Introduction to Time Series Analysis

Handout 2: Stationary Processes. Wold Decomposition and ARMA processes

Laura Mayoral

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- This lecture introduces the basic linear models for stationary processes.

- Most economic variables are non-stationary.

- However, stationary linear models are used as building blocks in more complicated nonlinear and/or non-stationary models.

Roadmap

- The Wold decomposition
- From the Wold decomposition to the ARMA representation
- MA processes and invertibility
- AR processes, stationarity and causality
- ARMA, invertibility and causality.

The Wold Decomposition

Wold theorem in words:

Any stationary process {Zt} can be expressed as a sum of two components:

- a stochastic component: a linear combination of lags of a white noise process.

- a deterministic component, uncorrelated with the latter stochastic component.

The Wold Decomposition

If $\{Z_t\}$ is a nondeterministic stationary time series, then:

$$Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} + V_t = \Psi(L) a_t + V_t,$$

where

1. $\psi_0 = 1$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$,
2. $a_t = Z_t - P(Z_t \mid Z_{t-1}, Z_{t-2}, \ldots)$, where $P(. \mid .)$ denotes linear projection.
3. $\{a_t\}$ is $WN(0, \sigma^2)$, with $\sigma^2 > 0$,
3. $Cov(a_s, V_t) = 0$ for all s and t,
4. The ψ_i's and the a's are unique.
5. $\{V_t\}$ is deterministic.

Importance of the Wold Decomposition

• This theorem implies that any stationary process can be written as a linear combination of a lagged values of a white noise process (this is the MA(∞) representation).

• By inverting the corresponding polynomial, we can obtain a representation of Zt that depends on past values of the variable and the contemporaneous value of the white noise.

• This is the AR(∞) representation of Zt.

• We will see that the AR representation can be estimated using standard methods: OLS!

• Problem: we might need to estimate a lot of parameters.

• ARMA models: they are an approximation to former representations that tries to be more parsimonious (=less parameters)

Birth of ARMA(p,q) models

Under general conditions the infinite lag polynomial of the Wold Decomposition can be approximated by the ratio of two finite lag polynomials:

$$\Psi(L) \approx \frac{\Theta_q(L)}{\Phi_p(L)}$$

Therefore

$$Z_t = \Psi(L)a_t \approx \frac{\Theta_q(L)}{\Phi_p(L)} a_t \ ,$$

$$\Phi_p(L) Z_t = \Theta_q(L) a_t$$

$$(1 - \phi_1 L - \ldots - \phi_p L^p) Z_t = (1 + \theta_1 L + \ldots + \theta_q L^q) a_t$$

$$\underbrace{Z_t - \phi_1 Z_{t-1} - \ldots - \phi_p Z_{t-p}}_{\textbf{AR(p)}} = \underbrace{a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}}_{\textbf{MA(q)}}$$

MA(q) processes

Moving Average of order 1, MA(1)

Let $\{a_t\}$ a zero-mean white noise process $a_t \rightarrow (0, \sigma_a^2)$

$$Z_t = \mu + a_t + \theta a_{t-1} \quad \rightarrow MA(1)$$

- Expectation

$$E(Z_t) = \mu + E(a_t) + \theta E(a_{t-1}) = \mu$$

- Variance

$$Var(Z_t) = E(Z_t - \mu)^2 = E(a_t + \theta a_{t-1})^2 =$$

$$= E(a_t^2 + \theta^2 a_{t-1}^2 + 2\theta a_t a_{t-1}) = \sigma_a^2(1 + \theta^2)$$

Autocovariance

1st. order

$$E(Z_t - \mu)(Z_{t-1} - \mu) = E(a_t + \theta a_{t-1})(a_{t-1} + \theta a_{t-2}) =$$

$$= E(a_t a_{t-1} + \theta a_{t-1}^2 + \theta a_t a_{t-2} + \theta^2 a_{t-1} a_{t-2}) = \theta \sigma_a^2$$

-Autocovariance of higher order

$$E(Z_t - \mu)(Z_{t-j} - \mu) = E(a_t + \theta a_{t-1})(a_{t-j} + \theta a_{t-j-1}) =$$

$$= E(a_t a_{t-j} + \theta a_{t-1} a_{t-j} + \theta a_t a_{t-j-1} + \theta^2 a_{t-1} a_{t-j-1}) = 0 \quad j > 1$$

- Autocorrelation

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2}$$

$$\rho_j = 0 \quad j > 1$$

Partial autocorrelation

$$\phi_{11} = \rho_1 = \frac{-\theta_1}{1+\theta_1^2} = \frac{-\theta_1(1-\theta_1^2)}{1-\theta_1^4}$$

$$\phi_{22} = -\frac{\rho_1^2}{1-\rho_1^2} = \frac{-\theta_1^2}{1+\theta_1^2+\theta_1^4} = \frac{-\theta_1^2(1-\theta_1^2)}{1-\theta_1^6}$$

$$\phi_{33} = \frac{\rho_1^3}{1-2\rho_1^2} = \frac{-\theta_1^3}{1+\theta_1^2+\theta_1^4+\theta_1^6} = \frac{-\theta_1^3(1-\theta_1^2)}{(1-\theta_1^8)}.$$

In general,

$$\phi_{kk} = \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}}, \qquad \text{for } k \geq 1.$$

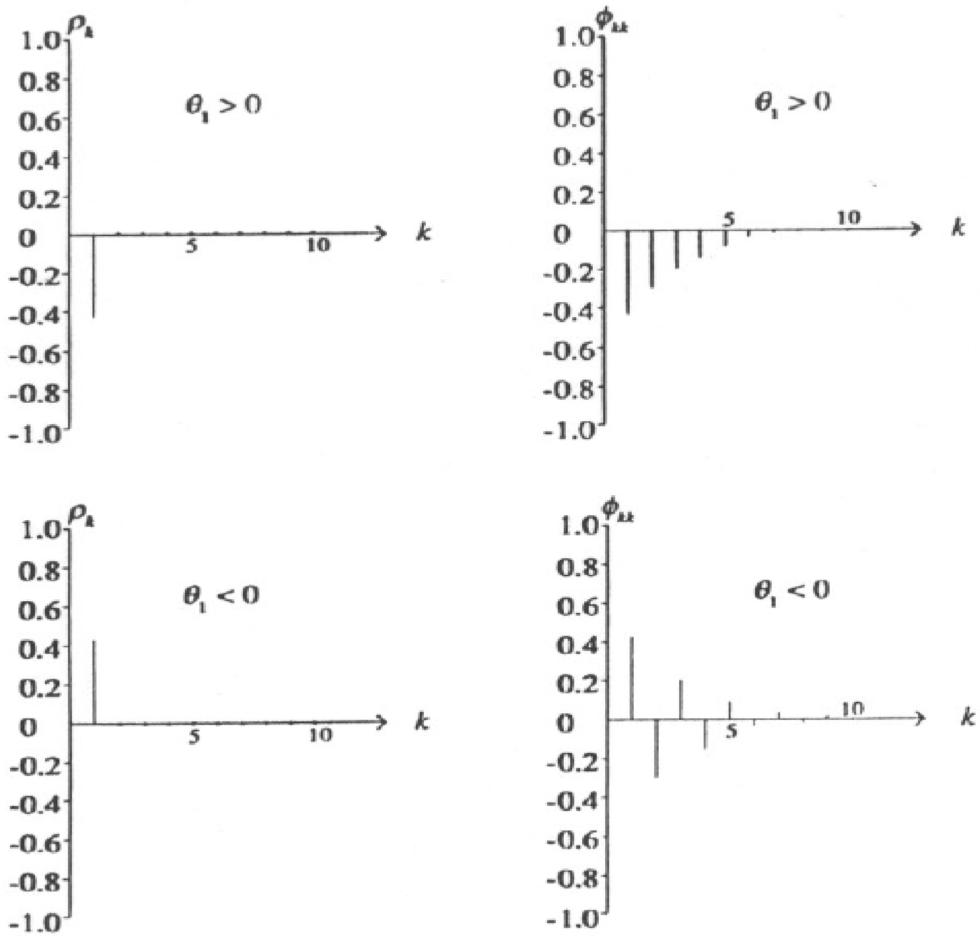

Fig. 3.10 ACF and PACF of MA(1) processes: $\dot{Z}_t = (1 - \theta B)a_t$.

Contrary to its ACF, which cuts off after lag 1, the PACF of an MA(1) model tails off exponentially in one of two forms depending on the sign of θ_1 (hence on the sign of ρ_1). If alternating in sign, it begins with a positive value; otherwise, it decays on the negative side, as shown in Figure 3.10. We also note that $|\phi_{kk}| < 1/2$.

MA(1): Stationarity and Ergodicity

Stationarity
MA(1) process is always covariance-stationary because

$$E(Z_t) = \mu \quad Var(Z_t) = (1+\theta^2)\sigma^2$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2}$$

$$\rho_j = 0 \quad j > 1$$

Ergodicity
MA(1) process is ergodic for first and second moments because

$$\sum_{j=0}^{\infty} \left|\gamma_j\right| = \sigma^2(1+\theta^2) + \left|\theta\sigma^2\right| < \infty$$

If a_t were Gaussian, then Z_t would be ergodic for all moments

MA(q) processes

A process is MA(q) if it can be written as a linear combination of q lags of a white noise process.

$$Z_t = \mu + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}$$

First and Second moments of a MA(q)

$$E(Z_t) = \mu$$

$$\gamma_0 = \text{var}(Z_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_a^2$$

$$\gamma_j = E(a_t + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q})(a_{t-j} + \theta_1 a_{t-j-1} + \cdots + \theta_q a_{t-j-q})$$

$$\gamma_j = \begin{cases} (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \cdots + \theta_q \theta_{q-j})\sigma^2 \text{ for } j \le q \\ 0 \quad \text{for } j > q \end{cases}$$

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \cdots + \theta_q \theta_{q-j}}{\sum_{i=1}^{q} \theta_i^2}$$

Example MA(2)

$$\rho_1 = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \quad \rho_2 = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} \quad \rho_3 = \rho_4 = \cdots = \rho_k = 0$$

Invertibility: definition

- A MA(q) process is said to be *invertible* if there exists a sequence of constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$a_t = \sum_{j=0}^{\infty} \pi_j Z_{t-j}, \qquad t = 0, \pm 1, \ldots$$

- In other words, Z_t is invertible if it admits and autoregressive representation.

Necessary and Sufficient Conditions for Invertibility

Theorem:

Let $\{Z_t\}$ *be a MA(q). Then* $\{Z_t\}$ *is invertible if and only if* $\theta(x) \neq 0$ for all $x \in C$ such that $|x| \leq 1$.

The coefficients $\{\pi_j\}$ *are determined by the relation:*

$$\pi(x) = \sum_{j=0}^{\infty} \pi_j x^j = \frac{1}{\theta(x)}, \quad |x| \leq 1.$$

MA processes are not uniquely identified

Consider the autocorrelation function of these two MA(1) processes:

$$Z_t = \mu + a_t + \theta a_{t-1} \qquad Z^*_t = \mu + a^*_t + (1/\theta) a^*_{t-1}$$

The autocorrelation functions are:

$$1)\ \rho_1 = \frac{\theta}{1+\theta^2}$$

$$2)\ \rho^*_1 = \frac{1/\theta}{1+(1/\theta)^2} = \frac{\theta}{1+\theta^2}$$

MA processes are not uniquely identified, II

- Then, these two processes show identical correlation pattern: The MA coefficient is not uniquely identified.

- In other words: any MA(1) process has two representations (one with MA parameter larger than 1, and the other, with MA parameter smaller than 1).

MA processes are not uniquely identified, III

•This means that each MA(1) has two representations: one that is invertible, another one that is not.

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•We prefer representations that are *invertible so* we will choose the representation with **θ<1.**

MA processes are not uniquely identified, IV

- The same problem is present for MA(q) processes.

- In this case, one needs to look at the roots of the MA(q) polynomial: roots smaller than 1 imply non-invertibity.

- There is always an invertible representation, obtained by inverting the root that is smaller than 1.

Exercise

Consider a MA(1) process with a MA coefficient equal to 1.3

1) Is it stationary? Is it ergodic?

2) is it invertible? If it is not, suggest an alternative representation that has identical autocorrelation structure and is invertible

MA(infinite)

This is the most general MA process.

It contains and infinite number of lags of a white noise process.

$$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j} \qquad \psi_0 = 1$$

MA(infinite): moments

$$E(Z_t) = \mu, \quad Var(Z_t) = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i^2$$

$$\gamma_j = E\left[(Z_t - \mu)(Z_{t-j} - \mu)\right] = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+j}$$

$$\rho_j = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+j}}{\sum_{i=0}^{\infty} \psi_i^2}$$

MA(infinite): stationarity condition

Notice that in order to define the second order moments we need

$$\sum_{i=0}^{\infty} \psi_i^2 < \infty$$

The process is covariance-stationary provided the former condition holds.

Some interesting results

Proposition 1.

$$\sum_{i=0}^{\infty} |\psi_i| < \infty \Rightarrow \sum_{i=0}^{\infty} \psi_i^2 < \infty$$

(absolutely summable) (square summable)

Proposition 2.

$$\sum_{i=0}^{\infty} |\psi_i| < \infty \Rightarrow \sum_{i=0}^{\infty} |\gamma_i| < \infty$$

Ergodic for second moments

Proof 1. $\sum_{i=0}^{\infty} |\psi_i| < \infty \Rightarrow \sum_{i=0}^{\infty} \psi_i^2 < \infty$

If $\sum_{i=0}^{\infty} |\psi_i| < \infty \Rightarrow \exists\, N < \infty$ such that $|\psi_i| < 1 \quad \forall i \geq N$

$$\psi_i^2 < |\psi_i| \quad \forall i \geq N \Rightarrow \sum_{i=N}^{\infty} \psi_i^2 < \sum_{i=N}^{\infty} |\psi_i|$$

Now,

$$\sum_{i=0}^{\infty} \psi_i^2 = \sum_{i=0}^{N-1} \psi_i^2 + \sum_{i=N}^{\infty} \psi_i^2 < \underbrace{\sum_{i=0}^{N-1} \psi_i^2}_{(1)} + \underbrace{\sum_{i=N}^{\infty} |\psi_i|}_{(2)}$$

(1) It is finite because N is finite

(2) It is finite because is absolutely summable

then

The picture can't be displayed.

Proof 2.

$$\gamma_j = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+j}$$

$$\left|\gamma_j\right| = \sigma^2 \left| \sum_{i=0}^{\infty} \psi_i \psi_{i+j} \right| \le \sigma^2 \sum_{i=0}^{\infty} \left|\psi_i \psi_{i+j}\right|$$

$$\sum_{j=0}^{\infty} \left|\gamma_j\right| \le \sigma^2 \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left|\psi_i \psi_{i+j}\right| = \sigma^2 \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left|\psi_i\right| \left|\psi_{i+j}\right| =$$

$$= \sigma^2 \sum_{i=0}^{\infty} \left|\psi_i\right| \sum_{j=0}^{\infty} \left|\psi_{i+j}\right| < \sigma^2 \sum_{i=0}^{\infty} \left|\psi_i\right| M < \sigma^2 M^2 < \infty$$

because by assumption $\sum_{j=0}^{\infty} \left|\psi_{i+j}\right| < M$

AR(p) processes

AR(1) process

An autoregressive process Z is a function of its own past and a contemporaneous value of a white noice sequence

$$Z_t = c + \phi Z_{t-1} + a_t$$

AR(1): Stationarity

AR(1) process is stationary if $|\phi| < 1$

$$Z_t = c + \phi c + \phi^2 Z_{t-2} + \phi a_{t-1} + a_t =$$

$$= \underbrace{c(1 + \phi + \phi^2 + \cdots)}_{\text{geometric progression}} + \underbrace{a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots}_{MA(\infty)}$$

if $|\phi| < 1 \Rightarrow$

(1) $1 + \phi + \phi^2 + \cdots = \dfrac{1}{1-\phi}$ bounded sequence

(2) $\sum_{j=0}^{\infty} \psi^2{}_j = \sum_{j=0}^{\infty} \phi^{2j} = \dfrac{1}{1-\phi^2} < \infty$ if $|\phi| < 1$

$\sum_{j=0}^{\infty} \psi^2{}_j < \infty$ is a sufficient condition for stationarity

AR(1): First and second order moments

Mean of a stationary AR(1)

$$Z_t = \frac{c}{1-\phi} + a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots$$

$$\mu = E(Z_t) = \frac{c}{1-\phi}$$

Variance of a stationary AR(1)

$$\gamma_0 = \left(1 + \phi^2 + \phi^4 + \cdots\right)\sigma^2 = \frac{1}{1-\phi^2}\sigma_a{}^2$$

Autocovariance of a stationary AR(1)

- You need to solve a system of equations:

$$\gamma_j = E\left[(Z_t - \mu)(Z_{t-j} - \mu)\right] = E\left[\left(\phi(Z_{t-1} - \mu) + a_t\right)(Z_{t-j} - \mu)\right] =$$

$$= \phi E\left[(Z_{t-1} - \mu)(Z_{t-j} - \mu) + a_t(Z_{t-j} - \mu)\right] = \phi\gamma_{j-1}$$

$$\gamma_j = \phi\gamma_{j-1} \qquad j \geq 1$$

Autocorrelation of a stationary AR(1)

$$\rho_j = \frac{\gamma_j}{\gamma_o} = \phi\frac{\gamma_{j-1}}{\gamma_0} = \phi\rho_{j-1} \qquad j \geq 1$$

$$\rho_j = \phi^2\rho_{j-2} = \phi^3\rho_{j-3} = \cdots = \phi^j\rho_0 = \phi^j$$

AR(1): Partial autocorrelation function

PACF: from Yule-Walker equations

$$\phi_{11} = \rho_1 = \phi$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\phi^2 - \phi^2}{1 - \rho_1^2} = 0$$

$$\phi_{kk} = 0 \quad k \geq 2$$

AR(1): Ergodicity

A stationary AR(1) process is ergodic for first and second moments.

Show this as an exercise.

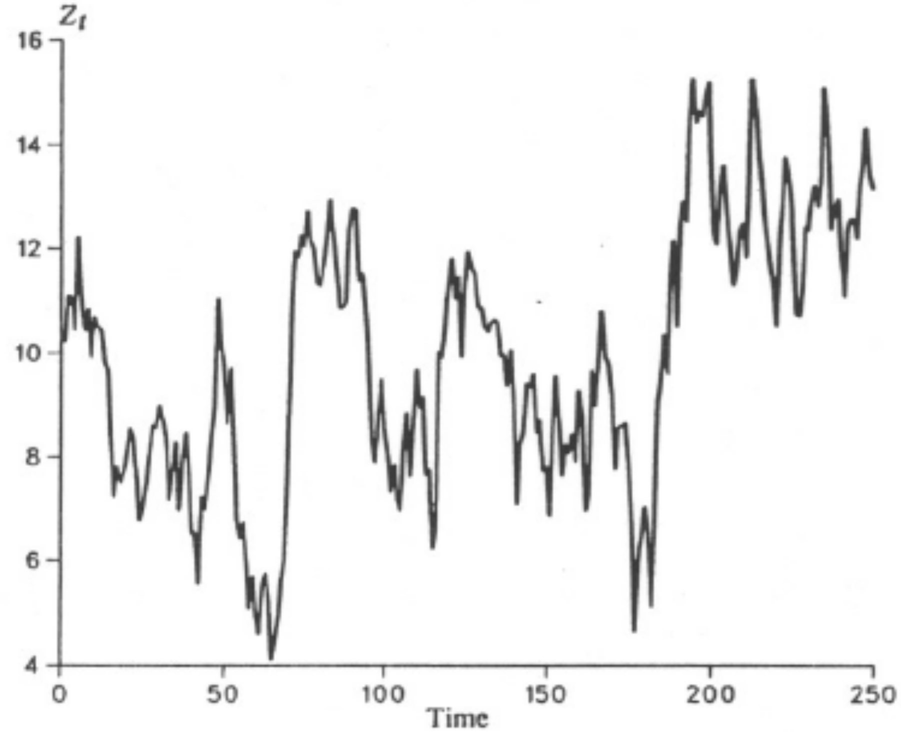


Fig. 3.2 A simulated AR(1) series, $(1 - .9B)(Z_t - 10) = a_t$.

Table 3.1 Sample ACF and sample PACF for a simulated series from $(1 - .9B)(Z_t - 10) = a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	.88	.76	.67	.57	.48	.40	.34	.28	.21	.17
St.E.	.06	.10	.12	.14	.14	.15	.16	.16	.16	.16
$\hat{\phi}_{kk}$	.88	.01	−.01	−.11	.02	−.01	.01	−.02	−.06	.05
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

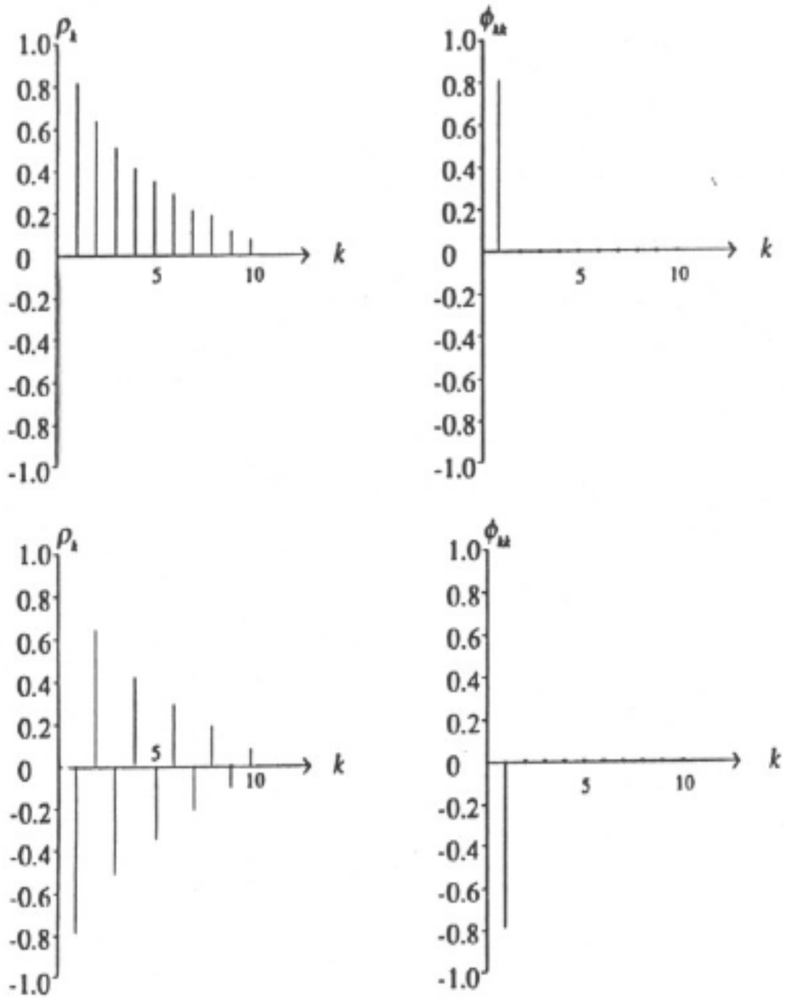


Fig. 3.1 ACF and PACF of the AR(1) process: $(1-\phi B)\dot{Z}_t = a_t$.

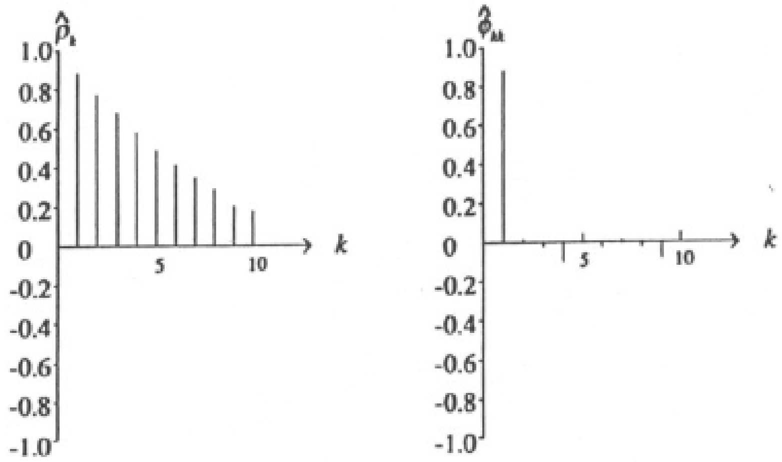


Fig. 3.3 Sample ACF and sample PACF of a simulated AR(1) series: $(1 - .9B)(Z_t - 10) = a_t$.

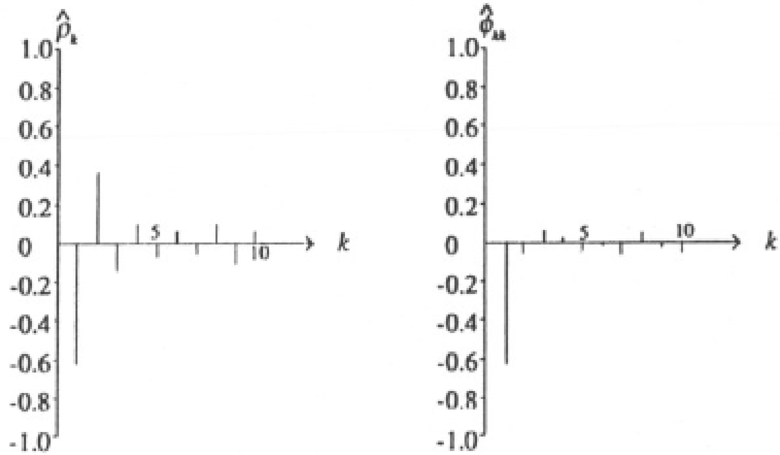


Fig. 3.5 Sample ACF and sample PACF of a simulated AR(1) series $(1 + .65B)(Z_t - 10) = a_t$.

Causality and Stationarity

Consider the AR(1) process, $Z_t = \phi_1 Z_{t-1} + a_t$

Iterating we obtain

$$Z_t = a_t + \phi_1 a_t + \ldots + \phi_1^{\,k} a_{t-k} + \phi_1 Z_{t-k-1}.$$

If $|\phi_1| < 1$ we showed that

$$Z_t = \sum_{j=0}^{\infty} \phi_1^{\,j} a_{t-j}$$

This cannot be done if $|\phi_1| \geq 1$, (no mean-square convergence)

However, in this case one could write

$$Z_t = \phi_1^{-1} Z_{t+1} - \phi_1^{-1} a_{t+1}$$

Then, $Z_t = -\sum_{j=0}^{\infty} \phi_1^{-j} a_{t+j}$

and this is a stationary representation of Z_t.

Causality and Stationarity, II

However, this stationary representation depends on future values of a_t

It is customary to restrict attention to AR(1) processes with $|\phi_1| < 1$

Such processes are called stationary but also CAUSAL, or future-indepent AR representations.

Causality and Stationarity, III

Definition: An AR(p) process defined by the equation

$$\phi_p(L)Z_t = a_t$$

is said to be ***causal***, or a ***causal function of*** $\{\mathbf{a_t}\}$,

if there exists a sequence of constants

$\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}, \quad t = 0, \pm 1, \ldots$

- A necessary and sufficient condition for causality is

$$\phi(x) \neq 0 \text{ for all } x \in C \text{ such that } |x| \leq 1.$$

AR(2)

$$Z_t = c + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

Stationarity $\longrightarrow$ Study of the roots of the characteristic equation

Let $1/\alpha_1$ and $1/\alpha_2$ be the roots of the AR polynomial

such that $1 - \phi_1 L - \phi_2 L^2 = (1 - \alpha_1 L)(1 - \alpha_2 L)$.

Then, $\phi_1 = \alpha_1 + \alpha_2$ and $\phi_2 = -(\alpha_1 \alpha_2)$

Z_t is stationary iff : $|\alpha_i| < 1, i = \{1,2\}$.

First and Second order moments

Mean of AR(2)

$$E(Z_t) = c + \phi_1 E(Z_t) + \phi_2 E(Z_t) \Rightarrow$$

$$E(Z_t) = \mu = \frac{c}{1 - \phi_1 - \phi_2}$$

Variance

$$\gamma_0 = E(Z_t - \mu)^2 = \phi_1 E(Z_{t-1} - \mu)(Z_t - \mu) + \phi_2 E(Z_{t-2} - \mu)\left(Z_t - \mu\right) + E(Z_t - \mu)a_t$$

$$\gamma_0 = \phi_1 \gamma_{-1} + \phi_2 \gamma_{-2} + \sigma^2{}_a$$

$$\gamma_0 = \phi_1 \rho_1 \gamma_0 + \phi_2 \rho_2 \gamma_0 + \sigma^2{}_a$$

$$\gamma_0 = \frac{\sigma^2{}_a}{1 - \phi_1 \rho_1 - \phi_2 \rho_2}$$

Autocorrelation function

$$\gamma_j = E(Z_t - \mu)(Z_{t-j} - \mu) = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \quad j \geq 1$$

Difference equation:

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} \quad j \geq 1$$

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} \qquad j \geq 1$$

$$\left.\begin{array}{ll} j = 1 & \rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 \\ j = 2 & \rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0 \end{array}\right\} \rightarrow \left.\begin{array}{l} \rho_1 = \dfrac{\phi_1}{1 - \phi_2} \\ \\ \rho_2 = \dfrac{\phi_1^{\ 2}}{1 - \phi_2} + \phi_2 \end{array}\right\}$$

$$j = 3 \qquad \rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1$$

Partial Autocorrelations

Partial autocorrelations: from Yule-Walker equations

$$\phi_{11} = \rho_1 = \frac{\phi_1}{1-\phi_2}; \quad \phi_{22} = \frac{\rho_2 - \rho_1^{\;2}}{1-\rho_1^{\;2}}; \quad \phi_{33} = 0$$

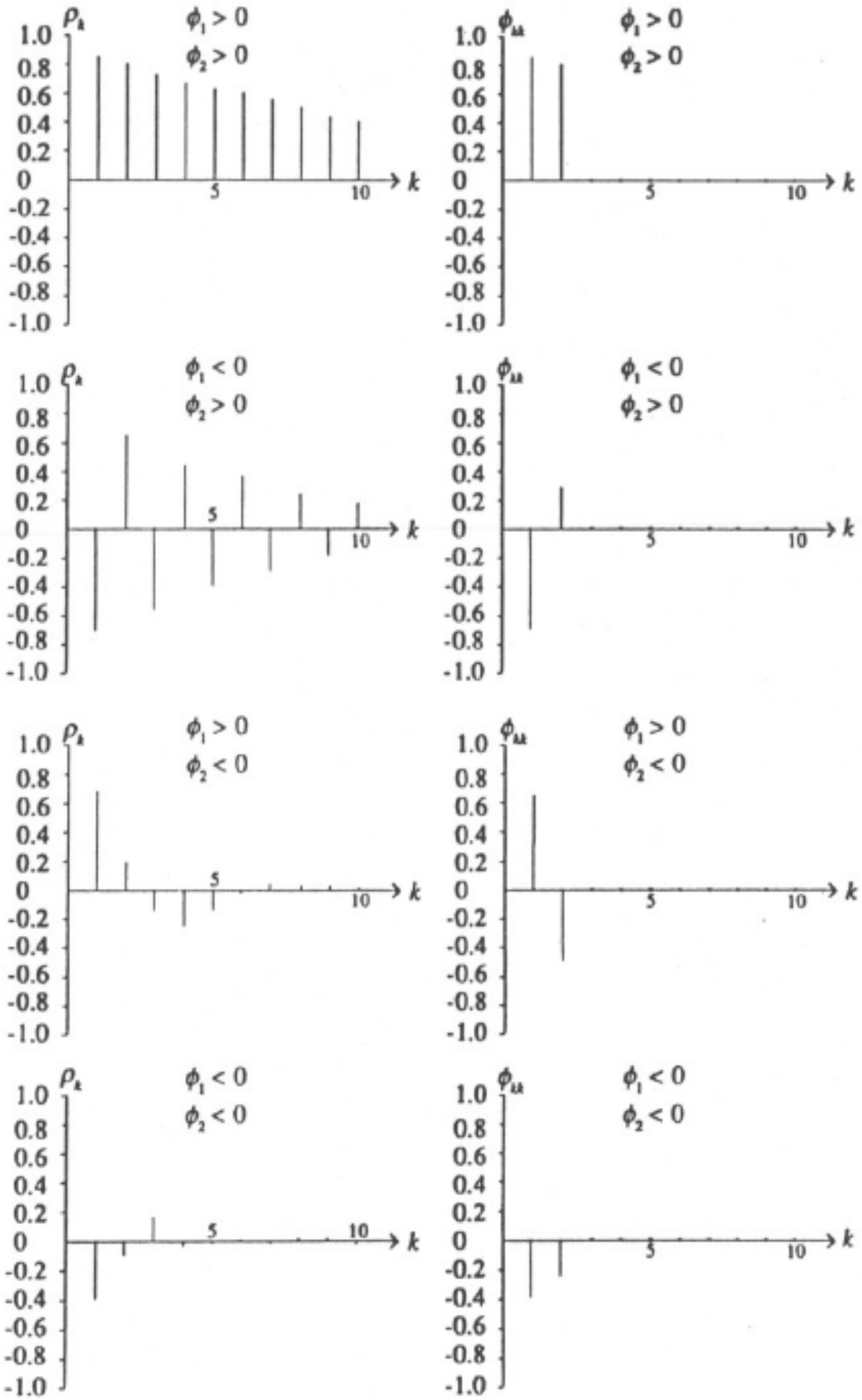


Fig. 3.7 ACF and PACF of AR(2) process: $(1 - \phi_1 B - \phi_2 B^2)\dot{Z}_t = a_t$.

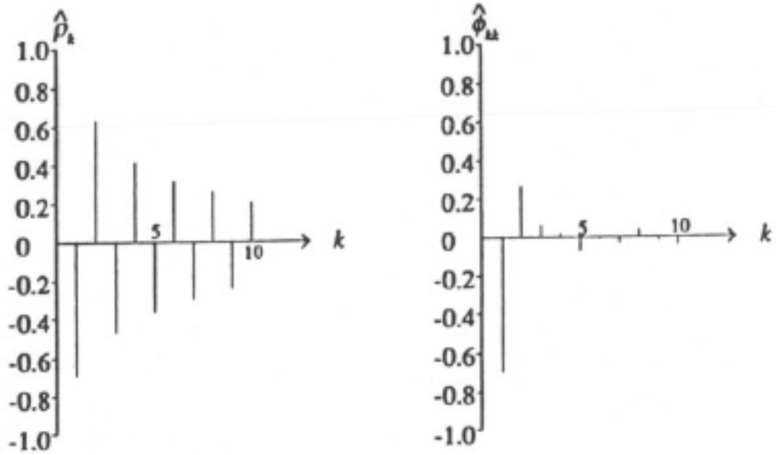
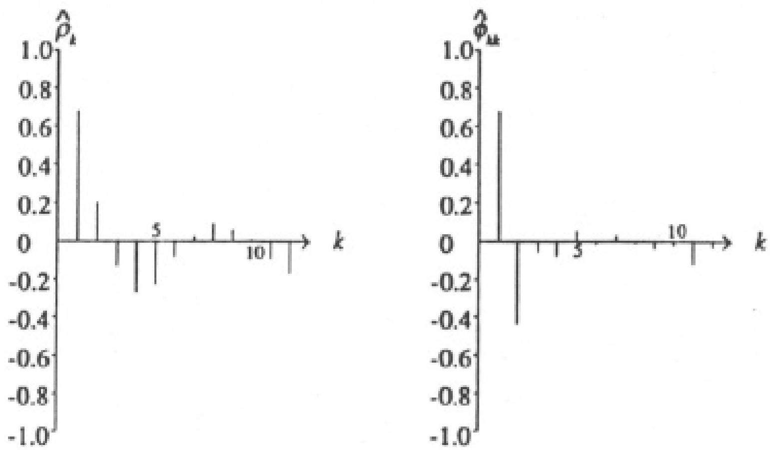


Fig. 3.8 Sample ACF and sample PACF of a simulated AR(2) series: $(1 + .5B - .3B^2)Z_t = a_t$.



(complex roots)

Fig. 3.9 Sample ACF and sample PACF of a simulated AR(2) series: $(1 - B + .5B^2)Z_t = a_t$.

AR(p) process

$$Z_t = c + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \ldots\ldots.\phi_p Z_{t-p} + a_t$$

Causality

All p roots of the characteristic equation outside of the unit circle

Second order moments

Autocorrelation Function

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \ldots\ldots\phi_p\rho_{k-p}$$

$$\left.\begin{array}{l} \rho_1 = \phi_1\rho_0 + \phi_2\rho_1 + \ldots\ldots\phi_p\rho_{p-1} \\ \rho_2 = \phi_1\rho_{11} + \phi_2\rho_0 + \ldots\ldots\phi_p\rho_{p-2} \\ \vdots \\ \rho_p = \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \ldots\ldots\phi_p\rho_0 \end{array}\right\}$$

System of equations.
The first p autocorrelations:
p unknowns and p equations

ACF decays as mixture of exponentials and/or damped sine waves,
Depending on real/complex roots

PACF

$$\phi_{kk} = 0 \quad \text{for} \quad k > p$$

Relationship between AR(p) and MA(q)

Stationary *AR(p)*

$$\Phi_p(L)Z_t = a_t \quad \Phi_p(L) = (1 - \phi_1 L - \phi_2 L^2 - \dots \phi_p L^p)$$

$$\frac{1}{\Phi_p(L)} = \Psi(L) \Rightarrow \Phi_p(L)\Psi(L) = 1$$

$$Z_t = \frac{1}{\Phi_p(L)} a_t = \Psi(L) a_t \quad \Psi(L) = (1 + \psi_1 L + \psi_2 L^2 + \dots)$$

Relationship between AR(p) and MA(q), II

Invertible *MA(q)*

$$Z_t = \Theta_q(L)a_t \quad \Theta_q(L) = (1 - \theta_1 L - \theta_2 L^2 - \ldots.\theta_q L^q)$$

$$\frac{1}{\Theta_q(L)} = \Pi(L) \Rightarrow \Theta_q(L)\Pi(L) = 1$$

$$\Pi(L)Z_t = \frac{1}{\Theta_q(L)} Z_t = a_t \quad \Pi(L) = (1 + \pi_1 L + \pi_2 L^2 + \ldots.)$$

ARMA(p,q) Processes

ARMA(p,q)

$$\Phi_p(L)Z_t = \Theta_q(L)a_t$$

$$\text{Invertibility} \rightarrow \text{roots of } \Theta_q(x) = 0 \quad |x| > 1$$

$$\text{Stationarity} \rightarrow \text{roots of } \Phi_\mathrm{p}(x) = 0 \quad |x| > 1$$

$$\text{Pure AR representation} \rightarrow \Pi(L)Z_t = \frac{\Phi_p(L)}{\Theta_q(L)} Z_t = a_t$$

$$\text{Pure MA representation} \rightarrow \mathrm{Z}_\mathrm{t} = \frac{\Theta_q(L)}{\Phi_p(L)} a_t = \Psi(L)a_t$$

ARMA(1,1)

$$(1-\phi L)Z_t = (1-\theta L)a_t$$

stationarity $\rightarrow |\phi| < 1$

invertibility $\rightarrow |\theta| < 1$

pure AR form $\rightarrow \Pi(\mathrm{L})\mathrm{Z}_\mathrm{t} = a_t \quad \pi_j = (\phi - \theta)\theta^{j-1} \quad j \geq 1$

pure MA form $\rightarrow Z_t = \Psi(\mathrm{L})a_t \quad \psi_j = (\phi - \theta)\phi^{j-1} \quad j \geq 1$

ACF of ARMA(1,1)

$$Z_t Z_{t-k} = \phi Z_{t-1} Z_{t-k} + a_t Z_{t-k} - \theta a_{t-1} Z_{t-k}$$

taking expectations

$$\gamma_k = \phi \gamma_{k-1} + E(a_t Z_{t-k}) - \theta E(a_{t-1} Z_{t-k})$$

you get this system of equations

$$k = 0 \quad E(a_t Z_t) = \sigma^2{}_a \qquad E(a_{t-1} Z_t) = (\phi - \theta)\sigma_a{}^2$$

$$\gamma_0 = \phi \gamma_1 + \sigma_a{}^2 - \theta(\phi - \theta)\sigma_a{}^2$$

$$k = 1 \quad \gamma_1 = \phi \gamma_0 - \theta \sigma_a{}^2$$

$$k \geq 2 \quad \gamma_k = \phi \gamma_{k-1}$$

$$\begin{cases} \text{system of 2 equations and 2 unknowns} \\ \text{solve for } \gamma_0 \text{ and } \gamma_1 \end{cases}$$

ACF

$$\rho_k = \begin{cases} 1 & k = 0 \\ \dfrac{(\phi - \theta)\left(1 - \phi\theta\right)}{1 + \theta^2 - 2\phi\theta} & k = 1 \\ \phi\rho_{k-1} & k \geq 2 \end{cases}$$

PACF

$$MA(1) \subset ARMA(1,1)$$

exponential decay

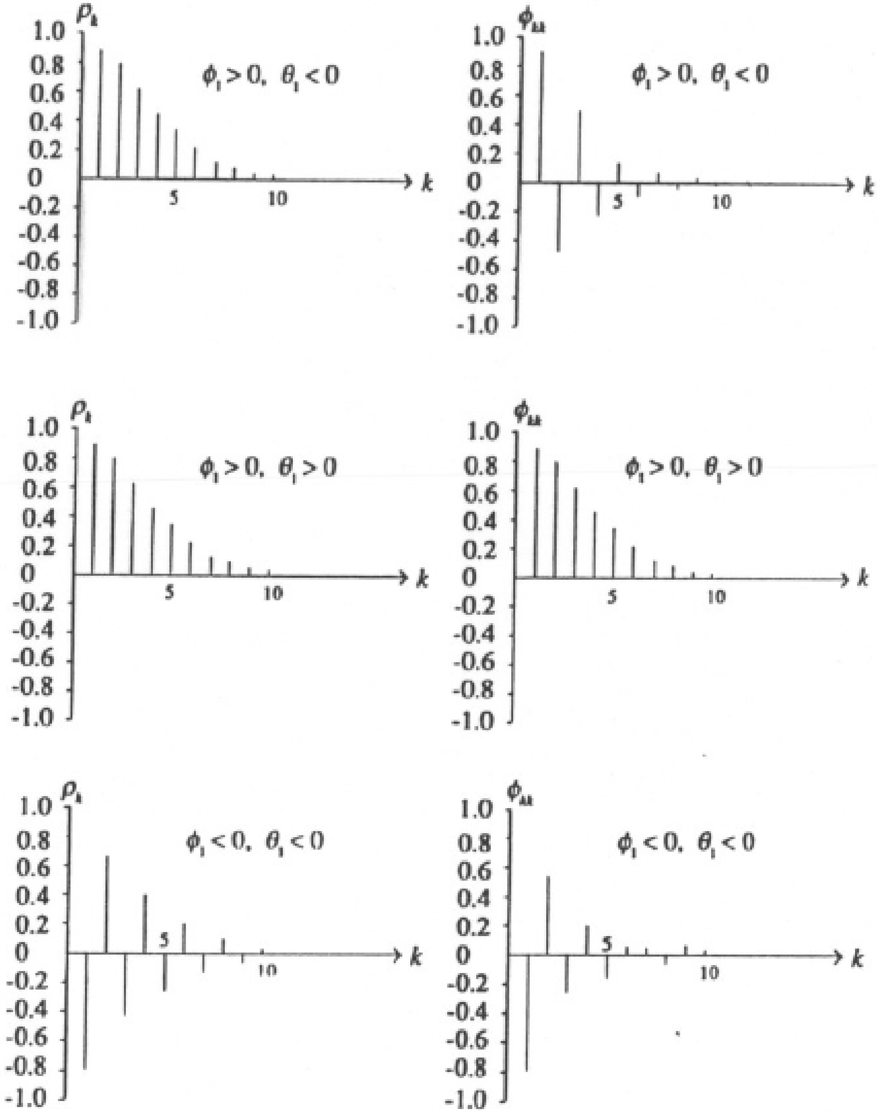


Fig. 3.14 ACF and PACF of ARMA(1, 1) model $(1 - \phi_1 B)\dot{Z}_t = (1 - \theta_1 B)a_t$.

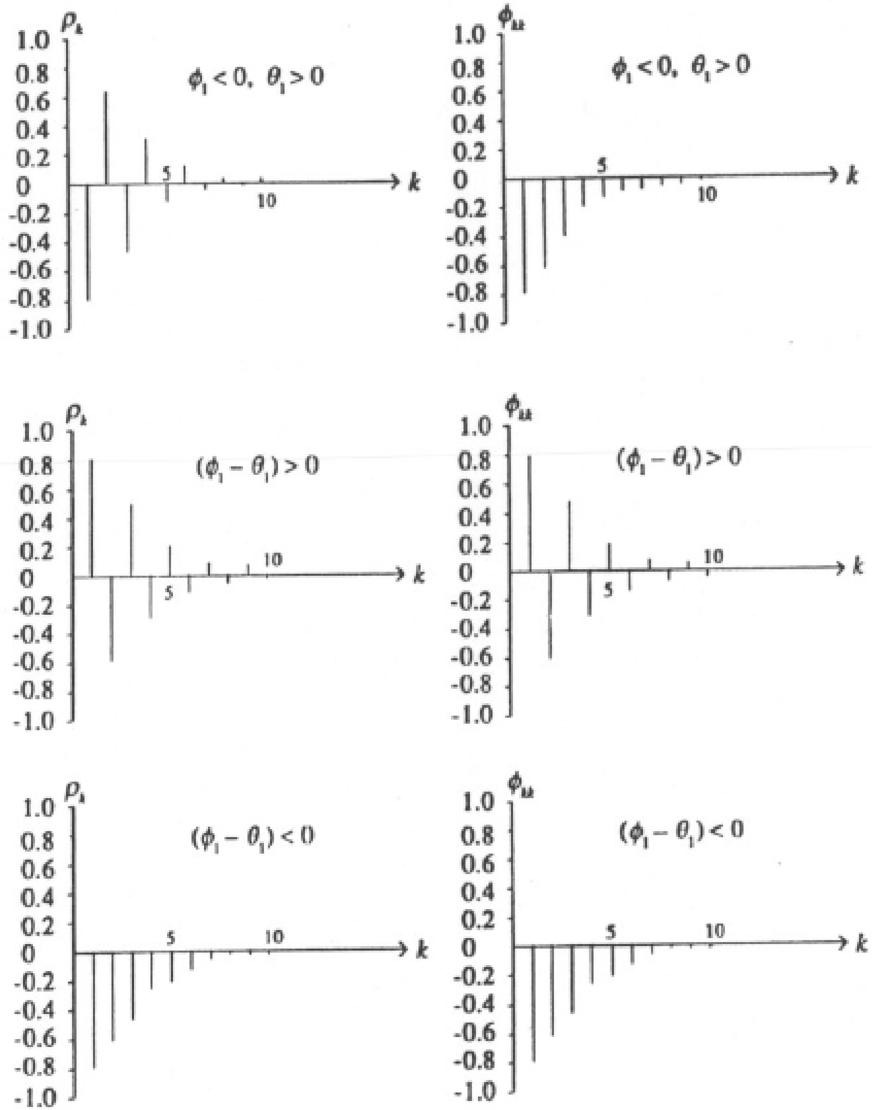
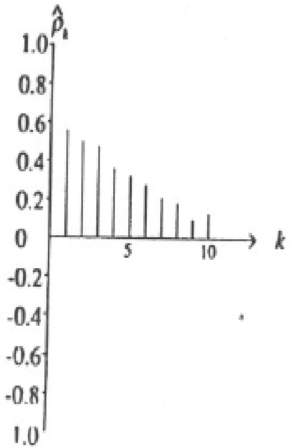


Fig. 3.14 *(continued)*

Table 3.7 Sample ACF and sample PACF for a simulated ARMA(1, 1) series from $(1 - .9B)Z_t = (1 - .5B)a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	.57	.50	.47	.35	.31	.25	.21	.18	.10	.12
St.E.	.06	.08	.09	.10	.11	.11	.11	.11	.11	.11
$\hat{\phi}_{kk}$	.57	.26	.18	−.03	.01	−.01	.01	.01	−.08	.05
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06



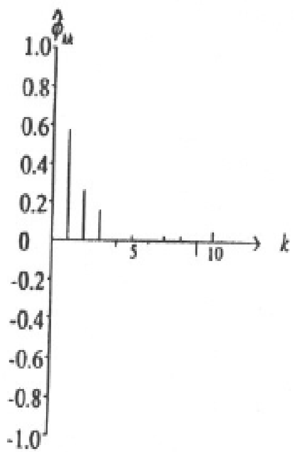


Table 3.8 Sample ACF and sample PACF for a simulated series of the ARMA(1, 1) process: $(1 - .6B)Z_t = (1 - .5B)a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	.10	.05	.09	.00	−.02	.02	−.02	.04	−.04	.01
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06
$\hat{\phi}_{kk}$	.10	.04	.08	−.02	−.02	.01	−.02	.05	−.05	.02
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06