Chapter 3
Discrete Probability Distributions

3.1 The Binomial Distribution

3.2 The Geometric and Negative Binomial Distributions

3.3 The Hypergeometric Distribution

3.4 The Poisson Distribution

3.5 The Multinomial Distribution

3.1 The Binomial Distribution
3.1.1 Bernoulli Random Variables

- To model a process that has only two possible outcomes.

- The outcomes are labeled 0 and 1

- The random variable is defined by the parameter p, $0 \leq p \leq 1$, which is the probability that the outcome is 1.
- Bernoulli distribution, $Ber(p)$

$$f(x;p) = p^x(1-p)^{1-x}, \qquad x = 0,1.$$

- $E(X) = p$

  $Var(X) = p(1-p).$

3.1.2 Definition of the Binomial Distribution

- Consider an experiment consisting of
  n Bernoulli trials $X_1, \cdots, X_n$ that are independent and
  that each has a constant probability p of success.

- Then the total number of successes X, that is $X = \sum_{i=1}^{n} X_i$, is a random variable that has a Binomial distribution with parameters n and p, which is written

$$\mathrm{X} \sim B(n, p).$$

3.1.2 Definition of the Binomial Distribution

- The probability mass function of a $B(n, p)$ random variable is

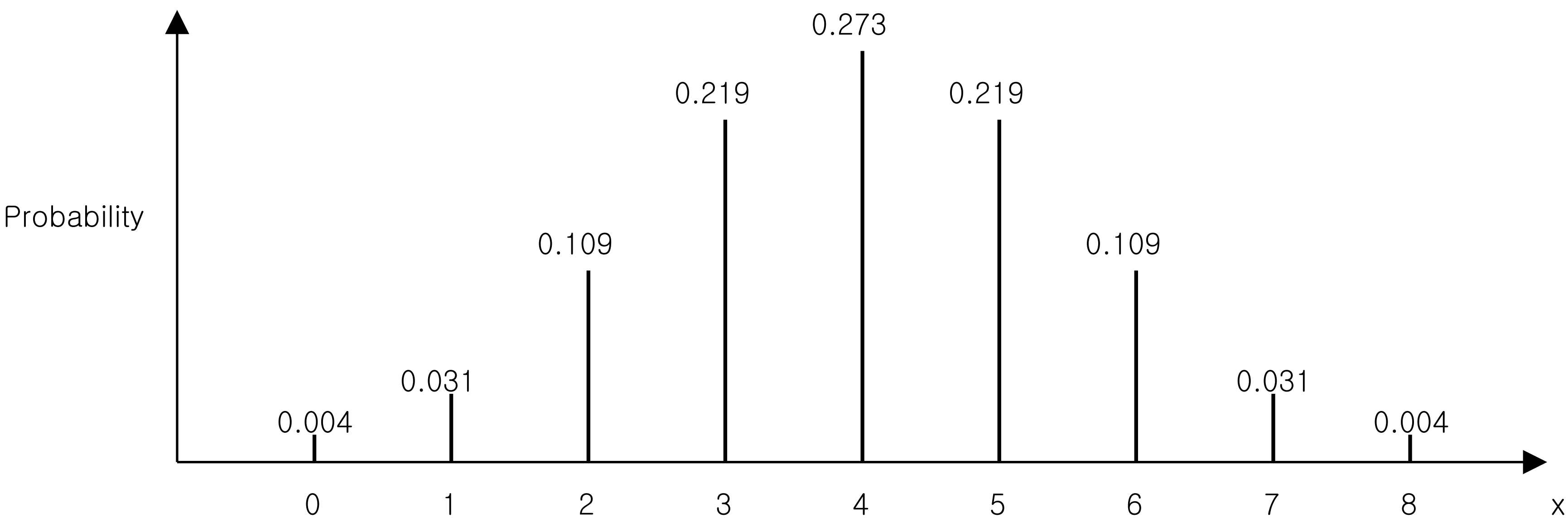
$$f(x; n, p) = \binom{n}{x} p^x q^{n-x}, \qquad x = 0,1, \cdots, n.$$

- $E(X) = np.$

  $Var(X) = np(1 - p).$

3.1.2 Definition of the Binomial Distribution

- Example. $X \sim B(8, 0.5)$



x	0	1	2	3	4	5	6	7	8
$P(X \leq x)$	0.004	0.035	0.144	0.363	0.636	0.855	0.965	0.996	1.000

Example 24 : Air Force Scrambles

- 16 planes.

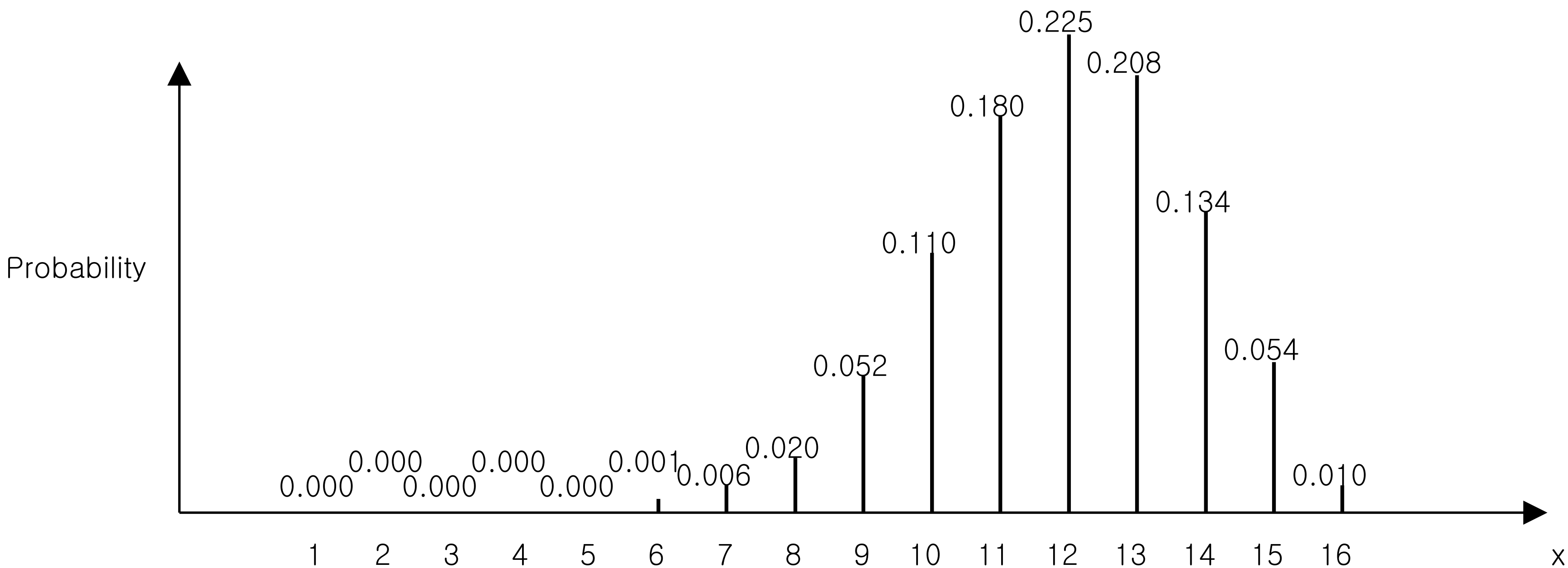
- A probability of 0.25 that the engines of a particular plane will not start at a given attempt.

- Then, the number of planes successfully launched has a binomial distribution with parameters n = 16 and p = 0.75

- The expected number of plane launched is

$$E(X) = np = 16 \times 0.75 = 12$$

Example 24 : Air Force Scrambles



x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$P(X \leq x)$	0.000	0.000	0.000	0.000	0.000	0.001	0.007	0.027	0.079	0.189	0.369	0.594	0.802	0.936	0.990	1.000

Proportion of successes in Bernoulli Trials

- Let $X \sim B(n, p)$
- Then, if $Y = \frac{X}{n}$, then

$$E(Y) = p \quad \text{and} \quad Var(Y) = \frac{p(1-p)}{n}.$$

3.2 The Geometric and Negative Binomial Distributions
3.2.1 Definition of the Geometric Distribution

- The number of trials up to and including the first success in a sequence of independent Bernoulli trials with a constant success probability p has a geometric distribution with parameter p.

- The probability mass function is

$$P(X = x) = (1 - p)^{x-1}p, \qquad x = 1,2,\cdots.$$

3.2.1 Definition of the Geometric Distribution

- The cumulative distribution function is

$$P(X \leq x) = 1 - (1-p)^x.$$

- $E(X) = \frac{1}{p}$

  $Var(X) = \frac{1-p}{p^2}$

$$
\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} xp(1-p)^{x-1} \\
&= p\sum_{x=1}^{\infty} x(1-p)^{x-1} \\
&\Downarrow\ q = 1-p \\
&= p\sum_{x=1}^{\infty} xq^{x-1} \\
&= p\frac{d}{dq}\sum_{x=1}^{\infty} q^{x} \\
&= p\frac{d}{dq}\frac{q}{1-q} \\
&= p\frac{(1-q)+q}{(1-q)^2} \\
&= 1/p.
\end{aligned}
$$

$$
\begin{aligned}
E(X(X-1)) &= \sum_{x=1}^{\infty} x(x-1)p(1-p)^{x-1} \\
&= p\sum_{x=2}^{\infty} x(x-1)(1-p)^{x-1} \\
&\Downarrow\ q = 1-p \\
&= p\sum_{x=2}^{\infty} x(x-1)q^{x-1} \\
&\Downarrow\ y = x-1 \\
&= pq\frac{d^2}{dq^2}\sum_{x=1}^{\infty} q^{(y+1)} \\
&= \frac{2q}{p^2}.
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= [E(X) + E(X(X-1))] - E(X) \\
&= \frac{1-p}{p^2}.
\end{aligned}
$$

Example 24 : Air Force Scrambles

- If the mechanics are unsuccessful in starting the engines, then they must wait 5 minutes before trying again.
- The distribution of the number of attempt needed to start a plane's engine is a geometric distribution with p = 0.75.
- The probability that the engines start on the third attempt is

$$P(X = 3) = 0.25^2 \times 0.75 = 0.047.$$

Example 24 : Air Force Scrambles

- The probability that the plane is launched within 10 minutes of the first attempt to start the engines is

$$P(X \leq 3) = 1 - 0.25^3 = 0.984.$$

- The expected number of attempts required to start the engines is

$$E(X) = \frac{1}{p} = \frac{1}{0.75} = 1.33.$$

3.2.2 Definition of the Negative Binomial Distribution

- The number X of trials up to and including the $\boldsymbol{r}$ th success in a sequence of independent Bernoulli trials with a constant success probability p has a negative binomial distribution with parameter p.
- The probability mass function X is

$$P(X=x)=\binom{x-1}{r-1}(1-p)^{x-r}p^r, \quad x=r,r+1,\cdots.$$

- $E(X)=\frac{r}{p}$.

  $Var(X)=\frac{r(1-p)}{p^2}$.

Example 12 : Personnel Recruitment

- Suppose that a company wishes to hire 3 new workers and each applicant interviewed has a probability of 0.6 of being found acceptable.
- The distribution of the total number of applicants that the company needs to interview is the Negative Binomial distribution with parameter p = 0.6 and r = 3.
- The probability that exactly 6 applicants need to be interviewed is

$$P(X = 6) = \binom{5}{2} 0.4^3 0.6^3 = 0.138$$

Example 12 : Personnel Recruitment

- If the company has a budget that allows up to 6 applicants to be interviewed, then the probability that the budget is sufficient is

$$P(X \leq 6) = \sum_{i=3}^{6} P(X = i) = 0.820$$

- The expected number of interviews required is

$$E(X) = \frac{r}{p} = \frac{3}{0.6} = 5$$

3.3 The Hypergeometric Distribution
3.3.1 Definition of the Hypergeometric Distribution

- Consider a collection of N items of which r are of a certain kind.
- If one of the items is chosen at random, the probability that it is of the special kind is clearly
$$p = \frac{r}{N}.$$
- Consequently, if n items are chosen at random with replacement, then clearly the distribution of X, the number of defective items chosen, is Binomial with parameters n and p.

3.3.1 Definition of the Hypergeometric Distribution

- However, if n items are chosen at random without replacement, then the distribution of X is the hypergeometric distribution.
- The hypergeometric distribution has a probability mass function given by

$$f(x; N, n, r) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}},$$

$$\max\{0, n-(N-r)\} \leqslant x \leqslant \min\{n, r\}.$$

3.3.1 Definition of the Hypergeometric Distribution

- $E(X) = n\frac{r}{N}$

  $$Var(X) = \frac{N-n}{N-1} n \frac{r}{N} (1 - \frac{r}{N})$$

- Comparison with $B(n, p)$ when $p = \frac{r}{N}$

  $$E_B(X) = E_H(X) = np$$

  $$\sigma_B^2(X) = npq \geq \sigma_H^2(X) = \frac{N-n}{N-1} npq$$

Computation of Mean

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} x \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} x \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} \frac{r\binom{r-1}{x-1}\binom{(N-1)-(r-1)}{(n-1)-(x-1)}}{\frac{N}{n}\binom{N-1}{n-1}} \\
&\Downarrow\ y = x-1 \\
&= \sum_{y=0}^{n-1} \frac{r\binom{r-1}{y}\binom{(N-1)-(r-1)}{(n-1)-y}}{\frac{N}{n}\binom{N-1}{n-1}} \\
&= n\frac{r}{N}.
\end{aligned}
$$

Computation of Variance

$$E\big(X(X-1)\big) = \sum_{x=0}^{n} x(x-1)\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=2}^{n} r(r-1)\frac{\binom{r-2}{x-2}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$\Downarrow y = x-2 \text{ (change of variable)}$$

$$= \sum_{y=0}^{n-2} r(r-1)\frac{\binom{r-2}{y}\binom{(N-2)-(r-2)}{(n-2)-y}}{N(N-1)\binom{N-2}{n-2}/n(n-1)}$$

$$= r(r-1)\frac{n(n-1)}{N(N-1)}$$

$$E(X^2) = E\big(X(X-1)\big) + E^2(X) = r(r-1)\frac{n(n-1)}{N(N-1)} + n\frac{r}{N}$$

$$Var(X) = E(X^2) - E^2(X) = \frac{N-n}{N-1}n\frac{r(N-r)}{N^2}$$

3.4 The Poisson Distribution
3.4.1 Definition of the Poisson Distribution

- The distribution useful for representing
  - the number of defects in an item
  - the number of radioactive particles emitted by a substance
  - the number of telephone calls received by an operator with a certain time limit

- That is, the number of “events” that occur within certain specified boundaries of space or time.

3.4.1 Definition of the Poisson Distribution

- A random variable X distributed as a Poisson random variable with parameter λ which is written
$$\mathrm{X} \sim P(\lambda)$$
has a probability mass function
$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!},\ x = 0,1,2,\cdots.$$

- $\mathrm{E(X)} = \mathrm{Var(X)} = \lambda.$

Computation for the mean and variance of a Poisson random variable

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\
&\Downarrow\ y = x - 1 \\
&= \sum_{y=0}^{\infty} \lambda \frac{e^{-\lambda}\lambda^y}{y!} \\
&= \lambda.
\end{aligned}
$$

The variance can be obtained by

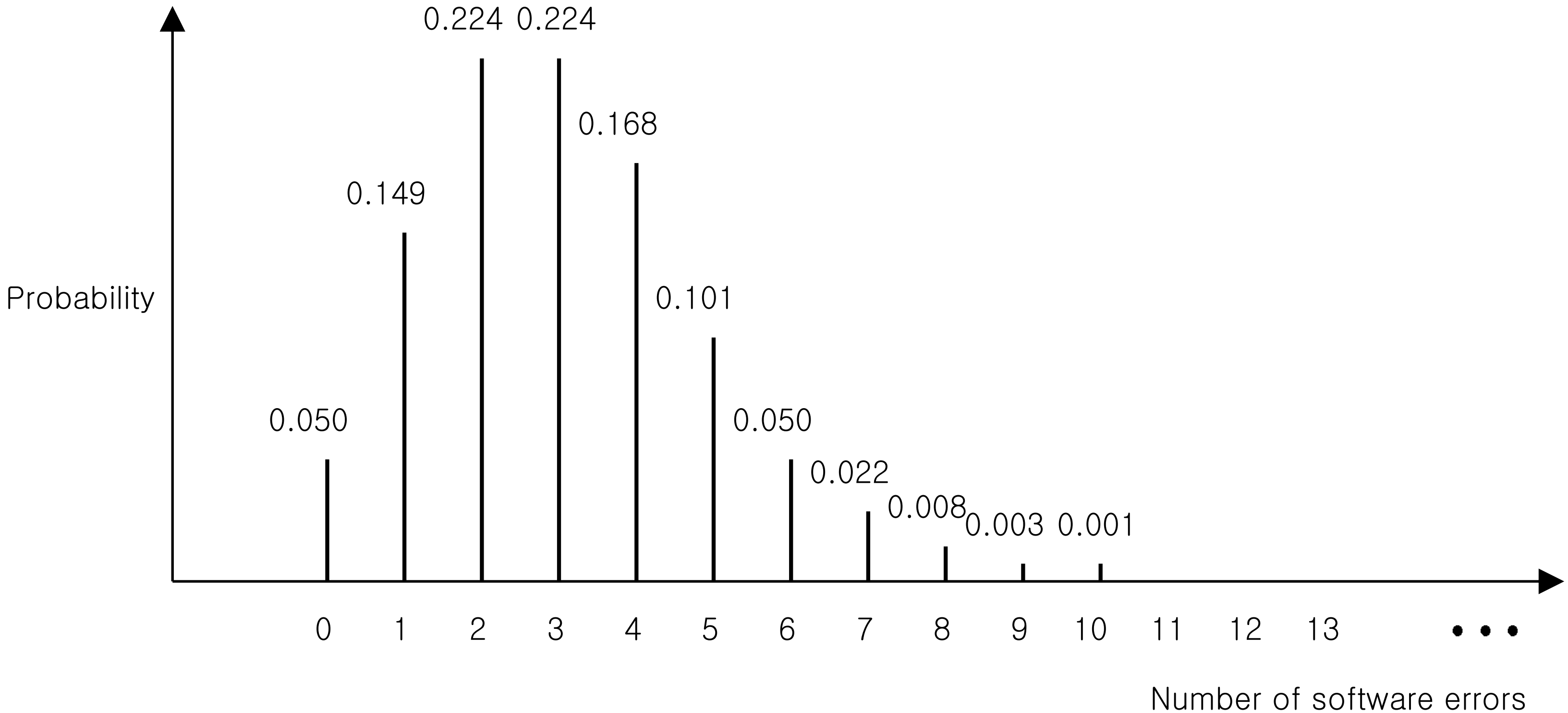
$Var(X) = E(X^2) - \mu_X^2$ where $\mu_X = E(X)$.

$E(X^2) = E\big(X(X-1)\big) + \mu_X$.

$E\big(X(X-1)\big)$ can be computed in the same way as for $E(X)$.

Example 3 : Software Errors

- Suppose that the number of errors (X) in a piece of software has a Poisson distribution with parameter λ=3.



3.5 The Multinomial Distribution
3.5.1 Definition of the Multinomial Distribution

- Consider a sequence of **n independent trials** where each individual trial can have **k outcomes** that occur with constant probability value $\mathrm{p}_1, \mathrm{p}_2, \cdots, p_k$ with

$$\mathrm{p}_1 + \mathrm{p}_2 + \cdots + p_k = 1.$$

- The random variables, $\mathrm{X}_1, \mathrm{X}_2, \cdots, X_k$, with $\sum_{i=1}^{k} X_i = n$, that count the numbers of occurrences of the k respective outcomes are said to have a **multinomial** distribution.

3.5.1 Definition of the Multinomial Distribution

- The joint probability mass function of $(X_1, \cdots, X_k)$ is

$$f(x_1, x_2, \cdots, x_k; p_1, \cdots, p_k, n) = \binom{n}{x_1, x_2, \cdots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

with $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} p_i = 1$

and write

$$(X_1, \cdots, X_k) \sim M_k(p_1, \cdots, p_k, n)$$

3.5.1 Definition of the Multinomial Distribution

- The random variables $X_1, \ldots, X_k$ have the expectation and variance given by

$$E(X_i) = np_i. \qquad Var(X_i) = np_i(1 - p_i).$$

Example 1 : Machine Breakdowns

Suppose that the machine breakdowns are attributable to electrical faults, mechanical faults, and operator misuse, and these causes occur with probabilities of 0.2, 0.5, and 0.3, respectively. The engineer is interested in predicting the causes of the next ten breakdowns.

X_1: the number of breakdowns due to electrical reasons.
X_2: the number of breakdowns due to mechanical reasons.
X_3: the number of breakdowns due to operator misuse.

Example 1 : Machine Breakdowns

- $X_1 + X_2 + X_3 = 10$

- The probability that there will be 3 electrical breakdowns, 5 mechanical breakdowns, and 2 misuse breakdowns is

$$f(3,5,2;0.2,0.5,0.3,10) = \binom{10}{3,5,2} 0.2^3 0.5^5 0.3^2.$$

- If the engineer is interested in the probability of there being no more two electrical breakdowns, then this can be calculated by noting that $X_1 \sim B(10, 0.2)$.

Chapter Summary

3.1 The Bernoulli and Binomial Distributions

3.2 The Geometric and Negative Binomial Distributions

3.3 The Hypergeometric Distribution

3.4 The Poisson Distribution

3.5 The Multinomial Distribution