RESOLUÇÃO · LISTA · REGRA DA CADEIA

(01) Em cada caso, verifique se a função real u, satisfaz à equação à derivadas parciais indicada:

ⓐ $u = f(x,y)$; $x(r,s) = r^3 - s^3$; $y(r,s) = s^3 - r^3$; $s^2 \frac{\partial u}{\partial r} + r^2 \frac{\partial u}{\partial s} = 0$;

Solução: usemos o quadro (01):

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \therefore \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}(3r^2) + \frac{\partial u}{\partial y}(-3r^2) \therefore s^2\frac{\partial u}{\partial r} = 3s^2r^2\frac{\partial u}{\partial x} - 3s^2r^2\frac{\partial u}{\partial y};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \therefore \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}(-3s^2) + \frac{\partial u}{\partial y}(3s^2) \therefore r^2\frac{\partial u}{\partial r} = -3r^2s^2\frac{\partial u}{\partial x} + 3r^2s^2\frac{\partial u}{\partial y};$$

Logo, $s^2\frac{\partial u}{\partial r} + r^2\frac{\partial u}{\partial r} = 3s^2r^2\frac{\partial u}{\partial x} - 3s^2r^2\frac{\partial u}{\partial y} - 3r^2s^2\frac{\partial u}{\partial x} + 3r^2s^2\frac{\partial u}{\partial y} = 0$;

Sim, satisfaz;

ⓑ $u = f(x,y)$; $x(r,s) = r+s$; $y(r,s) = r-s$; $\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\partial u}{\partial r}\frac{\partial u}{\partial s}$;

Solução: novamente usaremos o quadro (01):

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \therefore \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.(1) + \frac{\partial u}{\partial y}.(1) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \therefore \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}.(1) + \frac{\partial u}{\partial y}.(-1) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y};$$

Logo, $\frac{\partial u}{\partial r}\frac{\partial u}{\partial s} = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2$; Sim, satisfaz;

ⓒ $u = f(x,y,z)$; $x(r,s,t) = r-s$; $y(r,s,t) = s-t$; $z(r,s,t) = t-r$; $\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0$;

Solução: usaremos o quadro (05):

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial u}{\partial x}.(1) + \frac{\partial u}{\partial y}.(0) + \frac{\partial u}{\partial z}.(-1) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial z};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} = \frac{\partial u}{\partial x}.(-1) + \frac{\partial u}{\partial y}.(1) + \frac{\partial u}{\partial z}.(0) = -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y};$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial u}{\partial x}.(0) + \frac{\partial u}{\partial y}.(-1) + \frac{\partial u}{\partial z}.(1) = -\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z};$$

Logo, $\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$;

Sim, satisfaz;

(d) $u = f(x,y)$; $x(r,s) = \dfrac{s-r}{rs}$; $y(s,t) = \dfrac{t-s}{st}$; $r,s,t \neq 0$; $r^2\dfrac{\partial u}{\partial r} + s^2\dfrac{\partial u}{\partial s} + t^2\dfrac{\partial u}{\partial t} = 0$;

Solução: em primeiro lugar, como r, s e t são não-nulos, escrevamos:

$$x(r,s) = \frac{s}{rs} - \frac{r}{rs} = \frac{1}{r} - \frac{1}{s}; \quad y(r,s) = \frac{t-s}{st} = \frac{t}{st} - \frac{s}{st} = \frac{1}{s} - \frac{1}{t};$$

Em segundo lugar, observemos que nenhum dos quadros da teoria se encaixa, exatamente, em nossa situação.

Vamos então criar nosso quadro:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{-1}{r^2}\right);$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x}\left(\frac{1}{s^2}\right) + \frac{\partial u}{\partial y}\left(\frac{-1}{s^2}\right);$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial y}\cdot\left(\frac{1}{t^2}\right);$$

Logo, $r^2\dfrac{\partial u}{\partial r} + s^2\dfrac{\partial u}{\partial s} + t^2\dfrac{\partial u}{\partial t} = r^2\left(\dfrac{\partial u}{\partial x}\cdot\left(\dfrac{-1}{r^2}\right)\right) + s^2\left(\dfrac{\partial u}{\partial x}\cdot\left(\dfrac{1}{s^2}\right) - \dfrac{\partial u}{\partial y}\left(\dfrac{1}{s^2}\right)\right) + t^2\left(\dfrac{\partial u}{\partial y}\cdot\left(\dfrac{1}{t^2}\right)\right)$;

$$= -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} = 0; \text{ Sim, satisfaz;}$$

(e) $u = f(ar+bs)$; a e $b \in \mathbb{R}$; $b\dfrac{\partial u}{\partial r} - a\dfrac{\partial u}{\partial s} = 0$;

Solução: começamos introduzindo a variável $x(r,s) = ar+bs$; Usaremos o quadro (03);

Assim, ficamos com $u = f(x)$; $x(r,s) = ar+bs$;

$$\frac{\partial u}{\partial r} = \frac{du}{dx}\frac{\partial x}{\partial r} = \frac{du}{dx}\cdot a; \qquad \text{Logo, } b\frac{\partial u}{\partial r} - a\frac{\partial u}{\partial s} = ba\frac{du}{dx} - ab\frac{du}{dx} = 0;$$

$$\frac{\partial u}{\partial s} = \frac{du}{dx}\frac{\partial x}{\partial s} = \frac{du}{dx}\cdot b; \qquad \text{Sim, satisfaz;}$$

(f) $u(r,s) = s + f(r^2 - s^2)$; $s\dfrac{\partial u}{\partial r} + r\dfrac{\partial u}{\partial s} = r$;

Solução: vamos introduzir, $v = f(x)$ com $x(r,s) = r^2 - s^2$;

Então, $\dfrac{\partial u}{\partial r} = \dfrac{\partial (s)}{\partial r} + \dfrac{\partial v}{\partial r} = \dfrac{\partial v}{\partial r}$; e $\dfrac{\partial u}{\partial s} = \dfrac{\partial (s)}{\partial s} + \dfrac{\partial v}{\partial s} = 1 + \dfrac{\partial v}{\partial s}$;

Usaremos, novamente, o quadro (03) para $v = f(x)$;

Falta descobrirmos $\dfrac{\partial v}{\partial r}$ e $\dfrac{\partial v}{\partial s}$;

$$\frac{\partial v}{\partial r} = \frac{dv}{dx}\cdot\frac{\partial x}{\partial r} = 2r\frac{dv}{dx}; \qquad \frac{\partial v}{\partial s} = \frac{dv}{dx}\cdot\frac{\partial x}{\partial s} = -2s\frac{dv}{dx};$$

$$s\frac{\partial u}{\partial r} + r\frac{\partial u}{\partial s} = s\frac{\partial v}{\partial r} + r\left(1 + \frac{\partial v}{\partial s}\right)$$

$$= s\left(2r\frac{dv}{dx}\right) + r + r\left(-2s\frac{dv}{dx}\right)$$

$$= 2rs\frac{dv}{dx} + r - 2rs\frac{dv}{dx} = r;$$

Sim, satisfaz;

(8) $u(r,s) = rs + f(r^2+s^2)$; $s\dfrac{\partial u}{\partial r} - r\dfrac{\partial u}{\partial s} = s^2 - r^2$;

Solução: vamos introduzir, $v = f(x)$ com $x(r,s) = r^2+s^2$;

Então, $\dfrac{\partial u}{\partial r} = \dfrac{\partial(rs)}{\partial r} + \dfrac{\partial v}{\partial r} = s + \dfrac{\partial v}{\partial r}$; e $\dfrac{\partial u}{\partial s} = \dfrac{\partial(rs)}{\partial s} + \dfrac{\partial v}{\partial s} = r + \dfrac{\partial v}{\partial s}$;

E, para encontrarmos $\dfrac{\partial v}{\partial r}$ e $\dfrac{\partial v}{\partial s}$ vamos usar o quadro (03):

$$\left.\begin{aligned}\frac{\partial v}{\partial r} &= \frac{dv}{dx}\frac{\partial x}{\partial r} = 2r\frac{dv}{dx};\\ \frac{\partial v}{\partial s} &= \frac{dv}{dx}\frac{\partial x}{\partial s} = 2s\frac{dv}{dx};\end{aligned}\right\} s\frac{\partial u}{\partial r} - r\frac{\partial u}{\partial s} = s\left(s + \frac{\partial v}{\partial r}\right) - r\left(r + \frac{\partial v}{\partial s}\right)$$

$$= s^2 + 2rs\frac{dv}{dx} - r^2 - 2rs\frac{dv}{dx} = s^2 - r^2;$$

Sim, satisfaz;

(02) Sejam $u = f(x,y)$; $x(r,\theta) = r\cos\theta$; $y(r,\theta) = r\,\text{sen}\,\theta$; Mostre que:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2;$$

Solução: usaremos o quadro (01):

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\text{sen}\,\theta;$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\,\text{sen}\,\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = r\left[\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\text{sen}\,\theta\right];$$

Então:

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\text{sen}\,\theta\right)^2 + \frac{1}{r^2}\left[r^2\left(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\text{sen}\,\theta\right)^2\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \underline{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\,\text{sen}\,\theta} + \left(\frac{\partial u}{\partial y}\right)^2\text{sen}^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta \underline{- 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\,\text{sen}\,\theta} + \left(\frac{\partial u}{\partial x}\right)^2\text{sen}^2\theta$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta + \text{sen}^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\text{sen}^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2;$$

(03) [Equações de Cauchy-Riemann em coordenadas polares;]

Sejam $u(x,y)$ e $v(x,y)$ funções reais satisfazendo às equações de Cauchy (1789-1857) - Riemann (1826-1866). Mostre que se $x(r,\theta) = r\cos\theta$ e $y(r,\theta) = r\,\text{sen}\,\theta$, então: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ e $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$;

Solução:

Pelo que acabamos de ver na Questão (02) a Regra da Cadeia nos diz, que:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\,\text{sen}\,\theta;$$

$$\frac{\partial v}{\partial \theta} = r\left(\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\,\text{sen}\,\theta\right);$$

E Cauchy-Riemann nos diz, que: $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \end{cases}$

Logo, $\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\,\text{sen}\,\theta = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\,\text{sen}\,\theta = \frac{\partial u}{\partial r}$;

Bem como,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\,\text{sen}\,\theta;$$

$$\frac{\partial u}{\partial \theta} = r\left(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\,\text{sen}\,\theta\right);$$

Logo, $-\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\,\text{sen}\,\theta = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\,\text{sen}\,\theta = \frac{\partial v}{\partial r}$;

(04) [Fórmula de Leibniz;]

Sejam $w = F(u,v) = \int_v^u p(t)\,dt$; $u = g(x)$; $v = h(x)$; Mostre a seguinte fórmula devida à Leibniz (1646-1716): $\frac{dw}{dx} = p(g(x))g'(x) - p(h(x))h'(x)$;

Solução: Usaremos o quadro (07) e o Teorema Fundamental do Cálculo:

$$\frac{dw}{dx} = \frac{\partial w}{\partial u}\frac{du}{dx} + \frac{\partial w}{\partial v}\frac{dv}{dx} = \frac{\partial\left(\int_v^u p(t)dt\right)}{\partial u}.g'(x) + \frac{\partial\left(\int_v^u p(t)dt\right)}{\partial v}.h'(x)$$

$$= p(u).g'(x) + \frac{\partial\left(-\int_u^v p(t)dt\right)}{\partial v}.h'(x)$$

$$= p(g(x))g'(x) - p(v)h'(x)$$

$$= p(g(x))g'(x) - p(h(x))h'(x);$$

05 [Teorema de Euler para Funções homogêneas;]

Diz-se que $F(x,y)$ é homogênea de grau n, $n \geq 0$ um inteiro, quando $F(tx,ty) = t^n F(x,y)$. Mostre a seguinte igualdade, devida à Euler (1707-1783);

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF(x,y);$$

Solução: Vamos denotar $w = F(tx,ty) = t^n F(x,y)$;

- Então, de $w = t^n F(x,y)$ vemos que $\frac{\partial w}{\partial t} = n t^{n-1} F(x,y)$; (I)
- Agora para encontrarmos $\frac{\partial w}{\partial t}$ na expressão $w = F(tx,ty)$, introduzamos as variáveis $u(t,x) = tx$ e $v(t,y) = ty$. Então:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial w}{\partial u}x + \frac{\partial w}{\partial v}y; \text{ (II)}$$

Logo, concluímos de (I) e (II) que para todos t, x e y reais, temos:

$$x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} = n t^{n-1} F(x,y);$$

O que Euler, genialmente, percebeu é que, em particular, esta igualdade vale para $t=1$ e para todos x e y reais, e que nesse caso $u = x$ e $v = y$. Logo ela se torna: $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nF(x,y)$;

06 Em cada caso, mostre que se a equação:

ⓐ $F(x,y) = 0$, definir y como uma função diferenciável de x e $F_y \neq 0$, então $\frac{dy}{dx} = \frac{-F_x}{F_y}$;

Solução: Como y é função derivável de x, o quadro **07** nos informa:

De $F(x,y) = 0$, temos: $\frac{d\,F(x,y)}{dx} = 0 \therefore \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \therefore F_x + F_y\frac{dy}{dx} = 0 \therefore$

- E como $F_y \neq 0$, $\frac{dy}{dx} = \frac{-F_x}{F_y}$;

ⓑ $F(x,y,z) = 0$, definir z como uma função diferenciável de x e y e $F_z \neq 0$, então: $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$; e $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$;

Solução: como z é função diferenciável de x e y, o quadro **02** nos informa:

- De $F(x,y,z) = 0 \therefore \frac{\partial(F(x,y,z))}{\partial x} = 0 \therefore \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \therefore F_x + F_z\frac{\partial z}{\partial x} = 0$;

  E como, $F_z \neq 0$, então: $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$;

- De $F(x,y,z) = 0 \therefore \frac{\partial(F(x,y,z))}{\partial y} = 0 \therefore \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial(y)}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \therefore F_y + F_z\frac{\partial z}{\partial y} = 0$;

  E como, $F_z \neq 0$, temos: $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$;

ANEXO.4. UMA PROVA DA REGRA DA CADEIA

Vejamos a demonstração do quadro 01 da Regra da Cadeia. As demonstrações dos demais quadros seguem, exatamente, os mesmos argumentos.

REGRA DA CADEIA. QUADRO 01. Sejam $u=f(x,y)$ diferenciável e $x=g(r,s)$ e $y=h(r,s)$ tais que existam $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial r}$ e $\frac{\partial y}{\partial s}$. Então:

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}; \\[2ex] \dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}; \end{cases}$$

Prova: como $u=f(x,y)$ é diferenciável, temos:

$$f(x_0+\Delta x, y_0+\Delta y) - f(x,y) = \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \varepsilon_1(\Delta x,\Delta y)\Delta x + \varepsilon_2(\Delta x,\Delta y)\Delta y; \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_i(\Delta x,\Delta y)=0,\ i=1,2;$$

Usando a notação $u=f(x,y)$ e abstraindo (x_0,y_0) em $\frac{\partial f}{\partial x}(x_0,y_0)$ e $\frac{\partial f}{\partial y}(x_0,y_0)$, temos:

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1(\Delta x,\Delta y)\Delta x + \varepsilon_2(\Delta x,\Delta y)\Delta y; \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_i(\Delta x,\Delta y) = 0, \text{ para } i=1 \text{ e } 2;$$

Agora, para deduzirmos a 1ª equação do Quadro, dividamos os dois membros da igualdade acima, por Δr, obtendo: $\frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta r} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta r} + \varepsilon_1(\Delta x,\Delta y)\frac{\Delta x}{\Delta r} + \varepsilon_2(\Delta x,\Delta y)\frac{\Delta y}{\Delta r}$, com $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_i(\Delta x,\Delta y)=0$; para $i=1$ e 2;

Ocorre que: $\lim\limits_{\Delta r\to 0}\frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial r}$; $\lim\limits_{\Delta r\to 0}\frac{\Delta x}{\Delta r} = \frac{\partial x}{\partial r}$; e $\lim\limits_{\Delta r\to 0}\frac{\Delta y}{\Delta r} = \frac{\partial y}{\partial r}$;

E como existem $\frac{\partial x}{\partial r}$ e $\frac{\partial y}{\partial r}$, cada uma das funções $x=g(r,s)$ e $y=h(r,s)$, quando consideradas como função de uma única variável r, é contínua com relação à esta única variável. Logo: $\lim\limits_{\Delta r\to 0}\Delta x = \lim\limits_{\Delta r\to 0}[x(r+\Delta r,s) - x(r,s)] = 0$; bem como: $\lim\limits_{\Delta r\to 0}\Delta y = \lim\limits_{\Delta r\to 0}[y(r+\Delta r,s) - y(r,s)] = 0$; Ou seja quando $\Delta r\to 0$ temos: $(\Delta x,\Delta y)\to(0,0)$;

Este fato acarreta que: $\lim\limits_{\Delta r\to 0}\varepsilon_i(\Delta x,\Delta y) = \lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_i(\Delta x,\Delta y) = 0$, para $i=1$ e 2;

Então,

$$\frac{\partial u}{\partial r} = \lim_{\Delta r\to 0}\frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial x}\underbrace{\lim_{\Delta r\to 0}\frac{\Delta x}{\Delta r}}_{=\frac{\partial x}{\partial r}} + \frac{\partial u}{\partial y}\underbrace{\lim_{\Delta r\to 0}\frac{\Delta y}{\Delta r}}_{=\frac{\partial y}{\partial r}} + \underbrace{\lim_{\Delta r\to 0}\left[\varepsilon_1(\Delta x,\Delta y)\frac{\Delta x}{\Delta r}\right]}_{=0\cdot\frac{\partial x}{\partial r}=0} + \underbrace{\lim_{\Delta r\to 0}\left[\varepsilon_2(\Delta x,\Delta y)\frac{\Delta y}{\Delta r}\right]}_{=0\cdot\frac{\partial y}{\partial r}=0}$$

Portanto, $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$;

A segunda equação é obtida de forma exatamente igual, apenas dividindo os dois membros da igualdade:

$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1(\Delta x,\Delta y)\Delta x + \varepsilon_2(\Delta x,\Delta y)\Delta y$, $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_i(\Delta x,\Delta y)=0$, $i=1$ e 2, por Δs;