Theoretical Determination of the Minimum Drag of Airfoils at Supersonic Speeds

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ABSTRACT

Consider a thin wing in frictionless flow and suppose the plan form of the wing and also the total lift to be given. The drag of the wing will then depend on the way in which the lift is distributed over its surface. In a previous paper, it was shown that the minimum drag occurs when the superposition of the induced disturbance fields in forward and reversed motion results in a constant value of the induced downwash at all points of the wing surface. Similar problems involving the ideal distribution of thickness over the surface were found to lead to similar conditions governing the distribution of pressure in the superimposed or "combined" flow field.

The present paper describes a method for determining mathematically the combined disturbance field, and in certain cases the minimum drag, of wings at supersonic speeds. The simplest analytic example is provided by the wing of elliptic plan form, which achieves its minimum drag when the lift is distributed uniformly over the surface. With a symmetrical distribution of thickness, the requirement of minimum drag for a given total volume is found to lead to profiles of constant curvature.

INTRODUCTION

IN THE THEORY OF WINGS at subsonic speeds, it is shown that the production of lift by a wing of finite span gives rise to a drag force that depends on the distribution of lift over the span. This component of the drag, which arises in frictionless motion, may be related to the energy required for the continual extension of the two-dimensional field of motion induced by the wake of trailing vortices. Alternatively, by examining conditions in the vicinity of the wing sections, the drag may be related to the downward inclination of the air stream induced at the position of the wing by the action of the trailing vortices. Following the latter concept, the drag arising from the lift at subsonic speeds has been termed the "induced drag." It was shown by Munk[1] that this drag is a minimum for a given lift and a given span when the induced downwash is constant at all points of an equivalent lifting line, or vortex, having the same spanwise distribution of lift as the wing. It was further shown by Munk that the induced drag is actually independent of the chordwise distribution of lift.

At supersonic speeds an additional component of drag arises because of the formation of waves by the airfoil, and in this case the drag depends on both the spanwise and chordwise distributions of lift or, in other words, on the actual distribution of lifting pressure over the surface of the wing. In order to extend Munk's problem to wings at supersonic speeds it was necessary therefore to consider not merely the span as given but the actual shape of the wing in plan view. The problem could then be stated in the following form: For a given plan form s and a given total lift L, what distribution of the lift L over the surface s results in the minimum drag?

Furthermore, at supersonic speeds a certain drag arises from the thickness of the airfoil independently of the lift. The two components of drag may, however, be considered separately and later added in any desired combination. To isolate the effect of lift, as distinct from the effect of thickness, it is sufficient to replace the wing by its mean surface, which is supposed to be warped or cambered in whatever way may be required to cause the specified distribution of lift. On the other hand, the drag arising from the thickness may be determined by considering the thickness to be symmetrically disposed above and below a flat mean surface having no lift. In this way additional problems involving the ideal distribution of thickness over the plan form become apparent.

In reference 2 it was shown that all distributions of lift having the minimum drag for a given plan form and a given total lift are characterized by a single condition. If we suppose the wing with its given distribution of lift to be held fixed in a stream of velocity V, then there will arise in the vicinity of the wing and its wake additional small disturbance velocities u, v, and w. Now let the direction of the stream be reversed, but suppose that the curvature and inclination of the surface is so modified as to maintain the original distribution of lift. A new field of disturbance velocities u, v, and w will appear. The wake of trailing vortices will have the same form as before, but the wake will now extend from the wing in the opposite direction. We may now superimpose the two fields of disturbance velocities and obtain by this means a "combined disturbance field," associated with the given distribution of lift. For the drag to be a minimum, the downwash in the combined disturbance field must be constant at all points of the wing surface s.

If the velocity of flight is subsonic, the superposition of the two fields can be shown to result in a two-dimensional field of motion identical in form to the velocity

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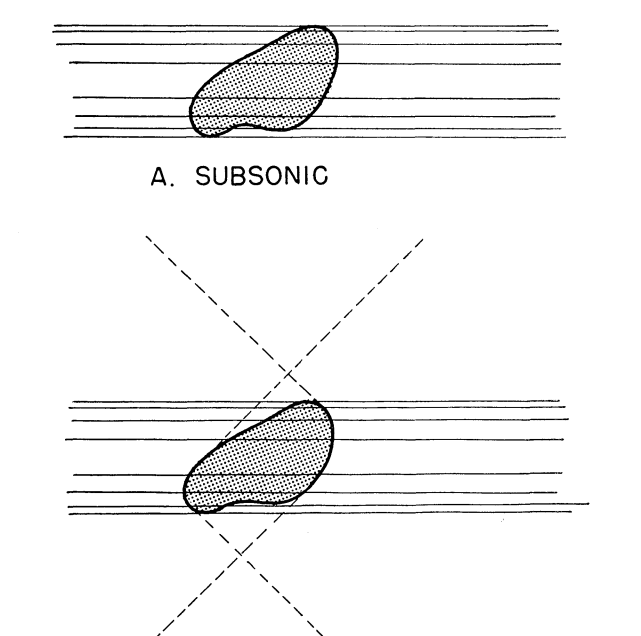

FIG. 1. Lifting surfaces with superimposed disturbance fields.

field of the vortex wake. For distributions of lift having the minimum drag, the downwash induced by the vortex wake in its own plane is a constant, and it will be evident that in this case the downwash is constant not only within the plan form of the wing but at all points of the vortex ribbon ahead of and behind the wing (Fig. 1A).

At supersonic speeds the combination of the forward and reversed disturbance fields again produces an infinite, parallel vortex ribbon, but the field is no longer two-dimensional in character and is bounded by two overlapping zones of influence or wave fronts, as illustrated in Fig. 1B.

By treating the problem of thickness in a similar manner, it was also shown in reference 2 that the minimum drag for a given frontal area of the wing occurs when the pressure in the combined flow field is constant at all points of the wing surface. Similarly, consideration of the minimum drag consistent with a given total volume led to the requirement of a constant streamwise gradient of the pressure in the combined flow.

The present paper describes a method for determining the combined disturbance fields associated with given distributions of lift or thickness. The basic idea of the method is to represent the elementary solutions of the flow equation, such as the solutions for the source and for the horseshoe vortex, by contour integrals, following forms introduced by Whittaker[3] and Bergman[4] rather than the usual forms. The distribution of sources or dipoles over the wing surface is then represented by a triple integral, in which the surface integral, after a change in the order of integration, represents a distribution of two-dimensional disturbances over the surface. The three-dimensional flow is thus obtained finally by the superposition of elementary two-dimensional flows. As will be shown, this method enables the calculation of three-dimensional wing flows that satisfy the conditions for minimum drag and provides, as examples, formulas for the minimum drag of wings of elliptic plan form at supersonic speed.

Preliminary Considerations

As is well known in the thin-airfoil theory, the lift distribution over a thin cambered wing or lifting surface appears as the resultant of two equal, opposite pressures over the upper and lower surfaces. With the pressure disturbance given by

$$\Delta p = -\rho u V \tag{1}$$

where u is the longitudinal perturbation velocity, we obtain, for the local lift,

$$l(x,y) = 2\rho V u(x,y,z); \quad z \to +0 \tag{2}$$

The velocity u is discontinuous across the lifting surface and is to be evaluated on the upper side ($z \to +0$). In the lifting case, the downwash velocity $w(x,y)$ is continuous over the whole plane of the wing. The drag is given by

$$D = \int_s\int l(x,y)\ \frac{w(x,y)}{V}\, dx\, dy \tag{3}$$

In certain cases, the lift density l and the stream inclination w/V may approach infinite values around the edges of the surface. The integral (3) must then be evaluated by a suitable limiting process, as described in reference 5.

It has been shown by von Kármán[6] and Hayes[7] that the drag of a given distribution of lift is unchanged by a reversal of the direction of motion. Hence, the drag of a specified distribution of lift may be calculated from the corresponding distribution of downwash in either direction of motion or from the combined downwash $\bar{w}$, as indicated by the following formulas:

$$D = \frac{1}{V}\int_s\int l\overrightarrow{w}\, dx\, dy = \frac{1}{V}\int_s\int l\overleftarrow{w}\, dx\, dy = \frac{1}{2V}\int_s\int l\bar{w}\, dx\, dy \tag{4}$$

For the minimum drag the combined downwash $\bar{w}$ will be constant over s, and we have

$$D = (1/2)L(\bar{w}/V) \tag{5}$$

where L is the total lift.

Turning to the case of a prescribed distribution of thickness with no lift, it is noted that the velocity u and the pressure as given by Eq. (1) are continuous across the upper and lower sides of the mean plane, but the velocity w has a discontinuity related to the equal and opposite slopes of the upper and lower wing surfaces. If $t(x,y)$ is the prescribed thickness of the wing at the point (x,y), we have

$$w(x,y) = -(1/2)Vt'(x,y); \quad z \to +0 \tag{6}$$

where t' denotes dt/dx. It can be readily verified that the velocity $\bar{w}$, in the combined flow field, vanishes at all points of the wing plan form. A value of $\bar{u}$ remains, however, and determines the drag through the relation

$$D = -\frac{\rho V}{2} \int_s \int t'\bar{u}\, dx\, dy \tag{7}$$

As shown in reference 2 the requirement of minimum drag with various specifications on the maximum thickness or the volume of the wing leads to conditions of the form

$$\bar{u} = \text{constant} \tag{8}$$

or

$$\partial \bar{u}/\partial x = \text{constant} \tag{9}$$

over all or part of the wing plan form.

General Form of the Solution

The field of disturbance velocities surrounding the airfoil will be characterized by a velocity potential satisfying the well-known differential equation

$$(M^2 - 1)\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \tag{10}$$

The same differential equation holds for the disturbance field in either direction of motion, as well as for the combination of the two fields. Since the fields for different Mach Numbers differ only by an affine transformation, it will be convenient to perform the calculations for $M = \sqrt{2}$.

As may be shown by direct differentiation, the resulting equation possesses the primary solutions[8]

$$\varphi = F(\alpha x - \beta y - \gamma z) \tag{11}$$

where F is an arbitrary, differentiable function and α, β, and γ are parameters determined so that

$$\alpha^2 - \beta^2 - \gamma^2 = 0 \tag{12}$$

Through Eq. (12), α, β, and γ may be made to depend on a single complex parameter λ. Writing $\lambda = e^{i\theta}$ and setting

$$\left.\begin{aligned} \alpha &= 1 \\ \beta &= \cos\theta = (1/2)\,[(1/\lambda) + \lambda] \\ \gamma &= \sin\theta = (i/2)\,[(1/\lambda) - \lambda] \end{aligned}\right\} \tag{13}$$

Eq. (12) is satisfied for values of λ extending over the entire complex plane. For real values of θ ($|\lambda| = 1$), Eq. (11) becomes

$$\varphi = F(x - y\cos\theta - z\sin\theta) \tag{14}$$

and the solution is seen to represent a plane wave of arbitrary form F. The wave front lies at an angle of 45° to the x axis ($M = \sqrt{2}$) but is inclined at an angle θ in the y,z plane. On the other hand, for large values of λ we have

$$\varphi = F[-(\lambda/2)(y - iz)] \tag{15}$$

and the solution here represents a cylindrical or two-dimensional flow with its axis parallel to x. This two-dimensional field is evidently a solution of

$$-\varphi_{yy} - \varphi_{zz} = 0 \tag{16}$$

with φ_{xx} separately equal to zero.

A general solution of Eq. (10) may be constructed by superimposing a number of solutions of the form (11) for various values of the parameter λ. It is clear that the form of the function F need not be the same for all values of λ, so that F may depend on the two variables $\alpha x - \beta y - \gamma z$ and λ. Thus we obtain

$$\varphi = \oint F(\alpha x - \beta y - \gamma z, \lambda)\, d\lambda \tag{17}$$

where C is some contour in the λ plane. Eq. (17) is closely analogous to Whittaker's solution of the Laplaces equation[3] and belongs to the more general class of integral operators studied by Bergman.[4]

Expressions for Elementary Disturbance Fields

As an example of Eq. (17), we may construct the well-known solution for the supersonic point source by means of the superposition of plane waves. This solution can be represented in terms of real values of θ as follows:

$$\varphi_s = \frac{1}{4\pi^2}\int_0^{2\pi} \frac{d\theta}{x - y\cos\theta - z\sin\theta} = \begin{cases} 0, \text{ unless } x^2 > y^2 + z^2 \\ \dfrac{1}{2\pi R} \text{ for } x > 0 \\ \dfrac{-1}{2\pi R} \text{ for } x < 0 \end{cases} \tag{18}$$

(See references 9 and 10.) Here, $R = \sqrt{x^2 - y^2 - z^2}$ and is assumed to have a positive real part. The equation $R = 0$ represents the Mach cone, which extends both ahead of and behind the point source. The integral (18) shows a "zone of silence" in the space between the fore cone and the rear cone.

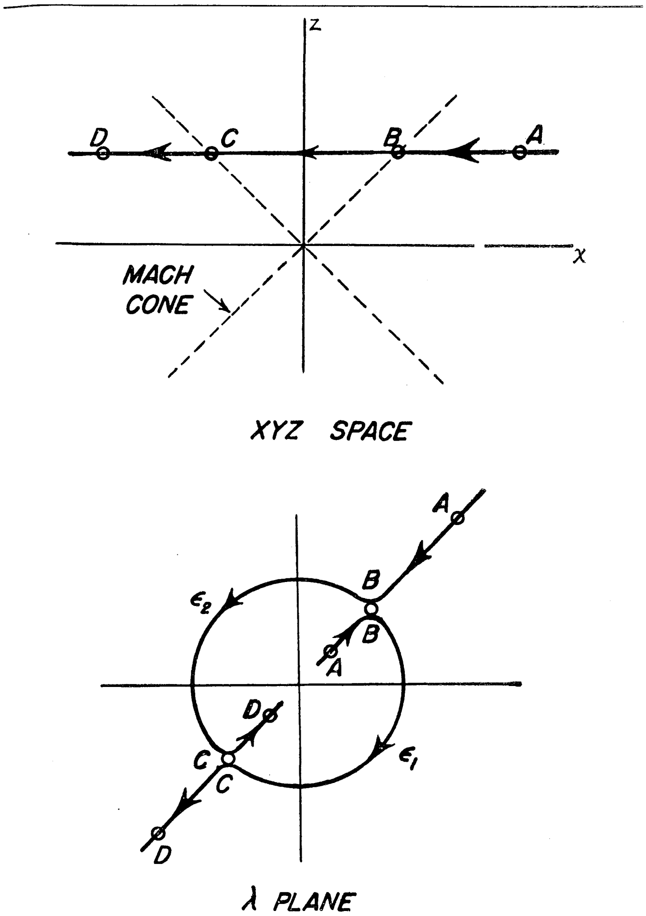

FIG. 2. Course of ϵ_1 and ϵ_2 during variation of point XYZ.

For the purpose of constructing the solution for a complete wing, it is found desirable to represent the elementary solutions in terms of the complex parameter $\lambda = e^{i\theta}$ and to select a contour C which avoids points on the unit circle. It will appear later that the contour C can be selected in a way that simplifies the integration of the elementary solutions over the wing surface.

In terms of λ the potential of the source becomes

$$\varphi_s = \frac{1}{4\pi^2} \oint \frac{d\lambda}{i\lambda\,(\alpha x - \beta y - \gamma z)} \tag{19}$$

The integral may now be evaluated by the method of residues. After expressing α, β, and γ in terms of λ with the aid of Eqs. (13), we have

$$\alpha x - \beta y - \gamma z = (1/2\lambda)\,[2\lambda x - (1 + \lambda^2)y - i(1 - \lambda^2)z] \tag{20}$$

The quantity in the brackets is a quadratic in γ and may be factored so that

$$\alpha x - \beta y - \gamma z = -\frac{(y - iz)\,(\lambda - \epsilon_1)\,(\lambda - \epsilon_2)}{2\lambda} \tag{21}$$

where

$$\left.\begin{aligned} \epsilon_1 &= \frac{x - R}{y - iz} = \frac{y + iz}{x + R} \\ \epsilon_2 &= \frac{x + R}{y - iz} = \frac{y + iz}{x - R} \\ R &= \sqrt{x^2 - y^2 - z^2} \end{aligned}\right\} \tag{22}$$

with the real part of R positive. The integral (19) may now be written

$$\varphi_s = \frac{i}{2\pi^2} \oint \frac{d\lambda}{(y - iz)\,(\lambda - \epsilon_1)\,(\lambda - \epsilon_2)} \tag{23}$$

To evaluate the integral (23) by the method of residues, it is necessary to investigate the positions of the poles ϵ_1 and ϵ_2 in the λ plane as functions of the coordinates x,y,z. Fig. 2 illustrates this correspondence. The three-dimensional x,y,z manifold is represented on the two-dimensional complex plane by the identification of points with rays. This representation was used by Busemann in his conical-flow theory.[11] Each ray drawn from the origin in (x,y,z) space appears as two points ϵ_1 and ϵ_2 in the λ plane. For rays drawn toward the positive direction of x and lying inside the Mach cone ($y^2 + z^2 < x^2$), the point ϵ_1 will lie inside the unit circle, $|\lambda| < 1$, while the point ϵ_2 will lie outside this circle at the reciprocal radius—that is,

$$\epsilon_2 = 1/\bar{\epsilon}_1 \tag{24}$$

Investigation of Eq. (22) shows that for each point (x,y,z) in the space outside the Mach cone ($y^2 + z^2 > x^2$), both ϵ_1 and ϵ_2 lie exactly on the unit circle. Thus, ϵ_1 and ϵ_2 project the space (x,y,z) inside the Mach cone on a surface, while the whole portion of the space (x,y,z) in the "zone of silence" outside the Mach cone is represented on a single line—i.e., the circle $|\lambda| = 1$.

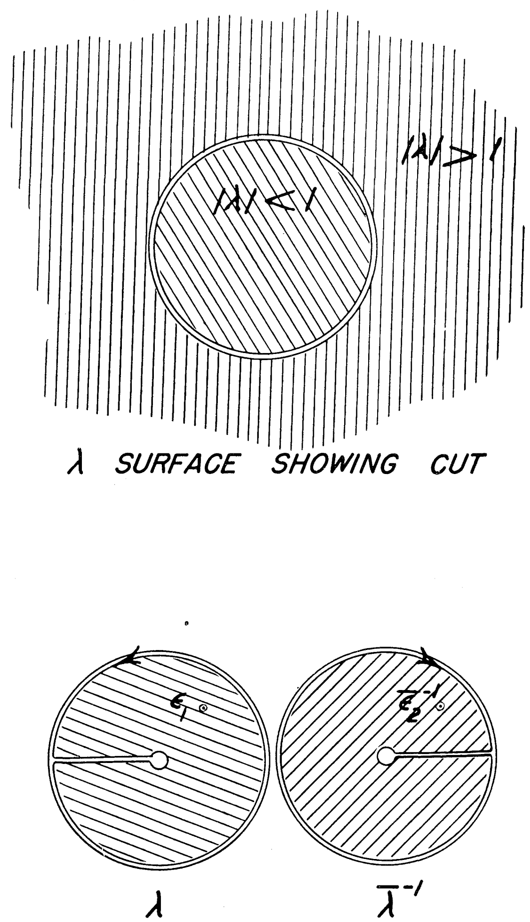

FIG. 3. Representation of λ surface showing two parts of con tour C.

This suggests cutting the λ surface into two pieces around the unit circle (see Fig. 3). The gap between the two portions of the surface can then represent the gap between the forward cone of disturbance and the rearward cone of disturbance in the (x,y,z) space. If the outer portion of the λ surface is now mapped onto a second area bounded by a unit circle by means of points at inverse radii, the two circular areas can be related to the circular cross sections of the Mach cone at $x = \pm 1$. Each point on the interior of the unit circle $|\lambda| < 1$ then corresponds to a point $(y + iz)/x$, defined by a ray drawn through the "Mach circle" at $x = +1$. The negative ray that pierces the Mach circle at $x = -1$ then corresponds to a point ϵ_1 on the second surface, $1/|\lambda| < 1$. The center of each surface corresponds to the center of the Mach cone. When both the fore cone and rear cone are considered, it is found that each piece of the λ surface is covered twice, once by ϵ_1 and again by ϵ_2.

Now consider the evaluation of Eq. (19) when the contour is drawn just inside the unit circle, enclosing the surface $|\lambda| < 1$ in the positive direction. For positive values of x such that $x^2 > y^2 + z^2$, the contour will enclose ϵ_1 so that the value of the integral is

$$\frac{-1}{\pi} \frac{1}{(y - iz)\,(\epsilon_1 - \epsilon_2)} = \frac{1}{2\pi R} \tag{25}$$

For points outside the Mach cone, both ϵ_1 and ϵ_2 lie in the gap between the two portions of the λ surface and are outside the contour, so that the value of the integral is zero. Proceeding toward negative values of x, as soon as the point (x,y,z) reaches the upstream Mach cone, the pole ϵ_1 moves onto the second portion of the λ surface, while the pole ϵ_2 now appears inside the contour $|\lambda| < 1$. The value of the integral is now

$$-\frac{1}{\pi} \frac{1}{(y - iz)\,(\epsilon_2 - \epsilon_1)} = \frac{-1}{2\pi R} \tag{26}$$

Exactly the same determination of φ_s results from a contour in the negative direction enclosing the remainder of the surface, $1/|\gamma| < 1$. As later calculations will show, changes in the order of integration and differentiation are simplified if the contour C is extended around both portions of the λ surface and if the points $\lambda = 0$ and $1/\bar{\lambda} = 0$ (i.e., poles of β and γ) are excluded by small circles. The two parts of the contour C are shown in Fig. 3.

To describe the flow in the region around a lifting surface we need an expression for the potential field of a "horseshoe vortex" representing the disturbance caused by an element of lift at the origin. This expression may be obtained by the familiar process of integrating the expression for the source in the x direction and then differentiating in the z direction. These operations performed on the integrand of Eq. (19) yield the factor

$$-\gamma d\lambda / i\alpha\lambda = d\beta \tag{27}$$

so that the expression for the horseshoe vortex becomes

$$\varphi = \frac{-1}{8\pi^2} \oint \frac{d\beta}{\alpha x - \beta y - \gamma z} = \begin{cases} 0, \text{ unless } x^2 > y^2 + z^2 \\ \dfrac{+xz}{2\pi R(y^2 + z^2)} \text{ for } x > 0 \\ \dfrac{-xz}{2\pi R(y^2 + z^2)} \text{ for } x < 0 \end{cases} \tag{28}$$

The field is not that of a single horseshoe but of a closely spaced vortex pair extending to infinity in both directions along the x axis. Outside the cone $R = 0$, the disturbance is zero.

Combined Disturbance Field of a Lifting Surface

The combined disturbance field for an entire lifting surface s is obtained by superimposing elementary solutions of the form (28). This superposition amounts to a double integration of elementary horseshoe vortices over the surface, the strength of the vortices at each point being determined by the local lift $l(x_1,y_1)$. After introducing appropriate constants and changing the order of integration so that the contour integral is performed last, the expression for the combined potential $\bar{\varphi}$ of the lifting surface becomes

$$\bar{\varphi} = \frac{-1}{8\pi^2 \rho V} \oint \int_s \int \frac{l(x_1,y_1)\, dx_1\, dy_1}{\alpha(x - x_1) - \beta(y - y_1) - \gamma z}\, d\beta \tag{29}$$

It will be shown later that the double integral over the surface s in Eq. (29) yields a two-dimensional complex potential function, the form of this function depending on the parameter β. The final integration over β (or) λ then yields the three-dimensional disturbance. By analogy to well-known formulas in two-dimensional potential theory, the integration over the surface s may be defined in such a way as to permit differentiation under the integral. Performing this differentiation for the velocity components $\bar{u}$ and $\bar{w}$, we obtain from Eq. (29)

$$\frac{\bar{u}}{V} = \frac{1}{8\pi^2 \rho V^2} \oint \int_s \int \frac{l(x_1,y_1)\, dx_1\, dy_1}{[\alpha(x - x_1) - \beta(y - y_1) - \gamma z]^2} \alpha\, d\beta \tag{30}$$

$$\frac{\bar{w}}{V} = \frac{1}{8\pi^2 \rho V^2} \oint \int_s \int \frac{l(x_1,y_1)\, dx_1\, dy_1}{[\alpha(x - x_1) - \beta(y - y_1) - \gamma z]^2} \gamma\, d\beta \tag{31}$$

The component $\bar{u}/V$ must, of course, vanish at every point of the wing surface.

Relation to Two-Dimensional Flow Theory

It may now be shown that Eq. (31) represents the downwash in the three-dimensional flow by the superposition of the downwash of infinitely many two-dimensional flows. Each two-dimensional flow is associated with an oblique strip drawn in the plane of the wing and having its edges tangent to the outline of the wing plan form. It will appear that the downwash contributed by each strip is given by the familiar "lifting line" formula, the loading on each equivalent lifting line being obtained by an integration of the surface loading on the wing in an oblique direction (see Fig. 4).

For those parts of the contour consisting of the small circles around $\lambda = 0$ and $1/\bar{\lambda} = 0$, the equivalent loading is simply the spanwise loading. It will be evident that these parts of the contour yield the vortex drag of the lifting surface. Considering first the loop around $\lambda = 0$, we have

$$\beta \rightrightarrows 1/2\lambda, \qquad \gamma \rightrightarrows i/2\lambda \tag{32}$$

$$\alpha(x - x_1) - \beta(y - y_1) - \gamma z \rightrightarrows (1/2\lambda)\,[y_1 - (y + iz)] \tag{33}$$

since $\alpha(x - x_1)$ is negligible by comparison. Eq. (31) now becomes

$$\frac{\bar{w}}{V} = \frac{-1}{8\pi^2 \rho V^2} \oint \int_s \int \frac{l(x_1,y_1)\, dx_1\, dy_1}{[y_1 - (y + iz)]^2} \frac{i\, d\lambda}{\lambda} \tag{34}$$

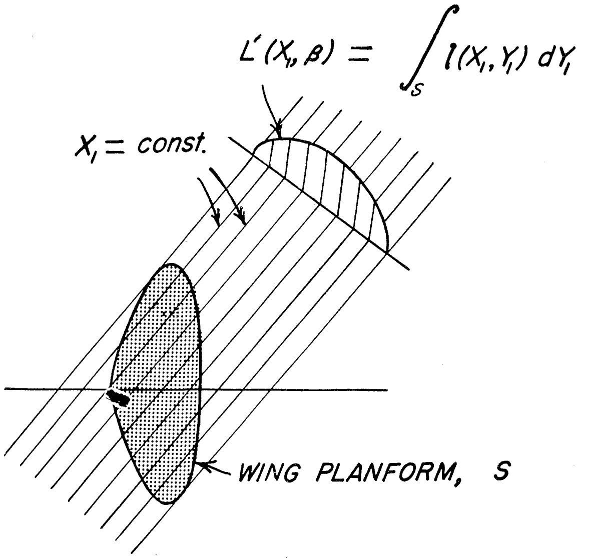

FIG. 4. Integration of surface loading along oblique lines X constant.

The integration along x_1 may now be performed directly and results in the spanwise loading, which may be denoted by

$$\int_s l(x_1,y_1)\, dx_1 = L'(y_1) \tag{35}$$

Now the quantity $l(x_1,y_1)$ represents the lifting pressure, and we may replace it by a quantity that takes equal and opposite values on the upper and lower sides of the wing surface. Similarly, the integrated quantity $L'(y_1)$ may be supposed to take equal and opposite values on the upper and lower sides of the strip, which, for $\lambda = 0$, coincides with vortex wake of the wing. In fact, if we write

$$L'(y_1)/\rho V = \Delta F(y_1) \tag{36}$$

it is clear that ΔF represents the discontinuity in the real part of the two-dimensional complex potential function associated with the trailing vortex wake of the wing.

Instead of integrating ΔF across the strip, we may integrate, using the values of $F = \pm(1/2)\Delta F$, in the positive direction on the upper side and continue in the negative direction on the lower side, forming a closed contour around the strip. Assuming that the function F is smooth and continuous at all points except the points of the strip and does not have a pole at infinity, the contour around the strip may be deformed so as to encircle the point $y + iz$. Then we have, by Cauchy's formula,

$$\frac{1}{\rho V} \int_s \int \frac{l(x_1,y_1)\, dx_1\, dy_1}{[y_1 - (y + iz)]^2} = \oint \frac{F(y_1)\, dy_1}{[y_1 - (y + iz)]^2} = 2\pi i F'(y + iz) \tag{37}$$

The quantity $F'(y + iz)$ is obviously the complex velocity function associated with the vortex wake. Because of the factor i, the real part represents the downwash and the imaginary part represents the lateral velocity v. However, by considering the small loop around the point $1/\bar{\lambda} = 0$ on the second portion of the λ surface, it is found that the quantity $2\pi i F'(y - iz)$ arises instead of Eq. (37). Hence, for the integration in the λ plane around both loops, the lateral velocity v, which is an odd function of z and discontinuous across the strip, vanishes, leaving a real value of the downwash w which is continuous throughout the field.

Returning to Eq. (34), we have, for the integration with respect to the parameter λ,

$$\oint i\, d\lambda/\lambda = 2\pi \tag{38}$$

The same value results for each of the small loops $\lambda \to 0$ and $1/\bar{\lambda} \to 0$. The downwash contributed by these portions of the contour then becomes

$$\frac{\bar{w}}{V} = \frac{-1}{2\pi\rho V^2}\,\text{R.P.} \int_s \frac{L'(y_1)\, dy_1}{[y_1 - (y + iz)]^2} \tag{39}$$

which is equivalent to Prandtl's formula for the downwash of the trailing vortex wake.

It will now be shown that those portions of the contour C near $|\lambda| = 1$ yield the wave drag of the lifting surface. To illustrate this relation, we replace $\alpha x - \beta y - \gamma z$ by the single variable

$$X = \alpha x - \beta y - \gamma z \tag{40}$$

and for the variable point on the wing we introduce

$$X_1 = \alpha x_1 - \beta y_1 \tag{41}$$

together with the orthogonal variable

$$Y_1 = (\beta x_1 + \alpha y_1)/(\alpha^2 + \beta^2) \tag{42}$$

The factor $\alpha^2 + \beta^2$ in the latter expression preserves the elementary area. Eq. (31) may now be written

$$\frac{\bar{w}}{V} = \frac{1}{8\pi^2\rho V^2} \oint \int_s\int \frac{l(X_1,Y_1)\, dX_1\, dY_1}{(X - X_1)^2}\,\gamma\, d\beta \tag{43}$$

and the integration with respect to Y_1 may be performed directly by writing

$$\int_s l(X_1,Y_1)\, dY_1 = L'(X_1,\beta) = \rho V \Delta F(X_1,\beta) \tag{44}$$

For $|\lambda| = 1$ the lines X_1 = constant correspond to the intercepts of a system of plane waves at 45° to the x axis ($M = \sqrt{2}$) and at the angle $\theta = \cos^{-1}\beta$ in the yz plane [see Eqs. (13)]. As the angle θ varies from 0 to 2π, the intersections of the plane waves with the lifting surface change their inclination between ±45°. For these various values of θ, Eq. (44) will correspond to an integration of the surface loading of the wing along various oblique directions, as illustrated in Fig. 4. As may be seen by introducing Eq. (44) in the surface integral in Eq. (43), the downwash contributed by each one of these integrated loadings is given by the familiar "lifting line" formula

$$2\pi i\, F'(X,\beta) = \oint \frac{F(X_1,\beta)}{(X - X_1)^2}\, dX_1 \tag{45}$$

In this form the surface integral in Eq. (31) can be recognized as the expression for the downwash arising from a given distribution of lift in two-dimensional flow. In particular, it is known that, if

$$L'(X_1,\beta) = \rho V \Delta F(X_1,\beta)$$

is an ellipse, the downwash, which corresponds to the imaginary part of $F'(X,\beta)$, will be a constant. Hence, if the lift is distributed over the wing in such a way that

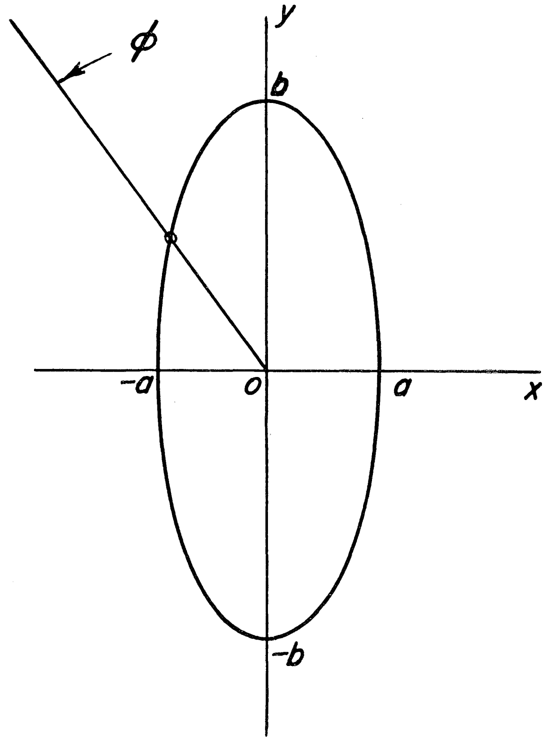

FIG. 5. Elliptic wing.

the integrated loading in every oblique direction between ±45° is elliptical, then the downwash contributed by the outer parts of the contour C (i.e., $|\lambda| \to 1$) will be constant over the entire plan form. If, in addition, the *spanwise* loading is elliptical, then the final integrated value of the downwash will be a constant.

Minimum Drag of Elliptic Wings at Supersonic Speed

The foregoing discussion indicated that the downwash will be constant over the plan form if the integrated loading in every oblique direction is elliptical. This condition is, of course, merely a sufficient, and not a necessary, one. The condition is met in the case of the elliptic plan form having a uniform surface distribution of lift. Hence it is concluded that such a uniform surface loading yields the minimum drag in the case of the elliptic wing.

The downwash of the elliptic wing may be calculated directly from Eq. (31). To evaluate the surface integral first, we write

$$\int_s\int \frac{l(x_1,y_1)\, dx_1\, dy_1}{[\alpha(x - x_1) - \beta(y - y_1) - \gamma z]^2} = 2\pi i \rho V F'(X,\beta) \tag{46}$$

After introducing $l = l_0$ and integrating over x_1, there is obtained

$$F'(X,\beta) = \frac{1}{2\pi i \rho V}\int_{-y_s}^{+y_s} \frac{1}{\alpha}\left(\frac{l_0\, dy_1}{X - \alpha x_1 + \beta y_1}\right)_{-x_s}^{+x_s} \tag{47}$$

The subscript s has been introduced to denote the limits of x_1,y_1 corresponding to the edges of the plan form.

For the ellipse with semiaxes a and b, we have, in parametric form (see Fig. 5),

$$y_1 = b \cos \phi; \quad x_s = -a \sin \phi \tag{48}$$

and the integral (47) becomes

$$F'(X,\beta) = \frac{l_0 b}{2\pi i \rho V \alpha} \int_0^{2\pi} \frac{\sin \phi \, d\phi}{X + b\beta \cos \phi + a\alpha \sin \phi} \tag{49}$$

This form is obviously similar to Eq. (18), the complex numbers X, $b\beta$, and $a\alpha$ taking the place of the real numbers x, y, and z in that equation.

The evaluation of integrals of this form has been given by Jacobi, and a complete discussion will be found in reference 10 (see also reference 9). There are two determinations depending on the location of the poles of the integrand, e_1 and e_2, where

$$\left.\begin{aligned} e_1 &= \frac{-X - \sqrt{X^2 - (b\beta)^2 - (a\alpha)^2}}{b\beta - ia\alpha} \\ e_2 &= \frac{-X + \sqrt{X^2 - (b\beta)^2 - (a\alpha)^2}}{b\beta - ia\alpha} \end{aligned}\right\} \tag{50}$$

For real values of $\beta = \cos \theta$, the equation

$$X^2 - (b\beta)^2 - (a\alpha)^2 = 0 \tag{51}$$

determines those values of X which correspond to planes tangent to the edge of the elliptic disc. For points (x,y) inside the elliptic disc,

$$(x^2/a^2) + (y^2/b^2) < 1 \tag{52}$$

and in this case we have the general formula

$$\int_0^{2\pi} \frac{\cos n\phi \, d\phi}{X + b\beta \cos \phi + a\alpha \sin \phi} = \int_0^{2\pi} \frac{i \sin n\phi \, d\phi}{X + b\beta \cos \phi + a\alpha \sin \phi} = \pi \frac{e_1^n - e_2^n}{\sqrt{X^2 - (b\beta)^2 - (a\alpha)^2}} \tag{53}$$

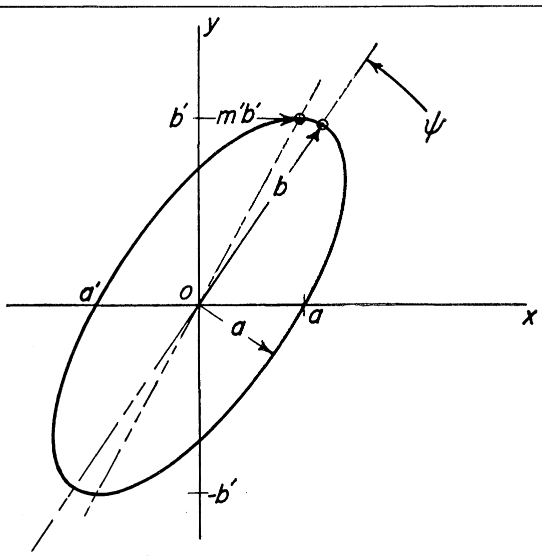

FIG. 6. Oblique ellipse.

Eq. (46) corresponds to $n = 1$,* and the formulas give

$$F'(X,\beta) = \frac{l_0 b}{\rho V \alpha} \frac{1}{b\beta - ia\alpha} \tag{54}$$

For the assumed constant surface loading $F'(X,\beta)$ is thus independent of X, and of x and y, over the surface of the ellipse. Hence the downwash will be constant at these points. The value of the downwash is obtained by introducing the value (54) for (46) in Eq. (31) after making use of the relations

$$\alpha = 1; \quad \gamma = \sqrt{1 - \beta^2} \tag{55}$$

Thus,

$$\frac{\bar{w}}{V} = \frac{-l_0 b}{4\pi i \rho V^2} \oint \frac{\sqrt{1 - \beta^2}}{b\beta - ia\alpha} d\beta = \frac{l_0}{\rho V^2} \sqrt{1 + \frac{a^2}{b^2}} \tag{56}$$

Since the value $\beta = i(a/b)$ corresponds to two distinct poles, one on each portion of the λ surface, the value of the integral is twice the residue at this point.

The drag is now given by

$$D_{min.} = (1/2)\,(\bar{w}/V)L \tag{57}$$

where L is the total lift equal to $l_0 S$ for a wing of area S. The formula for the minimum drag of the elliptic wing then becomes

$$D_{min.} = \frac{L^2}{4(\rho/2) V^2 S} \sqrt{1 + \frac{a^2}{b^2}} \tag{58}$$

This relation applies at $M = \sqrt{2}$. The variation of drag with Mach Number may be incorporated in Eq. (55) by applying the Prandtl-Glauert rule. The resulting formula in terms of the coefficients C_L and C_D is

$$C_{D_{min.}} = \sqrt{M^2 - 1}\, C_L{}^2 \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{\pi A'}\right)^2} \tag{59}$$

where $A' = \sqrt{M^2 - 1}(A)$ and A is the aspect ratio.

When the aspect ratio is large, Eq. (59) approaches

$$C_D = (\sqrt{M^2 - 1}/4) C_L{}^2 \tag{60}$$

which is the value given by the Ackeret theory for a flat wing in two-dimensional flow. On the other hand, when A' is small, the wave drag becomes negligible in comparison to the vortex drag, which is given by the well-known formula

$$C_D = C_L{}^2/\pi A \tag{61}$$

Elliptic Wing at an Angle of Yaw

Eq. (56) indicates that the drag of an elliptic wing of finite aspect ratio is greater than that given by the Ackeret theory (for infinite aspect ratio). Smaller values of the drag can be obtained, however, by placing the wing at an angle of yaw.

* It will be evident that the formulas for $n > 1$ provide extensions to cases of variable loading over the elliptic wing.

The treatment of the yawed wing follows the preceding analysis with only minor modifications. Again the minimum drag occurs when the lift is distributed uniformly over the ellipse. With the symbols defined as in Fig. 6, the equation of the yawed ellipse becomes

$$x_s = m'y_1 \pm (a'/b')\sqrt{b'^2 - y_1^2} \tag{62}$$

The poles of the integrand in the equation corresponding to Eq. (53) now appear, where

$$\beta = \alpha m' + i(a'/b') \tag{63}$$

and there are two distinct poles, one on each portion of the λ surface. The value of the integral (31) reduces to

$$\frac{\bar{w}}{V} = \frac{L}{\rho V^2 S}\,\text{R.P.}\sqrt{1 - \left(m' + i\frac{a'}{b'}\right)^2} \tag{64}$$

and we obtain

$$C_{D_{min.}} = \frac{C_L^2}{4}\,\text{R.P.}\sqrt{1 - \left(m' + i\frac{a'}{b'}\right)^2} \tag{65}$$

for the minimum drag of the yawed elliptic wing at $M = \sqrt{2}$.

The variation of drag coefficient with angle of yaw ψ is shown in Fig. 7 for ellipses of various proportions.

The limit $a'/b' \rightarrow 0$ corresponds to infinite aspect ratio, and in this case the expression for the drag coefficient reduces to

$$C_{D_{min.}} = (C_L^2/4)\,\text{R.P.}\sqrt{1 - m'^2} \tag{66}$$

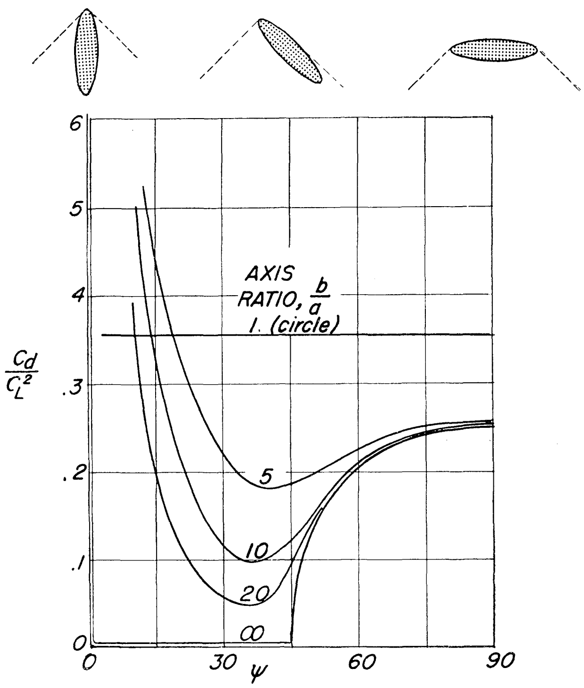

FIG. 7. Minimum drag of elliptic wings at various angles of yaw.

The change of $\sqrt{1 - m'^2}$ from a real number to an imaginary number as m' passes through 1 shows the disappearance of the wave drag when the wing of infinite aspect ratio is yawed behind the Mach cone.

Minimum Drag Due to Thickness

To represent the effect of a symmetrical distribution of thickness, we superimpose elementary solutions of the form (19), corresponding to sources, over the plan form. The expressions for the horizontal and vertical velocities in the combined disturbance field then become

$$\frac{\bar{u}}{V} = \frac{-1}{4\pi^2}\oint\int_s\int \frac{(1/2)t'(x_1,y_1)\,dx_1\,dy_1}{[\alpha(x - x_1) - \beta(y - y_1) - \gamma z]^2}\,\frac{i\,d\lambda}{\lambda} \tag{67}$$

$$\frac{\bar{w}}{V} = \frac{1}{4\pi^2}\oint\int_s\int \frac{(1/2)t'(x_1,y_1)\,dx_1\,dy_1}{[\alpha(x - x_1) - \beta(y - y_1) - \gamma z]^2}\,\frac{i\gamma\,d\lambda}{\lambda} \tag{68}$$

where $t'(x_1,y_1) = \partial t/\partial x_1$ and $(1/2)t$ denotes one-half the thickness of the airfoil.

In the case of a symmetrical distribution of thickness, the combined downwash $\bar{w}$ vanishes over the wing surface, while a value of $\bar{u}$ remains. The combined pressure distribution is given by the relation

$$\overline{\Delta p} = -\rho\bar{u}V \tag{69}$$

The evaluation of the integral (64) is again especially simple in the case of the elliptic plan form. If $t'(x_1,y_1)$ is assumed constant—corresponding to a constant source density over the surface—evaluation of Eq. (64) yields a constant value of $\bar{u}$. The drag therefore has the minimum value for a given frontal area. Since the sections are simple flat-sided wedges, the airfoil does not close at the trailing edge, and the figure is a semi-infinite body rather than a wing. Since the calculations are similar to those given for the lifting wing, they need not be repeated. The result is

$$C_{D_{min.}} = t'^2\,[1/\sqrt{1 + (a^2/b^2)}] \tag{70}$$

If the source intensity is assumed to vary linearly with x_1 so that $t''(x_1,y_1)$ is a constant, it is found that $\partial\bar{u}/\partial x$ is constant over the area of the wing. In this case the distribution of thickness yields the minimum drag consistent with a given volume—i.e.,

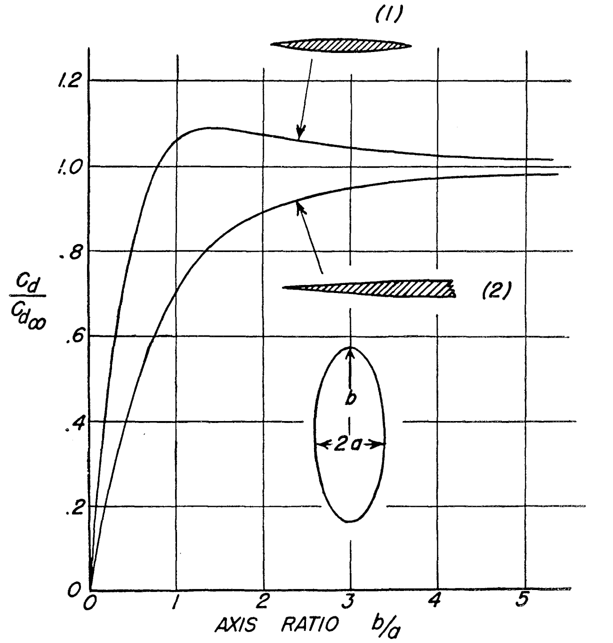

FIG. 8. Minimum drag of elliptic wings having (1) a given volume and (2) a given base area.

$$C_{D_{min.}} = \frac{t_0^2}{a^2}\left(\frac{1}{\sqrt{1 + (a^2/b^2)}}\right)\left(1 + 2\frac{a^2}{b^2}\right) \quad (71)$$

In this case the sections have a constant curvature in the stream direction. Eqs. (70) and (71) are plotted in Fig. 8.

In each case the area of the cross sections has the same distribution along x as that of the corresponding optimum body of revolution. It is interesting to note that each of these figures yields the minimum drag for all distributions of thickness within the space defined by the intersection of its forward and reversed characteristic envelopes.

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