Matrices and Matrix Operations


Matrices

Definition: A matrix is a rectangular array of numbers. The numbers in the array are called the entries of the matrix.
The size of a matrix M is written in terms of the number of its **rows** and the number of its **columns**. A 2x3 matrix has 2 rows and 3 columns

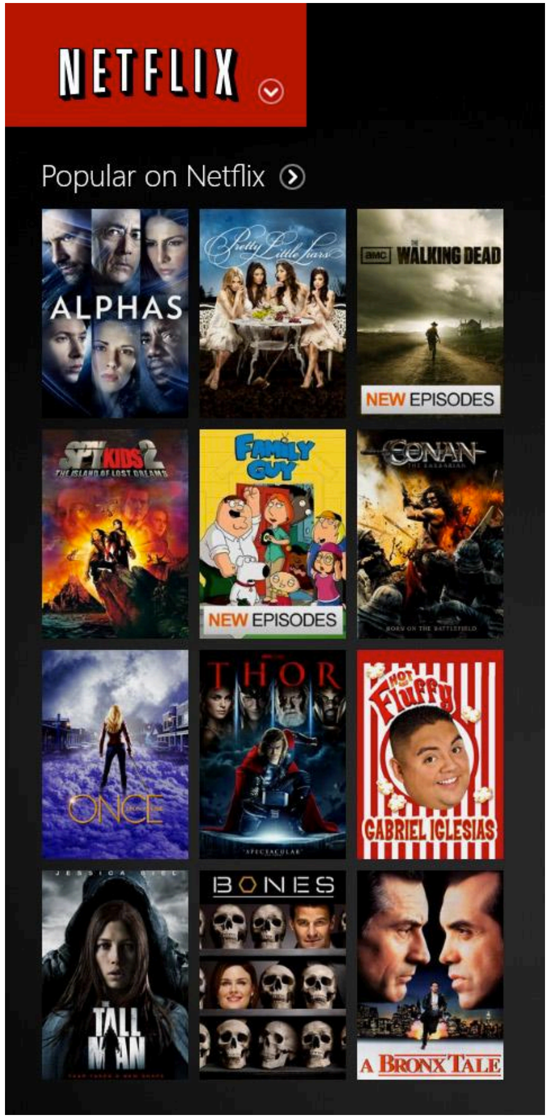
$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

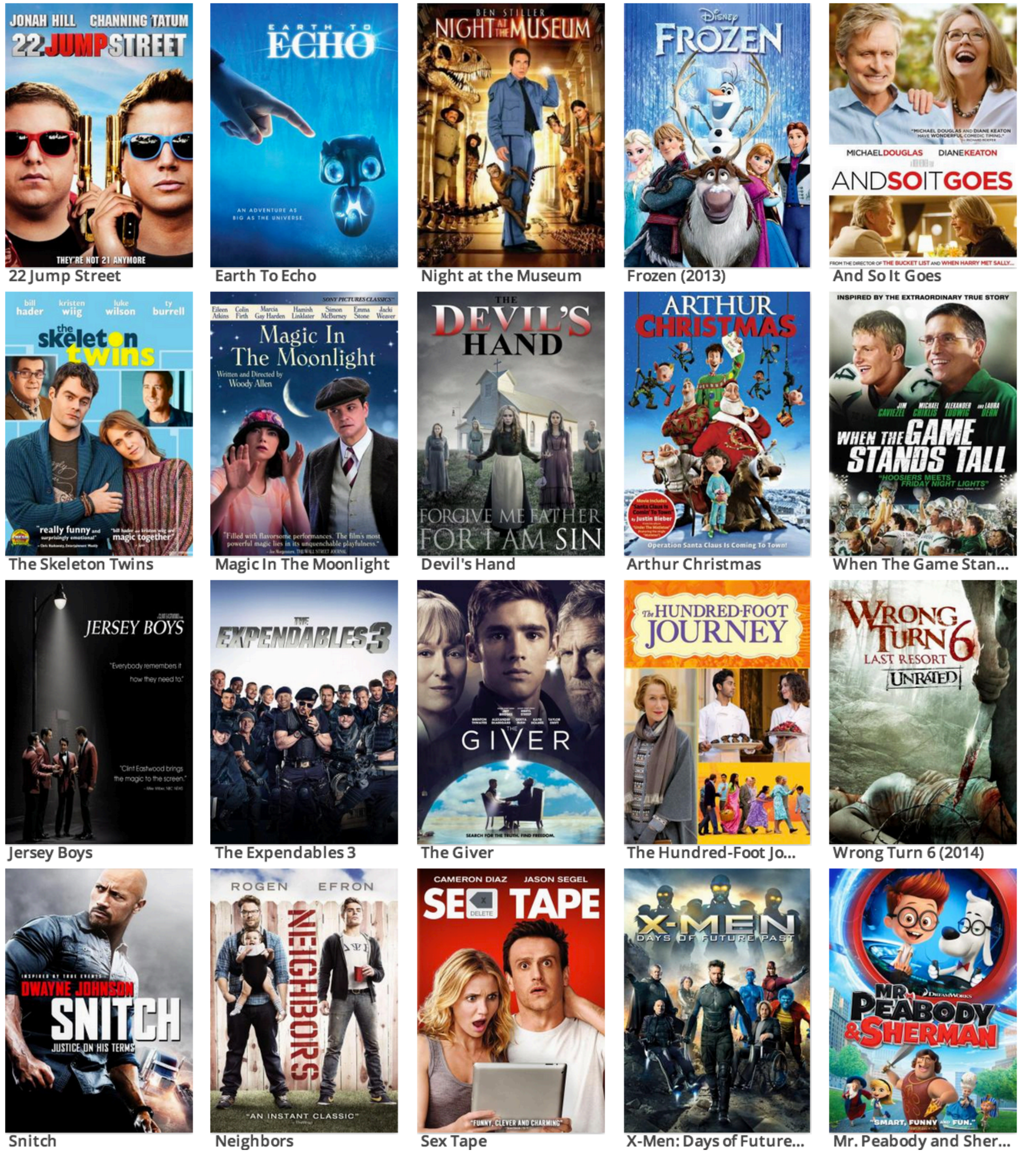
A matrix with only one row is called a row vector, and a matrix with only one column is called a column vector

The entry that occurs in row *i* and column *j* of a matrix A, will be denoted by a_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrices in the wild





Matrices Operations

Let A and B be two matrices of the same dimension.

- A + B: **add** the corresponding entries of A and B
- A – B: **subtract** the corresponding entries of B from those of A
- cA (**scalar multiplication**): multiply each entry of A by the constant c

A matrix A with *n* number of rows and *n* number of columns is said to be a **square matrix** of order *n*

The shaded entries a_{11}, a_{22}, … , a_{nn} are said to be in the **main diagonal** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The **zero matrix** is a matrix whose entries are all equal to 0.

Simple example

The A-Plus auto parts store chain has two outlets, one in Vancouver and one in Quebec. Among other things, it sells wiper blades, windshield cleaning fluid, and floor mats. The monthly sales of these items at the two stores for two months are given in the following tables:

January Sales

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

February Sales

	Vancouver	Quebec
Wiper Blades	23	12
Cleaning Fluid (bottles)	8	12
Floor Mats	4	5

Simple example (cont.)

Use matrix arithmetic to calculate the change in sales of each product in each store from January to February.

$$F - J = \begin{bmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 0 \\ -4 & 1 \end{bmatrix}$$

Thus, the change in sales of each product is the following:

	Vancouver	Quebec
Wiper Blades	3	−3
Cleaning Fluid (bottles)	−2	0
Floor Mats	−4	1

Historical Note

The concept of matrix multiplication is due to the German mathematician **Gotthold Eisenstein**, who introduced the idea around 1844 to simplify the process of making substitutions in linear systems. The idea was then expanded on and formalized by Cayley in his Memoir on the Theory of Matrices that was published in 1858. Eisenstein was a pupil of Gauss, who ranked him as the equal of Isaac Newton and Archimedes. However, Eisenstein, suffering from bad health his entire life, died at age 30, so his potential was never realized.

http://www-history.mcs.st-andrews.ac.uk/Biographies/Eisenstein.html

Examples

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

D + *E*

$$\begin{bmatrix} 1+6 & 5+1 & 2+3 \\ -1+(-1) & 0+1 & 1+2 \\ 3+4 & 2+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$$

4E – 2D

$$\begin{bmatrix} 4\cdot 6 & 4\cdot 1 & 4\cdot 3 \\ 4\cdot(-1) & 4\cdot 1 & 4\cdot 2 \\ 4\cdot 4 & 4\cdot 1 & 4\cdot 3 \end{bmatrix} - \begin{bmatrix} 2\cdot 1 & 2\cdot 5 & 2\cdot 2 \\ 2\cdot(-1) & 2\cdot 0 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 24-2 & 4-10 & 12-4 \\ -4-(-2) & 4-0 & 8-2 \\ 16-6 & 4-4 & 12-8 \end{bmatrix} = \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$$

Multiplications of Matrices

DEFINITION If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the *product* AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

$$\begin{array}{ccc} A & B & AB \\ m \times r & r \times n \; = & m \times n \end{array}$$

Inside

Outside

Multiplications of Matrices: Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

$$\begin{aligned}
(1 \cdot 4) &+ (2 \cdot 0) + (4 \cdot 2) = 12 \\
(1 \cdot 1) &- (2 \cdot 1) + (4 \cdot 7) = 27 \\
(1 \cdot 4) &+ (2 \cdot 3) + (4 \cdot 5) = 30 \\
(2 \cdot 4) &+ (6 \cdot 0) + (0 \cdot 2) = 8 \\
(2 \cdot 1) &- (6 \cdot 1) + (0 \cdot 7) = -4 \\
(2 \cdot 3) &+ (6 \cdot 1) + (0 \cdot 2) = 12
\end{aligned}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Transpose of a matrix

If A is an $m \times n$ matrix, then its **transpose** is the $n \times m$ matrix obtained by writing its rows as columns, so that the ith row of the original matrix becomes the ith column of the transpose. We denote the transpose of the matrix A by A^T.

Visualizing Transposition

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 \\ -3 & 0 & 1 \end{bmatrix}$$

Transpose: properties

Properties of Transposition

If A and B are $m \times n$ matrices, then the following hold:

$$(A+B)^T = A^T + B^T$$
$$(cA)^T = c(A^T)$$
$$(A^T)^T = A.$$

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Identity Matrix

The $n \times n$ identity matrix I is the matrix with 1s down the **main diagonal** (the diagonal starting at the top left) and 0s everywhere else. In symbols,

$$I_{ii} = 1, \quad \text{and}$$
$$I_{ij} = 0 \quad \text{if } i \neq j.$$

Multiplication Example

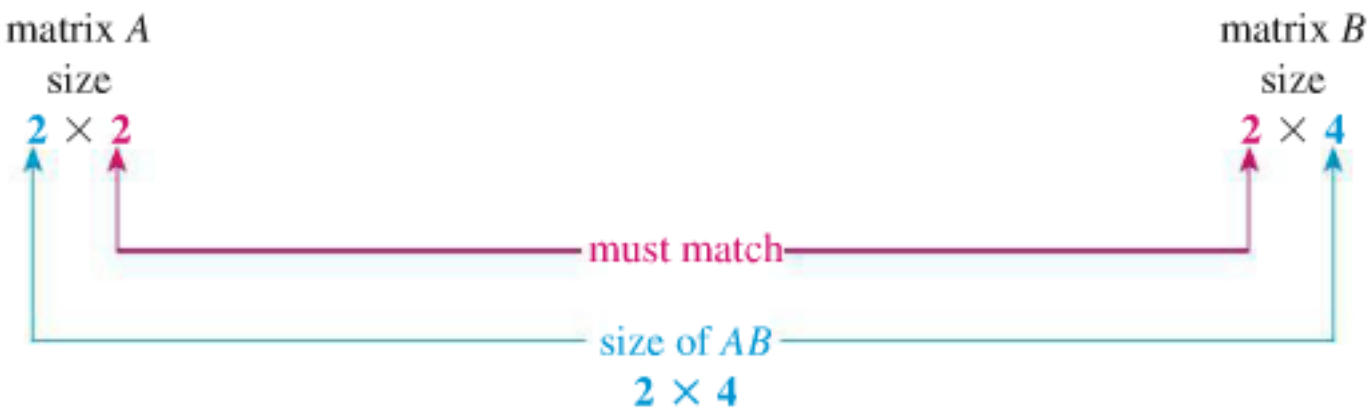
To understand the reasoning behind the definition of matrix multiplication, look again at the EZ Life Company. Suppose sofas and chairs of the same model are often sold as sets, with matrix W showing the number of each model set in each warehouse:

$$\begin{array}{l c} & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{array}{l} \text{New York} \\ \text{Chicago} \\ \text{San Francisco} \end{array} & \begin{bmatrix} 10 & 7 & 3 \\ 5 & 9 & 6 \\ 4 & 8 & 2 \end{bmatrix} \end{array} = W.$$

If the selling price of a model A set is \$800, of a model B set is \$1000, and of a model C set is \$1200, find the total value of the sets in the New York warehouse as follows:

$$\overset{W}{\begin{bmatrix} \boxed{10 \quad 7 \quad 3} \\ 5 \quad 9 \quad 6 \\ 4 \quad 8 \quad 2 \end{bmatrix}} \overset{P}{\begin{bmatrix} 800 \\ 1000 \\ 1200 \end{bmatrix}} \qquad \underset{\substack{\text{Product} \\ \text{of first} \\ \text{entries}}}{10(800)} \underset{+}{+} \underset{\substack{\text{Product} \\ \text{of second} \\ \text{entries}}}{7(1000)} \underset{+}{+} \underset{\substack{\text{Product} \\ \text{of third} \\ \text{entries}}}{3(1200)} = \overset{V}{\begin{bmatrix} 18{,}600 \\ 20{,}200 \\ 13{,}600 \end{bmatrix}}.$$

Multiplication: properties



For any matrices A, B, and C such that all the indicated sums and products exist, matrix multiplication is associative and distributive:

$$A(BC) = (AB)C; A(B + C) = AB + AC; (B + C)A = BA + CA.$$

If A and B are matrices such that the products AB and BA exist,

AB may not equal BA.

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hilbert Matrix

A Hilbert matrix, introduced by Hilbert (1894), is a square matrix with entries being the unit fractions

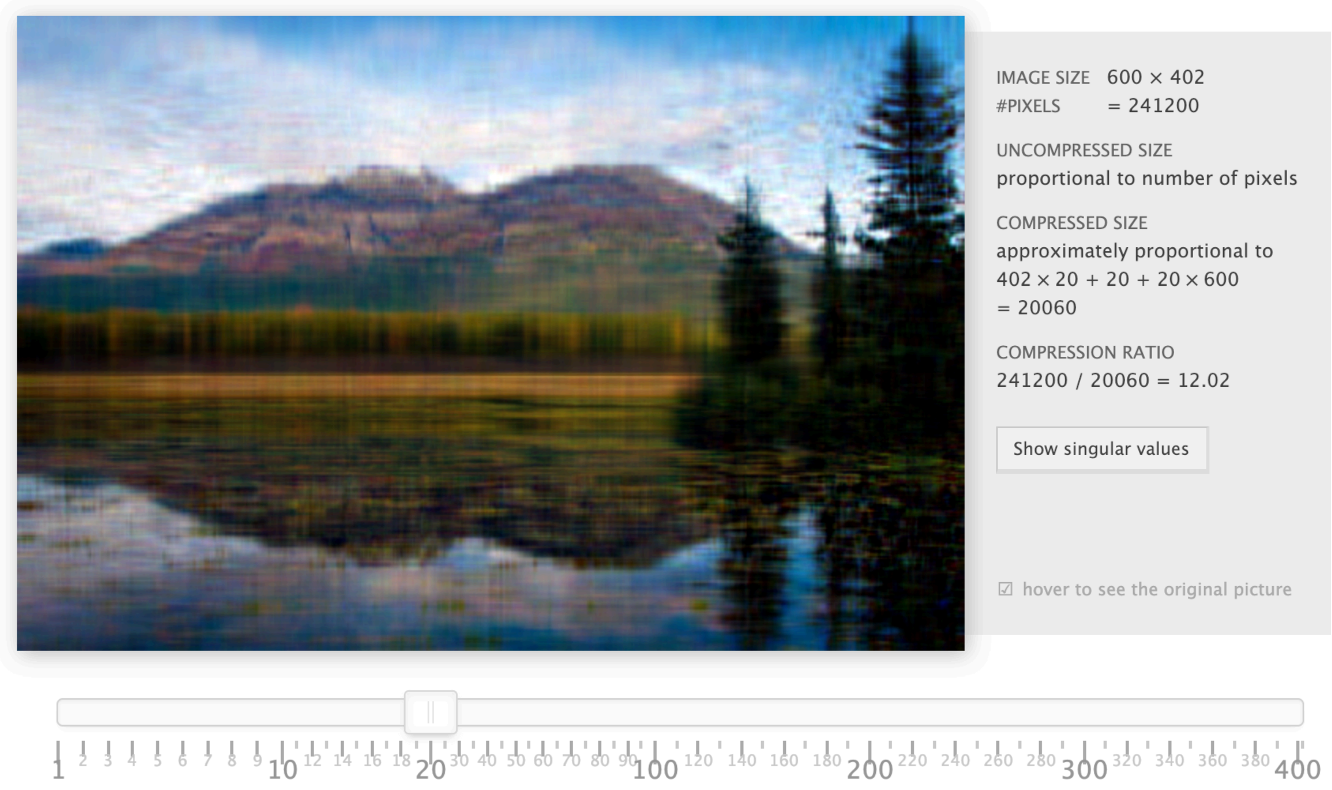
$$H_{ij} = \frac{1}{i+j-1}$$

Example: the 5x5 Hilbert matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

Matrix Factorization: SVD

Image Compression with **S**ingular **V**alue **D**ecomposition (**SVD**)



http://timbaumann.info/svd-image-compression-demo/