Chapter 2. Random Variables

2.1 Discrete Random Variables

2.2 Continuous Random Variables

2.3 The Expectation of a Random Variable

2.4 The Variance of a Random Variable

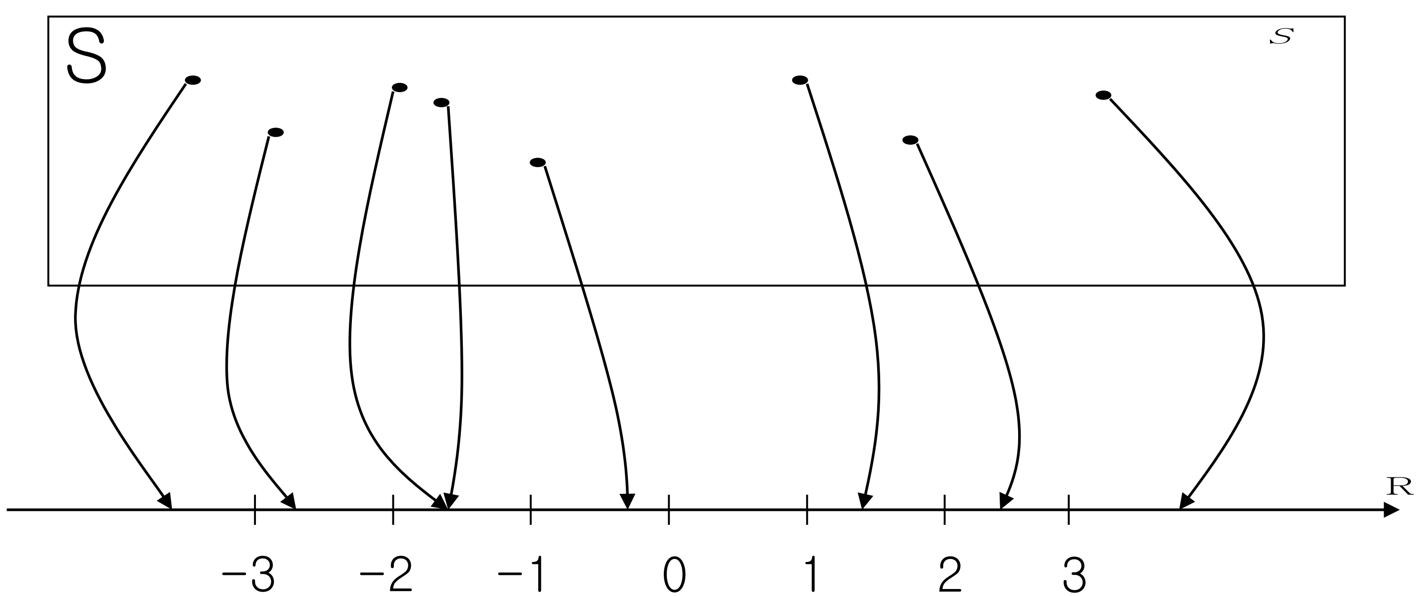
2.5 Jointly Distributed Random Variables

2.6 Combinations and Functions of Random Variables

2.1 Discrete Random Variable

2.1.1 Definition of a Random Variable

- Random variable X
  - Defined as a mapping from a sample space S to a real line R
  - A numerical value *X(w)* is mapped to each outcome *w* of a particular experiment



Example: Construction of a random variable

- X: the sum of the two dice scores
- X: the product of the two dice scores
- X: $\dfrac{the\ score\ of\ the\ first\ die}{the\ score\ of\ the\ second\ die}$

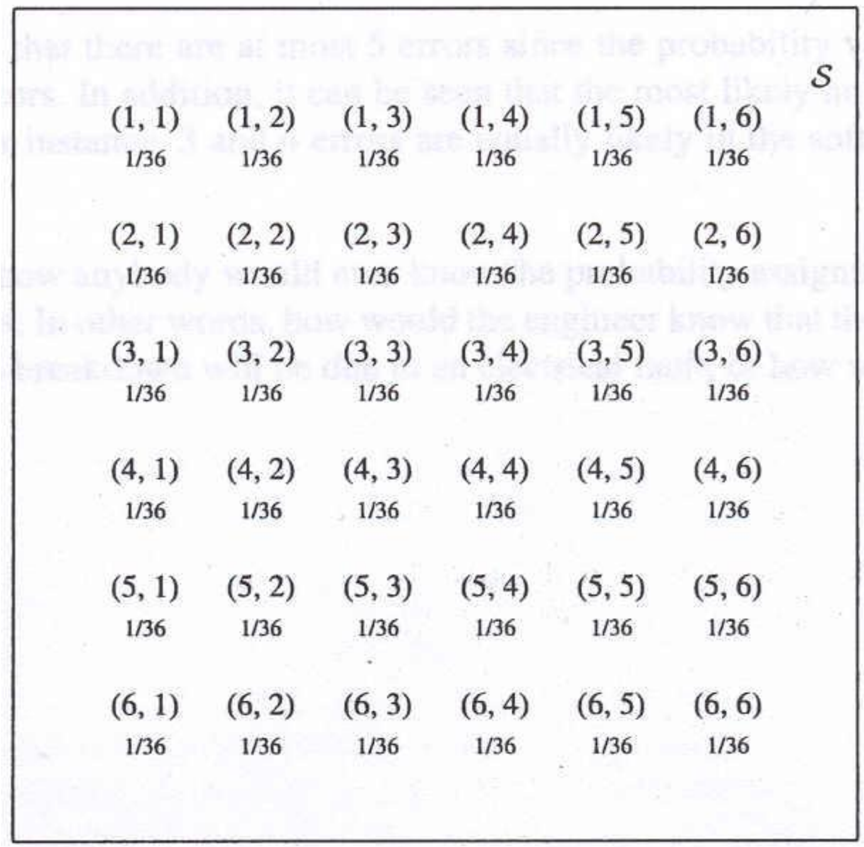


FIGURE 1.10 ●
Probability values for rolling two dice

2.1.1 Definition of a Random Variable

Example 1 : Machine Breakdowns

- Sample space : $S = \{electirical, mechanical, misuse\}$
- Each of these failures may be associated with a repair cost
- State space : $\{50, 200, 350\}$
- Cost is a random variable taking the values of 50, 200, or 350.

2.1.2 Probability Mass Function

- Probability Mass Function (p.m.f.)
  - A set of probability value p_i assigned to each of the values taken by the discrete random variable x_i
  - $p \leq p_i \leq 1$ and $\sum_i p_i = 1.$
  - Probability : $P(X = x_i) = p_i.$

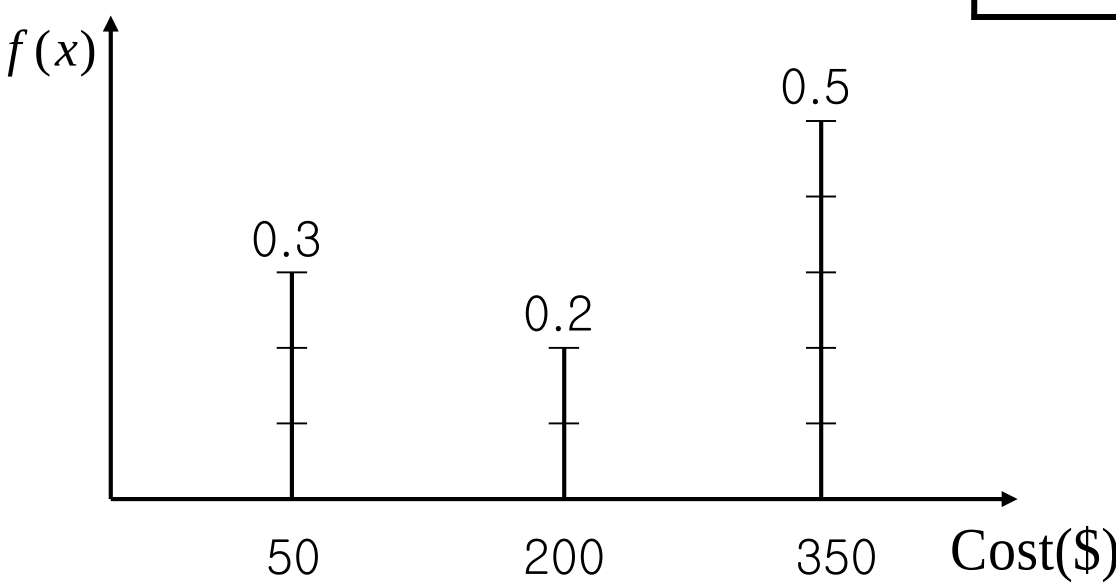
2.1.2 Probability Mass Function

- Example 1 : Machine Breakdowns

  P (cost=50)=0.3, P (cost=200)=0.2, P (cost=350)=0.5

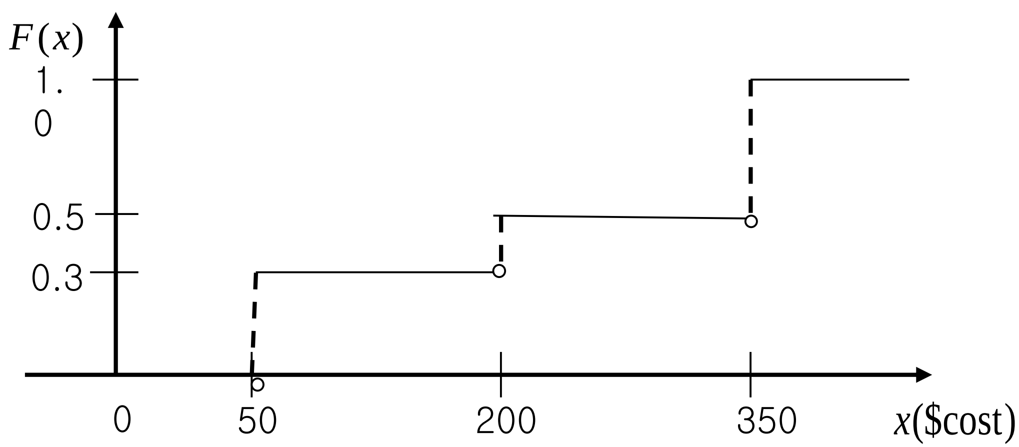
  0.3 + 0.2 + 0.5 =1

x_i	50	200	350
p_i	0.3	0.2	0.5



2.1.3 Cumulative Distribution Function

- Cumulative Distribution Function(CDF)
  $F(x) = P(X \leq x).$



2.2 Continuous Random Variables
2.2.1 Example of Continuous Random Variables

Example 14 : Metal Cylinder Production

Suppose that the random variable X is the diameter of a randomly chosen cylinder manufactured by the company. Since this random variable can take any value between 49.5 and 50.5, it is a continuous random variable.

2.2.2 Probability Density Function

- Probability Density Function (pdf)
  Probabilistic properties of a continuous random variable

$$f(x) \geq 0$$
$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

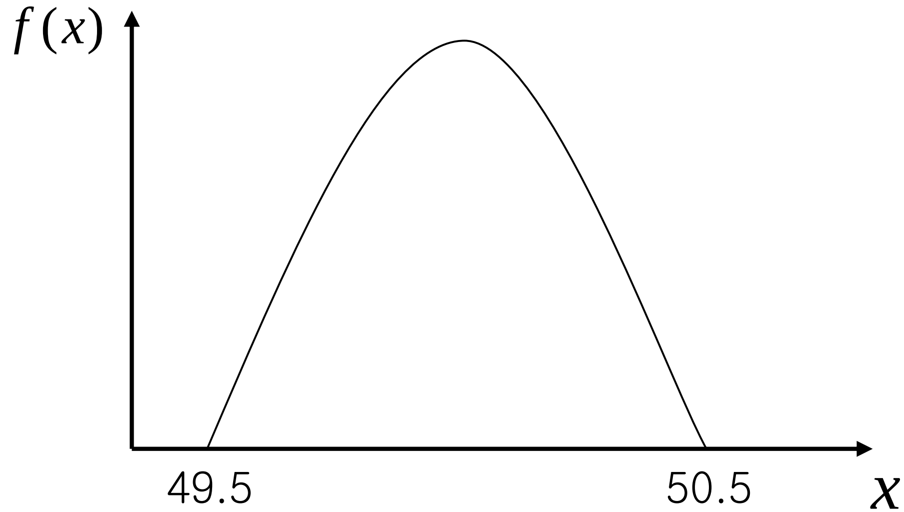
2.2.2 Probability Density Function

Example 14

Suppose that the diameter of a metal cylinder has a pdf

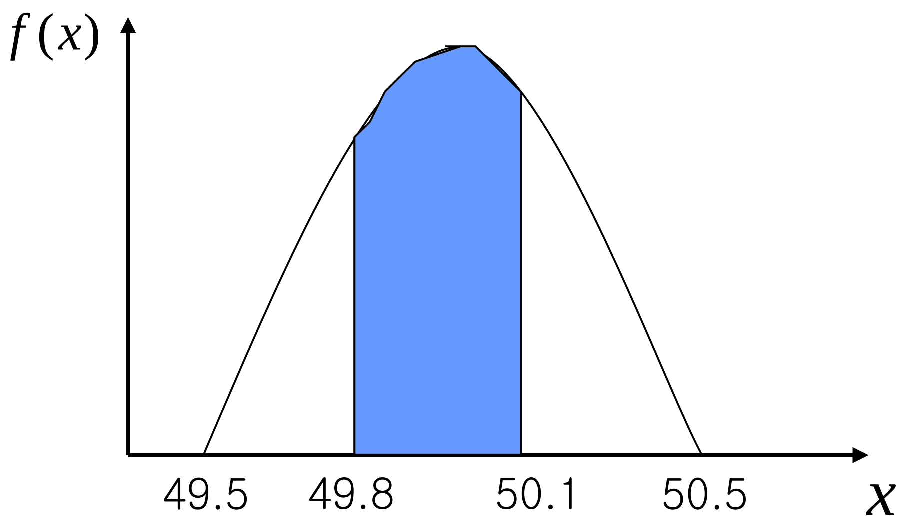
$$f(x) = \begin{cases} 1.5 - 6(x-50)^2 & \text{for } 49.5 \le x \le 50.5 \\ 0, & \text{elsewhere} \end{cases}$$

Check if $f(x)$ is a pdf.



2.2.2 Probability Density Function

- The probability that a metal cylinder has a diameter between 49.8 and 50.1 mm can be calculated as
- $P(49.8 \leq X \leq 50.1) = \int_{49.8}^{50.1} f(x)\ dx = 0.432.$



2.2.3 Cumulative Distribution Function

- Cumulative Distribution Function for continuous R.V.
  - $\mathrm{F(x)} = \int_{-\infty}^{x} f(y)\,dy$
  - $f(x) = \frac{dF(x)}{dx}$
  - $P(a < X \le b) = F(b) - F(a)$
  - $P(a \le X \le b) = P(a < X \le b)$ = $\mathrm{P(a<X<b)}$

2.3 The Expectation of a Random Variable
2.3.1 Expectations of Discrete Random Variables

- Expectation of a discrete random variable with p.m.f
  - $E(X) = \sum_i p_i x_i$

- Expectation of a continuous random variable with p.d.f $f(x)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\ dx$$

- The expected value of a random variable is also called the mean of the random variable

2.3.1 Expectations of Discrete Random Variables

Example 1 (discrete random variable)

The expected repair cost is

$E(cost) = 50 \times 0.3 + 200 \times 0.2 + 350 \times 0.5 = 230(\$).$

2.3.2 Expectations of Continuous Random Variables

Example 14 (continuous random variable)

The expected diameter of a metal cylinder is

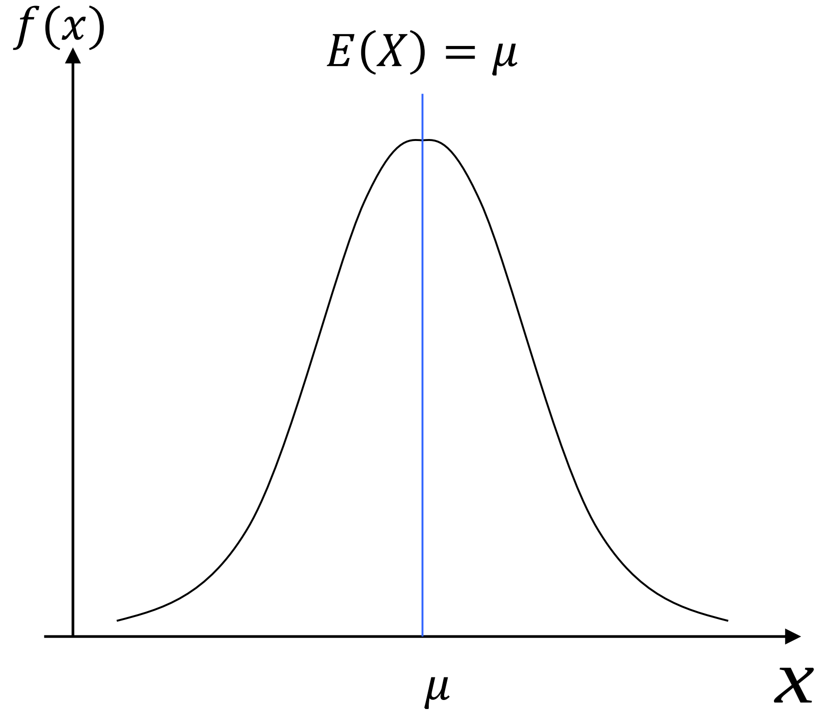
$$E(X) = \int_{49.5}^{50.5} x(1.5 - 6(x - 50)^2)\, dx = 50.$$

2.3.2 Expectations of Continuous Random Variables

- Symmetric Random Variables
  - If x has a pdf $\mathrm{f}(x)$ that is symmetric about a point μ so that

$$f(\mu + x) = f(\mu - x).$$

  Then, $E(X) = \mu$, the point of symmetry.



$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{\mu} xf(x)dx + \int_{\mu}^{\infty} xf(x)dx \\
&\Downarrow\ y = 2\mu - x \\
&= \int_{-\infty}^{\mu} xf(x)dx - \int_{\mu}^{-\infty} (2\mu - y)f(y)dy \\
&= \int_{-\infty}^{\mu} xf(x)dx + \int_{-\infty}^{\mu} (2\mu - y)f(y)dy \\
&= \int_{-\infty}^{\mu} 2\mu f(y)dy + \int_{-\infty}^{\mu} xf(x)dx - \int_{-\infty}^{\mu} yf(y)dy \\
&= 2\mu \int_{-\infty}^{\mu} f(y)dy \\
&= 2\mu \cdot \frac{1}{2} = \mu.
\end{aligned}
$$

2.3.3 Medians of Random Variables

- Median

  For a random variable X, its median is the value x such that $$F(x) = 0.5.$$

2.4 The variance of a Random Variable
2.4.1 Definition and Interpretation of Variance

- Variance(σ^2)

$$Var(X) = E(\left(X - E(X)\right)^2)$$

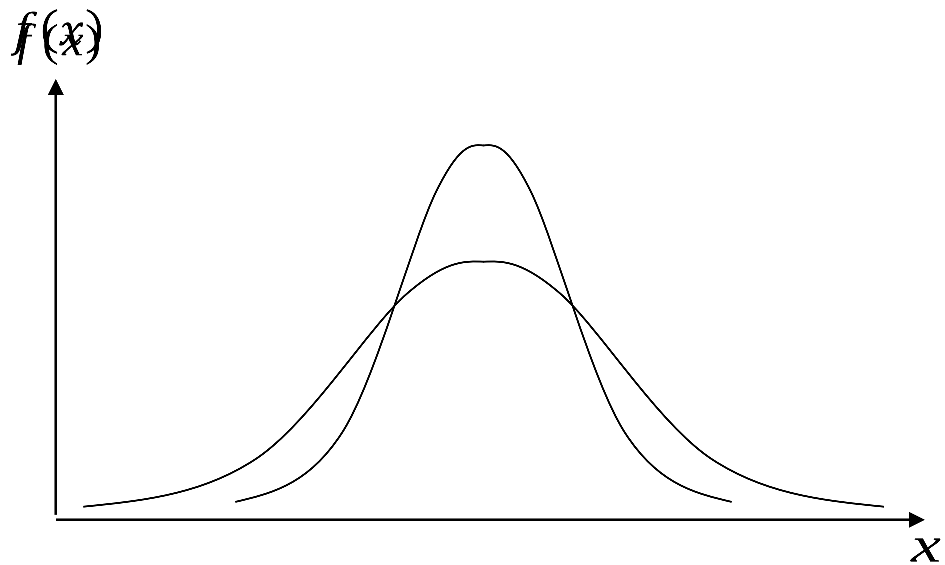
- A positive quantity that measures the spread of the distribution of the random variable about its mean value
- It is sometimes convenient to use the result

$$Var(X) = E(X^2) - \mu^2$$

- Standard Deviation(σ)

2.4.1 Definition and Interpretation of Variance

Two distribution with identical mean values but different variances



2.4.3 Chebyshev's Inequality

- Chebyshev's Inequality

  If a random variable X has a mean μ and a variance σ^2, then

$$P(\mu - c\,\sigma \le X \le \mu + c\,\sigma) \ge 1 - \frac{1}{c^2} \quad \text{for} \quad c \ge 1$$

2.4.4 Quantiles of Random Variables

- Quantiles of Random variables
  The p-th quantile x of a random variable X
  $$F(x) = p.$$
- Upper quartile(Q_3)
  The 75th percentile of the distribution
- Lower quartile(Q_1)
  The 25th percentile of the distribution
- Interquartile range(IQR)
  The distance between the two quartiles

2.4.4 Quantiles of Random Variables

- Example 14

  $F(x) = 1.5x - 2(x-50)^3 - 74.5$ for $49.5 \leq x \leq 50.5$.
  - Upper quartile : $\mathrm{F(x)} = 0.75$. $\mathrm{x} = 50.17$.
  - Lower quartile : $\mathrm{F(x)} = 0.25$. $\mathrm{x} =$49.83
  - Interquartile range : 50.17 - 49.83=0.34.

2.5 Jointly Distributed Random Variables
2.5.1 Jointly Distributed Random Variables

- Joint Probability Distributions
  - Discrete

    $$\mathrm{P}\left(\mathrm{X} = \mathrm{x}_i,\ \mathrm{Y} = \mathrm{y_j}\right) = \mathrm{p_{ij}} \geq 0$$

    satisfying $\sum_i \sum_j p_{ij} = 1.$

  - Continuous

    $\mathrm{f(x,y)} \geq 0$ satisfying $\int \int f(x,y)dxdy = 1.$

2.5.1 Jointly Distributed Random Variables

- Joint Cumulative Distribution Function

$$F(x, y) = P(X \le x, Y \le y)$$

  - Discrete

$$\mathrm{F}(a, b) = \sum_{i:x_i \le a} \sum_{j:y_j \le b} p_{ij}.$$

  - Continuous

$$\mathrm{F}(a, b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)dxdy$$

2.5.1 Jointly Distributed Random Variables

- Example 19 : Air Conditioner Maintenance
  - A company that services air conditioner units in residences and office blocks is interested in how to schedule its technicians in the most efficient manner
  - The random variable X, taking the values 1,2,3 and 4, is the service time in hours
  - The random variable Y, taking the values 1,2 and 3, is the number of air conditioner units

2.5.1 Jointly Distributed Random Variables

Y= number of units	X=service time			
	1	2	3	4
1	0.12	0.08	0.07	0.05
2	0.08	0.15	0.21	0.13
3	0.01	0.01	0.02	0.07

- Joint p.m.f

$$\sum_i \sum_j p_{ij} = 0.12 + \cdots + 0.07 = 1.00.$$

- Joint cumulative distribution function

$$\mathrm{F}(2,2) = \mathrm{p}_{11} + \mathrm{p}_{12} + \mathrm{p}_{21} + \mathrm{p}_{22} = 0.43.$$

2.5.2 Marginal Probability Distributions

- Marginal probability distribution
  - Obtained by summing or integrating the joint probability distribution over the values of the other random variable
  - Discrete

$$\mathrm{P}(\mathrm{X} = \mathrm{x_i}) = \mathrm{p_{i+}} = \sum_j p_{ij}.$$

  - Continuous

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

Example 20: (mineral deposits)

- x and y are the zinc and iron contents in ore samples, respectively.

$$f(x, y) = \frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000}, \qquad 0.5 \le x \le 1.5, \qquad 20 \le y \le 35.$$

- Joint pdf:

$$f(x, y)$$

- Marginal pdf's of X and Y:

2.5.3 Conditional Probability Distributions

- Conditional probability distributions
  - The probabilistic properties of the random variable X under the knowledge provided by the value of Y
  - Discrete

$$f_{X|Y}(x_i|y_j) = P(X = x_i|Y = y_j) = \frac{p_{ij}}{p_{+j}}.$$

  - Continuous

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

- The conditional probability distribution is also a probability distribution.

2.5.3 Conditional Probability Distributions

Example 19

- Marginal probability distribution of Y

- Conditional distribution of X given $Y = y_j$

Y= number of units	X=service time			
	1	2	3	4
1	0.12	0.08	0.07	0.05
2	0.08	0.15	0.21	0.13
3	0.01	0.01	0.02	0.07

2.5.3a Computation of E(g(X,Y))

Let $g(x, y)$ be a function of x and y. For example,

$\mathrm{g(x, y) = 2x + 5y}$, $\mathrm{g(x, y) = 3xy}$, or $\mathrm{g(x, y) = \frac{1}{x} + \frac{1}{y}}$.

- For discrete case

$$E\big(g(X, Y)\big) = \sum_{x,y} g(x, y) f(x, y).$$

- For continuous case

$$E\big(g(X, Y)\big) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy .$$

2.5.4 Independence and Covariance

- Two random variables **X** and **Y** are said to be independent if

$$f(x, y) = f_X(x) f_Y(y) \quad \text{for all } x \text{ and } y.$$

If X and Y are independent, $f(x, y)$ is factorized by a factor of x only and a factor of y only.

2.5.4 Independence and Covariance

- Covariance

$$Cov(X,Y) = E\big(X - E(X)\big)\big(Y - E(Y)\big).$$

- It is sometimes convenient to use that

$$Cov(X,Y) = E(XY) - E(X)E(Y).$$

- May take any positive or negative number
- Independent random variables have a covariance of zero
- What if the covariance is zero? Are X and Y independent?

2.5.4 Independence and Covariance

Example 19 (Air conditioner maintenance)

$E(X) = 2.59.$ $E(Y) = 1.79.$

$E(XY) = 4.86.$

$Cov(X, Y) = 0.224.$

- What happens to the covariance if X is replaced with $60 \times X$?

Y= number of units	X=service time			
	1	2	3	4
1	0.12	0.08	0.07	0.05
2	0.08	0.15	0.21	0.13
3	0.01	0.01	0.02	0.07

2.5.4 Independence and Covariance

- Correlation (ρ_{XY}):

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}.$$

- $-1 \leq \rho_{XY} \leq 1$. $\quad -1 \leq \rho_{XY} \leq 1$
- The correlation is invariant to linear transformations of X and Y.

Proof of $-1 \le \rho_{XY} \le 1$.

$$
\begin{aligned}
Q &= E[\big((X-\mu_X)+t(Y-\mu_Y)\big)^2] \\
&= E(X-\mu_X)^2 + 2tE[(X-\mu_X)(Y-\mu_Y)] + t^2E(Y-\mu_Y)^2 \\
&= \sigma_X^2 + 2t\,Cov(X,Y) + t^2\sigma_Y^2 \ .
\end{aligned}
$$

$Q \ge 0$ for all $-\infty < t < \infty$.

So, if we regard Q as a quadratic function of t,

the discriminant $Cov(X,Y)^2 - \sigma_X^2\sigma_Y^2 \le 0$.

This implies $-1 \le \rho_{XY} \le 1$

2.5.4 Independence and Covariance

- Example 19: (Air conditioner maintenance)

$Var(X) = 1.162$. $Var(Y) = 0.384$.

$\rho_{XY} = 0.34$.

Y= number of units	X=service time			
	1	2	3	4
1	0.12	0.08	0.07	0.05
2	0.08	0.15	0.21	0.13
3	0.01	0.01	0.02	0.07

- Interpretation of the value of ρ_{XY}.

2.6 Combinations and Functions of Random Variables

2.6.1 Linear Functions of Random Variables

- Linear Functions of a Random Variable

  If X is a random variable and $Y = aX + b$ for some numbers $a, b \in R$ then $E(Y) = aE(X) + b$ and $\mathrm{Var(Y)} = \mathrm{a^2 Var(X)}$.

- Standardization

  If a random variable X has an expectation of μ and a variance of σ^2,

$$Y = \frac{X-\mu}{\sigma}$$

has an expectation of zero and a variance of one.

2.6.1 Linear Functions of Random Variables

- Sums of Random Variables
  - If X_1 and X_2 are two random variables, then

    $$\mathrm{E}(\mathrm{X}_1 + X_2) = E(X_1) + E(X_2)$$

    and

    $$\mathrm{Var}(\mathrm{X}_1 + X_2) = Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2).$$

  - If X_1 and X_2 are independent, then

    $$\mathrm{Var}(\mathrm{X}_1 + X_2) = Var(X_1) + Var(X_2).$$

2.6.2 Linear Combinations of Random Variables

- Linear Combinations of Random Variables
  - If $X_1, \cdots, X_n$ is a sequence of random variables and $a_1, \cdots, a_n$ and b are constants, then
    $$E(a_1X_1 + \cdots + a_nX_n + b) = a_1E(X_1) + \cdots + a_nE(X_n) + b.$$
  - If, in addition, the random variables are independent, then
    $$Var(a_1X_1 + \cdots + a_nX_n + b) = a_1^2Var(X_1) + \cdots + a_n^2Var(X_n).$$

2.6.2 Linear Combinations of Random Variables

- Averaging Independent Random Variables
  - Suppose that $X_1, \cdots X_n$ is a sequence of independent random variables with an expectation μ and a variance σ^2.
  - Let $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$.

  Then $E(\overline{X}) = \mu$

  and $Var(\overline{X}) = \frac{\sigma^2}{n}$.

2.6.3 Nonlinear Functions of a Random Variable

- A nonlinear function of a random variable X is another random variable Y=g(X) for some nonlinear function g.

  For example, $Y = X^2$, $Y = \sqrt{X}$, $Y = e^X$.

- Let's consider $Y = X^2$ where X is a continuous random variable.

  $E(Y) = \int g(x)f(x)dx = \int x^2 f(x)dx.$

  If $f(x) = 0.5$ for $-1 \leq x \leq 1$ and 0 otherwise.

  Then

$$E(Y) = \int_{-1}^{1} 0.5x^2\, dx = \frac{1}{3}.$$

Chapter Summary

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Independence and Covariance

2.6 Combinations and Functions of Random Variables