HW6 · 20224314 강현규.

7.2.4.

(a) $Var(\hat{\mu}_1) = Var\left(\frac{X_1+X_2}{2}\right) = \frac{1}{4}\left(Var(X_1) + Var(X_2) + 2\cdot Cov(X_1,X_2)\right)$

$= \frac{1}{4}\left(8 + 2Cov(X_1,X_2)\right)$

* $Cov(X_1,X_2)$에 대한 가정, 또는 값이 없어 그대로 둠.

(b) $Var(\mu) = Var(pX_1 + (1-p)X_2)$

$= p^2\cdot Var(X_1) + 2p(1-p)Cov(X_1,X_2) + (1-p)^2\cdot X_2$.

$= p^2 + 2p(1-p)\cdot Cov(X_1,X_2) + 7(1-p)^2$.

* $\frac{\partial}{\partial p} Var(\mu) = 2p + 2(1-p)\cdot Cov(X_1,X_2) - 2p\,Cov(X_1,X_2) + 14(p-1)$

$= 16p - 14 + Cov(X_1,X_2)(2-4p)$

$= 4p(4 - Cov(X_1,X_2)) + 2Cov(X_1,X_2) - 14$.

when $p = \frac{7 - Cov(X_1,X_2)}{2\cdot(4 - Cov(X_1,X_2))}$, $Var(\mu)$ is minimized

(c) relative efficiency of $\hat{\mu}_1 = \frac{Var(\theta)}{Var(\hat{\mu}_1)}$ where θ is smallest variance

$Var(X_1)$ is smallest since other things are combination of X_1, X_2,

it will bigger than X_1.

$\therefore$ relative efficiency of $\hat{\mu}_1$ to $\theta = \frac{1}{\frac{1}{2}(4 + Cov(X_1,X_2))} = \frac{2}{4 + Cov(X_1,X_2)}$

7.2.8

(a) $\text{bias} = E(\hat{p}) - p$

$= E\left(\frac{X}{11}\right) - p$

$= \frac{1}{11} \cdot 10 \cdot p - p = -\frac{1}{11} p$

(b) $Var(\hat{p}) = Var\left(\frac{X}{11}\right)$

$= \frac{1}{11^2} \cdot 10 \cdot p(1-p)$

(c) $MSE(\hat{p}) = E\left[(\hat{p} - p)^2\right]$

$= Var(\hat{p}) + \text{bias}^2(\hat{p})$

$= \frac{10}{121} p(1-p) + \left(-\frac{1}{11} p\right)^2$

$= \frac{10p - 10p^2 + p^2}{121} = \frac{10p - 9p^2}{121}$

(d) $MSE\left(\frac{X}{10}\right) = E\left[\left(\frac{X}{10} - p\right)^2\right]$

$= Var\left(\frac{X}{10}\right) + \text{bias}^2\left(\frac{X}{10}\right)$

$* \; Var\left(\frac{X}{10}\right) = \frac{p(1-p)}{10}, \quad \text{bias}\left(\frac{X}{10}\right) = 0$

$= \frac{p(1-p)}{10}$

$\Rightarrow MSE(\hat{p}) - MSE\left(\frac{X}{10}\right) = \frac{10p - 9p^2}{121} - \frac{p(1-p)}{10}$

$= \frac{100p - 90p^2 - 121p + 121p^2}{121 \cdot 10} = \frac{31p^2 - 21p}{1210} = \frac{p(31p - 21)}{1210}$

when $p \le \frac{21}{31}$, $MSE(\hat{p}) < MSE\left(\frac{X}{10}\right)$

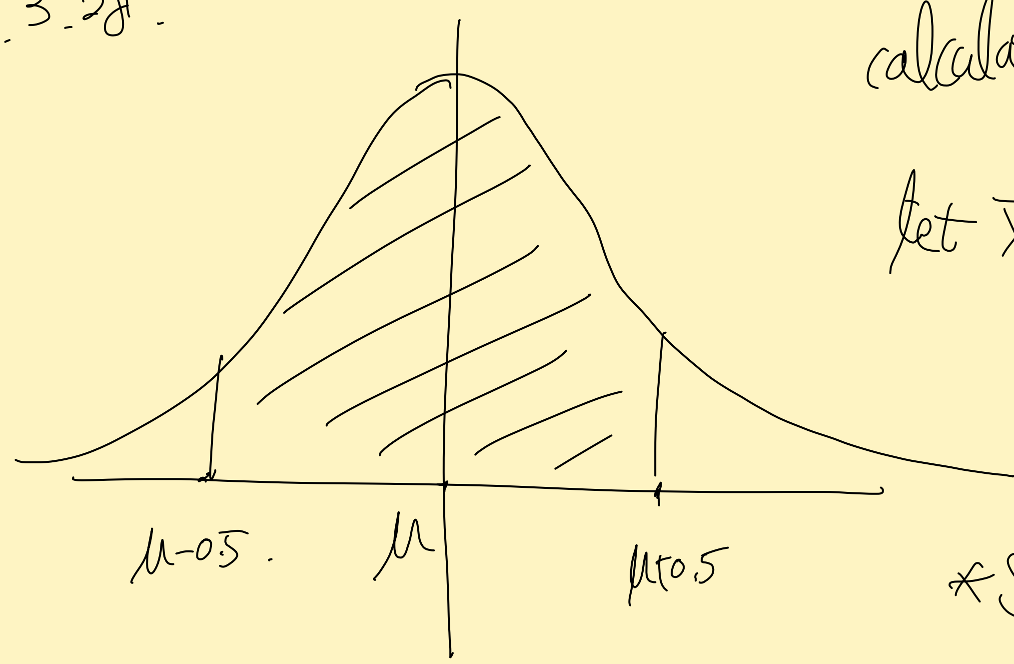
7.3.8. let $\hat{p} = \frac{234}{450}$

$$sd(\hat{p}) = \sqrt{Var(\hat{p})} = \sqrt{\frac{\hat{p}\cdot(1-\hat{p})}{n}}$$

$$= \sqrt{\frac{1}{450}\cdot\frac{234}{450}\cdot\frac{216}{450}}$$

$$\approx 0.0235$$

7.3.28.



calculate $P(\mu-0.5 \le \frac{\sum_{i=1}^{n} x_i}{5} \le \mu+0.5)$

let $\bar{X} = \sum_{i=1}^{n} x_i$

then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

$* S = \sigma = 0.84$

(a) when $n=5$, $\frac{\bar{X}-\mu}{0.84/\sqrt{5}} \sim t_4$

$$\Rightarrow \int_{-2.236}^{2.236} t_4 \approx 0.911$$

(b) when $n=10$, $\frac{\bar{X}-\mu}{0.84/\sqrt{10}} \sim t_9$

$$\Rightarrow \int_{-3.162}^{3.162} t_9 \approx 0.988$$

(c) find n len prob at least 99%, let $t = \frac{\bar{X}-\mu}{0.84/\sqrt{n}} = 0.595\times\sqrt{n}$.

when $n=11$, $\int_{-3.316}^{3.316} t_{10} \approx 0.992 > 0.99$.

7.4.2.

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\cdot du} = \frac{1}{B(\alpha,\beta)} \cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}.$$

since. $E[f(x;\alpha,\beta)] = \frac{\alpha}{\alpha+\beta} = 0.782$ ... ①

$$Var[f(x;\alpha,\beta)] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.0083. \quad ...②$$

by ①. $\alpha = (\alpha+\beta)\times 0.782.$ $\Rightarrow \beta = \frac{0.218}{0.782}\cdot\alpha$

by ②, $\alpha\beta = 0.0083(\alpha+\beta)^2\cdot(\alpha+\beta+1)$

$$\Leftrightarrow \frac{0.218}{0.782}\alpha^2 = 0.0083\cdot\left(\frac{\alpha}{0.782}\right)^2\cdot\left(\frac{\alpha}{0.782}+1\right)$$

$$\Leftrightarrow 0.218\cdot 0.782^2\cdot\alpha^2 - 0.0083\,\alpha^2(\alpha+0.782) = 0$$

$$\Leftrightarrow \alpha^2(0.611524 - 0.0083\alpha - 0.0064906) = 0.$$

$$\therefore \alpha = \frac{0.605034}{0.0083} \approx 72.895$$

$\beta \approx 20.32127$

7.4.4.

$$L(x_1, \cdots, x_k, p_1, \cdots, p_k) = \frac{n!}{x_1! \cdots x_k!} \cdot p_1^{x_1} \cdots p_k^{x_k}$$

when condition that $p_1 + \cdots + p_k = 1$

$$\Rightarrow \log L(x_1, \cdots, x_k, p_1, \cdots, p_k) = \log\left(\frac{n!}{x_1! \cdots x_k!}\right) + x_1 \cdot \log p_1 + \cdots + x_k \cdot \log p_k.$$

since $\sum_{i=1}^{k} p_i = 1$,

let $$\mathcal{L}(p_1 \cdots p_k, \lambda) = \log L(x_1 \cdots x_k, p_1, \cdots, p_k) + \lambda\left(1 - \sum_{i=1}^{k} p_i\right)$$

$$\frac{\partial}{\partial p_1}\mathcal{L}(p_1 \cdots p_k, \lambda) = \frac{x_1}{p_1} - \lambda = 0.$$

$\vdots$

$$\frac{\partial}{\partial p_k}\mathcal{L}(p_1, \cdots p_k, \lambda) = \frac{x_k}{p_k} - \lambda = 0$$

$$\Rightarrow \frac{\partial}{\partial p_i}\mathcal{L}(p_1, \cdots, p_k, \lambda) = 0$$

$$\Leftrightarrow p_1 = \frac{x_1}{\lambda}$$

since $\sum_{i=1}^{k} p_i = 1 = \sum_{i=1}^{k} \frac{x_i}{\lambda} = \frac{1}{\lambda}\sum_{i=1}^{k} x_i = \frac{n}{\lambda}$, $\therefore \lambda = n.$

$$\therefore \hat{p}_i = \frac{x_1}{\lambda} = \frac{x_1}{n}$$

나머지 풀이는 Jupyter notebook 에서