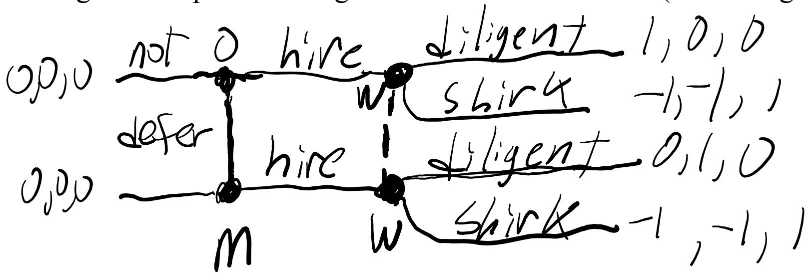
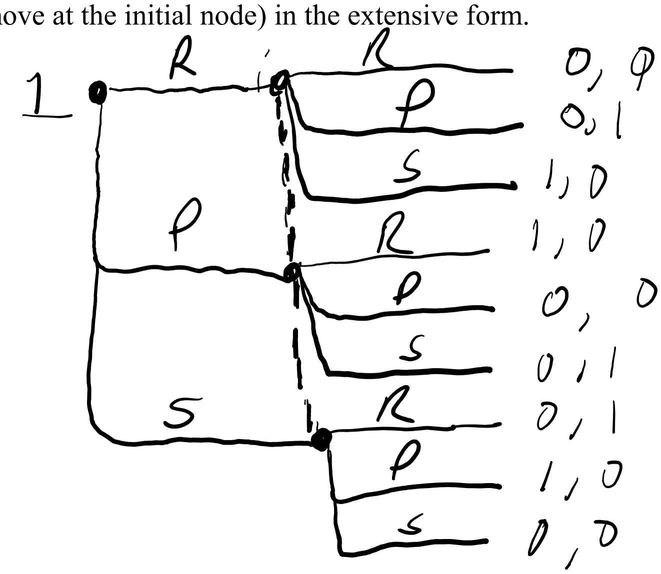
Game Theory Solutions – Ch 2 – Ch 27

2.2**.** Consider the following strategic situation concerning the owner of a firm (O), the manager of the firm (M), and a potential worker (W). The owner first decides whether to hire the worker, to refuse to hire the worker, or to let the manager make the decision. If the owner lets the manager make the decision, then the manager must choose between hiring the worker or not hiring the worker. If the worker is hired, then he or she chooses between working diligently and shirking. Assume that the worker does not know whether he or she was hired by the manager or the owner when he or she makes this decision. If the worker is not hired, then all three players get a payoff of 0. If the worker is hired and shirks, then the owner and manager each get a payoff of −1, whereas the worker gets 1. If the worker is hired by the owner and works diligently, then the owner gets a payoff of 1, the manager gets 0, and the worker gets 0. If the worker is hired by the manager and works diligently, then the owner gets 0, the manager gets 1, and the worker gets 1. Represent this game in the extensive form (draw the game tree).

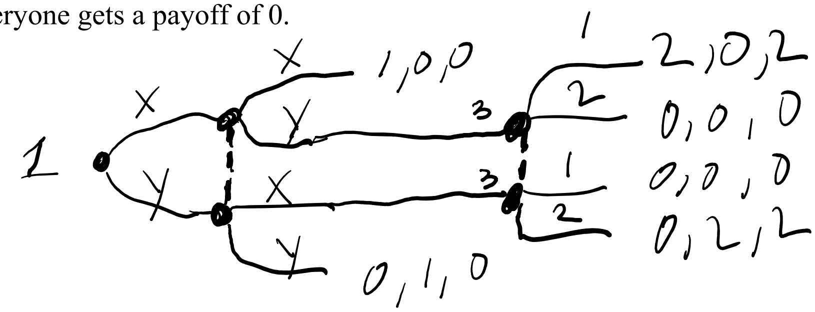


2.4. The following game is routinely played by youngsters—and adults as well—throughout the world. Two players simultaneously throw their right arms up and down to the count of "one, two, three." (Nothing strategic happens as they do this.) On the count of three, each player quickly forms his or her hand into the shape of either a rock, a piece of paper, or a pair of scissors. Abbreviate these shapes as R, P, and S, respectively. The players make this choice at the same time. If the players pick the same shape, then the game ends in a tie. Otherwise, one of the players wins and the other loses. The winner is determined by the following rule: rock beats scissors, scissors beats paper, and paper beats rock. Each player obtains a payoff of 1 if he or she wins, −1 if he or she loses, and 0 if he or she ties. Represent this game in the extensive form. Also discuss the relevance of the order of play (which of the players has the move at the initial node) in the extensive form.



If the game was sequential, the second mover would always win.

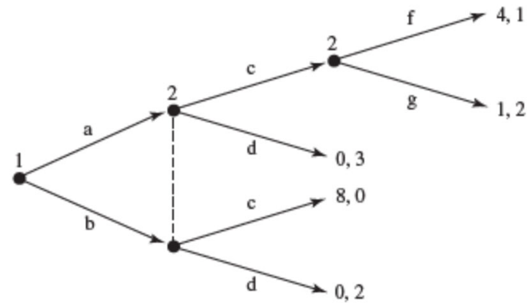
2.6 Represent the following game in the extensive form. There are three players, numbered 1, 2, and 3. At the beginning of the game, players 1 and 2 simultaneously make decisions, each choosing between "X" and "Y." If they both choose "X," then the game ends and the payoff vector is (1, 0, 0); that is, player 1 gets 1, player 2 gets 0, and player 3 gets 0. If they both choose "Y," then the game ends and the payoff vector is (0, 1, 0); that is, player 2 gets 1 and the other players get 0. If one player chooses "X" while the other chooses "Y," then player 3 must guess which of the players selected "X"; that is, player 3 must choose between "1" and "2." Player 3 makes his selection knowing only that the game did not end after the choices of players 1 and 2. If player 3 guesses correctly, then he and the player who selected "X" each obtains a payoff of 2, and the player who selected "Y" gets 0. If player 3 guesses incorrectly, then everyone gets a payoff of 0.



3.2 Suppose a manager and a worker interact as follows. The manager decides whether to hire or not hire the worker. If the manager does not hire the worker, then the game ends. When hired, the worker chooses to exert either high effort or low effort. On observing the worker's effort, the manager chooses to retain or fire the worker. In this game, does "not hire" describe a strategy for the manager? Explain.

No: A strategy must specify what to choose in all possible information sets, wheter or not they will be reached if that strategy is played with fidelity.

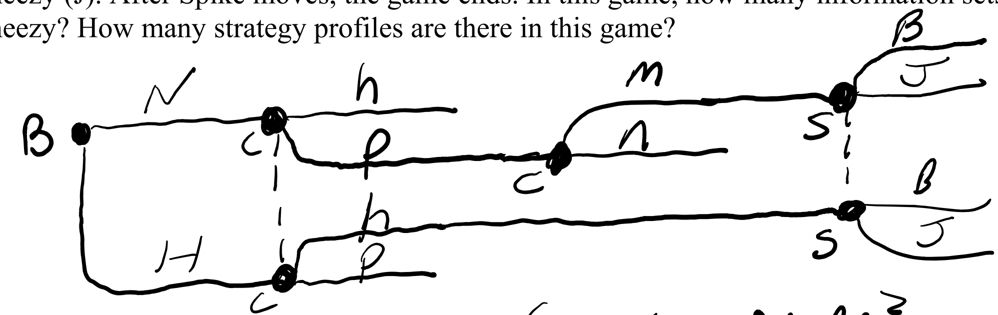
3.4 In the extensive-form game that follows, how many strategies does player 2 have?



cf, cg, df dg

Four.

3.8 Consider the following strategic setting involving a cat named Baker, a mouse named Cheezy, and a dog named Spike. Baker's objective is to catch Cheezy while avoiding Spike; Cheezy wants to tease Baker but avoid getting caught; Spike wants to rest and is unhappy when he is disturbed. In the morning, Baker and Cheezy simultaneously decide what activity to engage in. Baker can either nap (N) or hunt (H), where hunting involves moving Spike's bone. Cheezy can either hide (h) or play (p). If nap and hide are chosen, then the game ends. The game also will end immediately if hunt and play are chosen, in which case Baker captures Cheezy. On the other hand, if nap and play are chosen, then Cheezy observes that Baker is napping and must decide whether to move Spike's bone (m) or not (n). If he chooses to not move the bone, then the game ends. Finally, in the event that Spike's bone was moved (either by Baker choosing to hunt or by Cheezy moving it later), then Spike learns that his bone was moved but does not observe who moved it; in this contingency, Spike must choose whether to punish Baker (B) or punish Cheezy (J). After Spike moves, the game ends. In this game, how many information sets are there for Cheezy? How many strategy profiles are there in this game?



$\{N, H\} \times \{B, J\} \times \{hm, hn, pm, pn\}$

$2 \times 2 \times 4$

Cheesy has 2 information sets.
There are 16 strategy profiles.

4.2 Suppose we have a game where $S_1 = \{H, L\}$ and $S_2 = \{X, Y\}$. If player 1 plays H, then her payoff is z regardless of player 2's choice of strategy; player 1's other payoff numbers are $u_1(L, X) = 0$ and $u_1(L, Y) = 10$. You may choose any payoff numbers you like for player 2 because we will only be concerned with player l's payoff.

(a) Draw the normal form of this game.

1 \ 2	X	Y
H	Z,	Z,
L	0,	10,

(b) If player 1's belief is $\theta_2 = (1/2, 1/2)$, what is player 1's expected payoff of playing H? What is his expected payoff of playing L? For what value of z is player 1 indifferent between playing H and L?

$u_1(H|\theta_2) = Z \qquad u_1(L|\theta_2) = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 10 = 5$

Indifferent if $Z = 5$

(c) Suppose $\theta_2 = (1/3, 2/3)$. Find player l's expected payoff of playing L.

$u_1(L|\theta_2) = \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 10 = 20/3$

4.3 Evaluate the following payoffs for the game pictured here:

1 \ 2	I	O
OA	2, 2	2, 2
OB	2, 2	2, 2
IA	4, 2	1, 3
IB	3, 4	1, 3

(a) $u_1(\sigma_1, I)$ for $\sigma_1 = (1/4, 1/4, 1/4, 1/4)$

$\frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 4 + \frac{1}{4}\cdot 3 = 11/4$

(b) $u_2(\sigma_1, O)$ for $\sigma_1 = (1/8, 1/4, 1/4, 3/8)$

$\frac{1}{8}\cdot 2 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 3 + \frac{3}{8}\cdot 3 = 21/8$

(c) $u_1(\sigma_1, \sigma_2)$ for $\sigma_1 = (1/4, 1/4, 1/4, 1/4)$, $\sigma_2 = (1/3, 2/3)$

$\frac{1}{4}\left(\frac{1}{3}\cdot 2 + \frac{2}{3}\cdot 2\right) + \frac{1}{4}\left(\frac{1}{3}\cdot 2 + \frac{2}{3}\cdot 2\right) + \frac{1}{4}\left(\frac{1}{3}\cdot 4 + \frac{2}{3}\cdot 1\right) + \frac{1}{4}\left(\frac{1}{3}\cdot 3 + \frac{2}{3}\cdot 1\right)$
$= 23/12$

(d) $u_1(\sigma_1, \sigma_2)$ for $\sigma_1 = (0, 1/3, 1/6, 1/2)$, $\sigma_2 = (2/3, 1/3)$

$0 + \frac{1}{3}\left(\frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 2\right) + \frac{1}{6}\left(\frac{2}{3}\cdot 4 + \frac{1}{3}\cdot 1\right) + \frac{1}{2}\left(\frac{2}{3}\cdot 3 + \frac{1}{3}\cdot 1\right) = 7/3$

4.4. For each of the classic normal-form games (see Figure 3.4), find $u1(\sigma1, \sigma2)$ and $u2(\sigma1, s2)$ for $\sigma1 = (1/2, 1/2)$ and $\sigma2 = (1/2, 1/2)$.

Note: Only do Matching pennies, Battle of the Sexes, and Hawk Dove.

1 \ 2	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Matching Pennies

$u_1 = \frac{1}{4}(1-1+1-1) = 0$

$u_2 = \frac{1}{4}(-1+1-1+1) = 0$

1 \ 2	Opera	Movie
Opera	2, 1	0, 0
Movie	0, 0	1, 2

Battle of the Sexes

$u_1 = \frac{1}{4}(2+0+0+1) = 3/4$

$u_2 = \frac{1}{4}(1+0+0+2) = 3/4$

1 \ 2	H	D
H	0, 0	3, 1
D	1, 3	2, 2

Hawk-Dove/Chicken

$u_1 = \frac{1}{4}(0+3+1+2) = 3/2$

$u_2 = \frac{1}{4}(0+1+3+2) = 3/2$

6.2 For the game in Exercise 1 of Chapter 4, determine the following sets of best responses.

1 \ 2	L	C	R
U	10, 0	0, 10	3, 3
M	2, 10	10, 2	6, 4
D	3, 3	4, 6	6, 6

(a) $BR1(\theta2)$ for $\theta2 = (1/3, 1/3, 1/3)$

$U_1(U) = 13/3$ $\quad U_1(M) = 18/3$

$U_1(D) = 13/3$ $\quad BR(\theta_2)\ \{M\}$

(b) $BR2(\theta1)$ for $\theta1 = (0, 1/3, 2/3)$

$U_2(L) = 16/3$ $\quad U_2(C) = 14/3$ $\quad U_2(R) = 16/3$

$BR_2(\theta_1) = \{L, R\}$

(c) $BR1(\theta2)$ for $\theta2 = (5/9, 4/9, 0)$

$U_1(U) = 50/9$ $\quad U_1(M) = 50/9$ $\quad U_1(R) = 39/9$

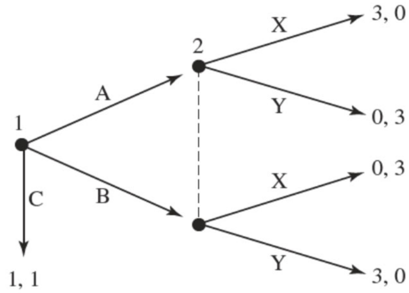
$BR_1(\theta_2) = \{U, M\}$

(d) $BR2(\theta1)$ for $\theta1 = (1/3, 1/6, 1/2)$

$U_2(L) = 19/6$ $\quad U_2(C) = 40/6$ $\quad U_2(R) = 28/6$

$BR_2(\theta_1) = \{C\}$

6.6 In the game pictured here, is it ever rational for player 1 to select strategy C? Why?



Let p denote player 1's belief about the probability of X.

$u_1(C,p) \geq u_1(A,p)$

$1 \geq 3p + 0(1-p)$

$1/3 \geq p$

$u_1(C,p) \geq u_1(B,p)$

$1 \geq 0p + 3(1-p)$

$1 \geq 3 - 3p$

$p \geq 2/3$

No, because p cannot satisfy both inequalities.

6.7 In the normal-form game pictured below, is player 1's strategy M dominated? If so, describe a strategy that dominates it. If not, describe a belief to which M is a best response.

1 \ 2	X	Y
K	9, 2	1, 0
L	1, 0	6, 1
M	3, 2	4, 2

$\sigma_1 = (1/3, 2/3, 0)$

$u_1(\sigma_1, X) = \frac{1}{3} \cdot 9 + \frac{2}{3} \cdot 1 = 11/3 > 3$

$u_1(\sigma_1, Y) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 6 = 13/3 > 4$

M is Dominated by $\sigma_1 = (1/3, 2/3, 0)$

7.2 Suppose that you manage a firm and are engaged in a dispute with one of your employees. The process of dispute resolution is modeled by the following game, where your employee chooses either to "settle" or to "be tough in negotiation," and you choose either to "hire an attorney" or to "give in." In the cells of the matrix, your payoff is listed second; x is a number that both you and the employee know. Under what conditions can you rationalize selection of "give in"? Explain what you must believe for this to be the case.

Employee \ You	Give in	Hire attorney
Settle	1, 2	0, 1
Be tough	3, 0	x, 1

$\theta = (p, 1-p)$ $\quad u_2(G) = 2p \quad u_2(A) = 1 \quad p \geq 1/2$ is required for the player 2 to give in.

x must not exceed 0 for this to be true, or "tough" dominates, and you hire the attorney.

7.3 Find the set of rationalizable strategies for the following game. Note that each player has more than one dominated strategy. Discuss why, in the iterative process of deleting dominated strategies, the order in which dominated strategies are deleted does not matter.

1 \ 2	a	b	c	d
w	5, 4	4, 4	4, 5	12, 2
x	3, 7	8, 7	5, 8	10, 6
y	2, 10	7, 6	4, 6	9, 5
z	4, 4	5, 9	4, 10	10, 9

x dominates y

$\frac{2}{3}w + \frac{1}{3}x$ dominates z

c dominates d

$\frac{9}{10}c + \frac{1}{10}a$ dominated b

$R^1 = \{w, x\} \times \{a, c\}$

c dominates a

$R^2 = \{w, x\} \times \{c\}$

x dominates w $\quad$ so $s = (x, c)$

In any given iteration, each player considers all strategies not eliminated in the last iteration. Then, in the next iteration, all strategies eliminated for either player are eliminated. Since all are eliminated between iterations, "order" has little meaning within any given round.

7.4. Imagine that there are three major network-affiliate television stations in Turlock, California: RBC, CBC, and MBC. All three stations have the option of airing the evening network news program live at 6:00 p.m. or in a delayed broadcast at 7:00 p.m. Each station's objective is to maximize its viewing audience in order to maximize its advertising revenue. The following normal-form representation describes the share of Turlock's total population that is "captured" by each station as a function of the times at which the news programs are aired. The stations make their choices simultaneously. The payoffs are listed according to the order RBC, CBC, MBC. Find the set of rationalizable strategies in this game.

MBC: 6:00

RBC \ CBC	6:00	7:00
6:00	14, 24, 32	8, 30, 27
7:00	30, 16, 24	13, 12, 50

MBC: 7:00

RBC \ CBC	6:00	7:00
6:00	16, 24, 30	30, 16, 24
7:00	30, 23, 14	14, 24, 32

6 dominates 7 for MBC

$R^1 = \{6,7\} \times \{6,7\} \times \{6\}$

7 dominates 6 for RBC

$R^2 = \{7\} \times \{6,7\} \times \{6\}$

6 dominates 7 for CBC

$R^3 = \{7\} \times \{6\} \times \{6\}$

$s = (7, 7, 6)$

7.6 Suppose that in some two-player game, s_1 is a rationalizable strategy for player 1. If, in addition, you know that s_1 is a best response to s_2, can you conclude that s_2 is a rationalizable strategy for player 2? Explain.

No. s_1 might be rationalizable because it is a best response to some other strategy, s_2', that is rationalizable, making s_1 rationalizable even if s_2 is not.

7.7. Consider a guessing game with ten players, numbered 1 through 10. Simultaneously and independently, the players select integers between 0 and 10. Thus player i's strategy space is $S_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, for $i = 1, 2, \ldots, 10$. The payoffs are determined as follows: First, the average of the players' selections is calculated and denoted a. That is, $a = (s_1 + s_2 + \ldots + s_{10})/10$ where s_i denotes player i's selection, for $i = 1, 2, \ldots, 10$. Then, player i's payoff is given by $u_i = (a - i - 1)s_i$. What is the set of rationalizable strategies for each player in this game?

– $a \le 10$, so $a - 10 - 1 \le -1$, so $s_{10} = 0$ dominates.

Now $a \le 9$, so $s_9 = 0$ dominates

Now $a \le 8$, so $s_8 = 0$ dominates

$\vdots$

Now $a \le 1$, so $s_1 = 0$ dominates

$s = (0, 0, \ldots, 0)$

8.2. Consider a location game with nine regions like the one discussed in this chapter. But instead of having the customers distributed uniformly across the nine regions, suppose that region 1 has a different number of customers than the other regions. Specifically, suppose that regions 2 though 9 each has ten customers, whereas region 1 has *x* customers. For what values of *x* does the strategy of locating in region 2 dominate locating in region 1?

		Player 2 Location								
		1	2	3	4	5	6	7	8	9
Player 1 Location	1	$\frac{X}{2}+40$	X	$X+5$	$X+10$	$X+15$	$X+20$	$X+25$	$X+30$	$X+35$
	2	80.	$\frac{X}{2}+40$	$X+10$	$X+15$	$X+20$	$X+25$	$X+30$	$X+25$	$X+20$

only player 1's payoffs shown

$$80 > \frac{X}{2} + 40 \qquad X/2 + 40 > X$$
$$40 > X/2 \qquad 40 > X/2$$
$$80 > X \qquad 80 > X$$

8.6. Consider a location game with five regions on the beach in which a vendor can locate. The regions are arranged on a straight line, as in the original game discussed in the text. Instead of there being two vendors, as with the original game, suppose there are *three* vendors who simultaneously and independently select on which of the five regions to locate. There are thirty consumers in each of the five regions; each consumer will walk to the nearest vendor and purchase a soda, generating a $1.00 profit for the vendor. Assume that if some consumers are the same distance from the two or three nearest vendors, then these consumers are split equally between these vendors.

(a) Can you rationalize the strategy of locating in region 1?

(b) If your answer to part (a) is "yes," describe a belief that makes locating at region 1 a best response. If your answer is "no," find a strategy that strictly dominates playing strategy 1.

A mix of 2 & 3, $\sigma_i = (0, p, 1-p, 0, 0)$ dominates 1. It's tedious to show. You must write Player i's payoffs for every possible choice of the opponents. Below I show player 3's payoffs.

2 & 3 each weakly dominate 1. So, the mix strictly dominates 1.

The ties for player 3 are highlighted below.

3 choose s = 1

1 \ 2	1	2	3	4	5
1	50	15	25	40	55
2	15	30	30	30	30
3	25	30	40	45	45
4	40	30	45	70	75
5	55	30	45	75	100

3 choose s = 2

1 \ 2	1	2	3	4	5
1	120	60	30	45	75
2	60	50	30	40	55
3	30	30	60	60	60
4	45	40	60	70	75
5	75	35	60	75	100

3 choose s = 3

1 \ 2	1	2	3	4	5
1	100	90	55	45	60
2	90	90	45	30	45
3	55	45	50	45	55
4	45	30	45	90	90
5	60	45	55	90	00

8.7. Consider a game in which, simultaneously, player 1 selects a number $x \in [2, 8]$ and player 2 selects a number $y \in [2, 8]$. The payoffs are given by:
$u_1(x, y) = 2xy - x^2$
$u_2(x, y) = 4xy - y^2$.
Calculate the rationalizable strategy profiles for this game.

$\partial u_1/\partial x = 2\bar{y} - 2x = 0 \rightarrow BR_1(\bar{y}) = \bar{y}$

$\partial u_2/\partial y = 4\bar{x} - 2y = 0 \rightarrow BR_2(\bar{x}) = \begin{cases} 2\bar{x} & \bar{x} \leq 4 \\ 8 & \bar{x} > 4 \end{cases}$

- Suppose $\bar{x} = 2$, then $y = 4$
- $\bar{y} = 4$, then $x = 4$
- $\bar{x} = 4$, $y = 8$
- $\bar{y} = 8$ $x = 8$

So $s = (8, 8)$ is the only rationalizable strategy profile.

8.8 Finish the analysis of the "social unrest" model by showing that for any $a > 2$, the only rationalizable strategy profile is for all players to protest. Here is a helpful general step: Suppose that is it common knowledge that all players above y will protest, so $x \geq 1 - y$. Find a cutoff player number $f(y)$ with the property that given $x \geq 1 - y$, every player above $f(y)$ strictly prefers to protest.

i protests if
$8x - 4 + \alpha i > 4x - 2$
$\alpha i > 2 - 4x$
Suppose $E(x) = \bar{x} = 0$
This becomes
$i > 2/\alpha$
If $\alpha > 2$, $2/\alpha < 1$
So, some will protest even if $\bar{x} = 0$.
Therefore $\bar{x} = 0$ is not rationalizable.

Suppose now $\bar{x} = 1 - 2/\alpha$, which is the least rationalizable belief in R^1.
Now i protests if
$\alpha i > 2 - 4(1 - 2/\alpha)$
$\alpha i > -2 + 8/\alpha$
$i > -2/\alpha + 8/\alpha^2$
The least rationalizable belief is now
$\bar{x} = 1 + 2/\alpha - 8/\alpha^2$
$= 1 - \frac{2}{\alpha}\left(\frac{4}{\alpha} - 1\right) > 1 - 2/\alpha$
This continues, with x growing each round until
$\alpha(1 - x) = 2 - 4x$
$\alpha = 2 + \alpha x - 4x$
$\alpha - 2 = x(\alpha - 4)$
$x = \frac{\alpha - 2}{\alpha - 4} > 1$. So, $x = 1$.

9.2 Find the Nash equilibria of and the set of rationalizable strategies for the games in Exercise 1 at the end of Chapter 6.

1 \ 2	L	R
A	3, 3	2, 0
B	4, 1	8, –1

(a)

1 \ 2	L	C	R
U	5, 9	0, 1	4, 3
M	3, 2	0, 9	1, 1
D	2, 8	0, 1	8, 4

(b)

1 \ 2	W	X	Y	Z
U	3, 6	4, 10	5, 0	0, 8
M	2, 6	3, 3	4, 10	1, 1
D	1, 5	2, 9	3, 0	4, 6

(c)

1 \ 2	L	R
U	1, 1	0, 0
D	0, 0	5, 5

(d)

a) B dominates A and L dominates R. So (B, L) is the only NE and the only rationalizable strategy profile.

b) L dominates R. Nothing else is dominated, so all else is rationalizable. With R eliminated, U weakly dominates M and D, so (U, L) is a reasonable prediction. Both (U L) and (M, C) are NE, but (M, C) is not a very sensible NE.

c) $\sigma_1 = (2/3, 0, 1/3)$ dominates M, X dominates Z

R^1 U dominates D, X dominates W.

R^2 X dominates Y. (U, X) is the only rationalizable strategy profile, and the only N.E.

d) Any profile is rationalizable. (U, L) & (D, R) are both N.E.

9.4. Compute the Nash equilibria of the following location game. There are two people who simultaneously select numbers between zero and one. Suppose player 1 chooses s_1 and player 2 chooses s_2. If $s_i < s_j$, then player i gets a payoff of $(s_i + s_j)/2$ and player j obtains $1 - (s_i + s_j)/2$, for $i = 1, 2$. If $s_1=s_2$, then both players get a payoff of 1/2.

If $s_1 < s_2$, u_1 is increasing in s_1, so $BR_1(s_2)$ is never less than s_2.

If $s_1 > s_2$, u_1 is decreasing in s_1, so $BR(s_2)$ is never more than s_2.

Consider $s_1 = s_2 < 1/2$, so that $u_1 = u_2 = 1/2$

$s_1 = s_2 + \varepsilon$ (ε is tiny) $\Rightarrow u_1 \approx 1 - \frac{2s_2}{2} > 1/2$

so $s_1 = s_2 < 1/2$ is not a NE

Consider $s_1 = s_2 > 1/2$, so that $u_1 = u_2 = 1/2$.

$s_1 = s_2 - \varepsilon \Rightarrow u_1 \approx 2s_2/2 > 1/2$.

So $s_1 = s_2 > 1/2$ is not a NE.

Consider $s_1 = s_2 = 1/2$, so that $u_1 = u_2 = 1/2$

Now, $s_1 = s_2 + \varepsilon \Rightarrow u_1 = 1 - \frac{1}{2} - \frac{\varepsilon}{2} < 1/2$

$s_1 = s_2 - \varepsilon \Rightarrow u_1 = 1/2 - \frac{\varepsilon}{2} < 1/2$

So $s_1 = s_2 = 1/2$ is the N.E.

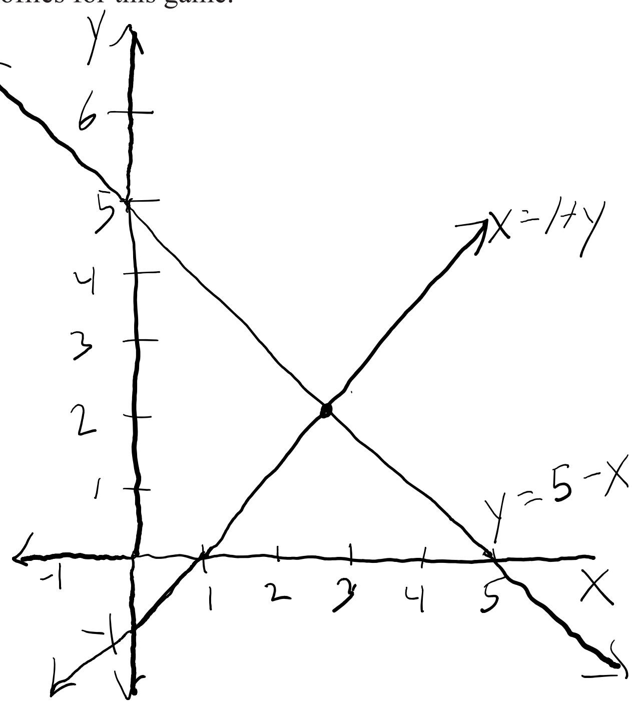
9.6. Consider a game in which, simultaneously, player 1 selects any real number x and player 2 selects any real number y. The payoffs are given by:
$u_1(x, y) = 2x - x^2 + 2xy$ $u_2(x, y) = 10y - 2xy - y^2$.

a. Calculate and graph each player's best-response function as a function of the opposing player's pure strategy.
b. Find and report the Nash equilibria of the game.
c. Determine the rationalizable strategy profiles for this game.

$$\frac{\partial u_1}{\partial x} = 2 - 2x + 2y = 0$$
$$x = 1 + y$$

$$\frac{\partial u_2}{\partial y} = 10 - 2y - 2x = 0$$
$$y = 5 - x$$

$x = 1 + (5 - x)$ $\quad$ $y = 5 - 3$
$2x = 6$ $\quad$ $y = 2$
$x = 3$



Note first there are no known bounds to start from. Setting that aside, suppose x_L is a known lower bound for x. Then y below $5 - x_L$ is eliminated. So, x above $1+5-x_L$ can be eliminated. Then y below $x_L - 1$ is eliminated. Then x below $1 + x_L - 1 = x_L$ is eliminated. This elimination did not alter the lower bound of player 1's rationalizable strategies.

9.10. Is the following statement true or false? If it is true, explain why. If it is false, provide a game that illustrates that it is false. "If a Nash equilibrium is not strict, then it is not efficient."

B

A	L	R
U	2, 2	0, 2
D	2, 0	1, 1

False. The profile (U,B) is a N.E. but not a strict N.E., and it is efficient.

9.11. This exercise asks you to consider what happens when players choose their actions by a simple rule of thumb instead of by reasoning. Suppose that two players play a specific finite simultaneous-move game many times. The first time the game is played, each player selects a pure strategy at random. If player *i* has m_i strategies, then she plays each strategy s_i with probability $1/m_i$. At all subsequent times at which the game is played, however, each player *i* plays a best response to the pure strategy actually chosen by the other player the *previous* time the game was played. If player *i* has *k* strategies that are best responses, then she randomizes among them, playing each strategy with probability $1/k$.

a. Suppose that the game being played is a prisoners' dilemma. Explain what will happen over time.

In the second round, each plays their dominant strategy, confess, because that is the only best response. This continues forever.

b. Next suppose that the game being played is the battle of the sexes. In the long run, as the game is played over and over, does play always settle down to a Nash equilibrium? Explain.

Not if one of the 2 NE profiles is not played the first time. They forever go where the other was last time. So if they were in different places the first time, they spend forever looking for eachother.

c. What if, by chance, the players happen to play a strict Nash equilibrium the first time they play the game? What will be played in the future? Explain how the assumption of a strict Nash equilibrium, rather than a nonstrict Nash equilibrium, makes a difference here.

The same profile. Because for each player, every other strategy s_i' yields a lower payoff against s_{-i}: $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \forall s_i' \neq s_i \in S_i$.
The strict inequality is what makes it so no one would randomly deviate to a different best response.

d. Suppose that, for the game being played, a particular strategy s_i is not rationalizable. Is it possible that this strategy would be played in the long run? Explain carefully.

No. Only strategies that are best responses to something will be selected. by any player.

9.15. Suppose you know the following for a particular three-player game: The space of strategy profiles S is finite. Also, for every $s \in S$, it is the case that $u_2(s)=3u_1(s)$, $u_3(s)=[u_1(s)]^2$, and $u_1(s)\in[0,1]$.

a. Must this game have a Nash equilibrium? Explain your answer.

b. Must this game have an efficient Nash equilibrium? Explain your answer.

c. Suppose that in addition to the information given above, you know that s^* is a Nash equilibrium of the game. Must s^* be an efficient strategy profile? Explain your answer; if you answer "no," then provide a counterexample.

a) Yes. Each player chooses what is best for player 1 given the choices of the other players. The profile that maximizes u_1 is therefore always a NE.

b) The equilibrium in (a) is efficient.

c) If there are only strict N.E., the one in (a) is efficient <u>and</u> the ony NE.

There may be inefficient non-strict NE.

	L	R
U	1, 3, 1	0, 0, 0
D	0, 0, 0	0, 0, 0

	L	R
U	0, 0, 0	0, 0, 0
D	0, 0, 0	½, 1, ¼

A   B

3

(D, R, B) is an inefficient N.E.

Chapter 10

1. Consider a more general Cournot model than the one presented in this chapter. Suppose there are n firms. The firms simultaneously and independently select quantities to bring to the market. Firm i 's quantity is denoted q_i, which is constrained to be greater than or equal to zero. All of the units of the good are sold, but the prevailing market price depends on the total quantity in the industry, which is $Q = \sum_i q_i$. Suppose the price is given by $p = a - bQ$ and suppose each firm produces with marginal cost c. There is no fixed cost for the firms. Assume $a > c > 0$ and $b > 0$. Note that firm i 's profit is given by $u_i = p(Q)q_i - cq_i = (a - bQ)q_i - cq_i$. Defining Q_{-i} as the sum of the quantities produced by all firms except firm i, we have $u_i = (a - bq_i - bQ_{-i})q_i - cq_i$. Each firm maximizes its own profit.

(a) Represent this game in the normal form by describing the strategy spaces and payoff functions.

(b) Find firm i 's best-response function as a function of Q_{-i}. Graph this function.

(c) Compute the Nash equilibrium of this game. Report the equilibrium quantities, price, and total output. (Hint: Summing the best-response functions over the different players will help.) What happens to the equilibrium price and the firm's profits as n becomes large?

(d) Show that for the Cournot duopoly game ($n = 2$), the set of rationalizable strategies coincides with the Nash equilibrium.

a) $u_i = (a - bq_i - bQ_{-i} - c)q_i$

$q_i \in \mathbb{R}^+ \ \forall i$

b) $\frac{\partial u_i}{\partial q_i} = a - bQ_{-i} - c - 2bq_i = 0$

$q_i = \frac{a-c}{2b} - \frac{Q_{-i}}{2}$

c) using symmetry

$q_i = \frac{a-c}{2b} - \frac{n-1}{2} q_i$

$\left(\frac{n+1}{2}\right) q_i = \frac{a-c}{2b}$

$q_i = \frac{a-c}{(n+1)b}$

$Q = \left(\frac{n}{n+1}\right)\left(\frac{a-c}{b}\right)$

$p = a - \cancel{b}\left(\frac{n}{n+1}\right)\frac{(a-c)}{\cancel{b}}$

$= \frac{(n+1-n)a}{n+1} + \frac{n}{n+1}c$

$= \frac{a-c}{n+1} + c$

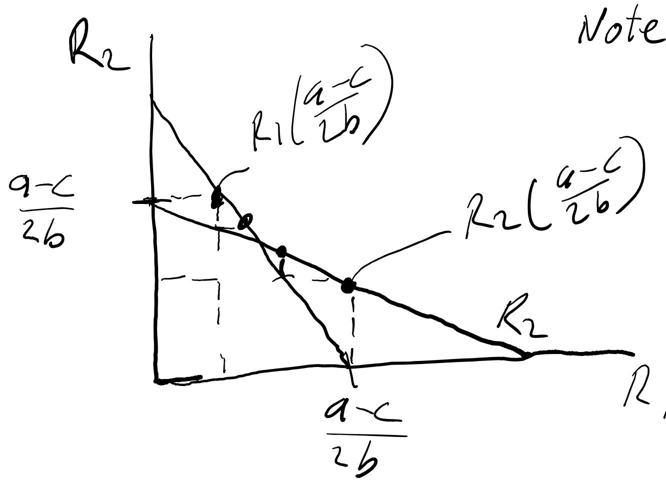
$\lim_{n\to\infty} p = c$

d) If $n=2$, $q_i = \frac{a-c}{3b}$ in the NE.

Rationalizability

Recall $R_i = \frac{a-c}{2b} - \frac{q_{-i}}{2}$

So, 0 and $\frac{a-c}{2b}$ serve as initial bounds.



Note $R_i\left(\frac{a-c}{2b}\right) > 0$, narrowing the rationalizable set at the first round.

Visually, this continues at each round, converging to the NE at the intersection of the Best Response functions.

4. Consider the game between a criminal and the government described in this chapter.
(a) Write the first-order conditions that define the players' best-response functions and solve them to find the best-response functions. Graph the best-response functions.
(b) Compute the Nash equilibrium of this game.
(c) Explain how the equilibrium levels of crime and enforcement change as *c* increases.

a) $u_G = -xc^4 - y^2x^{-1}$

$\frac{\partial u_G}{\partial x} = -c^4 + \left(y/x\right)^2 = 0$

$y/x = c^2$

$x = y/c^2$

$u_C = y^{1/2}(1+xy)^{-1}$

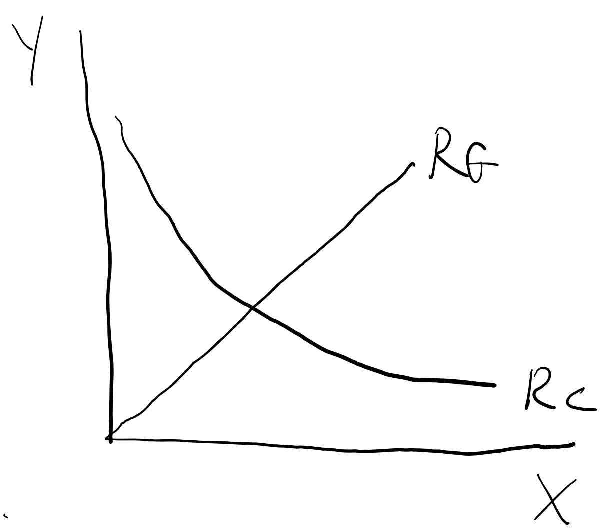
$\frac{\partial u_C}{\partial y} = \frac{1}{2}y^{-1/2}(1+xy)^{-1} - y^{1/2}(1+xy)^{-2}x = 0$

$\frac{1}{2}y^{-1/2}(1+xy)^{-1} = y^{1/2}(1+xy)^{-2}x$

$1+xy = 2xy$

$xy = 1$

$y = 1/x$



b) $x = \frac{1}{c^2 x}$

$x^2 = 1/c^2$

$x = 1/c$

$y = c$

c) $\frac{dx}{dc} = -1/c^2 < 0$

$\frac{dy}{dc} = 1 > 0$

5. In the years 2000 and 2001, the bubble burst for many Internet and computer firms. As they closed shop, some of the firms had to liquidate sizable assets, such as inventories of products. Suppose eToys is going out of business and the company seeks a buyer for a truckload of Elmo dolls in its warehouse. Imagine that eToys holds an auction on eBay to sell the dolls and that two retailers (players 1 and 2) will bid for them. The rules of the auction are as follows: the retailers simultaneously and independently submit sealed bids and then eToys gives the merchandise to the highest bidder, who must pay his bid. It is common knowledge that the retailer who obtains the load of dolls can resell the load for a total of \$15,000. Thus, if player i wins the auction with bid b_i, then player i 's payoff is \$15,000 − b_i. The losing retailer gets a payoff of \$0. If the retailers make the same bids ($b_1 = b_2$), then eToys declares each player the winner with probability 1/2, in which case player i obtains an expected payoff of (1/2)(15,000 − b_i). What will be the winning bid in the Nash equilibrium of this auction game? If you can, describe the equilibrium strategies and briefly explain why this is an equilibrium. (Hint: This is similar to the Bertrand game.)

ANSWER
The winning bid is \$15,000, which is bid by both players. For any expected bid by the opponent that is less than that, a, the best response is to bid slightly more, a+e where e us a tiny increment, so as to win and make a profit of 15000-a-e. More than 15000 is never bid. Any bid $b_i \leq 15000$ is a best response to 15000, so the best responses intersect at 15000.

6. Imagine that a zealous prosecutor (P) has accused a defendant (D) of committing a crime. Suppose that the trial involves evidence production by both parties and that by producing evidence, a litigant increases the probability of winning the trial. Specifically, suppose that the probability that the defendant wins is given by $e_D/(e_D + e_P)$, where e_D is the expenditure on evidence production by the defendant and e_P is the expenditure on evidence production by the prosecutor. Assume that e_D and e_P are greater than or equal to 0. The defendant must pay 8 if he is found guilty, whereas he pays 0 if he is found innocent. The prosecutor receives 8 if she wins and 0 if she loses the case.
(a) Represent this game in normal form.
(b) Write the first-order condition and derive the best-response function for each player.
(c) Find the Nash equilibrium of this game. What is the probability that the defendant wins in equilibrium?
(d) Is this outcome efficient? Why?

$e_D, e_P \in \mathbb{R}_+^2$

$U_D = \frac{-8e_P}{e_D+e_P} - e_D$ $\qquad$ $U_P = \frac{-8e_D}{e_D+e_P} - e_P$

$\frac{\partial U_D}{\partial e_D} = \frac{8e_P}{(e_D+e_P)^2} - 1 = 0$ $\qquad$ $\frac{\partial U_P}{\partial e_P} = \frac{8e_D}{(e_D+e_P)^2} - 1 = 0$

$8e_P = (e_D+e_P)^2$ $\qquad$ $8e_D = (e_D+e_P)^2$

$e_D = (8e_P)^{1/2} - e_P$ $\qquad$ $e_P = (8e_D)^{1/2} - e_D$

From the FOC, $e_P = e_D$

$e_D = (8e_D)^{1/2} - e_D \rightarrow 4e_D^2 = 8e_D \rightarrow e_D = 2 \rightarrow e_P = 2$

wins $) = 1/2$. This

PCD is not efficient because co-coord, for example, gives the same probability, but costs much less.

15. An island has two reefs that are suitable for fishing, and there are twenty fishers who simultaneously and independently choose at which of the two reefs (1 or 2) to fish. Each fisher can fish at only one reef. The total number of fish harvested at a single reef depends on the number of fishers who choose to fish there. The total catch is equally divided between the fishers at the reef. At reef 1, the total harvest is given by $f_1(r_1) = 8r_1 - r_1^2/2$, where r_1 is the number of fishers who select reef 1. For reef 2, the total catch is $f_2(r_2) = 4r_2$, where r_2 is the number of fishers who choose reef 2. Assume that each fisher wants to maximize the number of fish that he or she catches.

(a) Find the Nash equilibrium of this game. In equilibrium, what is the total number of fish caught?

(b) The chief of the island asks his economics advisor whether this arrangement is efficient (i.e., whether the equilibrium allocation of fishers among reefs maximizes the number of fish caught). What is the answer to the chief's question? What is the efficient number of fishers at each reef?

(c) The chief decides to require a fishing license for reef 1, which would require each fisher who fishes there to pay the chief x fish. Find the Nash equilibrium of the resulting location-choice game between the fishers. Is there a value of x such that the equilibrium choices of the fishers results in an efficient outcome? If so, what is this value of x?

a)
$$\frac{8r_1 - r_1^2/2}{r_1} = 4$$
$$8 - r_1/2 = 4$$
$$r_1/2 = 4$$
$$r_1 = 8$$
$$r_2 = 12$$
$$f_1 + f_2 = 64 - 64/2 + 48$$
$$= 32 + 48 = 80$$

b)
$$f_1 + f_2 = 8r_1 - r_1^2/2 + 4(20 - r_1)$$
$$= 80 + 4r_1 - r_1^2/2$$
$$\frac{\partial f_T}{\partial r_1} = 4 - r_1 = 0$$
$$\left.\begin{matrix} r_1 = 4 \\ r_2 = 16 \end{matrix}\right\} \text{efficient}$$

$r_1 = 8$ is <u>NOT</u> efficient.

c)
$$\frac{8r_1 - r_1^2/2}{r_1} - x = 4$$
$$8 - r_1/2 - x = 4$$
$$r_1 = 8 - 2x$$
$$8 - 2x = 4$$
$$2x = 4$$
$$x = 2$$

Chapter 11

3. Consider another version of the lobbying game introduced in this chapter. Suppose the payoffs are the same as presented earlier, except in the case in which firm X lobbies and firm Y does not lobby. In this case, suppose the government's decision yields x to firm X and zero to firm Y. Assume that $x > 25$. The normal form of this game is pictured here.

		q	
	X \ Y	L	N
p	L	−5, −5	x−15, 0
	N	0, 15	10, 10

(a) Designate the (pure-strategy) Nash equilibria of this game (if it has any).
(b) Compute the mixed-strategy Nash equilibrium of the game.
(c) Given the mixed-strategy equilibrium computed in part (b), what is the probability that the government makes a decision that favors firm X? (It is the probability that (L, N) occurs.)
(d) As x rises, does the probability that the government makes a decision favoring firm X rise or fall? Is this good from an economic standpoint?

$-5q + (1-q)(X-15) = 0\cdot q + 10(1-q)$

b) $X - 15 - qX + 10q = 10 - 10q$

$20q - Xq = 25$

$q = \frac{25-X}{20-X}$

$-5p + 15(1-p) = 0p + 10(1-p)$

$15 - 20p = 10 - 10p$

$10p = 5$

$p = 1/2$

$20 - X \quad -25 + X$

c) $P(L,N) = p\cdot(1-q)$

$= \frac{1}{2}\left(\frac{-5}{20-X}\right)$

$= \frac{5}{2X-40}$

d) In the mixed eq, it falls, but (L, N) is better the larger is X.

5. This exercise explores how, in a mixed-strategy equilibrium, players must put positive probability only on best responses. Consider the game in the following figure.

1 \ 2	L	M (m)	R (r)
U	x, x	$x, 0$	$x, 0$
(c) C	$0, x$	$2, 0$	$0, 2$
(d) D	$0, x$	$0, 2$	$2, 0$

Compute the pure-strategy and mixed-strategy Nash equilibria for this game, and note how they depend on x. In particular, what is the difference between $x > 1$ and $x < 1$?

If all three are played, Player 1's indifference requires:

$$x = 0(1-m-r) + 2m + 0r = 0(1-m-r) + 0m + 2r$$

$m = r = x/2$. Since this is symmetric, we have

$\sigma_1 = (1-x, x/2, x/2)$ and $\sigma_2 = (1-x, x/2, x/2)$

- If $x > 1$, U, L is the only equilibrium, since probabilities are between 0 and 1.
- If $x < 0$, U & L are dominated, so $\sigma_1 = (0, 1/2, 1/2)$ and $\sigma_2 = (0, 1/2, 1/2)$ is the only equilibrium.
- If $0 \le x \le 1$, (U, L) is a pure strategy equilibrium, and $\sigma_1 = (1-x, x/2, x/2)$ $\sigma_2 = (1-x, x/2, x/2)$ is a mixed strategy equilibrium.

6. Determine all of the Nash equilibria (pure-strategy and mixed-strategy equilibria) of the following games.

(d)

1 \ 2	A (q)	B ($1-q$)
A (p)	1, 4	2, 0
B ($1-p$)	0, 8	3, 9

$1q+2(1-q)=0q+3(1-q)$

$2-q=3-3q$

$4q=1$

$q=1/4$

$4p+8(1-p)=0p+9(1-p)$

$8-4p=9-9p$

$5p=1$

$p=1/5$

(e)

1 \ 2	A (q)	B ($1-q$)
A (p)	2, 2	0, 0
B ($1-p$)	0, 0	3, 4

$2q+0(1-q)=0q+3(1-q)$

$2q=3-3q$

$5q=3$

$q=3/5$

$2p+0(1-p)=0p+4(1-p)$

$2p=4-4p$

$6p=4$

$p=2/3$

f)

1 \ 2	L (0)	M (q)	R ($1-q$)
U (p)	8, 1	0, 2	4, 3
C ($1-p$)	3, 1	4, 4	0, 0
D (0)	5, 0	3, 3	1, 4

$4q+0(1-q)=3q+1(1-q)$

$4q=2q+1$

$q=1/2$

$2p+4(1-p)=4p+0(1-p)$

$4-2p=4p$

$6p=4$

$p=2/3$

8. Consider the following social problem. A pedestrian is hit by a car and lies injured on the road. There are *n* people in the vicinity of the accident. The injured pedestrian requires immediate medical attention, which will be forthcoming if at least one of the *n* people calls for help. Simultaneously and independently, each of the *n* bystanders decides whether or not to call for help (by dialing 911 on a cell phone or pay phone). Each bystander obtains v units of utility if someone (anyone) calls for help. Those who call for help pay a personal cost of *c*. That is, if person *i* calls for help, then he obtains the payoff $v - c$; if person *i* does not call but at least one other person calls, then person *i* gets v; finally, if none of the *n* people calls for help, then person *i* obtains zero. Assume $v > c$.

(a) Find the symmetric Nash equilibrium of this *n*-player normal-form game. (Hint: The equilibrium is in mixed strategies. In your analysis, let *p* be the probability that a person does not call for help.)

(b) Compute the probability that at least one person calls for help in equilibrium. (This is the probability that the injured pedestrian gets medical attention.) Note how this depends on *n*. Is this a perverse or intuitive result?

a) $V - C = (1 - p^{n-1})V$

$p^{n-1} = C/V$

$p = (C/V)^{\frac{1}{n-1}}$

b) $P(X > 0) = 1 - P(0)$

$= 1 - (C/V)^{\frac{n}{n-1}}$

Since $C/V < 1$, as n grows, $P(X > 0)$ falls as n increases. This is both perverse and intuitive. Perverse because there are so many who could help. Intuitive because if there is only 1 person, of course they call. But, if there are many people, everyone just counts on others to do it.

In the limit, $P(X > 0) \rightarrow 1 - C/V$.

13. Consider the following three-player team production problem. Simultaneously and independently, each player chooses between exerting effort (E) or not exerting effort (N). Exerting effort imposes a cost of 2 on the player who exerts effort. If two or more of the players exert effort, each player receives a benefit of 4 regardless of whether she herself exerted effort. Otherwise, each player receives zero benefit. The payoff to each player is her realized benefit less the cost of her effort (if she exerted effort). For instance, if player 1 selects N and players 2 and 3 both select E, then the payoff vector is (4, 2, 2). If player 1 selects E and players 2 and 3 both select N, then the payoff vector is (−2, 0, 0).

(a) Is there a pure-strategy equilibrium in which all three players exert effort? Explain why or why not.

(b) Find a symmetric mixed-strategy Nash equilibrium of this game. Let p denote the probability that an individual player selects N.

a) No. Any player could unilaterally improve from $u = 4-2$ to $u = 4-0$ by not exerting effort.

b) $u_i(E) = u_i(N)$

$4(1-p^2) + 0 \cdot p^2 - 2 = 4(1-p)^2 + 0(1-(1-p)^2)$

$2 - 4p^2 = 4 + 4p^2 - 8p$

$8p^2 - 8p + 2 = 0$

$4p(1-p) = 1$

$p = 1/2$

Chapter 12

1. Determine which of the following games are strictly competitive.

(a)

1 \ 2	X	Y	Z
A	2, 9	6, 5	7, 4
B	5, 6	8, 2	3, 8
C	9, 1	4, 7	7, 3

(b)

1 \ 2	X	Y	Z
A	8, 1	7, 2	3, 6
B	9, 0	2, 8	4, 5
C	7, 2	8, 1	6, 4

(c)

1 \ 2	X	Y
A	1, 1	2, 0
B	0, 2	1, 1

(d)

1 \ 2	X	Y
U	1, 1	2, 2
D	4, 4	3, 3

2. Find the players' security strategies for the games pictured in Exercise 1. In Red.

5. The guided exercise in this chapter demonstrates that not all security strategies are rationalizable. Find an example in which player 1's security strategy is dominated (in the first round of the rationalizability construction). For your example, what is the relation between player 1's security level and maxmin level?

1 \ 2	L	C	R
U	4, 4	0, 0	2, 1
M	0, 0	4, 4	2, 1
D	1, 2	1, 2	1, 1

D & R are the secure strategies. They are dominated by a mix of U & M and L & R. The maxmin is $\sigma_1 = (1/2, 1/2, 0)$ & $\sigma_2 = (1/2, 1/2, 0)$, which guarantee a payoff of 2 in expectation, while the secure strategy pays only 1.

Chapter 13

1. Consider a contractual setting in which the technology of the relationship is given by the following underlying game:

1 \ 2	I	N
I	5, 5	−1, 1
N	7, −1	0, 0

Suppose an external enforcer will compel transfer a from player 2 to player 1 if (N, I) is played, transfer b from player 2 to player 1 if (I, N) is played, and transfer g from player 2 to player 1 if (N, N) is played. The players wish to support the investment outcome (I, I).

(a) Suppose there is limited verifiability, so that $a = b = g$ is required. Assume that this number is set by the players' contract. Write the matrix representing the induced game and determine whether (I, I) can be enforced. Explain your answer.

(b) Suppose there is full verifiability, but that a, b, and g represent reliance damages imposed by the court. Write the matrix representing the induced game and determine whether (I, I) can be enforced. Explain your answer.

a)

1 \ 2	I	N
I	5, 5	$-1+\alpha, 1-\alpha$
N	$7+\alpha, -1-\alpha$	$\alpha, -\alpha$

$5 > 7+\alpha$ $\quad$ $5 > 1-\alpha$

$-2 > \alpha$ $\quad$ $\alpha > -4$

$-4 < \alpha < -2$ works

b)

	I	N
I	5, 5	0, 0
N	6, 0	0, 0

$5 < 6$, so, no.

2. Consider a contractual setting in which the technology of the relationship is given by the following partnership game:

1 \ 2	I	N
I	4, 4	–4, 9
N	2, –4	0, 0

Suppose the players contract in a setting of court-imposed breach remedies. The players can write a formal contract specifying the strategy profile they intend to play; the court observes their behavior in the underlying game and, if one or both of them cheated, imposes a breach transfer. The players wish to support the investment outcome (I, I).

(a) Write the matrix representing the induced game under the assumption that the court imposes *expectation damages.* Can a contract specifying (I, I) be enforced? Explain your answer.
(b) Write the matrix representing the induced game under the assumption that the court imposes *restitution damages.* Can a contract specifying (I, I) be enforced?
(c) Write the matrix representing the induced game under the assumption that the court imposes *reliance damages.* Can a contract specifying (I, I) be enforced with reliance transfers? Explain your answer.
(d) Suppose litigation is costly. When a contract is breached, each player has to pay a court fee of *c* in addition to the *reliance* transfer imposed by the court. What is the induced game in this case?
(e) Under what condition on *c* can (I, I) be enforced with reliance transfers and court costs?
(f) Continue to assume the setting of part (d). Suppose the court intervenes after a breach only if the plaintiff brings suit. For what values of *c* does the plaintiff have the incentive to sue?
(g) How does your answer to part (e) change if the court forces the losing party to pay all court costs?

a)

	I	N
I	4, 4	4, 1
N	−6, 4	0, 0

Yes, as a NE.

b)

	I	N
I	4, 4	5, 0
N	0, −2	0, 0

Yes, as a NE.

c)

	I	N
I	4, 4	0, 5
N	−2, 0	0, 0

No, not the NE.

f) same as e.

g) Need only $Z_1 > X_1 + Y_2$, $Z_2 > X_2 + Y_1$ which is more likely to hold.

d) Generally, the induced game is:

	I	N
I	Z_1, Z_2	$-C, X_2+Y_1-C$
N	$X_1+Y_2-C, -C$	0, 0

e) For (I, I) to be a NE:
$Z_1 > X_1 + Y_2 - C$
$Z_2 > X_2 + Y_1 - C$

12. Consider the following two-player team production problem. Each player i chooses a level of effort $a_i \geq 0$ at a personal cost of a_i^2. The players select their effort levels simultaneously and independently. Efforts a_1 and a_2 generate revenue of $r = 4(a_1 + a_2)$. There is limited verifiability in that the external enforcer (court) can verify only the revenue generated by the players, not the players' individual effort levels. Therefore, the players are limited to revenue-sharing contracts, which can be represented by two functions $f_1 : [0,\infty) \rightarrow [0, \infty)$ and $f_2 : [0, \infty) \rightarrow [0, \infty)$. For each player i, $f_i(r)$ is the monetary amount given to player i when the revenue is r. We require $f_1(r) + f_2(r) \leq r$ for every r.

Call a contract *balanced* if, for every revenue level r, it is the case that $f_1(r) + f_2(r) = r$. That is, the revenue is completely allocated between the players. A contract is *unbalanced* if $f_1(r) + f_2(r) < r$ for some value of r, which means that some of the revenue is destroyed or otherwise wasted.

(a) What are the efficient effort levels, which maximize the joint value $4(a1 + a2) - a_1^2 - a_2^2$?

(b) Suppose that the players have a revenue-sharing contract specifying that each player gets half of the revenue. That is, player i gets $(r/2) - a_i^2 = 2(a1 + a2) - a_i^2$. What is the Nash equilibrium of the effort-selection game?

(c) Next consider more general contracts. Can you find a balanced contract that would induce the efficient effort levels as a Nash equilibrium? If so, describe such a contract. If not, see if you can provide a proof of this result.

(d) Can you find an unbalanced contract that would induce the efficient effort levels as a Nash equilibrium? If so, describe such a contract. If not, provide a proof as best you can. [Consider the answer assuming i) the contract is enforced even if neither party brings suit alleging breach, and ii) the contract is only enforced if one or the other brings suit, otherwise the two evenly split the revenue.]

(e) Would the issues of balanced transfers matter if the court could verify the players' effort levels? Explain.

a) $\frac{\partial V}{\partial a_1} = 4 - 2a_1 = 0$

$a_1 = 2$

Same for a_2

b) $\frac{\partial u_1}{\partial a_1} = 2 - 2a_1 = 0$

$a_1 = 1$

Same for a_2

d) At the efficient effort levels, $r = 16$.

$f_i = \begin{cases} 8 & r = 16 \\ 0 & \text{otherwise} \end{cases}$

works. However, no one would bring suit to enforce that, since they would get 0.

e) No, because the contract would specify effort, not revenue.

c) Individual choice requires:

$\frac{\partial f_1}{\partial r} 4 = 2a_1, \quad \frac{\partial f_2}{\partial r} 4 = 2a_2$

Efficiency requires

$4 = 2a_1, \quad 4 = 2a_2$

Combining those requires

(1) $\frac{\partial f_1}{r} = 1$ and $\frac{\partial f_2}{\partial r} = 1$

Balance requires $f_1(r) + f_2(r) = r$ hence

(2) $\frac{\partial f_1}{\partial r} + \frac{\partial f_2}{\partial r} = 1.$

But, both (1) & (2) can't be true at the same time.

Chapter 14

6. Consider a variant of the game described in Exercise 4. Suppose that the firms move sequentially rather than simultaneously. First, firm 1 selects its quantity q_1, and this is observed by firm 2. Then, firm 2 selects its quantity q_2, and the payoffs are determined as in Exercise 4, so that firm i's payoff is $(12-q_i-q_j)q_i$. As noted in Exercise 6 of Chapter 3, this type of game is called the *Stackelberg duopoly model*. This exercise asks you to find some of the Nash equilibria of the game. Further analysis appears in Chapter 15.

Note that firm 1's strategy in this game is a single number q_1. Also note that firm 2's strategy can be expressed as a function that maps firm 1's quantity q_1 into firm 2's quantity q_2. That is, considering q_1, $q_2 \in [0, 12]$, we can write firm 2's strategy as a function $s_2 : [0, 12] \to [0, 12]$. After firm 1 selects a specific quantity q_1, firm 2 would select $q_2 = s_2(q_1)$.

(a) Draw the extensive form of this game.

(b) Consider the strategy profile (q_1, s_2), where $q_1 = 2$ and s_2 is defined as follows:

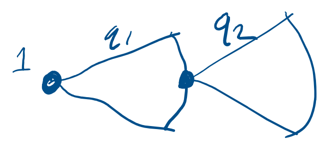
$$s_2(q_1) = \begin{cases} 5 & \text{if } q_1 = 2 \\ 12 - q_1 & \text{if } q_1 \neq 2 \end{cases}.$$

That is, firm 2 selects $q_2 = 5$ in the event that firm 1 chooses $q_1 = 2$; otherwise, firm 2 picks the quantity that drives the price to zero. Verify that these strategies form a Nash equilibrium of the game. Do this by describing the payoffs players would get from deviating.

(c) Show that for any $x \in [0, 12]$, there is a Nash equilibrium of the game in which $q_1 = x$ and $s_2(x) = (12-x)/2$. Describe the equilibrium strategy profile (fully describe s_2) and explain why it is an equilibrium.

(d) Are there any Nash equilibria (q_1, s_2) for which $s_2(q_1) \neq (12-q_1)/2$? Explain why or why not.

a)



b) $R_2 = (12-q_2)/2$

$R_2(2) = (12-2)/2 = 5$

$u_1(2) = (12-7)\cdot 2 = 10$

$u_2(q_2 \neq 2) = 0 \cdot q_2 = 0$

c) $s_2(q_1) = \begin{cases} 12-q_1 & q_1 \neq X \\ (12-X)/2 & q_1 = X \end{cases}$

$u_1(X) = (12-X)X > 0$

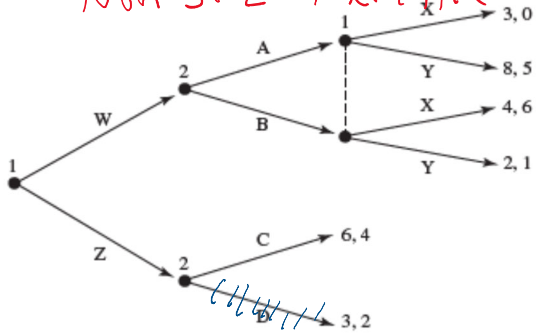
$u_1(q_1 \neq X) = 0q_1 = 0$

$R_2(X) = (12-X)/2$

d) No. As the last mover, firm 2 maximizes profit where $q_2 = (12-q_1)/2$, any deviation would reduce their profit.

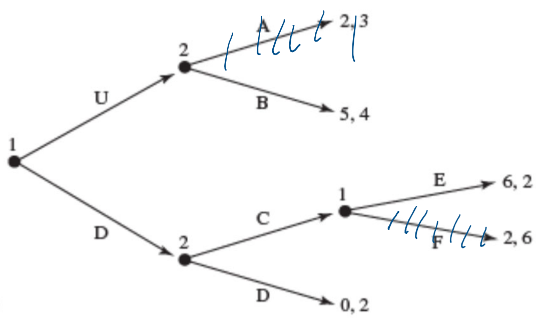
Chapter 15

2. Compute the Nash equilibria and subgame perfect equilibria for the following games. Do so by writing the normal-form matrices for each game and its subgames. Which Nash equilibria are not subgame perfect? Non SPE indicated with Red



(a)

	AC	AD	BC	BD
WX	3,0	3,0	4,6	4,6
WY	8,5	8,5	2,1	2,1
ZX	6,4	3,2	6,4	3,2
ZY	6,4	3,2	6,4	3,2

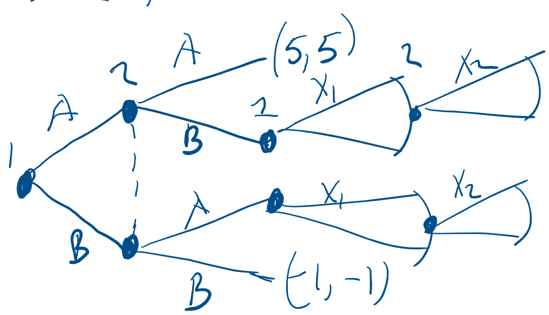


(b)

	AC	AD	BC	BD
UE	2,3	2,3	5,4	5,4
UF	2,3	2,3	5,4	5,4
DE	6,2	0,2	6,2	0,2
DF	2,6	0,2	2,6	0,2

8. Imagine a game in which players 1 and 2 simultaneously and independently select A or B. If they both select A, then the game ends and the payoff vector is (5, 5). If they both select B, then the game ends with the payoff vector (−1, −1). If one of the players chooses A while the other selects B, then the game continues and the players are required simultaneously and independently to select positive numbers. After these decisions, the game ends and each player receives the payoff (x1 + x2)/(1 + x1 + x2) , where x1 is the positive number chosen by player 1 and x2 is the positive number chosen by player 2.
(a) Describe the strategy spaces of the players.
(b) Compute the Nash equilibria of this game.
(c) Determine the subgame perfect equilibria.

a) $\{A, B\} \times \mathbb{R}_+ \times \mathbb{R}_+$



$\frac{x_1 + x_2}{1 + x_1 + x_2} = z_1 \quad 0 < z_1 < 1$

$\frac{x_1 + x_2}{1 + x_1 + x_2} = z_2 \quad 0 < z_2 < 1$

b) B is never part of a NE, because defecting against either A or B yields a higher payoff. ($A\ a_1$, $A\ a_2$) is a NE for any value of $a_i > 0$.

c) But, in the subgame, each wants to select the largest possible value of x_i, which is not bounded. So any ($A\ a_i$) with $a_i < \infty$ is not a SPE.

12. Suppose players 1 and 2 will play a prisoners' dilemma. Prior to interacting in the prisoners' dilemma, simultaneously each player i announces a binding penalty p_i that this player commits to pay the other player j in the event that player i defects and player j cooperates. Assume that these commitments are binding. Thus, after the announcements, the players effectively play the following induced game.

1 \ 2	C	D
C	5, 5	p_2, $8 - p_2$
D	$7 - p_1$, p_1	1, 1

(**a**) What values of p_1 and p_2 are needed to make (C, C) a Nash equilibrium of the induced game?
(**b**) What values of p_1 and p_2 will induce play of (C, C) and would arise in a subgame perfect equilibrium of the entire game (penalty announcements followed by the prisoners' dilemma)? Explain.
(**c**) Compare the unilateral commitments described here with contracts (as developed in Chapter 13).

a) $5 \geq 7 - p_1$ $\quad 5 \geq 8 - p_2$
$p_1 \geq 2$ $\quad p_2 \geq 3$

b) $p_1 = 2$, $p_2 = 3$, because $5 > 1$ and if a mistake were made, one would not want to pay more.

c) They function the same way, in that they support the jointly efficient NE. But, they are not written jointly to maximize the joint payoff, but to maximize individual utility.

Chapter 16

5. Consider a slight variation of the dynamic monopoly game analyzed in this chapter. Suppose there is only one high-type customer (Hal) and only one low-type customer (Laurie).

(a) Analyze this game and explain why $p_2 = 200$ is not optimal if Hal does not purchase a monitor in period 1. Find the optimal pricing scheme for Tony. Discuss whether Tony would gain from being able to commit to not selling monitors in period 2.

	Period 1	Period 2
Benefit to Hal	1200	300
Benefit to Laurie	500	200

(b) Finally, analyze the game with one of each type of customer and ownership benefits given in the following figure. In this case, would Tony gain from being able to commit to not selling monitors in period 2?

a) Because $500 > 200+200$, $P_2 = 500$ if Hal did not buy @ $t=1$.
Hal buys @ $t=1$ if $1700 - P_1 \geq 500 - 500$
$P_1 \leq 1700$, so $P_2 = 200$,
$\pi = 1700 + 200 = 1900$.

Tony would not gain from commitment.

b) Now if Hal does not buy @ $t=1$,
$300 < 200+200$, so $P_2 = 200$.
Hal Buys @ $t=1$ if $1500 - P_1 \geq 300 - 200$
$P_1 \leq 1400$.

$\pi = 1600$. Tony would gain if he could commit to $P_2 = P_1$.

16.X1. Endogenous Timing in Differentiated Bertrand Competition

Two firms engage in price competition. They sell somewhat differentiated products (think Nike and adidas), which are substitutes but not perfect substitutes. The demand for firm $i \in \{1,2\}$ is $q_i = 0.5 - p_i + 0.5p_{-i}$. For simplicity, assume per unit operating costs are 0, or equivalently, that price is measured in terms of its difference from cost.

Here is the intuition for the strategic market situation captured by these demand curves. If both firms set prices of 0, total sales are 1, so we are scaling units so 1 is the largest reasonable number of sales. An increase in a firm's price leads customers to purchase less of the firm's product—some drop out of the market and others switch brands. In this case half switch and half drop out.

a) Find the Nash Equilibrium prices, and associated payoffs, assuming prices are chosen simultaneously and independently. Draw the reaction functions and label the equilibrium.

b) Find the SPE prices, and associated payoffs, assuming firm 1 chooses price first in a sequential game. Label this outcome in the reaction function diagram, and, intuitively explain the nature of the difference form the simultaneous move game.

c) Assume the game spans four discrete time periods. In the first period each firm simultaneously and independently announces whether they will post their price in period 2 or period 3. Once this announcement is made the firms are committed to that timing, as they are to their prices once posted. In the fourth period, sales and payoffs are realized at the posted prices. Draw the extensive form of this game and find the pure strategy SPE. Is there an equilibrium where players strategies regarding when to announce prices are mixed? If so, what is it? Comment on anything interesting about the timing in this game and how it related to first and second mover advantages.

a) $u_i = (1/2 + 1/2\, p_{-i} - p_i)\, p_i$

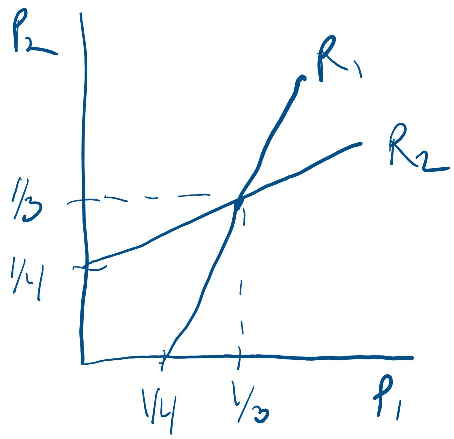
$\frac{\partial u_i}{\partial p_i} = 1/2 + 1/2\, p_{-i} - 2p_i = 0$

$p_i = (1 + p_{-i})/4$

$4p = 1 + p$

$3p = 1$

$p = 1/3$



$u_i = (1/2 - 1/6)\,1/3$

$= 1/9$

b) $u_1 = (1/2 + \frac{1}{2}(1 + p_1)/4 - p_1)\, p_1$

$= (1/2 + 1/8 + p_1/8 - p_1)\, p_1$

$= (5/8 - 7/8\, p_1)\, p_1$

$\frac{\partial u_1}{\partial p_1} = 5/8 - 14/8\, p_1 = 0$

$p_1 = 5/14 = 20/56$

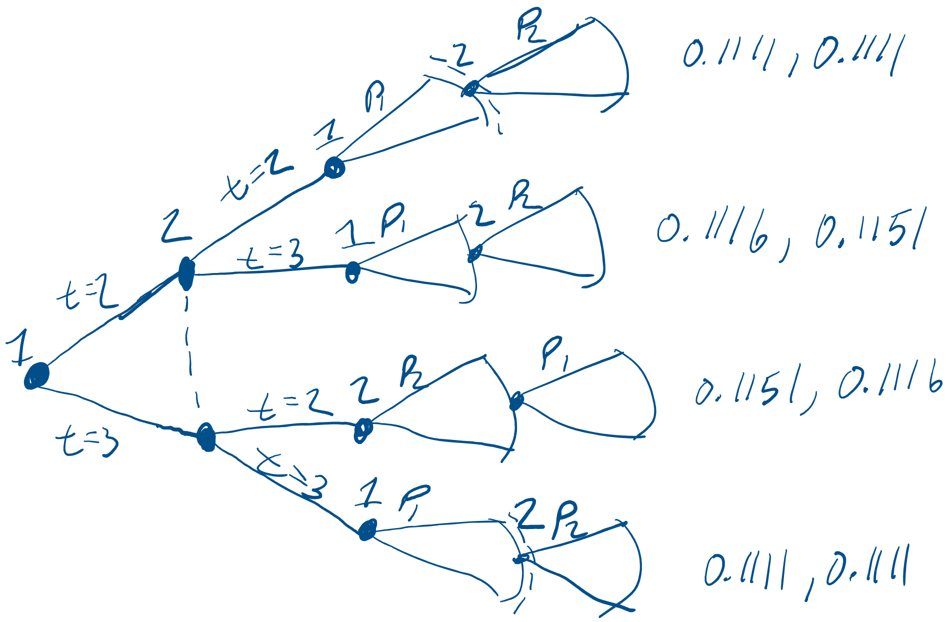
$p_2 = \frac{1}{4}\, 19/14 = 19/56$

$u_1 = (1/2 + 19/112 - 20/56)\, 19/56$

$= 0.1116$

$u_2 = (1/2 + 10/56 - 19/56)\, 20/56$

$u_2 = 0.1151$



Using continuation values to capture the pricing stage, the normal form can be represented as follows:

Player 1 Announces at \ Player 2 Announces at	t=2	t=3	
t=2	0.1111, 0.1111	0.1116, 0.1151	p
t=3	0.1151, 0.1116	0.1111, 0.1111	1-p
	q	1-q	

There is no NE in which the two announce at the same time.

$$p\,0.1111 + (1-p)\,0.1116 = p\,0.1151 + (1-p)\,0.1111$$
$$0.1116 - p\,0.0005 = 0.1111 + p\,0.004$$
$$0.0045p = 0.0005$$
$$p = 1/9$$

$$u_i = \frac{1}{81}\cdot 0.1111 + \frac{64}{81}\cdot 0.1111 + 8/81\cdot 0.1116 + 8/81\cdot 0.1151$$
$$= 0.1115$$

$$0.1111q + 0.1116(1-q) = 0.1151q + 0.1111(1-q)$$
$$0.1116 - 0.0005q = 0.1111 + 0.004q$$
$$0.0045q = 0.0005$$
$$q = 1/9$$

There is a symmetric mixed NE, but it is worse for both players than either of the two pure strategy NE.

Both want to move last, but it is better to move first than to move at the same time.

Chapter 22

2. Find conditions on the discount factor under which cooperation can be supported in the infinitely repeated games with the following stage games.

(a)

1 \ 2	C	D
C	2, 2	0, 4
D	4, 0	1, 1

(b)

1 \ 2	C	D
C	3, 4	0, 7
D	5, 0	1, 2

(c)

1 \ 2	C	D
C	3, 2	0, 1
D	7, 0	2, 1

Use the grim-trigger strategy profile.

a) $\frac{2}{1-\delta} \geq 4 + 1\frac{\delta}{1-\delta}$

$2 \geq 4 - 4\delta + \delta$

$3\delta \geq 2$

$\delta \geq 2/3$

b) Player 1

$\frac{3}{1-\delta} \geq 5 + 1\frac{\delta}{1-\delta}$

$3 \geq 5 - 5\delta + \delta$

$4\delta \geq 2$

$\delta \geq 1/2$

Player 2

$\frac{4}{1-\delta} \geq 7 + 2\frac{\delta}{1-\delta}$

$4 \geq 7 - 7\delta + 2\delta$

$5\delta \geq 3$

$\delta \geq 3/5$

c) Player 1

$\frac{3}{1-\delta} \geq 7 + 2\frac{\delta}{1-\delta}$

$3 \geq 7 - 7\delta + 2\delta$

$5\delta \geq 4$

$\delta \geq 4/5$

Player 2

$2 > 1$, so player 2 would never defect.

3. Consider the following stage game.

(a) Find and report all of the (pure-strategy) Nash equilibria of this game.

(b) Consider the two-period repeated game in which this stage game is played twice and the repeated-game payoffs are simply the sum of the payoffs in each of the two periods. Is there a subgame perfect equilibrium of this repeated game in which (A, X) is played in the first period? If so, fully describe the equilibrium. If not, explain why.

1 \ 2	X	Y
A	5, 6	0, 0
B	8, 2	2, 2

Is A B(X) B(Y) a BR to X X(A) Y(B)?

$5+8 > 8+2$, yes

$13 > 10$

Is X X(A) Y(B) a BR to A B(X) B(Y)?

$6+2 > 0+2$, yes.

$8 > 2$

(A B(X) B(Y), X X(A) Y(B)) is a SPE

4. If its stage game has exactly one Nash equilibrium, how many subgame perfect equilibria does a two-period, repeated game have? Explain. Would your answer change if there were *T* periods, where *T* is any finite integer?

One, because there is no credible threat of punishing defection by playing to lower payoff NE in the last stage. This is true regardless of T, cooperation unravels without a credible threat to punish defection.

10. Consider a repeated game between a supplier (player 1) and a buyer (player 2). These two parties interact over an infinite number of periods. In each period, player 1 chooses a quality level $q \in [0, 5]$ at cost q. Simultaneously, player 2 decides whether to purchase the good at a fixed price of 6. If player 2 purchases, then the stage-game payoffs are $6 - q$ for player 1 and $2q - 6$ for player 2. Here, player 2 is getting a benefit of $2q$. If player 2 does not purchase, then the stage-game payoffs are $-q$ for player 1 and 0 for player 2. Suppose that both players have discount factor d.

(a) Calculate the efficient quality level under the assumption that transfers are possible (so you should look at the sum of payoffs).

(b) For sufficiently large d, does this game have a subgame perfect Nash equilibrium that yields the efficient outcome in each period? If so, describe the equilibrium strategies and determine how large d must be for this equilibrium to exist.

(a) $u_1 + u_2 = 2q - q = q$, maxed @ $q = 5$.

(b) Note $q=0$, do not buy is the NE of the stage game, with payoffs 0, 0.

If $q = 5$, $u_1 = 1$, $u_2 = 4$. So:

Seller: $1 \frac{1}{1-\delta} > 6 + 0 \cdot \frac{\delta}{1-\delta}$

$1 > 6 - 6\delta$

$6\delta > 5$

$\delta > 5/6$

Buyer: $6\frac{1}{1-\delta} > 0 + 0 \cdot \frac{\delta}{1-\delta}$

Any δ works.

If $\delta > 5/6$, Buy at $t=1$ at t if $q_{t-k} = 5$ $\forall k$ else do not buy, and $q = 5$ @ $t=1$ & if the buyer has always purchased, else $q_t = 0$ is a NE.

11. This is an extension of the previous exercise. Consider the following stage game between a manager (also called the "Principal") and a worker (the "Agent"). Let the manager be player 1 and the worker be player 2. Simultaneously, the manager chooses a bonus payment $p \in [0, \infty)$ and the worker chooses an effort level $a \in [0, \infty)$. The stage-game payoffs are $u_1(p, a) = 4a - p$ and $u_2(p, a) = p - a^2$.

(a) Determine the efficient effort level for the worker.

(b) Find the Nash equilibrium of the stage game.

(c) Suppose the stage game is to be played twice (a two-period repeated game) and there is no discounting. Find all of the subgame perfect equilibria.

(d) Suppose the stage game is to be played infinitely many times in succession (an infinitely repeated game) and assume that the players share the discount factor $d < 1$. Find conditions on the discount factor under which there is a subgame perfect equilibrium featuring selection of the efficient effort level in each period (on the equilibrium path).

a) $u_1 + u_2 = 4a - a^2$

$\frac{\partial V}{\partial a} = 4 - 2a$

$a = 2$

b) $a = 0, \; P = 0$

c) $a = 0, \; P = 0$ both times

d) Principal

$(8 - P)\frac{1}{1-\delta} \geq 8 + 0\frac{\delta}{1-\delta}$

$8 - P \geq 8 - 8\delta$

$8\delta \geq P$

Agent

$(P - 4)\frac{1}{1-\delta} \geq P + 0\frac{\delta}{1-\delta}$

$P - 4 \geq P - P\delta$

$P\delta \geq 4$

$P \geq 4/\delta$

$8\delta \geq P \geq 4/\delta$

$\delta^2 \geq 1/2$

$\delta \geq \sqrt{1/2} \approx 0.7$

Chapter 23

1. Consider the Bertrand oligopoly model, where n firms simultaneously and independently select their prices, $p_1, p_2, \ldots, p_n$, in a market. (These prices are greater than or equal to 0.) Consumers observe these prices and only purchase from the firm (or firms) with the lowest price p, according to the demand curve $Q = 110 - p$, where $p = \min\{p_1, p_2, \ldots, p_n\}$. That is, the firm with the lowest price gets all of the sales. If the lowest price is offered by more than one firm, then these firms equally share the quantity demanded Q. Assume that firms must supply the quantities demanded of them and that production takes place at a constant cost of 10 per unit. (That is, the cost function for each firm is $c(q) = 10q$.) Determining the Nash equilibrium of this game was the subject of a previous exercise.
(a) Suppose that this game is infinitely repeated. (The firms play the game each period, for an infinite number of periods.) Define d as the discount factor for the firms. Imagine that the firms wish to sustain a collusive arrangement in which they all select the monopoly price $p_M = 60$ each period. What strategies might support this behavior in equilibrium? (Do not solve for conditions under which equilibrium occurs. Just explain what the strategies are. Remember, this requires specifying how the firms punish each other. Use the Nash equilibrium price as punishment.)
(b) Derive a condition on n and d that guarantees that collusion can be sustained.
(c) What does your answer to part (b) imply about the optimal size of cartels?

a) $P_{it} = 60$ @ $t = 1$

$P_{it} = 60$ @ $t > 1$ if $P_{i\,t-k} = 60 \ \forall \ 1 < k < t$

else $P_{it} = 10.$

b) $Q^c = (110 - 60) = 50$

$u_i^c = 50 \cdot 50/n = \frac{2500}{n}$

$P_i^D = 60 - \varepsilon \approx 60$

(ε is the tiniest price increment)

$u_i^D \approx 50 \cdot 50 = 2500$

$\frac{2500}{n} \frac{1}{1-\delta} \geq 2500 + 0 \frac{\delta}{1-\delta}$: For cooperate to be a NE

$1 \geq n - n\delta$

$n\delta \geq n - 1$

$\delta \geq 1 - \frac{1}{n}$

c) Smaller is better. If n is very large, cooperation is not possible.

6. This exercise addresses the notion of goodwill between generations. Consider an "overlapping generations" environment, whereby an infinite-period game is played by successive generations of persons. Specifically, imagine a family comprising an infinite sequence of players, each of whom lives for two periods. At the beginning of period t, player t is born; he is young in period t, he is old in period $t + 1$, and he dies at the end of period $t + 1$. Thus, in any given period t, both player t and player $t - 1$ are alive. Assume that the game starts in period 1 with an old player 0 and a young player 1.

When each player is young, he starts the period with one unit of wealth. An old player begins the period with no wealth. Wealth cannot be saved across periods, but each player t can, while young, give a share of his wealth to the next older player $t - 1$. Thus, consumption of the old players depends on gifts from the young players. Player t consumes whatever he does not give to player $t - 1$. Let x_t denote the share of wealth that player t gives to player $t - 1$. Player t's payoff is given by $(1 - x_t) + 2x_{t+1}$, meaning that players prefer to consume when they are old.

(a) Show that there is a subgame perfect equilibrium in which each player consumes all of his wealth when he is young; that is, $x_t = 0$ for each t.

(b) Show that there is a subgame perfect equilibrium in which the young give all of their wealth to the old. In this equilibrium, how is a player punished if he deviates?

(c) Compare the payoffs of the equilibria in parts (a) and (b).

a) Suppose player K gives $a_K > 0$ to player $K-1$.

$u_K = (1-a) + 2(0) = 1-a$

Suppose player K gives 0 to player $K-1$.

$u_K = 1 > 1-a$. This is a NE.

b) Strategies

$x_t = 1$ if $x_{t-1} = 1$

$x_t = 0$ if $x_{t-1} < 1$

Utility if coop: $u_t = 0+2$

Utility if defect and give $a < 1$:

$u_t = a + 0 < 1$.

$2 > a$,

So this is an equilibrium.

c) $2 > 1$.

8. Consider the prisoners' dilemma stage game pictured here:
The following questions ask you to consider various discrete-time environments in which people meet and play the stage game. For each environment, you are to determine the extent to which cooperation can be sustained in a subgame perfect equilibrium.

Row player \ Column player	C	D
C	3, 3	0, 5
D	5, 0	2, 2

(a) Suppose that two players interact in an infinitely repeated game, with the stage game shown, and that the players share the discount factor *d* ∈ (0, 1). Under what conditions on *d* is there a subgame perfect equilibrium that supports both players selecting C each period?

$$3\frac{1}{1-\delta} \geq 5 + 2\frac{\delta}{1-\delta}$$

$$3 \geq 5 - 5\delta + 2\delta$$

$$3\delta \geq 2$$

$$\delta \geq 2/3$$

(b) Suppose that there is a society of *k* individuals. In each period, two people from the society are randomly selected to play the stage game. Individuals are equally likely to be selected. Furthermore, the random draw that determines who is selected in a given period is independent of the outcomes in previous periods. Thus, an individual has a probability $2/k$ of being selected to play the stage game in any given period. One of the people selected plays the role of Row player in the stage game, whereas the other person plays the role of Column player. (The stage game is symmetric, so the roles do not matter.) Those who are selected obtain the payoffs of the stage game, whereas the individuals who were not selected get 0 for this period. Everyone discounts the future using discount factor *d*.

Assume that the individuals can identify each other. That is, individuals know the names and faces of everyone else in the society.

Assume also that before the stage game is played in a given period, everyone observes who is selected to play. Further, everyone observes the outcome of each period's stage game. Is there a sense in which it is more difficult to sustain cooperation in this random-matching setting than was the case in part (a)? Explain why or why not. Calculate the cutoff value of *d* under which cooperation can be sustained.

$$3 + \frac{2\delta}{k(1-\delta)}3 \geq 5 + \frac{2\delta}{k(1-\delta)}2$$

$$2\delta \geq 2k - 2k\delta$$

$$\delta(1+k) \geq k$$

$$\delta \geq \frac{k}{1+k}$$

It is more difficult because there will be less to lose in the future since an individual plays only 2/K of future plays.

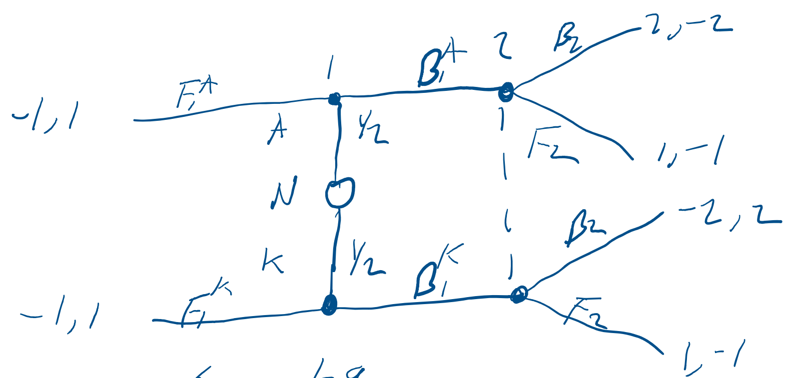
(c) Consider the same setting as in part (b) *except assume that the individuals only observe their own history of play*. As before, individuals can identify each other. However, an individual only knows when he is selected to play the stage game; if he is not selected to play in a given period, he observes neither who was selected to play nor the outcome of the stage game. When an individual is selected to play the stage game, he learns whom he is matched with; then the stage game is played, and the two matched individuals observe the outcome. Explain why cooperation is more difficult to sustain in this setting than in the setting of part (b). If you can, calculate the cutoff value of d under which cooperation can be sustained.

Cooperation is much harder to sustain because you can only be punished if you are matched with someone you played against before.

I suggest not wirrying about the calculation for this one.

Chapter 24

1. Here is a description of the simplest poker game. There are two players and only two cards in the deck, an Ace (A) and a King (K). First, the deck is shuffled and one of the two cards is dealt to player 1. That is, nature chooses the card for player 1. It is the Ace with probability 1/2 and the King with probability 1/2. *Player 2 does not receive a card*. Player 1 observes his card and then chooses whether to bid (B) or fold (F). If he folds, then the game ends with player 1 getting a payoff of −1 and player 2 getting a payoff of 1 (that is, player 1 loses his ante to player 2). If player 1 bids, then player 2 must decide whether to bid or fold. When player 2 makes this decision, she knows that player 1 bid, but she has not observed player 1's card. The game ends after player 2's action. If player 2 folds, then the payoff vector is (1, −1), meaning player 1 gets 1 and player 2 gets −1. If player 2 bids, then the payoff depends on player 1's card; if player 1 holds the Ace, then the payoff vector is (2, −2); if player 1 holds the King, then the payoff vector is (−2, 2). Represent this game in the extensive form and in the Bayesian normal form.



		q	1-q
		B	F
p	$B_A B_K$	0, 0	1, -1
1-p	$B_A F_K$	0.5, -0.5	0, 0
0	$F_A B_K$	-1.5, 1.5	0, 0
0	$F_A F_B$	-1, 1	-1, 1

In the BNE, both mix

$0p - 0.5(1-p) = -1p + 0(1-p)$

$-0.5 + 0.5p = -p$

$1.5p = 0.5$

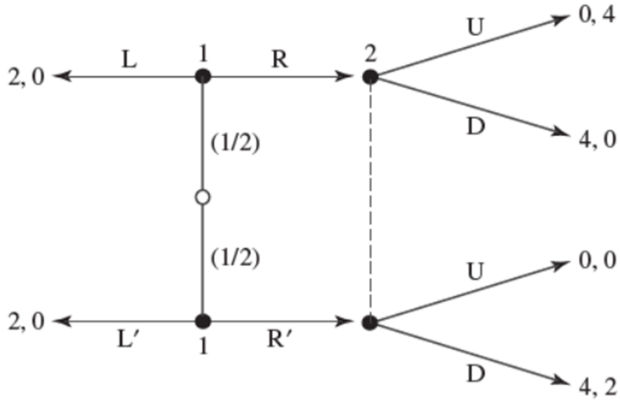
$p = 1/3$

$0q + 1(1-q) = 0.5q + 0(1-q)$

$1 = 1.5q$

$q = 2/3$

3. Represent the following game in the Bayesian normal form.



	U	D
RR′	0, 2	4, 1
RL′	1, 2	3, 0
LR′	1, 0	3, 1
LL′	2, 0	2, 0

Chapter 26

5. Consider a differentiated duopoly market in which firms compete by selecting prices and produce to fill orders. Let p_1 be the price chosen by firm 1 and let p_2 be the price of firm 2. Let q_1 and q_2 denote the quantities demanded (and produced) by the two firms. Suppose that the demand for firm 1 is given by $q_1=22-2p_1+p_2$, and the demand for firm 2 is given by $q_2=22-2p_2+p_1$. Firm 1 produces at a constant marginal cost of 10 and no fixed cost. Firm 2 produces at a constant marginal cost of c and no fixed cost. The payoffs are the firms' individual profits.
(a) The firms' strategies are their prices. Represent the normal form by writing the firms' payoff functions.
(b) Calculate the firms' best-response functions.
(c) Suppose that $c=10$ so the firms are identical (the game is symmetric). Calculate the Nash equilibrium prices.
(d) Now suppose that firm 1 does not know firm 2's marginal cost c. With probability 1/2 nature picks $c=14$, and with probability 1/2 nature picks $c=6$. Firm 2 knows its own cost (that is, it observes nature's move), but firm 1 only knows that firm 2's marginal cost is either 6 or 14 (with equal probabilities). Calculate the best-response functions of player 1 and the two types ($c=6$ and $c=14$) of player 2 and calculate the Bayesian Nash equilibrium quantities.

(a) $u_1 = (p_1-10)(22-2p_1+p_2)$ $\quad$ $u_2 = (p_2-c)(22-2p_2+p_1)$

(b) $\frac{\partial u_1}{\partial p_1} = 22-2p_1+p_2-2p_1+20=0$ $\quad$ $\frac{\partial u_2}{\partial p_2} = 22-2p_2+p_1-2p_2+2c=0$

$42+p_2 = 4p_1$ $\quad$ $22+2c+p_1 = 4p_2$

$p_1 = 10.5 + p_2/4$ $\quad$ $p_2 = 5.5 + c/2 + p_1/4$

(c) $p_1 = 10.5 + p_1/4$

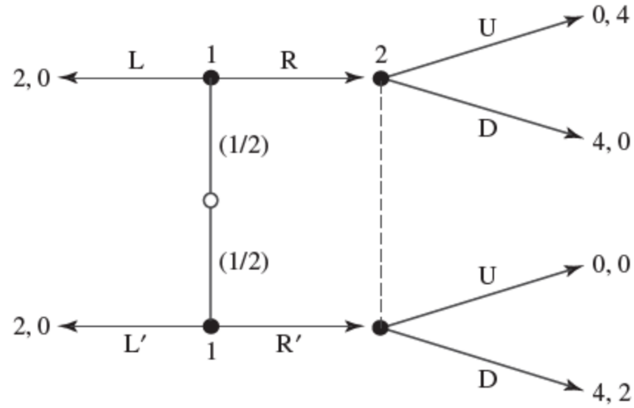
$(3/4)p_1 = 10.5$

$p_1 = 42/3 = 14 = p_2$

(d) $u_1 = (p_1-10)(22-2p_1-\frac{1}{2}p_{2,6}-\frac{1}{2}p_{2,14})$

$$\left.\begin{aligned} p_1 &= 10.5 + p_{2,6}/8 + p_{2,14}/8 \\ p_{2,6} &= 8.5 + p_1/4 \\ p_{2,14} &= 12.5 + p_1/4 \end{aligned}\right\} \text{Solving} \rightarrow \begin{aligned} p_1 &= 14 \\ p_{2,6} &= 12 \\ p_{2,14} &= 16 \end{aligned}$$

6. Find the Bayesian Nash equilibrium of the game pictured here. Note that Exercise 3 of Chapter 24 asked you to convert this into the normal form.



The NE is (LL′; U).
See ex 24.3

9. Consider the simple poker game described in Exercise 1 of Chapter 24, where there are just two cards in the deck and one card is dealt to player 1. This game has a single Nash equilibrium (perhaps in mixed strategies). Calculate and report the equilibrium strategy profile. Explain whether bluffing occurs in equilibrium.

See ex. 24.1. The equilibrium is in mixed strategies. Player 1 sometimes bids with a King (one third of the time).

Chapter 27

1. Regarding the trade game played by Jerry and Freddie that was analyzed in this chapter, are there values of p such that no equilibrium exists? Are there values of p such that the equilibrium entails *no* trade whatsoever?

- There is always an equilibrium. The question is only what, if anything, is traded, and if the equilibrium is in pure or mixed strategies.
- If $p < 2000$, Freddie would never trade.
- If $p > 1000 + 2000q$, Jerry would never trade

4. Suppose that a person (the "seller") wishes to sell a single desk. Ten people are interested in buying the desk: Ann, Bill, Colin, Dave, Ellen, Frank, Gale, Hal, Irwin, and Jim. Each of the potential buyers would derive some utility from owning the desk, and this utility is measured in dollar terms by the buyer's "value." The valuations of the 10 potential buyers are shown in the following table. Each bidder knows his or her own valuation of owning the desk. Using the appropriate concepts of rationality, answer these questions:

Ann	Bill	Colin	Dave	Ellen	Frank	Gale	Hal	Irwin	Jim
45	53	92	61	26	78	82	70	65	56

(a) If the seller holds a second-price, sealed-bid auction, who will win the auction and how much will this person pay?
(b) Suppose that the bidders' valuations are common knowledge among them. That is, it is common knowledge that each bidder knows the valuations of all of the other bidders. Suppose that the seller does not observe the bidders' valuations directly and knows only that they are all between 0 and 100. If the seller holds a first-price, sealed-bid auction, who will win the desk and how much will he or she have to pay? (Think about the Nash equilibrium in the bidding game. The analysis of this game is a bit different from the [more complicated] analysis of the first-price auction in this chapter because here the bidders know one another's valuations.)
(c) Now suppose that the seller knows that the buyers' valuations are 45, 53, 92, 61, 26, 78, 82, 70, 65, and 56, but the seller does not know exactly which buyer has which valuation. The buyers know their own valuations but not one another's valuations. Suppose that the seller runs the following auction: She first announces a *reserve price* $\underline{p}$. Then simultaneously and independently the players select their bids; if a player bids below $\underline{p}$, then this player is disqualified from the auction and therefore cannot win. The highest bidder wins the desk and has to pay the amount of his or her bid. This is called a "sealed-bid, first price auction with a reserve price." What is the optimal reserve price $\underline{p}$ for the seller to announce? Who will win the auction? And what will the winning bid be?

a) Colin wins and pays \$82.
b) Colin wins, and pays 82 (or 83 to ensure he wins).
c) $\underline{p} = 92$, Colin wins and pays 92.