MA - II

Assignment - IV

Q.① a, b → 2 fixed vectors

$r = xi + yj + zk$

$r^2 = x^2 + y^2 + z^2$

(i) $\nabla(r^n) = n r^{n-2} \bar{r}$ for any integer n

Sol^n:- $r = \sqrt{x^2+y^2+z^2}$

$$\nabla r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z}$$

$$= \frac{n r^{n-1} x}{\sqrt{x^2+y^2+z^2}} i + \frac{n r^{n-1} y}{\sqrt{x^2+y^2+z^2}} j + \frac{n r^{n-1} z}{\sqrt{x^2+y^2+z^2}} k$$

$$= n r^{n-2}(xi + yj + zk)$$

$$\boxed{\nabla r^n = n r^{n-2} \bar{r}}$$

Hence, proved

(ii) $\bar{a}\cdot\nabla\left(\frac{1}{r}\right) = -\left(\frac{\bar{a}\cdot\bar{r}}{r^3}\right)$.

By using property (i)

$$\nabla\left(\frac{1}{r}\right) = -r^{-3}\bar{r}$$

$$\bar{a}\cdot\nabla\left(\frac{1}{r}\right) = \bar{a}\cdot(-r^{-3}\bar{r}) = -r^{-3}(\bar{a}\cdot\bar{r}) = -\left(\frac{\bar{a}\cdot\bar{r}}{r^3}\right).$$

Hence, proved

(iii) $$b\cdot\nabla\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{3(a\cdot r)(b\cdot r)}{r^5} - \frac{a\cdot b}{r^3}$$

$$\nabla\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = i\frac{\partial\left(a\cdot\nabla\left(\frac{1}{r}\right)\right)}{\partial x} + j\frac{\partial\left(a\cdot\nabla\left(\frac{1}{r}\right)\right)}{\partial y} + k\frac{\partial\left(a\cdot\nabla\left(\frac{1}{r}\right)\right)}{\partial z}$$

let $\bar{a} = a_1 i + a_2 j + a_3 k$

$$a\cdot\nabla\left(\frac{1}{r}\right) = -\frac{a\cdot r}{r^3} = -\frac{(a_1x + a_2y + a_3z)}{r^3}$$

$$\frac{\partial}{\partial x}\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{\partial}{\partial x}\left(\frac{-a_1x}{(\sqrt{x^2+y^2+z^2})^3} - \frac{a_2y+a_3z}{(\sqrt{x^2+y^2+z^2})^3}\right)$$

$\cancel{x^2+y^2+z^2=t^2}$

$\cancel{2x\,dx = 2t\,dt}$

$$-\frac{a_1}{r^3} + \frac{3a_1x^2}{r^4\,r} + \frac{3(a_2y+a_3z)x}{r^4\cdot r}$$

$$\nabla\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{3(a\cdot r)\bar{r}}{r^5} - \frac{a}{r^3}$$

$$b\cdot\nabla\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = b\cdot\left(\frac{3(a\cdot r)\bar{r}}{r^5} - \frac{a}{r^3}\right)$$

$$b\cdot\nabla\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{3(a\cdot r)(b\cdot r)}{r^5} - \frac{a\cdot b}{r^3}$$

Hence, proved

Q.2 $f, g \rightarrow$ 2 scalar f^n on $\mathbb{R}^m$

(i) $\nabla(fg) = f\nabla g + g\nabla f$

$\left\{ \hat{a}_1 \rightarrow \right.$ unit vector in x_1 axis's dir^n

$$\nabla(fg) = \hat{a}_1 \frac{\partial(fg)}{\partial x_1} + \hat{a}_2 \frac{\partial(fg)}{\partial x_2} + \dots + \hat{a}_m \frac{\partial(fg)}{\partial x_m}$$

$$\nabla(fg) = \hat{a}_1\left(f\frac{\partial g}{\partial x_1} + g\frac{\partial f}{\partial x_1}\right) + \hat{a}_2\left(f\frac{\partial g}{\partial x_2} + g\frac{\partial f}{\partial x_2}\right) + \dots + \hat{a}_m\left(f\frac{\partial g}{\partial x_m} + g\frac{\partial f}{\partial x_m}\right)$$

$$\nabla(fg) = \left(f\left(\hat{a}_1\frac{\partial g}{\partial x_1} + \hat{a}_2\frac{\partial g}{\partial x_2} + \dots + \hat{a}_m\frac{\partial g}{\partial x_m}\right)\right) + \left(g\left(\hat{a}_1\frac{\partial f}{\partial x_1} + \hat{a}_2\frac{\partial f}{\partial x_2} + \dots + \hat{a}_m\left(\frac{\partial f}{\partial x_m}\right)\right)\right)$$

$$\hat{a}_1\frac{\partial g}{\partial x_1} + \hat{a}_2\frac{\partial g}{\partial x_2} + \dots + \hat{a}_m\frac{\partial g}{\partial x_m} = \nabla g$$

$$\hat{a}_1\frac{\partial f}{\partial x_1} + \hat{a}_2\frac{\partial f}{\partial x_2} + \dots + \hat{a}_m\frac{\partial f}{\partial x_m} = \nabla f$$

$$\boxed{\nabla(fg) = f\nabla g + g\nabla f}$$

Hence, proved

(ii) $\nabla f^n = n f^{n-1} \nabla f$

$$\nabla f^n = \hat{a}_1 \frac{\partial f^n}{\partial x_1} + \hat{a}_2 \frac{\partial f^n}{\partial x_2} + \dots + \hat{a}_m \frac{\partial f^n}{\partial x_m}$$

$$= \hat{a}_1 (n f^{n-1}) \frac{\partial f}{\partial x_1} + \hat{a}_2 (n f^{n-1}) \frac{\partial f}{\partial x_2} + \dots + \hat{a}_m (n f^{n-1}) \frac{\partial f}{\partial x_m}$$

$$= (n f^{n-1}) \left(\hat{a}_1 \frac{\partial f}{\partial x_1} + \hat{a}_2 \frac{\partial f}{\partial x_2} + \dots + \hat{a}_m \frac{\partial f}{\partial x_m} \right)$$

$$\boxed{\nabla f^n = n f^{n-1} \nabla f}$$

Hence, proved

(iii) $\nabla \left(\frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2}$ whenever $g \neq 0$

$$\nabla \left(f/g \right) = \hat{a}_1 \frac{\partial}{\partial x_1} (f/g) + \hat{a}_2 \frac{\partial}{\partial x_2} (f/g) + \dots + \hat{a}_m \frac{\partial}{\partial x_m} (f/g)$$

$$\nabla (f/g) = \hat{a}_1 \left(\frac{g \frac{\partial f}{\partial x_1} - f \frac{\partial g}{\partial x_1}}{g^2} \right) + \hat{a}_2 \left(\frac{g \frac{\partial f}{\partial x_2} - f \frac{\partial g}{\partial x_1}}{g^2} \right) + \dots + \hat{a}_m \left(\frac{g \frac{\partial f}{\partial x_m} - f \frac{\partial g}{\partial x_m}}{g^2} \right)$$

$$\nabla (f/g) = \frac{1}{g^2} \left(g \left(\hat{a}_1 \frac{\partial f}{\partial x_1} + \hat{a}_2 \frac{\partial f}{\partial x_2} + \dots + \hat{a}_m \frac{\partial f}{\partial x_m} \right) - f \left(\hat{a}_1 \frac{\partial g}{\partial x_1} + \hat{a}_2 \frac{\partial g}{\partial x_2} + \dots + \hat{a}_m \frac{\partial g}{\partial x_m} \right) \right)$$

$$\cancel{+ \frac{1}{g^2} \left(g \left(\hat{a}_1 \right. \right.}$$

$$\nabla (f/g) = \frac{1}{g^2} \left(g \nabla f - f \nabla g \right)$$

$$\boxed{\nabla (f/g) = \frac{g \nabla f - f \nabla g}{g^2}}$$

Hence, proved

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Q. 3

let $v = v_1 i + v_2 j + v_3 k$

(i) $\nabla \cdot (fv) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (fv)$

$\nabla \cdot (fv) = \frac{\partial (fv_1)}{\partial x} + \frac{\partial (fv_2)}{\partial y} + \frac{\partial (fv_3)}{\partial z}$

$\nabla \cdot (fv) = f\frac{\partial v_1}{\partial x} + v_1\frac{\partial f}{\partial x} + f\frac{\partial v_2}{\partial y} + v_2\frac{\partial f}{\partial y} + f\frac{\partial v_3}{\partial z} + v_3\frac{\partial f}{\partial z}$

$\nabla \cdot (fv) = f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + v\left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right)$

$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = \nabla \cdot v$

$v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y} + v_3\frac{\partial f}{\partial z} = v(\nabla f) = (\nabla f) \cdot v$

$v_1 i\frac{\partial f}{\partial x} + v_2 j\frac{\partial f}{\partial y} + v_3 k\frac{\partial f}{\partial z}$

$$\boxed{\nabla \cdot (fv) = f\nabla \cdot v + \nabla f \cdot v}$$

Hence, proved

(ii) $\nabla \times (fv) = f(\nabla \times v) + (\nabla f) \times v$

let $v = v_1 i + v_2 j + v_3 k$

$$\nabla \times (fv) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fv_1 & fv_2 & fv_3 \end{vmatrix}$$

$\nabla \times (fv) = i\left(\frac{\partial (fv_3)}{\partial y} - \frac{\partial (fv_3)}{\partial z}\right) - j\left(\frac{\partial (fv_3)}{\partial x} - \frac{\partial (fv_1)}{\partial z}\right) + k\left(\frac{\partial (fv_2)}{\partial x} - \frac{\partial (fv_1)}{\partial y}\right)$

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$$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}V_3 - \frac{\partial}{\partial z}V_2\right) - j\left(\frac{\partial}{\partial x}V_3 - \frac{\partial}{\partial z}V_1\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\nabla f \times V = \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(V_3\frac{\partial f}{\partial y} - V_2\frac{\partial f}{\partial z}\right)i - \left(V_3\frac{\partial f}{\partial x} - V_1\frac{\partial f}{\partial z}\right)j + \left(V_2\frac{\partial f}{\partial x} - V_1\frac{\partial f}{\partial y}\right)k$$

$$\nabla \times (fV) = i\left(\frac{\partial}{\partial y}(fV_3) - \frac{\partial}{\partial z}(fV_2)\right) - j\left(\frac{\partial}{\partial x}(fV_3) - \frac{\partial}{\partial z}(fV_1)\right) + k\left(\frac{\partial}{\partial x}(fV_2) - \frac{\partial (fV_1)}{\partial y}\right)$$

$$\nabla \times (fV) = i\left(f\frac{\partial V_3}{\partial y} + V_3\frac{\partial f}{\partial y} - f\frac{\partial V_2}{\partial z} - V_2\frac{\partial f}{\partial z}\right) - j\left(f\frac{\partial V_3}{\partial x} + V_3\frac{\partial f}{\partial x} - f\frac{\partial V_1}{\partial z} - V_1\frac{\partial f}{\partial z}\right)$$
$$+ k\left(f\frac{\partial V_2}{\partial x} + V_2\frac{\partial f}{\partial x} - f\frac{\partial V_1}{\partial y} - V_1\frac{\partial f}{\partial y}\right)$$

$$= f\left[i\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - j\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$+ i\left(V_3\frac{\partial f}{\partial y} - V_2\frac{\partial f}{\partial z}\right) - j\left(V_3\frac{\partial f}{\partial x} - V_1\frac{\partial f}{\partial z}\right) + k\left(V_2\frac{\partial f}{\partial x} - V_1\frac{\partial f}{\partial y}\right)$$

$$\boxed{\nabla \times (fV) = f(\nabla \times V) + (\nabla f) \times V}$$

Hence, Proved

$$\nabla\cdot\nabla\left(\frac{\partial}{\partial x}i\right)+\left(\frac{\partial}{\partial y}j\right)+\left(\frac{\partial}{\partial z}k\right)$$

$$=\boxed{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}}$$

(iii) $\nabla\times(\nabla\times V)=\nabla(\nabla\cdot V)-(\nabla\cdot\nabla)V$

L.H.S. $=\nabla\times\nabla\times V$

$$\nabla\times V=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$=i\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)+j\left(-\frac{\partial V_3}{\partial x}+\frac{\partial V_1}{\partial z}\right)+k\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)$$

$$\nabla\times(\nabla\times V)=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right) & \left(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right) & \left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right) \end{vmatrix}$$

$$\nabla\times(\nabla\times V)=j\left(\frac{\partial^2 V_2}{\partial y\partial x}-\frac{\partial^2 V_1}{\partial y^2}-\frac{\partial V_1}{\partial z^2}+\frac{\partial^2 V_3}{\partial z\partial x}\right)$$
$$-j\left(\frac{\partial^2 V_2}{\partial x^2}-\frac{\partial^2 V_1}{\partial x\partial y}-\frac{\partial^2 V_3}{\partial z\partial y}+\frac{\partial^2 V_2}{\partial z^2}\right)$$
$$+k\left(\frac{\partial^2 V_1}{\partial x\partial z}-\frac{\partial^2 V_3}{\partial x^2}-\frac{\partial^2 V_3}{\partial y^2}+\frac{\partial^2 V_2}{\partial y\partial z}\right)$$

R.H.S. $=\nabla(\nabla\cdot V)-(\nabla\cdot\nabla)V$

$$(\nabla\cdot\nabla)V=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(V_1 i+V_2 j+V_3 k)$$

$$=\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}+\frac{\partial^2 V_1}{\partial z^2}\right)i+\left(\frac{\partial^2 V_2}{\partial x^2}+\frac{\partial^2 V_2}{\partial y^2}+\frac{\partial^2 V_2}{\partial z^2}\right)j$$
$$+\left(\frac{\partial^2 V_3}{\partial x^2}+\frac{\partial^2 V_3}{\partial y^2}+\frac{\partial^2 V_3}{\partial z^2}\right)k$$

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$\nabla(\nabla \cdot V)$

$$\nabla \cdot V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(V_1 i + V_2 j + V_3 k\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$\nabla(\nabla \cdot V) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$= i\left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial x \partial y} + \frac{\partial^2 V_3}{\partial x \partial z}\right)$$
$$+ j\left(\frac{\partial^2 V_1}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_3}{\partial y \partial z}\right)$$
$$+ k\left(\frac{\partial^2 V_1}{\partial z \partial x} + \frac{\partial^2 V_2}{\partial z \partial y} + \frac{\partial^2 V_3}{\partial z^2}\right)$$

$$R.H.S = \nabla(\nabla \cdot V) - (\nabla \cdot \nabla) V$$

$$= i\left(\frac{\partial^2 V_2}{\partial x \partial y} + \frac{\partial^2 V_3}{\partial x \partial z} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right)$$
$$+ j\left(\frac{\partial^2 V_1}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial y \partial z} - \frac{\partial^2 V_2}{\partial x^2} - \frac{\partial^2 V_2}{\partial z^2}\right)$$
$$+ k\left(\frac{\partial^2 V_1}{\partial z \partial x} + \frac{\partial^2 V_2}{\partial z \partial y} - \frac{\partial^2 V_3}{\partial x^2} - \frac{\partial^2 V_3}{\partial y^2}\right)$$

L.H.S. = R.H.S.

Hence, proved

(iv) $\nabla\cdot(f\nabla g) - \nabla\cdot(g\nabla f) = f\nabla^2 g - g\nabla^2 f$

For LHS:-

$$f\nabla g = f\left(\frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k\right)$$

$$\nabla\cdot(f\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} \quad \to (1)$$

Similarly,

$$\nabla\cdot(g\nabla f) = g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \quad \to (2)$$

(1) − (2)

$$L.H.S. = \nabla\cdot(f\nabla g) - \nabla\cdot(g\nabla f) = f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) - g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\nabla\cdot\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \quad \& \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$L.H.S. = f\nabla^2 g - g\nabla^2 f$$

$$RHS = f\nabla^2 g - g\nabla^2 f$$

$$LHS = RHS$$

Hence, proved

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$$V = V_1 i + V_2 j + V_3 k$$

(V) $\nabla \cdot (\nabla \times V) = 0$

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\nabla \times V = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) i - j\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times V) = \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_1}{\partial y \partial z} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}$$

$$\boxed{\nabla \cdot (\nabla \times V) = 0}$$

Or

$$\nabla \cdot (\nabla \times V) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

Hence, proved

determinant is zero when two rows are same

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(vi) $\nabla \times (\nabla f) = 0$

$$\nabla f = i\left(\frac{\partial f}{\partial x}\right) + j\left(\frac{\partial f}{\partial y}\right) + k\left(\frac{\partial f}{\partial z}\right)$$

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) i - j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$$\boxed{\nabla \times \nabla f = 0}$$

Hence, Proved

(vii) $\nabla \cdot (g\nabla f \times f\nabla g) = 0.$

For LHS

$$g\nabla f = i\, g\frac{\partial f}{\partial x} + j\, g\frac{\partial f}{\partial y} + k\, g\frac{\partial f}{\partial z}$$

$$f\nabla g = i\, f\frac{\partial g}{\partial x} + j\, f\frac{\partial g}{\partial y} + k\, f\frac{\partial g}{\partial z}$$

$$g\nabla f \times f\nabla g = \begin{vmatrix} i & j & k \\ g\,\partial f/\partial x & g\,\partial f/\partial y & g\,\partial f/\partial z \\ f\,\partial g/\partial x & f\,\partial g/\partial y & f\,\partial g/\partial z \end{vmatrix}$$

$$= i\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) - j\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) + k\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

$$\nabla\cdot\left(g\nabla f \times f\nabla g\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(i\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) + j\left(fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right) + k\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial y}\left(fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial z}\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}\right) - \frac{\partial}{\partial x}\left(fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial y}\left(fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) - \frac{\partial}{\partial y}\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial z}\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial z}\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$x \cdot f$

$f \cdot x$

$$= \frac{\partial}{\partial x}\cancel{\left(fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)} - \frac{\partial}{\partial x}\cancel{\left(fg\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)}$$
$$+ \frac{\partial}{\partial y}\cancel{\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)} - \frac{\partial}{\partial y}\cancel{\left(fg\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)}$$
$$+ \frac{\partial}{\partial z}\cancel{\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)} - \frac{\partial}{\partial z}\cancel{\left(fg\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)}$$

LHS $= 0$

RHS $= 0$

LHS = RHS

Hence, proved

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Q.4 $r = xi + yj + zk$ $\quad r = |r| = \sqrt{x^2+y^2+z^2}$

(i) $\nabla^2 f = \nabla \cdot (\nabla f(r)) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$

LHS $= \nabla \cdot (\nabla f(r))$

$\nabla f(r) = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$ $\quad \left\{ g = \frac{\partial f}{\partial r} = \frac{df}{dr} \right.$

$= \frac{gx}{r} i + \frac{gy}{r} j + \frac{gz}{r} k$

$\frac{x}{\sqrt{x^2+y^2+z^2}}$

$\nabla f(r) = \frac{g}{r}\bar{r}$

$\nabla \cdot (\nabla f(r)) = \cancel{\frac{g}{r}}\ \frac{\partial}{\partial x}\left(\frac{gx}{r}\right) + \frac{\partial}{\partial y}\left(\frac{gy}{r}\right) + \frac{\partial}{\partial z}\left(\frac{gz}{r}\right)$

$= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} + g\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + g\frac{\partial}{\partial y}\left(\frac{y}{r}\right) + g\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$

$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r - x^2/r}{r^2} = \frac{y^2+z^2}{r^3}$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \left(\frac{y^2+z^2}{r^3}\right)g + \left(\frac{x^2+z^2}{r^3}\right)g + \left(\frac{x^2+y^2}{r^3}\right)g$

$= \frac{d^2 f}{dr^2} + \frac{2\cancel{(x^2+y^2+z^2)}}{r\,\cancel{r^2}}g$

$= \frac{d^2 f}{dr^2} + \frac{2}{r}g$

LHS $= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$

RHS $= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$

LHS = RHS

Hence, proved

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(ii) $\nabla.(r^n \bar{r}) = (n+3) r^n$

LHS $= \nabla.(r^n \bar{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(r^n x i + r^n y j + r^n z k\right)$

$= \frac{\partial (r^n x)}{\partial x} + \frac{\partial (r^n y)}{\partial y} + \frac{\partial (r^n z)}{\partial z}$

$= r^n\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) + x\frac{\partial r^n}{\partial x} + y\frac{\partial r^n}{\partial y} + z\frac{\partial r^n}{\partial z}$

$= 3r^n + \frac{x^2 n r^{n-1}}{r} + \frac{y^2 n r^{n-1}}{r} + \frac{z^2 n r^{n-1}}{r}$

$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$

$= 3r^n + n r^{n-2} r^2$

$= 3r^n + n r^n$

LHS $= (n+3) r^n$

RHS $= (n+3) r^n$

LHS = RHS

Hence, proved

(iii) $\nabla \times (r^n \bar{r}) = 0$

LHS $= \nabla \times (r^n \bar{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$

$= i\left(\frac{\partial (r^n z)}{\partial y} - \frac{\partial (r^n y)}{\partial z}\right) - j\left(\frac{\partial (r^n z)}{\partial x} - \frac{\partial (r^n x)}{\partial z}\right) + k\left(\frac{\partial (r^n y)}{\partial x} - \frac{\partial (r^n x)}{\partial y}\right)$

$= i\left(\cancel{nz r^{n-2} y - ny r^{n-2} z}\right) - j\left(\cancel{nz r^{n-2} x - nx r^{n-2} z}\right) + k\left(\cancel{ny r^{n-2} x - nx r^{n-2} y}\right)$

LHS $= 0$
RHS $= 0$ LHS = RHS

Hence, proved

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(ii) $\nabla \cdot \left(\nabla \frac{1}{r}\right) = 0$ for $r \neq 0$

By the property.

$\nabla(r^n) = n r^{n-2} \bar{r}$

$\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

$\underline{L.H.S.} = \nabla \cdot \left(\nabla \frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(\frac{-x}{r^3} i - \frac{y}{r^3} j - \frac{z}{r^3} k\right)$

$= -\left(\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right)$

$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$

$= -\left(\frac{r^2 - 3x^2}{r^5} + \frac{r^2 - 3y^2}{r^5} + \frac{r^2 - 3z^2}{r^5}\right)$

$\frac{r^3 - \frac{3r^2 x^2}{r}}{r^6}$

$= -\left(\frac{3r^2 - 3(x^2+y^2+z^2)}{r^5}\right)$ $\{x^2+y^2+z^2 = r^2$

$= -\left(\frac{3r^2 - 3r^2}{r^5}\right)$

LHS $= 0$

RHS $= 0$

LHS = RHS

Hence; proved

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Q.6)

(i) $\omega \rightarrow$ vector field with constant dirn
$\nabla \times \omega \neq 0$

To prove: $(\nabla \times \omega) \perp \omega$ always

Let $\omega = \omega_1 i + \omega_2 j + \omega_3 k$

If two vectors are perpendicular then their scalar product is always zero.

means, $\omega \cdot (\nabla \times \omega) = 0$

$$\nabla \times \omega = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$\nabla \times \omega = i\left(\frac{\partial \omega_3}{\partial y} - \frac{\partial \omega_2}{\partial z}\right) - j\left(\frac{\partial \omega_3}{\partial x} - \frac{\partial \omega_1}{\partial z}\right) + k\left(\frac{\partial \omega_2}{\partial x} - \frac{\partial \omega_1}{\partial y}\right)$$

$$\omega \cdot (\nabla \times \omega) = \omega_1 \frac{\partial \omega_3}{\partial y} - \omega_1 \frac{\partial \omega_2}{\partial z} - \omega_2 \frac{\partial \omega_3}{\partial x} + \omega_2 \frac{\partial \omega_1}{\partial z} + \omega_3 \frac{\partial \omega_2}{\partial x} - \omega_3 \frac{\partial \omega_1}{\partial y}$$

By scalar triple pdt:

$$\omega \cdot (\nabla \times \omega) = \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix} = 0$$

det is zero whenever 2 rows are same.

Hence, proved. ω is orthogonal to $\nabla \times \omega$

$$w = a_1 i + a_2 j + a_3 k$$
$$r = xi + yj + zk$$

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(ii) If $V = \omega \times r$ for a constant vector ω, prove that $\nabla \times V = 2\omega$

$$V = \omega \times r = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y) i + (a_3 x - a_1 z) j + (a_1 y - a_2 x) k$$

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (a_2 z - a_3 y) & (a_3 x - a_1 z) & (a_1 y - a_2 x) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(a_1 y - a_2 x) - \frac{\partial}{\partial z}(a_3 x - a_1 z)\right) i - j\left(\frac{\partial}{\partial x}(a_1 y - a_2 x) - \frac{\partial}{\partial z}(a_2 z - a_3 y)\right)$$
$$+ k\left(\frac{\partial}{\partial x}(a_3 x - a_1 z) - \frac{\partial}{\partial y}(a_2 z - a_3 y)\right)$$

$$= \left(\frac{\partial (a_1 y)}{\partial y} + \frac{\partial (a_1 z)}{\partial z}\right) i + j\left(\frac{\partial (a_2 z)}{\partial z} + \frac{\partial (a_2 x)}{\partial x}\right)$$
$$+ k\left(\frac{\partial (a_3 x)}{\partial x} + \frac{\partial (a_3 y)}{\partial y}\right)$$

$$= 2a_1 i + 2a_2 j + 2a_3 k$$
$$= 2(a_1 i + a_2 j + a_3 k)$$

$$\boxed{\nabla \times \omega = 2\omega}$$

Hence, proved

(iii) If $\rho V = \nabla p$ where $\rho \neq 0$ & $p \to$ scalar f^n prove that

$$V \cdot (\nabla \times V) = 0.$$

~~let~~

$$\nabla p = \frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k$$

$$V = \frac{1}{\rho} \nabla p = \frac{1}{\rho}\frac{\partial p}{\partial x} i + \frac{1}{\rho}\frac{\partial p}{\partial y} j + \frac{1}{\rho}\frac{\partial p}{\partial z} k$$

By scalar triple pdt:

$$V \cdot (\nabla \times V) = \begin{vmatrix} \frac{1}{\rho}\frac{\partial p}{\partial x} & \frac{1}{\rho}\frac{\partial p}{\partial y} & \frac{1}{\rho}\frac{\partial p}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{\rho}\frac{\partial p}{\partial x} & \frac{1}{\rho}\frac{\partial p}{\partial y} & \frac{1}{\rho}\frac{\partial p}{\partial z} \end{vmatrix} = 0$$

Determinant is zero whenever two or more rows are same.

Hence, proved

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Q.⑥ Calculate

$$f(x,y) = (x^2-2xy)\,i + (y^2-2xy)\,j$$

from $(-1,1)$ to $(1,1)$ along $y = x^2$

$$f(x,y) = (t^2-2t^3)\,i + (t^4-2t^3)\,j \qquad \begin{cases} x = t \\ y = t^2 \end{cases}$$

$$C : \bar{r} = t\,i + j \qquad -1 \le t \le 1$$

$$\int_C f \cdot d\bar{r} = \int_C f \cdot \frac{d\bar{r}}{dt}\,dt$$

$$= \int_{-1}^{1} \left((t^2-2t^3)\,i + (t^4-2t^3)\,j\right) \cdot (i)\,dt$$

$$\left.\frac{d\bar{r}}{dt} = i\right\} \qquad = \int_{-1}^{1} (t^2-2t^3)\,dt$$

$$= \left.\frac{t^3}{3} - \frac{t^4}{2}\right|_{-1}^{1}$$

$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{-1}{3} - \frac{1}{2}\right) = \frac{2}{3}$$ Ans

$\cos^2\theta \sin\theta$ $\quad \cos\theta = t$, $\sin\theta\, d\theta = -dt$

$\int t^2 dt = -\frac{t^3}{3} = \frac{\cos^3\theta}{3}$

$\int (1-t^2)\, s$ $\quad \int(-1+t^2)\,dt = \frac{t^3}{3} - t$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $\quad \frac{\theta}{2} + \frac{\sin 2\theta}{4}$

$\frac{\sin 2\theta}{2} = -\frac{\cos 2\theta}{4}$

Q.(1) $f(x,y) = (x^2+y^2)\,i + (x-y)\,j$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in counter clockwise dir^n

$x = a\cos\theta$

$y = b\sin\theta$

$f(x,y) = (a^2\cos^2\theta + b^2\sin^2\theta)\,i + (a\cos\theta - b\sin\theta)\,j$

$\bar{r} = a\cos\theta\, i + b\sin\theta\, j$

$\frac{d\bar{r}}{d\theta} = -a\sin\theta\, i + b\cos\theta\, j \qquad \{ 0 \le \theta \le 2\pi$

$$\int_C f(x,y)\cdot dr = \int_C f(x,y)\cdot \frac{dr}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \left((a^2\cos^2\theta + b^2\sin^2\theta)\, i + (a\cos\theta - b\sin\theta)\, j \right)\cdot(-a\sin\theta\, i + b\cos\theta\, j)\, d\theta$$

$$= \int_0^{2\pi} \left(-a^3\cos^2\theta\sin\theta - ab^2\sin^3\theta + \cancel{a^2\cos\theta\sin\theta}\; ab\cos^2\theta - b^2\sin\theta\cos\theta \right) d\theta$$

$$= \left(a^3\frac{\cos^3\theta}{3} - ab^2\left(\frac{\cos^3\theta}{3} - \cos\theta\right) + ab\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + b^2\frac{\cos 2\theta}{4} \right)_0^{2\pi}$$

$$= \left(\cancel{\frac{a^3}{3}} + \cancel{\frac{2}{3}ab^2} + \pi ab + \cancel{\frac{b^2}{4}} \right) - \left(\cancel{\frac{a^3}{3}} + \cancel{\frac{2}{3}ab^2} + \cancel{\frac{b^2}{4}} \right)$$

$$= \pi ab$$

Ans

Q.8 $\oint_C \frac{(x+y)\,dx - (x-y)\,dy}{x^2+y^2} = \oint_C \frac{a(\cos\theta+\sin\theta)\,dx}{a^2} - \frac{a(\cos\theta-\sin\theta)}{a^2}dy$

Curve $x^2+y^2=a^2$
in counter clockwise dirⁿ $\{0 \le \theta \le 2\pi$

$x = a\cos\theta$

$y = a\sin\theta$

$C: \bar{r} = a\cos\theta\, i + a\sin\theta\, j$ $\quad\begin{cases} dx = -a\sin\theta\, d\theta \\ dy = a\cos\theta\, d\theta \end{cases}$

$\frac{d\bar{r}}{d\theta} = -a\sin\theta\, i + a\cos\theta\, j$

$\Rightarrow \oint_C \left(\frac{(x+y)}{a^2}\frac{dx}{d\theta}d\theta - \frac{(x-y)}{a^2}\frac{dy}{d\theta}d\theta \right)$

$\oint \to \frac{1}{a^2}\oint_0^{2\pi} \left(a(\cos\theta+\sin\theta)(-a\sin\theta)\,d\theta - a(\cos\theta-\sin\theta)(a\cos\theta)\,d\theta \right)$

$\oint_0^{2\pi} \left((-\sin\theta\cos\theta - \sin^2\theta)\,d\theta - (\cos^2\theta - \sin\theta\cos\theta) \right) d\theta$

$\oint_0^{2\pi} (-1)\,d\theta = -2\pi$

$\int \left(-(\cos\theta+\sin\theta)\sin\theta\, d\theta - (\cos\theta-\sin\theta)\cos\theta\, d\theta \right)$

$\left(-\sin\theta\cancel{\cos\theta} - \sin^2\theta - \cos^2\theta + \sin\theta\cancel{\cos\theta} \right) d\theta$

$\int_0^{2\pi} (-1)\,d\theta = -2\pi$ Ans

Q. (1) $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

(i) $\nabla \cdot F(x,y)$

$$\nabla \cdot F(x,y) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \cancel{+k\frac{\partial}{\partial z}}\right) \cdot \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$\frac{2yx}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = \frac{4xy}{(x^2+y^2)^2}$$ Ans.

(ii) $\nabla \times F(x,y)$

$$\nabla \times F(x,y) = \cancel{\begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{matrix}}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0.i - j(0) + k\left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right)$$

$$= \left(\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right)k$$

$$= \left(\frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{(x^2+y^2)^2}\right)k = 0$$ Ans

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(iii) line integeral $F(x,y)$ along curve

$C(t) = (a\cos t, b\sin t), \quad a \neq b, \quad 0 \leq t \leq 2\pi$

$$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = \left(\frac{-b\sin t}{a^2\cos^2 t + b^2\sin^2 t}, \frac{a\cos t}{a^2\cos^2 t + b^2\sin^2 t}\right)$$

$\bar{r} = xi + yj$

$\bar{r} = a\cos t\, i + b\sin t\, j$

$\frac{d\bar{r}}{dt} = -a\sin t\, i + b\cos t\, j$

$$\int_C F \cdot d\bar{r} = \int_0^{2\pi} \left(\frac{-b\sin t}{a^2\cos^2 t + b^2\sin^2 t}, \frac{a\cos t}{a^2\cos^2 t + b^2\sin^2 t}\right) \cdot (-a\sin t, b\cos t)\, dt$$

$$= \int_0^{2\pi} \left(\frac{ab\sin^2 t}{a^2\cos^2 t + b^2\sin^2 t} + \frac{ab\cos^2 t}{a^2\cos^2 t + b^2\sin^2 t}\right) dt$$

$$= ab\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2\sin^2 t}\, dt$$

$$= ab\int_0^{2\pi} \frac{\sec^2 t\, dt}{a^2 + b^2\tan^2 t}$$

$\tan t = u$

$\sec^2 t\, dt = du$

$$\Rightarrow \frac{a}{b}\int_0^{2\pi} \frac{du}{\frac{a^2}{b^2} + u^2}$$

$$\rightarrow \frac{\cancel{a}}{b}\frac{\cancel{b^2}}{\cancel{a^2}} \tan^{-1}\frac{bu}{a}\Big|_0^{2\pi}$$

$$\Rightarrow \tan^{-1}\left(\frac{b\tan t}{a}\right)\Big|_0^{2\pi}$$

$$\Rightarrow 0 - 0 = \underline{\underline{0}} \quad \text{Ans}$$

$$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$$

$$\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$$

$4+4\cos^2 t+8\cos t$
$4+4\sin^2 t+8\sin t$

(iv) $C(t)=(2+2\cos t,\ 2+2\sin t),\quad 0\le t\le 2\pi$

$$F(x,y)=\left(-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)\equiv\left(\frac{-(2+2\sin t)}{12+8(\sin t+\cos t)},\ \frac{2+2\cos t}{12+8(\sin t+\cos t)}\right)$$

$$\bar{r}=(2+2\cos t)\,i+(2+2\sin t)\,j$$

$$\frac{d\bar{r}}{dt}=-2\sin t\,i+2\cos t\,j$$

$$\int \frac{4\cancel{\cos^2 t}+8+8(\cos t+\sin t)}{12+3}$$

$$\int_0^{2\pi}\left(\frac{+(2+2\sin t)2\sin t}{12+8(\sin t+\cos t)}+\frac{(2+2\cos t)2\cos t}{12+8(\sin t+\cos t)}\right)dt$$

$$\int_0^{2\pi}\frac{4\sin t+4\sin^2 t+4\cos t+4\cos^2 t}{12+8(\sin t+\cos t)}\,dt$$

$$\int_0^{2\pi}\frac{4+4(\sin t+\cos t)}{12+8(\sin t+\cos t)}\,dt$$

$$\int_0^{2\pi}\frac{1+(\sin t+\cos t)}{3+2(\sin t+\cos t)}\,dt$$

$$\tan\frac{t}{2}=u$$

$$\sec^2\frac{t}{2}\,dt=2du$$

$$\int_0^{2\pi}\left(\frac{1}{3+2(\sin t+\cos t)}+\frac{\sin t+\cos t}{3+2(\sin t+\cos t)}\right)dt$$

$$\frac{\sec^2\frac{t}{2}\,dt}{3+3\tan^2\frac{t}{2}+4\tan\frac{t}{2}+2-2\tan^2\frac{t}{2}}+\frac{\left(2\tan\frac{t}{2}+1-\tan^2\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt}{\left(5+\tan^2\frac{t}{2}+4\tan\frac{t}{2}\right)\left(1+\tan^2\frac{t}{2}\right)}$$

$$\frac{2\,du}{5+u^2+4u}+\frac{\left(2\tan\frac{t}{2}+1-\tan^2\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt}{\tan^4\frac{t}{2}+4\tan^3\frac{t}{2}+6\tan^2\frac{t}{2}+4\tan\frac{t}{2}+5}$$

$$\frac{2\,du}{(u+2)^2+1}+\frac{2(2u+1-u^2)\,du}{u^4+4u^3+6u^2+4u+5}$$

$$2\tan^{-1}(u+2)$$

u^2-2u-1

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$$\int_0^{2\pi} \frac{1+\sin t+\cos t}{3+2(\sin t+\cos t)}\,dt$$

$0\ \pi \rightarrow -1$ 1

$\pi\ 2\pi \rightarrow 1$ -1

-1 -1

$\sin t-\cos t = u$

$(\cos t+\sin t)\,dt = du$

$$\int \frac{dt}{3+2(s+c)} + \int \frac{(\sin t+\cos t)\,dt}{3+2(\sin t+\cos t)}$$

$(s+c)^2 = 1+2sc$

$(s-c)^2 = 1-2sc$

$+4sc$

$$\cancel{\left[2\tan^{-1}\left(\tan\frac{t}{2}+2\right)\right]_0^{2\pi}} + \int \frac{du}{3+2(-u^2+2)}$$

$$\int \frac{du}{-2u^2+7}$$

$1-2sc = u^2$

$2(1-u^2) = 4sc$

$$-\frac{1}{2}\int \frac{du}{u^2-7/2} \quad \cancel{= \frac{-1}{2}\frac{1}{7}\ln\left|\frac{2u-7}{2u+7}\right|}$$

u^2 $-2u^2+2$

$$= -\frac{1}{2}\frac{1}{7}\ln\left|\frac{2u-7}{2u+7}\right|\Bigg|_0^{2\pi}$$

$$-\frac{1}{14}\left(\cancel{\ln\left|\frac{-9}{5}\right|} - \cancel{\ln\left|\frac{-9}{5}\right|}\right) = 0$$

Ans $= 0$

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Q. 10 $F = (2xy + z^3)i + x^2 j + 3xz^2 k$ on $\mathbb{R}^3$

(i) $F = \nabla \phi \Rightarrow \phi = ?$

(ii) work done from $(1, 2, 3)$ to $(4, 5, 7)$

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(9) $\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = F$

$\frac{\partial \phi}{\partial x} = 2xy + z^3 \Rightarrow \phi_1 = x^2y + xz^3$ ✓

$\frac{\partial \phi}{\partial y} = x^2 \Rightarrow \phi_2 = x^2y$

$\frac{\partial \phi}{\partial z} = 3xz^2 \Rightarrow \phi_3 = xz^3$ ✗

ϕ_2 & ϕ_3 are not acceptable because on taking ϕ_3 or ϕ_2 as potential function value of F ~~~~ is coming out ~~~~ a different value ~~~~ ~~~~ from the value given in the question.

Hence, ϕ_1 is acceptable.

So, potential f^n

$$\boxed{\phi = x^2y + xz^3}$$

(ii) $\bar{r} = (1, 2, 3) + t(3, 3, 4) \quad \{0 \le t \le 1$

$= (1+3t,\ 2+3t,\ 3+4t)$

$\frac{d\bar{r}}{dt} = (3, 3, 4)$

$(1+3t)(16t^2 + 9 + 24t)$

$48t^3 + 16t^2 + 72t^2 + 27t + 24t + 9$

$F(t) = 2(9t^2 + 9t + 2 + 27 + 64t^3 + 144t^2 + 108t)\, i$
$+ (9t^2 + 1 + 6t)\, j + 3(48t^3 + 88t^2 + 51t + 9)\, k$

$$\int_c F \cdot \frac{dr}{dt}\, dt = \int_0^1 \left(6(64t^3 + 153t^2 + 117t + 29) + (27t^2 + 18t + 3) + 12(48t^3 + 88t^2 + 51t + 9) \right) dt$$

$$= \int_0^1 \left(384t^3 + 918t^2 + 702t + 174 + 27t^2 + 18t + 3 + 576t^3 + 1056t^2 + 612t + 108 \right) dt$$

$$= \int_0^1 (960t^3 + 2001t^2 + 1332t + 285)\, dt$$

$$= \frac{960}{4}t^4 + \frac{2001}{3}t^3 + \frac{1332}{2}t^2 + 285t \Big|_0^1$$

$$= 240 + 667 + 666 + 285$$

$$= 1858 \quad \text{Ans}$$