Mimetic Operator Discretization 1D:
Exploration of classical iterative methods and preconditioners

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Abstract

This work performs experiments looking for preconditioned iterative methods for sparse linear systems that overcome stagnation for the numerical resolution of the one-dimensional Poisson equation with Robin boundary conditions, discretized with a second-order mimetic method, based on the 1D Castillo-Grone mimetic operator. By refining the grid, only an appropriate preconditioner will allow the associate linear system iterative method to achieve convergence.

Keywords: adaptive iterative methods, Castillo-Grone mimetic operator, Krylov Subspace methods, preconditioners, sparse linear systems.

Introduction

The Castillo-Grone method [1] allows for the construction of discrete differential operators, e.g. divergence, gradient, which are $O(h^k)$ accurate (for any step size h and every even k) at the interior points and the domain boundary. The aim of these mimetic operators is to satisfy in the discrete sense the vectorial, global conservation laws that the continuum model do in order to make the problem more accurate to physical constraints.

Goals

- To build non-singular matrices from second-order mimetic operators for one-dimensional boundary conditions.
- To compare the results, efficiency, and order of convergence between different implemented iterative methods for linear systems and preconditioners.

Methodology

Consider the one-dimensional Poisson equation on a uniform grid [2],

$$\nabla^2 u(x) = f(x) \quad on \quad [0,1] \tag{1}$$

subject to Robin boundary conditions

$$\alpha f(0) - \beta f'(0) = -1; \quad \alpha f(1) + \beta f'(1) = 0,$$

where

$$f(x) = \frac{\lambda^2 e^{\lambda x}}{e^{\lambda x} - 1}, \quad \alpha = -e^{\lambda}, \quad \beta = \frac{e^{\lambda} - 1}{\lambda}, \quad \lambda = -1.$$

The analytical solution to the problem is

$$u^*(x) = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}.$$

To solve the linear system $Au = f$ associated with this problem, we used the restarted GMRES with $m = 10$, and the BiCGStab as iterative methods. Also, the following preconditioners: Jacobi and succesive over-relaxation (SOR) with $\omega = 1$, are used to try to improve the convergence (relative residual norm$||b - Au_j||/||b||$ less than 10^{-6}).

Numerical results

	Jacobi			SOR ($\omega = 1$)		
h	Condest	GMRES(10)	BiCGStab	Condest	GMRES(10)	BiCGStab
0.20	311.9	0.002184	0.002184	190.6	0.002184	0.002184
0.10	1073.0	0.000537	0.000545	662.5	0.000543	0.00541
0.05	3948.4	0.000126	0.00136	2421.5	**1.008095**	0.00132
0.01	91903.2	**1.165131**	0.000004	55928.2	**1.142317**	0.000010

Table 1: Relative error $||u - u^*||/||u^*||$ for different iterative solvers and preconditioners.

Numbers in red, bold face indicate that the combination does not reach convergence.

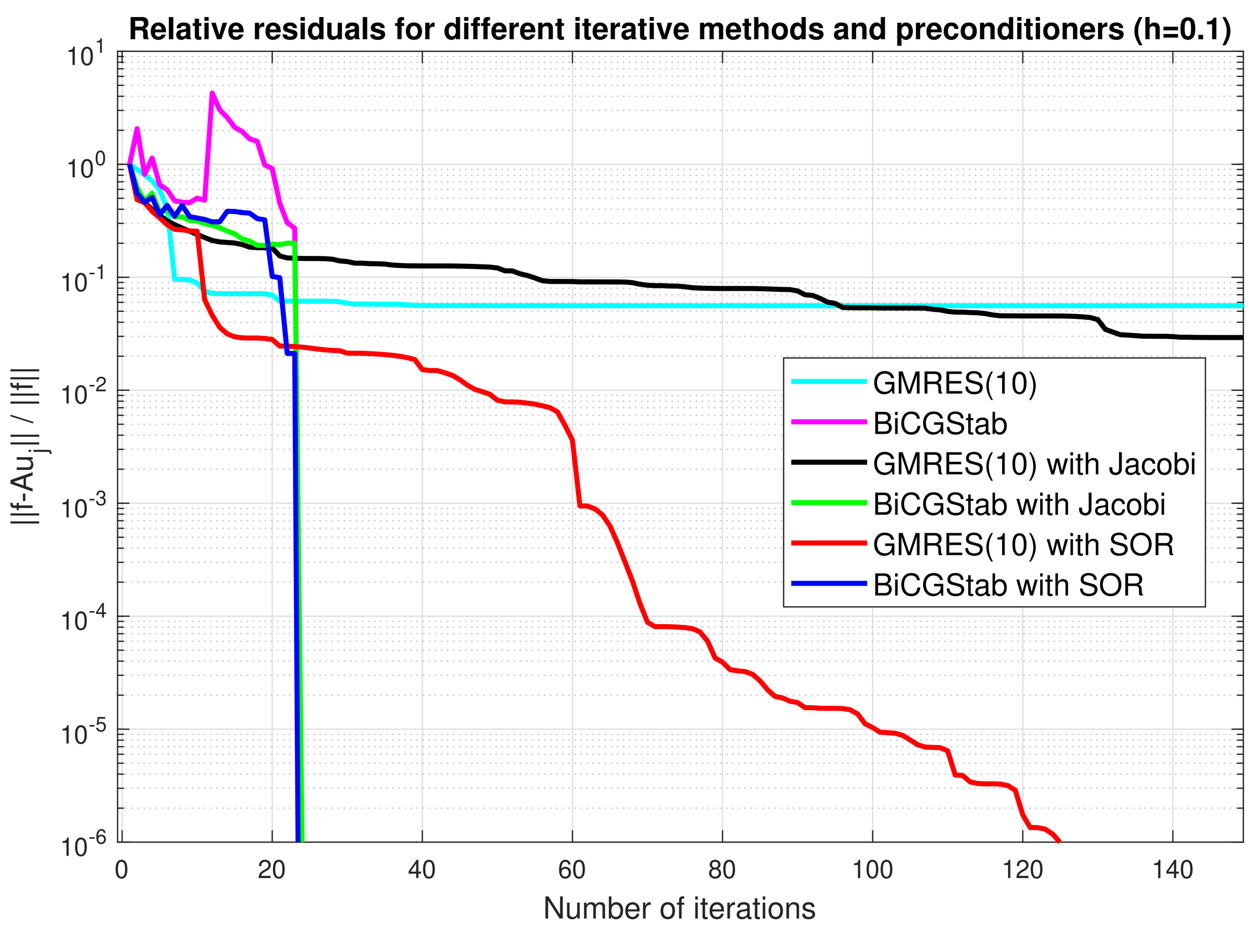

Figure 1: Relative residuals for GMRES(10) and BiCGStab using step size $h = 0.1$.

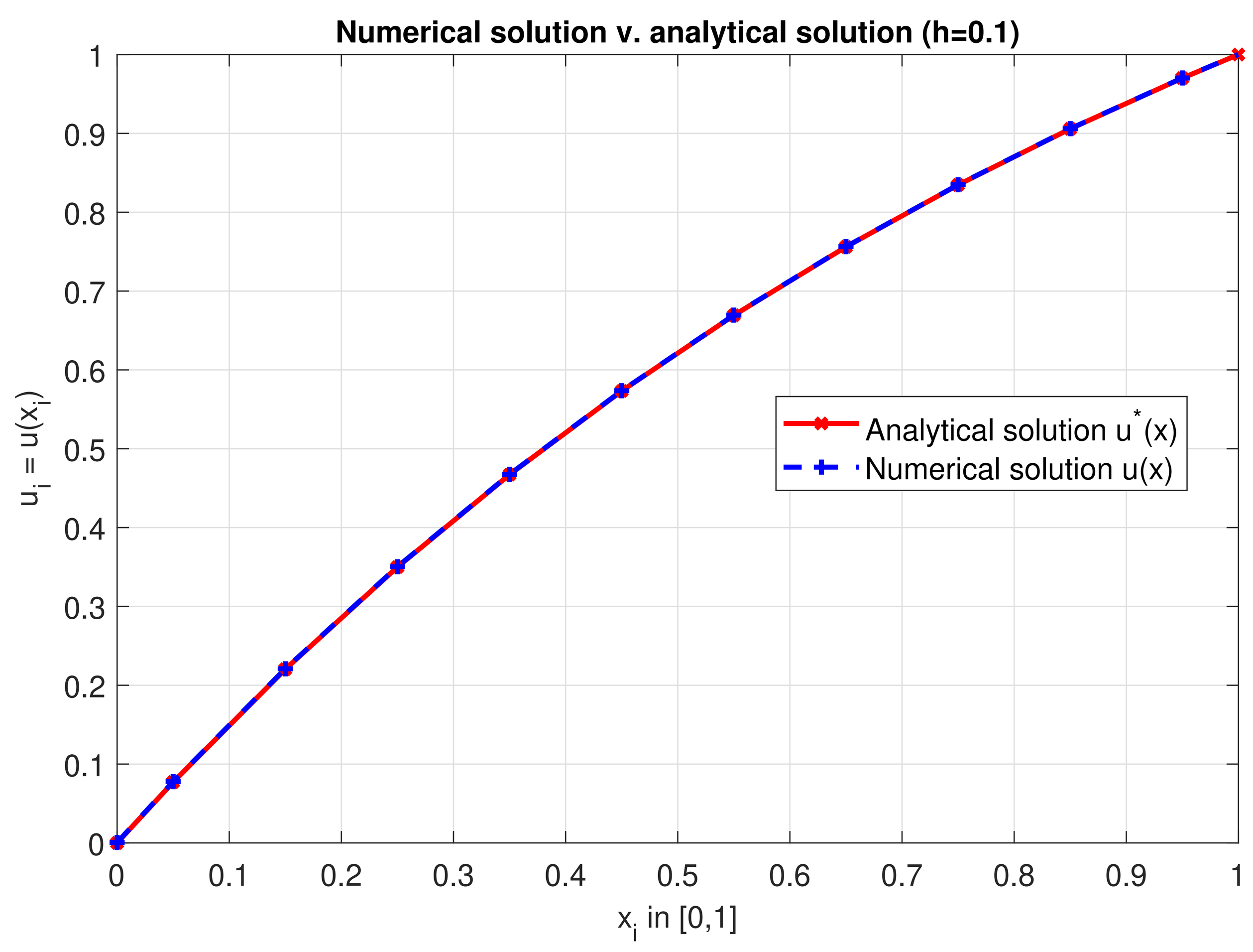

Figure 2: Numerical v. analytical solution of the 1D Poisson equation using step size $h = 0.1$.

Conclusion

For smaller step size h, only an appropriate preconditioner will allow convergence regardless of the chosen iterative method. Ongoing research work is analyzing the matrix structure for higher orders and dimensions (2D, 3D). Adaptive iterative methods and modern, block preconditioning methods may be useful to solve the stagnation problem.

References

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