

Statements of Examinable Theorems

SF2745 Advanced Complex Analysis

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Remark. Theorem's for which the proof must be supplied are marked with (*).

Theorem 1. *Under stereographic projection, circles and lines in \mathbb{C} correspond to circles in \mathbb{S}^2 .*

Theorem 2 (M test *). *If $|a_n(z - z_0)^n| \leq M_n$ for $|z - z_0| \leq r$ and if $\sum M_n < \infty$, then $\sum a_n(z - z_0)^n$ converges absolutely and uniformly in $\{z : |z - z_0| \leq r\}$.*

Theorem 3 (Root test). *Suppose $\sum a_n(z - z_0)^n$ is a formal power series. Let*

$$R = \liminf_{n \rightarrow \infty} |a_n|^{-1/n} = \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{1/n}} \in [0, \infty].$$

Then $\sum a_n(z - z_0)^n$

- (a) converges absolutely in $D(R, z_0)$,*
- (b) converges uniformly in $\overline{D(r, z_0)}$ for all $r \in (0, R)$,*
- (c) diverges in $\mathbb{C} \setminus \overline{D(R, z_0)}$.*

Theorem 4 (Uniqueness *). *If f is analytic in a region Ω then either $f \equiv 0$ or the zeros of f are isolated.*

Theorem 5 (Power series expansion *). *An analytic function f has derivatives of all orders. If f is equal to a convergent power series on $D(r, z_0)$ then the power series is given by*

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

for $z \in D(r, z_0)$.

Theorem 6 (Maximum principle). *If f is analytic on a region Ω , then unless f is constant, there is no $z_0 \in \Omega$ such that $f(z_0) = \sup_{z \in \Omega} |f(z)|$.*

Theorem 7 (Openness of NC analytic maps). *If $E \subseteq \Omega$ is an open set, and f is a non-constant analytic map on Ω , then $f(E)$ is open.*

Theorem 8 (Liouville *). *If f is an entire function, and f is bounded, then f is constant.*

Theorem 9 (Schwarz lemma *). *If f is analytic on \mathbb{D} , $f(0) = 0$, and $|f(z)| \leq 1$ on \mathbb{D} , then $|f(z)| < |z|$ for all $z \in \mathbb{D}$, and $f'(0) \leq 1$. Moreover, if $|f(z)| = |z|$ for any non-zero $z \in \mathbb{D}$, or if $|f'(0)| = 1$, then there is $c \in \partial\mathbb{D}$ such that $f(z) = cz$.*

Theorem 10 (*). Let C be a positively oriented circle of radius r centered at z_0 , then

$$\int_C \frac{dz}{z-a} = \begin{cases} 1, & |a - z_0| < r, \\ 0, & |a - z_0| > r. \end{cases}$$

Theorem 11 (Cauchy simple *). Let D be a disc of radius r centered at z_0 , and suppose f is holomorphic on D , then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw$$

for all $z \in D$.

Theorem 12. A function f is holomorphic on a region Ω if and only if it is analytic on Ω . Moreover, the series expansion for f centered at $z_0 \in \Omega$ has the radius of the largest disc centered at z_0 and contained in Ω as its radius of convergence.

Theorem 13 (Cauchy estimate and derivatives *). Let D be a disc of radius r centered at z_0 , and suppose f is holomorphic on D , then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

for all $n \geq 0$. And

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{w \in \partial D} |f(w)|$$

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Theorem 14 (Morea). If f is continuous in an open disc B , and if

$$\int_{\partial R} f dz = 0$$

for all closed rectangles $R \subset B$ with sides parallel to the axes, then f is analytic on B .

Theorem 15 (Runge). If f is analytic on a compact set K , and if $\varepsilon > 0$, then there is a rational function r so that

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

Theorem 16 (Weierstrass). Suppose $\{f_n\}$ is a collection of analytic functions on a region Ω such that $f_n \rightarrow f$ uniformly on compact subsets of Ω . Then f is analytic in Ω . Moreover $f'_n \rightarrow f'$ uniformly on compact subsets of Ω .

Theorem 17 (Cauchy integral formula). Suppose γ is cycle contained in a region Ω , and suppose

$$\int_{\gamma} \frac{dw}{w-a} = 0$$

for all $a \notin \Omega$. If f is analytic on Ω and $z \in \mathbb{C} \setminus \gamma$ then

$$\int_{\gamma} \frac{f(w)}{w-z} dw = f(z) \int_{\gamma} \frac{dw}{w-z}.$$

Theorem 18 (Primitive, log, on SCR). *Suppose f is analytic on a simply-connected region Ω . Then*

- (i) $\int_{\gamma} f = 0$ for all closed curves $\gamma \subset \Omega$,
- (ii) there exists a function F analytic on Ω such that $F' = f$,
- (iii) if also $f \neq 0$ for all $z \in \Omega$, then there exists a function g on Ω such that $f = e^g$.

Theorem 19 (Laurent). *Suppose f is analytic on $A = \{z : r < |z-a| < R\}$. Then there is a unique sequence $\{a_n\} \subset \mathbb{C}$ so that*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z-a)^n,$$

where the series converges uniformly and absolutely on compact subsets of A . Moreover,

$$a_n = \frac{1}{2\pi i} \int_{C_s} \frac{f(w)}{(w-a)^{n+1}} dw,$$

where C_s is a positively oriented circle centered at a with radius $s \in (r, R)$.

Theorem 20 (Density near essential *). *If f is analytic in $U = \{z : 0 < |z-b| < \delta\}$ and if b is an essential singularity of f , then $f(U)$ is dense in \mathbb{C} .*

Theorem 21 (Argument Principle *). *Suppose f is meromorphic in a region Ω , with zeros z_j and poles p_k . Suppose γ is a cycle with $\gamma \sim 0$ in Ω , and suppose no zero or pole appears on γ . Then*

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k).$$

Theorem 22 (LFT circles). *LFTs map "circles" onto "circles" and "discs" onto "discs".*

Theorem 23 (LFT three points *). *Given z_1, z_2, z_3 , distinct points in \mathbb{C}^* and w_1, w_2, w_3 distinct points in \mathbb{C}^* , there is a unique LFT, T , such that $T(z_i) = w_i$, for $i = 1, 2, 3$.*

Theorem 24 (Max principle *). *Let u be subharmonic on a region Ω . Then, unless u is constant, there is no $z \in \Omega$ such that $u(z) = \sup_{w \in \Omega} u(w)$.*

Theorem 25 (Poisson integral). *Let g be a continuous function defined on $\partial\mathbb{D}$, the solution to the Dirichlet problem on \mathbb{D} with boundary values g is given by*

$$u(z) = PI(g)(z) = \int_0^{2\pi} P_z(t) g(e^{it}) dt = \int_{0^{2\pi}} \frac{1}{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} g(e^{it}) dt.$$

Theorem 26 (Cauchy-Riemann). *Suppose Ω is a region, and that $u, v : \Omega \rightarrow \mathbb{R}$ are C^1 . Then u and v satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x,$$

if and only if $f = u + iv$ is holomorphic on Ω .

Theorem 27 (Lindelöf max *). *Suppose Ω is a region and that $\{\xi_1, \dots, \xi_n\}$ is a finite subset of $\partial\Omega$, not equal to $\partial\Omega$. If u is subharmonic on Ω with $u \leq M < \infty$ on Ω and if*

$$\limsup_{z \in \Omega \rightarrow \xi} u(z) \leq m,$$

for all $\xi \in \Omega \setminus \{\xi_1, \dots, \xi_n\}$, then $u \leq m$ on Ω .

Theorem 28 (Harnack inequality). *Suppose u is a positive harmonic function on \mathbb{D} . Then*

$$\frac{1-r}{1+r}u(0) \leq u(z) \leq \frac{1+r}{1-r}u(0),$$

for $r = |z| < 1$.

Theorem 29 (Harnack's principle). *Suppose $\{u_n\}$ are harmonic on a region Ω such that $u_n(z) \leq u_{n+1}(z)$ for all $z \in \Omega$. Then either*

(i) $\lim_n u_n = u$ exists and is harmonic on Ω , or

(ii) $\lim_n u_n = \infty$ everywhere,

where convergence is uniform on compact subsets of Ω . In case (ii), this means that given compact $K \subset \Omega$ and $M < \infty$, there is n_0 such that $u_n > M$ for all $z \in K$, and $n > n_0$.

Theorem 30 (Schwarz reflection). *Suppose Ω is a region which is symmetric about \mathbb{R} . Set $\Omega^+ = \Omega \cap \mathbb{H}$ and $\Omega^- = \Omega \setminus \overline{\mathbb{H}}$. If ν is harmonic on Ω^+ , continuous on $\Omega^+ \cup (\Omega \cap \mathbb{R})$ and equal to 0 on $\Omega \cap \mathbb{R}$, then the function defined by*

$$V(z) = \begin{cases} \nu(z), & z \in \Omega^+ \\ -\nu(\bar{z}), & z \in \Omega^- \end{cases}$$

is harmonic on Ω . If also $\nu(z) = \operatorname{Im} f(z)$, where f is analytic on Ω^+ , then the function

$$g(z) = \begin{cases} f(z), & z \in \Omega^+ \\ \overline{f(\bar{z})}, & z \in \Omega^- \end{cases}$$

extends to be analytic in Ω .

Theorem 31 (Schwarz-Christoffel). *Suppose Ω is a bounded simply-connected region whose positively oriented boundary $\partial\Omega$ is a polygon with vertices v_1, \dots, v_n . Suppose the tangent direction on $\partial\Omega$ increases by $\pi\alpha_j$ at vertex v_j , $\alpha_j \in (-1, 1)$. Then there exists $x_1 < x_2 < \dots < x_n$ and constants c_1, c_2 so that*

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^n (\zeta - x_j)^{-\alpha_j} d\zeta + c_2$$

is a conformal map of \mathbb{H} onto Ω , where the integral is along any curve γ_z in \mathbb{H} from i to z .

Theorem 32 (Residue Theorem *). *Suppose f is analytic in Ω except for isolated singularities at a_1, \dots, a_n . If γ is a cycle in Ω with $\gamma \sim 0$ and $a_j \notin \gamma$, $j = 1, \dots, n$, then*

$$\int_{\gamma} f = 2\pi i \sum_k n(\gamma, a_k) \text{Res}_{a_k} f.$$

Theorem 33 (Arzela-Ascoli). *A family \mathcal{F} of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

- (i) \mathcal{F} is equicontinuous on Ω , and
- (ii) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of \mathbb{C} .

Theorem 34 (Normal analytic families). *The following are equivalent for a family \mathcal{F} of analytic functions on a region Ω*

- (i) \mathcal{F} is normal on Ω ;
- (ii) \mathcal{F} is locally bounded on Ω ;
- (iii) $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of \mathbb{C} .

Theorem 35 (Riemann Mapping Theorem *). *Suppose $\Omega \subset \mathbb{C}$ is simply-connected and $\Omega \neq \mathbb{C}$. Then there exists a one-to-one map f of Ω onto \mathbb{D} . If $z_0 \in \Omega$, then there is a unique such map with $f(z_0) = 0$ and $f'(z_0) > 0$.*

Theorem 36 (Mittag-Leffler). *Suppose $b_k \in \Omega \rightarrow \partial\Omega$ with $b_k \neq b_j$ if $k \neq j$. Set*

$$S_k(z) = \sum_{j=1}^{n_k} \frac{c_{j,k}}{(z - b_k)^j},$$

where each n_k is a positive integer and $c_{j,k} \in \mathbb{C}$. Then there is a function meromorphic in Ω with singular parts S_k at b_k , $k = 1, 2, \dots$, and no other singularities in Ω .

Theorem 37 (Weierstrass Product Theorem). *Suppose Ω is a bounded region. If $\{b_j\} \subset \Omega$ with $b_j \rightarrow \partial\Omega$, and if n_j are positive integers, then there exists an analytic function f on Ω such that f has a zero of order exactly n_j at b_j , $j = 1, 2, \dots$, and no other zeros in Ω .*

Theorem 38 (Jensen's formula). *Suppose f is meromorphic on $|z| \leq R$ with zeros $\{a_k\}$ and poles $\{b_j\}$. Suppose also that 0 is not a zero or a pole of f . Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(Re^{it})| dt = \log |f(0)| + \sum_{a_k < R} \log \frac{R}{|a_k|} - \sum_{b_j < R} \log \frac{R}{|b_j|}.$$

Theorem 39 (Monodromy). *Suppose Ω is simply-connected and suppose f_0 is defined and analytic in a neighbourhood of $b \in \Omega$. If f_0 can be analytically continued along all curves in Ω beginning at b then there is an analytic function f on Ω so that $f = f_0$ in a neighbourhood of b .*

Theorem 40 (Green's function *). *Suppose $p_0 \in W$ and suppose $z : U \rightarrow \mathbb{D}$ is a coordinate function such that $z(p_0) = 0$. If $g_W(p, p_0)$ exists, then*

$$\begin{aligned} g_W(p, p_0) &> 0 \quad \text{for } p \in W \setminus \{p_0\}, \text{ and,} \\ g_W(p, p_0) + \log |z(p)| &\text{ extends to be harmonic in } U. \end{aligned}$$

Theorem 41 (Green is symmetric if $W = \mathbb{D}$). *Suppose W is a Riemann surface for which Green's function g_W with pole at p exists, for some $p \in W$, and suppose $W^* = \mathbb{D}$. Then g_W with pole at q exists for all $q \in W$, and*

$$g_W(p, q) = g_W(q, p).$$

Theorem 42 (Uniformisation case 1 *). *If W is a simply-connected Riemann surface then the following are equivalent:*

- (i) $g_W(p, p_0)$ exists for some $p_0 \in W$,
- (ii) $g_W(p, p_0)$ exists for all $p_0 \in W$, and
- (iii) there is a one-to-one analytic map φ of W onto \mathbb{D} .

Moreover, if g_W exists, then

$$g_W(p_1, p_0) = g_W(p_0, p_1),$$

and $g_W(p, p_0) = -\log |\varphi(p)|$, where $\varphi(p_0) = 0$.

Theorem 43 (Green is symmetric general case). *Suppose W is a Riemann surface for which Green's function g_W with pole at p exists, for some $p \in W$. Then g_W with pole q exists, for all $q \in W$, and*

$$g_W(p, q) = g_W(q, p).$$

Theorem 44 (Uniformisation case 2). *Suppose W is a simply-connected Riemann surface for which Green's function does not exist. If W is compact, then there is a one-to-one analytic map of W onto \mathbb{C}^* . If W is not compact, then there is a one-to-one analytic map of W onto \mathbb{C} .*