1 Notes on Matrix Derivation

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Matrix Algebra for OLS Estimator

Big Picture

- Matrix algebra can produce compact notation
- Some programs are matrix oriented
- Excel is a matrix

Dependent Variable

- dependent var is an $n \times 1$ column vector

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

- subscript = index
- use bold typeface

Independent Variables

- k independent vars and a constant term
- thus, $n \times (k+1)$ matrix size

Linear Regression Model

- define B as a $(k+1) \times 1$ vector of coefficients
  U as an $n \times 1$ vector of error terms

  ↳ linear multiple regression in matrix form: $Y = XB + U$
- keep track of the dimensions

First order condition of applying RSS

- OLS estimators are residual sum squares (RSS)

$$\frac{dRSS}{dB_j} = 0 \Rightarrow \sum_{i=1}^{n} x_{ij}\hat{U}_i = 0, \quad (j = 0, 1, \ldots, k)$$

↳ $\hat{U}$ = residual

- system of k+1 equations written as $X'\hat{U} = 0 \rightarrow (k+1) \times 1$ vector of 0

  ↳ transpose of X

OLS Estimators in Matrix Form

- $\hat{B}$ is a $(k+1) \times 1$ vector of OLS estimates

$$X'U = 0$$
$$X'(Y - X\hat{B}) = 0$$
$$X'Y = (X'X)\hat{B}$$
$$\hat{B} = (X'X)^{-1}(X'Y)$$

An important result

$$\hat{B} = (X'X)^{-1}(X'Y) = (X'X)^{-1}(X'(XB+U)) = B + (X'X)^{-1}(X'U)$$

- $\hat{B}$ in general differs from B due to the error U
- B is an unknown constant
- distribution of $\hat{B}$ is the sampling distribution

Statistical Properties of OLS estimator I

- under certain assumptions, the OLS estimator is unbiased

Statistical Properties of OLS estimator II

- most likely $\hat{B}$ is biased for two reasons:
  1) data is not independent
  2) $E(U|X) \neq 0$ which can be contributed to an omitted variable, simultaneity, and measurement error

Statistical Properties of OLS Estimator III

only valid if homoskedasticity holds

$$E((\hat{B} - B)(\hat{B} - B)'|X) = \sigma^2 (X'X)^{-1}$$

Heteroskedasticity

$$E((\hat{B} - B)(\hat{B} - B)'|X) = (X'X)^{-1}(X'\Omega X)(X'X)^{-1}$$

Ω = diagonal matrix

White Sandwich Estimator

$$X'\hat{\Omega}X = \sum_{i=1}^{n} \hat{U}_i^2 X_i X_i'$$

$$(X'X)^{-1}(X'\hat{\Omega}X)(X'X)^{-1}$$

Predicted Values

$$P \equiv X(X'X)^{-1}X'$$

↳ projection matrix

$P = P'$ $\quad PP = P$ $\quad PX = X$

Residuals

$$\hat{U} = Y - \hat{Y} = (I - P)Y = MY$$

$$M \equiv I - P$$

$M = M'$ $\quad MM = M$ $\quad PM = 0$

Frisch Waugh Theorem I