Definitions used in Brezis' *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (first edition)

Gustaf Bjurstam
bjurstam@kth.se

Preliminaries – not in the book

Definition 0.1. Let E be a vector space over $\mathbb{R}$. A *functional* is a function $f : A \to \mathbb{R}$ where A is some subspace of E.

1 The Hahn-Banach Theorems. Introduction to Conjugate Convex Functions

Definition 1.1. Let E be a vector space over $\mathbb{R}$. A *Minkowski functional* is a function $p : E \to \mathbb{R}$ satisfying

$$p(\lambda x) = \lambda p(x), \qquad \forall x \in E \text{ and } \lambda > 0. \tag{1}$$

$$p(x+y) \leq p(x) + p(y), \qquad \forall x, y \in E. \tag{2}$$

Definition 1.2. Let P be a set with a (partial) order relation $\leq$. A subset $Q \subseteq P$ is *totally ordered* if for any pair (a, b) in Q at least one of $a \leq b$ and $b \leq a$ holds.

Definition 1.3. Let P be a set with a partial order relation $\leq$, and let $Q \subset P$. We say that $c \in P$ is an *upper bound* for Q if $a \leq c$ for all $a \in Q$. We say that $m \in P$ is a *maximal element* of P if there is no element $x \in P \setminus \{m\}$ such that $m \leq x$. If every totally ordered subset Q of P has an upper bound, we call P *inductive*.

Definition 1.4. Let E be a real normed vector space. We denote by $E^\star$ the *dual space* of E, that is, the set of all continuous linear functionals on E. The *dual norm* is defined by

$$\|f\|_{E^\star} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} f(x).$$

Given $f \in E^\star$ and $x \in E$ we may write $\langle f, x\rangle$ instead of $f(x)$; we say that $\langle , \rangle$ is the *scalar product for the duality* $E^\star, E$.

Definition 1.5. Let E be a normed vector space over $\mathbb{R}$. For every $x_0 \in E$, we set

$$F(x_0) = \left\{ f_0 \in E^\star : \|f_0\| = \|x_0\| \text{ and } \langle f_0, x_0\rangle = \|x_0\|^2 \right\}.$$

The map $x_0 \mapsto F(x_0)$ is called the *duality map* of E into $E^\star$.

Definition 1.6. Let E be a real vector space. An *affine hyperplane* is a subset H of E of the form $H = \{x \in E : f(x) = \alpha\}$ where f is a linear functional not necessarily in $E^\star$, and $\alpha \in \mathbb{R}$ is a given constant. We write $H = [f = \alpha]$ and say that $f = \alpha$ is the equation of H.

Definition 1.7. Let E be a normed vector space. Let $A, B \subset E$, we say that the hyperplane $H = [f = \alpha]$ *separates* A and B if $f(x) \leq \alpha$ for all $x \in A$ and $f(x) \geq \alpha$ for all $x \in B$. If there is $\varepsilon > 0$ such that $f(x) \leq \alpha - \varepsilon, \forall x \in A$ and $f(x) \geq \alpha + \varepsilon, \forall x \in B$, we say that H *strictly separates* A and B.

Definition 1.8. Let E be a normed vector space. We say that $A \subset E$ is convex if $tx + (1-t)x \in A$ for all $x, y \in A$ and $t \in [0, 1]$.

Definition 1.9. Let E be a normed vector space, and let $C \subset E$ be an open convex set with $0 \in C$. For every $x \in E$ set $p(x) = \inf\left\{\alpha : \alpha^{-1}x \in C\right\}$. We call p the *gauge of* C or the *Minkowski functional of* C.

Definition 1.10. Let E be a normed vector space. The *bidual* $E^{\star\star}$ is the dual of $E^\star$ with norm

$$\|\xi\|_{E^{\star\star}} = \sup_{\substack{f \in E^\star \\ \|f\| \leq 1}} \langle \xi, f \rangle .$$

The *canonical injection* $J : E \to E^{\star\star}$ is defined as $Jx = \langle f, x\rangle$, this

Definition 1.11. An *isometry* is a map f between metric spaces A and B such that $d_A(x, y) = d_B(f(x), f(y))$ for all $x, y \in A$.

Definition 1.12. Let E be a normed vector space. If the canonical injection $J : E \to E^{\star\star}$ is surjective, then we say that E is *reflexive* and identify E with $E^{\star\star}$.

Definition 1.13. Let E be a normed vector space, and suppose M is a linear subspace of E. We set

$$M^\perp = \{f \in E^\star : \langle f, x\rangle = 0 \forall x \in M\} .$$

If N is a linear subspace of $E^\star$, we set

$$N^\perp = \{x \in E : \langle f, x\rangle = 0 \forall f \in N\} .$$

Note that $N^\perp$ is a subset of E and not $E^{\star\star}$. We call $M^\perp$ (or $N^\perp$) *the space orthogonal to* M *(or* N*)*.

Definition 1.14. Let E be a set, and let $\varphi : E \to (-\infty, \infty]$. We denote $D(\varphi) = \{x \in E : \varphi(x) < \infty\}$, and define the *epigraph* of φ as $\operatorname{epi}\varphi = \{[x, \lambda] \in E \times \mathbb{R} : \varphi(x) \leq \lambda\}$.