RESOLUÇÃO. LISTA. INTEGRAIS TRIPLAS.

01 Em cada caso, calcule a Integral Tripla Iterada dada:

a) $$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}} z\Big/_0^{\sqrt{9-x^2}} dy\,dx =$$

$$=\int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dy\,dx = \int_0^3 \sqrt{9-x^2}\int_0^{\sqrt{9-x^2}} dy\,dx =$$

$$=\int_0^3 \sqrt{9-x^2}\; y\Big/_0^{\sqrt{9-x^2}} dx = \int_0^3 (9-x^2)\,dx = \left(9x-\frac{x^3}{3}\right)\Big/_0^3 = (27-9) = 18;$$

b) $$\int_0^{\pi/2}\int_0^{y}\int_0^{x} \cos(x+y+z)\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^{y} \operatorname{sen}(x+y+z)\Big/_0^{x} dx\,dy =$$

$$=\int_0^{\pi/2}\int_0^{y}\left[\operatorname{sen}(2x+y)-\operatorname{sen}(x+y)\right]dx\,dy = \int_0^{\pi/2}\left[-\frac{1}{2}\cos(2x+y)+\cos(x+y)\right]\Big/_0^{y} dy =$$

$$=\int_0^{\pi/2}\left[\left(-\frac{1}{2}\cos 3y+\cos(2y)\right)-\left(-\frac{1}{2}\cos y+\cos y\right)\right]dy =$$

$$=\int_0^{\pi/2}\left(-\frac{1}{2}\cos 3y+\cos 2y-\frac{1}{2}\cos y\right)dy$$

$$=\left(-\frac{1}{6}\operatorname{sen}3y+\frac{1}{2}\operatorname{sen}2y-\frac{1}{2}\operatorname{sen}y\right)\Big/_0^{\pi/2} = \left(-\frac{1}{6}\right)(-1)+0-\frac{1}{2} = \frac{1}{6}-\frac{1}{2} = -\frac{1}{3};$$

(c) $\displaystyle\int_0^1\int_1^{\sqrt{e}}\int_1^{e} z e^z \ln y \frac{(\ln x)^2}{x}\,dx\,dy\,dz = \int_0^1 z e^z \int_1^{\sqrt{e}} \ln y \int_1^{e} (\ln x)^2 \frac{dx}{x}\,dy\,dz =$

$\displaystyle = \int_0^1 z e^z \int_1^{\sqrt{e}} \ln y \left.\frac{(\ln x)^3}{3}\right|_1^{e} dy\,dz = \int_0^1 z e^z \int_1^{\sqrt{e}} \ln y\left(\frac{(\ln e)^3}{3} - \frac{(\ln 1)^3}{3}\right) dy\,dz =$

$\displaystyle = \int_0^1 z e^z \int_1^{\sqrt{e}} \ln y \cdot \frac{1}{3}\,dy\,dz = \frac{1}{3}\int_0^1 z e^z \left.(y\ln y - y)\right|_1^{\sqrt{e}} dz =$

$\boxed{\begin{array}{l} u = \ln y \therefore du = \frac{dy}{y} \\ dv = dy \therefore v = y \end{array}}$

$\displaystyle = \frac{1}{3}\int_0^1 z e^z \left[(\sqrt{e}\ln\sqrt{e} - \sqrt{e}) - (1\ln 1 - 1)\right] dz = \frac{1}{3}\left(\sqrt{e}\cdot\frac{1}{2}\ln e - \sqrt{e} + 1\right)\int_0^1 z e^z\,dz =$

$\displaystyle = \frac{1}{3}\left[\frac{1}{2}\sqrt{e} - \sqrt{e} + 1\right]\left[\left. z e^z\right|_0^1 - \int_0^1 e^z\,dz\right] =$

$\boxed{\begin{array}{l} u = z \therefore du = dz \\ dv = e^z dz \therefore v = e^z \end{array}}$

$\displaystyle = \frac{1}{3}\left(1 - \frac{\sqrt{e}}{2}\right)\left[(1.e^1 - 0.e^0) - \left. e^z\right|_0^1\right] = \frac{1}{3}\left(1 - \frac{\sqrt{e}}{2}\right)\left[e - (e-1)\right] = \frac{1}{3}\left(1 - \frac{\sqrt{e}}{2}\right);$

(d) $\displaystyle\int_0^2\int_0^y\int_0^{\sqrt{3}z} \frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_0^2\int_0^y\int_0^{\sqrt{3}z} \frac{z}{z^2\left(\left(\frac{x}{z}\right)^2+1\right)}\,dx\,dz\,dy =$

$\displaystyle = \int_0^2\int_0^y\int_0^{\sqrt{3}z} \frac{1}{\left(1+\left(\frac{x}{z}\right)^2\right)}\cdot\frac{dx}{z}\,dz\,dy = \int_0^2\int_0^y \left.\operatorname{arctg}\left(\frac{x}{z}\right)\right|_0^{\sqrt{3}z} dz\,dy =$

$\displaystyle = \int_0^2\int_0^y \left(\operatorname{arctg}\left(\frac{\sqrt{3}z}{z}\right) - \underbrace{\operatorname{arctg} 0}_{=0}\right) dz\,dy = \operatorname{arctg}\sqrt{3}\int_0^2\int_0^y dz\,dy =$

$\displaystyle = \frac{\pi}{3}\int_0^2 \left. z\right|_0^y dy = \frac{\pi}{3}\int_0^2 y\,dy = \left.\frac{\pi}{6} y^2\right|_0^2 = \frac{4\pi}{6} = \frac{2\pi}{3};$

$$② \int_0^{\pi}\int_0^{1}\int_0^{\sqrt{1-z^2}} z\,\text{sen}x\,dy\,dz\,dx = \int_0^{\pi}\text{sen}x\int_0^{1} z\int_0^{\sqrt{1-z^2}} dy\,dz\,dx =$$

$$= \int_0^{\pi}\text{sen}x\int_0^{1} z\, y\Big/_0^{\sqrt{1-z^2}} dz\,dx = \int_0^{\pi}\text{sen}x\int_0^{1} z\sqrt{1-z^2}\,\frac{(-2dz)}{(-2)}\,dx =$$

$$= -\frac{1}{2}\int_0^{\pi}\text{sen}x\cdot\frac{2}{3}(1-z^2)^{3/2}\Big/_0^{1} dx = -\frac{1}{3}\int_0^{\pi}\text{sen}x\,(0-1)\,dx =$$

$$= \frac{1}{3}(-\cos x)\Big/_0^{\pi} = -\frac{1}{3}(\cos\pi-\cos 0) = -\frac{1}{3}(-1-1) = \frac{2}{3};$$

$$⑦ \int_{\pi/4}^{\pi/2}\int_{z}^{\pi/2}\int_0^{xz}\cos\left(\frac{y}{z}\right)dy\,dx\,dz = \int_{\pi/4}^{\pi/2}\int_{z}^{\pi/2}\int_0^{xz} z\cos\left(\frac{y}{z}\right)\frac{dy}{z}\,dx\,dz =$$

$$= \int_{\pi/4}^{\pi/2} z\int_{z}^{\pi/2}\text{sen}\left(\frac{y}{z}\right)\Big/_0^{xz} dx\,dz = \int_{\pi/4}^{\pi/2} z\int_{z}\left[\text{sen}\left(\frac{xz}{z}\right)-\text{sen}\,0\right]dx\,dz =$$

$$= \int_{\pi/4}^{\pi/2} z\int_{z}^{\pi/2}\text{sen}x\,dx\,dz = \int_{\pi/4} z\,(-\cos\cancel{x})\Big/_{z}^{\pi/2} dz = -\int_{\pi/4}^{\pi/2} z\left(\cos\frac{\pi}{2}-\cos z\right)dz =$$

$$= \int_{\pi/4}^{\pi/2} z\cos z\,dz = z\,\text{sen}z\Big/_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}\text{sen}z\,dz =$$

$$\boxed{\begin{array}{l} u = z \therefore du = dz \\ V = \cos z\,dz \therefore V = \text{sen}z \end{array}}$$

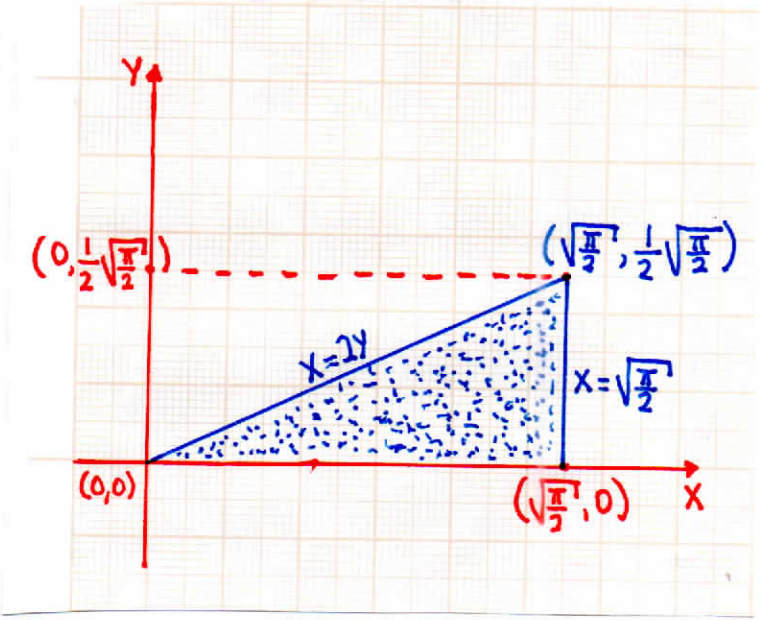
$$= \left[\frac{\pi}{2}\text{sen}\frac{\pi}{2} - \frac{\pi}{4}\text{sen}\frac{\pi}{4}\right] + \cos z\Big/_{\pi/4}^{\pi/2} =$$

$$= \frac{\pi}{2} - \frac{\pi\sqrt{2}}{8} + \left(\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right) =$$

$$= \frac{\pi}{2} - \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{2}}{2} = \frac{(4-\sqrt{2})\pi - 4\sqrt{2}}{8};$$

02) Em cada caso, use o Teorema de Fubini a fim de possibilitar o cálculo, por meio de funções elementares, da Integral Tripla Iterada dada:

a) $\displaystyle\int_1^4\int_0^{\frac{1}{2}\sqrt{\frac{\pi}{2}}}\int_{2y}^{\sqrt{\frac{\pi}{2}}}\frac{\cos(x^2)}{\sqrt{z}}\,dx\,dy\,dz;$



Solução: como a integral

$\int\cos(x^2)dx$ não tem primitiva,

por meio de funções elementares, vamos inverter a ordem dxdy para dydx; ficaremos com:

$$\int_1^4\int_0^{\frac{1}{2}\sqrt{\frac{\pi}{2}}}\int_{2y}^{\sqrt{\frac{\pi}{2}}}\frac{\cos(x^2)}{\sqrt{z}}\,dx\,dy\,dz=\int_1^4\frac{1}{\sqrt{z}}\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{x/2}\cos(x^2)\,dy\,dx\,dz=$$

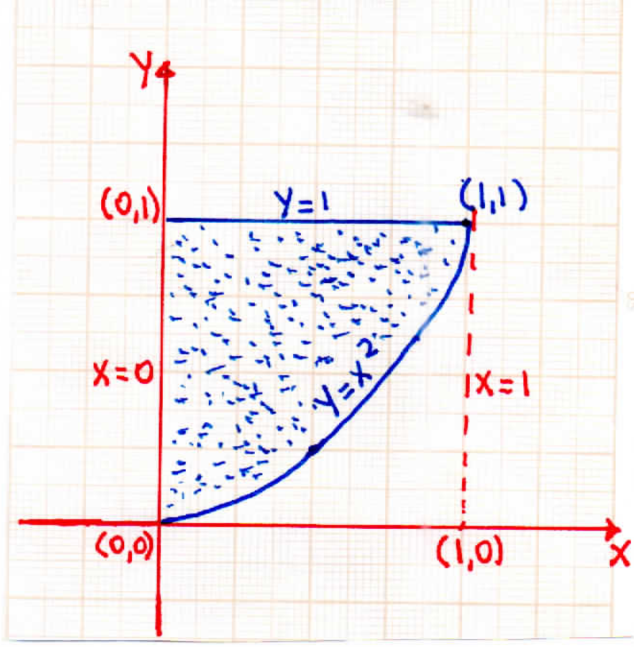
$$=\int_1^4\frac{1}{\sqrt{z}}\int_0^{\sqrt{\frac{\pi}{2}}}\cos(x^2)\int_0^{x/2}dy\,dx\,dz=\int_1^4\frac{1}{\sqrt{z}}\int_0^{\sqrt{\frac{\pi}{2}}}\cos(x^2)\,y\Big|_0^{x/2}dx\,dz=$$

$$=\int_1^4\frac{1}{\sqrt{z}}\int_0^{\sqrt{\frac{\pi}{2}}}\cos(x^2).\frac{x}{2}\,dx\,dz=\frac{1}{4}\int_1^4\frac{1}{\sqrt{z}}\int_0^{\sqrt{\frac{\pi}{2}}}\cos(x^2)(2x\,dx)\,dz=$$

$$=\frac{1}{4}\int_1^4\frac{1}{\sqrt{z}}\,\mathrm{sen}(x^2)\Big|_0^{\sqrt{\frac{\pi}{2}}}dz=\frac{1}{4}\int_1^4\frac{1}{\sqrt{z}}\left(\mathrm{sen}\,\frac{\pi}{2}-\mathrm{sen}\,0\right)dz=$$

$$=\frac{1}{4}\int_1^4 z^{-1/2}dz=\frac{1}{4}.2z^{1/2}\Big|_1^4=\frac{1}{2}\left(4^{1/2}-1^{1/2}\right)=\frac{1}{2};$$

ⓑ $\int_0^1\int_0^1\int_{x^2}^1 12xz e^{zy^2}\,dy\,dx\,dz;$



Solução: como a integral $\int e^{zy^2}dy$ não tem primitiva, por meio de Funções elementares, vamos inverter a ordem dydx para dxdy; Ficaremos com:

$$\int_0^1\int_0^1\int_{x^2}^1 12xz e^{zy^2}\,dy\,dx\,dz = 12\int_0^1\int_0^1\int_0^{\sqrt{y}} xz e^{zy^2}\,dx\,dy\,dz$$

$$= 12\int_0^1\int_0^1 z e^{zy^2}\int_0^{\sqrt{y}} x\,dx\,dy\,dz = 12\int_0^1\int_0^1 z e^{zy^2}\cdot\frac{x^2}{2}\Big|_0^{\sqrt{y}}dy\,dz =$$

$$= 6\int_0^1\int_0^1 z e^{zy^2}((\sqrt{y})^2-0^2)\,dy\,dz = 3\int_0^1\int_0^1 e^{zy^2}(2yz\,dy)\,dz =$$

$$= 3\int_0^1 e^{zy^2}\Big|_0^1 dz = 3\int_0^1 (e^z-1)\,dz = 3(e^z-z)\Big|_0^1 = 3[(e-1)-(1-0)] = 3(e-2);$$

03 Em cada caso, indique a região $\mathcal{E}$ do espaço-xyz que:

ⓐ minimize o resultado da integral $\iiint_{\mathcal{E}}(4x^2+4y^2+z^2-4)\,dV;$

Solução:

A região $\mathcal{E}$ será aquela constituída por todos os pontos $(x,y,z)\in R^3$ tais que: $4x^2+4y^2+z^2-4\le 0 \therefore 4x^2+4y^2+z^2\le 4;$

Ou seja: $x^2+y^2+\frac{z^2}{4}\le 1;$

Portanto, $\mathcal{E}$ é a região de R^3 delimitada pelo elipsóide:

$$x^2+y^2+\frac{z^2}{4}=1;$$

(b) maximize o resultado da integral $\iiint_{E}(1-x^2-2y^2-3z^2)\,dV$;

<u>Solução</u>: A região E será aquela constituída por todos os pontos $(x,y,z)\in\mathbb{R}^3$ tais que: $1-x^2-2y^2-3z^2\geq 0 \therefore x^2+2y^2+3z^2\leq 1$;

Portanto, E é a região delimitada pelo elipsóide:

$$x^2+\frac{y^2}{(1/2)}+\frac{z^2}{(1/3)}=1;$$

(04) Encontre os dois valores de a, tais que:

$$\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y} dz\,dy\,dx=\frac{4}{15};$$

<u>Solução</u>: vamos calcular a integral em função da incógnita a, e depois descobrir os valores de a:

$$\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y} dz\,dy\,dx=\int_0^1\int_0^{4-a-x^2} z\Big/_a^{4-x^2-y}\,dy\,dx=$$

$$=\int_0^1\int_0^{4-a-x^2}(4-x^2-y-a)\,dy\,dx=\int_0^1\int_0^{4-a-x^2}((4-a-x^2)-y)\,dy\,dx=$$

$$=\int_0^1\left[(4-a-x^2)y-\frac{1}{2}y^2\right]\Big/_0^{4-a-x^2}dx=\int_0^1\left[(4-a-x^2)^2-\frac{1}{2}(4-a-x^2)^2\right]dx$$

$$=\frac{1}{2}\int_0^1\left[(4-a)^2-2(4-a)x^2+x^4\right]dx=\frac{1}{2}\left[(4-a)^2x-\frac{2}{3}(4-a)x^3+\frac{x^5}{5}\right]\Big/_0^1=$$

$$=\frac{1}{2}\left[(4-a)^2-\frac{2}{3}(4-a)+\frac{1}{5}\right]=\frac{1}{2}\left[\frac{15(4-a)^2-10(4-a)+3}{15}\right];$$

E igualando este resultado à $\frac{4}{15}$, obtemos

$$15(16-8a+a^2)-40+10a+3=8 \therefore 240-120a+15a^2-40+10a-5=0 \therefore$$

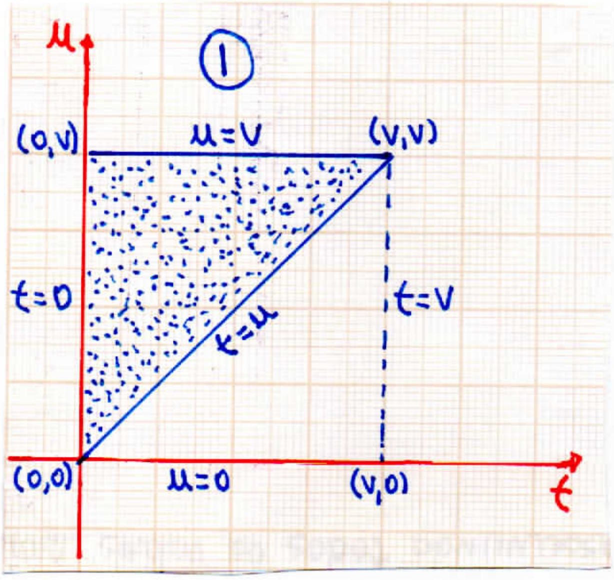
$$15a^2-110a+195=0 \therefore 3a^2-22a+39=0 \therefore a=\frac{22\pm\sqrt{484-468}}{6}\therefore$$

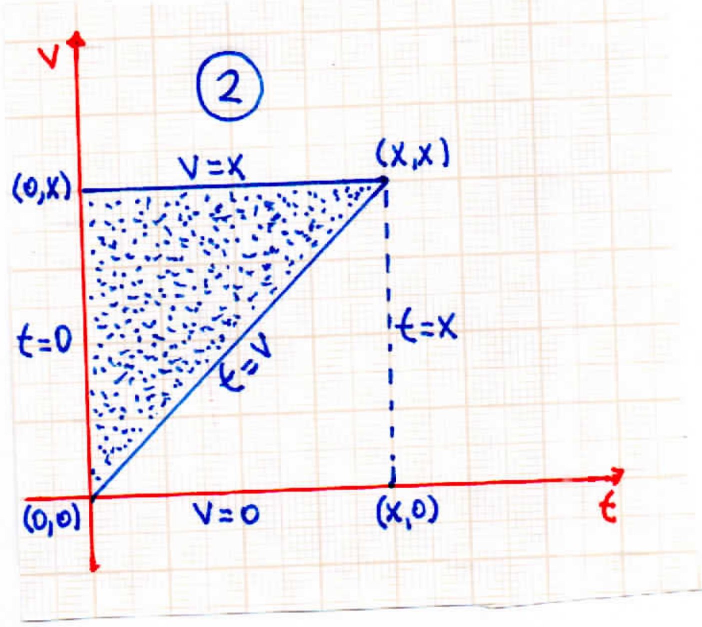
$$a=\frac{22\pm\sqrt{16}}{6}\therefore\begin{cases}a=\frac{22+4}{6}=\frac{13}{3};\\ a=\frac{22-4}{6}=3;\end{cases}$$

05 Sabendo que as hipóteses do Teorema de Fubini (1879-1943), estão satisfeitas use-o para mostrar que:

$$\int_0^x\int_0^v\int_0^u e^{m(x-t)}.F(t)\,dt\,du\,dv = \int_0^x \frac{(x-t)^2}{2}.e^{m(x-t)}.F(t)\,dt;$$

Solução:





$$\int_0^x\left[\int_0^v\int_0^u e^{m(x-t)}.F(t)\,dt\,du\right]dv \overset{1}{=} \int_0^x\left[\int_0^v\int_t^v e^{m(x-t)}.F(t)\,du\,dt\right]dv =$$

$$= \int_0^x\int_0^v e^{m(x-t)}.F(t)\,u\Big/_t^v\,dv = \int_0^x\int_0^v e^{m(x-t)}.F(t)(v-t)\,dt\,dv \overset{2}{=}$$

$$\overset{2}{=} \int_0^x\int_t^x e^{m(x-t)}.F(t)(v-t)\,dv\,dt = \int_0^x e^{m(x-t)}.F(t)\int_t^x (v-t)\,dv\,dt =$$

$$= \int_0^x e^{m(x-t)}.F(t).\frac{1}{2}(v-t)^2\Big/_t^x\,dt = \int_0^x e^{m(x-t)}.F(t).\frac{1}{2}\left[(x-t)^2-(t-t)^2\right]dt$$

$$= \int_0^x \frac{(x-t)^2}{2}.e^{m(x-t)}.F(t)\,dt;$$