Examinable Definitions in SF2745 Advanced Complex Analysis

Gustaf Bjurstam
bjurstam@kth.se

Definition 0.1. Identify $\mathbb{C}$ with the plane $\{(x, y, 0) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3$. The **stereographic projection** of $z = x + iy$ is the unique point on the unit sphere intersecting the line defined by the two points $(x, y, 0)$ and $(0, 0, 1)$. Let $\pi : \mathbb{C} \to \mathbb{S}^2$ be the function which maps $z \in \mathbb{C}$ to its stereographic projection $z^* \in \mathbb{S}^2$. We then have

$$\pi(x + iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

Definition 0.2. The formal power series $f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ is called a **convergent power series centered at** z_0 if there is an $r > 0$ such that the series converges for all z such that $|z - z_0| < r$. The largest possible such r is called the **radius of convergence** of the series.

Definition 0.3. A function f is **analytic at** z_0 if f has a power series expansion valid in a neighbourhood of z_0.

Definition 0.4. A **region** is a connected open set.

Definition 0.5. The **mean-value property** for analytic functions f states that, if f is analytic at z_0 and the radius of convergence is $r_0 > 0$, then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$ for all $r < r_0$.

Definition 0.6. We say that f is **locally conformal** if it preserves angles (including direction) between curves.

Definition 0.7. Let $\zeta \in \partial\mathbb{D}$, if $\sum_{n \geq 0} a_z \zeta^n$ converges, and if Γ is any Stoltz angle at ζ, then we have **non-tangential convergence** if

$$\lim_{z \in \Gamma \to \zeta} \sum_{n \geq 0} a_n z^n = \sum_{n \geq 0} a_z \zeta^n.$$

Definition 0.8 (Curve integral)**.** If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentialbe curve, and if f is a complex valued function defined on (the image of) γ, then

$$\int_\gamma f(z)\, dz \equiv \int_a^b f(\gamma(t))\gamma'(t)\, dt. \tag{1}$$

Definition 0.9. If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentialbe curve, then the **length** of γ is defined to be $\ell(\gamma) = |\gamma| \int_\gamma |dz| = \int_a^b |\gamma'(t)|\, dt$.

Definition 0.10. A complex-valued function is said to be **holomorphic** on an open set U if

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for all $z \in U$ and is continuous on U. A function f is said to be holomorphic on a set S if it is holomorphic on an open set $U \supset S$.

Definition 0.11. If γ is a cycle, then the **index** or **winding number** of γ about a is

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - a},$$

for $a \notin \gamma$.

Definition 0.12. Closed curves γ_1 and γ_2 are **homologous** in a region Ω if $n(\gamma_1 - \gamma_2, a) = 0$ for all $a \notin \Omega$, an we then write $\gamma_1 \sim \gamma_2$.

Definition 0.13 (Classification of singularities)**.** If f has an isolated singularity at b, and $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - b)^n$ sufficiently close to b, then we say that

(a) b is a **removeable singularity** if $a_n = 0$ for $n < 0$,

(b) b is a **zero of order** $n_0 > 0$ if for $n < n_0$ we have $a_n = 0$,

(c) b is a **pole of order** $n_0 > 0$ if for $n < -n_0$ $a_n = 0$, and

(d) b is an essential singularity if for any $n_0 > 0$ there is $n < -n_0$ such that $a_n \neq 0$.

Definition 0.14. If f is analytic in a region Ω, except for at isolated poles in Ω, we say that f is meromorphic in Ω.

Definition 0.15. A **linear fractional transformation, LFT,** is a map of the form

$$T(z) = \frac{az + b}{cz + d}.$$

Definition 0.16. The **Cayley transform** is the map $C(z) = \frac{z-i}{z+i}$, and maps the upper half plane to the unit disc.

Definition 0.17. The **principal branch** of the logarithm is the one which takes arguments in $(-\pi, \pi)$. We may also take $\log(-1) = \pi i$.

Definition 0.18. The **Joukovski map** is the function $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

Definition 0.19. A function $u : \Omega \to \mathbb{R}$ is called **harmonic** on the region $\Omega \subset \mathbb{C}$ if, for each $z \in \Omega$, there is $r_z > 0$ such that, for all $r < r_z$

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\, dt.$$

That is, u satisfies the **mean value property**.

Definition 0.20. A function $u : \Omega \to [-\infty, \infty)$ is called **subharmonic** on the region $\Omega \subset \mathbb{C}$ if, for each $z \in \Omega$, there is $r_z > 0$ such that, for all $r < r_z$

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\, dt.$$

That is, u satisfies the **mean value inequality**.

Definition 0.21. The **Poisson kernel** is given by

$$P_z(t) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2},$$

and $u = PI(g) \equiv \int P_z(t) g(e^{it}) dt$ is the **Poisson integral of** g.

Definition 0.22. The kernel given by $f_z(t) = \frac{e^{it}+z}{e^{it}-z}$ is called the **Herglotz kernel**, the **Herglotz integral** is the function $\frac{1}{2\pi} \int f_z(t) u(e^{it}) dt$. When u is harmonic on $\mathbb{D}$, then the Herglotz integral is the unique analytc function with real part u and imaginary part 0 at $z = 0$.

Definition 0.23. If u is harmonic on a region Ω, then a **harmonic conjugate of** u is any function v such that $u + iv$ is analytic on Ω.

Definition 0.24. Let $I = (0, 1)$. An open analytic arc γ contained in the boundary of a region Ω is called a **one-sided arc** if there exists a function g which is one-to-one and analytic in a neighbourhood N of I, with $g(I) = \gamma$, and $g(N \cap \mathbb{H}) \subset \Omega$ and $g(N \setminus \overline{\mathbb{H}}) \subset \mathbb{C} \setminus \Omega$. If $g(N \setminus I) \subset \Omega$ then γ is called a **two-sided arc**.

Definition 0.25. If f is analytic in $\{z : 0 < |z - a| < \delta\}$ for some $\delta > 0$, then the **residue of** f **at** a, is the coefficient of $\frac{1}{z-a}$ in the Laurent series expansion of f about $z = a$.

Definition 0.26. A familily $\mathcal{F}$ of function on a region $\Omega \subset \mathbb{C}$ is said to be **normal on** Ω provided every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

Definition 0.27. A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is

(a) **equicontinuous at** $w \in E$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $z \in E$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$;

(b) **equicontinuous on** E if it is equicontinuous at each $w \in E$;

(c) **uniformly equicontinuous on** E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $z, w \in E$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$.

Definition 0.28. A family $\mathcal{F}$ of continuous functions is said to be **locally bounded** on Ω if for each $w \in \Omega$ there is a $\delta > 0$ and $M < \infty$ so that if $|z - w| < \delta$ then $|f(z)| < M$ for all $f \in \mathcal{F}$.

Definition 0.29. Suppose f is meromorphic in a region Ω, with a pole of order M at $b \in \Omega$. Then **the singular part of** f **at** b is $S_b(z) = \sum_{k=-M}^{-1} c_k(z-b)^k$, where $\sum_{k\in\mathbb{Z}} c_k(z-b)^k$ is the Laurent series for f at b.

Definition 0.30. Let $C(\partial\Omega)$ denote the set of continuous functions on the boundary in $\mathbb{C}^*$ of a region Ω. The **Dirichlet problem** on Ω for a function $f \in C(\partial\Omega)$ is to find a harmonic function u on Ω that is continuous on $\overline{\Omega}$ and equal to f on $\partial\Omega$.

Definition 0.31. A family $\mathcal{F}$ of subharmonic functions on a region Ω is called a **Perron family** if it satisfies

(i) if $v_1, v_2 \in \mathcal{F}$ then $\max(v_1, v_2) \in \mathcal{F}$,

(ii) if $v \in \mathcal{F}$ and D is a disc with $\overline{D} \subset \Omega$, and if $v > -\infty$ on ∂D, then $v_D \in \mathcal{F}$, and

(iii) for each $z \in \Omega$, there exists $v \in \mathcal{F}$ such that $v(z) > -\infty$.

Remark. For a subharmonic function v and a disc D centered at c and of radius r, v_D is defined by $v_D(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z-c|^2/r^2}{|e^{it}-(z-c)/r|^2} v(c + re^{it})\,dt$.

Definition 0.32. If $\Omega \subset \mathbb{C}^*$ is a region, and if f is a real-valued function on $\partial\Omega$ with $|f| \leq M < \infty$ on $\partial\Omega$, set

$$\mathcal{F}_f = \{v \text{ subharmonic on } \Omega : \limsup_{z\in\Omega\to\zeta} v_(z) \leq f(\zeta),\ \text{for all } \zeta \in \partial\Omega\}.$$

Then $u_f(z) \equiv \sup_{u\in\mathcal{F}_f} u(z)$ is harmonic in Ω. The function u_f is called the **Perron solution to the Dirichlet problem** on Ω for the function f.

Definition 0.33. If $\Omega \subset \mathbb{C}^*$ is a region and if $\zeta_0 \in \partial\Omega$ then b is called a **local barrier at ζ_0 for the region Ω** provided

(i) b is defined and is subharmonic on $\Omega \cap D$ for some open disc D containing ζ_0,

(ii) $b(z) < 0$ for $z \in \Omega \cap D$, and

(iii) $\lim_{z \in \Omega \to \zeta_0} b(z) = 0$.

Definition 0.34. If there exist a local barrier at $\zeta_0 \in \partial\Omega$ then ζ_0 is called a **regular point of $\partial\Omega$**. Otherwise ζ_0 is called an **irregular point of $\partial\Omega$**. If every $\zeta \in \partial\Omega$ is a regular point, then Ω is called a **regular region**.

Definition 0.35. If $\gamma : [0, 1] \to \mathbb{C}$ is a curve, and if f_0 is analytic in a neighbourhood of $\gamma(0)$, then an **analytic continuation of f_0 along γ** is a finite sequence $f_1, \ldots, f_n$ of functions where $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ is a partition of $[0, 1]$ and f_j is defined and analytic in a neighbourhood of $\gamma([t_j, t_{j+1}])$, $j = 0, \ldots, n$ such that $f_j = f_{j+1}$ in a neighbourhood of $\gamma(t_{j+1})$, $j = 0, \ldots, n-1$.

Definition 0.36. A **Riemann surface** is a connected Hausdorff space W, together with a collection of open subsets $U_\alpha \subset W$ and functions $z_\alpha : U_\alpha \to \mathbb{C}$ such that

(i) $W = \bigcup U_\alpha$,

(ii) z_α is a homeomorphism of U_α onto $\mathbb{D}$, and

(iii) if $U_\alpha \cap U_\beta \neq \emptyset$ then $z_\beta \circ z_\alpha^{-1}$ is analytic on $z_\alpha(U_\alpha \cap U_\beta)$.

The functions z_α are called **coordinate functions or maps** and the sets U_α are called **coordinate charts or discs**. Functions of the form $z_\beta \circ z_\alpha^{-1}$ are called **transition maps**.

Definition 0.37. Let W be a Riemann surface with charts (U_α, z_α). Fix some $b \in W$. Let $[\,\gamma\,]$ be the equivalence class of curves homotopic to a curve $\gamma \subset W$. Let

$$W^* = \Big\{ [\,\gamma\,] : \gamma(0) = b \Big\}.$$

Define $\pi : W^* \to W$ by $[\,\gamma\,] \mapsto \gamma(1)$. Let U_α be a chart at $c \in W$, γ be curve in W from b to c, and let

$$U_\alpha^* = \Big\{ [\,\gamma\sigma_d\,] : d \in U_\alpha \Big\}$$

where σ_d is a curve in U_α from c to d. Define $z_\alpha^* : U_\alpha^* \to \mathbb{D}$ by $z_\alpha^* = z_\alpha \circ \pi$. Give W^* a topology by declaring each U_α^* open. Then W^* is a Riemann surface with charts (U_α^*, z_α^*) called the **universal covering surface of W** and π is called the **universal covering map**.

Definition 0.38. If σ is a closed curve in W such that $\sigma(0) = \sigma(1) = b \in W$, then $M_{[\sigma]} : W^* \to W^*$ defined by $M_{[\sigma]}([\gamma]) = [\sigma\gamma]$ is called a **deck transformation**. The deck transformations form a group under composition, this group is called **the fundamental group of W at b**.

Definition 0.39. If W_1 and W_2 are Riemann surfaces, then $f : W_1 \to W_2$ is said to be **analytic** if $w_\beta \circ f \circ z_\alpha^{-1}$ is analyitc for each coordinate function z_α on W_1 and w_β on W_2, wherever it is defined.

Definition 0.40. Let W be a Riemann surface and fix some $p_0 \in W$. Let $z : U \to \mathbb{D}$ be a coordinate function such that $z(p_0) = 0$. Let $\mathcal{F}_{p_0}$ be the collection of subharmonic functions v on $W \setminus \{p_0\}$ satisfying $v = 0$ on $W \setminus K$ for some compact proper subset of W, and $\limsup_{p \to p_0}(v(p) + \log|z(p)|) < \infty$. Then F_{p_0} is a Perron family on $W \setminus \{p_0\}$. Set $g_w(p, p_0) = \sup\{v(p) : v \in \mathcal{F}_{p_0}\}$, then by Harnack's theorem we have two cases

(i) $g_W(p, p_0)$ is harmonic in $W \setminus \{p_0\}$, or

(ii) $g_W(p, p_0) = +\infty$ for all $p \in W \setminus \{p_0\}$.

In the first case we say that g_W is **Green's function on W with pole at** p_0, and in the second case **Green's function with pole at p_0 does not exist on W**.