

# Examinable Definitions in SF2745 Advanced Complex Analysis

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**Definition 1.** Identify  $\mathbb{C}$  with the plane  $\{(x, y, 0) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3$ . The **stereographic projection** of  $z = x + iy$  is the unique point on the unit sphere intersecting the line defined by the two points  $(x, y, 0)$  and  $(0, 0, 1)$ . Let  $\pi : \mathbb{C} \rightarrow \mathbb{S}^2$  be the function which maps  $z \in \mathbb{C}$  to its stereographic projection  $z^* \in \mathbb{S}^2$ . We then have

$$\pi(x + iy) = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

**Definition 2.** The formal power series  $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$  is called a **convergent power series centered at**  $z_0$  if there is an  $r > 0$  such that the series converges for all  $z$  such that  $|z - z_0| < r$ . The largest possible such  $r$  is called the **radius of convergence** of the series.

**Definition 3.** A function  $f$  is **analytic at**  $z_0$  if  $f$  has a power series expansion valid in a neighbourhood of  $z_0$ .

**Definition 4.** A **region** is a connected open set.

**Definition 5.** The **mean-value property** for analytic functions  $f$  states that, if  $f$  is analytic at  $z_0$  and the radius of convergence is  $r_0 > 0$ , then  $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$  for all  $r < r_0$ .

**Definition 6.** We say that  $f$  is **locally conformal** if it preserves angles (including direction) between curves.

**Definition 7.** Let  $\zeta \in \partial\mathbb{D}$ , if  $\sum_{n \geq 0} a_n \zeta^n$  converges, and if  $\Gamma$  is any Stoltz angle at  $\zeta$ , then we have **non-tangential convergence** if

$$\lim_{z \in \Gamma \rightarrow \zeta} \sum_{n \geq 0} a_n z^n = \sum_{n \geq 0} a_n \zeta^n.$$

**Definition 8** (Curve integral). If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a piecewise continuously differentialbe curve, and if  $f$  is a complex valued function defined on (the image of)  $\gamma$ , then

$$\int_{\gamma} f(z) dz \equiv \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (1)$$

**Definition 9.** If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a piecewise continuously differentialbe curve, then the **length** of  $\gamma$  is defined to be  $\ell(\gamma) = |\gamma| \int_{\gamma} |dz| = \int_a^b |\gamma'(t)| dt$ .

**Definition 10.** A complex-valued function is said to be **holomorphic** on an open set  $U$  if

$$f'(z) = \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$$

exists for all  $z \in U$  and is continuous on  $U$ . A function  $f$  is said to be holomorphic on a set  $S$  if it is holomorphic on an open set  $U \supset S$ .

**Definition 11.** If  $\gamma$  is a cycle, then the **index** or **winding number** of  $\gamma$  about  $a$  is

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - a},$$

for  $a \notin \gamma$ .

**Definition 12.** Closed curves  $\gamma_1$  and  $\gamma_2$  are **homologous** in a region  $\Omega$  if  $n(\gamma_1 - \gamma_2, a) = 0$  for all  $a \notin \Omega$ , and we then write  $\gamma_1 \sim \gamma_2$ .

**Definition 13** (Classification of singularities). If  $f$  has an isolated singularity at  $b$ , and  $f(z) = \sum_{n \in \mathbb{Z}} a_n(z - b)^n$  sufficiently close to  $b$ , then we say that

- (a)  $b$  is a **removeable singularity** if  $a_n = 0$  for  $n < 0$ ,
- (b)  $b$  is a **zero of order**  $n_0 > 0$  if for  $n < n_0$  we have  $a_n = 0$ ,
- (c)  $b$  is a **pole of order**  $n_0 > 0$  if for  $n < -n_0$   $a_n = 0$ , and
- (d)  $b$  is an essential singularity if for any  $n_0 > 0$  there is  $n < -n_0$  such that  $a_n \neq 0$ .

**Definition 14.** If  $f$  is analytic in a region  $\Omega$ , except for at isolated poles in  $\Omega$ , we say that  $f$  is meromorphic in  $\Omega$ .

**Definition 15.** A **linear fractional transformation, LFT**, is a map of the form

$$T(z) = \frac{az + b}{cz + d}.$$

**Definition 16.** The **Cayley transform** is the map  $C(z) = \frac{z-i}{z+i}$ , and maps the upper half plane to the unit disc.

**Definition 17.** The **principal branch** of the logarithm is the one which takes arguments in  $(-\pi, \pi)$ . We may also take  $\log(-1) = \pi i$ .

**Definition 18.** The **Joukovski map** is the function  $w(z) = \frac{1}{2} \left( z + \frac{1}{z} \right)$ .

**Definition 19.** A function  $u : \Omega \rightarrow \mathbb{R}$  is called **harmonic** on the region  $\Omega \subset \mathbb{C}$  if, for each  $z \in \Omega$ , there is  $r_z > 0$  such that, for all  $r < r_z$

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt.$$

That is,  $u$  satisfies the **mean value property**.

**Definition 20.** A function  $u : \Omega \rightarrow [-\infty, \infty)$  is called **subharmonic** on the region  $\Omega \subset \mathbb{C}$  if, for each  $z \in \Omega$ , there is  $r_z > 0$  such that, for all  $r < r_z$

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt.$$

That is,  $u$  satisfies the **mean value inequality**.

**Definition 21.** The **Poisson kernel** is given by

$$P_z(t) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2},$$

and  $u = PI(g) \equiv \int P_z(t)g(e^{it})dt$  is the **Poisson integral** of  $g$ .

**Definition 22.** The kernel given by  $f_z(t) = \frac{e^{it} + z}{e^{it} - z}$  is called the **Herglotz kernel**, the **Herglotz integral** is the function  $\frac{1}{2\pi} \int f_z(t)u(e^{it})dt$ . When  $u$  is harmonic on  $\mathbb{D}$ , then the Herglotz integral is the unique analytic function with real part  $u$  and imaginary part 0 at  $z = 0$ .

**Definition 23.** If  $u$  is harmonic on a region  $\Omega$ , then a **harmonic conjugate** of  $u$  is any function  $v$  such that  $u + iv$  is analytic on  $\Omega$ .

**Definition 24.** Let  $I = (0, 1)$ . An open analytic arc  $\gamma$  contained in the boundary of a region  $\Omega$  is called a **one-sided arc** if there exists a function  $g$  which is one-to-one and analytic in a neighbourhood  $N$  of  $I$ , with  $g(I) = \gamma$ , and  $g(N \cap \mathbb{H}) \subset \Omega$  and  $g(N \setminus \overline{\mathbb{H}}) \subset \mathbb{C} \setminus \Omega$ . If  $g(N \setminus I) \subset \Omega$  then  $\gamma$  is called a **two-sided arc**.

**Definition 25.** If  $f$  is analytic in  $\{z : 0 < |z - a| < \delta\}$  for some  $\delta > 0$ , then the **residue of  $f$  at  $a$** , is the coefficient of  $\frac{1}{z-a}$  in the Laurent series expansion of  $f$  about  $z = a$ .

**Definition 26.** A family  $\mathcal{F}$  of function on a region  $\Omega \subset \mathbb{C}$  is said to be **normal on  $\Omega$**  provided every sequence  $\{f_n\} \subset \mathcal{F}$  contains a subsequence which converges uniformly on compact subsets of  $\Omega$ .

**Definition 27.** A family of functions  $\mathcal{F}$  defined on a set  $E \subset \mathbb{C}$  is

- (a) **equicontinuous at**  $w \in E$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $z \in E$  and  $|z - w| < \delta$ , then  $|f(z) - f(w)| < \varepsilon$  for all  $f \in \mathcal{F}$ ;
- (b) **equicontinuous on**  $E$  if it is equicontinuous at each  $w \in E$ ;
- (c) **uniformly equicontinuous on**  $E$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $z, w \in E$  and  $|z - w| < \delta$ , then  $|f(z) - f(w)| < \varepsilon$  for all  $f \in \mathcal{F}$ .

**Definition 28.** A family  $\mathcal{F}$  of continuous functions is said to be **locally bounded** on  $\Omega$  if for each  $w \in \Omega$  there is a  $\delta > 0$  and  $M < \infty$  so that if  $|z - w| < \delta$  then  $|f(z)| < M$  for all  $f \in \mathcal{F}$ .

**Definition 29.** Suppose  $f$  is meromorphic in a region  $\Omega$ , with a pole of order  $M$  at  $b \in \Omega$ . Then **the singular part of  $f$  at  $b$**  is  $S_b(z) = \sum_{k=-M}^{-1} c_k(z-b)^k$ , where  $\sum_{k \in \mathbb{Z}} c_k(z-b)^k$  is the Laurent series for  $f$  at  $b$ .

**Definition 30.** Let  $C(\partial\Omega)$  denote the set of continuous functions on the boundary in  $\mathbb{C}^*$  of a region  $\Omega$ . The **Dirichlet problem** on  $\Omega$  for a function  $f \in C(\partial\Omega)$  is to find a harmonic function  $u$  on  $\Omega$  that is continuous on  $\overline{\Omega}$  and equal to  $f$  on  $\partial\Omega$ .

**Definition 31.** A family  $\mathcal{F}$  of subharmonic functions on a region  $\Omega$  is called a **Perron family** if it satisfies

- (i) if  $v_1, v_2 \in \mathcal{F}$  then  $\max(v_1, v_2) \in \mathcal{F}$ ,
- (ii) if  $v \in \mathcal{F}$  and  $D$  is a disc with  $\overline{D} \subset \Omega$ , and if  $v > -\infty$  on  $\partial D$ , then  $v_D \in \mathcal{F}$ , and
- (iii) for each  $z \in \Omega$ , there exists  $v \in \mathcal{F}$  such that  $v(z) > -\infty$ .

**Remark.** For a subharmonic function  $v$  and a disc  $D$  centered at  $c$  and of radius  $r$ ,  $v_D$  is defined by  $v_D(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z-c|^2/r^2}{|e^{it}-(z-c)/r|^2} v(c + re^{it}) dt$ .

**Definition 32.** If  $\Omega \subset \mathbb{C}^*$  is a region, and if  $f$  is a real-valued function on  $\partial\Omega$  with  $|f| \leq M < \infty$  on  $\partial\Omega$ , set

$$\mathcal{F}_f = \{v \text{ subharmonic on } \Omega : \limsup_{z \in \Omega \rightarrow \zeta} v(z) \leq f(\zeta), \text{ for all } \zeta \in \partial\Omega\}.$$

Then  $u_f(z) \equiv \sup_{u \in \mathcal{F}_f} u(z)$  is harmonic in  $\Omega$ . The function  $u_f$  is called the **Perron solution to the Dirichlet problem** on  $\Omega$  for the function  $f$ .

**Definition 33.** If  $\Omega \subset \mathbb{C}^*$  is a region and if  $\zeta_0 \in \partial\Omega$  then  $b$  is called a **local barrier at  $\zeta_0$  for the region  $\Omega$**  provided

- (i)  $b$  is defined and is subharmonic on  $\Omega \cap D$  for some open disc  $D$  containing  $\zeta_0$ ,
- (ii)  $b(z) < 0$  for  $z \in \Omega \cap D$ , and
- (iii)  $\lim_{z \in \Omega \rightarrow \zeta_0} b(z) = 0$ .

**Definition 34.** If there exist a local barrier at  $\zeta_0 \in \partial\Omega$  then  $\zeta_0$  is called a **regular point of  $\partial\Omega$** . Otherwise  $\zeta_0$  is called an **irregular point of  $\partial\Omega$** . If every  $\zeta \in \partial\Omega$  is a regular point, then  $\Omega$  is called a **regular region**.

**Definition 35.** If  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is a curve, and if  $f_0$  is analytic in a neighbourhood of  $\gamma(0)$ , then an **analytic continuation of  $f_0$  along  $\gamma$**  is a finite sequence  $f_1, \dots, f_n$  of functions where  $0 = t_0 < t_1 < \dots < t_{n+1} = 1$  is a partition of  $[0, 1]$  and  $f_j$  is defined and analytic in a neighbourhood of  $\gamma([t_j, t_{j+1}])$ ,  $j = 0, \dots, n$  such that  $f_j = f_{j+1}$  in a neighbourhood of  $\gamma(t_{j+1})$ ,  $j = 0, \dots, n-1$ .

**Definition 36.** A **Riemann surface** is a connected Hausdorff space  $W$ , together with a collection of open subsets  $U_\alpha \subset W$  and functions  $z_\alpha : U_\alpha \rightarrow \mathbb{C}$  such that

- (i)  $W = \bigcup U_\alpha$ ,
- (ii)  $z_\alpha$  is a homeomorphism of  $U_\alpha$  onto  $\mathbb{D}$ , and
- (iii) if  $U_\alpha \cap U_\beta \neq \emptyset$  then  $z_\beta \circ z_\alpha^{-1}$  is analytic on  $z_\alpha(U_\alpha \cap U_\beta)$ .

The functions  $z_\alpha$  are called **coordinate functions or maps** and the sets  $U_\alpha$  are called **coordinate charts or discs**. Functions of the form  $z_\beta \circ z_\alpha^{-1}$  are called **transition maps**.

**Definition 37.** Let  $W$  be a Riemann surface with charts  $(U_\alpha, z_\alpha)$ . Fix some  $b \in W$ . Let  $[\gamma]$  be the equivalence class of curves homotopic to a curve  $\gamma \subset W$ . Let

$$W^* = \{[\gamma] : \gamma(0) = b\}.$$

Define  $\pi : W^* \rightarrow W$  by  $[\gamma] \mapsto \gamma(1)$ . Let  $U_\alpha$  be a chart at  $c \in W$ ,  $\gamma$  be curve in  $W$  from  $b$  to  $c$ , and let

$$U_\alpha^* = \{[\gamma\sigma_d] : d \in U_\alpha\}$$

where  $\sigma_d$  is a curve in  $U_\alpha$  from  $c$  to  $d$ . Define  $z_\alpha^* : U_\alpha^* \rightarrow \mathbb{D}$  by  $z_\alpha^* = z_\alpha \circ \pi$ . Give  $W^*$  a topology by declaring each  $U_\alpha^*$  open. Then  $W^*$  is a Riemann surface with charts  $(U_\alpha^*, z_\alpha^*)$  called the **universal covering surface of  $W$**  and  $\pi$  is called the **universal covering map**.

**Definition 38.** If  $\sigma$  is a closed curve in  $W$  such that  $\sigma(0) = \sigma(1) = b \in W$ , then  $M_{[\sigma]} : W^* \rightarrow W^*$  defined by  $M_{[\sigma]}([\gamma]) = [\sigma\gamma]$  is called a **deck transformation**. The deck transformations form a group under composition, this group is called **the fundamental group of  $W$  at  $b$** .

**Definition 39.** If  $W_1$  and  $W_2$  are Riemann surfaces, then  $f : W_1 \rightarrow W_2$  is said to be **analytic** if  $w_\beta \circ f \circ z_\alpha^{-1}$  is analytic for each coordinate function  $z_\alpha$  on  $W_1$  and  $w_\beta$  on  $W_2$ , wherever it is defined.

**Definition 40.** Let  $W$  be a Riemann surface and fix some  $p_0 \in W$ . Let  $z : U \rightarrow \mathbb{D}$  be a coordinate function such that  $z(p_0) = 0$ . Let  $\mathcal{F}_{p_0}$  be the collection of subharmonic functions  $v$  on  $W \setminus \{p_0\}$  satisfying  $v = 0$  on  $W \setminus K$  for some compact proper subset of  $W$ , and  $\limsup_{p \rightarrow p_0} (v(p) + \log |z(p)|) < \infty$ . Then  $\mathcal{F}_{p_0}$  is a Perron family on  $W \setminus \{p_0\}$ . Set  $g_w(p, p_0) = \sup\{v(p) : v \in \mathcal{F}_{p_0}\}$ , then by Harnack's theorem we have two cases

- (i)  $g_w(p, p_0)$  is harmonic in  $W \setminus \{p_0\}$ , or
- (ii)  $g_w(p, p_0) = +\infty$  for all  $p \in W \setminus \{p_0\}$ .

In the first case we say that  $g_w$  is **Green's function on  $W$  with pole at  $p_0$** , and in the second case **Green's function with pole at  $p_0$  does not exist on  $W$** .