Economic Analysis for Technologists – Spring 2016 - Exam 1 - Solutions

1. Optimal Capacity and Real Option Value

Your firm is bringing a new product to market under the following conditions:

- Inverse demand will be $p = 400 - 0.05q$ in the initial year (*p* is price per unit and *q* is quantity sold) and for every year thereafter that the product has not yet become obsolete.
- The probability the product becomes obsolete after each year is *f*, so the probability it will be produced another year following *x* years of production is $(1-f)^x$.
- The rate of return on alternative safe investments (the firm's discount rate) is *r*.
- A onetime capacity investment of $500\bar{q}$ is required before the first production year to construct a production line capable of producing $q \le \bar{q}$ annually.
- For simplicity, the capacity investment does not depreciate until the product becomes obsolete, at which point it becomes completely worthless.
- The firm makes decisions in a risk neutral manner.
- Initial capacity costs is paid at time t=0. First year revenue and operating costs are received and paid at time t=1. Second year revenue and costs are realized at time t=2. And so on.

Note, if *a*<1, $\sum_{x=1}^{\infty} a^x = \frac{a}{1-a}$. So, when you calculate the expected present value of all future revenue to set up the expected present value of the flow of profits, $\sum_{x=1}^{\infty}\left(\frac{1-f}{1+r}\right)^x = \frac{1-f}{r+f}$.

A) Assume operating cost is \$100 (every unit produced costs \$100 in addition to capacity cost), f=0.05, and r=0.05. Set up the optimization problem and solve for the optimum annual price, sales level, and corresponding expected present value of profit.

Answer

I'll work the problem taking r and f as variables, and just plug them in later, to part C easier, and to illustrate the process better..

Letting π represent annual sales less operating cost, the expected present value of profit is:

$$V = -500q + \frac{(1-f)^0}{(1+r)^1}\pi + \frac{(1-f)^1}{(1+r)^2}\pi + \frac{(1-f)^0}{(1+r)^1}\pi + \cdots$$

To make this easier to work with, factor 1/(1-f) out of the infinite series.

$$V = -500q + \frac{1}{1-f}\left(\frac{(1-f)^1}{(1+r)^1}\pi + \frac{(1-f)^2}{(1+r)^2}\pi + \frac{(1-f)^3}{(1+r)^1}\pi + \cdots\right)$$

Now just simplify.

$$V = \pi \frac{1}{1-f} \sum_{t=1}^{\infty} \left(\frac{1-f}{1+r} \right)^t - 500q$$

$$V = \pi \left(\frac{1}{1-f} \right) \left(\frac{1-f}{r+f} \right) - 500q$$

$$V = \frac{1}{r+f} \pi - 500q$$

Substitute $\pi = (400 - 0.05q)q - 100q = (300 - 0.05q)q$.

$$V = \left(\frac{1}{r+f} \right) (300 - 0.05q)q - 500q.$$

Differentiation gives the first order condition.

$$\frac{dV}{dq} = \left(\frac{1}{r+f} \right) (300 - 0.1q) - 500 = 0.$$

Solving:

$$\begin{aligned} 300 - 0.1q &= 500(r+f) \\ q &= 3000 - 5000(r+f) \end{aligned}.$$

Note q would be smaller if f were larger, so obsolescence was expected sooner, and if r were larger, so that future production was less valuable relative to present expenditures on capacity.

Setting r=0.05 and f=0.05 gives: $q = 3000 - 500 = 2500$, $p = 275$, and $\pi = (275 - 100)2500 = 437500$. So: $V = 10 \times 437500 - 500 \times 2500 = 3125000$.

B) Suppose that, because the product is new, uncertainty about operating cost will not be resolved until the start of the first production year. With probability 0.7 it will be \$115 and otherwise it will be \$65. Since the expected operating cost is \$100, at first glance you might be tempted to think this alone does not change the answer from (A), but it does.

Now, suppose you can pay \$500 per unit capacity at time t=0 for capacity construction started one year before production starts as in (A), and also add a clause to the contract with the company building your production facility giving you an option to put in place additional capacity for \$600 per unit by the start of the second year of production, provided you exercise the option by the 3rd month of the first production year. How much does that option increase the expected present value of profits, again assuming f=0.05, and r=0.05? Assume the second capacity payment, if you exercise the option, is made at time t=1.

Answer

First, what would be the best possible outcome without the option? There is just one capacity, call it q_H, representing the higher production target when operating cost is low. Of course, it need not all be used if operating cost is high, in which case only q_L units of capacity are used. Assuming we will get $q_H > q_L$ in the solution, expected present value is then:

$$V=\left(0.7\left(285-0.05q_L\right)q_L+0.3\left(335-0.05q_H\right)q_H\right)\left(\frac{\left(1-f\right)^0}{\left(1+r\right)^1}+\frac{\left(1-f\right)^1}{\left(1+r\right)^2}+\ldots\right)-500q_H$$

$$V=\left(0.7\left(285-0.05q_L\right)q_L+0.3\left(335-0.05q_H\right)q_H\right)\left(\frac{1}{1-f}\right)\left(\frac{\left(1-f\right)^1}{\left(1+r\right)^1}+\frac{\left(1-f\right)^2}{\left(1+r\right)^2}+\ldots\right)-500q_H$$

$$V=\left(0.7\left(285-0.05q_L\right)q_L+0.3\left(335-0.05q_H\right)q_H\right)\left(\frac{1}{r+f}\right)-500q_H$$

Substituting values for r and f:

$$V=\left(7\left(285-0.05q_L\right)q_L+3\left(335-0.05q_H\right)q_H\right)-500q_H$$

Maximizing gives

$$\frac{\partial V}{\partial q_H}=3\left(335-0.1q_H\right)-500=0$$

$$q_H=3350-5000/3=1683$$

$$\frac{\partial V}{\partial q_L}=7\left(285-0.1q_L\right)=0$$

$$q_L=2850$$

This breaks the assumption we made that $q_H>q_L$. So, we have to reformulate, acknowledging that if cost is high, we can't sell more than capacity. So,

$$V=\frac{1}{r+f}\left(\left(400-0.05q\right)q-\left(0.7\cdot115-0.3\cdot65\right)q\right)-500q$$

$$V=\frac{1}{r+f}\left(300-0.05q\right)q-500q$$

This, it turns out, is exactly the same objective function as in part A, when operating cost was known to be \$100 per unit. But, we can only proceed this way having first shown that $q_H>q_L$ did not hold in the solution. From A, then, $V=3125000$

Now on to the option. First, note you would exercise the option only if operating cost turns out to be \$65—you want to make more if operating cost is lower. Let q_L represent the initial capacity investment, the lower quantity you will produce if operating cost is high. Let q_H be the higher capacity you will want in place to support higher production if operating cost turns out to be low. Thus q_H-q_L is the additional capacity purchased if operating cost turns out to be low. We will just assume that the solution will yield $q_H>q_L$, and build the problem accordingly, only go back and add it as a constraint if that turns out to be an incorrect assumption in the end.

Second, note that in period 1, if cost turns out low, you will just be stuck with lower capacity than you would like, until the additional capacity comes online for year 2.

Expected present value is then:

$$V = 0.7\left(285 - 0.05q_L\right)q_L\left(\frac{\left(1-f\right)^0}{\left(1+r\right)^1} + \frac{\left(1-f\right)^1}{\left(1+r\right)^2} + \ldots\right) - 500q_L$$
$$+\frac{0.3}{1+r}\left(\left(335 - 0.05q_L\right)q_L - 600\left(q_H - q_L\right)\right)$$
$$+0.3\left(335 - 0.05q_H\right)q_H\left(\frac{\left(1-f\right)^1}{\left(1+r\right)^2} + \frac{\left(1-f\right)^2}{\left(1+r\right)^3} + \ldots\right)$$

The first line just says that, with probability 0.7, your initial low capacity choice will turn out to be all you want. Revenue is discounted by r, and from the second year on there is some chance the product will be obsolete, reflected by f.

The second line says that, with probability 0.3, in year 1 you will be stuck with low capacity in production even though you would like more, but will pay to have more capacity, and that is discounted by one year.

The third line says that, with 0.3 probability, from year 2 on you will have higher capacity and so produce more at a lower operating cost, but that has to be adjusted for discounting and by the annual probability of obsolescence.

Now factor 1/(1-f) from the first infinite series and 1/(1+r) from the second.

$$V = 0.7\left(285 - 0.05q_L\right)q_L\left(\frac{1}{1-f}\right)\left(\frac{\left(1-f\right)^1}{\left(1+r\right)^1} + \frac{\left(1-f\right)^2}{\left(1+r\right)^2} + \ldots\right) - 500q_L$$
$$+\frac{0.3}{1+r}\left(\left(335 - 0.05q_L\right)q_L - 600\left(q_H - q_L\right)\right)$$
$$+0.3\left(335 - 0.05q_H\right)q_H\left(\frac{1}{1+r}\right)\left(\frac{\left(1-f\right)^1}{\left(1+r\right)^1} + \frac{\left(1-f\right)^2}{\left(1+r\right)^2} + \ldots\right)$$

Now simplify.

$$V = 0.7\left(285 - 0.05q_L\right)q_L\frac{1}{r+f} - 500q_L$$
$$+\frac{0.3}{1+r}\left(\left(335 - 0.05q_L\right)q_L - 600\left(q_H - q_L\right)\right)$$
$$+0.3\left(335 - 0.05q_H\right)q_H\left(\frac{1}{1+r}\right)\left(\frac{1-f}{r+f}\right)$$

That is the objective function, as a function of r and f. Substituting for them gives:

$$V = 7\left(285 - 0.05q_L\right)q_L - 500q_L$$
$$+0.286\left(\left(335 - 0.05q_L\right)q_L - 600\left(q_H - q_L\right)\right)$$
$$+2.714\left(335 - 0.05q_H\right)q_H$$

Maximizing, letting π_L represent profit per period with low capacity and high cost, π_H represent profit per period with high capacity and low cost, and π' represent profit in year 1 if cost is low but capacity is also low, gives:

$$\frac{\partial V}{\partial q_L} = 7\left(285 - 0.1q_L\right) - 500 + 0.286\left(335 - 0.1q_L + 600\right) = 0$$

$$q_L = \left(7 \cdot 285 + 0.286 \cdot 935 - 500\right)/0.7286 = 2419$$

$$p_L = 400 - 0.05 \cdot 2419 = 279.05$$

$$\pi_L = \left(279.05 - 115\right)2419 = 396837$$

$$\pi' = \left(279.05 - 65\right)2419 = 517787$$

$$\frac{\partial V}{\partial q_H} = 2.714\left(335 - 0.1q_H\right) - 0.286 \cdot 600 = 0$$

$$q_H = 3350 - 2.86 \cdot 600/2.714 = 2718$$

$$p_H = 400 - 0.05 \cdot 2718 = 264.10$$

$$\pi_H = \left(264.1 - 65\right)2718 = 541154$$

$$V = 7 \cdot 396837 - 500 \cdot 2419 + 0.286\left(517787 - 600\left(2718 - 2419\right)\right) + 2.714 \cdot 541154 = 3133830$$

So, the expected present value of profit increased by $\Delta V = 3133830 - 3125000 = 8830$

C) How do changes in f and r influence the profit maximizing choice of capacity, the expected present value of profits, and the value of the option from (B)? Explain intuitively and in plain English.

<u>Answer</u>

In (A) we saw that, without the option, q would be smaller if f were larger, so obsolescence was expected sooner, and if r were larger, so that future production was less valuable relative to present expenditures on capacity. From that it follows V is higher with lower values of f and r. Basically, the future is more valuable if f and r are lower, meaning continuation is more likely (there is more likely to be a future), and the firm is more patient.

The impact of r and f on the option value is slightly more subtle, because increases in r and f would decrease V both with and without the option, so it is not immediately clear what happens to the difference in V. We could approach the problem purely mathematically, but that is not needed. Observe that the option allows one to fit production more closely to the actual cost circumstances over the whole production horizon. We showed above that that is valuable. From there, it is easy to see that the more important the future (lower r and f), the more important having that option will be.

2. John Rawls and Economic Justice

John Rawls (1921-2002) is regarded among the greatest political and moral philosophers of all time. He reasoned about economic and social justice as seen from behind a veil of ignorance. That is, imagine a situation in which no one knows anything about what station they will occupy in life. Income, occupation, parents, race, health, personality, productivity, skill, and all such things are unknown. All anyone knows is that they will occupy some (random) position in the society they design, and thus all individuals are exactly equal in expectation. What kind of society would they want to build? Since they are all identical and equal behind the veil of ignorance, their choice would be unanimous. In that sense, a world so ordered would be just, or fair. (That does not mean that, once the individuals find who they are in that world they chose, they would not then wish for a different world, but then it is too late.)

He thought that, from behind the veil of ignorance, choices of such individuals would be well approximated by the maximin principle, meaning society should be arranged to maximize the wellbeing of its least advantaged member. This became his basic criterion of social and economic justice. Expressing it more formally, letting S be the set of members of society, indexed by i, and U_i be the utility of individual i, the social welfare function (W) we should seek to maximize is $W = \underset{i \in S}{Min}\left(U_i\right)$. Thus the name maximin.

A) Consider 3 possible scenarios for 16 students splitting an allocation of pizza. Every student prefers more slices, up to a total of 3, and there are no salient differences between students other than the number of slices they will get.

1) There is 1 pizza, each student gets one-half of a slice.

2) There are 3 pizzas, randomly 10 students get 1 slice, 4 get 2 slices, and 2 get 3 slices.

3) There are 4 pizzas, randomly 1 student gets 0 slices, 13 get 2 slices, and 2 get 3 slices.

By the maximin principle, scenario 2 is preferred, because the worst off student gets one slice of pizza, rather than 0 or 0.5.

Suppose that, from behind the veil of ignorance, everyone chooses rationally—meaning preferences over probability distributions exhibit completeness, continuity, transitivity, and independence, and so can be represented by a von Neuman-Morgenstern utility function (u). Since everyone is identical behind the veil of ignorance, each has the same utility function, and so they will rank possible worlds identically. Normalize the utility of zero slices to 0 ($u(0)=0$).

i) Show that if $u\left(2\right)\frac{10}{9} > u\left(1\right)$, scenario 3 is unanimously preferred to scenario 2 from behind the veil of ignorance.

ii) Sketch a utility function for which the individual is indifferent between scenario 2 and scenario 3, that is for which $u\left(2\right) = \frac{10}{9}u\left(1\right)$. As u is only unique to an affine transformation, I suggest normalizing $u(1)$ to 1 for the sketch.

iii) What does this suggest about the relation between the maximin criterion and risk aversion? In particular, does it seem likely that, from behind the veil of ignorance, preferences of (rational) individuals would correspond quite closely to the maximin criterion?

Answer

i) The expected utility of option 3 is at least as high as option 2 if:

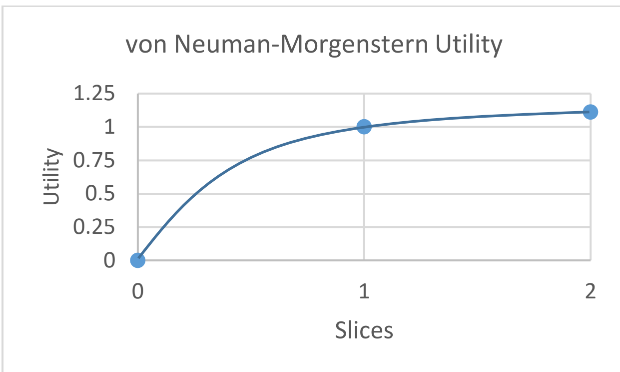
$$\frac{1}{16}u(0)+\frac{0}{16}u(1)+\frac{13}{16}u(2)+\frac{2}{16}u(3)>\frac{10}{16}u(1)+\frac{4}{16}u(2)+\frac{2}{16}u(3)\text{, or if:}$$

$$u(0)+13u(2)+2u(3)>10u(1)+4u(2)+2u(3).$$

Using u(0)=0 and collecting terms:

$$9u(2)>10u(1)\text{, or } u(2)>\frac{10}{9}u(1).$$

ii) For the sketch, then, u(0)=0, u(1)=1, and u(2)=10/9:



iii) This sketch shows a very concave the utility function, or, high risk aversion. Avoiding 0 slices is crucial, and a second slice is worth next to nothing comparatively. If the increase in utility of going from the first to second slice is not so small compared to the utility of going from no slices to the first slice, option 3 is better. This suggests that if people are willing to accept a small chance at a worse outcome in exchange for a large chance at a much better outcome, choices from behind the veil of ignorance will not conform particularly closely to the minimax principle.

B) Rawlsian notions of justice are often used, sometimes in a logically correct way and sometimes in a logically incorrect way, to argue for income redistribution. In this problem, you will analyze a redistribution scheme to see if it is consistent with Rawlsian notions of justice.

There are 2 individuals. Both produce output for consumption through their work (labor). Let q_i represent the amount individual i works. The amount worked reflects intensity as well as time—working very intensely for 30 hours may be more work than goofing off for 60 hours. The disutility of work is $0.5q_i^2$ for both types. This reflects that, as more effort is allocated to work, there is increasingly less chance to enjoy one's family or leisure activities. Let y_1 represent individual 1's income and y_2 represent individual 2's income. With that as context, the crucial information to work the problem is as follows.

- Individual 1's utility is $u_1 = y_1 - 0.5q_1^2$.
- Individual 2's utility is $u_2 = y_2 - 0.5q_2^2$.
- Individual 1 produces 1 unit of output per unit of work.
- Individual 2 produces $\alpha < 1$ units of output per unit of work.

i) If each earns income equal to their production, find each individual's utility maximizing choice of q and use it to find their income and utility.

ii) Suppose income is redistributed so both earn $y = 0.5q_1 + 0.5\alpha q_2$. Find each individual's utility maximizing choice of q and use it to get income and utility. This one is a bit more complicated than (i) in that each individual's income depends on the other individual's work. But, each individual chooses only their own level of work, and treats the other's as a constant in making their own work decision. So, when you take the derivative of one's utility function to find their optimal work level, you would, in general, find their level of work, but *as a function of both* their productivity (1 or α) and the other's level of work. You would have to solve these 2 equations simultaneously for the 2 unknowns. You would, in general, find the solutions for the amount worked differ from those in (i), and that both depend on α. In this case, though, I have made the example simple enough that incomes will depend on both individual's work decision, but the work decision of one does not impact the work decision of the other. Just be aware this is a simplification of the general situation.

iii) Does individual 2 receive more or less income with redistribution than without? If it depends on the value of α, for which values of α does individual 2 receive more income with redistribution than without?

iv) Does individual 2 receive more or less utility with redistribution than without? If it depends on the value of α, for which values of α does individual 2 receive more utility with redistribution than without?

v) Which individual is worse off under redistribution?

vi) Putting all that together, discuss whether this income redistribution scheme is consistent with a Rawlsian notion of social and economic justice.

<u>*Answer*</u>

$$u_1 = q_1 - 0.5q_1^2 \qquad u_2 = \alpha q_2 - 0.5q_2^2$$

$$\frac{\partial u_1}{\partial q_1} = 1 - q_1 = 0 \qquad \frac{\partial u_2}{\partial q_2} = \alpha - q_2 = 0$$

i) $q_1 = 1$ and $q_2 = \alpha$

$$y_1 = 1 \qquad y_2 = \alpha^2$$

$$u_1 = 1 - 0.5 = 0.5 \qquad u_2 = \alpha^2 - 0.5\alpha^2 = 0.5\alpha^2$$

$$u_1 = 0.5q_1 + 0.5\alpha q_2 - 0.5q_1^2 \qquad u_2 = 0.5q_1 + 0.5\alpha q_2 - 0.5q_2^2$$

ii) $\dfrac{\partial u_1}{\partial q_1} = 0.5 - q_1 = 0$ and $\dfrac{\partial u_2}{\partial q_2} = 0.5\alpha - q_2 = 0$

$q_1 = 0.5$ $\qquad$ $q_2 = 0.5\alpha$

so

$$\begin{aligned} y &= 0.25 + 0.25\alpha^2 \\ u_1 &= 0.25 + 0.25\alpha^2 - 0.125 \\ &= 0.125 + 0.25\alpha^2 \\ u_2 &= 0.25 + 0.25\alpha^2 - 0.125\alpha^2 \\ &= 0.25 + 0.125\alpha^2 \end{aligned}$$

iii) Individual 2 receives more income under redistribution if:

$$0.25 + 0.25\alpha^2 > \alpha^2$$
$$0.25 > 0.75\alpha^2$$
$$1/3 > \alpha^2$$
$$\alpha < \sqrt{1/3} = 0.58$$

Individual 2 receives more income under redistribution if their earning power is less than 58% of individual 1's.

iv) Individual 2 receives more utility under redistribution if:

$$0.25 + 0.125\alpha^2 > 0.5\alpha^2$$
$$0.25 > 0.375\alpha^2$$
$$2/3 > \alpha^2$$
$$\alpha < \sqrt{2/3} = 0.82$$

Individual 2 experiences a higher utility under redistribution if their earning power is less than 82% of individual 1's.

v) The difference between the utilities of individual 1 and individual 2 is:

$$\begin{aligned} u_1 - u_2 &= 0.125 + 0.25\alpha^2 - 0.25 - 0.125\alpha^2 \\ &= 0.125\alpha^2 - 0.125 \\ &= 0.125\left(\alpha^2 - 1\right) < 0 \end{aligned}$$

So, under redistribution, individual 1 is now worse off than individual 2.

vi) This particularly extreme form of redistribution is not Rawlsian. It has redistributed so heavily that it actually reverses the positions of the individuals. Individual 1 has become the disadvantaged one, not individual 2, because while their incomes are the same, individual 1 incurs $1/\alpha^2$ times more disutility of work (e.g., if α=0.5, individual 1 incurs 4 times the disutility of work of individual 2. Further, if individual 1 is only modestly less productive than individual 2, this scheme actually makes both worse off, in both utility and in income, though their lower incomes are equal.