Overview of Mathematical Tools for Intermediate Microeconomics

Christopher Flinn
Department of Economics
New York University

August 2002

Contents

Economics is a quantitative social science and to appreciate its usefulness in problem solving requires us to make limited use of some results from the differential calculus. These notes are to serve as an overview of definitions and concepts that we will utilize repeatedly during the semester, particularly in the process of solving problems and in the rigorous statements of concepts and definitions.

1 Functions

Central to virtually all economic arguments is the notion of a function. For example, in the study of consumer choice we typically begin the analysis with the specification of a utility function, from which we later derive a system of demand functions, which can be used in conjunction with the utility function to define an indirect utility function. When studying firm behavior, we work with production functions, from which we define (many types of) cost functions, factor demand functions, firm supply functions, and industry supply functions. You get the idea - functions are with us every step of the way, whether we use calculus or simply graphical arguments in analyzing the problems we confront in economics.

Definition 1 *A univariate (real) function is a rule associating a unique real number $y = f(x)$ with each element x belonging to a set X.*

Thus a function is a rule that associates, for every value x in some set X, a unique outcome y. The particularly important characteristic that we stress is that there is a unique value of y associated with each value of x. There can, in general, be different values of x that yield the same value of y however. When a function also has the property that for every value of y there exists a unique value of x we say that the function is *1-1* or *invertible.*

Example 2 *Let $y = 3 + 2x$. For every value of x there exists a unique value of y, so that this is clearly a function. Note that it is also an invertible function since we can write*

$$\begin{aligned} y &= 3 + 2x \\ \Rightarrow x &= \frac{y-3}{2}. \end{aligned}$$

Thus there exists a unique value of y for every value of x.

Example 3 *Let $y = x^2$. As required, for every value of x there exists a unique value of y. However for every positive value of y there exists two possible values of x given by $\sqrt{y}$ and -$\sqrt{y}$. Hence this is a function, but not an invertible one.*

2 Limits

Let a function $f(x)$ be defined in terms of x. We say that the limit of the function as x becomes arbitrarily close to some value x_0 is A, or

$$\lim_{x \to x_0} f(x) = A.$$

Technically this means that for any $\varepsilon > 0$ there exists a δ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - A| < \varepsilon,$$

where $|z|$ denotes the absolute value of z (i.e. $|z| = z$ if $z \geq 0$ and $|z| = -z$ if $z < 0$).

The limit of a function at a point x_0 need not be equal to the value of the function at that point, that is

$$\lim_{x \to x_0} f(x) = A$$

does not necessarily imply that

$$f(x_0) = A.$$

In fact, it may be that $f(x_0)$ is not even defined.

Example 4 *Let $y = f(x) = 3 + 2x$. Then as $x \to 3$, $y \to 9$, or the limit of $f(x) = 9$ as $x \to 3$. This is shown by demonstrating that $|f(x) - 9| < \varepsilon$ for any ε if $|x - 3|$ is sufficiently small. Note that*

$$\begin{aligned} |f(x) - 9| &= |3 + 2x - 9| \\ &= |2x - 6| = 2|x - 3|. \end{aligned}$$

If we choose $\delta = \varepsilon/2$, then if $|x - 3| < \delta$,

$$2|x - 3| = |2x - 6| = |f(x) - 9| < 2\delta = \varepsilon.$$

Thus for any ε, no matter how small, by setting $\delta = \varepsilon/2$, $|x - 3| < \delta$ implies that $|f(x) - 9| < \varepsilon$. Then by definition

$$\lim_{x \to 3} f(x) = 9.$$

It so happens that in this case

$$f(3) = 9,$$

but these are not the same thing.

Definition 5 *When $\lim_{x \to x_0} f(x) = f(x_0)$ we say that the function is continuous at x_0.*

Then our example $f(x) = 3 + 2x$ is continuous at the value $x_0 = 3$. You should be able to convince yourself that this function is continuous everywhere on the real line $R \equiv (-\infty, \infty)$. An example of a function that is not continuous everywhere on R is the following.

Example 6 *Let the function f be defined as*

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}.$$

First, consider the limit of $f(x)$ for any point not equal to 0, say for example at $x = 2$. Now $f(2) = 4$. For any $\varepsilon > 0$, $|f(x) - 4| < \varepsilon$ is equivalent to $|x^2 - 4| < \varepsilon$, or $|x - 2||x + 2| < \varepsilon$. At points near $x = 2$, $|x + 2| \to 4$. It is clearly the case that $|x + 2| < 5$, say, for x sufficiently close to 2. Then let $\delta = \varepsilon/5$. Then $|x - 2| < \delta$ implies that $\varepsilon > 5|x - 2| > |x + 2||x - 2| = |f(x) - 4|$. Thus there exists a δ (in this case we have chosen $\delta = \varepsilon/5$) such that if x is within δ of $x_0 = 2$ (that is, $|x - 2| < \delta$), then $f(x)$ is within ε of the value 4, and this holds for any choice of ε. We also note that since $f(2) = 4$, the function is continuous at 2.

We note that the function is continuous in this example is continuous at all points other than $x_0 = 0$. To see this, note the following.

Example 7 *(Continued) The limit of $f(x)$ at 0 is given by*

$$\lim_{x \to 0} f(x) = 0.$$

This is the case because for any arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x| < \delta$ implies that $|x^2 - 0| < \varepsilon$. However, since $f(0) = 1$, we have

$$\lim_{x \to 0} f(x) \neq f(0),$$

so that the function is not continuous at the point $x_0 = 0$. As we remarked above, it is continuous everywhere else on R.

On an intuitive level, we say that a function is continuous if the function can be drawn without ever lifting one's pencil from the sheet of paper. In Example 4 the function was a straight line, and this function obviously can be drawn without ever lifting one's pencil (although it would take a long time to draw the line since it extends indefinitely in either direction). On the other hand, in Example 6 we have to lift our pencil at the value $x = 0$, where we have to move up to plot the point $f(0) = 1$. Thus this function is not continuous at that one point.

To rigorously define continuity, and, in the next section, differentiability, it will be useful to introduce the concepts of the left hand limit and the right hand limit. These are only distinguished by the fact that the left hand limit of the function at some point x_0 is defined by taking increasing large values of x that become arbitrarily close to x_0, while the right hand limit of the function is obtained by taking increasingly smaller values of x that become arbitrarily close to the point x_0. Up to this point we have been describing the limiting operation implicitly in terms of the right hand limit. Now let's explicitly distinguish between the two.

Definition 8 *Let $\Delta x > 0$. At a point x_0 the left hand limit of the function f is defined as*

$$L_1(x_0) = \lim_{\Delta x \to 0} f(x_0 - \Delta x),$$

and the right hand limit is defined as

$$R_1(x_0) = \lim_{\Delta x \to 0} f(x_0 + \Delta x).$$

We then have the following result.

Definition 9 *A function f is continuous at the point x_0 if and only if*

$$L_1(x_0) = R_1(x_0) = f(x_0).$$

We say that the function is continuous everywhere on its domain if $L_1(x) = R_1(x)$ for all x in the domain of f.

3 Derivatives of a Univariate Function

Armed with the definition of a function, continuity, and limits, we have all the ingredients to define the derivatives of a univariate function. A derivative is essentially just the rate of a change of a function evaluated at some point. The section heading speaks of derivatives in the plural since we can speak of the rate of change of the function itself (first derivative), the rate of change of the rate of change of the function (the second derivative), etc. Don't become worried, we shall never need to use anything more than the second derivative in this course (and that rarely).

Consider a function $f(x)$ which is continuous at the the point x_0. We can of course "perturb" the value of x_0 by a small amount, let's say by Δx. Δx could be positive or negative, but without any loss of generality let's say that it is positive. Thus the "new" value of x is x_0 plus the change, or $x_0 + \Delta x$. Now we can consider how much the function changes when we move from the point x_0 to the point $x_0 + \Delta x$. The function change is given by

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

Thus the rate of change (or average change) in the function is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Definition 10 *The (first) derivative of the function $f(x)$ evaluated at the point x_0 is defined as this rate of change as the perturbation Δx becomes arbitrarily small, or*

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{1}$$

For [1] to be well-defined requires that the function f have certain properties at the point x_0. First, it must be continuous at x_0, but in addition must have the further property that it be differentiable at x_0. For the function f to be differentiable at x_0 requires that the operation described on the right hand side of [1] be the same whether the point x_0 is approached from "the left" or from "the right." Say that Δx is always positive, which is the convention we have adopted. Then the limit represented on the right hand side of [1] represents the limit of the quotient from the right (taking successively smaller values of Δx but such that $x_0 + \Delta x$ is always greater than x_0). The left hand limit can be defined as

$$L_2(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x},$$

and the right hand limit can be defined as

$$R_2(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Then we have the following definition of differentiability at the point x_0.

Definition 11 *The function f is (first order) differentiable at the point x_0 in its domain if and only if $L_2(x_0) = R_2(x_0)$.*

We say that the function f is (first order) differentiable everywhere on its domain if $L_2(x) = R_2(x)$ for all points x in its domain.

Example 12 *Consider the function*

$$f(x) = \begin{cases} 0 & if \quad x < 2 \\ x - 2 & if \quad x \geq 2 \end{cases} .$$

We first note that this function is continuous everywhere on R. Now consider the derivative of this function at the point $x_0 = 2$. The left hand derivative would be defined as

$$\begin{aligned} & \lim_{\Delta x \to 0} \frac{f(2 - \Delta x) - f(2)}{\Delta x} \\ = & \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0. \end{aligned}$$

The right hand side derivative at the point $x_0 = 2$ would instead be

$$\begin{aligned} & \lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} \\ = & \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1. \end{aligned}$$

Thus the left hand side and right hand side derivatives are different at the point $x_0 = 2$, and so the function is not differentiable there. You should convince yourself that this function is differentiable at every other point in R, however.

For functions f that are differentiable at a point x_0, we can now consider some examples of the computation of first derivatives.

Example 13 *(Quadratic function) Let $f(x) = a + bx + cx^2$, where a, b, and c are fixed scalars. This function is everywhere differentiable on the real line R. Let's consider the computation of the derivative of f at the point $x_0 = 2$. Then we have*

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=2} &= \lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[a + b(2 + \Delta x) + c(2 + \Delta x)^2] - [a + b2 + c2^2]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{b\Delta x + 4c\Delta x + c(\Delta x)^2}{\Delta x} \\ &= b + 4c + \lim_{\Delta x \to 0} c \frac{(\Delta x)^2}{\Delta x} \\ &= b + 4c + c \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\Delta x} \\ &= b + 4c. \end{aligned}$$

[Note: Convince yourself that $\lim_{\Delta x \to 0} \frac{(\Delta x)^n}{\Delta x} = 0$ for all $n > 1$.]

Example 14 *(General polynomial) Let $f(x) = x^n$. Then*

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_x &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + ...] - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}(\Delta x)^3 + ...]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x}{\Delta x} + \lim_{\Delta x \to 0} \frac{n(n-1)x^{n-2}(\Delta x)^2}{\Delta x} + \lim_{\Delta x \to 0} \frac{n(n-1)(n-2)x^{n-3}(\Delta x)^3}{\Delta x} + ... \\
&= nx^{n-1} + 0 + 0 + ... \, .
\end{aligned}$$

This last line follows from the fact that

$$\lim_{\Delta x \to 0} \frac{(\Delta x)^r}{(\Delta x)} = 0, \; r > 1.$$

Differentiation is a linear operation. That means that if our function is modified to include a multiplicative constant k, the derivative of the new function is

$$\frac{d(kx^n)}{dx} = knx^{n-1}.$$

By the same token, if we define another polynomial function, for example

$$g(x) = lx^m,$$

then the derivative of the function

$$f(x) + g(x) = kx^n + lx^m$$

is just the sum of the derivatives of each, or

$$\begin{aligned}
\frac{d(f(x)+g(x))}{dx} &= \frac{df(x)}{dx} + \frac{dg(x)}{dx} \\
&= knx^{n-1} + lmx^{m-1}.
\end{aligned}$$

Using this result we see that the derivative of

$$f(x) = 6x^4 + 3x^2 - 1.5x + 2$$

is

$$f'(x) \equiv \frac{df(x)}{dx} = 24x^3 + 6x - 1.5.$$

To this point we have only consider first derivatives of functions. While we will rarely use them, for completeness we also define higher order derivatives of the function f. Just as the first derivative can be considered the rate of change in the function at some particular point, the second derivative is defined as the rate of change in the first derivative at some particular point in the domain of f. That is, the second derivative is defined as the derivative of the first derivative of the function, the third derivative is the derivative of the second derivative of the function, and so on. Let us formally consider the definition of the second derivative.

Definition 15 *The second derivative of the function $f(x)$ at some point x_0 is defined as*

$$\frac{d^2 f(x)}{dx^2} = \frac{df'(x)}{dx},$$

where $f'(x) = \frac{df(x)}{dx}$. Of course, we presume that the function is (at least) second order differentiable at the point x.

Example 16 *Consider the function $f(x) = 6x^4 + 3x^2 - 1.5x + 2$. This function is differentiable up to any order, so we don't have to worry about whether first or second derivatives exist. As we saw above, the first derivative is*

$$f'(x) = 24x^3 + 6x - 1.5.$$

The second derivative of this function is just the derivative of $f'(x)$, so we apply the same principles for differentiating a polynomial function as we did to derive the first derivative. Then we find

$$f''(x) = \frac{d^2 f(x)}{dx^2} = 72x^2 + 6.$$

The third derivative of the function would be $144x$, the fourth derivative would be 144, and the fifth and all higher order derivatives would be equal to 0.

We shall use second order derivatives solely to check whether the solutions we obtain to the optimization problems we will treat correspond to local maxima, local minima, or neither.

4 Some Rules of Differentiation

In this section we simply list some of the most common rules of differentiation that you are likely to need this semester. For completeness, the list includes a few that won't be used as well. We will discuss derivatives of the functions that we will most commonly encounter in the last section of this handout.

In the following expressions a and n represent constants while $f(x)$ and $g(x)$ are unspecified (differentiable) functions of the single variable x.

1. $\frac{d}{dx}(a) = 0$.
2. $\frac{d}{dx}(x) = 1$.
3. $\frac{d}{dx}(ax) = a$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$
5. $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$.
6. $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{1}{g(x)}\frac{df(x)}{dx} - \frac{f(x)}{g(x)^2}\frac{dg(x)}{dx}$
7. $\frac{d}{dx}(f(x))^n = (n-1)f(x)^{n-1}\frac{df(x)}{dx}$
8. $\frac{d}{dx}(f(g(x)) = \frac{d}{dg(x)}(f(g(x))) \cdot \frac{d}{dx}(g(x))$ (Chain Rule)
9. $\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)}\frac{d}{dx}(f(x))$
10. $\frac{d}{dx}(\exp(f(x))) = \exp(f(x)) \cdot \frac{d}{dx}(f(x))$.

Note that in equation 9 the symbol "ln" represents logarithm to the base e, where e is the natural number approximately equal to 2.718. Similarly, in expression 10 the symbol "exp" represents e, and signifies the the term following is the exponent of e. That is, $\exp(x) = e^x$.

These derivative expressions are all that you will be using in this course, and for that matter represent those relationships most frequently used in performing applied economics at any level.

5 Maximization and Minimization of Univariate Functions

In this Section we introduce the idea of maximization and minimization of functions of one variable. We only consider unconstrained optimization (the term "optimization" encompasses maximization and minimization) here to introduce ideas. The main tool of applied economics is constrained optimization of multivariate functions, a topic treated below. We will only very quickly introduce the subject and will skip over most details. The development is decidedly nonrigorous and only treats special cases likely to be of interest to us in this course.

Let $f(x)$ be a univariate function of x, and assume that x is differentiable everywhere on its domain (which let's assume is the entire real line, denote by R). Let's introduce the subject by considering some simple examples.

Example 17 *Let us consider finding the value of x that maximizes the function $f(x) = 2x$. Now the first derivative of this function is $d(2x)/dx = 2$. This means that for every unit that x is increased the function is increased by 2 units. Clearly the maximum of the function is obtained by taking x to be as large as possible. This is an example in which the maximization (or minimization) problem is not well-posed since the solution to the problem is not finite.*

Example 18 *Consider the maximization of the function $g(x) = 5 + x - 2x^2$. In maximizing this function we don't run into the same problem as we had in the case of $f(x) = 2x$, since clearly as x gets indefinitely large or indefinitely small ($x \to -\infty$) the function $g(x)$ goes to $-\infty$ as well. In Figure 1.a we plot the function and you can note its behavior. In Figure 1.b we plot the derivative of the function as well. We observe the following behavior. There exists one maximum value of the function, and this value is equal to 5.125, and the value of x that is associated with the maximum function value is equal to .25.*

We note that at the value of $x = .25$ the derivative of the function is 0. To the left of that value, i.e., for $x < .25$, the derivative of the function is positive, which indicates that by increasing x we can increase the function value. To the right of .25, i.e., for values of $x > .25$ the derivative is negative, indicating that further increases in x decrease the function value. Then the value of x that maximizes the function is that value at which the derivative of the function is 0.

Say that a function $f(x)$ is continuously differentiable on some interval D, where D could be the entire real line. Let's say that we are attempting to locate the global maximum of f, where by the global maximum we mean that there exists an x^* (assume it is unique) such that $f(x^*) > f(y)$ for all $y \in D$, $y \neq x^*$. The function f can also possess local maxima (the point x^* will always be a local as well as a global maximum). A local maximum is defined as follows. Consider a point $\hat{x}$. Then $\hat{x}$ is a local maximum if $f(\hat{x}) > f(z)$, $z \in (\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2)$, $z \neq \hat{x}$, for some $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $(\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_2) \subseteq D$.

On the interval D, let's define the set of points $E = \{\tau_1, ..., \tau_m\}$ such that the following condition holds

$$f'(\tau_i) \equiv \left.\frac{df(x)}{dx}\right|_{\tau_i} = 0,\ i = 1, ..., m.$$

Since f is continuously differentiable, it also possesses second derivatives. Consider the set of second derivatives of f evaluated at the points $(\tau_1, ..., \tau_m)$,

$$\{f''(\tau_1), ..., f''(\tau_m)\}.$$

Condition 19 *The set of points $E_1 \subseteq E$ such that*

$$x \in E_1 \Leftrightarrow f''(x) < 0$$

contains all of the local maxima of f on the interval D.

Condition 20 *The set of points $E_2 \subseteq E$ such that*

$$x \in E_2 \Leftrightarrow f''(x) > 0$$

contains all of the local minima of f on the interval D.

Condition 21 *The set of points $E_3 \subseteq E$ such that*

$$x \in E_2 \Leftrightarrow f''(x) = 0$$

contains all of the inflection points of f on the interval D.

Clearly the sets E_1, E_2, and E_3 contain no points in common and $E_1 \cup E_2 \cup E_3 = E$ (i.e., E_1, E_2, and E_3 constitute a partition of E).

The conditions presented above are extremely useful ones in practice. It is often the case that the functions we work with in economic applications are very well-behaved - that is the reason we choose to work with them. For example, if we work with a quadratic function of x defined on the entire real line we know immediately that the function will either possess (1) a unique, finite maximum and no finite minimum or (2) a unique, finite minimum and no finite maximum. To see this, write the "general" form of the quadratic function as

$$f(x) = a + bx + cx^2,$$

where a, b, and c are scalars that can be of any sign, but we require that $b \neq 0$ and $c \neq 0$. Begin by defining the set of values $\{\tau_1, ..., \tau_m\}$ of this function where the first derivative is 0. Note that

$$f'(x) = b + 2cx,$$

and now find the values of x where this function is equal to 0. It is clear that there is only one value of x that solves this equation, so that $m = 1$. The solution is

$$\tau_1 = -\frac{b}{2c}.$$

The second derivative of the function evaluated at τ_1 is

$$f''(\tau_1) = 2c.$$

Thus τ_1 corresponds to a maximum if and only if $c < 0$. It corresponds to a minimum when $c > 0$. In either case, since there is only one element of E, the extreme point located is unique.

Say that the function to be maximized is quadratic and that $c > 0$. In this case we know that f possesses one global minimum value on R, and the value at which the minimum is obtained is equal to $-b/2c$. What about the maximum value attained by this function? Clearly, as x becomes indefinitely large the value of f becomes indefinitely large (since the term cx^2 will eventually "dominate" the term bx in determining the function value no matter what the sign or absolute size of b). In this case we say that the function is unbounded on R, and there exists no value of x that yields the maximum function value. We'll say in this case that the maximum of the function doesn't exist.

For some functions, like the one we will look at in the next example, there can exist multiple local maxima and/or minima, i.e., the sets E_1 and or E_2 contain more than one element. Say that we are interested in locating the global maximum of the function f and the set E_1 contains more than one element. The only way to accomplish the task is the "brute force" method of evaluating the function at each of the elements of E_1 and then define the global maximum as that element of E_1 associated with the largest function value. Formally, x^* is the global maximum if $x^* \in E_1$ and

$$f(x^*) > f(y), \text{ all } y \in E_1, y \neq x^*.$$

The global minimum is analogously defined.

Example 22 *Let us attempt to locate the global maximum and minimum of the cubic equation*

$$f(x) = a + bx + cx^2 + dx^3.$$

Now the first order condition is

$$f'(\tau_i) = 0 = b + 2c\tau_i + 3d\tau_i^2.$$

This is a quadratic equation, and therefore it has two solutions, which are given by

$$\tau_1^* = \frac{-2c + \sqrt{4c^2 - 12bd}}{2b}; \tau_1^* = \frac{-2c - \sqrt{4c^2 - 12bd}}{2b}.$$

Both solutions will be real numbers if $4c^2 > 12bd$*, and let us suppose that this is the case.*

Now to see whether the solutions correspond to local maxima, local minima, or inflection points we have to evaluate the second derivatives of the function at τ_1^* *and* τ_2^**. As long as both solutions are real, then the second derivative of the function evaluated at one of the solutions will be positive and the other negative. Therefore, the maximum of the function and the minimum of the function would appear to be unique. This is only apparently true however, since if the domain of the function is* R*, there is no finite maximum of minimum. Since as* x *gets large the function is dominated by the term* dx^3*, if* $d > 0$*, for example, the function becomes indefinitely large as* $x \to \infty$ *and becomes indefinitely small as* $x \to -\infty$*. Thus the first order conditions determine the maximum and minimum of the function only if the domain of the function* D *is restricted appropriately. See the associated figure for the details.*

6 Multivariate Functions

A multivariate function is identical to a univariate function except for the fact that it has more than one “argument.” In our course we will begin by studying consumer theory, and virtually always we will deal with the case in which an individual’s “utility” is determined by their level of consumption of two goods, say x_1 and x_2. We will write this as

$$u = U(x_1, x_2),$$

where U is the utility function and u denotes the level of utility the consumer attains when consuming the level x_1 of the first good and x_2 of the second. When we assert that U is a function, we mean that for each pair of values (x_1, x_2) there exists one and only one value of the function U, that is, the quantity u is unique for each pair of values (x_1, x_2).

In general a multivariate function is defined with respect to n arguments as

$$y = f(x_1, x_2, ..., x_n),$$

where n is a positive integer greater than 1. Thus we could define our utility function over n goods instead of 2, which would make our analysis more "realistic," without question, but complicates things and doesn't add any insights at the level of analysis we will be operating in this course.

When we work with multivariate functions we will often be interested in the relationship between the outcome measure y and one of the arguments of the function holding the other argument (or arguments) of the function fixed at some given values. By holding all other arguments fixed at some preassigned values, the function becomes a univariate one since only one of the arguments is allowed to freely vary.

Example 23 *Consider the linear combination of x_1 and x_2*

$$y = a + bx_1 + cx_2,$$

where a, b, and c are given non-zero constants. This is clearly a function, since for any pair (x_1, x_2) there is a unique value of y. If we fix the value of x_2 at some constant, say 5 for example, then we have

$$y = (a + c \cdot 5) + bx_1,$$

which is a univariate function of x_1 now. Clearly when setting different values of x_2 we will get different univariate functions of x_1, but the important thing to note is that they will all be functions in the rigorous sense of the word. We can of course do the same thing if we fix x_1 at a given value and allow x_2 to freely vary instead.

In the case of univariate functions we called a function $y = f(x)$ 1-1 if not only was there a unique value of y for each value of x, but also each value of y implied a unique value of x. In such a case we can also call the function f "invertible." For a "well-behaved" multivariate function there is no "group" invertibility notion - that is, though we require that each set of values $(x_1, ..., x_n)$ be associated with a unique value of y, we cannot (in general) have that a particular value of y implies a unique set of values $(x_1, ..., x_n)$.

We can, however, define a multivariate function as 1-1 in an argument x_i conditional on the values of all of the other arguments. For simplicity, we will continue to limit our attention to the case in which there are only two arguments. Let's begin with the linear example above.

Example 24 *(Continued) When x_2 is fixed at a value k, for example, the (conditional on $x_2 = k$) linear function is given by*

$$\begin{aligned} y &= a + bx_1 + ck \\ y &= (a + ck) + bx_1. \end{aligned}$$

By conditioning on the value $x_2 = k$ we have defined a new linear function with a constant term equal to $a + ck$ and a variable term bx_1. From our earlier discussion, we know that this new *relationship is 1-1 between y and x_1 no matter what value of k we choose to set x_2 equal to. Thus this "conditional" function (on $x_2 = k$) is invertible. We can switch the role of x_1 and x_2 and define a conditional (on $x_1 = k$) function in which x_2 is the only variable. This conditional relationship is invertible as well.*

Example 25 *Consider a function we will often use in the course,*

$$y = Ax_1^{\alpha}x_2^{\beta},$$

where A, α, and β are all positive as are the two arguments x_1 and x_2. This is clearly a function, since any pair (x_1, x_2) yields a unique value of y. The (conditional) function is invertible in each argument. For example, fix $x_1 = k$, so that the conditional function (of x_2) is

$$y = Ak^{\alpha}x_2^{\beta},$$

which we can write as

$$x_2 = \left[\frac{y}{Ak^{\alpha}}\right]^{1/\beta},$$

which is a monotonic (increasing) function of y for any value of k.

7 Partial Derivatives of Multivariate Functions

Assume that the function $y = f(x_1, x_2)$ is continuous and differentiable everywhere on R^2 [$R^2 = R \times R$, which means that both arguments can be any real number] in both arguments, x_1 and x_2. When computing the partial derivative of the function we have to specify two values for x_1 and x_2 to take. Let's say that we want to evaluate the partial derivative of f with respect to the first argument at the point $(x_1 = 2, x_2 = -3)$. Since we are taking the partial derivative with respect to the first argument, we are holding constant the value of the second argument at $x_2 = -3$. Given this value of x_2, we want to assess how the function changes as we change x_1 by a small amount given that we start at the value $x_1 = 2$. Thus when we specify values for all of the arguments of the function in computing a partial derivative with respect to the k^{th} argument, we are holding the values of all other variables besides the k^{th} constant at the values specified, and we are seeing how the function changes with respect to small changes in the k^{th} starting from the value x_k.

Formally the partial derivative of the function f with respect to argument k is given as follows:

Definition 26 *The partial derivative of $f(x_1, ..., x_m)$ with respect to x_k at $(\tilde{x}_1, ..., \tilde{x}_m)$ is*

$$\left.\frac{\partial f(x_1, ..., x_m)}{\partial x_k}\right|_{x_1=\tilde{x}_1,...,x_m=\tilde{x}_m} = \lim_{\varepsilon \to 0} \frac{f(\tilde{x}_1, ..., \tilde{x}_{k-1}, \tilde{x}_k + \varepsilon, \tilde{x}_{k+1}, ..., \tilde{x}_m) - f(\tilde{x}_1, ..., \tilde{x}_m)}{\varepsilon},$$

where $\varepsilon > 0$.

Example 27 *Consider the quadratic function*

$$f(x_1, x_2) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_1^2 + \alpha_4 x_2^2 + \alpha_5 x_1 x_2.$$

The first partial of f with respect to its first argument evaluated at the point $(x_1 = 2, x_2 = 3)$ is given by

$$\begin{aligned} f_1(2,3) &\equiv \left.\frac{\partial f(x_1,x_2)}{\partial x_1}\right|_{x_1=2,x_2=3} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{[\alpha_0+\alpha_1(2+\varepsilon)+\alpha_2 3+\alpha_3(2+\varepsilon)^2+\alpha_4 3^2+\alpha_5(2+\varepsilon)\cdot 3] \\ &\quad -[\alpha_0+\alpha_1 2+\alpha_2 3+\alpha_3 2^2+\alpha_4 3^2+\alpha_5 2\cdot 3]\} \\ &= \lim_{\varepsilon\to 0}\frac{\alpha_1\varepsilon+\alpha_3\{[4+4\varepsilon+\varepsilon^2]-4\}+\alpha_5 3\varepsilon}{\varepsilon} = \alpha_1+4\alpha_3+3\alpha_5. \end{aligned}$$

The last line follows from the fact that $\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{\varepsilon}=0$.

Perhaps it is most helpful to think of the partial differentiation process as follows. Essentially what we are doing is thinking of the multivariate function as a univariate function - a function of only the one variable that we are computing the derivative with respect to. The other variables in the function are treated as constants for the purpose of computing the partial derivative, and are fixed at the values specified $(\tilde{x}_1, ..., \tilde{x}_{k-1}, \tilde{x}_{k+1}, ..., \tilde{x}_m)$ if we are computing the partial derivative with respect to x_k. After these variables have been fixed at these values, the function f is essentially a univariate function and we can apply the rules for differentiating univariate functions provided above to compute the desired partial derivative.

Example 28 *Consider a slightly generalized version of the (Cobb-Douglas) function that we looked at previously, and let*

$$f(x_1,x_2,x_3) = Ax_1^{\alpha}x_2^{\beta}x_3^{\delta},$$

where x_1, x_2, x_3 assume positive values and A, α, β, and δ are all positive constants. First consider the partial derivative of this function with respect to its first argument evaluated at $(x_1 = 2, x_2 = 3, x_3 = 1)$. Begin by conditioning on the values of $x_2 = 3$ and $x_3 = 1$, so that we now have the univariate function

$$\tilde{f}(x_1|x_2=2,x_3=1) = \{A3^{\beta}1^{\delta}\}x_1^{\alpha}.$$

The derivative of this univariate function, when evaluating at the value $x_1 = 2$, is

$$\begin{aligned} f_1(2,3,1) &= \{A3^{\beta}1^{\delta}\}\left.\alpha x_1^{\alpha-1}\right|_{x_1=2} \\ &= \{A3^{\beta}1^{\delta}\}\alpha 2^{\alpha-1}. \end{aligned}$$

The partial derivatives of this function evaluated at the same values of x_1, x_2, x_3 with respect to the variables x_2 and x_3 are given by:

$$\begin{aligned} f_2(2,3,1) &= \{A2^{\alpha}1^{\delta}\}\beta 3^{\beta-1} \\ f_3(2,3,1) &= \{A2^{\alpha}3^{\beta}\}\delta 1^{\delta-1}. \end{aligned}$$

Typically we write the partial derivative of a function in a generic form, where by this we mean we write it as a function of the variables $x_1, x_2, ..., x_m$ without supplying specific values for these variables. Clearly the partial derivative of a function is simply another function, and is often a function of all of the same arguments as the original function (it can be a function of fewer depending on the form of f and can certainly never be a function of more).

Example 29 *(Continued) The generic form, so to speak, of the previous example is given as follows. The partial derivatives of f with respect to each of its arguments are given by*

$$\begin{aligned} f_1(x_1, x_2, x_3) &= \{Ax_2^{\beta}x_3^{\delta}\}\alpha x_1^{\alpha-1} \\ f_2(x_1, x_2, x_3) &= \{Ax_1^{\alpha}x_3^{\delta}\}\beta x_2^{\beta-1} \\ f_3(x_1, x_2, x_3) &= \{Ax_1^{\alpha}x_2^{\beta}\}\delta x_3^{\delta-1}. \end{aligned}$$

To this point we have only discussed first partial derivatives of functions, but just as in the case of univariate functions, we can define higher order partial derivatives as well. While we will not be making much use of these in the course, for completeness we will define second-order partial derivatives. A second-order partial derivative is simply a partial derivative of a first partial, just as a second derivative of a univariate function was the derivative of the first derivative. Because partial derivatives are defined with respect to multivariate functions, we can define "own" second partials - which are the partial derivative with respect to x_k of the first partial derivative with respect to x_k - and "cross" partials - which are the partial derivatives with respect to any variable other than x_k of the first partial derivative with respect to x_k.

Definition 30 *Let $f(x_1, ..., x_m)$ be a continuously differentiable function. Then the first partials of the function are defined by*

$$f_i(\tilde{x}_1, ..., \tilde{x}_m) \equiv \left.\frac{\partial f(x_1, ..., x_m)}{\partial x_i}\right|_{x_1=\tilde{x}_1,...,x_m=\tilde{x}_m}, \ i = 1, ..., m,$$

and the second partials of the function are defined by

$$\begin{aligned} f_{ij}(\tilde{x}_1, ..., \tilde{x}_m) &\equiv \left.\frac{\partial^2 f(x_1, ..., x_m)}{\partial x_j \partial x_i}\right|_{x_1=\tilde{x}_1,...,x_m=\tilde{x}_m} \\ &= \left.\left[\frac{\partial}{\partial x_j}\frac{\partial f(x_1, ..., x_m)}{\partial x_i}\right]\right|_{x_1=\tilde{x}_1,...,x_m=\tilde{x}_m}, \ i = 1, ..., m; \ j = 1, ..., m. \end{aligned}$$

The terms f_{ii}, $i = 1, ..., m$ are referred to as "own" partials and the terms f_{ij}, $i = 1, ..., m$, $j = 1, ..., m$, $i \neq j$ are referred to as "cross" partials. Note that $f_{ij} = f_{ji}$ for all i, j.

The matrix of second partial derivatives of the function f evaluated at the values $(\tilde{x}_1, ..., \tilde{x}_m)$ is sometimes referred to as the Hessian. When considering optimization of a univariate function $f(x)$, we noted that solution(s) to the equation $f'(\hat{x}) = 0$ corresponded to maxima or minima depending on whether the second derivative of the function was negative or positive when evaluated at $\hat{x}$. When optimizing the multivariate function $f(x_1, ..., x_m)$, any proposed solution $\hat{x} = (\hat{x}_1, ..., \hat{x}_m)$ will be classified as yielding a local maximum or minimum depending on properties of the Hessian evaluated at $\hat{x}$. This is the primary use of second partials in applied microeconomic research.

Example 31 *Consider the function* $f(x_1, x_2) = \alpha \ln(x_1) + \beta x_2$*, where* $x_1 > 0$*. Then the first partial derivatives are*

$$\begin{aligned} f_1(x_1, x_2) &= \frac{\alpha}{x_1} \\ f_2(x_1, x_2) &= \beta \end{aligned}$$

The second partial derivatives of this function are

$$\begin{array}{cc} f_{11}(x_1, x_2) = -\frac{\alpha}{x_1^2} & f_{12}(x_1, x_2) = 0 \\ f_{21}(x_1, x_2) = 0 & f_{22}(x_1, x_2) = 0 \end{array}.$$

A bivariate function for which the cross partial derivatives are 0 for all values of x_1 and x_2 is called additively separable in its arguments. This means that the function $f(x_1, x_2)$ can be written as the sum of two univariate functions, one a function of x_1 and the other a function of x_2.

Example 32 *Let*

$$f(x_1, x_2) = x_1^{\alpha} x_2^{\beta},$$

where $x_1 > 0$ *and* $x_2 > 0$*. Then the first partials are*

$$\begin{aligned} f_1(x_1, x_2) &= \alpha x_1^{\alpha-1} x_2^{\beta} \\ f_2(x_1, x_2) &= \beta x_1^{\alpha} x_2^{\beta-1} \end{aligned}$$

and the second partials are given by

$$\begin{array}{cc} f_{11}(x_1, x_2) = \alpha(\alpha - 1) x_1^{\alpha-2} x_2^{\beta} & f_{12}(x_1, x_2) = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} \\ f_{21}(x_1, x_2) = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} & f_{22}(x_1, x_2) = \beta(\beta - 1) x_1^{\alpha} x_2^{\beta-2}. \end{array}$$

In this example note that $f_{12} = f_{21}$, as always, and that the function is not additively separable since $f_{12} \neq 0$ for some values of (x_1, x_2).

8 Total Derivatives

When we compute a partial derivative of a multivariate function with respect to one of its arguments x_i, we find the effect of a small change in x_i on the function value starting from the point of evaluation $(\tilde{x}_1, ..., \tilde{x}_m)$. When computing the total derivative, instead, we look at the simultaneous impact on the function value of allowing all of the arguments of the functions to change by a small amount.

It is important to emphasize that the expression we use for the total derivative is only valid when each of the arguments of the function change by small amounts. If this were not the case, we could not write the function in the linear form that we do.

Partial derivatives play a role in writing the total derivative, of course. For small changes in each of the arguments, denoted by dx_i for the i^{th} argument of the function, we can write

$$dy = f_1(x_1, ..., x_m)dx_1 + ... + f_m(x_1, ..., x_m)dx_m.$$

Of course, the change in the function value, dy, in general will depend on the point of evaluation of the total derivative $(x_1, ..., x_m)$.

Example 33 *Compute the total derivative of the function*

$$y = \exp(\beta x_1 + x_2) + \ln(x_2).$$

The partial derivatives of the function are

$$\begin{aligned} f_1(x_1, x_2) &= \beta \exp(\beta x_1 + x_2) \\ f_2(x_1, x_2) &= \exp(\beta x_1 + x_2) + \frac{1}{x_2}. \end{aligned}$$

Therefore the total derivative of the function is

$$dy = \{\beta \exp(\beta x_1 + x_2))dx_1 + \{\exp(\beta x_1 + x_2) + \frac{1}{x_2}\}dx_2.$$

We will most often use the total derivative to define implicit relationships between the "independent" variables, the x. Consider the bivariate utility function

$$u = Ax_1^\alpha x_2^\beta.$$

The total derivative of this function tells us how utility changes when we change x_1 and x_2 simultaneously by small amounts (starting from the generic values x_1 and x_2). The total derivative is

$$\begin{aligned} du &= A\alpha x_1^{\alpha-1} x_2^\beta dx_1 + A\beta x_1^\alpha x_2^{\beta-1} dx_2 \\ &= \alpha \frac{u}{x_1} dx_1 + \beta \frac{u}{x_2} dx_2 \\ &= u(\frac{\alpha}{x_1} dx_1 + \frac{\beta}{x_2} dx_2). \end{aligned}$$

Now an indifference curve is a locus of points (in the space $x_1 - x_2$) such that all points (x_1, x_2) yield the same level of utility. Say that x_1 and x_2 are two consumption goods, and that the consumer currently has 2 units of x_1 and 3 units of x_2. If we increase the amount of x_1 by a small amount, how much would we have to decrease the amount of x_2 consumed by the individual to keep her level of welfare constant?

We want to determine the values of dx_1 and dx_2 required to keep utility constant beginning from the point $x_1 = 2$, $x_2 = 3$. Generically we write

$$\begin{aligned} 0 &= du = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2 \\ &\Rightarrow \frac{dx_2}{dx_1} = -\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}. \end{aligned}$$

Note that since x_1 and x_2 are both "goods," the partial derivatives on the right hand side of the function are positive, which means that the left hand side of the expression is negative. This just reflects the fact that if both are goods, to keep utility constant we must reduce the consumption of one if we increase the consumption of the other.

For our specific functional form, we have

$$\begin{aligned}
0 &= du = u(\frac{\alpha}{x_1}dx_1 + \frac{\beta}{x_2}dx_2) \\
\Rightarrow 0 &= \frac{\alpha}{x_1}dx_1 + \frac{\beta}{x_2}dx_2 \\
\Rightarrow & \frac{dx_2}{dx_1} = -\frac{\alpha x_2}{\beta x_1}.
\end{aligned}$$

At the point $x_1 = 2$, $x_2 = 3$ then the slope of the indifference curve is

$$-\frac{3\alpha}{2\beta}.$$

9 Maximization and Minimization of Multivariate Functions

The majority of the course, and of the study of economics in general, consists of solving constrained optimization problems. We begin the course with the study of the consumer, and endow her with a set of preferences that we will assume can be represented by a utility function. This utility function "measures" how happy she is at any level of consumption of the goods that provide her with satisfaction. We will assume that there are only two goods, and we will discuss the bivariate case throughout this section. The multivariate is a fairly straightforward extension, and we shall deal almost exclusively with the bivariate case throughout the course.

Let the individual's utility function be given by $u = u(x_1, x_2)$, and assume that both x_1 and x_2 are goods, so that the first partial derivatives of u with respect to both x_1 and x_2 are nonnegative for all positive values of x_1 and x_2. If we allow the individual to maximize her utility in such a case, she should choose $x_1 = \infty$ and $x_2 = \infty$. This doesn't lead to very interesting possibilities for analysis.

Instead, we impose the realistic constraint that the goods have positive prices, denoted by p_1 and p_2 and that the individual has a finite amount of income, I, to spend on them. Most of the time in this course we will only be considering static optimization problem - that is, we look at the consumer's (or producer's) choices when there is no tomorrow. In terms of the consumer's problem of utility maximization, this means that savings have no value, so that to maximize her consumption the consumer should spend all of the resources she currently has on one or both of the goods available. Then her choices of x_1 and x_2 should satisfy the following equality:

$$I = p_1x_1 + p_2x_2$$

The budget constraint is actually

$$I \geq p_1x_1 + p_2x_2,$$

but as long as both commodities are goods, there is no satiation, and the consumer is rational, we can assume the relationship holds with strict equality. Then the constrained maximization problem of the consumer is formally stated as

$$\begin{aligned}
V(p_1, p_2, I) &= \max_{x_1,x_2} u(x_1, x_2) \\
\text{s.t. } I &= p_1x_1 + p_2x_2.
\end{aligned}$$

There are two ways to go about solving this problem. We will begin with what might be the more intuitive approach. When the multivariate optimization problem involves only two variables, this first "brute force" approach is effective and is no more time-consuming to implement then is the Lagrange multiplier method that we will turn to next.

Optimization of functions of one variable, which we briefly described above is pretty straightforward to carry out. The idea of the first method is to make the two variable optimization problem into a one variable problem by substituting the constraint into the function to be maximized (or minimized, as the case may be). When the budget constraint holds with strict equality, then given I, the two goods prices, and the consumption level of one of the goods, we know exactly how much of the other good must be consumed (so as to exhaust the consumers income). That is, we can rewrite the budget constraint as

$$x_2 = \frac{1}{p_2}(I - p_1 x_1).$$

In this way we view x_2 as a function of the exogenous constraints the consumer faces (prices and I) and their choice of x_1. Now substitute this constraint into the consumer's objective function (i.e., their utility function) to get $u(x_1, \frac{1}{p_2}(I - p_1 x_1))$. The trade-offs between consuming x_1 rather than x_2 given the constraints are now "built into" the objective function, so we have converted the two variable constrained maximization problem into a one variable unconstrained maximization problem, something that is much easier to compute.

Formally, the maximization problem becomes

$$V(p_1, p_2, I) = \max_{x_1} u(x_1, \frac{1}{p_2}(I - p_1 x_1)).$$

Solving the above equation gives us the optimal value of x_1, which we denote x_1^*. To find the optimal consumption level of x_2, just substitute this value into the budget constraint to get

$$x_2^* = \frac{1}{p_2}(I - p_1 x_1^*).$$

Example 34 *Assume that an individual's utility function is given by*

$$u = \alpha_1 \ln x_1 + \alpha_2 \ln x_2.$$

We use the method of substitution to find the optimal quantities consumed as follows. First we substitute the budget constraint into the "objective function" to get

$$\max_{x_1} \alpha_1 \ln x_1 + \alpha_2 \ln\left(\frac{1}{p_2}(I - p_1 x_1)\right).$$

To find the maximum of this univariate objective function with respect to its one argument, x_1, we find the first derivative and solve for the value of x_1, x_1^*, at which this derivative is zero. The derivative of the function is

$$\begin{aligned}\frac{du}{dx_1} &= \frac{\alpha_1}{x_1} + \frac{\alpha_2}{\left(\frac{1}{p_2}(I - p_1 x_1)\right)}\left(-\frac{p_1}{p_2}\right) \\ &= \frac{\alpha_1}{x_1} - \frac{\alpha_2 p_1}{(I - p_1 x_1)},\end{aligned}$$

so the value of x_1 that sets this derivative equal to 0 is given by

$$\begin{aligned} 0 &= \frac{\alpha_1}{x_1^*} - \frac{\alpha_2 p_1}{(I - p_1 x_1^*)} \\ &\Rightarrow \alpha_1 (I - p_1 x_1^*) = \alpha_2 p_1 x_1^* \\ &\Rightarrow x_1^* = \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \frac{I}{p_1}. \end{aligned}$$

Given x_1^*, the optimal consumption level of x_2 is given by

$$\begin{aligned} x_2^* &= \frac{1}{p_2}(I - p_1 x_1^*) \\ \Rightarrow x_2^* &= \frac{1}{p_2}(I - p_1 \left[\frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{I}{p_1}\right]) \\ &= \frac{\alpha_2}{(\alpha_1 + \alpha_2)} \frac{I}{p_2}. \end{aligned}$$

We know that the solution x_1^* corresponds to a maximum since the second derivative of the objective function is negative (for all values of x_1).

We now consider the general approach to solving constrained optimization problems, that of Lagrange multipliers. Once again, the trick is to make the constrained optimization problem an unconstrained one. We will write the general approach out for a general multivariate function before specializing to the bivariate case. For simplicity, however, we will only consider the one constraint case. The method is very general and can be extended to incorporate many constraints simultaneously.

Consider the constrained maximization problem

$$\begin{aligned} & \max_{x_1, \ldots, x_m} u(x_1, \ldots, x_m) \\ \text{s.t. } I &= p_1 x_1 + \ldots + p_m x_m. \end{aligned}$$

We rewrite this as an unconstrained optimization problem as follows:

$$\max_{x_1, \ldots, x_m, \lambda} u(x_1, \ldots, x_m) + \lambda(I - p_1 x_1 - \ldots - p_m x_m).$$

The function $u(x_1, \ldots, x_m) + \lambda(I - p_1 x_1 - \ldots - p_m x_m)$ is called the Lagrangian, and the coefficient λ is termed the Lagrange muliplier - since it multiplies the constraint. Note that λ is treated as an undetermined value. We have converted the m variable constrained optimization problem into an $m + 1$ variable unconstrained optimization problem.

As long as the function u is continuously differentiable, the solutions to the unconstrained optimization problem are given by the solutions to the first order conditions. The system of equations that determines these solutions is given by

$$\begin{aligned} 0 &= u_1(x_1^*, \ldots, x_m^*) - \lambda p_1 \\ &\vdots \\ 0 &= u_m(x_1^*, \ldots, x_m^*) - \lambda p_m \\ 0 &= I - p_1 x_1^* - \ldots - p_m x_m^* \end{aligned}$$

The last equation results from the differentiation of the Lagrangian with respect to the variable λ.

This is a system of $m+1$ equations in $m+1$ unknowns $(x_1^*, ..., x_m^*, \lambda)$ so when u is well-behaved there exists a unique solution to the problem. As long as the the matrix of second partial derivatives has the appropriate properties we can show that the solution corresponds to a maximum.

Example 35 *We will repeat the previous example using the method of Lagrange this time. The Lagrangian for this problem is written as*

$$\alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \lambda(I - p_1 x_1 - p_2 x_2).$$

Taking first partials and setting them to 0 *yields the three equations*

$$\begin{aligned} 0 &= \frac{\alpha_1}{x_1^*} - \lambda p_1 \\ 0 &= \frac{\alpha_2}{x_2^*} - \lambda p_2 \\ 0 &= I - p_1 x_1^* - p_2 x_2^* \end{aligned}$$

Manipulating the first equation yields the relationship

$$\lambda = \frac{\alpha_1}{p_1 x_1^*}.$$

Substituting this into the second equation we get

$$\begin{aligned} \frac{\alpha_2}{x_2^*} &= \frac{\alpha_1 p_2}{p_1 x_1^*} \\ &\Rightarrow \; x_2^* = \frac{\alpha_2 p_1}{\alpha_1 p_2} x_1^*. \end{aligned}$$

If we substitute this relationship into the third equation we find

$$\begin{aligned} 0 &= I - p_1 x_1^* - p_2 \frac{\alpha_2 p_1}{\alpha_1 p_2} x_1^* \\ &\Rightarrow \; 0 = I - p_1 x_1^* (1 + \frac{\alpha_2}{\alpha_1}) \\ &\Rightarrow \; x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{I}{p_1} \end{aligned}$$

Upon substitution we find the same value for x_2^* as before, naturally. With x_1^* or x_2^*, we can solve for λ and find

$$\begin{aligned} \lambda &= \frac{\alpha_1}{p_1 \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{I}{p_1} \right)} \\ &= \frac{\alpha_1 + \alpha_2}{I}. \end{aligned}$$

Typically we set $\alpha_1 + \alpha_2 = 1$, in which case $\lambda = I^{-1}$.

While the coefficient λ may appear to be of little substantive interest, in fact it has an interpretation. It measures "how binding" the constraint actually is, or in other words, how valuable a small relaxation in the constraint would be. In terms of utility maximization problems, we call λ the marginal utility of wealth. Note that in the case of the functional form used in the example (Cobb-Douglas), the marginal utility of wealth is the inverse of income. As income increases the value of "slackening" the constraint decreases, reflecting the fact that the marginal utility of consumption of both goods is decreasing in the level of consumption. When a constraint is not binding, the value of λ is 0, reflecting the fact that relaxing the constraint has no value since it does not impinge on the consumer's choices.

Obviously the same techniques can be used in conjunction with solving the firm's cost minimization problem, for example.

10 Some Important Functions for Economists

In solving problems or providing illustrative examples when conducting theoretical exercises, economists repeatedly tend to use the same convenient functional forms. In this class we almost exclusively use four functional forms to represent utility functions and production functions, which are the building blocks of consumer demand theory and the theory of the firm, respectively. These functional forms are the following.

10.1 Linear

We say that y is a linear function of its arguments $(x_1, ..., x_m)$ if the function can be written as

$$y = \alpha_0 + \alpha_1 x_1 + ... + \alpha_m x_m.$$

Note that this function is continuously differentiable in its arguments, and that

$$\frac{\partial y}{\partial x_i} = \alpha_i, \text{ for all } i = 1, ..., m,$$

and

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = 0, \text{ for all } i, j = 1, ..., m.$$

10.2 Fixed Coefficients

The fixed coefficients, or Leontief (after its originator, who taught at NYU for many years), function is given by

$$y = \min(\alpha_1 x_1, \alpha_2 x_2).$$

It is differentiable except at the set of points for which

$$\begin{aligned} \alpha_1 x_1 &= \alpha_2 x_2 \\ \Rightarrow x_2 &= \frac{\alpha_1}{\alpha_2} x_1. \end{aligned}$$

The first partials of the function are

$$\frac{dy}{dx_1} = \begin{cases} 0 & \text{if} \quad \alpha_1 x_1 > \alpha_2 x_2 \\ \text{not defined} & \text{if} \quad \alpha_1 x_1 = \alpha_2 x_2 \\ \alpha_1 & \text{if} \quad \alpha_1 x_1 < \alpha_2 x_2 \end{cases}$$

$$\frac{dy}{dx_2} = \begin{cases} \alpha_2 & \text{if} \quad \alpha_1 x_1 > \alpha_2 x_2 \\ \text{not defined} & \text{if} \quad \alpha_1 x_1 = \alpha_2 x_2 \\ 0 & \text{if} \quad \alpha_1 x_1 < \alpha_2 x_2 \end{cases}.$$

The Leontief production or utility function is used mainly for conducting applied theoretical analysis.

10.3 Cobb-Douglas

This function, named after its two originators, both of which were economists and one a U.S. Senator, is perhaps the most widely used functional form by applied theorists and empiricists alike. It is given by

$$y = Ax_1^{\alpha_1} \cdots x_m^{\alpha_m}, \tag{2}$$

where all the arguments of the function are restricted to be nonnegative and A and the α_i are positive constants. In this form the function is used to represent the utility function of a consumer or a production function of a firm. Since from the point of view of consumer demand analysis it makes no difference whether a consumer has utility u or $g(u)$, where g is a monotonic increasing function, when representing utility it is not uncommon to posit

$$u = \alpha_1 \ln x_1 + ... + \alpha_m \ln x_m.$$

This is also referred to as a Cobb-Douglas utility function.

The first derivatives of [2] are given by

$$\begin{aligned} \frac{\partial y}{\partial x_i} &= A\alpha_i x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_m^{\alpha_m} \\ &= \alpha_i \frac{y}{x_i}, \; i = 1, ..., m. \end{aligned}$$

Note that these partial derivatives are all positive. The second partials are given by

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \begin{cases} A\alpha_i \alpha_j \frac{y}{x_i x_j} & \text{if} \quad i \neq j \\ A\alpha_j(\alpha_j - 1)\frac{y}{x_j^2} & \text{if} \quad i = j \end{cases}$$

The "own" second partials are also all positive. The "cross" partials can be positive or negative. However when $f(x)$ represents a utility function we typically adopt the assumption that $\sum_i \alpha_i = 1$, which implies that each $\alpha_i \in (0, 1)$. In this case the cross partials are all negative.

10.4 Quasi-Linear

This is a somewhat peculiar function, but one that is useful for constructing applied microeconomics examples and for some theoretical work. It is typically used to represent utility (as opposed to a production function), and has the form

$$u = \alpha_1 \ln x_1 + \alpha_2 x_2.$$

We can see that it looks a bit like a cross between a Cobb-Douglas utility function and a linear one.

The first partials of the function are

$$\begin{aligned} u_1(x_1, x_2) &= \frac{\alpha_1}{x_1} \\ u_2(x_1, x_2) &= \alpha_2, \end{aligned}$$

and the second partials are given by

$$\begin{array}{cc} u_{11}(x_1, x_2) = -\frac{\alpha_1}{x_1^2} & u_{12}(x_1, x_2) = 0 \\ u_{21}(x_1, x_2) = 0 & u_{22}(x_1, x_2) = 0 \end{array}.$$

Figure 1.a
Function Value
$f(x) = 5 + x - 2x^2$

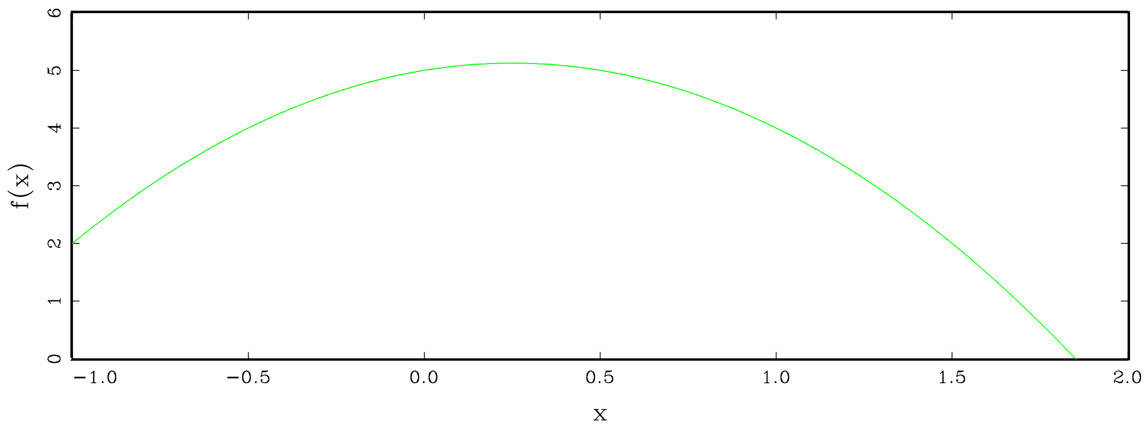

Figure 1.b
First Derivative
$f'(x) = 1 - 4x$