· MUDANÇA DE VARIÁVEIS · COORDENADAS: POLARES; CILÍNDRICAS; E ESFÉRICAS ·

(I) TEOREMA DE JACOBI (1804-1851) · [MUDANÇA DE VARIÁVEIS EM INTEGRAIS DUPLAS] ·

Seja $x(u,v)$ e $y(u,v)$ uma mudança de variáveis, inversível, com derivadas parciais de 1ª ordem $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$ e $\frac{\partial y}{\partial v}$, contínuas, a qual transforme uma região S, no plano-uv, numa região R, no plano-xy; então: $\iint_R f(x,y)\,dy\,dx = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$, onde $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$ e não muda de sinal em S;

(II) APLICAÇÃO · MUDANÇA DE COORDENADAS CARTESIANAS PARA COORDENADAS POLARES ·

Temos: $x(r,\theta) = r\cos\theta$ e $y(r,\theta) = r\,\text{sen}\,\theta$; Logo: $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\,\text{sen}^2\theta = r$;

Logo: $\iint_R f(x,y)\,dy\,dx = \iint_S f(x(r,\theta),y(r,\theta))\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \iint_S f(r\cos\theta, r\,\text{sen}\,\theta)\,|r|\,dr\,d\theta$;

EXEMPLOS. Em cada caso, faça a mudança de coordenadas cartesianas para polares e calcule a integral:

ⓐ $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dy\,dx = \int_0^{2\pi}\int_0^{a} e^{-(r\cos\theta)^2-(r\,\text{sen}\,\theta)^2}|r|\,dr\,d\theta = \int_0^{2\pi}\int_0^{a} e^{-r^2}\,r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}\int_0^{a} e^{-r^2}(-2r\,dr)\,d\theta =$
$(a>0)$

$= -\frac{1}{2}\int_0^{2\pi} e^{-r^2}\Big|_0^{a} d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-e^{0})\,d\theta = \frac{(1-e^{-a^2})}{2}\int_0^{2\pi} d\theta = \frac{(1-e^{-a^2})}{2}\,\theta\Big|_0^{2\pi} = (1-e^{-a^2})\pi$;

ⓑ $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} dy\,dx = \int_0^{\pi}\int_0^{3} |r|\,dr\,d\theta = \int_0^{\pi}\int_0^{3} r\,dr\,d\theta = \int_0^{\pi} \frac{1}{2}r^2\Big|_0^{3} d\theta = \int_0^{\pi}\frac{1}{2}(3^2-0^2)\,d\theta = \frac{9}{2}\int_0^{\pi} d\theta = \frac{9}{2}\,\theta\Big|_0^{\pi} = \frac{9\pi}{2}$;

Quando devemos mudar de coordenadas cartesianas para coordenadas polares? A resposta a esta pergunta é: (i) quando a região de integração for o interior, ou parte do interior, de uma circunferência; ou (ii) quando a expressão "x^2+y^2" fizer parte da definição de $f(x,y)$;

Vejamos mais um exemplo:

$$\text{(c)}\ \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/4}\int_0^2 \frac{1}{1+r}\cdot|r|\,dr\,d\theta = \int_0^{\pi/4}\int_0^2 \frac{r}{1+r}\,dr\,d\theta = \int_0^{\pi/4}\int_0^2 \left[1-\frac{1}{1+r}\right]dr\,d\theta =$$

$$= \int_0^{\pi/4} (r-\ln(1+r))\Big|_0^2\,d\theta = \int_0^{\pi/4}\left[(2-\ln 3)-(0-\ln 1)\right]d\theta = (2-\ln 3)\int_0^{\pi/4} d\theta =$$

$$= (2-\ln 3)\,\theta\Big|_0^{\pi/4} = (2-\ln 3)\left(\frac{\pi}{4}-0\right) = \frac{(2-\ln 3)\pi}{4};$$

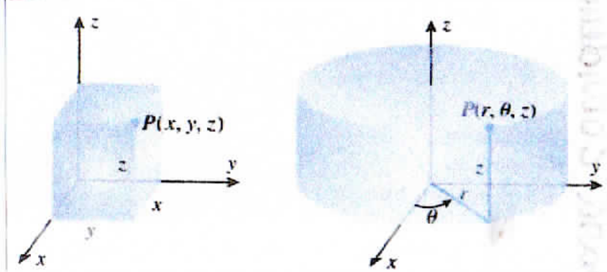
(III) TEOREMA DE JACOBI. [MUDANÇA DE VARIÁVEIS EM INTEGRAIS TRIPLAS].

Seja $x(u,v,w)$; $y(u,v,w)$; e $z(u,v,w)$; uma mudança de variáveis, invertível, com derivadas parciais de 1ª ordem $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ e $\frac{\partial z}{\partial w}$, contínuas, a qual transforme uma região G, no espaço-uvw, numa região S no espaço-xyz, então:

$$\iiint_S f(x,y,z)\,dz\,dy\,dx = \iiint_G f(x(u,v,w),y(u,v,w),z(u,v,w))\cdot\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du,$$ onde: $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$ e não muda de sinal em G;

(IV) APLICAÇÃO. MUDANÇA DE COORDENADAS CARTESIANAS PARA COORDENADAS CILÍNDRICAS.

A Figura ao lado, esclarece bem as relações entre as coordenadas cartesianas x, y e z e as coordenadas cilíndricas r, θ e z; no plano-xy as relações entre x e y e r e θ são as mesmas das coordenadas polares, ou seja:



$x=r\cos\theta$ e $y=r\,\text{sen}\,\theta$; e como $z=z$, em coordenadas cilíndricas, temos:

$x(r,\theta,z)=r\cos\theta$; $y(r,\theta,z)=r\,\text{sen}\,\theta$; e $z(r,\theta,z)=z$;

Logo: $\dfrac{\partial(x,y,r)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\,\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\,\text{sen}^2\theta = r;$

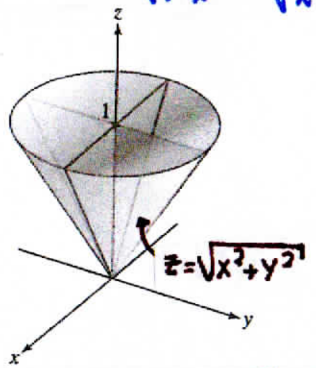
Então, $\displaystyle\iiint_S F(x,y,z)\,dz\,dy\,dx = \iiint_G F(r\cos\theta, r\,\text{sen}\,\theta, z)\,|r|\,dz\,dr\,d\theta;$

Devemos mudar de coordenadas cartesianas x, y e z para as coordenadas cilíndricas r, θ e z, quando a região de integração S for um interior, ou parte, do interior de um cilindro, ou, quando a expre-ssão "x^2+y^2" fizer parte da definição de $F(x,y,z)$; vejamos exemplos:

d) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)^{3/2}\,dz\,dy\,dx = \int_0^{2\pi}\int_0^1\int_r^1 (r^2)^{3/2}\cdot|r|\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_r^1 r^3\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^4\int_r^1 dz\,dr\,d\theta =$

$\displaystyle= \int_0^{2\pi}\int_0^1 r^4 z\Big|_r^1 dr\,d\theta = \int_0^{2\pi}\int_0^1 r^4(1-r)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^4-r^5)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^5}{5}-\frac{r^6}{6}\right)\Big|_0^1 d\theta =$

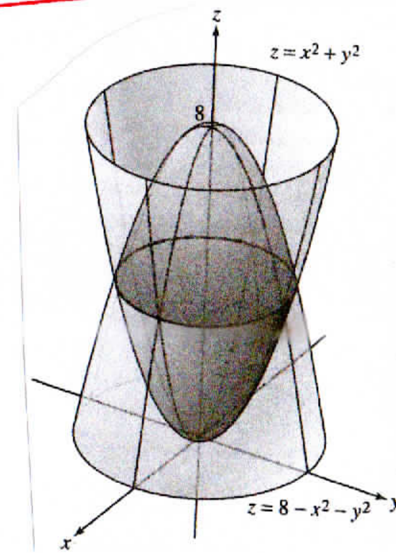
$\displaystyle= \int_0^{2\pi}\left[\left(\frac{1}{5}-\frac{1}{6}\right)-(0-0)\right]d\theta = \frac{(6-5)}{30}\int_0^{2\pi} d\theta = \frac{1}{30}\,\theta\Big|_0^{2\pi} = \frac{1}{30}(2\pi-0) = \frac{\pi}{15};$



e) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} |r|\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\int_{r^2}^{8-r^2} dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\,z\Big|_{r^2}^{8-r^2} dr\,d\theta =$

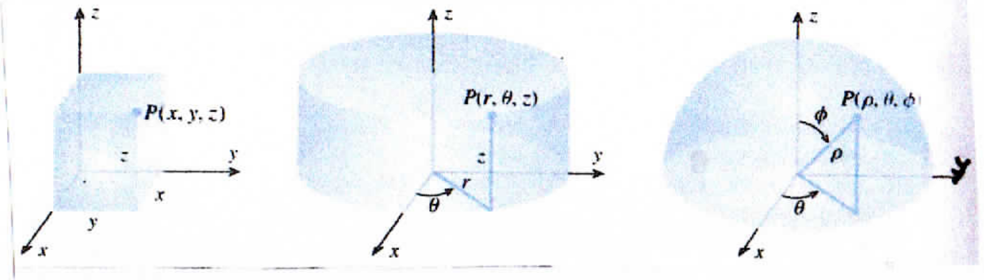
$\displaystyle= \int_0^{2\pi}\int_0^2 r(8-r^2-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (8r-2r^3)\,dr\,d\theta = \int_0^{2\pi}\left(4r^2-\frac{r^4}{2}\right)\Big|_0^2 d\theta =$

$\displaystyle= \int_0^{2\pi}[(16-8)-(0-0)]\,d\theta = 8\int_0^{2\pi} d\theta = 8\,\theta\Big|_0^{2\pi} = 8(2\pi-0) = 16\pi;$

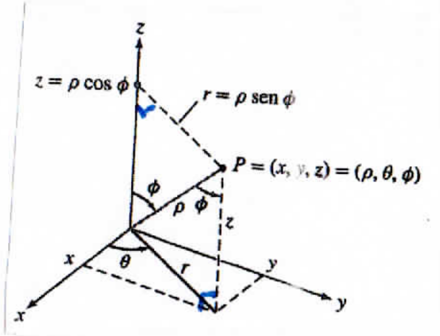


(V) APLICAÇÃO. MUDANÇA DE COORDENADAS CARTESIANAS PARA COORDENADAS ESFÉRICAS.

As Figuras ao lado, mostram um mesmo ponto no espaço, representado em: coordenadas cartesianas x, y e z; coordenadas cilíndricas r, θ e z; e coordenadas esféricas ρ, θ e ϕ;



Já a Figura ao lado, esclarece bem as relações entre as coordenadas cartesianas x, y e z e as coordenadas esféricas ρ, θ e ϕ; ρ é a distância do ponto (x,y,z) à origem $(0,0,0)$; ou seja: $\rho = \sqrt{x^2+y^2+z^2}$; já θ continua sendo o mesmo ângulo, no plano-xy, tanto das coordenadas polares, quanto das cilíndricas; $0 \leq \theta \leq 2\pi$; a novidade é o ângulo ϕ; que é medido partindo do eixo-z positivo, ou seja $\phi = 0$, e vai até o eixo-z negativo, quando $\phi = \pi$; portanto: $0 \leq \phi \leq \pi$; se por P tiramos uma perpendicular ao eixo-z, formamos um triângulo retângulo, com hipotenusa ρ e catetos: $z = \rho\cos\phi$ e $r = \rho\,\text{sen}\,\phi$; e este r é o mesmo r de coordenadas polares; portanto: $x = r\cos\theta = \rho\,\text{sen}\,\phi\cos\theta$ e $y = r\,\text{sen}\,\theta = \rho\,\text{sen}\,\phi\,\text{sen}\,\theta$; temos portanto:



$x(\rho,\theta,\phi) = \rho\,\text{sen}\,\phi\cos\theta$; $y(\rho,\theta,\phi) = \rho\,\text{sen}\,\phi\,\text{sen}\,\theta$; e $z(\rho,\theta,\phi) = \rho\cos\phi$; Encontremos, então, o Jacobiano:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \text{sen}\,\phi\cos\theta & -\rho\,\text{sen}\,\phi\,\text{sen}\,\theta & \rho\cos\phi\cos\theta \\ \text{sen}\,\phi\,\text{sen}\,\theta & \rho\,\text{sen}\,\phi\cos\theta & \rho\cos\phi\,\text{sen}\,\theta \\ \cos\phi & 0 & -\rho\,\text{sen}\,\phi \end{vmatrix}$$

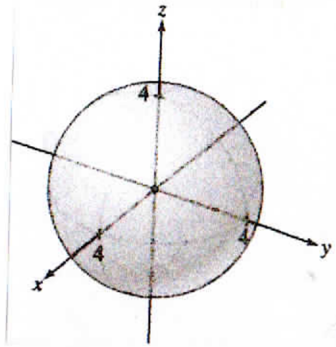
; Ou seja:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \cos\phi\left[(-\rho^2)\,\text{sen}\,\phi\cos\phi(\text{sen}^2\theta+\cos^2\theta)\right] + (-\rho\,\text{sen}\,\phi)\left[\rho\,\text{sen}^2\phi\,(\cos^2\theta+\text{sen}^2\theta)\right] = -\rho^2\,\text{sen}\,\phi(\cos^2\phi+\text{sen}^2\phi) = -\rho^2\,\text{sen}\,\phi;$$

Logo, $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = |-\rho^2\,\text{sen}\,\phi| = \rho^2\,\text{sen}\,\phi$;

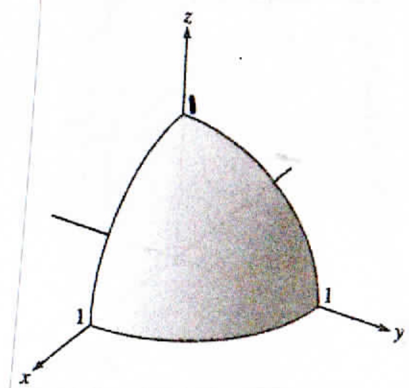
Então, $\iiint\limits_S F(x,y,z)\,dz\,dy\,dx = \iiint\limits_G F(\rho\,\text{sen}\,\phi\cos\theta,\ \rho\,\text{sen}\,\phi\,\text{sen}\,\theta,\ \rho\cos\phi)\,\rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta;$

Vejamos exemplos:



(f) $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} dz\,dy\,dx = \int_0^{2\pi}\int_0^{\pi}\int_0^{4} \rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\text{sen}\,\phi\int_0^{4}\rho^2\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\text{sen}\,\phi\,\frac{\rho^3}{3}\Big|_0^4\,d\phi\,d\theta =$

$= \frac{64}{3}\int_0^{2\pi}\int_0^{\pi}\text{sen}\,\phi\,d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi}\,d\theta = \frac{64}{3}(1-(-1))\int_0^{2\pi} d\theta = \frac{128}{3}\theta\Big|_0^{2\pi} = \frac{256\pi}{3};$



(g) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{(1+x^2+y^2+z^2)}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1}{1+\rho^2}\cdot\rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\text{sen}\,\phi\int_0^1\frac{\rho^2}{1+\rho^2}\,d\rho\,d\phi\,d\theta =$

$= \int_0^{\pi/2}\int_0^{\pi/2}\text{sen}\,\phi\int_0^1\left(1-\frac{1}{1+\rho^2}\right)d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\text{sen}\,\phi\,(\rho-\text{arctg}\,\rho)\Big|_0^1\,d\phi\,d\theta =$

$= \int_0^{\pi/2}\int_0^{\pi/2}\text{sen}\,\phi\,[(1-\text{arctg}\,1)-(0-\text{arctg}\,0)]\,d\phi\,d\theta =$

$= \int_0^{\pi/2}\int_0^{\pi/2}\text{sen}\,\phi\left(1-\frac{\pi}{4}\right)d\phi\,d\theta = \left(1-\frac{\pi}{4}\right)\int_0^{\pi/2} -\cos\phi\Big|_0^{\pi/2}\,d\theta = \left(1-\frac{\pi}{4}\right)(0-(-1))\int_0^{\pi/2} d\theta = \left(1-\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{2}-\frac{\pi^2}{8};$

(h) $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dx\,dy = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho\,\rho^2\,\text{sen}\,\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2}\text{sen}\,\phi\int_0^2\rho^3\,d\rho\,d\phi\,d\theta =$

$= \int_0^{2\pi}\int_0^{\pi/2}\text{sen}\,\phi\,\frac{\rho^4}{4}\Big|_0^2\,d\phi\,d\theta = 4\int_0^{2\pi}(-\cos\phi)\Big|_0^{\pi/2}\,d\theta = 4(0-(-1))\,\theta\Big|_0^{2\pi} = 8\pi;$