74.

ON A GEOMETRICAL TREATMENT OF A THEOREM IN NUMBERS.

[*Johns Hopkins University Circulars*, I. (1882), p. 209.]

THE author made some remarks additional to those made on the same subject at the preceding meeting of the seminarium. In a plane reticulation four cases present themselves, namely, a line may be drawn through a line of nodes, or through a solitary node, or parallel to a line of nodes, or so as neither to pass through any node nor to be parallel to a line of nodes. In the third case the distance of the nodes of nearest approach is constant: in the second and fourth cases it approximates continually to zero. So in a solid reticulation eight cases present themselves, namely, four in addition to those last detailed: for without lying in a nodal plane, the line of flight may (α) pass through a single node, or (β) it may be parallel to a line of nodes, or (γ) it may be parallel to a nodal plane but not to a nodal line, or (δ) it may not pass through any node. In case (β) the distance of the nodes of nearest approach is constant; in case (γ) it approximates to a constant finite limit: in cases (α) and (δ) it approximates to zero.

There are thus four cases in all for which the distance from the nodes of nearest approach is a continually decreasing infinitesimal, namely: two for which the line of flight does not pass through any node, and two for which it does pass through a node—these latter two being those which serve to establish the theorem relating to the non-existence of trebly periodic functions.

The author further drew attention to the singular metamorphosis undergone by the geometrical setting forth of this theorem. It may be put under the form of asserting that a trilateral whose three sides are conditioned to be exact multiples of, and parallel to, three given straight lines lying in a plane may either be made to form a closed triangle or else such that the line closing the trilateral shall be less than any assigned quantity. On the other hand, the very same fact lends itself to, and is absolutely equivalent in substance to the statement that an arrow let fly from a node of a solid reticulation whether it speed along a nodal plane or be shot miscellaneously at the stars must (the law of gravity being supposed to be suspended) pass *indefinitely near* an infinite number of nodes in the course of its flight. The corresponding theorem for space of five dimensions serves to show that Quaternion Functions cannot have a higher than a quadruple periodicity.

75.

ON THE PROPERTIES OF A SPLIT MATRIX.

[*Johns Hopkins University Circulars*, I. (1882), pp. 210, 211.]

SUPPOSE a square matrix split into two sets of lines which need not be contiguous and may be called ranges, say ABC, $DEFG$. Let the sum of the products of the corresponding elements of any two lines be called their product. It is well known (see Salmon's *Higher Alg.*, 3rd Ed., p. 82) that if the product of each line in the first range by every line in the other is zero, the opposite complete minors of the two ranges will be in a constant ratio, say in the ratio $l:\lambda$. Call the content of the matrix Δ: then it follows, if S, Σ denote the sums of the squares of the complete minors in the two ranges respectively, that

$$\frac{\lambda}{l} S = \frac{l}{\lambda} \Sigma = \Delta.$$

But by a theorem of Cauchy concerning rectangular matrices S is equal to the determinant $(A, B, C)^2$, that is, to the determinant

$$\begin{vmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{vmatrix}$$

and similarly $$\Sigma = (D, E, F, G)^2$$

so that $$\lambda^2 : l^2 :: (D, E, F, G)^2 : (A, B, C)^2$$

and $$S\Sigma = \Delta^2.$$

Suppose now that the product of *every* two lines in the entire matrix is zero. Then into whatever two ranges the matrix be divided the ratio $\lambda^2 : l^2$ (since all but the diagonal terms in the matrices which express the ratio $l^2 : \lambda^2$ vanish) will be expressed by the ratio of one simple product to another: thus for example for the ranges $ABC : DEFG$

$$\lambda^2 : l^2 :: D^2 . E^2 . F^2 . G^2 : A^2 . B^2 . C^2; \text{ also } \Delta^2 = A^2 . B^2 . C^2 . D^2 . E^2 . F^2 . G^2.$$

If we now further suppose that the sum of the squares of the elements in each line is unity, that is, that

$$A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = G^2 = 1,$$

it will follow that every minor whatever divided by its opposite will be equal to Δ $\left(\text{for on the hypothesis made, } \frac{\lambda}{l} = \frac{\Delta}{S} = \Delta\right)$.

Also Δ will be plus or minus unity since $\Delta^2 = 1$. Thus it is seen that we may pass by a natural transition from the theory of a split to that of an orthogonal or self-reciprocal matrix—to show which was the principal motive to the present communication. It is by aid of the theorem of the *split matrix* that I prove a remarkable theorem in Multiple Algebra, namely, that if the product of two matrices of the same order is a complete null, the sum of the nullities of the two factors must be at least equal to the order of the matrix—the nullity of a matrix of the order ω being regarded as unity, when its determinant simply is zero, as 2 when each first minor simply is zero, as 3 when each second minor is zero ... as $(\omega - 1)$ when each quadratic minor is zero and as ω (or absolute) when every element is zero. This theorem again is included in the more general and precise one following—*If any number of matrices of the same order be multiplied together, the nullity of their product is not less than the nullity of any single factor and not greater than the sum of the nullities of all the several factors.*

In Professor Cayley's memoir on Matrices (*Phil. Trans.*, 1858) the very important proposition is stated that if

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{matrix}$$

be any matrix of substitution, say m (here taken by way of illustration of the order 4) the determinant

$$\begin{vmatrix} a - m & b & c & d \\ a' & b' - m & c' & d' \\ a'' & b'' & c'' - m & d'' \\ a''' & b''' & c''' & d''' - m \end{vmatrix}$$

is identically zero; or in other words, its nullity is complete. By means of the above theorem it may be shown that the nullity of any i distinct algebraical factors of such matrix is equal to i, i having any value from unity up to the number which expresses the order of the matrix, inclusive.

76.

A WORD ON NONIONS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 241, 242;
II. (1883), p. 46.]

IN my lectures on Multiple Algebra I showed that if u, v are two matrices of the second order, and if the determinant of the matrix $(z+yv+xu)$ be written as

$$z^2 + 2bxz + 2cyz + dx^2 + 2exy + fy^2$$

then the necessary and sufficient conditions for the equation $vu + uv = 0$ are the following, namely,

$$b = 0, \quad c = 0, \quad e = 0.$$

If to these conditions we superadd $d = 1, f = 1$, and write $uv = w$, then

$$u^2 = -1, \quad v^2 = -1, \quad w^2 = -1, \quad uv = -vu = w, \quad vw = -wv = u, \quad wu = -uw = v;$$

and $1, u, v, w$ form a quaternion system. The conditions above stated will be satisfied if

$$\text{Det.}\,(z + yv + xu) = z^2 + y^2 + x^2,$$

which will obviously be the case if

$$v = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad u = \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix},$$

where $\theta = \sqrt{(-1)}$. For then

$$z + yv + xu = \begin{vmatrix} z & y + x\theta \\ -y + x\theta & z \end{vmatrix}.$$

Hence the matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix} \begin{vmatrix} -\theta & 0 \\ 0 & \theta \end{vmatrix}$$

construed as complex quantities are a linear transformation of the ordinary quaternion system 1, i, j, k; that is to say, if we form the multiplication table

	λ	μ	ν	τ
λ	λ	μ	0	0
μ	0	0	λ	μ
ν	ν	τ	0	0
τ	0	0	ν	τ

$$\lambda+\tau=1 \qquad -\mu+\nu=i$$

$$-\theta\lambda+\theta\tau=k \qquad \theta\mu+\theta\nu=j.$$

Since u, v contain between them 8 letters subject to the satisfaction of 5 conditions, the most general values of λ, μ, ν, τ ought to contain 3 arbitrary constants; but it is well-known that any particular (i, j, k) system may be superseded by a $\lambda\,(i', j', k')$ system, where i', j', k' are orthogonally related linear functions of i, j, k; and as this substitution introduces just 3 arbitrary constants, we may, by aid of it, pass from the system of matrices above given, to the most general form. The general expression for the matrices containing 3 arbitrary constants may also be found directly by the method given in my lectures, which will be reproduced in the memoir on Multiple Algebra in the *Mathematical Journal.* What goes before is by way of introduction to the *word* on Nonions which follows.

Just as the necessary and sufficient condition that u, v, two matrices of the second order, may satisfy the equations $vu=-uv$, $u^2=1$, $v^2=1$, is that the determinant to $z+yv+xu$ may be $z^2+y^2+x^2$, so I have proved that the necessary and sufficient condition, in order that we may have $vu=\rho uv$, $u^3=1$, $v^3=1$ (u, v being matrices of the third order, and ρ an imaginary cube root of unity) is that the determinant to $z+yu+xv$ may be $z^3+y^3+x^3$; but if we make

$$u=\begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix}, \qquad v=\begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix},$$

then
$$z+yu+xv=\begin{vmatrix} z & 0 & y+x \\ \rho y+\rho^2 x & z & 0 \\ 0 & \rho^2 y+\rho x & z \end{vmatrix}$$

of which the determinant is

$$z^3+(y+x)(\rho y+\rho^2 x)(\rho^2 y+\rho x)=z^3+y^3+x^3.$$

Hence there will be a system of Nonions (precisely analogous to the known system of quaternions) represented by the 9 matrices

$$\begin{array}{ccccc} & & 1 & & \\ & u & & v & \\ u^2 & & uv & & v^2 \\ & u^2v & & uv^2 & \\ & & u^2v^2 & & \end{array}$$

and just as in the preceding case the 8 terms ± 1, $\pm u$, $\pm v$, $\pm uv$ form a closed group, so here the 27 terms obtained by multiplying each of the above 9 by $1, \rho, \rho^2$ will form a closed group. The values of the 9 matrices will easily be found to be

$$\begin{array}{ccccc}
 & & \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} & & \\
 & \begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} & \\
\begin{vmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho \\ 1 & 0 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho \\ \rho & 0 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^2 \\ 1 & 0 & 0 \end{vmatrix} \\
 & \begin{vmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} & & \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{vmatrix} & \\
 & & \begin{vmatrix} 0 & 0 & \rho \\ \rho & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} & &
\end{array}$$

These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870).

I will only stay to observe that as the condition of the Determinant to $z + uy + vx$ (which for general values of u, v is a general cubic with the coefficient of z^3 unity) assuming the form $z^3 + y^3 + x^3$, implies the satisfaction of 9 conditions, and as u, v between them contain 18 constants, the most general form of a system of Nonions must contain $18 - 9$, or 9 arbitrary constants; but how these can be obtained from the particular form of the system above given, remains open for further examination.

[*Note.* For the remark made above] "These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870)," read "Mr C. S. Peirce informs me that these forms can be derived from his *Logic of Relatives*, 1870." I know nothing whatever of the fact of my own personal knowledge*. I have not read the paper referred to, and am not

* I have also a great repugnance to being made to speak of Algebras in the plural; I would as lief acknowledge a plurality of Gods as of Algebras.

acquainted with its contents. The mistake originated in my having left instructions for Mr Peirce to be invited to supply in my final copy for the press, such reference as he might think called for. He will be doing a service to Algebra by showing in these columns how he derives my forms from his logic*. The application of Algebra to Logic is now an old tale—the application of Logic to Algebra marks a far more advanced stadium in the evolution of the human intellect; the same may be said as regards the application by Descartes of Analysis to Geometry, and the reverse application by Eisenstein, Dirichlet, Cauchy, Riemann, and others, of Geometry to Analysis—so that if Mr Peirce accomplishes the task proposed to him (his ability to do which I do not call into question), he will have raised himself as far above the level of the ordinary Algebraic logicians as Riemann's mathematical stand-point tops that of Descartes.

It is but justice to Boole's memory to recall the fact that, in one of his papers in the *Philosophical Transactions,* he has made a reverse use of logic to establish a certain theorem concerning inequalities, which is very far from obvious, and which I think he states it took him ten years to deduce from purely algebraical considerations, having previously seen it through logical spectacles—I mean, by the aids to vision afforded him by his logical calculus: this theorem I believe (or at least did so when it was present to my mind) must of necessity admit of a much more comprehensive form of statement.

* I had understood Mr Peirce to say that these forms were actually contained in his memoir.

77.

ON MECHANICAL INVOLUTION.

[*Johns Hopkins University Circulars*, I. (1882), pp. 242, 243.]

MANY years ago I gave in the *Comptes Rendus* of the Institute of France, one or more geometrical constructions of the problem of Mechanical Involution.

When forces can be introduced along six given lines in space whose statical sum is zero, a certain geometrical condition must be fulfilled by the 6 lines which are then said to be in involution. If two homographic pencils of rays in different planes have two corresponding rays coincident (but their centres apart), any six lines, each of which cuts two corresponding rays, will form an involution system. In the communication to the Society I showed that the analytical condition of involution might be expressed by means of equating to zero a certain compound determinant. I have found since that this determinant is given by Cayley in the *Cam. Phil. Soc. Tr.* 1861, part 2.

Let 1, 2, 3, 4, 5, 6 be the six lines and on each of them let two arbitrary points be taken; let the quadri-planar coordinates of the two arbitrary points on any of the lines, say j, be called j_x, j_y, j_z, j_t; j_x', j_y', j_z', j_t', respectively, the condition of involution referred to will be

$$\begin{vmatrix} & 1.2 & 1.3 & 1.4 & 1.5 & 1.6 \\ 2.1 & & 2.3 & 2.4 & 2.5 & 2.6 \\ 3.1 & 3.2 & & 3.4 & 3.5 & 3.6 \\ 4.1 & 4.2 & 4.3 & & 4.5 & 4.6 \\ 5.1 & 5.2 & 5.3 & 5.4 & & 5.6 \\ 6.1 & 6.2 & 6.3 & 6.4 & 6.5 & \end{vmatrix} = 0$$

where any binary combination $ij = ji$, and where either of them represents the determinant

$$\begin{vmatrix} i_x, & i_y, & i_z, & i_t \\ i_x', & i_y', & i_z', & i_t' \\ j_x, & j_y, & j_z, & j_t \\ j_x', & j_y', & j_z', & j_t' \end{vmatrix}$$

Six lines in involution represent indifferently lines along which forces or axes of couples can be introduced, whose statical sum is zero. Consequently such a system is the analogue in space at one and the same time to three force-lines converging to a point, or to three points in a line regarded as centres of moments, in a plane. But *in plano* the concurrence of right lines is the polar property to the collineation of points. Hence we ought to expect that the polar reciprocal in respect to any quadric of an involution system, should also be an involution system; and such is obviously the case by virtue of the fact that the correspondence of the rays in the two homographic pencils, referred to above, will not be affected when for each ray in either pencil is substituted its polar in respect to any quadric. (A direct proof will be found in the *American Mathematical Journal*, Vol. IV., part 4 *.) I concluded with pointing out the analogy between the problem of Mechanical Involution and what I call Algebraical Involution, which takes place when x, y being each of them matrices of the order ω, a linear equation connects the ω^2 ground-forms represented by the distinct terms of the product

$$(1, x, x^2, \dots x^{\omega-1} \between 1, y, y^2, \dots y^{\omega-1}).$$

Mechanical involution in a plane, in 3-dimensional, in 4-dimensional space, etc., is the analogue of algebraical involution between two matrices of the order 2, 3, 4, etc.; the $\frac{1}{2}(\omega^2+\omega)$ directions in ω-dimensional space being the analogues of the ω^2 ground-forms of matrices of the order ω. Each of the two problems consists of two parts: to obtain the condition of involution being the one part, to assign the relative magnitudes, in the one case, of the forces which cause their statical sum to vanish, and in the other case of the coefficients which enter into the linear function, the other part of the problem. The form of the solution of this second part of the algebraical problem (subject only to a certain ambiguity) has been given in my lectures, and will appear in the Memoir on Multiple Algebra in the *American Journal of Mathematics*; but the former part of the algebraical problem, that is, the determination of the *condition* of Algebraical Involution, except for the case of matrices of the second order, I have not yet succeeded in solving.

[* Cf. p. 560, above.]

78.

ON CROCCHI'S THEOREM.

[*Johns Hopkins University Circulars*, II. (1883), p. 2.]

In *Battaglini's Journal* for July, 1880, Signor Crocchi has given a theorem which may be stated in the following terms. If s_i, σ_i, h_i denote respectively the sum of the elementary combinations, of the powers, and of the homogeneous products each of the ith order of any number of elements, then h_i is the same function of σ_1, $\bar{\sigma}_2$, σ_3, $\bar{\sigma}_4$, ... that s_i is of σ_1, σ_2, σ_3, σ_4, ...

Signor Crocchi's proof is very elegant but a little circuitous. An instantaneous proof may be derived from the relation of reciprocity which connects s and h, namely, that if

$$h_i = f(s_1, s_2, \ldots s_i) \text{ then } s_i = f(h_1, h_2, \ldots h_i),$$

which is an immediate deduction from the well-known fact that

$$(1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots)(1 - h_1 y + h_2 y^2 - h_3 y^3 + \ldots) = 1.$$

For from this relation spring the equations

$$s_1 - h_1 = 0, \quad s_2 - h_1 s_1 + h_2 = 0, \quad s_3 - h_1 s_2 + h_2 s_1 - h_3 = 0 \ldots$$

which equations continue unaltered when the letters s and h are interchanged; for when such interchange takes place, the functions equated to zero of an even rank remain unaltered and those of an odd rank merely change their sign.

Returning to the immediate object in view, if $a, b, c, \ldots$ are the elements subject to the s, h, σ symbols, we may write

$$\Sigma \log(1 + ay) = \log(1 + s_1 y + s_2 y^2 + s_3 y^3 + s_4 y^4 + \ldots)$$

or,

$$\Sigma \log(1 - ay) = -\log(1 + h_1 y + h_2 y^2 + h_3 y^3 + h_4 y^4 + \ldots).$$

The first equation by differentiation performed in each side gives

$$\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \sigma_4 y^3 + \ldots = \frac{s_1 + 2s_2 y + 3s_3 y^2 + 4s_4 y^3 + \ldots}{1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots},$$

and similarly the second equation gives

$$\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \sigma_4 y^3 + \ldots = \frac{h_1 + 2h_2 y + 3h_3 y^2 + 4h_4 y^3 + \ldots}{1 + h_1 y + h_2 y^2 + h_3 y^3 + \ldots},$$

that is, $(\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \ldots)(1 + s_1 y + s_2 y^2 + \ldots) = s_1 + 2s_2 y + 3s_3 y^2 + \ldots$

and $\quad (\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \ldots)(1 + h_1 y + h_2 y^2 + \ldots) = h_1 + 2h_2 y + 3h_3 y^2 + \ldots.$

By comparison of coefficients of the powers of y, the first of these two equations affords the means of finding any σ in terms of the s quantities, and the second of these any σ in terms of the h quantities. But if we change s into h and $\sigma_2, \sigma_4, \ldots$ into $-\sigma_2, -\sigma_4, \ldots$ the first equation becomes the second. Hence if

$$s = f(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots)$$

$$h = f(\sigma_1, \bar{\sigma}_2, \sigma_3, \bar{\sigma}_4, \ldots). \quad \text{Q.E.D.}$$

It is not without interest to set out the reciprocity of the 6 relations which exist between s, σ, h. The synoptical scheme of such reciprocity may be exhibited symbolically as follows:

$$h/s = s/h, \quad h/\sigma = s/\pm\sigma, \quad \sigma/h = \pm\sigma/s.$$

As an illustration of the second of these symbolic equalities take

$$s_3 = \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \qquad s_4 = \frac{\sigma_1^4 - 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 - 6\sigma_4}{24};$$

the corresponding equations are

$$h_3 = \frac{\sigma_1^3 + 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \qquad h_4 = \frac{\sigma_1^4 + 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 + 6\sigma_4}{24},$$

and it is worthy of observation that the sum of the numerical coefficients is always (as in the above examples) zero for the function of the σ quantities which gives an s of any order, and unity for the function of the same which expresses any h*.

* This statement is proved instantaneously by taking one of the elements equal to unity and all the rest zero; and the latter part of it gives a new proof of Cauchy's theorem which he obtains by a consideration of all the possible cyclic representations of the substitutions of n elements. The theorem is that if n elements be divided in every possible way into λ set of l, μ set of m, ν set of n ... elements, then

$$\Sigma \frac{1}{\pi\lambda\pi\mu\pi\nu \ldots l^\lambda m^\mu n^\nu \ldots} = 1.$$

For we know by a theorem of Waring that

$$s_n = \Sigma \pm \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots} \sigma_l^{\lambda} \sigma_m^{\mu} \ldots.$$

Hence by Crocchi's theorem the sum of the coefficients in h_n expressed in σ's is equal to

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m \ldots}$$

but it is also equal to unity. Cauchy's theorem is therefore proved.

Frequent occasion presents itself (especially in the theory of numbers) for expressing any s in terms of σ's, but probably up to the time when Signor Crocchi wrote on the subject there had never been any occasion to express h in terms of the σ's: for had such occasion ever arisen it seems almost impossible that the relation between the two corresponding sets of formulæ could have escaped observation.

In some recent researches, however, of the writer of this note on the irreducible semi-invariants of a quantic of an unlimited order, it becomes indispensable to convert homogeneous products into sums of powers, and Crocchi's theorem comes into play. (See sec. 4 of Article on Subinvariants, *Am. Math. Journ.*, Vol. v., part 2 [p. 597, above].)

The relation $\sigma/h = \pm\, \sigma/s$ is interesting under the point of view that virtually it contains an example of a sort of *invariance* of form which may possibly contain within itself the germ of an important theory. It informs us that if, in the function of h's which expresses any σ, in lieu of each h the function of s quantities to which it is equal be substituted, the form of the σ function will remain unchanged, except that when the order of the σ is an even number, its algebraical sign is reversed. Thus, for example,

$$\sigma_3 = h_1^3 - 3h_1h_2 + 3h_3, \quad h_1 = s_1, \quad h_2 = s_1^2 - s_2, \quad h_3 = s_1^3 - 2s_1s_2 + s_3.$$

Consequently if we write $\phi = x^3 - 3xy + 3z$, and for y and z substitute $x^2 - y$, $x^3 - 2xy + z$, respectively, the value of ϕ remains unaltered. So in like manner if we write

$$\phi = x^4 - 4x^2y + 4xz + 2y^2 - 4t,$$

and substitute for y, z, t;

$$x^2 - y, \quad x^3 - 2xy + z, \quad x^4 - 3x^2y + 2xz + y^2 - t,$$

respectively, no change ensues in ϕ except that it undergoes a change of sign.

So in general the σ functions with even and those with odd subindices may be regarded as the analogues of symmetrical and skew-invariants, respectively.

Again in the formula for s the sign *plus* or *minus* depends on the oddness or evenness of $\lambda + \mu + \ldots$. Hence if in

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots}$$

only those values of λ, μ, ... are admitted which make $\lambda + \mu + \ldots$ always odd or always even, either sum so formed will be equal to $\frac{1}{2}$, because the difference of the two sums is zero and their sum unity.

This theorem can, of course, be deduced like the former one from the method of cycles applied now, not to the entire number of the substitutions, but to that half of them which correspond to the *alternate group* of each, of which the number of representative cycles (monomial ones included) is always odd or else always even, according as the number of elements is one or the other.

79.

ON CERTAIN SUCCESSIONS OF INTEGERS THAT CANNOT BE INDEFINITELY CONTINUED.

[*Johns Hopkins University Circulars*, II. (1883), pp. 2, 3.]

A SUCCESSION of decreasing integers we know cannot be indefinitely continued, but there are also successions of increasing integers subject to certain stated conditions, but otherwise arbitrary, which are similarly incapable of indefinite extension.

The following is a simple instance of the kind. Suppose integers to be written down one after the other, no one of which is a multiple of any other, nor the sum of a multiple of any other and of a multiple of a specified one. *Such a succession cannot be indefinitely continued.*

Let a be the specified integer.

(1) Suppose that all the other integers of the succession are prime to a.

Then if b be any other of the integers, the equation $ax + by = c$ is soluble in integers if c is greater than ab, as follows at once from the consideration that the numbers $c - b, c - 2b, c - 3b \ldots c - ab$ must be all distinct residues to the modulus a, inasmuch as the difference of any two of them being of the form $(i - j)b$ where $i - j$ is less than and b prime to a, cannot be divisible by a.

But if the succession could be indefinitely produced, it must contain a number greater than ab. Hence the theorem is proved for the case where a is prime to every other integer in the succession.

(2) Suppose the theorem to be true for the case where the quotient of a divided by i prime numbers (not necessarily all distinct) is prime to all the other terms of the series: it must be true when the number of such prime numbers is $i + 1$. For let p be one of them and $a = pa'$, consider all the terms of the succession divisible by p apart from the rest.

Let pa', pb', pc' ... be those terms. By the law of the succession the equation $pa'x+pb'y=pc'$ cannot be satisfied for any values of b', c', and consequently $a'x+b'y=c'$ cannot be satisfied.

Hence by hypothesis since a' divided by i factors is prime to b', c' ... the succession of terms divisible by p must be finite in number, and this will be true for every factor p. Hence the succession b, c, ... will contain only a finite number of terms having any factor in common with a. Moreover the succession containing a and terms prime to a exclusively, must also be finite by the preceding case. Consequently the whole succession will be finite, and the theorem if assumed to be true for $i=0$, or any positive integer, is true for $i+1$.

But by the preceding case the proposition is true when $i=0$. Hence it is true universally.

In the long footnote to the Article on Subinvariants in Vol. v., pp. 92, 93 of the *Am. Journal of Math.*, will be found given the mode of extending this theorem to the case of successions of complex integers or multiplets, when a proper restriction is laid upon the ratios to one another of the simple numbers which constitute the multiplets, and a possible connexion pointed out between the finiteness of such successions and that of the system of ground-forms to a binary quantic [p. 580, above].

80.

ON THE FUNDAMENTAL THEOREM IN THE NEW METHOD OF PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), p. 22.]

THE new method of partitions which I gave to the world more than a quarter of a century ago is an application of a theorem which, I think it must be conceded, is, after Newton's Binomial Theorem, the most important organic theorem which exists in the whole range of the Old Algebra. What Newton's theorem effects for the development of *radical*—that theorem accomplishes for the development of *fractional* forms of algebraical functions.

One (but not the most perfect) form in which it can be presented is the following. If Fx be any proper algebraical fraction in x, whose infinity roots (that is, the values of x which make Fx infinite) are $a, b, \dots l$, quantities all supposed to differ from zero, then the coefficient of x^n for any value of n will be the residue, that is, the coefficient of $\frac{1}{x}$ in

$$\Sigma\,(\lambda^{-n}e^{nx})\,F(\lambda e^{-x}) \quad [\lambda = a, b, \dots, l].$$

By supposing Fx broken up into proper simple fractions of the form $\Sigma\,\frac{fx}{(a-x)^i}$ it is very easy to see that the theorem will be true in general if true for $\frac{fx}{(a-x)^i}$, and from this it is but a step to see that the theorem will be true in general if true for the simplest form of rational function, that is, $\frac{1}{(1-x)^i}$.

All then that remains to do is to show that the coefficient of x^n in this fraction is the same as the coefficient of $\frac{1}{x}$ in $\frac{e^{nx}}{(1-e^{-x})^i}$ which may be done as follows:

$$\frac{1}{(1-e^{-x})^i} = \left(1-\frac{\delta_x}{1}\right)\left(1-\frac{\delta_x}{2}\right)\left(1-\frac{\delta_x}{3}\right)\cdots\left(1-\frac{\delta_x}{(i-1)}\right)\left(\frac{1}{1-e^{-x}}\right)$$

$$= (1 - A\delta_x + B\delta_x^2 - C\delta_x^3 \ldots)\left(\frac{1}{x}+\ldots\right)$$

$$= \left(\frac{1}{x}+\frac{A}{x^2}+\frac{1.2B}{x^3}+\frac{1.2.3C}{x^4}+\ldots\right) + \text{positive powers of } x.$$

Therefore the coefficient of $\frac{1}{x}$ in

$$\frac{1+nx+\frac{n^2}{1.2}x^2+\frac{n^3}{1.2.3}x^3+\ldots}{(1-e^{-x})^i}$$

$$= (1 + An + Bn^2 + Cn^3 + \ldots)$$

$$= \left(1+\frac{n}{1}\right)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\cdots\left(1+\frac{n}{i-1}\right)$$

$$= \frac{(n+1)(n+2)\ldots(n+i-1)}{1,\ 2,\ \ldots\ (i-1)} = \text{coefficient of } x^n \text{ in } \frac{1}{(1-x)^i}. \quad \text{Q.E.D.}$$

This method of proof, however, is not the simplest or best; as soon as we mould the theorem into a form most easily admitting of being expressed in general terms that very form itself suggests a simpler (nay, so to say, an instantaneous) proof, and moreover relaxes an unnecessarily stringent condition in the previous statement of the theorem.

Of course by a finite infinity root of a function no one can fail to understand a value of the variable differing from zero which makes the function infinite. This then is the true statement of the theorem in general terms.

In any proper-fractional function developed in ascending powers of a variable, the constant term is equal to the Residue (with its sign changed) of a sum of functions obtained by substituting in the given function in place of the variable the product of each, in succession, of its finite infinity roots into the exponential of the variable.

That is to say, if we take the proper-fraction

$$Fx = \frac{\phi x}{x^i (x-a)^j (x-b)^k \ldots (x-l)^\omega},$$

the constant term (with its sign changed) in this fraction developed in ascending powers of x is the same as the Residue of $\Sigma F(\lambda e^x)\,[\lambda = a, b, \ldots l]$.

To prove this it is only necessary to suppose the fraction Fx separated into simple partial fractions with constant numerators and the theorem becomes self-evident*.

* It must, however, previously be shown that the residue of $\frac{1}{(1-e^x)^i}$, where i is a positive

It follows, therefore, writing n in place of i that the coefficient of x^n in ascending-power series for the fraction

$$Gx = \left(\frac{\phi x}{(x-a)^j \ldots (x-l)^\omega}\right)$$

will be the Residue with its sign changed of $\Sigma\,(a^{-n}e^{-nx})\,G\,(ae^x)$, or which is the same thing is the Residue of $\Sigma a^{-n}e^{nx}\,G\,(ae^{-x})$, which theorem we now see is true not merely for the case where G is a proper-fraction, that is, is a function of x whose degree is a negative integer, but remains true when the degree of G is any number inferior to n, for when that condition is satisfied $\frac{G}{x^n}$ is a proper fraction, which is all that is required in order for the parent theorem to apply.

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integer, is the same as that of $\frac{1}{1-e^x}$, that is, is -1; this becomes obvious from the consideration that the change of i into $i+1$ alters the quantity to be residuated by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{(1-e^x)^i}$ divided by i, of which the residue is necessarily zero—that being true for the differential derivative of any series of powers of a variable.

81.

NOTE ON THE PAPER OF MR DURFEE'S.

[*Johns Hopkins University Circulars*, II. (1883), pp. 23, 24; 42, 43.]

MR DURFEE'S very elegant and interesting theorem above given may, by help of Euler's law of reciprocity, be expressed in the following terms.

Let fx and ϕx represent respectively:

$$\frac{1}{1-x}+\frac{x^4}{(1-x)(1-x^2)(1-x^3)}$$
$$+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}+\ldots$$
$$+\frac{x^{2i^2+2i}}{(1-x)\ldots(1-x^{2i+1})}+\ldots$$

and
$$\frac{x^2}{(1-x)(1-x^2)}+\frac{x^8}{(1-x)(1-x^2)(1-x^3)(1-x^4)}+\ldots$$
$$+\frac{x^{2i^2}}{(1-x)\ldots(1-x^{2i})}+\ldots,$$

then the number of self-conjugate partitions of $2m+1$ and of $2m$ are the coefficients of x^m in the ascending expansions of fx, ϕx, respectively.

Thus, suppose $2m+1=13$, the coefficient of x^6 in fx developed, that is, $\frac{6}{1,}+\frac{2}{1,\,2,}$ or 3 is the number of self-conjugate partitions of 13.

These will be found to be 7 1 1 1 1 1 1, 4 4 3 2, 5 3 3 1 1. To find the conjugate to any partition $a, b, c \ldots, l$, the most expeditious method is to find n_i, the number of the elements in the partition not less than i: $n_1, n_2, \ldots, n_l$ (l being supposed to be the largest value of any element) will then be its opposite.

Thus, for example, for the partition 5 3 3 1 1, $n_1=5$, $n_2=3$, $n_3=3$, $n_4=1$, $n_5=1$, and $n_1\ n_2\ n_3\ n_4\ n_5$ reproduces 5 3 3 1 1.

If $2m = 12$ we have to find the value of $\frac{4}{1,\ 2,}$ which is again 3, and the 3 self-conjugate or self-opposite partitions of 12 will be seen to be 4 4 2 2; 5 3 2 1 1; 6 2 1 1 1 1.

In M. Faà de Bruno's tables of symmetric functions, which are only complete for the case of equations of not higher than the 11th degree, the number of self-conjugate partitions which appear among the headings and sidings of the tables is either 1 or 2, and it was therefore reasonable to try the effect of making arrangement of the partitions such as to bring the self-conjugate or pair of self-conjugate partitions into the centre of the line or column; but as soon as that degree is passed such a kind of principle (the rule founded upon which M. de Bruno does not state) becomes *prima facie* inapplicable at all events without undergoing modifications of which at present we know nothing.

Thus M. de Bruno's tables end just where his proposed principle of arrangement becomes inapplicable, stopping short at the case of the 12th degree, which has since been tabulated by Mr Durfee in the *American Journal of Mathematics.*

The term "opposite" or "conjugate" is used by Mr Durfee in the sense in which I am in the habit of employing it to signify the relation between what M. Faà de Bruno calls *combinaisons associées.* I think it right to recall attention to the fact that the credit of calling into being this kind of conjugate relation, is due to Dr Ferrers (the present Master of Gonville and Caius College, Cambridge), who some 30 years ago or more was the first to apply it to obtain an intuitive proof of Euler's great law of reciprocity, the very same as that which I have here employed to transform Mr Durfee's theorem. Euler demonstrated his law by help of his favourite instrument of generating functions.

By instituting in the case of combinations of *unrepeated* elements quite another and more exquisite kind of conjugate relation applicable to all such with the exception of those which belong to the infinite succession 1, 2, 2 3, 3 4, 3 4 5, 4 5 6, 4 5 6 7, 5 6 7 8, Mr Franklin, of this University, succeeded in finding an instantaneous demonstration of another well-known but very much more recondite theorem in partitions, also due to Euler, expressible by the statement that the indefinite product

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots$$

has for its development

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \ldots,$$

where the indices are the complete series of direct and retrograde pentagonal numbers.

By a singular oversight in my note in the last *Circular*, I omitted to state that Mr Durfee's rule is tantamount to affirming that the number of self-conjugate partitions or (which is the same thing) of symmetrical partition graphs for n, is the coefficient of x^n in the series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})} + \ldots$$

and since this series is identical with the infinite product

$$(1+x)(1+x^3)(1+x^5)\ldots$$

the number of self-conjugate partitions is the number of ways of distributing n into unrepeated odd-integers, a result which can be obtained directly by regarding any symmetrical partition graph as made up of a set of successively diminishing equilateral elbows or say carpenters' rules, each of which necessarily contains an odd number of points: the number of such elbows for any given graph will be the same as the number of points in the side of Mr Durfee's square nucleus, and consequently we have an intuitive proof of the theorem that the infinite product

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

is equal to the infinite series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i + \ldots$$

because the coefficient of $a^i x^n$ is the same in both expressions. By a similar method I obtain an intuitive and almost instantaneous solution of the problem to expand in infinite series the infinite products which express a Theta Function and its *reciprocal*, and many other questions of a similar nature.

It was the anticipation of the parallelism between the expressions for the number of special partitions in the unrepeated-numbers and the repeated-numbers theories which led me to find *à priori* the partition-into-odd-integers expression for the number of self-conjugate partitions, and thus started me on the track of the graphical method of transforming infinite products into infinite series: the light of analogy may sometimes "lead astray" but it is more often "light from heaven."