Statements of Examinable Theorems
SF2745 Advanced Complex Analysis

Gustaf Bjurstam
bjurstam@kth.se

Remark. Theorem's for which the proof must be supplied are marked with $(*)$.

Theorem 0.1. *Under stereographic projection, circles and lines in $\mathbb{C}$ correspond to circles in $\mathbb{S}^2$.*

Theorem 0.2 (M test $*$)**.** *If $|a_n(z-z_0)^n| \leq M_n$ for $|z-z_0| \leq r$ and if $\sum M_n < \infty$, then $\sum a_n(z-z_0)^n$ converges absolutely and uniformly in $\{z : |z-z_0| \leq r\}$.*

Theorem 0.3 (Root test)**.** *Suppose $\sum a_n(z-z_0)^n$ is a formal power series. Let*

$$R = \liminf_{n\to\infty} |a_n|^{-1/n} = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \in [0, \infty].$$

Then $\sum a_n(z-z_0)^n$

(a) coverges absolutely in $D(R, z_0)$,

(b) converges uniformly in $\overline{D(r, z_0}$ for all $r \in (0, R)$,

(c) diverges in $\mathbb{C} \setminus \overline{D(R, z_0)}$.

Theorem 0.4 (Uniqueness $*$)**.** *If f is analytic in a region Ω then either $f \equiv 0$ or the zeros of f are isolated.*

Theorem 0.5 (Power series expansion $*$)**.** *An analytic function f has derivatives of all orders. If f is equal to a convergent power series on $D(r, z_0)$ then the power series is given by*

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

for $z \in D(r, z_0)$.

Theorem 0.6 (Maximum principle)**.** *If f is analytic on a region Ω, then unless f is constant, there is no $z_0 \in \Omega$ such that $f(z_0) = \sup_{z\in\Omega} f(z)$.*

Theorem 0.7 (Openness of NC analytic maps)**.** *If $E \subseteq \Omega$ is an open set, and f is a non-constant analytic map on Ω, then $f(E)$ is open.*

Theorem 0.8 (Liouville $*$)**.** *If f is an entire function, and f is bounded, then f is constant.*

Theorem 0.9 (Schwarz lemma $*$)**.** *If f is analytic on $\mathbb{D}$, $f(0) = 0$, and $|f(z)| \leq 1$ on $\mathbb{D}$, then $|f(z)| < |z|$ for all $z \in \mathbb{D}$, and $f'(0) \leq 1$. Moreover, if $|f(z)| = |z|$ for any non-zero $z \in \mathbb{D}$, or if $|f'(0)| = 1$, then there is $c \in \partial\mathbb{D}$ such that $f(z) = cz$.*

Theorem 0.10 $(*)$**.** *Let C be a positivley oriented circle of radius r centered at z_0, then*

$$\int_C \frac{dz}{z-a} = \begin{cases} 1, & |a-z_0| < r, \\ 0, & |a-z_0| > r. \end{cases}$$

Theorem 0.11 (Cauchy simple $*$). *Let D be a disc of radius r centered at z_0, and suppose f is holomorphic on D, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw$$

for all $z \in D$.

Theorem 0.12. *A function f is holomorphic on a region Ω if and only if it is analytic on Ω. Moreover, the series expansion for f centered at $z_0 \in \Omega$ has the radius of the largest disc centered at z_0 and contained in Ω as its radius of convergence.*

Theorem 0.13 (Cauchy estimate and derivatives $*$). *Let D be a disc of radius r centered at z_0, and suppose f is holomorphic on D, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

for all $n \geq 0$. And

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{w \in \partial D} |f(w)|$$

.

Theorem 0.14 (Morea). *If f is continuous in an open disc B, and if*

$$\int_{\partial R} f \, dz = 0$$

for all closed rectangels $R \subset B$ with sides parallel to the axes, then f is analytic on B.

Theorem 0.15 (Runge). *If f is analytic on a compact set K, and if $\varepsilon > 0$, then there is a rational function r so that*

$$\sup_{z \in K} |f(z) - r(z)| < \varepsilon.$$

Theorem 0.16 (Weirstrass). *Suppose $\{f_n\}$ is a collection of analytic functions on a regiona Ω such that $f_n \to f$ uniformly on compact subsets of Ω. Then f is analytic in Ω. Moreover $f_n' \to f'$ uniformly on compact subsets of Ω.*

Theorem 0.17 (Cauchy integral formula). *Suppose γ is cycle contained in a region Ω, and suppose*

$$\int_\gamma \frac{dw}{w-a} = 0$$

for all $a \notin \Omega$. If f is analytic on Ω and $z \in \mathbb{C} \setminus \gamma$ then

$$\int_\gamma \frac{f(w)}{w-z} \, dw = f(z) \int_\gamma \frac{dw}{w-z}.$$

Theorem 0.18 (Primitive, log, on SCR). *Suppose f is analytic on a simply-connected region Ω. Then* *(i)* $\int_\gamma f = 0$ *for all closed curves* $\gamma \subset \Omega$,

(ii) there exists a function F *analytic on* Ω *such that* $F' = f$,

(iii) if also $f \neq 0$ *for all* $z \in \Omega$, *then there exists a function* g *on* Ω *such that* $f = e^g$.

Theorem 0.19 (Laurent)**.** *Suppose* f *is analytic on* $A = \{z : r < |z - a| < R\}$. *Then there is a unique sequence* $\{a_n\} \subset \mathbb{C}$ *so that*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - a)^n,$$

where the series converges uniformly and absolutely on compact subsets of A. *Moreover,*

$$a_n = \frac{1}{2\pi i} \int_{C_s} \frac{f(w)}{(w - a)^{n+1}} \, dw,$$

where C_s *is a positively oriented circle centered at* a *with radius* $s \in (r, R)$.

Theorem 0.20 (Density near essential $*$)**.** *If* f *is analytic in* $U = \{z : 0 < |z - b| < \delta\}$ *and if* b *is an essential singularity of* f, *them* $f(U)$ *is dense in* $\mathbb{C}$.

Theorem 0.21 (Argument Principle $*$)**.** *Suppose* f *is meromorphic in a region* Ω, *with zeros* z_j *and poles* p_k. *Suppose* γ *is a cycle with* $\gamma \sim 0$ *in* Ω, *and suppose no zero or pole appears on* γ. *Then*

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k).$$

Theorem 0.22 (LFT circles)**.** *LFTs map "circles" onto "circles" and "discs" onto "discs".*

Theorem 0.23 (LFT three points $*$)**.** *Given* z_1, z_2, z_3, *distinct points in* $\mathbb{C}^*$ *and* w_1, w_2, w_3 *distinct points in* $\mathbb{C}^*$, *there is a unique LFT,* T, *such that* $T(z_i) = w_i$, *for* $i = 1, 2, 3$.

Theorem 0.24 (Max principle $*$)**.** *Let* u *be subharmonic on a region* Ω. *Then, unless* u *is constant, there is no* $z \in \Omega$ *such that* $u(z) = \sup_{w \in \Omega} u(w)$.

Theorem 0.25 (Poisson integral)**.** *Let* g *be a continuous function defined on* $\partial\mathbb{D}$, *the solution to the Dirichlet problem on* $\mathbb{D}$ *with boundary values* g *is given by*

$$u(z) = PI(g)(z) = \int_0^{2\pi} P_z(t) g(e^{it}) \, dt = \int_{0^{2\pi}} \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} g(e^{it}) \, dt.$$

Theorem 0.26 (Cauchy-Riemann)**.** *Suppose* Ω *is a region, and that* $u, v : \Omega \to \mathbb{R}$ *are* C^1. *Then* u *and* v *satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \qquad u_y = -v_x,$$

if and only if $f = u + iv$ *is holomorphic on* Ω.

Theorem 0.27 (Lindelöf max $*$). *Suppose Ω is a region and that $\{\xi_1, \dots, \xi_n\}$ is a finite subset of $\partial\Omega$, not equal to $\partial\Omega$. If u is subharmonic on Ω with $u \le M < \infty$ on Ω and if*

$$\limsup_{z \in \Omega \to \xi} u(z) \le m,$$

for all $\xi \in \Omega \setminus \{\xi_1, \dots, \xi_n\}$, then $u \le m$ on Ω.

Theorem 0.28 (Harnack inequality). *Suppose u is a positive harmonic function on $\mathbb{D}$. Then*

$$\frac{1-r}{1+r} u(0) \le u(z) \le \frac{1+r}{1-r} u(0),$$

for $r = |z| < 1$.

Theorem 0.29 (Harnack's principle). *Suppose $\{u_n\}$ are harmonic on a region Ω such that $u_n(z) \le u_{n+1}(z)$ for all $z \in \Omega$. Then either*

(i) $\lim_n u_n = u$ *exists and is harmonic on Ω, or*

(ii) $\lim_n u_n = \infty$ *everywhere,*

where convergence is uniform on compact subsets of Ω. In case (ii), this means that given compact $K \subset \Omega$ and $M < \infty$, there is n_0 such that $u_n > m$ for all $z \in K$, and $n > n_0$.

Theorem 0.30 (Schwarz reflection). *Suppose Ω is a region which is symmetric about $\mathbb{R}$. Set $\Omega^+ = \Omega \cap \mathbb{H}$ and $\Omega^- = \Omega \setminus \overline{\mathbb{H}}$. If ν is harmonic on Ω^+, continuous on $\Omega^+ \cup (\Omega \cap \mathbb{R})$ and equal to 0 on $\Omega \cap \mathbb{R}$, then the function defined by*

$$V(z) = \begin{cases} \nu(z), & z \in \Omega \setminus \Omega^- \\ -\nu(\overline{z}), & z \in \Omega^- \end{cases}$$

is harmonic on Ω. If also $\nu(z) = \Im f(z)$, where f is analytic on Ω^+, then the function

$$g(z) = \begin{cases} f(z), & z \in \Omega^+ \\ \overline{f(\overline{z})}, & z \in \Omega^- \end{cases}$$

extends to be analytic in Ω.

Theorem 0.31 (Schwarz-Christoffel). *Suppose Ω is a bounded simply-connected region whose positively oriented boundary $\partial\Omega$ is a polygon with verticies $v_1, \dots, v_n$. Suppose the tangen direction on $\partial\Omega$ increases by $\pi\alpha_j$ at vertex v_j, $\alpha_j \in (-1, 1)$. Then there exists $x_1 < x_2 < \cdots < x_n$ and constants c_1, c_2 so that*

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^{n} (\zeta - x_j)^{-\alpha_j} \, d\zeta + c_2$$

is a conformal map of $\mathbb{H}$ onto Ω, where the integral is along any curve γ_z in $\mathbb{H}$ from i to z.

Theorem 0.32 (Residue Theorem $*$)**.** *Suppose f is analytic in Ω except for isolated singularities at $a_1, \ldots, a_n$. If γ is a cycle in Ω with $\gamma \sim 0$ and $a_j \notin \gamma$, $j = 1, \ldots, n$, then*

$$\int_\gamma f = 2\pi i \sum_k n(\gamma, a_k) Res_{a_k} f.$$

Theorem 0.33 (Arzela-Ascoli)**.** *A family $\mathcal{F}$ of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

(i) $\mathcal{F}$ is equicontinuous on Ω, and

(ii) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Theorem 0.34 (Normal analytic families)**.** *The following are equivalent for a family $\mathcal{F}$ of analytic functions on a region Ω*

(i) $\mathcal{F}$ is normal on Ω;

(ii) $\mathcal{F}$ is locally bounded on Ω;

(iii) $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Theorem 0.35 (Riemann Mapping Theorem $*$)**.** *Suppose $\Omega \subset \mathbb{C}$ is simply-connected and $\Omega \neq \mathbb{C}$. Then there exists a one-to-one map f of Ω onto $\mathbb{D}$. If $z_0 \in \Omega$, then there is a unique such map with $f(z_0) = 0$ and $f'(z_0) > 0$.*

Theorem 0.36 (Mittag-Leffler)**.** *Suppose $b_k \in \Omega \to \partial\Omega$ with $b_k \neq b_j$ if $k \neq j$. Set*

$$S_k(z) = \sum_{j=1}^{n_k} \frac{c_{j,k}}{(z - b_k)^j},$$

where each n_k is a positive integer and $c_{j,k} \in \mathbb{C}$. Then there is a function meromorphic in Ω with singular parts S_k at b_k, $k = 1, 2, \ldots,$ and no other singularities in Ω.

Theorem 0.37 (Weierstrass Product Theorem)**.** *Suppose Ω is a bounded region. If $\{b_j\} \subset \Omega$ with $b_j \to \partial\Omega$, and if n_j are positive integers, then there exists an analytic function f on Ω such that f has a zero of order exactly n_j at b_j, $j = 1, 2, \ldots,$ and no other zeros in Ω.*

Theorem 0.38 (Jensen's formula)**.** *Suppose f is meromorphic on $|z| \leq R$ with zeros $\{a_k\}$ and poles $\{b_j\}$. Suppose also that 0 is not a zero or a pole of f. Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(Re^{it})| \, dt = \log |f(0)| + \sum_{a_k < R} \log \frac{R}{|a_k|} - \sum_{b_j < R} \log \frac{R}{|b_j|}.$$

Theorem 0.39 (Monodromy)**.** *Suppose Ω is simply-connected and suppose f_0 is defined and anakytic in a neighbourhood of $b \in \Omega$. If f_0 can be analytically continued along all curves in Ω beginning at b then there is an analytic function f on Ω so that $f = f_0$ in a neighbourhood of b.*

Theorem 0.40 (Green's function $*$). *Suppose $p_0 \in W$ and suppose $z : U \to \mathbb{D}$ is a coordinate function such that $z(p_0) = 0$. If $g_W(p, p_0)$ is exists, then*

$$\begin{aligned} &g_W(p, p_0) > 0 \quad \textit{for } p \in W \setminus \{p_0\}, \ \textit{and,} \\ &g_W(p, p_0) + \log |z(p)| \quad \textit{extends to be harmonic in } U. \end{aligned}$$

Theorem 0.41 (Green is symmetric if $W = \mathbb{D}$). *Suppose W is a Riemann surface for which Green's function g_W with pole at p exists, for some $p \in W$, and suppose $W^* = \mathbb{D}$. Then g_W with pole at q exists for all $q \in W$, and*

$$g_W(p, q) = g_W(q, p).$$

Theorem 0.42 (Uniformisation case 1 $*$). *If W is a simply-connected Riemann surface then the following are equivalent:*

(i) $g_W(p, p_0)$ *exists for some $p_0 \in W$,*

(ii) $g_W(p, p_0)$ *exists for all $p_0 \in W$, and*

(iii) there is a one-to-one analytic map φ of W onto $\mathbb{D}$.

Moreover, if g_W exists, then

$$g_W(p_1, p_0) = g_W(p_0, p_1),$$

and $g_W(p, p_0) = -\log |\varphi(p)|$, where $\varphi(p_0) = 0$.

Theorem 0.43 (Green is symmetric general case). *Suppose W is a Riemann surface for which Green's function g_W with pole at p exists, for some $p \in W$. Then g_W with pole q exists, for all $q \in W$, and*

$$g_W(p, q) = g_W(q, p).$$

Theorem 0.44 (Uniformisation case 2). *Suppose W is a simply-connected Riemann surface for which Green's function does not exist. If W is compact, then there is a one-to-one analytic map of W onto $\mathbb{C}^*$. If W is not compact, then there is a one-to.one analytic moap of W onto $\mathbb{C}$.*