Se $X \sim Beta(a,b)$

$$\int_0^1 \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}\,dx = \frac{1}{B(a,b)} \underbrace{\int_0^1 x^{a-1}(1-x)^{b-1}\,dx}$$

$$= \frac{B(a,b)}{B(a,b)} = 1 \quad //$$

Se $a=b=1 \Rightarrow X \sim B(1,1) \equiv \mathcal{U}(0,1)$

$$\frac{1}{B(1,1)} x^0 (1-x)^0 \mathbb{1}_{(0,1)}(x) = \frac{\cancel{\Gamma(2)}^{1}}{\cancel{\Gamma(1)\Gamma(1)}} \mathbb{1}_{(0,1)}(x) = \mathbb{1}_{(0,1)}(x)$$

Se $X \sim \mathcal{U}(a,b)$ se $a=0$ e $b=1$

$$f(x) = \frac{1}{b-a} \mathbb{1}_{(a,b)}(x) = \mathbb{1}_{(0,1)}(x)$$

$$\mathbb{E}(X) = \frac{1}{B(a,b)} \int_0^1 x^{a} (1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)} \underbrace{\int_0^1 x^{(a+1)-1}(1-x)^{b-1}\,dx}$$

$$= \frac{B(a+1,b)}{B(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}$$

$$= \frac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}\cancel{\Gamma(b)}} \frac{a\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{(a+b)\cancel{\Gamma(a+b)}} = \frac{a}{a+b} \quad //$$

$$\mathbb{E}(X^k) = \frac{1}{B(a,b)} \int_0^1 x^k x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)} \int_0^1 x^{(a+k)-1}(1-x)^{b-1}\,dx$$

$$= \frac{B(a+k,b)}{B(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\cancel{\Gamma(b)}} \frac{\Gamma(a+k)\cancel{\Gamma(b)}}{\Gamma(a+b+k)}$$

$$= \frac{\Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}$$

$$\mathbb{E}(X^2) = \frac{\Gamma(a+b)\Gamma(a+2)}{\Gamma(a)\Gamma(a+b+2)} = \frac{\Gamma(a+b)(a+1)\Gamma(a+1)}{\Gamma(a)(a+b+1)\Gamma(a+b+1)}$$

$$= \frac{\cancel{\Gamma(a+b)}(a+1)a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}(a+b+1)(a+b)\cancel{\Gamma(a+b)}} = \frac{a(a+1)}{(a+b)(a+b+1)} \quad /$$

Logo,

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2} = \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)}$$

$$= \frac{\cancel{a^3} + \cancel{a^2} + \cancel{a^2 b} + ab - \cancel{a^3} - \cancel{a^2 b} - \cancel{a^2}}{(a+b)^2(a+b+1)}$$

$$= \frac{ab}{(a+b)^2(a+b+1)} \quad //$$