Time-ordering techniques and solution of differential difference equation appearing in quantum optics

Cite as: Journal of Mathematical Physics **27**, 772 (1986); https://doi.org/10.1063/1.527182
Submitted: 05 August 1985 . Accepted: 03 October 1985 . Published Online: 04 June 1998

G. Dattoli, J. Gallardo, and A. Torre

Time-ordering techniques and solution of differential difference equation appearing in quantum optics

G. Dattoli, J. Gallardo,[a] and A. Torre[b]
ENEA, Dip. TIB, Divisione Fisica Applicata, C.R.E. Frascati, C.P. 65, 00044 Frascati, Rome, Italy

(Received 5 August 1985; accepted for publication 3 October 1985)

Time-ordering techniques based on the Magnus expansion and the Wei–Norman algebraic procedure are discussed and their relevance and usefulness to quantum optics are stressed.

I. INTRODUCTION

This paper has a twofold motivation: (a) to discuss relatively unknown time-ordering techniques, and (b) to show that these techniques are a useful tool to solve a large class of differential finite difference equations, too.

The problem of time-ordering expansion is as old as quantum mechanics and the most common treatment of it is the Feynman–Dyson[1] diagrammatic technique. However, alternative rigorous procedures, apparently not widely known, have been developed through the years by Magnus[2] and Wei and Norman.[3] These techniques offer definite advantages with respect to the well-known Feynman–Dyson[1] expansion and are tailored to be suited for a class of Hamiltonian operators appearing in many problems of quantum optics.

The considerations we develop here are general enough to be applied to diverse physical problems such as two-level molecular dynamics, stimulated Compton scattering, and the acousto-optic effect.

Let us briefly review the problems underlying the operator time evolution and time-ordering expansion. From elementary quantum mechanics[4] the evolution of the wave function of a physical system driven by a time-dependent Hamiltonian operator can be found formally by writing the solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=\hat{H}(t)\psi \tag{1.1}$$

as

$$\psi(t)=\hat{U}(t)\psi(0), \tag{1.2}$$

where $\hat{U}(t)$ is the time-evolution operator obeying the equation

$$i\hbar\frac{\partial\hat{U}(t)}{\partial t}=\hat{H}(t)\hat{U}(t),\quad \hat{U}(0)=\hat{1}. \tag{1.3}$$

If the operator $\hat{H}$ is time independent or if it commutes with itself at different times ($[\hat{H}(t),\hat{H}(t')]=0$), the solution of (1.3) is straightforward, i.e.,

$$\hat{U}(t)=\exp\left[-\frac{i}{\hbar}\int_0^t\hat{H}(t')dt'\right]\hat{U}(0). \tag{1.4}$$

If the operator $\hat{H}(t)$ does not commute at different times, time-ordering problems arise and the solution of (1.3) cannot be expressed in the simple form (1.4).

The technique most frequently adopted to deal with the evolution of $\hat{U}(t)$ is the use of the Feynman–Dyson[1] expansion

$$\left\{\exp\left[\frac{i}{\hbar}\int_0^t\hat{H}(t')dt'\right]\right\}_+ =1-\frac{i}{\hbar}\int_0^t dt_1\,\hat{H}(t_1) +\left(\frac{i}{\hbar}\right)^2\int_0^t dt_1\int_0^{t_1}dt_2\,\hat{H}(t_1)\hat{H}(t_2)+\cdots. \tag{1.5}$$

Where $\{\cdot\}_+$ denotes the time ordering and plays the role of the "chronological" operator. The expansion (1.5) is a perturbation series with all the practical disadvantages of the perturbative expansion. Indeed as noticed elsewhere,[5] the operator $\hat{U}^{(n)}(t)$, obtained by truncating the series, is no more a unitary operator, furthermore it is expected to be accurate for a small time interval or when $\hat{H}(t)$ can be treated as a perturbation. In many problems $H(t)$ cannot be considered as a perturbation or an accurate evaluation requires an excessively large number of infinitesimal orders. However, it must be stressed that in many cases (1.5) can be easily handled and each term can be usefully understood in terms of the symbolic Feynman diagrams.[1]

To go beyond the expansion (1.5) we require at least (a) a functional form of $\hat{U}(t)$ which preserves the unitary nature of the evolution operator; and (b) an exact form of the operator without any recourse to perturbation, or if perturbation is needed, a method that allows the expansion at any higher order.

Two methods, essentially complementary, have been proposed that satisfy the above requirements.

The first due to Magnus[2] consists in writing

$$\hat{U}(t)=\exp\{\hat{A}(t)\},\quad \hat{A}(0)=0, \tag{1.6}$$

where $\hat{A}(t)$ is a functional of $\hat{H}(t)$, more precisely an infinite series whose nth term is a sum of integrals of n-fold multiple commutators of $\hat{H}(t)$.

This method is now briefly reviewed, we follow a simpler but rigorous version due to Pechukas and Light.[5] The search for the time-displacement operator expressed in the form (1.6) meets both the requirements (a) and (b) and is an immediate generalization to a more complicated case of the corresponding expression (1.4).

According to Ref. 5, for the time derivative of $\hat{U}(t)$,

[a] Permanent address: University of California at Santa Barbara, Quantum Institute, Santa Barbara, California 93106.
[b] Also at Istituto Nazionale di Fisica Nucleare, Sezione Napoli, Napoli, Italy.

 0022-2488/86/030772-09$02.50

$$\frac{d}{dt}\hat{U}(t) = \left[\frac{e^{\operatorname{ad}\hat{A}}-1}{\operatorname{ad}A}\frac{d}{dt}\hat{A}\right]\hat{U}(t), \tag{1.7}$$

and the evolution equation (1.3), one immediately obtains

$$\frac{d\hat{A}(t)}{dt} = \left[\frac{\operatorname{ad}\hat{A}}{e^{\operatorname{ad}\hat{A}}-1}\right]\frac{\hat{H}}{i\hbar}. \tag{1.8}$$

(The operator $\operatorname{ad}\hat{A}$ is a linear operator defined as $(\operatorname{ad}\hat{A})\hat{X} = [\hat{A},\hat{X}]$.) Expanding in series of the operator on the right-hand side of (1.8) we get

$$\frac{d\hat{A}(t)}{dt} = \left(1 - \frac{1}{2}\operatorname{ad}\hat{A} + \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{(2n)!}B_n(\operatorname{ad}\hat{A})^{2n}\right)\frac{\hat{H}}{i\hbar}, \tag{1.9}$$

where the B_n are the Bernoulli numbers $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$,... . Solving the above equation by iteration one ends up with the expression

$$\hat{A} = \sum_{n=1}^{\infty}\hat{A}_n, \tag{1.10}$$

where the $(n+1)$th term reads

$$\hat{A}_{n+1} = \int_0^t dt'\left[-\frac{1}{2}\operatorname{ad}\hat{A}_n + \frac{1}{12}\sum_{m=1}^{n-1}\operatorname{ad}\hat{A}_m\operatorname{ad}\hat{A}_{n-m}\right]\frac{\hat{H}}{i\hbar}. \tag{1.11}$$

The first four terms are

$$\begin{aligned}
\hat{A}_1(t) &= -\frac{i}{\hbar}\int_0^t \hat{H}(t')dt', \\
\hat{A}_2(t) &= -\frac{1}{2}\left(\frac{i}{\hbar}\right)^2\int_0^t dt'\int_0^{t'} dt''[\hat{H}(t''),\hat{H}(t')], \\
\hat{A}_3(t) &= -\frac{1}{6}\left(\frac{i}{\hbar}\right)^3\int_0^t dt'\int_0^{t'} dt''\int_0^{t''} dt''' \\
&\quad\times\{[\hat{H}(t'''),[\hat{H}(t''),\hat{H}(t')]] \\
&\quad+[[\hat{H}(t'''),H(t'')],H(t')]\}, \\
\hat{A}_4(t) &= -\frac{1}{12}\left(\frac{i}{\hbar}\right)^4\int_0^t dt'\int_0^{t'} dt''\int_0^{t''} dt'''\int_0^{t'''} dt'''' \\
&\quad\times\{[\hat{H}(t''''),[[\hat{H}(t'''),\hat{H}(t'')],\hat{H}(t')]] \\
&\quad+[[\hat{H}(t''''),[\hat{H}(t'''),\hat{H}(t'')]],\hat{H}(t')] \\
&\quad+[[\hat{H}(t''''),\hat{H}(t''')],[\hat{H}(t''),\hat{H}(t')]]\}.
\end{aligned} \tag{1.12}$$

We note that the structure of the Magnus expansion corresponds to the continuous version of the Baker–Hausdorff disentangling theorem.[6]

The second method we describe is the Wei–Norman algebraic procedure.[3] This technique is complementary to the previous one, in the sense that it works when the Hamiltonian operator $\hat{H}(t)$ can be expressed in terms of the generators of an n-dimensional finite Lie algebra. The Magnus expansion, on the other side, applies when the multiple commutators in (1.11) converge to a c number.

According to Ref. 3 we write the Hamiltonian as

$$\hat{H}(t) = \sum_{j=1}^{m\leqslant n} a_j(t)\hat{L}_j, \tag{1.13}$$

where the $\hat{L}_j$ are the generators of the Lie algebra, the $a_j(t)$ are linearly independent functions of t, and the index j runs from 1 to $m\leqslant n$, where n is the dimensionality of the algebra.

The form of the solution (1.4), being valid, within this framework in the case of a one-dimensional algebra, suggests the following generalization to the case (1.13)

$$\hat{U}(t) = \prod_{j=1}^{n}\exp[g_j(t)\hat{L}_j], \quad g_j(0) = 0. \tag{1.14}$$

The functions $g_j(t)$, entering the above expression, can be obtained from a set of nonlinear differential equations whose specific form depends on the $a_j(t)$, and the algebraic structure constants involved in replacing (1.14) in (1.3) immediately yields

$$\sum_{i=1}^{n}\dot{g}_i(t)\left[\prod_{j=1}^{i-1}\exp(g_j(t)\hat{L}_j)\right]\hat{L}_i\cdot\left[\prod_{j=i}^{n}\exp(g_j(t)\hat{L}_j)\right] = \sum_{i=1}^{n}a_i(t)\hat{L}_i\hat{U}(t). \tag{1.15}$$

After a postmultiplication by the inverse operator $\hat{U}^{-1}$ and the direct computation of the expression

$$\left[\prod_{j=1}^{i-1}\exp(g_j(t)\hat{L}_j)\right]\hat{L}_i\left[\prod_{j=i-1}^{1}\exp(-g_j(t)\hat{L}_j)\right] = \sum_{k=1}^{n}\xi_{ki}\hat{L}_k, \tag{1.16}$$

we find

$$\sum_{i=1}^{n}a_i(t)\hat{L}_i = \sum_{j=1}^{n}\sum_{i=j}^{n}\dot{g}_j(t)\xi_{ij}\hat{L}_i, \tag{1.17}$$

where the matrix elements $\xi_{i,j}$ depend on the algebraic structure constants and on the g functions.

The linear independence of the generators reduces (1.17) into the nth-order system of differential equations

$$\begin{pmatrix}a_1\\ a_n\end{pmatrix} = \begin{bmatrix}\xi_{1,1}\cdots\xi_{1,n}\\ \xi_{n,2}\cdots\xi_{n,n}\end{bmatrix}\begin{pmatrix}\dot{g}_1\\ \dot{g}_n\end{pmatrix}. \tag{1.18}$$

It is therefore clear that once the explicit form of $a_i(t)$ and $\xi_{i,k}$ are known one can determine the functions g_i solving a set of nonlinear differential equations. For the proof of invertibility of (1.18) see Ref. 3. Let us finally point out that it has been shown[3] that uncoupling theorem holds for all solvable Lie algebras and for the real "split three-dimensional" simple Lie algebra (see Sec. II).

The paper is organized as follows: In Sec. II we will discuss systems that allow exact solutions, in Sec. III we will discuss perturbation methods, and finally in Sec. IV we present some conclusive remarks.

II. EXACT SOLUTIONS

In this section we will apply the above-discussed techniques to specific cases of physical interest in quantum optics.

The first we consider is a Hamiltonian operator that is a generalization to the two-level case of the so-called Kano Hamiltonian,[7] namely ($\hbar = 1$)

$$\hat{H} = \omega(t)\hat{J}_3 + \Omega^*(t)\hat{J}_+ + \Omega(t)\hat{J}_- + \beta(t), \tag{2.1}$$

where $\omega(\tau)$, $\Omega^*(\tau)$, $\Omega(\tau)$, and $\beta(\tau)$ are time-dependent, complex nonsingular functions, furthermore the $\hat{J}$ operators obey the well-known angular momentum commutation relations

$$[\hat{J}_+,\hat{J}_-] = 2\hat{J}_3, \quad [\hat{J}_\pm,\hat{J}_3] = \mp\hat{J}_\pm \;. \tag{2.2}$$

Assuming, for simplicity, that the Hamiltonian (2.1) drives a system of coupled harmonic oscillators with n_+, n_- initial quanta in the upper and lower level, the more general time evolution of the state can be described by the wave function

$$|\psi(t)\rangle = \sum_{l=-n_+}^{n_-} C_l(t)|n_+ + l, n_- - l\rangle\,, \tag{2.3}$$

where l is an integer accounting for the number of exchanged photons and the $C_l(t)$ are time-dependent coefficients denoting the amplitude probabilities of l emissions at time t. The Schrödinger equation gives for the coefficients $C_l(t)$ the following motion equation:

$$\begin{aligned} i\frac{dC_l}{dt} &= \omega(t)\left[\frac{n_+ - n_-}{2} + l\right]C_l(t) + \beta(t)C_l(t) \\ &\quad + \Omega(t)\sqrt{(n_- - l)(n_+ + l + 1)}C_{l+1}(t) \qquad (2.4) \\ &\quad + \Omega^*(t)\sqrt{(n_+ + l)(n_- - l + 1)}C_{l-1}(t)\,, \\ C_l(0) &= c_l\,. \end{aligned}$$

(To deduce (2.4) we have used the Schwinger realization of the angular momentum algebra, namely (see Ref. 4) $\hat{J}_+ = \hat{a}_+^\dagger \hat{a}_-$, $\hat{J}_- = \hat{a}_-^\dagger \hat{a}_+$, $\hat{J}_3 = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$, where $\hat{a}_\pm$, $\hat{a}_\pm^\dagger$ are creation annhilation operators ($[\hat{a}_\pm^\dagger,\hat{a}_\pm] = -1\,[\hat{a}_+^\dagger,\hat{a}_-] = 0$). The initial condition should be $C_l(0) = \delta_{l,0}$, we have assumed a generic discrete function to discuss slightly a more general problem.)

This differential difference equation already has been studied in Ref. 8 where it was pointed out that it belongs to the family of Raman–Nath (RN) equations[9] (i.e., spherical or SU_2 RN). We must stress that, according to the discussion of the previous section, the introduction of Eq. (2.4) is not a necessary step. The analytical expression of the evolution operator indeed can be found by means of the Hamiltonian operator (2.1). We have introduced this rather artificial step to remark that the technique we discuss here is also a powerful tool to solve equations of the RN type.

Adopting the same procedure of Ref. 8, we use the transformation

$$\begin{aligned} C_l(t) &= (-i)^l \exp\left\{-i\int_0^t \omega(t')dt'\left[\frac{n_+ - n_-}{2} + l\right]\right\} \\ &\quad \times \exp\left[-i\int_0^t \beta(t')dt'\right]\cdot M_l\,, \end{aligned} \tag{2.5}$$

which, once inserted in (2.4), yields

$$\begin{aligned} \frac{d}{dt}M_l &= -\Omega(t)\exp\left\{-i\int_0^t dt'\,\omega(t')\right\} \\ &\quad \cdot\sqrt{(n_- - l)(n_+ + l + 1)}M_{l+1} \\ &\quad + \Omega^*(t)\exp\left\{+i\int_0^t dt'\,\omega(t')\right\} \\ &\quad \cdot\sqrt{(n_+ + l)(n_- - l + 1)}M_{l-1}\,, \end{aligned} \tag{2.6}$$

$$M_l(0) = \sum_k i^k c_k \delta_{l,k}\,.$$

We now can solve the problem of finding the explicit solution of $M_l(t)$ exploiting the Wei–Norman technique discussed in the previous section.

The structure of (2.5) suggests the following equation for the evolution operator:

$$\begin{aligned} \frac{d\hat{U}(t)}{dt} &= \hat{T}(t)\hat{U}(t), \quad \hat{U}(0) = \hat{1}\,, \\ \hat{T}(t) &= -\Omega(t)\exp\left\{-i\int_0^t dt'\,\omega(t')\right\}\hat{J}_- \qquad (2.7) \\ &\quad + \Omega^*(t)\exp\left\{+i\int_0^t dt'\,\omega(t')\right\}\hat{J}_+\,. \end{aligned}$$

According to (14) the explicit solution of (2.6) can be written as

$$\hat{U}(t) = \exp\{2h(t)\hat{J}_3\}\exp\{g(t)\hat{J}_+\}\exp\{-f(t)\hat{J}_-\}\hat{1}\,. \tag{2.8}$$

Before giving the differential equations from which one can derive the functions f, g, and h, we notice that, to calculate the functional form of (2.6), it will be sufficient to evaluate the following matrix element:

$$\begin{aligned} \langle l|\hat{U}(t)|k\rangle &= \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r[g(t)]^m\cdot[f(t)]^r}{m!r!} \\ &\quad \times\langle l|\exp[2h(t)\hat{J}_3]\hat{J}_+^m\hat{J}_-^r|k\rangle\,. \end{aligned} \tag{2.9}$$

After some algebra and exploiting the properties of the $\hat{J}_-$ operators we find

$$\begin{aligned} &\langle l|\hat{U}(t)|k\rangle \\ &\quad = \exp\{2h(t)[\tfrac{1}{2}(n_+ - n_-) + l]\}[g(t)]^{l-k} \\ &\quad \times\left[\binom{n_- - k}{l-k}\binom{n_+ + l}{l-k}\right]^{1/2} \\ &\quad \times{}_2F_1(-n_+ + k, n_- - k + 1; l - k + 1; f(t)\cdot g(t))\,. \end{aligned} \tag{2.10}$$

Combining (2.5), (2.6), and (2.10) we finally easily find

$$\begin{aligned} &C_l(t) \\ &\quad = \exp\left\{-i\int_0^t \beta(t')dt'\right\}\exp\left\{\left[\frac{1}{2}(n_+ - n_-) + l\right]\mathscr{H}(t)\right\} \\ &\quad \times\sum_k(-i)^{l-k}c_k[g(t)]^{l-k} \\ &\quad \times\left[\binom{n_- - k}{l-k}\binom{n_+ + l}{l-k}\right]^{1/2} \\ &\quad \times{}_2F_1(-n_+ + k, n_- - k + 1; l - k + 1; g(t)\cdot f(t))\,, \end{aligned} \tag{2.11}$$

where

$$\mathscr{H}(t) = 2h(t) - i\int_0^t \omega(t')dt' \tag{2.12}$$

and ${}_2F_1(\cdots)$ is the hypergeometric function.[10] The result (2.12) is a more general expression of that obtained in Ref. 8(b).

Let us now consider the problem of writing the differential equations satisfied by (f,g,h).

It is easy to derive from (1.18) and from the algebraic structure of the angular momentum operators the following equations:

$$\dot{h}(t) = \Omega(t)\exp\{\mathscr{H}(t)\}g(t),$$
$$\dot{g}(t) = \Omega^*(t)\exp\{-\mathscr{H}(t)\} - g(t)\cdot\dot{h}(t), \quad (2.13)$$
$$\dot{f}(t) = \Omega(t)\exp\{\mathscr{H}(t)\}, \quad h(0) = f(0) = g(0) = 0.$$

It can be shown that the solution of (2.13) depends on the single Riccati equation

$$\dot{u} - u^2 + r(t)u + q(t) = 0, \quad \dot{h}(t) = u(t), \quad u(0) = 0, \quad (2.14)$$

$$r(t) = -\frac{d}{dt}\ln\Omega(t) + i\omega(t), \quad q(t) = -|\Omega(t)|^2.$$

The solution of (2.14) can be explicitly written in a restricted number of cases. In the less general case $\omega(t) = \omega_0 = \text{const}$; $\Omega(t) = \Omega = \text{const}$ and real $\beta(t) = 0$, we easily get

$$h(t) = (i/2\omega_0 t - \ln(1-p(t))^{1/2} - i\arctan\left(\left(\frac{\omega_0}{\delta}\right)\tan\left(\frac{\delta t}{2}\right)\right),$$
$$g(t) = [p(t)]^{1/2}[1-p(t)]^{1/2} \times\exp\left\{i\arctan\left(\left(\frac{\omega_0}{\delta}\right)\tan\left(\frac{\delta t}{2}\right)\right)\right\}, \quad (2.15)$$
$$f(t) = \left[\frac{p(t)}{1-p(t)}\right]^{1/2}\exp\left\{-i\arctan\left(\left(\frac{\omega_0}{\delta}\right)\tan\left(\frac{\delta t}{2}\right)\right)\right\},$$
$$p(t) = \left(\Omega\frac{\sin\delta t/2}{\delta/2}\right)^2, \quad \delta = \sqrt{\omega_0^2 + 4\Omega^2}.$$

In the case of $n_+ = 0$ we get [see Ref. 8(a)]

$$C_l(t) = \binom{n_-}{l}^{1/2}\exp\left\{in_-\arctan\left(\frac{\omega_0}{\delta}\tan\frac{\delta t}{2}\right)\right\}(\alpha(t))^l \times(1-|\alpha(t)|^2)^{(n_- - l)/2}, \quad (2.16)$$
$$\alpha(t) = (-i)\left(\Omega\frac{\sin\delta t/2}{\delta/2}\right)\exp\left[-i\arctan\left(\frac{\omega_0}{\delta}\tan\left(\frac{\delta t}{2}\right)\right)\right].$$

The results obtained so far are very general. We can now discuss some interesting limiting cases, when the number of "excitation quanta" $n_\pm$ are very large.

A somewhat crude approach to the problem could be that of taking the asymptotic limits $n_\pm \to \infty$ in (2.11). This procedure gives the right functional form of the coefficient C_l but raises doubts on the correct expression of the functions f, g, and h. The appropriate and rigorous procedure requires the so-called group contraction method,[11] which will allow us to understand the intimate connection between the SU_2 algebra and its contraction to the "harmonic oscillator" and "shift" algebras.

We introduce a three-dimensional Lie algebra with generators $\hat{H}_1$, $\hat{H}_2$, and $\hat{H}_3$ with commutation relations[3]

$$[\hat{H}_1, \hat{H}_2] = 2\lambda\hat{H}_2,$$
$$[\hat{H}_1, \hat{H}_3] = -2\lambda\hat{H}_3, \quad (2.17)$$
$$[\hat{H}_2, \hat{H}_3] = -\delta\hat{H}_1,$$

where λ and δ are numbers which define the explicit form of the $\hat{H}$ operators. We leave, for the moment, the operators in (2.17) undefined and notice that an evolution operator driven by the $\hat{H}_s$ (we mean by this a Hamiltonian of the type $\hat{H} = [\omega(t)/2]\hat{H}_1 + \Omega^*(t)\,\hat{H}_2 - \Omega(t)\,\hat{H}_3)$ [see (2.8)] exhibits f, g, and h functions defined by the differential equations

$$\dot{h}(t) = \delta g(t)\dot{f}(t),$$
$$\dot{g}(t) = \Omega^*(t)\exp\{-\mathscr{H}_\lambda^{(t)}\} - \lambda g(t)\dot{h}(t), \quad (2.18)$$
$$\dot{f}(t) = \Omega(t)\exp\{\mathscr{H}_\lambda^{(t)}\},$$
$$\mathscr{H}_\lambda^{(t)} = 2\lambda h(t) - i\int_0^t \omega(t')dt'.$$

It is easy to understand that when $\lambda = \delta = 1$ the $\hat{H}$ operators can be identified with

$$\hat{H}_1 = 2\hat{J}_3, \quad \hat{H}_2 = \hat{J}_+, \quad \hat{H}_3 = -\hat{J}_-. \quad (2.19)$$

Furthermore if $\lambda = 0$ and $\delta = 1$ the SU_2 algebra contracts to the three-dimensional non-Abelian algebra with generators $\{\hat{a}^+, \hat{a}, \hat{I}\}$, where $\hat{a}^+$, $\hat{a}$ are creation annihilation operators. Finally when $\lambda = \delta = 0$ all our algebra collapses in a "shift algebra" with generators $\{\hat{E}^+, \hat{E}^-, \hat{I}\}$, where the $\hat{E}^\pm$ are shift operators.

Let us now discuss the cases $n_- \to \infty$ and $n_\pm \to \infty$.

A. Large number of lower level excitation quanta ($n_- \to \infty$)

In this case the SU_2 RN equation (2.4) reduces to the so-called harmonic RN equation[12]

$$i\frac{dC_l}{dt} = \omega(t)(n_+ + l)C_l + \bar{\Omega}^*(t)\sqrt{n_+ + l}\,C_{l-1} + \bar{\Omega}(t)\sqrt{n_+ + l + 1}\,C_{l+1} + \bar{\beta}(t)C_l,$$
$$\bar{\Omega}(t) = \Omega(t)\sqrt{n_-}, \quad \bar{\beta}(t) = \beta(t) - \frac{n_- + n_+}{2}\omega(t). \quad (2.20)$$

This expression suggests the following identification of the $\hat{H}$ operators:

$$\hat{H}_1 = -n_-\hat{1}, \quad \hat{H}_2 = \sqrt{n_-}\,\hat{a}^+, \quad \hat{H}_3 = -\sqrt{n_-}\,\hat{a}. \quad (2.21)$$

Therefore, setting $\lambda = 0, \delta = 1$ in (2.18) we immediately find the solutions

$$g(t) = \int_0^t \Omega^*(t')\exp\left[i\int_0^{t'}\omega(t'')dt''\right]dt',$$
$$f(t) = \int_0^t \Omega(t')\exp\left[-i\int_0^{t'}\omega(t'')dt''\right]dt', \quad (2.22)$$
$$h(t) = \frac{1}{2}\int_0^t g(t')\dot{f}(t')dt' - \frac{1}{2}\int_0^t [\dot{g}(t')f(t') - g(t')\dot{f}(t')]dt'.$$

Furthermore, using the well-known asymptotic properties of the hypergeometric function,[12] we find for C_l the expression

$$C_l(t)=\exp\left\{-i\int_0^t \bar\beta(t')dt'\right\}$$
$$\times\exp\{-i(n_+ + l)\int_0^t \omega(t')dt'\}\exp\left\{-\frac{1}{2}\alpha(t)\cdot\gamma(t)\right\}$$
$$\times\exp\left\{-\frac{1}{2i}\int_0^t [\bar\Omega^*(t')\gamma(t')+\bar\Omega(t')\alpha(t')]dt'\right\}$$

$$\times\sum_k c_k(-i)^{l-k}\sqrt{(n_+ + k)!(n_+ + l)!}$$
$$\times\mathscr{F}^{l-k}(t)L^{l-k}_{n_+ + k}(\alpha(t)\cdot\gamma(t)), \tag{2.23}$$

where the L^l_n are the generalized Laguerre polynomials and

$$\begin{aligned}\mathscr{F}(t)&=\int_0^t\bar\Omega^*(t')\exp\left\{+i\int_0^{t'}\omega(t'')dt''\right\}dt',\\ \alpha(t)&=-i\mathscr{F}(t)\exp\left\{-i\int_0^{t'}\omega(t'')dt''\right\},\\ \gamma(t)&=i\left[\int_0^t\bar\Omega(t')\exp\left\{-i\int_0^{t'}\omega(t'')dt''\right\}dt'\right]\\ &\quad\times\exp\left\{i\int_0^t\omega(t')dt'\right\}.\end{aligned}\tag{2.24}$$

It is easy to derive from (2.23) the solution already found in Ref. 12 when $C_l(0)=\delta_{l,0}, \beta(t)=0, \omega_0=$ const, and $\bar\Omega=\bar\Omega^*$ const, namely

$$\begin{aligned}C_l(t)&=\sqrt{\frac{n_+!}{(n_+ + l)!}}\alpha(t)^l\cdot e^{-in_+\omega_0 t}\\ &\quad\times\exp\left\{i\frac{\omega_0}{2}\int_0^t|\alpha(t')|^2dt'\right\}\\ &\quad\times\exp(-|\alpha(t)|^2/2)L^l_n[|\alpha(t)|^2],\\ \alpha(t)&=-i\left(\bar\Omega\frac{\sin\omega_0 t/2}{\omega_0/2}\right)e^{-i\omega_0 t/2}.\end{aligned}\tag{2.25}$$

B. Large number of upper and lower number of excitation quanta ($n_\pm\to\infty$)

Equation (2.4) reduces, in this hypothesis, to the so-called shift RN equation, namely

$$i\frac{dC_l}{dt}=\omega(t)lC_l+\bar\Omega(t)C_{l+1}+\bar\Omega^*(t)C_{l-1}+\bar\beta(t)C_l,$$
$$\bar\Omega=\sqrt{n_+n_-}\,\Omega(t),\quad \bar\beta=\beta(t)+(n_+ - n_-)/2\omega(t). \tag{2.26}$$

The identification of the $\hat H$ operators is straightforward:

$$\hat H_1=0,\quad \hat H_2=\sqrt{n_+n_-}\hat E^+,\quad \hat H_3=-\sqrt{n_+n_-}\hat E^-. \tag{2.27}$$

Therefore setting $\lambda=\delta=0$ in (36) and exploiting again the asymptotic properties of the hypergeometric function for large $n_\pm$ (see Ref. 10), we finally find

$$C_l=\exp\left(-i\int_0^t\bar\beta(t')dt'\right)$$
$$\times\sum_k c_k\exp\left(-ik\int_0^t\omega(t')dt'\right)\left[\frac{\alpha(t)}{\gamma(t)}\right]^{(l-k)/2}$$
$$\times J_{l-k}[2\sqrt{\alpha(t)\cdot\gamma(t)}], \tag{2.28}$$

where $J_l(\cdot)$ is the cylindrical Bessel function of the first kind and integer order, furthermore

$$\begin{aligned}\alpha(t)&=-i\left[\int_0^t\bar\Omega^*(t')\cdot\exp\left(-i\int_0^{t'}\omega(t'')dt''\right)\right]\\ &\quad\times\exp\left(-i\int_0^t\omega(t')dt'\right),\\ \gamma(t)&=i\left[\int_0^t\bar\Omega(t')\exp\left(+i\int_0^{t'}\omega(t'')dt''\right)\right]\\ &\quad\times\exp\left(+i\int_0^t\omega(t')dt'\right).\end{aligned}\tag{2.29}$$

It is easy to see that when $\beta(t)=0$, $\Omega(t)=\Omega^*(t)$ $=$ const $C_l(0)=\delta_{l,0}$ (2.29) reduces to the well-known solution[13]

$$C_l=(-i)^l\exp\left(-il\frac{\omega_0 t}{2}\right)J_l\left[2\bar\Omega\frac{\sin\omega_0 t/2}{\omega_0/2}\right]. \tag{2.30}$$

Before concluding this section we stress that the analysis we have presented is very general and based on the Wei–Norman technique. However, while this procedure is strictly necessary for the SU_2 algebra in the case of the "harmonic oscillator" algebra, the Magnus expansion is equally useful (see the Appendix).

III. PERTURBED SOLUTIONS

In the previous section we considered particularly significant cases that admit exact solutions. In this section we will discuss different situations where exact solutions are not available but nontrivial perturbed solutions may be obtained.

The analysis we develop in this section is relevant, e.g., to the evolution of quantum systems driven by Hamiltonians of the type

$$\hat H=\omega(t)\hat J_3+\epsilon(t)\hat J_3^2+[\Omega^*(t)\hat J_+ + \Omega(t)\hat J_-]+\beta(t), \tag{3.1}$$

where $\epsilon(t)$ is a nonsingular time-dependent function that can be treated as a perturbation.

To illustrate the method we shall restrict ourselves to the algebraically simpler case of ϵ, ω, and $\Omega=\Omega^*$ time-independent constants. However, we stress that the identical procedure applies to the Hamiltonian (3.2).

We now will consider the specific problem of the stimulated Thomson scattering of two counterpropagating electromagnetic waves.[14]

According to Ref. 14 this process can be described by a spherical RN equation of the type

$$\begin{aligned}i\frac{dC_l}{dt}&=\omega_0 lC_l+\epsilon l^2C_l+\Omega\big[\sqrt{(n_- - l)\,(n_+ l+1)}C_{l+1}(t)\\ &\quad+\sqrt{(n_- - l+1)\,(n_+ + l)}C_{l-1}(t)\big],\ C_l(0)=\delta_{l,0}.\end{aligned}\tag{3.2}$$

The main difference between the above equation and the one considered in the previous section is the presence of the quadratic term in l. However, if this term can be treated as a perturbation (as happens in many cases of physical interest), one can find a perturbed solution to first order in ϵ [(3.2)]. We proceed as in the previous section. Namely we introduce the function

$$C_l(t) = (-i)^l \exp\{-i(\omega_0 + \epsilon l)lt\} M_l(t), \tag{3.3}$$

which, once inserted in (3.2), gives the new expression

$$\begin{aligned}\frac{dM_l(t)}{dt} &= \Omega\{\sqrt{(n_+ + l)\,(n_- - l + 1)} \\ &\quad \times \exp[i(\omega_0 + \epsilon(2l-1))(t)]M_{l-1}(t) \\ &\quad - \sqrt{(n_- - l)\,(n_+ l + 1)} \\ &\quad \times \exp[-i(\omega_0 + \epsilon(2l+1))(t)]M_{l+1}(t)\}, \\ M_l(0) &= i^l \delta_{l,0}.\end{aligned} \tag{3.4}$$

The above expression, even if more complicated than those discussed in Sec. II, suggests the following structure for the motion equation of the evolution operator:

$$\begin{aligned}\frac{d\hat{U}}{dt} &= \hat{T}(t)\hat{U}, \\ \hat{T}(t) &= \Omega e^{i\omega_0 t} \exp\{i\epsilon[2\hat{J}_3 - (n_+ - n_-) - 1]t\}\hat{J}_+ \\ &\quad - \Omega e^{-i\omega_0 t} \exp\{-i\epsilon[2\hat{J}_3 - (n_+ - n_-) + 1]t\}\hat{J}_-.\end{aligned} \tag{3.5}$$

The presence of the $\hat{J}_3$ operator at the exponent, does not allow any exact solution of (3.5) in closed form.

However, expanding the exponents up to the first order in ϵ the $\hat{T}$ operator can be written as

$$\begin{aligned}\hat{T}(t) &\cong b^0(t)\hat{H}_2 + C^0(t)\hat{H}_3 + \epsilon[b^1(t)\hat{H}_2 \\ &\quad + C^1(t)\hat{H}_3 + p(t)\hat{H}_1\hat{H}_3 + q(t)\hat{H}_1\hat{H}_2],\end{aligned} \tag{3.6}$$

where the $\hat{H}$ are the operators introduced in (2.19):

$$\begin{aligned} &b^0(t) = \Omega e^{i\omega_0 t}, \quad b^1(t) = -t\Omega i(\Delta n + 1)e^{i\omega_0 t}, \\ &C^0(t) = \Omega e^{-i\omega_0 t}, \quad C^1(t) = -ti\Omega(\Delta n - 1)e^{-i\omega_0 t}, \\ &p(t) = \frac{\partial}{\partial\omega_0} C^0(t), \quad q(t) = \frac{\partial}{\partial\omega_0} b^0(t), \\ &\Delta n \equiv n_+ - n_-.\end{aligned} \tag{3.7}$$

A convenient form to find the time evloution of $\hat{U}(t)$ is

$$\begin{aligned}\hat{U} &= \exp\{\epsilon\zeta(t)\hat{H}_1^2\}\exp\{\epsilon\eta(t)\hat{H}_2^2\}\exp\{\epsilon\gamma(t)\hat{H}_3^2\} \\ &\quad \times \exp\{\epsilon\delta(t)\hat{H}_1\cdot\hat{H}_2\} \\ &\quad \times \exp\{\epsilon\theta(t)\hat{H}_1\hat{H}_3\}\exp\{\epsilon\lambda(t)\hat{H}_2\hat{H}_3\} \\ &\quad \times \exp\{g_1(t)\hat{H}_1\}\exp\{g_2(t)\hat{H}_2\}\exp\{g_3(t)\hat{H}_3\}\hat{I},\end{aligned} \tag{3.8}$$

where the functions in the exponents are specified by the following system of differential equations:

$$\begin{aligned} g_1(t) &= h(t) + \epsilon h^1(t), \\ g_2(t) &= g(t) + \epsilon g^1(t), \\ g_3(t) &= f(t) + \epsilon f^1(t).\end{aligned} \tag{3.9}$$

Here, $f(t)$, $g(t)$, and $h(t)$ are the functions defined in the previous section and furthermore

$$\begin{aligned} b^1(t) &= e^{2h(t)}[g^1(t)^2 + g(t)^2\dot{f}^1(t)] + 2b^0(t)h^1(t) \\ &\quad + 2C^0(t)g^1(t) - 2b^0(t)(2\zeta(t) \\ &\quad + \lambda(t)) + 2\eta(t)C^0(t), \\ C^1(t) &= 2b^0(t)\gamma(t) + e^{-2h(t)}[\dot{f}^1(t) - 2\dot{f}(t)h^1(t)] \\ &\quad - 4\zeta(t)C^0(t), \\ 0 &= \dot{h}^1(t) - \dot{f}^1(t)g(t) - \dot{f}(t)g^1(t) - 2\delta(t)C^0(t), \\ q(t) &= \dot{\delta}(t) + (4\zeta(t) + \lambda(t))b^0(t) - 2\eta(t)C^0(t), \\ p(t) &= \dot{\theta}(t) + 2\gamma(t)b^0(t) - C^0(t)(4\zeta(t) + \lambda(t)), \\ 0 &= \dot{\lambda}(t) + 2\theta(t)b^0(t) - 2\delta(t)C^0(t), \\ 0 &= \dot{\zeta}(t) + \theta(t)b^0(t) - \delta(t)C^0(t), \\ 0 &= \dot{\eta}(t) + 2\delta(t)b^0(t), \\ 0 &= \dot{\gamma}(t) - 2\theta(t)C^0(t).\end{aligned} \tag{3.10}$$

Finally, using the properties of the $\hat{H}$ operators we find the following expression for the $C_l(t)$ coefficients:

$$\begin{aligned} C_l(t) &= \left[\binom{n_-}{l}\binom{n_+ + l}{l}\right]^{1/2} \exp\left\{i(n_- - n_+)\left[\arctan\left(\frac{\omega_0}{\delta}\tan\frac{\delta t}{2}\right) - \frac{\omega_0 t}{2}\right]\right\}\alpha(t)^l(1 - |\alpha(t)|^2)^{(n_- - n_+ - l)/2} \\ &\quad \times\Big\{[1 - i\epsilon l^2 t + \epsilon(n_+ - n_- + 2l)\,(h^1(t) - i\alpha(t)\,(1 - |\alpha(t)|^2)^{1/2}\exp[2i\arctan(\omega_0/\delta\tan\delta t/2)]f^1(t)) \\ &\quad + \epsilon\lambda(t)(\tfrac{1}{2}(n_+ - n_- + 2l)^2 - (n_+ + l)\,(n_- - l + 1)]\,{}_2F_1(-n_+, n_- + 1; l + 1; |\alpha(t)|^2) \\ &\quad - i\epsilon\frac{\exp[-2i\arctan(\omega_0/\delta\tan\delta t/2)]}{\alpha(t)(1 - |\alpha(t)|^2)^{1/2}}\Big[(g^1(t) + \alpha(t)^2(1 - |\alpha(t)|^2) \\ &\quad \times\exp[4i\arctan(\omega_0/\delta\tan\delta t/2)]f^1(t))\cdot l\,{}_2F_1(-n_+, n_- + 1; l; |\alpha(t)|^2) \\ &\quad + \frac{(n_- - l)\,(n_+ + l + 1)}{(l+1)}\alpha(t)^2(1 - |\alpha(t)|^2)\exp[4i\arctan(\omega_0/\delta\tan\delta t/2)] \\ &\quad \times f^1(t)\,{}_2F_1(-n_+, n_- + 1; l + 2; |\alpha(t)|^2)\Big] - i\epsilon\frac{(n_+ - n_- + 2l)}{\alpha(t)(1 - |\alpha(t)|^2)^{1/2}} \\ &\quad \times\Big[l\delta(t)e^{-i\omega_0 t}(1 - |\alpha(t)|^2)\,{}_2F_1(-n_+, n_- + 1; l; |\alpha(t)|^2) \\ &\quad + \theta(t)e^{i\omega_0 t}\frac{n_- - l}{l+1}(n_+ + l + 1)\alpha(t)^2\,{}_2F_1(-n_+, n_- + 1; l + 2; |\alpha(t)|^2)\Big]\end{aligned}$$

$$-\epsilon\Big[\eta(t)e^{-2i\omega_0 t}\frac{(1-|\alpha(t)|^2)}{\alpha(t)^2}l(l+1)\,{}_2F_1(-n_+,n_-+1;l-1;|\alpha(t)|^2)\Big]$$

$$-\epsilon\gamma(t)e^{2i\omega_0 t}\frac{\alpha(t)^2}{(1-|\alpha(t)|^2)}\frac{(n_- -l)(n_- -l-1)(n_+ +l+1)(n_+ +l+2)}{(l+1)(l+2)}$$

$$\times{}_2F_1(-n_+,n_-+1;l+3;|\alpha(t)|^2)\Big\}. \tag{3.11}$$

The results we have derived show that a perturbed analysis of quantum systems driven by Hamiltonians like (3.1) in principle can be carried out analytically.

However, we must point out that (a) the system of differential equations (3.10) cannot be solved straight forwardly; and (b) the expression (3.11) is rather complicated and in the present framework further simplifications cannot be made.

In any case these results can be usefully exploited when physical assumptions allow some simplifications or as a test of numerical analysis.

Let us now discuss the "contraction" of Eq. (50) to (namely for large n_-)

$$i\frac{dC_l}{dt}=\omega_0 lC_l+\epsilon l^2C_l+\bar{\Omega}\big[\sqrt{n+l+1}\,C_{l+1}+\sqrt{n+l}\,C_{l-1}\big],\quad C_l(0)=c_l. \tag{3.12}$$

(This case is relevant to Hamiltonians of the type $\hat{H}=\omega_0\hat{a}^+\hat{a}+\epsilon(\hat{a}^+\hat{a})^2+\Omega[\hat{a}^++\hat{a}]$.) The perturbed solution of (3.12) can be found using the same technique of group contraction discussed in the previous section. We omit the details of the calculations and write directly the solution

$$C_l(t)=\exp\Big(\frac{i\omega_0}{2}\int_0^t|\alpha(\tau')|^2\,d\tau'\Big)\exp\Big(\frac{-1}{2}|\alpha(t)|^2\Big)\sum_k\sqrt{\frac{(n+k)!}{(n+l)!}}\exp(-ik\omega_0 t)\alpha(t)^{l-k}c_k\{A_{l-k}(t)+iD_{l-k}(t)\}, \tag{3.13}$$

where

$$\begin{aligned}A_{l-k}(t)=\;&L_{n+k}^{l-k}(\cdot)[1-\epsilon kC(t)]+\frac{\epsilon}{|\alpha(t)|}\frac{\partial}{\partial\omega_0}|\alpha(t)|\big[|\alpha(t)|^2L_{n+k-1}^{l-k+1}(\cdot)+(n+k+1)L_{n+k+1}^{l-k-1}(\cdot)\big]\\
&+\frac{\epsilon}{|\alpha(t)|}\Big[R(t)-\frac{2}{9}\frac{\partial}{\partial\omega_0}|\alpha(t)|^3\Big]\big[|\alpha(t)|^2L_{n+k-1}^{l-k+1}(\cdot)-(n+k+1)L_{n+k+1}^{l-k-1}(\cdot)\big]\\
&+\frac{2\epsilon}{|\alpha(t)}k\frac{\partial}{\partial\omega_0}|\alpha(t)|\big[n+k+1)L_{n+k+1}^{l-k-1}(\cdot)\\
&-|\alpha(t)|^2L_{n+k-1}^{l-k-1}(\cdot)\big]-\frac{\epsilon}{|\alpha(t)|}\frac{\partial}{\partial\omega_0}|\alpha(t)|\big[|\alpha(t)|^4L_{n+k-2}^{l-k+2}(\cdot)-(n+k+1)(n+k+2)L_{n+k+2}^{l-k-2}(\cdot)\big],\\
D_{l-k}(t)=\;&-\epsilon l^2t+\epsilon[2C(t)|\alpha(t)|^2+G(t)-7/6t\,|\alpha(t)|^4+(2(n+k)+1)(2C(t)-t\,|\alpha(t)|^2)]L_{n+k}^{l-k}(\cdot)\\
&-\epsilon(t/2)\big[|\alpha(t)|^2L_{n+k-1}^{l-k+1}(\cdot)-(n+k+1)L_{n+k+1}^{l-k-1}(\cdot)\big]+\epsilon[2t\,|\alpha(t)|^2-3C(t)]\\
&\times\big[|\alpha(t)|^2L_{n+k-1}^{l-k+1}(\cdot)+(n+k+1)L_{n+k+1}^{l-k-1}(\cdot)\big]\\
&+\epsilon tk\big[(n+k+1)L_{n+k+1}^{l-k-1}(\cdot)-L_{n+k-1}^{l-k+1}(\cdot)|\alpha(t)|^2\big]\\
&-\frac{\epsilon}{|\alpha(t)|^2}\Big[C(t)+\frac{t}{2}|\alpha(t)|^2\Big]\big[|\alpha(t)|^4L_{n+k-2}^{l-k+2}(\cdot)\\
&+(n+k+1)(n+k+2)L_{n+k+2}^{l-k-2}(\cdot)\big]\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}C(t)&=\Big(\frac{\bar{\Omega}}{\omega_0}\Big)^2\Big[t-\frac{\sin\omega_0 t}{\omega_0}\Big],\\
G(t)&=-\frac{2}{3}\Big(\frac{\bar{\Omega}}{\omega_0}\Big)^4\Big[-\frac{t}{4}\cos 2\omega_0 t-\frac{13}{4}t-2t\cos\omega_0 t+\frac{1}{6}\omega_0^2t^3+\frac{3}{4\omega_0}\sin 2\omega_0 t+\frac{4}{\omega_0}\sin\omega_0 t\Big],\\
R(t)&=-\frac{2}{3}\Big(\frac{\bar{\Omega}}{\omega_0}\Big)^3\Big[7t\cos\frac{\omega_0 t}{2}+2t\cos\omega_0 t\cos\frac{\omega_0 t}{2}-\frac{7}{2\omega_0}\sin\frac{3}{2}\omega_0 t-\frac{15}{2\omega_0}\sin\frac{\omega_0 t}{2}\Big].\end{aligned} \tag{3.15}$$

[The $L_n^l(\cdot)$ are the generalized Laguerre polynomials of argument $|\alpha(\tau)|^2$.] As final example we consider the further "contraction" of (3.2) when both $n_\pm$ are large quantities. In this case the spherical RN equation reduces to

$$i\frac{dC_l}{dt}=\omega_0 lC_l+\epsilon l^2C_l+\bar{\Omega}[C_{l+1}+C_{l-1}],\quad C_l(0)=c_l. \tag{3.16}$$

Using the same procedure leading to (3.12) we find

$$C_l=\sum_k(-i)^{l-k}e^{-i[\omega_0(l-k)t/2]}c_k\{A_{l-k}+iD_{l-k}\}, \tag{3.17}$$

where

$$A_{l-k} = J_{l-k}(\cdot) + \epsilon\frac{\partial}{\partial\omega_0}\left(\overline{\Omega}\frac{\sin\omega_0 t/2}{\omega_0/2}\right)[(2l-1)J_{l-k-1}(\cdot)$$

$$-(2l+1)J_{l-k+1}(\cdot)] + \frac{1}{2}\epsilon\overline{\Omega}^2\frac{\partial}{\partial\omega_0}\left(\overline{\Omega}\frac{\sin\omega_0 t/2}{\omega_0/2}\right)^2[J_{l-k+2}(\cdot) - J_{l-k-2}(\cdot)],$$

$$D_{l-k} = \left[-\epsilon l^2 t + 4\epsilon\frac{\overline{\Omega}^2}{\omega_0^2}\left(\frac{\sin\omega_0 t}{\omega_0} - t\right)\right]J_{l-k}(\cdot) + \frac{\epsilon t}{2}\left(\overline{\Omega}\frac{\sin\omega_0 t/2}{\omega_0/2}\right)[(2l-1)J_{l-k-1}(\cdot) + (2l+1)J_{l-k+1}(\cdot)]$$

$$+\epsilon\frac{\overline{\Omega}^2}{\omega_0^2}\left(t\cos\omega_0 t - \frac{\sin\omega_0 t}{\omega_0}\right)[J_{l-k-2}(\cdot) + J_{l-k+2}(\cdot)]. \tag{3.18}$$

The argument of the Bessel functions left indicated by $(\cdot)$ is $2\overline{\Omega}(\sin\omega_0 t/2)/(\omega_0/2)$.

IV. CONCLUSIONS

In this work we have discussed a rather general technique that can be usefully applied to a number of physical problems. Furthermore the techniques we have developed may be of interest even for mathematicians, since they amount to a useful tool to construct solutions for a large class of differential finite difference equations.

Concerning this last point we wish to add the following comment.

In some cases when the structure of the RN equation is particularly simple, namely when it can be derived from Hamiltonians that do not involve noncommuting operators [as, e.g., the $(\hat{E}^+, \hat{E}^-, \hat{I})$ generators] more direct methods can be used.

To give an example we reconsider the shift RN equation with constant coefficients (see Sec. II):

$$i\frac{dC_l}{dt} = \omega_0 l C_l + \Omega[C_{l+1} + C_{l-1}],\quad C_l(0) = \delta_{l,0}. \tag{4.1}$$

Using the transformation (2.5) we get

$$\frac{dM_l}{dx} = -e^{-i\beta x}M_{l+1} + e^{i\beta x}M_{l-1},$$
$$M_l(0) = i^l\delta_{l,0}\quad (x = \Omega t,\quad \beta = \omega_0/\Omega). \tag{4.2}$$

Multiplying both sides of (4.2) by s^l and summing over l we find

$$\frac{d\Gamma(x,s)}{dx} = \left(se^{+i\beta x} - \frac{1}{s}e^{-i\beta x}\right)\Gamma(x,s),$$
$$\Gamma(0,s) = 1,\quad \Gamma(x,s) = \sum_{l=-\infty}^{\infty} s^l M_l(x), \tag{4.3}$$
$$0 < |s| < \infty.$$

Equation (4.3) can be solved straightforwardly, namely

$$\Gamma(x,s) = \exp\left\{\frac{\sin\beta x/2}{\beta/2}\left[se^{i\beta x/2} - \frac{1}{s}e^{-i\beta x/2}\right]\right\}. \tag{4.4}$$

Therefore, using the Bessel generating function $[\sum_{l=-\infty}^{+\infty} t_l J_l(x) = e^{x/2(t-1/t)}]$ we easily find [see also Ref. (10)]

$$C_l(t) = (-i)^l e^{-il\omega_0 t/2} J_l[2\Omega\sin\omega_0 t/2/\omega_0/2]. \tag{4.5}$$

This is only a particularly simple example that shows that in a few selected cases simpler and more direct techniques are available. In any case when noncommuting operators are involved with time-dependent coefficients, time-ordering techniques, of the type discussed here, are a necessary step.

A further point, relevant to the differential finite difference equation, that we want to touch upon is the fact that we have discussed only homogeneous equations. We have not mentioned inhomogeneous cases which may arise treating perturbed solutions of nonlinear differential finite difference systems [to give an example, $i\,dC_l/dt = \Omega(C_{l+1} + C_{l-1}) + \xi|C_l|^2$, with ξ an expansion parameter]. We want to give a simple example that shows that even in this hypothesis exact solutions can be found.

The equation we consider is the following:

$$i\frac{dC_l}{dt} = \Omega(C_{l+1} + C_{l-1}) + f_l(t),\quad C_l(0) = \delta_{l,0}, \tag{4.6}$$

where $f_l(t)$ is a generic function depending both on the time and on the discrete index l.

It is easy to verify that the solution of (4.6) can be written as

$$C_l(t) = C_l^h(t)$$
$$+\sum_{m=-\infty}^{+\infty}(-i)\int_0^t C_{l-m}^h(t-t')f_m(t')dt', \tag{4.7}$$

where $C_l^h(t)$ is the solution of the homogeneous case $[C_l^h(t) = (-i)^l J_l(2\Omega t)]$.

In a forthcoming paper we will apply all the previously found results to particular physical problems.

ACKNOWLEDGMENTS

The authors recognize the contributions of M. Richetta for checking the calculations and A. Dipace for critical remarks.

It is a pleasure to thank A. Renieri, S. Solimeno, and C. D. Cantrell for kind interest and comment. One of the authors (G. Dattoli) acknowledges the warm hospitality of the Department of Physics of the Napoli University where part of this work was carried out.

J. Gallardo wishes to acknowledge the support of the Office of Naval Research under Contract No. N00014-80-C-0308.

APPENDIX: A NOTE ON THE MAGNUS EXPANSION

The Magnus method has been discussed in the Introduction but, in the following sections, we have been mainly concerned with the Wei–Norman method, which has the advantage of being more general.

We have exploited that method even for cases in which it is not strictly necessary. In this way, however, we have shown the intimate connection between apparently disconnected problems.

In this Appendix we will discuss some examples where the Magnus expansion directly applies.

We consider a Hamiltonian of the type

$$\hat{H} = \omega_0 \hat{a}^+ \hat{a} + \Omega(\hat{a}^+ + \hat{a}) \quad (\hbar = 1), \tag{A1}$$

from which we can write down an interaction Hamiltonian of the type

$$\hat{H}_{\text{int}} = \Omega e^{+i\omega_0 t \hat{a}^+ \hat{a}} (\hat{a}^+ + \hat{a}) e^{-i\omega_0 t \hat{a}^+ \hat{a}} = \Omega(\hat{a}^+ e^{+i\omega_0 t} + \hat{a} e^{-i\omega_0 t}). \tag{A2}$$

The equation of the evolution operator writes

$$i\frac{d\hat{U}}{dt} = \hat{H}_{\text{int}} \hat{U}, \quad \hat{U}(0) = \hat{I}, \tag{A3}$$

whose explicit solution, using the expressions (1.12), reads

$$\begin{aligned}\hat{U}(t) = {} & \exp\left\{ i\frac{\omega_0}{2} \int_0^t dt' \, \Omega^2 \left(\frac{\sin \omega_0 t'/2}{\omega_0/2} \right)^2 \right\} \\ & \times \exp\left\{ -\frac{1}{2}\Omega^2 \left(\frac{\sin \omega_0 t/2}{\omega_0/2} \right)^2 \right\} \\ & \times \exp\left[-i\Omega \left(\frac{\sin(\omega_0 t/2)}{\omega_0/2} \right) \hat{a}^+ e^{i(\omega_0 t/2)} \right] \\ & \times \exp\left[-i\Omega \left(\frac{\sin(\omega_0 t/2)}{\omega_0/2} \right) \hat{a} e^{-i(\omega_0 t/2)} \right]. \end{aligned} \tag{A4}$$

Using the above expression and assuming that we start from the vacuum, we easily get the following expression[12] for the evolution of $|\psi\rangle$:

$$|\psi\rangle = \sum_{l=0}^{\infty} \exp\left\{ i\frac{\omega_0}{2} \int_0^t |\alpha(\tau')^2 d\tau' \right\} \times [(\alpha(t))^l/\sqrt{l!}] e^{-(|\alpha(t)|^2)/2} |l\rangle. \tag{A5}$$

If we consider the Hamiltonian

$$\hat{H} = \omega_0 \hat{a}^+ \hat{a} + \epsilon(\hat{a}^+ \hat{a})^2 + \Omega(\hat{a}^+ + \hat{a}), \tag{A6}$$

the situation is considerably more complicated than before. However, following the same steps as before one finds

$$\begin{aligned}\hat{H}_{\text{int}} &= \Omega e^{+i[\omega_0 \hat{a}^+ \hat{a} + \epsilon(\hat{a}^+ \hat{a})^2]t} (\hat{a}^+ + \hat{a}) \times e^{-i[\omega_0 \hat{a}^+ \hat{a} + \epsilon(\hat{a}^+ \hat{a})^2]t} \\ &= \Omega\{\exp[i(-\omega_0 - \epsilon(2\hat{a}^+ \hat{a} + 1))t]\hat{a} + \exp[i(\omega_0 + \epsilon(2\hat{a}^+ \hat{a} - 1))t]\hat{a}^+\}. \end{aligned} \tag{A7}$$

If one is interested in a perturbed solution in ϵ, the Magnus procedure can be applied. The calculations are quite cumbersome, the Magnus series[11] must be calculated up to the fourth term and the results for $C_l(t)$ coincide, as they must, with (3.13) (for further comments and details see Ref. 12).

[1]R. P. Feyman, *Quantum Electrodynamics* (Benjamin, New York, 1961).
[2]N. Magnus, Commun. Pure Appl. Math. 7, 649 (1954).
[3]J. Wei and E. Norman, J. Math. Phys. A 4, 575 (1963).
[4]G. Baym, *Lectures on Quantum Mechanics* (Benjamin, New York, 1969).
[5]P. Pechukas and J. C. Light, J. Chem. Phys. 44, 3897 (1966).
[6]See, e.g., D. Finkelstein, Commun. Pure Appl. Math. 8, 245 (1955).
[7]Y. Kamo, Phys. Lett. A 56, 7 (1976).
[8](a) P. Bosco, G. Dattoli, and M. Richetta, J. Phys. A 17, L395 (1984); (b) P. Bosco, J. Gallardo, and G. Dattoli, J. Phys. A 17, 2739 (1984); (c) G. Dattoli, J. Gallardo, and A. Torre, Lett. Nuovo Cimento 42, 163 (1985).
[9]C. W. Raman and N. S. Nath, Proc. Ind. Acad. Sci. 2, 406 (1936).
[10]N. N. Lebedev, *Special Functions and Their Applications* (Dover, New York, 1972).
[11]R. Gilmore, *Lie Group, Lie Algebras and Some of Their Applications* (Wiley, New York, 1972).
[12]F. Ciocci, G. Dattoli, and M. Richetta, J. Phys. A 17, 1333 (1984); G. Dattoli, M. Richetta, and I. Pinto, Nuovo Cimento D 4, 293 (1984).
[13]B. Mache, Opt. Commun. 28, 131 (1979); M. V. Fedorov, Sov. Phys. JETP 46, 69 (1977); P. Bosco and G. Dattoli, J. Phys. A 16, 4409 (1984).
[14]G. Dattoli, J. Gallardo, and A. Torre, submitted for publication.