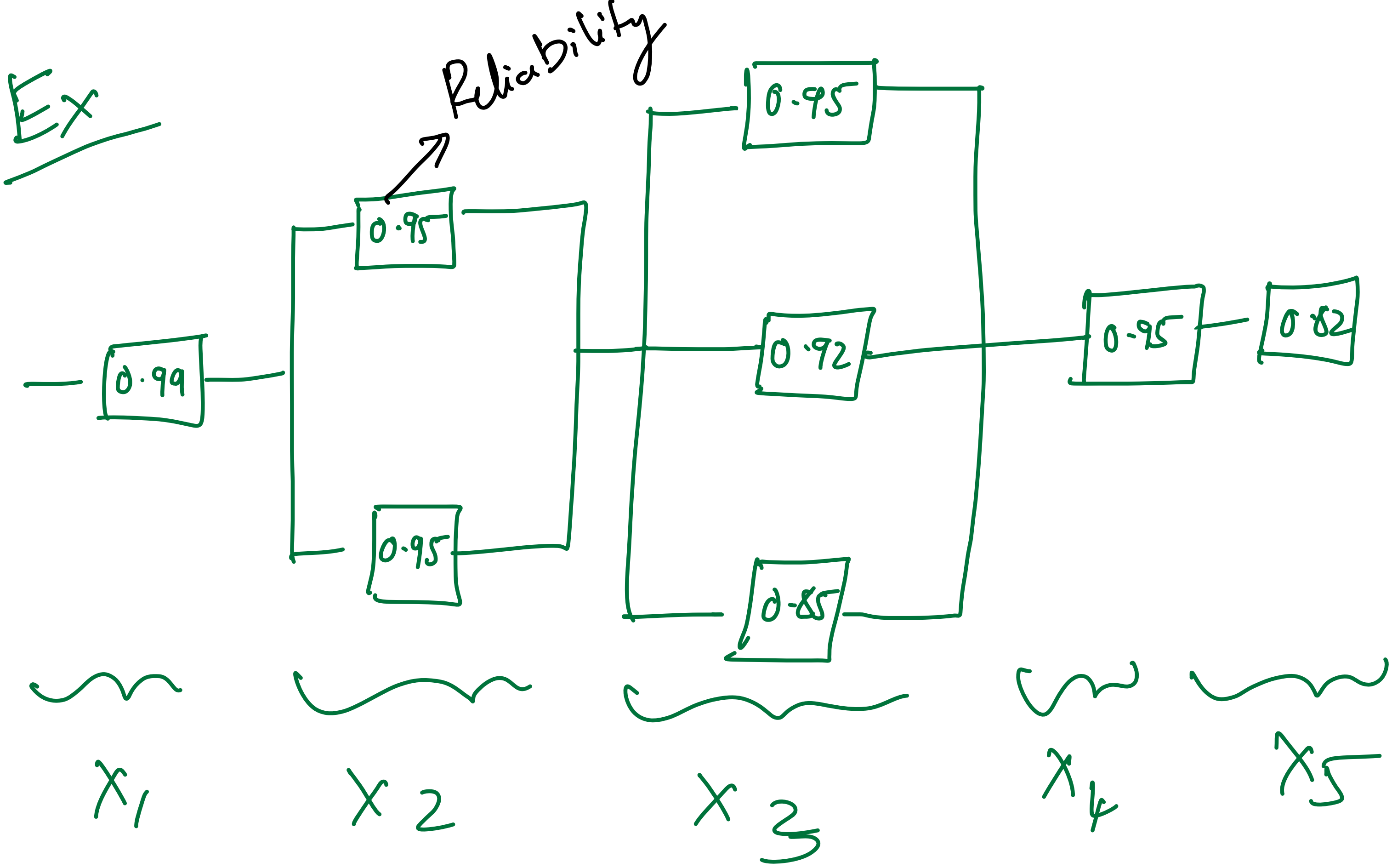
$$= 1 - P(X_1 \le t, X_2 \le t \dots X_k \le t)$$

$$= 1 - \prod_{i=1}^{k} P(X_i \le t)$$

$$= 1 - \prod_{i=1}^{k} \{1 - P(X_i > t)\}$$

$$\boxed{R_x(t) = 1 - \prod_{i=1}^{k} [1 - R_i(t)]}$$

Ex



$$R_x(t) = R_{x_1}(t)\ R_{x_2}(t)\ R_{x_3}(t)\ R_{x_4}(t)\ R_{x_5}(t)$$

$$R_{x_2}(t) = 0.9975$$

$$R_{x_3}(t) = 0.99952$$

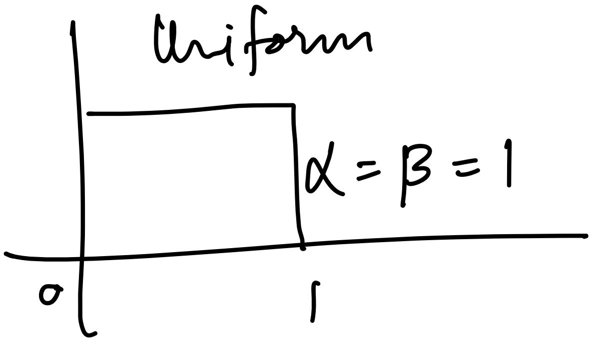
$$R_x(t) = 0.7689$$

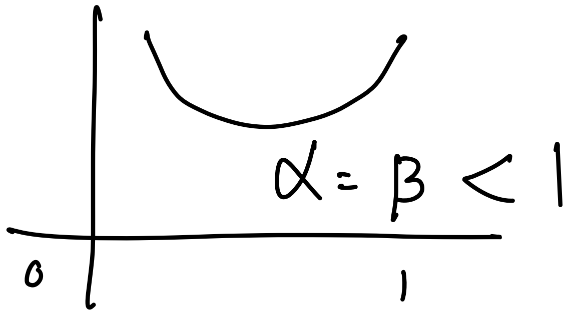
β Distribution

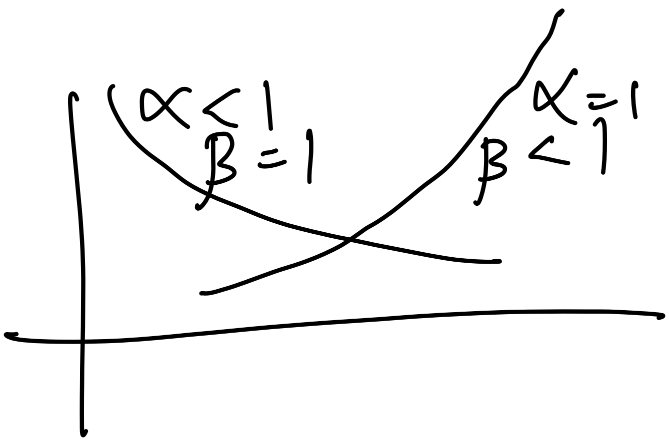
Bounded

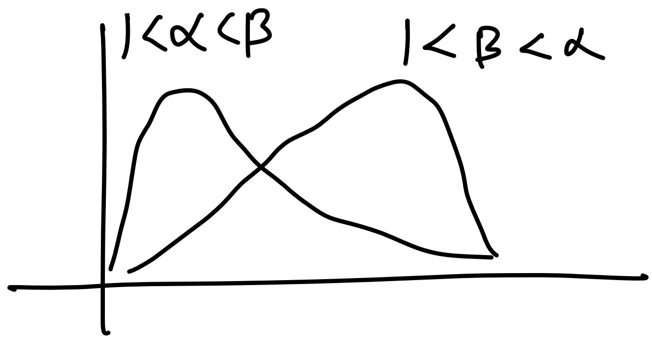
$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0<x<1$$

$$\alpha>0 \quad \beta>0$$









$\alpha = \beta > 1$

$$\boxed{\mu_k' = \frac{B(\alpha+k,\ \beta)}{B(\alpha,\beta)}}$$

$$\boxed{\mu_1' = E(x) = \frac{\alpha}{\alpha+\beta}}$$

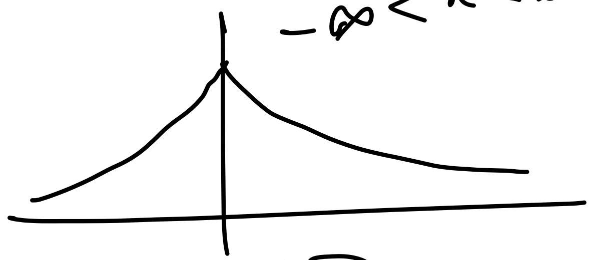
$$\mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\boxed{\mu_2 = V(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

Double Exponential

Laplace Distn

$$\boxed{f_X(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}}$$



$-\infty < x < \infty$

$\mu \in \mathbb{R}$

$\sigma > 0$

$$\boxed{E(X) = \mu}$$

$$\boxed{V(X) = 2\sigma^2}$$

Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)}$$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

$$\boxed{Z = \frac{x-\mu}{\sigma}}$$

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^k\right] = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^k \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)}$$

$$= \int_{-\infty}^{\infty} z^k \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = 0 \quad \text{If } k \text{ is odd}$$

If k is even $(k = 2m)$

$$= 2\int_0^\infty z^{2m} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= 2\int_0^\infty (2t)^m \frac{1}{\sqrt{2\pi}} e^{-t} \frac{1}{\sqrt{2t}} \, dt$$

$$t = \frac{z^2}{2}$$

$$dz = \frac{1}{\sqrt{2t}} \, dt$$

$$= \frac{2^m}{\sqrt{\pi}} \int_0^\infty t^{m-1/2} e^{-t} \, dt$$

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^k\right] = \frac{2^m}{\sqrt{\pi}} \Gamma\left(m + \frac{1}{2}\right)$$

$$= \frac{2^m}{\cancel{\sqrt{\pi}}} \left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right) \cdots \frac{3}{2} \, \frac{1}{2} \cancel{\sqrt{\pi}}$$

$$= (2m-1)(2m-3) \cdots 5 \cdot 3 \cdot 1$$

$$E\left[(x-\mu)^k\right] = 0 \quad \text{for } k \text{ odd}$$

All odd ordered central moment of a normal distribution is 0

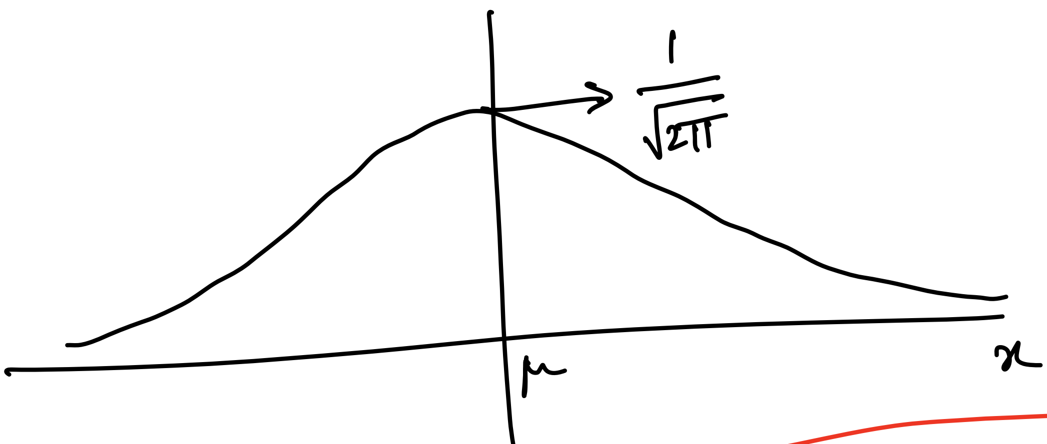
$$E\left[(x-\mu)^{2m}\right] = \sigma^{2m}(2m-1)(2m-3)\ldots 5.3.1$$

$\mu_2 = \sigma^2$

$\mu_4 = 3\sigma^4$

$\beta_1 = 0$

$\beta_2 = \dfrac{\mu_4}{\mu_2^2} - 3 = 0$



$$\boxed{M_x(t) = E\left(e^{tx}\right) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

Theorem:

If $x \sim N(\mu, \sigma^2)$ and $y = ax + b$ $a \neq 0$

then $y \sim N(a\mu + b, a^2\sigma^2)$

Linearity property of Normal Distribution

$N(0, 1) \longrightarrow$ Standard Normal

pdf

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$$

cdf

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\, dt$$

from Symmetry

$$1 - \Phi(z) = \Phi(-z)$$

$$\Phi(0) = 1/2$$

Functions of Random Variable

<u>Theorem</u>: Let X be a random defined on $(\Omega, \mathcal{B}, P)$. Let g be a measurable function $g: \mathbb{R} \to \mathbb{R}$.

Then $g(X)$ is <u>also a random variable</u>.

<u>Theorem</u>: Given a r.v X with cdf F_X with dist^n of r.v $Y = g(X)$, where g is measurable and determined.

Joint Distributions

$X = (X_1, X_2 \ldots X_k) : \Omega \longrightarrow \mathbb{R}^k$

$\downarrow$

measurable

Joint cdf

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

$$(x, y) \in \mathbb{R}^2$$

Both x and y are discrete

(i) $P(X = x_i, Y = y_j) = p_{X,Y}(x_i, y_i)$, $(x_i\ y_i) \in \mathcal{X} \times \mathcal{Y}$

(ii) $p(x_i, y_i) \geq 0 \quad \forall\ (x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$

(iii) $\sum_{x} \sum_{y} p_{xy}(x_i, y_j) = 1$

Ex

Suppose a car showroom has 10 cars out of which

5 → good cars

2 → Defective transmission

3 → Defective steering

2 cars are selected at random.

X → No. of cars with defective transmission.

Y → No. of cars with defective steering

Find probability distribution of $X Y$; $p_{XY}(x_i, y_i)$

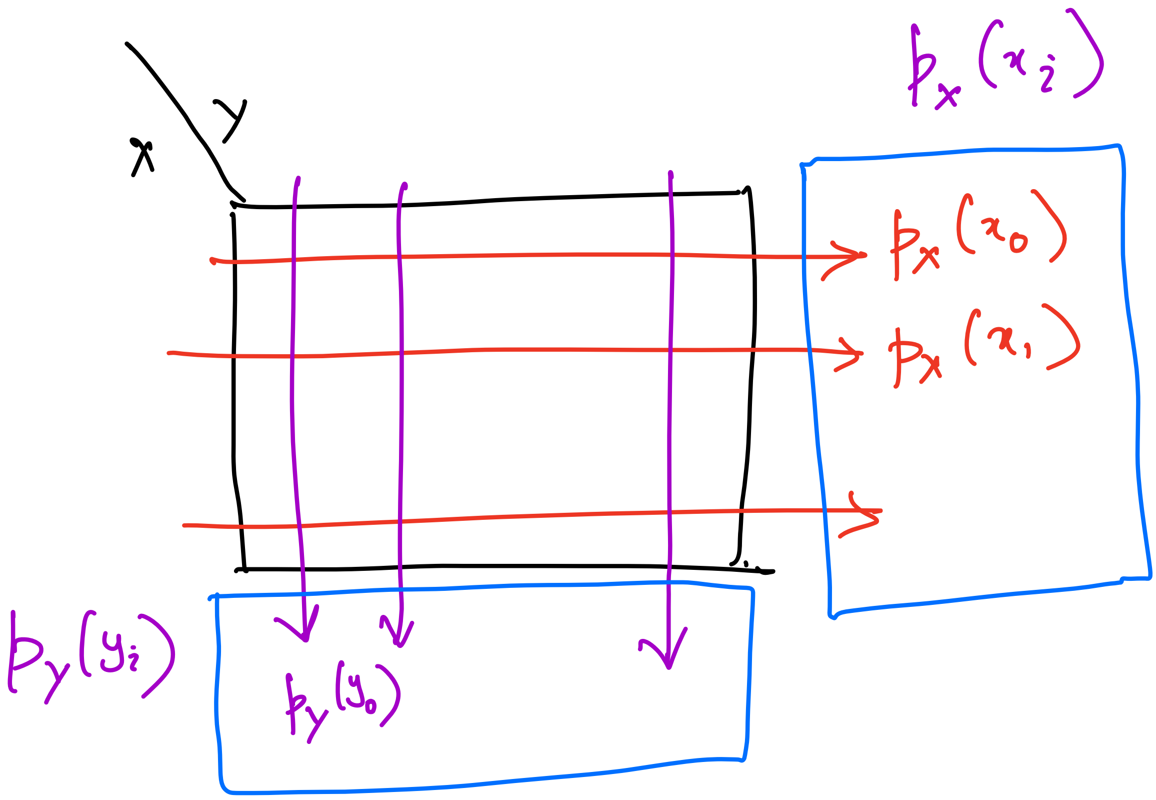
$$p_{XY}(0,0) = \frac{\binom{5}{2}}{\binom{10}{2}} \qquad p_{XY}(0,1) = \frac{\binom{5}{1}\binom{3}{1}}{\binom{10}{2}}$$

x \ y	0	1	2
0	10/45	15/45	3/45
1	10/45	6/45	0
2	1/45	0	0

Marginal distribution

$$p_X(x_i) = \sum_{y_j \in \mathcal{Y}} p_{XY}(x_i, y_j)$$

$$p_Y(y_i) = \sum_{x_i \in \mathcal{X}} p_{XY}(x_i, y_j)$$



The Conditional prob. of Y given $X = x_i$

$$p_{Y|X=x_i}(y_j) = \frac{p_{XY}(x_i, y_j)}{p_X(x_i)} \qquad y_j \in \mathcal{Y}$$

Ex:

$x \backslash y$	0	1	2
0	10/45	15/45	3/45
1	10/45	6/45	0
2	1/45	0	0

$$p_{Y|x=0}(0) = \frac{p_{xy}(0,0)}{p_X(0)} = \frac{10}{28}$$

$$p_{Y|x=0}(1) = \frac{p(0,1)}{p_X(0)} = \frac{15}{28}$$

Let x, y be Continuous

i) $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \underbrace{dx\, dy}_{dy\, dx} = 1$$

Marginal

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\, dx$$

Ex

$$f_{xy}(x,y) = \begin{cases} 10xy^2 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$$\int_0^1\int_0^y f_{xy}(x,y)\, dx\, dy = \int_0^1 5y^4\, dy = 1$$

$$f_X(x) = \int_x^1 10xy^2\,dy = \begin{cases} \frac{10}{3}x(1-x^3), & 0<x<1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \int_0^y 10xy^2\,dx = \begin{cases} 5y^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

Conditional pdf of X given Y = y

$$f_{X|Y=y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_Y(y) \neq 0$$

CDF

$$F_{XY}(x, y) = P(X \leq x, \; Y \leq y)$$

$$\lim_{y \to \infty} F_{XY}(x, y) = F_X(x)$$

$$\lim_{x \to \infty} F_{XY}(x, y) = F_Y(y)$$

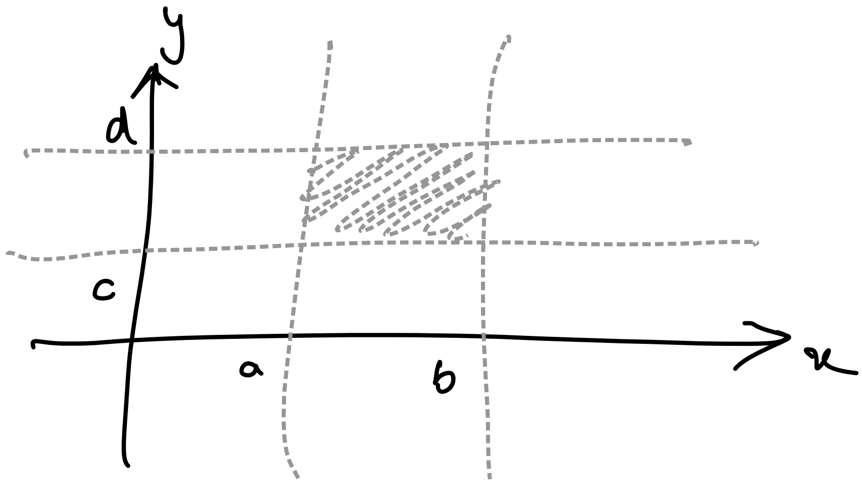
$$\lim_{x \to -\infty} F_{XY}(x, y) = 0 = \lim_{y \to -\infty} F_{XY}(x, y)$$

$F(x, y)$ is Non decreasing in each of x and y

$F(x, y)$ is continuous from right in each of x and y

$$P(a < X \le b,\ c < y \le d)$$

$$= F_{xy}(b, d) - F_{xy}(a, d) - F_{xy}(b, c) + F_{xy}(a, c)$$



Independence of Random Variable

cdf

$$F_{XY}(x,y) = F_X(x)\,F_Y(y) \quad \forall (x,y) \in \mathbb{R}^2$$

$$P((X,Y) \in B) = P(X \in A_1)\,P(Y \in A_2)$$

$B = A_1 \times A_2$ $\qquad A_1 \in \mathcal{X} \qquad A_2 \in \mathcal{Y}$

pdf

discrete

$$p_{XY}(x,y) = p_X(x)\,p_Y(y)$$

Continuous

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

Joint $\xrightarrow{\text{gives} ✓}$ Marginals

Marginals $\xrightarrow[\text{give}]{\text{Do not} ✗}$ Joints

Marginals $\xrightarrow{\text{gives} ✓}$ Joins if r.v are independent

Expectation

$$E[g(x,y)] = \underbrace{\sum\sum g(x_i, y_j)\, p_{xy}(x_i, y_j)}_{\text{convergent}}$$

$$E[g(x,y)] = \iint g(x,y)\, f_{x,y}(x,y)\, dx\, dy$$

Product Moment

$$\boxed{\mu'_{r,s} = E(x^r y^s)}$$ $(r,s)^{th}$ moment

$$\mu'_{1,1} = E(xy)$$

$$\mu'_{1,0} = E(x) = \mu_x$$

$$\mu'_{0,1} = E(y) = \mu_y$$

$$\boxed{\mu_{r,s} = E\left[(x-\mu_x)^r (y-\mu_y)^s\right]}$$

$(r,s)^{th}$ central product moment

$$\boxed{\mu_{1,1} = E\left[(X-\mu_X)(Y-\mu_Y)\right]} \rightarrow \underline{\underline{\text{Covariance}}}$$

$$\mu_{11} = E\left[XY - X\mu_Y - \mu_X Y + \mu_X \mu_Y\right]$$

$$\boxed{\mu_{11} = E(XY) - E(X)E(Y)}$$

If X, Y are independent
$Cov(X,Y) = 0$

<u>If the r.v are independent</u>

$$E(X^r Y^s) = E(X^r)E(Y^s)$$

$$E(X-\mu_X)^r (Y-\mu_Y)^s = E(X-\mu_X)^r E(Y-\mu_Y)^s$$

<u>Theorem</u> : X and Y are independent

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$$

Coefficient of Correlation

$$\rho_{xy} = \frac{Cov(x,y)}{S.d(x)\ s.d(y)} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

→ Gives the measure of linear relationship between x and y

If x, y are independent $\rho_{xy} = 0$
but the converse is not true

Consider random variables U and V such that $E(U) = 0$ $E(U^2) = 1$
$E(V) = 0$ $E(V^2) = 1$

Now consider the term

$$E\left[(U - V)^2\right] \geq 0$$

$$E(U^2 + V^2 - 2UV) \geq 0$$

$$E(UV) \leq 1$$

Similarly $E(U+V)^2 \geq 0$

$$E(U^2 + V^2 + 2UV) \geq 0$$

$$E(UV) \geq -1$$

if

$$\boxed{-1 \leq E(UV) \leq 1}$$

When will equality be attained

$$E(UV) = 1$$

$$(U - V)^2 = 0$$

if $P(U = V) = 1$

$$E(UV) = -1$$

$$(U + V)^2 = 0$$

if $P(U = -V) = 1$

Consider x, y

$E(x) = \mu_x \qquad E(y) = \mu_y$

$Var(x) = \sigma_x^2 \qquad Var(y) = \sigma_y^2$

Define

$$U = \frac{x - \mu_x}{\sigma_x} \qquad V = \frac{y - \mu_y}{\sigma_y}$$

$$E(U) = E\left(\frac{x - \mu_x}{\sigma_x}\right) = 0$$

$$E(U^2) = E\left(\frac{(x - \mu_x)^2}{\sigma_x^2}\right) = 1$$

$$E(V) = 0$$

$$E(V^2) = 0$$

From prev proof

$$\therefore \quad -1 \le E(UV) \le 1$$

$$E(UV) = E\left(\frac{X-\mu_x}{\sigma_x}\right)\left(\frac{Y-\mu_y}{\sigma_y}\right)$$

$$= \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \rho_{xy}$$

$$\therefore \quad \boxed{-1 \le \rho_{xy} \le 1}$$

$$\rho_{xy} = 1 \iff P\left(\frac{X-\mu_x}{\sigma_x} = \frac{Y-\mu_y}{\sigma_y}\right) = 1$$

or $P(\underline{X = aY + b}) = 1$ where $\underline{a > 0}$

↳ Perfectly linearly related in +ve direction

$$\rho_{xy} = -1 \iff P\left(\frac{X-\mu_x}{\sigma_x} = -\frac{Y-\mu_y}{\sigma_y}\right) = 1$$

or $P(\underline{X = aY + 1}) = 1$ where $\underline{a < 0}$

↳ Perfect linear relation in -ve direction

Joint Moment Generating Function

$$M_{XY}(s,t) = E\left(e^{sX+tY}\right)$$

Provided the expectation exists in the neighbourhood of $(0,0)$

Theorem

If X and Y are independent

$$\boxed{M_{XY}(s,t) = M_X(s)\,M_Y(t)} \quad \forall (s,t)$$

Theorem

If X and Y are independent

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

Bivariate Normal Distribution

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]\right)$$

Marginals are Normal distributions

$\hookrightarrow N(\mu_1, \sigma_1)$

$\hookrightarrow N(\mu_2, \sigma_2)$

Conditional Distribution

$$X \mid Y=y \sim N\left(\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right), \sigma_1^2(1-\rho^2)\right)$$

$$Y \mid X=x \sim N\left(\mu_2 + \rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right), \sigma_2^2(1-\rho^2)\right)$$

Theorem

If X, Y is bivariate normal then marginal and conditional are normal. Converse is also true

22, 21, 20

3 skip

Sampling Distribution

Population - collection of measurements

Sample - Subset of population

Random Sampling

Let $X_1, X_2 \dots X_n$ be n independent and identically distributed (iid) random variables each having the same prob. distⁿ $f(x)$. Then we say that $(X_1 \dots X_n)$ is a random sample from population with distⁿ $f(x)$. The joint distⁿ of $X_1 \dots X_n$ is

$$f(x_1 \dots x_n) = f(x_1)\, f(x_2) \dots f(x_n)$$

Statistic

A function of random sample

Sampling Distribution

→ Probability distribution of statistic is sampling distribution

Central Limit Theorem

Let $X_1, X_2 \ldots$ be a sequence of iid random variables with a mean of μ and variance $\sigma^2 \, (< \infty)$.

Let $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

Then the limiting distribution of $\boxed{\frac{\sqrt{n}\,(\overline{X}_n - \mu)}{\sigma}}$ is $\boxed{N(0,1)}$ as $n \to \infty$

Remark : In practice $\boxed{n \geq 30}$ is considered good.

Let $X_{11}\ X_{12}\ \ldots\ldots\ X_{1n_1}$ be iid r.v with mean μ_1 and variance σ_1^2

Let $X_{21}\ X_{22}\ \ldots\ldots\ X_{2n_2}$ be iid r.v with mean μ_2 and variance σ_2^2

$$\bar{X}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} X_{1i} \qquad \bar{X}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} X_{2i}$$

$$\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \xrightarrow[\substack{n_1 \to \infty \\ n_2 \to \infty}]{} N(0,1)$$

Chi-Square Distribution

A continuous r.v W is said to have chi-square distribution with n degrees of freedom if it has pdf given by

$$\boxed{f_W(w) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}}\, e^{-\frac{w}{2}}\, w^{\frac{n}{2}-1}}, \quad \begin{matrix} w > 0 \\ n > 0 \end{matrix}$$

$\hookrightarrow$ This is gamma distribution with parameters $\boxed{G\left(\frac{n}{2}, \frac{1}{2}\right)}$

$E(W) = n \longrightarrow$ mean = degrees of freedom

$Var(W) = 2n$

$\mu_k' = E(W^k) = n(n+2)\cdots(n+2(k-1))$

$$M_w(t) = (1-2t)^{-n/2}, \quad t < 1/2$$

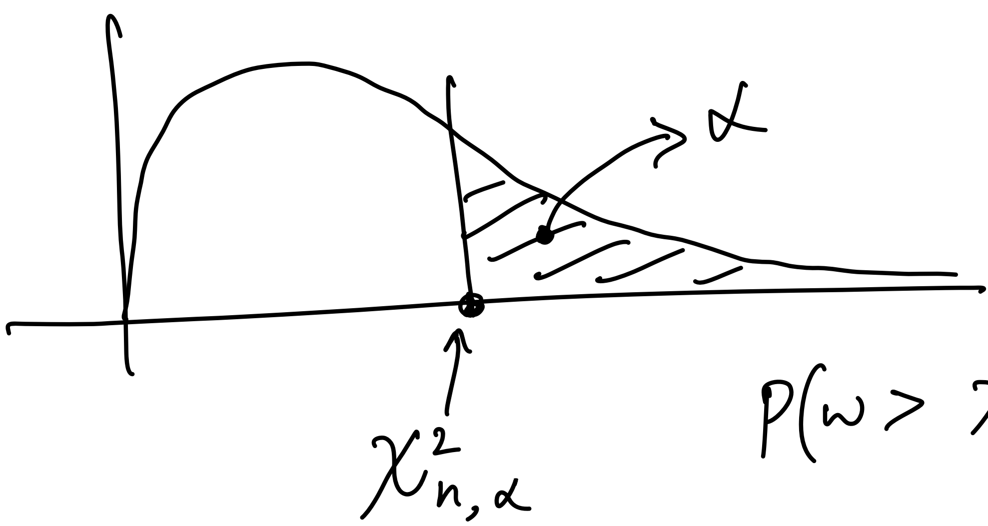
$$\mu_3 = 8n > 0$$

$$\beta_1 = \frac{8n}{(2n)^{3/2}} = \sqrt{\frac{8}{n}}$$

$$\mu_4 = 12n(n+4)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{12}{n} > 0$$ Higher than normal peak

$n \to \infty$ tends to normal



$$P(w > \chi^2_{n,\alpha}) = \alpha$$

If $W_1 \dots W_k$ are independent chi square r.v with $W_i \sim \chi^2_{n_i}$ $i = 1, 2 \dots k$

Then $\sum_{i=1}^{k} W_i \sim \chi^2_{\sum_{i=1}^{k} n_i}$

Distribution of Sum of chi Square is again a chi Square and the degrees of freedom is added

Watch Sampling Distion

1 second half.

$\boxed{\text{Inference}}$ → Estimation

→ Hypothesis testing

Estimation

→ Point Estimation

→ Interval Estimation

Point Estimation

$X_1\ X_2 \ldots\ X_n$ is a random sample from a population with distribution P_θ, θ

Parametric inference $\longrightarrow$ Model determined

Non Parametric inference $\longrightarrow$ No Model

Unbiased Estimator

A statistic $T(\underline{X})$ is said to be an unbiased estimator of $g(\theta)$ if

$$E_{\underline{\theta}}\left[T(\underline{X})\right] = g(\underline{\theta}) \quad \forall\ \underline{\theta} \in H$$

If $E_{\underline{\theta}}[T(\underline{x})] = g(\underline{\theta}) + b(\underline{\theta})$, then we say that $T(\underline{x})$ is biased for $g(\underline{\theta})$ and $b(\underline{\theta})$ is the bias of $T(\underline{x})$

Ex 1

$X \sim Bin(n, p)$ n is known

p — prob. of success

① $T(x) = \frac{X}{n}$

$E\left(\frac{X}{n}\right) = p$

So $\frac{X}{n}$ is unbiased estimator for mean

② $T(x) = \frac{X(X-1)}{n(n-1)}$

$$E[X(X-1)] = n(n-1)p^2$$

$$E\left[\frac{X(X-1)}{n(n-1)}\right] = p^2$$

$$\mathrm{Var}(X) = np(1-p)$$

$$= n(p - p^2)$$

$$= n\,E\left[\frac{X}{n} - \frac{X(X-1)}{n(n-1)}\right]$$

$$\Rightarrow E\left\{\frac{X(n-X)}{n-1}\right\} = np(1-p)$$

So $\dfrac{X(n-X)}{n-1}$ is an unbiased estimator for variability

Ex 2

Let $X_1 \dots X_n \sim \text{Poisson}(\lambda)$

① $E(X_1) = \lambda$ $\qquad \bar{X} = \frac{1}{n}\sum X_i$

$\therefore E(\bar{X}) = \lambda \longrightarrow \bar{X}$ is unbiased estimator of λ

②
$$T(X_1) = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{if } X_1 \neq 0 \end{cases}$$

Then $E[T(X_1)] = 1 \cdot P(X_1 = 0) + 0 \cdot P(X_1 \neq 0)$

$= e^{-\lambda}$

So $T(X_1)$ is an unbiased estimator of $e^{-\lambda}$

In general $T(X_i)$ is unbiased

Ex 3

$X_1, X_2 \ldots\ldots X_n \sim N(\mu, \sigma^2)$

$E(\bar{X}) = \mu$ $\qquad S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$

We know that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$E\left[\frac{(n-1)S^2}{\sigma^2}\right] = (n-1)$$

$\Rightarrow E(S^2) = \sigma^2$

So $\bar{X}$ and S^2 are unbiased estimators for μ and σ^2

Remark 1

Sometimes unbiased estimators may be absurd

Ex

$X \sim \text{Poisson}(\lambda) \quad g(\lambda) = e^{-3\lambda}, \quad 0 < e^{-3\lambda} < 1$

If $T(X) = (-2)^X$

$$E[T(X)] = \sum_{x=0}^{\infty} (-2)^x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(-2\lambda)^x}{x!} = e^{-\lambda} e^{-2\lambda}$$

$$= e^{-3\lambda}$$

$\therefore (-2)^X$ is unbiased estimator of $e^{-3\lambda}$

$(-2)^X$ can take values $\rightarrow 1, -2, 4, -8, 16 \ldots \ldots$

but $0 < e^{-3\lambda} < 1$

Remark 2

Sometimes unbiased estimate do not exist.

Ex

$X \sim Bin(n, p)$

(1) $g(p) = \frac{1}{p}$

Let $T(x)$ be unbiased estimator of $\frac{1}{p}$

then

$$E[T(x)] = \frac{1}{p} \quad \forall p \in [0, 1]$$

$$\underbrace{\sum_{x=0}^{n} T(x) \binom{n}{x} p^x (1-p)^{n-x}}_{\text{is a polynomial of degree atmost } p} = \underbrace{\frac{1}{p}}_{\text{No a polynomial}}$$

$\therefore$ Does not exist

Consistency

To denote n observations are used

An estimator $T_n = T(\underline{X})$ is said to be consistent for $g(\theta)$ if for every $\epsilon > 0$

$$P\left(\left| T_n - g(\theta) \right| \right.$$