9.2.8 Let x_i, y_i be the times take to kill i-th culture of a bacterium using a standard antibiotic and using a new antibiotic for $i = 1, \cdots, 8$, respectively. Also, we define $z_i = x_i - y_i$. Then, the sample average and the sample standard deviation of z_i are $\overline{z} = 1.375$ and $s = 1.785$ respectively. The t-statistic is

$$t = \frac{\sqrt{n}\overline{z}}{s} = \frac{\sqrt{8} \cdot 1.375}{1.785} = 2.179.$$

The p-value for the one-sided hypothesis testing problem

$$H_0 : \mu \leq 0 \quad \text{versus} \quad H_A : \mu > 0$$

is $P(X > 2.179)$. We note that X has a t-distribution with 7 degrees of freedom. Since $P(X > 2.179) = 0.033 < 0.05$, we reject H_0. Thus we conclude that there is some evidence that the new antibiotic is quicker than the standard antibiotic.

9.3.2 (a) Note that $s_p = \sqrt{\frac{(n-1)s_x^2+(m-1)s_y^2}{n+m-2}} = \sqrt{\frac{13 \cdot 4.30^2+13 \cdot 5.23^2}{26}} = 4.787$ and the degrees of freedom is $(n-1)+(m-1) = 26$. The confidence interval is given by

$$\left(\overline{x} - \overline{y} - t_{0.005,26} \cdot s_p\sqrt{\frac{1}{n} + \frac{1}{m}}, \overline{x} - \overline{y} + t_{0.005,26} \cdot s_p\sqrt{\frac{1}{n} + \frac{1}{m}}\right)$$

$$= (-14.028, -3.972).$$

(b) We use the the unpooled variance method to solve this problem. For the unpooled procedure, the appropriate degrees of freedom are

$$\nu = \frac{\left(\frac{s_x^2}{n} + \frac{s_y^2}{m}\right)^2}{\frac{s_x^4}{n^2(n-1)} + \frac{s_y^4}{m^2(m-1)}} = 25.063$$

which can be rounded down to $\nu = 25$. Thus the confidence interval is given by

$$\left(\overline{x} - \overline{y} - t_{0.005,25}\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}, \overline{x} - \overline{y} + t_{0.005,25}\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}\right) = (-14.043, -3.957).$$

(c) The t-statistic is

$$t = \frac{\overline{x} - \overline{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}} = -4.974.$$

The two-sided p-value is $2P(X > 4.974) = 0.00004$ where X has a t-distribution with $\nu = 25$ degrees of freedom. Since $0.00004 < 0.01$, we reject H_0.

9.3.6

(a) From the problem, $H_0 : \mu_A = \mu_B$ versus $H_A : \mu_A \neq \mu_B$. We remark that we use a pooled variance procedure. We have $s_p = \sqrt{\frac{(n-1)s_x^2+(m-1)s_y^2}{n+m-2}} = 0.131$ and $t = \frac{\overline{x}-\overline{y}}{s_p\sqrt{\frac{1}{n}+\frac{1}{m}}} = -2.765$. Since the two-sided p-value is $P(X > 2.765) = 0.007 < 0.01$, we reject H_0. Note that X has a t-distribution with $n+m-2 = 80$ degrees of freedom.

(b) The C.I. is given by

$$\left(\overline{x} - \overline{y} - t_{0.005,80} \cdot s_p\sqrt{\frac{1}{n} + \frac{1}{m}}, \overline{x} - \overline{y} + t_{0.005,80} \cdot s_p\sqrt{\frac{1}{n} + \frac{1}{m}}\right)$$
$$= (-0.156, -0.004).$$

9.3.10

(a) The z-statistic is

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} = \frac{5.782 - 6.443}{\sqrt{\frac{2.0^2}{38} + \frac{2.0^2}{40}}} = -1.459 \quad \textbf{(+3 points)}$$

The p-value is

$$p\text{-value} = \Phi(-1.459) = 0.072 \quad \textbf{(+2 points)}$$

(b) $z_{0.01} = 2.33$. Then, the end-point of a 99% one-sided confidence interval is

$$\bar{x} - \bar{y} + z_{0.01}\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}} = 5.782 - 6.443 + 2.33\sqrt{\frac{2.0^2}{38} + \frac{2.0^2}{40}} = 0.395$$

The 99% one-sided confidence interval is $(-\infty, 0.395)$ **(+5 points)**

9.3.14 From 9.3.2, a degree of freedom $\nu = 25$.

$$L_0 = 5 \geq 2t_{0.005,25}\sqrt{\frac{s_x^2}{n_0} + \frac{s_y^2}{n_0}} = 5.574\sqrt{\frac{4.30^2 + 5.23^2}{n_0}} \quad \textbf{(+5 points)}$$

$$n_0 \geq \frac{5.574^2(4.30^2 + 5.23^2)}{25} = 56.97 \quad \textbf{(+2 points)}$$

Then, $n_0 = 57$ is sufficient, so 43 additional samples are recommended. **(+3 points)**

9.3.22 Test $H_0 : \mu_{std} \geq \mu_{new}$ vs $H_A : \mu_{std} < \mu_{new}$ **(+2 points)**

By the general procedure,

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}} = \frac{56.43 - 62.11}{\sqrt{\frac{6.30^2}{14} + \frac{7.15^2}{20}}} = -2.446 \quad \textbf{(+2 points)}$$

$$\nu^* = \frac{\left(\frac{s_x^2}{n} + \frac{s_y^2}{m}\right)^2}{\frac{s_x^4}{n^2(n-1)} + \frac{s_y^4}{m^2(m-1)}} = \frac{\left(\frac{6.30^2}{14} + \frac{7.15^2}{20}\right)^2}{\frac{6.30^4}{14^2*13} + \frac{7.15^4}{20^2*19}} \approx 30 \quad \textbf{(+2 points)}$$

Then, p-value is

$$p\text{-value} = P(T < -2.446) = 0.0103 \quad \textbf{(+2 points)}$$

Since $\alpha = 0.05$, the null hypothesis is rejected and there is sufficient evidence that the new method has a larger breaking strength. **(+2 points)**

9.7.9 (a)

$$\frac{\sigma_A^2 S_y^2}{\sigma_B^2 S_x^2} = \frac{S_y^2/\sigma_B^2}{S_x^2/\sigma_A^2} \sim \frac{\chi_{n-1}^2/n-1}{\chi_{m-1}^2/m-1} \sim F_{m-1,n-1} \quad \textbf{(+3 points)}$$

(b) From (a), the first one is obvious **(+2 points)** and observe that

$$P\left(F_{1-\frac{\alpha}{2},m-1,n-1} \leq F_{m-1,n-1}\right) = 1 - \frac{\alpha}{2},$$

$$P\left(F_{m-1,n-1} \leq \frac{1}{F_{1-\frac{\alpha}{2},m-1,n-1}}\right) = 1 - \frac{\alpha}{2}.$$

Hence, by definition, $\frac{1}{F_{1-\frac{\alpha}{2},m-1,n-1}} = F_{\frac{\alpha}{2},n-1,m-1}$, which proves the second one **(+2 points)**.

(c) We can deduce (c) from (b)

$$\begin{aligned} &P\left(\frac{S_x^2}{S_y^2 F_{\alpha/2,n-1,m-1}} \leq \frac{\sigma_A^2}{\sigma_B^2} \leq \frac{S_x^2 F_{\alpha/2,n-1,m-1}}{S_y^2}\right) \\ =&P\left(\frac{1}{F_{\alpha/2,n-1,m-1}} \leq \frac{S_y^2\sigma_A^2}{S_x^2\sigma_B^2} \leq F_{\alpha/2,n-1,m-1}\right) = 1 - \alpha \quad \textbf{(+3 points)} \end{aligned}$$

9.7.10 By the previous problem, we can conclude that the desired confidence interval is (0.20, 1.01) or (0.98, 4,78) $\left(\text{This is for } \sigma_y^2/\sigma_x^2\right)$ **(+10 points)**.

9.7.14 Let x_i be the strength of the cement sample tested with procedure 1 and let y_i be the strength of the cement sample tested with procedure 2. We can construct the hypothesis $H_0 : \mu_1 - \mu_2 = 0$ versus $H_A : \mu_1 - \mu_2 \neq 0$ **(+3 points)**. Then the mean $\bar{z}$ of $z_i = x_i - y_i$ is -0.022 and the sample deviation s is 0.591. p-value is

$$2 \cdot P\left(t_8 \leq \frac{\sqrt{n}(\bar{z} - \mu)}{s}\right) = 2 \cdot P\left(t_8 \leq \frac{\sqrt{9}(-0.022)}{0.591}\right) = 2 \cdot P\left(t_8 \leq -0.112\right) = 0.913.$$

Since $0.913 > 0.05$, there is no evidence that two procedures provide different results on average **(+7 points)**.

9.7.22 Let x_i be the blood pressure with a standard method via a sphygmomanometer and y_i be the blood pressure with a new method based upon a simple finger monitor. From data, we know that $n = 15$ and sample mean of $z_i = x_i - y_i$ is 0.400 and the std s of z_i is 1.957. We can conduct the dependent t-test for the paired samples. Consider the hypothesis :

$$H_0 : \mu_z = 0 \quad \text{versus} \quad H_A : \mu_z \neq 0$$

Then the t-statistic and the p-value are given by

$$t = \frac{\sqrt{n}(\bar{z} - \mu_z)}{s} = 0.792 < t_{\alpha/2, n-1} = 2.145, \quad \text{p-value} = 2P(t_{n-1} > t) = 0.442 > \alpha,$$

where t_{n-1} follows the t-distribution with degree $n - 1$. We can not reject the null hypothesis at the significant level $\alpha = 0.05$. There is no evidence that two procedures provide different results on average.

- In this problem, the sample is not present in an ordinary unpaired testing situation. So, you should use a paired difference test **(+5 points)** to increase the statistical power.
- Conduct the test you suggest correctly. **(+5 points)**