Chapter 4.
Continuous Probability Distributions

4.1 The Uniform Distribution

4.2 The Exponential Distribution

4.3 The Gamma Distribution

4.4 The Weibull Distribution

4.5 The Beta Distribution

4.1 The Uniform Distribution
4.1.1 Definition of the Uniform Distribution

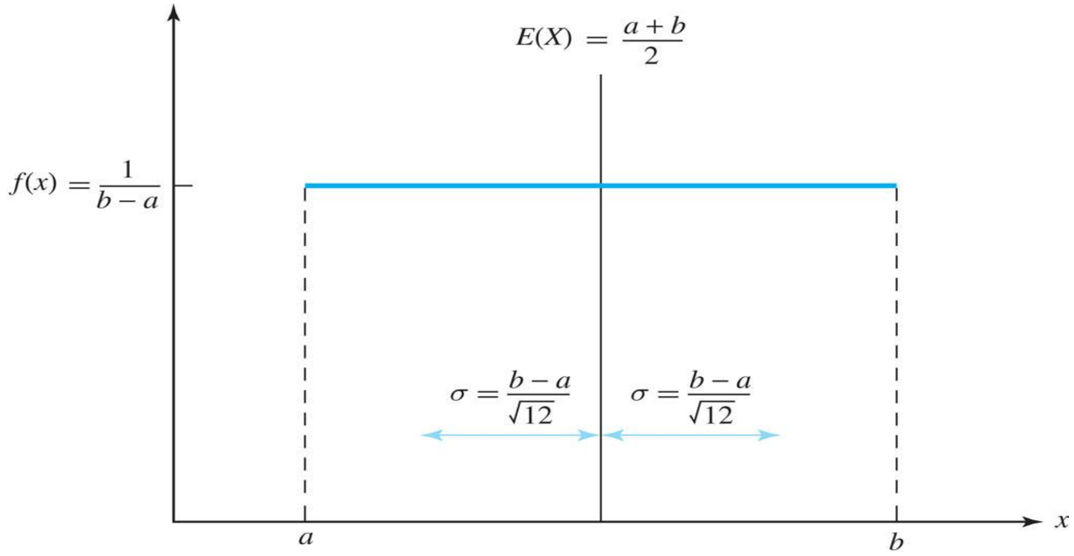
- **Uniform distribution,** $\mathrm{U}(a, b)$

$$f(x; a, b) = \frac{1}{b-a}, \qquad a \leq x \leq b.$$

- **The mean and variance of the uniform distribution are**

$$\mu = \frac{a+b}{2} \textbf{ and}$$

$$\sigma^2 = \frac{(b-a)^2}{12}$$



4.2 The Exponential Distribution
4.2.1 Definition of the Exponential Distribution

- Exponential distribution (Exp(λ))

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x > 0.$$

- Cdf:

$$F(x) = 1 - e^{-\lambda x}$$

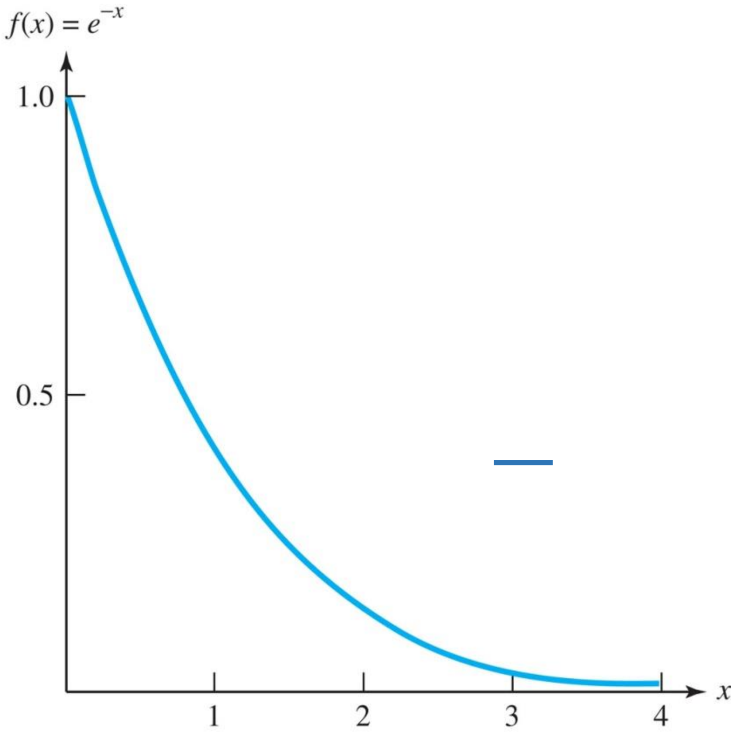
- Mean and variance:

$$E(X) = \frac{1}{\lambda}$$

$$Var(X) = \frac{1}{\lambda^2}.$$

Figure 4.3

Probability density function of an exponential distribution with parameter λ = 1



- The exponential distribution often arises, in practice, as being of **the amount of time until some specific event occurs**.

  For example,

  the amount of time until an earthquake occurs,

  the amount of time until a new war breaks out, or

  the amount of time until a telephone call at a certain telephone.

4.2.2. The memoryless property of the Exponential Distribution

- For any non-negative $\boldsymbol{x}$ and $\boldsymbol{y}$

$$\boldsymbol{P(X \geq x + y | X \geq x) = P(X \geq y)}$$

- This is equivalent to

$$\boldsymbol{P(X \geq x + y) = P(X \geq x)P(X \geq y)}$$

- Example:

Suppose that a number of miles that a car run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000 mile trip, what is the probability that he will be able to complete his trip without replacing the battery?

(Sol)

Let X be a random variable including the remaining lifetime (in thousand miles) of the battery. Then,

$$E(X) = \frac{1}{\lambda} = 10.$$

$$P(X \geq 5) = 1 - F(5) \approx 0.604$$

- Proposition:

If $\mathrm{X}_1, \cdots X_n$ are independent exponential random variables having respective parameters $\lambda_1, \cdots, \lambda_n$, then $\min\{\mathrm{X}_1 \cdots, X_n\}$ is the exponential random variable with parameter $\sum_{i=1}^{n} \lambda_i$.

(Proof)

$$P(\min(X_1, \cdots, X_n) > x) = P(X_1 > x, \cdots, X_n > x)$$

$$= \prod_{i=1}^{n} P(X_i > x)$$

$$= \prod_{i}^{n} \exp(-\lambda_i x)$$

$$= \exp(-\textstyle\sum_{i=1}^{n} \lambda_i x).$$

- Example:

A series system is one that all of its components to function in order for the system itself to be functional. For an *n* component series system in which the component lifetimes are independent exponential random variables with respective parameters $\lambda_1, \cdots, \lambda_n$. What is the probability the system serves for a time t?

(Sol)

Let X be a random variable indicating the system lifetime.

Then, X is an exponential random variable with parameter $\sum_{i=1}^{n} \lambda_i$.

Hence, $P(X \geq t) = \exp(-\sum_{i=1}^{n} \lambda_i t)$.

4.2.3 The Poisson process

- A stochastic process is a sequence of random events.
- A Poisson process with parameter λ is a stochastic process

  where the time (or space) intervals between event-occurrences follow the Exponential distribution with parameter λ.
- If X is the number of events occurring within a fixed time (or space) interval of length t, then

$$\mathrm{X} \sim Poi(\lambda t).$$

4.2.4 Examples of the Exponential Distribution

Example 32 (Steel Girder Fractures, p.195)

- 42 fractures on average on a 10m long girder

  So the between-fracture length is

  10/43=0.23m on average

- If the between-fracture length (X) follows an exponential distribution, how would you define the gap?

$$X \sim Exp'l(\lambda) \quad \text{with } \lambda = 43/10.$$

- How would you define the number of fractures (Y) per 1m steel girder?

$$\mathrm{Y} \sim \ Poi(\lambda).$$

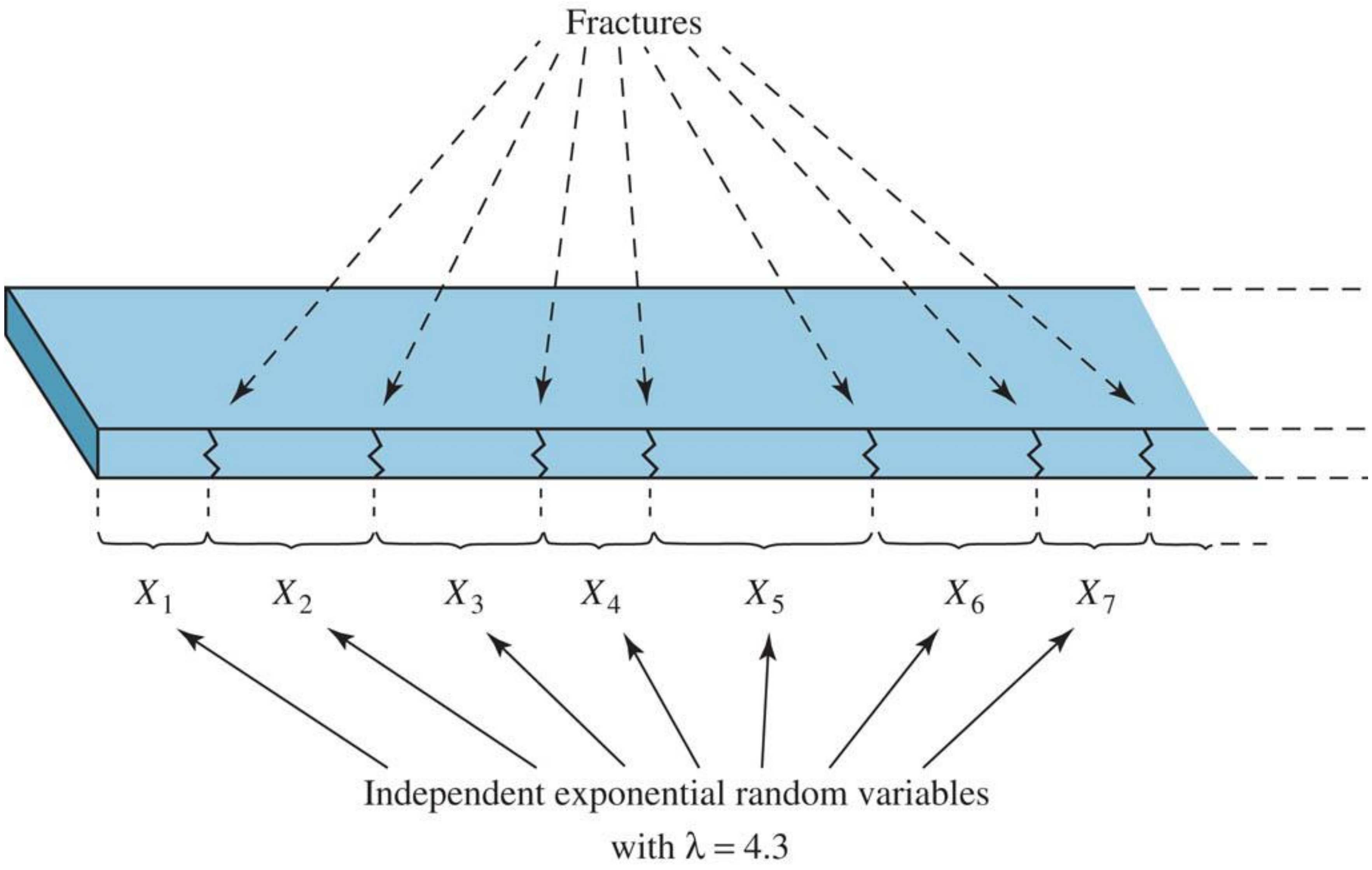


Figure 4.9 Poisson process modeling fracture locations on a steel girder

- P(the length of a gap is less than 10cm)=?

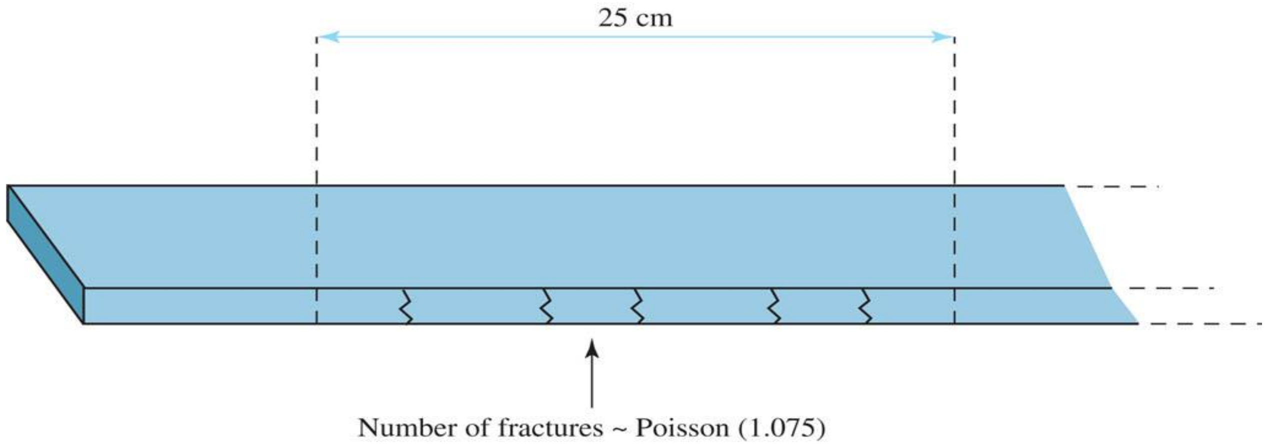
$$P(X \leq 0.1) = 1 - e^{-4.3\times 0.1} = 0.35$$

- P(a 25-cm segment of a girder contains at least two fractures)=?

The mean rate for a 25-cm segment is equal to $\lambda \times 0.25$.

$P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1) = 0.292.$

Here $P(Y = y) = \dfrac{e^{-\lambda/4}(\lambda/4)^y}{y!}$.



4.3 The Gamma Distribution
4.3.1 Definition of the Gamma distribution

- Useful for reliability theory and life-testing and has several important sister distributions
- The Gamma function:

$$\Gamma(k) = \int_0^\infty y^{k-1} e^{-y}\, dy \text{ for } k > 0.$$

- Let $X = \frac{Y}{\lambda}$ for λ>0. Then

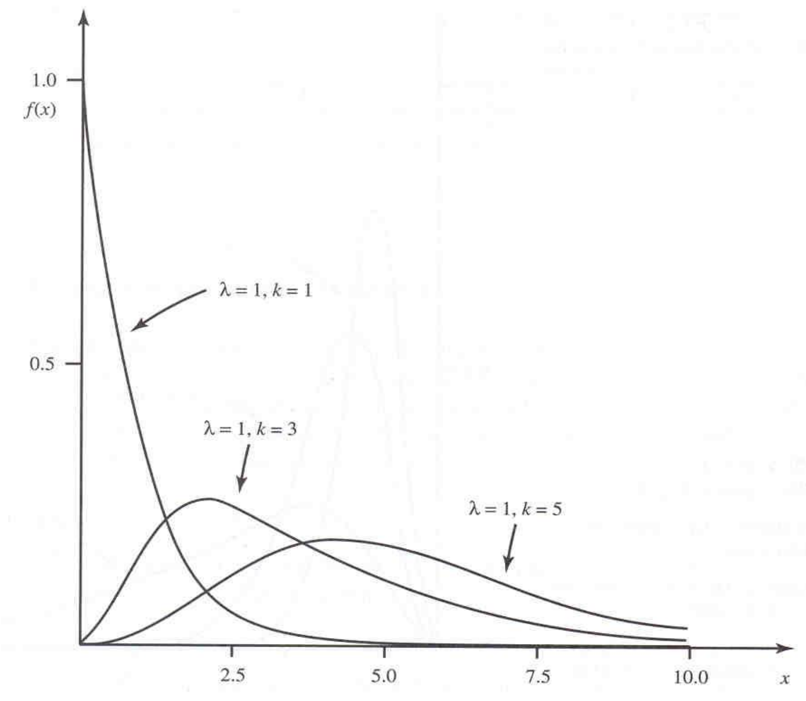
$$\Gamma(k) = \int_0^\infty \lambda^k x^{k-1} e^{-\lambda x}\, dx \text{ for } k > 0.$$

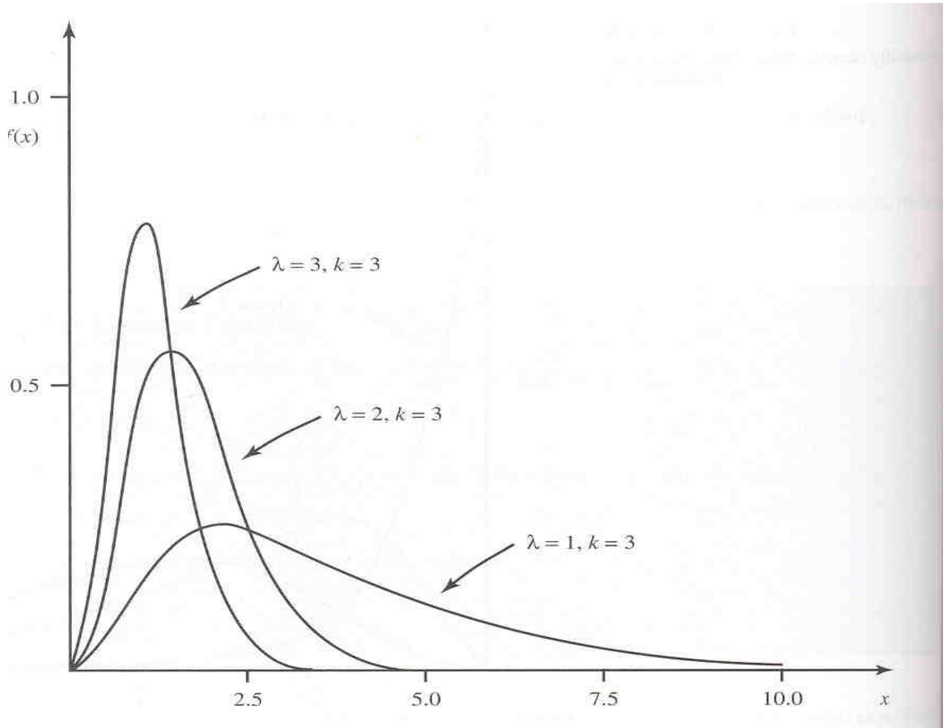
- Gamma distribution (Gam(k, λ)) with k>0 and λ>0

$$f(x; k, \lambda) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}, \quad x > 0.$$

- $E(X) = k/\lambda$ and $\mathrm{Var}(X) = k/\lambda^2$

Curves of Gamma pdf





$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} \quad \text{for} \quad x \geq 0$$

Properties of the Gamma function

- $\Gamma(\alpha+1) = \int_0^\infty x^\alpha\, e^{-x} dx = -x^\alpha e^{-x}|_0^\infty + \alpha \int_0^\infty x^{\alpha-1}\, e^{-x} dx$

  $= \ \alpha \int_0^\infty x^{\alpha-1}\, e^{-x} dx = \alpha\, \Gamma(\alpha).$
- $\Gamma(1) = 1.$
- $\Gamma(1/2) = \sqrt{\pi}$. (to be proved in Ch. 5)
- $\Gamma(n) = (n-1)!.$

Properties of Gamma random variables

- If $X_1 \cdots, X_n$ are independent Gamma random variables with respective parameters (k_i, λ), then

$$\sum_{i=1}^{n} X_i \sim \ \ Gam(\sum_{i}^{n} k_i, \lambda)$$

4.3.2 Examples of the Gamma distribution

Example 32 (Steel Girder Fractures) :

$X_1, \cdots, X_k$ iid with $Exp'l(\lambda)$. Then

$$X = \sum_{i=1}^{k} X_i \sim Gam(k, \lambda).$$

Y= the number of fractures within 1m of the girder

Then,

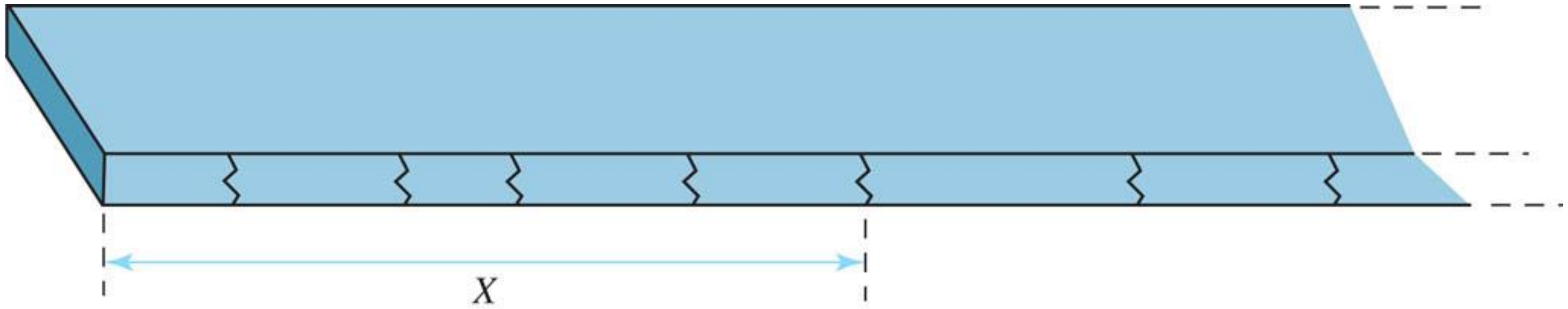
$Y \sim Poi(\lambda)$

$P(X \leq 1) = P(Y \geq k)$

X is a Gamma random variable with $k = 5$ and $\lambda = 4.3$.

$$\mathrm{E}(X) = \frac{k}{\lambda} = \frac{5}{4.3} = 1.16(m)$$
$P(\mathrm{X} \leq 1) = 0.4296$.

If we let $\mathrm{Y} \sim Poi(\lambda)$, $\quad P(Y \geq 5) = P(X \leq 1)$.

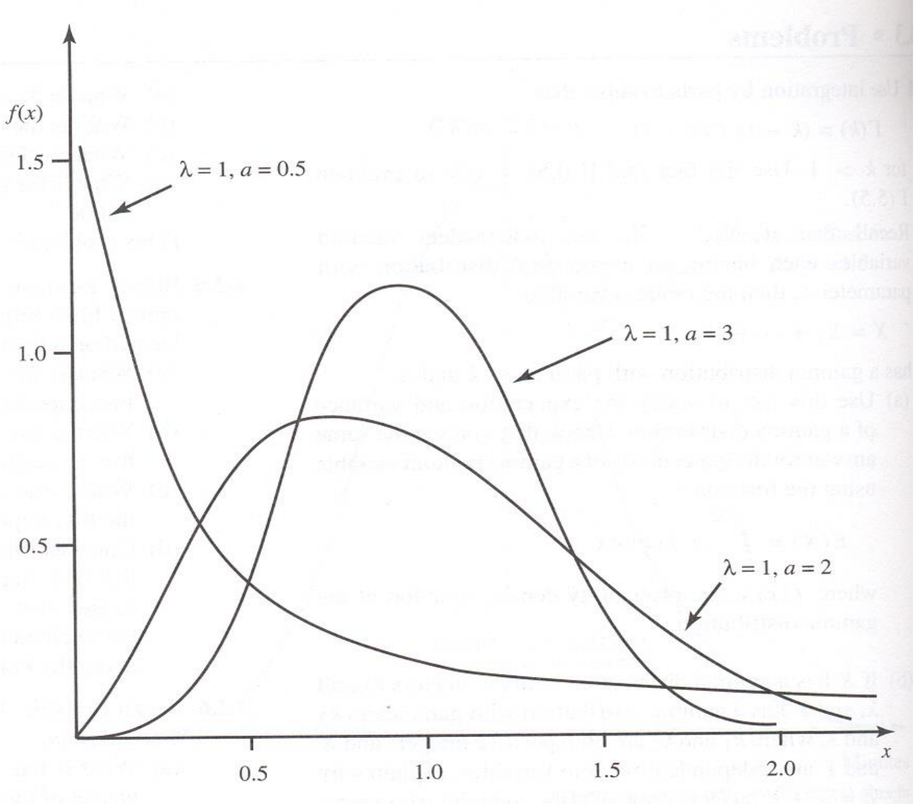


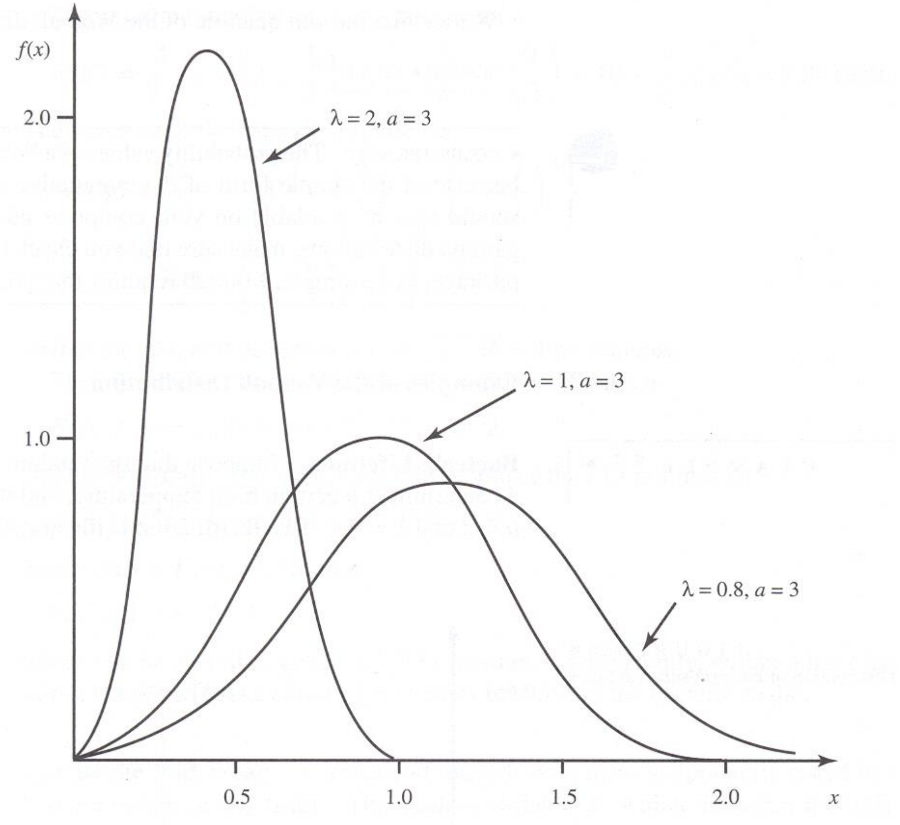
$X \sim$ Gamma $\quad k = 5, \lambda = 4.3$

4.4 Weibull distribution
4.4.1 Definition of the Weibull distribution

- Useful for modeling failure or waiting times
  [see Examples 33(Bacteria lifetimes) & 34(Car brake pad wear)]
- If X $\sim Exp'l(1)$ and $\mathrm{Y} = \frac{1}{\boldsymbol{\lambda}}\mathrm{X}^{1/\mathrm{a}}$ for a > 0, then
  $$\mathrm{Y} \sim Weibull(\boldsymbol{\lambda}, \boldsymbol{a}).$$
- The pdf of Weibull($\boldsymbol{\lambda}$, $\boldsymbol{a}$):
  $$f(y) = a\lambda(\lambda y)^{a-1}e^{-(\lambda y)^a} \quad \text{for a} > 0, \ \ \lambda > 0.$$
  If a = 1, the Weibull distribution is the same as the Exponential distribution with parameter λ.
- The mean and variance :
  $$E(Y) = \frac{1}{\lambda}\Gamma(1 + \frac{1}{a})$$
  $$Var(Y) = \frac{1}{\lambda^2}\{\Gamma\left(1 + \frac{2}{a}\right) - \left[\Gamma\left(1 + \frac{1}{a}\right)\right]^2\}$$

Curves of the Weibull distribution





$$f(x) = a\lambda(\lambda x)^{a-1} e^{-(\lambda x)^a}$$

4.5 The Beta distribution

- Useful for modeling proportions and personal probability

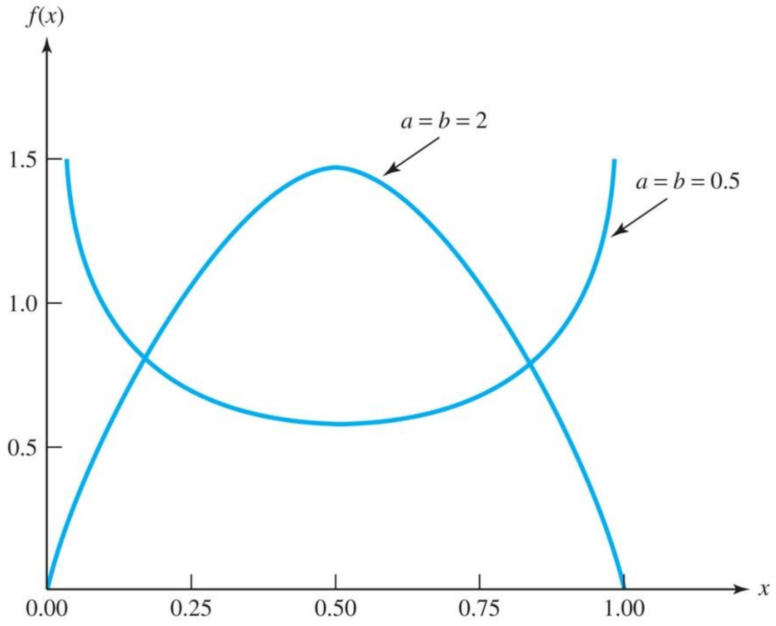
[See Examples 35(Stock prices) & 36(Bee colonies).]

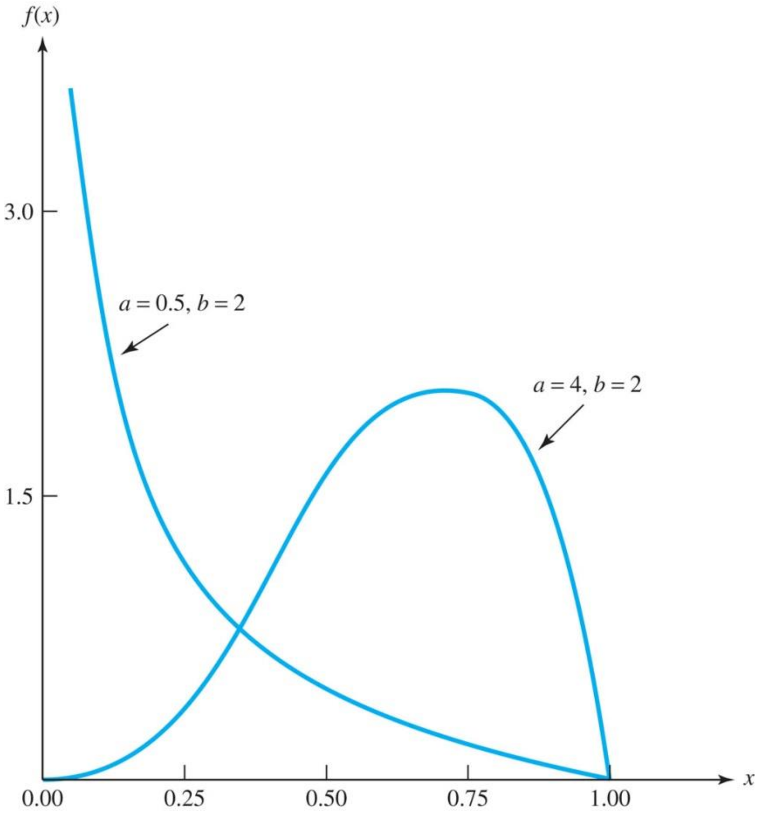
- Pdf:

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1.$$

- Mean and variance:

$$\mu = \frac{\alpha}{\alpha+\beta}. \quad \sigma^2 = \frac{1}{\alpha+\beta+1}\cdot\frac{\alpha\beta}{(\alpha+\beta)^2}$$





Probability density functions of the beta distribution

- Example: the Beta distribution and rainstorms.

  Data gathered by the U.S. Weather Service in Alberquerque, New Mexico, concern the fraction of the total rainfall falling during the first 5 minutes of storms occurring during both summer and non-summer seasons. The data for 14 non-summer storms can be described reasonably well by a standard beta distribution with a=2.0 and b=8.8.

- Let X be the fraction of the storm's rainfall falling during the first 5 minutes. Then, the probability that more than 20% of the storm's rainfall during the first 5 minutes is determined by

$$P(X > 0.2) = \frac{\Gamma(2.0 + 8.8)}{\Gamma(2.0)\Gamma(8.8)} \int_{0.2}^{1.0} u^{2.0-1}(1-u)^{8.8-1}\, du = 0.39$$

Chapter Summary

4.1 The Uniform Distribution

4.2 The Exponential Distribution and a Poisson Process

4.3 The Gamma Distribution

4.4 The Weibull Distribution

4.5 The Beta Distribution