Economic Analysis for Technologists – Spring 2020 - Exam 1 - Solution

Question 1

At peak time, inverse demand is $p_1 = 1 - 0.5q_1$ and at off peak time demand is $p_2 = 1 - q_2$.

a) Capacity cost is 0.1 per unit and operating cost is 0.05 per unit. Find the profit maximizing prices, quantities, and capacity and also maximum profit.

SOLUTION

This one is just like the class example. Assuming $q_1 > q_2$, profit is:

$$\pi = (1-0.5q_1)q_1 - (1-q_2)q_2 - 0.15q_1 - 0.05q_2$$

Setting MR=MC for both periods:

$$\begin{aligned} 1-q_1 &= 0.15 & \qquad & & 1-2q_2 &= 0.05 \\ q_1 &= 0.85 & & \text{and} & q_2 &= 0.5\times 0.95 = 0.475 \\ p_1 &= 1-0.5\times 0.85 = 0.575 & & & p_2 &= 1-0.475 = 0.525 \end{aligned}$$

Indeed, $q_1 > q_2$, so this satisfies the capacity constraint. From there:

$$\pi = (0.575-0.15)0.85 + (0.525-0.05)0.475 = 0.58688$$

b) Continuing from (a), suppose capacity must be chosen before operating cost is known with certainty. With probability 0.2 operating cost will be 0.01 and otherwise it will be 0.06. Find the profit maximizing prices, quantities, and capacity and also maximum expected profit.

SOLUTION

There are potentially four different quantities to consider, how much to sell at peak time if cost is low, q_{1L}, how much to sell at peak time if cost is high, q_{1H}, how much to sell at off peak time if cost is low, q_{2L}, and how much to sell at off peak time if cost is high, q_{2H}. It is apparent that $q_{1L} \geq q_{2L} \geq q_{2H}$, $q_{1L} \geq q_{1H} \geq q_{2H}$. Assuming q_{1L} is higher than all the others, profit is:

$$\begin{aligned} \pi = {} & 0.2\left[(1-0.5q_{1L})q_{1L} - 0.01q_{1L} + (1-q_{2L})q_{2L} - 0.01q_{2L}\right] \\ & +0.8\left[(1-0.5q_{1H})q_{1H} - 0.06q_{1H} + (1-q_{2H})q_{2H} - 0.06q_{2H}\right] \\ & -0.1q_{1L} \end{aligned}$$

Setting MR=MC for q_{1L} and q_{1H} gives:

$$\begin{aligned} 0.2(1-q_{1L}) &= 0.2\times 0.01 + 0.1 & \qquad & & 1-q_{1H} &= 0.06 \\ 1-q_{1L} &= 0.01 + 0.5 & & \text{and} & q_{1H} &= 0.94 \\ q_{1L} &= 0.49 \end{aligned}$$

This violates the assumption that $q_{1L} > q_{1H}$.

So, we must revise the problem so that $q_{1L} = q_{1H} = q_1$. Profit is then:

$$\begin{aligned} \pi = {} & (1-0.5q_1)q_1 - (0.2\times 0.01 + 0.8\times 0.06)q_1 - 0.1q_1 \\ & +0.2\left[(1-q_{2L})q_{2L} - 0.01q_{2L}\right] + 0.8\left[(1-q_{2H})q_{2H} - 0.06q_{2H}\right] \end{aligned}$$

Setting MR=MC for the three quantities gives:

$$\begin{aligned} 1-q_1 &= 0.15 \\ q_1 &= 0.85 \\ p_1 &= 1-0.5\times 0.85 \\ &= 0.575 \end{aligned}, \quad \begin{aligned} 1-2q_{2L} &= 0.01 \\ q_{2L} &= 0.5\times 0.99 \\ &= 0.495 \\ p_{2L} &= 1-0.495 \\ &= 0.505 \end{aligned} \text{, and} \quad \begin{aligned} 1-2q_{2L} &= 0.06 \\ q_{2L} &= 0.5\times 0.94 \\ &= 0.47 \\ p_{2L} &= 1-0.47 \\ &= 0.53 \end{aligned}$$

Since both q_{2L} and q_{2H} are less than $q_{1L}=q_{1H}$, this is feasible, so this is the profit maximizing production and sales plan. Profit is then:

$$\pi = (0.575-0.15)0.85 + 0.2(0.505-0.01)0.495 + 0.8(0.53-0.06)0.47 = 0.58698$$

c) Continuing from b, suppose the firm will operate for two periods. Capacity put in place before the first period will last both periods. The discount rate is 0.05. Capacity cost is 0.1 before the first period, as before. The firm could decide to put in place additional capacity before the second period at a cost of 0.12 per unit, at which time it will know the value of operating cost. After the second period of operation, all capacity may be sold for a scrap value of 0.03 per unit in a third period. Find the profit maximizing prices, quantities, and capacity for each period, and also the maximum expected present value of profit.

ANSWER

To see how to tackle this problem, we have to see that there are now potentially eight different quantities, q_{itc}, where i is peak or off peak (1 or 2), t is time (1 or 2), and c is cost (L or H for Low or High). For example, q_{12L} is the quantity to produce in period 2 for peak demand when cost is low.

Period 1 capacity must be at least as high as all period 1 quantities and period 2 capacity must be at least as high as all period 2 quantities. Start by assuming q_{11L} is higher than all other period 1 quantities and q_{12L} is higher than all other period 2 quantities. Note q_{12L}-q_{11L} is the extra capacity put in place for period 2. We must check that this is positive, not negative. If it turns out to be negative, we must go back and rework assuming q_{12L}=q_{22L}.

We can make a few simplifications immediately. First, as long as off-peak quantity is less than on peak quantity, q_{21L}=q_{22L} and q_{21H}=q_{22H}, so we will denote these q_{2tL} and q_{2tH} respectively. (If that assumption turns out to be wrong, we will have to revisit this.) Second, capacity is more valuable if it is used for both periods. So, the only way we would add capacity after learning operating cost is if operating cost is low. If we would add it regardless of operating cost, we would add it in the first period. This means, then, that q_{11H}=q_{12H}, which we will simply denote q_{1tH}.

With those definitions and assumptions, the problem is as follows:

$$\pi = 0.2\left(1-0.5q_{11L}-0.01\right)q_{11L} + \frac{0.2}{1.05}\left(1-0.5q_{12L}-0.01\right)q_{12L}$$
$$+0.8\left(1+\frac{1}{1.05}\right)\left(1-0.5q_{1tH}-0.06\right)q_{1tH}$$
$$+0.2\left(1+\frac{1}{1.05}\right)\left(1-0.5q_{2tL}-0.01\right)q_{2tL} + 0.8\left(1+\frac{1}{1.05}\right)\left(1-0.5q_{2tH}-0.06\right)q_{2tH}$$
$$-\left(0.1-\frac{0.03}{1.05^2}\right)q_{11L} - 0.2\left(\frac{0.12}{1.05}-\frac{0.03}{1.05^2}\right)\left(q_{12L}-q_{11L}\right)$$

The first order condition for q_{11L}, and its solution, are now:

$$0 = 0.2\left(0.99-q_{11L}\right) - \left(0.1-\frac{0.03}{1.05^2} - 0.2\left(\frac{0.12}{1.05}-\frac{0.03}{1.05^2}\right)\right)$$
$$q_{11L} = 0.99 - 0.5 + \frac{0.15}{1.05^2} + \frac{0.12}{1.05} - \frac{0.03}{1.05^2} = 0.713$$

The first order condition and solution for q_{12L} are as follows.

$$0 = \frac{0.2}{1.05}\left(\left(0.99-q_{12L}\right) - \frac{0.12}{1.05} + \frac{0.03}{1.05^2}\right)$$
$$q_{12L} = 0.903$$

The first order condition and solution for q_{1tH} are as follows.

$$0 = 0.94 - q_{1tH}$$
$$q_{1tH} = 0.94$$

This is wrong, because q_{1tH} exceeds capacity of 0.7131. So, we must impose that $q_{11L}=q_{1tH}$, making the problem:

$$\pi = 0.2\left(1-0.5q_{11L}-0.01\right)q_{11L} + 0.8\left(1+\frac{1}{1.05}\right)\left(1-0.5q_{11L}-0.06\right)q_{11L}$$
$$+\frac{0.2}{1.05}\left(1-0.5q_{12L}-0.01\right)q_{12L}$$
$$+0.2\left(1+\frac{1}{1.05}\right)\left(1-0.5q_{2tL}-0.01\right)q_{2tL} + 0.8\left(1+\frac{1}{1.05}\right)\left(1-0.5q_{2tH}-0.06\right)q_{2tH}$$
$$-\left(0.1-\frac{0.03}{1.05^2}\right)q_{11L} - 0.2\left(\frac{0.12}{1.05}-\frac{0.03}{1.05^2}\right)\left(q_{12L}-q_{11L}\right)$$

The first order condition for q_{11L}, and its solution, are now:

$$0 = 0.2\left(0.99-q_{11L}\right) + \frac{0.8\times 2.05}{1.05}\left(0.94-q_{11L}\right) - \left(0.1-\frac{0.03}{1.05^2} - 0.2\left(\frac{0.12}{1.05}-\frac{0.03}{1.05^2}\right)\right)$$
$$q_{11L} = \left(0.2\times 0.99 + \frac{0.8\times 2.05\times 0.94}{1.05} - 0.1 + \frac{0.024}{1.05^2} + \frac{0.024}{1.05}\right) \Big/ \left(0.2 + \frac{0.8\times 2.05}{1.05}\right)$$
$$q_{11L} = 0.914$$

The first order condition and solution for q_{12L} are as follows.

$$0=\frac{0.2}{1.05}\left(0.99-q_{12L}-\frac{0.12}{1.05}+\frac{0.03}{1.05^{2}}\right)$$

$q_{12L}=0.903$

This means the change in capacity is less than zero, 0.903-0.914<0, violating the assumptions of the problem. So, we must reformulate assuming no new capacity will be put in place. This means all peak periods will be against capacity, as in part b, the period itself does not matter. The problem is then as follows:

$$\pi=\frac{2.05}{1.05}\left(0.95-0.5q_{11L}\right)q_{1tL}-\left(0.1-\frac{0.03}{1.05^{2}}\right)q_{11t}$$
$$+0.2\frac{2.05}{1.05}\left(0.99-0.5q_{2tL}\right)q_{2tL}+0.8\frac{2.05}{1.05}\left(0.94-0.5q_{2tH}-0.06\right)q_{2tH}$$

The first order condition and solution for q_{1tL} are as follows.

$$\pi=\frac{2.05}{1.05}\left(0.95-0.5q_{1tL}\right)q_{1tL}-\left(0.1-\frac{0.03}{1.05^{2}}\right)q_{1tL}$$

$$0=\frac{2.05}{1.05}\left(0.95-q_{1tL}\right)-\left(0.1-\frac{0.03}{1.05^{2}}\right)$$

$$q_{1tL}=0.95-\frac{1.05}{2.05}\left(0.1-\frac{0.03}{1.05^{2}}\right)=0.913$$

$p_{1tL}=0.544$

The solution for off peak times will be just as it was for part b. Expected profit is then:

$$\pi=\left[2.05\left(0.544-0.05\right)-0.1+0.03/1.05^{2}\right]0.913$$
$$+2.05\left[0.2\left(0.505-0.01\right)0.495+0.8\left(0.53-0.06\right)0.47\right].$$
$$=1.252$$

Question 2

Suppose all individuals consume two things, goods, g, and services, s. Their utility function is $u(g,s) = \ln g + \ln s$. (Recall the derivative of $\ln x$ is $1/x$.) The price of goods is p_g, the price of services is p_s, and the individual's income is m. Since consumers are identical, it really does not matter how many there are, so let us just assume there is only one.

a) a) Assume p_s=2, p_g=1, and m=50. Find the consumer's choice of goods and services, and their level of utility. Illustrate the solution in a figure.

SOLUTION

We could set up the Lagrangian to do this formally, but let us simply set the MRS_{gs} equal to the price ratio to find where the indifferce curve will be tangent to the budget line and substitute from that into the equation for the budget line and then solve:

$$\frac{MU_g}{MU_x} = \frac{p_g}{p_s} \qquad \frac{s}{g} = \frac{1}{2} \qquad g = 2s$$

$$50 = g + 2s$$
$$50 = 4s$$
$$s = 12.5$$
$$g = 25$$
$$u = \ln(12.5) + \ln(25) = 5.745$$

See figure below.

b) Assume the government must provide some level of public services to keep society functioning, funded by total tax revenue, R. Suppose R=10 is required. To raise revenue, the government taxes goods, but not services, at a proportional rate of t_g (a sales tax). What tax rate is needed to finance government? What is the consumer's utility? Hint:

i) Start with the consumer's budget constraint, where the price of goods is now $p_g(1+t_g)$.
ii) Solve for the quantity of goods purchased as a function of the level of the tax rate, and from there revenue as a function of the tax rate. If you did it right, you will get $R = 25\frac{t_g}{1+t_g}$, but you must show how to derive that for credit.
iii) Use that to find the required tax rate, the new tax inclusive price, the new consumer purchases of goods and services, and from there the new utility level.

SOLUTION

$$\frac{MU_g}{MU_x} = \frac{(1+t_g)p_g}{p_s} \qquad \frac{s}{g} = \frac{1+t_g}{2} \qquad s = 0.5(1+t_g)g$$

$$50 = 2s + (1+t_g)g$$
$$50 = 2(1+t_g)g$$
$$g = 25/(1+t_g)$$
$$s = 12.5$$

$$R = t_g g$$
$$10 = 25t_g/(1+t_g)$$
$$1 + t_g = 2.5t_g$$
$$1.5t_g = 1$$
$$t_g = 2/3$$

$$g = 25/(5/3) = 15$$
$$u = \ln(12.5) + \ln(15)$$
$$= 5.234$$

c) Assume the government taxes both goods and services at proportionate rates t_g and t_s respectively. What tax rates are needed to finance government? What is the consumer's utility? Hint: If you did it right, you will find the tax rates are identical and considerably lower than they were in part b, and the consumer's utility is higher.

SOLUTION

$$\frac{MU_g}{MU_x} = \frac{(1+t_g)p_g}{(1+t_s)p_s}$$

$$\frac{s}{g} = \frac{1}{2}\frac{(1+t_g)}{(1+t_s)}$$

$$s = \frac{1}{2}\frac{(1+t_g)}{(1+t_s)}g$$

$$50 = (1+t_s)2s + (1+t_g)g$$

$$50 = 2(1+t_g)g$$

$$g = 25/(1+t_g)$$

$$s = 12.5/(1+t_s)$$

$$R = t_g g + t_s 2s$$

$$10 = 25t_g/(1+t_g) + 25t_s/(1+t_s)$$

$$0.4 = t_g/(1+t_g) + t_s/(1+t_s)$$

Any tax rates that satisfy the last equation above will work. So, the question becomes, what tax rate that raises sufficient revenue yields the highest utility. Utility is:

$$u = \ln(25/(1+t_g)) + \ln(12.5/(1+t_s))$$

$$u = 5.745 - \ln(1+t_g) - \ln(1+t_s)$$

So, this becomes a constrained optimization problem. The Lagrangian is:

$$L = 5.745 - \ln(1+t_g) - \ln(1+t_s) + \lambda\left[25t_g/(1+t_g) + 25t_s/(1+t_s) - 10\right]$$

The first two first order conditions are:

$$\frac{\partial L}{\partial t_g} = -\frac{1}{1+t_g} + \lambda\frac{1+t_g-t_g}{(1+t_g)^2} = 0 \quad \text{and} \quad \frac{\partial L}{\partial t_s} = -\frac{1}{1+t_s} + \lambda\frac{1+t_s-t_s}{(1+t_s)^2} = 0$$

$$1+t_g = \lambda \qquad\qquad 1+t_s = \lambda$$

From which it immediately follows that the tax rates are identical. Putting that into the government budget constraint gives:

$$10 = 50t/(1+t)$$

$$50t = 10 + 10t$$

$$40t = 10$$

$$t = 0.25$$

$$g = 25/1.25 = 20$$

$$s = 12.5/1.25 = 10$$

$$u = \ln(20) + \ln(10) = 5.298$$

d) Illustrate the answer to (b) and (c) in a new figure.
See below.

e) Explain intuitively *why* consumers are always happier for any given amount of total tax revenue if it is financed with similar tax rates on all items, rather than taxing only certain items. Carefully examining your figure from (d) will help you understand why this is so.

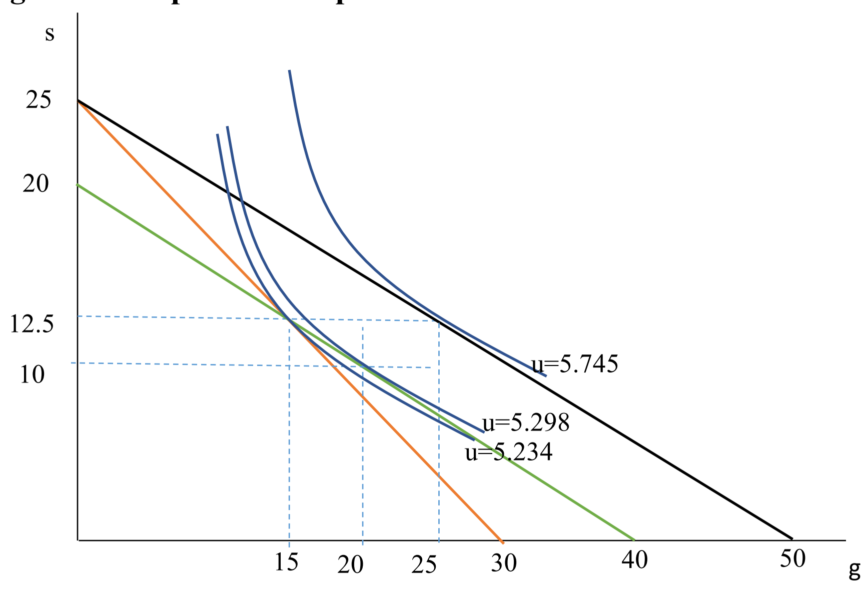
ANSWER

This is simply problem set 4, question 2c, only in reverse, with a tax instead of a subsidy, but a subsidy is only a negative tax. (Compare the figure below to the figure for that problem.) Taxing one good and not the other distorts consumer choice, increasing the cost of taxation.

To see that in the math of consumer optimization, note that to maximize utility, $\frac{MU_g}{MU_x} = \frac{(1+t_g)p_g}{(1+t_s)p_s}$. If the tax rates are the same, they cancel, and this simply becomes $\frac{MU_g}{MU_x} = \frac{p_g}{p_s}$, just like without the tax. Thus, when all goods are taxed at the same rate, consumers choices reflect the true underlying costs of the goods, they are not making choices driven only by decisions about taxes. Thus, for the same tax revenue, in the figure a higher utility can always be reached if both goods are taxed at the same rate, keeping the consumer's budget line parallel to their original budget line.

An important caveat about this is that all consumers are identical in this problem. If some consumers were poorer, and we did not have other means of redistribution, and poorer consumers consumed more of one good or the other, it might be optimal from a distributional point of view to tax more heavily the goods consumed relatively heavily by those that are wealthier. But if redistribution is handled through a progressive income tax, and/or at another level of government in a federal system, this argument is less important. Also, if some goods are harder to tax than others, or cause more tax avoidance costs than others, optimal tax rates would also differ. But, in general, it is almost never optimal to have zero tax rates on goods that are consumer substitutes with other goods taxed at a high rate.

Figure for all parts of the problem



f) In Florida, the state taxes sales on goods from brick and mortar retail establishments at a rate of 6% (localities may add additional sales taxes) but does not tax online purchases or purchases of services. Write a letter to the editor of a newspaper (e.g. the *Tampa Times* or *Lakeland Ledger*) making the case that Florida should replace the sales tax on goods from brick and mortar establishments with an equal tax rate on all goods, whether purchased online or in brick and mortar establishments, and on all services, that is lower than the current rate of 6% but still raises the same level of revenue. Explain why this would make not just owners of brick and mortar establishments better off, but also consumers and taxpayers (who are largely, but not exactly, the same people). Also explain why (you think) there is opposition to expanding the sales tax base to include online sales and services when doing so would clearly benefit the residents of the state overall.

ANSWER

(Yours should cover the same major points for the most credit.)

Florida taxes sales of most goods purchased at brick and mortar retail establishments at a rate of 6%. Sales tax is not paid on services. Sales tax is also not paid on internet purchases. Taxing services and internet purchases, and using the new revenues to reduce the sales tax rate, would not only help the owners of brick and mortar retail establishments, but also benefit Floridian's as consumers while making the state's tax system more robust.

To see why, ignore for the moment differences between customers and differences between the difficulties associated with taxing different types of goods. I will return to these below and show they do not change the conclusion. In a world without such differences, consumers should make choices based on the real costs of the items they purchase, not based on differing tax rates. Any decision made or not made purely because of the tax treatment of the transaction represents a social waste. Differential tax rates distort consumer choices, destroying value added. For example, consider a consumer who buys a $1,000 item online just to avoid the state sales tax of $60, but would otherwise have preferred to buy local for the better customer support. The loss of the opportunity to enjoy that better customer service is purely due to the inefficiency of taxing internet sales and brick and mortar sales differently. Clearly tax rates on all items purchased should be the same in such a world.

Now consider what happens when individuals differ in their level of income, as they do, and where the poor spend relatively more of their income on some items than others. In that case, we might wish to have different tax rates for equity reasons. However, in the United States, there is a progressive income tax at the Federal level precisely for such equity reasons. This means there is far less reason to distort consumer decisions with differential sales tax rates for reasons of equity. Moreover, it is doubtful the poor spend more of their money on services or on internet sales than they do on purchases from brick and mortar retail establishments. So, there is no equity argument against taxing services or internet sales at the state level.

Different tax rates may be justified when some goods are more susceptible to tax avoidance than others. Certainly, avoiding taxes on services may be easier than on purchases of goods, since it may be easier to keep some such transactions off the books. While perhaps things like handyman services may be subject to being billed off the books, many other corporately provided services would have a much harder time avoiding the sales tax. So, while a lower sales tax might be justified for services, surely a rate of zero is not. Regarding internet sales, And, it may be more difficult to administer taxes them, since it requires collecting revenues from out of state vendors shipping to Florida residents. However, nationally state governments are moving to cooperatively enable such collections.[1] Perhaps even more important, as internet sales grow, if this is not addressed, the tax on sales at brick and mortar retail establishments will have to continue to grow to keep up with revenue needs, further distorting choices.

Why then has Florida persisted in not taxing services or internet sales. Who knows, but two explanations seem likely to play a part. First is the simple knee jerk reaction against any new taxes. This fails to realize that narrowing the breadth of the tax base necessarily means raising the tax rates on what remains, which is even worse. Second is the fact that internet sales giants, and some large service industries, are politically powerful and in a position to invest considerable time and money resisting such taxes because. While broadening the tax base is clearly good for Floridian's and for proprietors of brick and mortar retail establishments, it would clearly cut into their profits, and they have the resources to resist it, so far effectively.

[1] https://www.ncsl.org/research/fiscal-policy/e-fairness-legislation-overview.aspx