Theorems from Brezis' *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (first edition)

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1 The Hahn-Banach Theorems. Introduction to Conjugate Convex Functions

Theorem 1.1 (Hahn-Banach, analytic). *Let E be a vector space over $\mathbb{R}$, and let $p : E \to \mathbb{R}$ be a Minkowski functional. Let G be a linear subspace of E and let $g : G \to \mathbb{R}$ be a linear functional such that $g(x) \leq p(x)$ for all $x \in G$. There exists a linear functional $f : E \to \mathbb{R}$ such that*

$$f(x) = g(x),\ \forall x \in G,$$

and

$$f(x) \leq p(x),\ \forall x \in E.$$

Lemma 1.1 (Zorn). *Every nonempty ordered set that is inductive has a maximal element.*

Corollary 1.2. *Let G be a linear subspace of the real vector space E. If $g \in G^\star$, then there exists $f \in E^\star$ that extends g and such that $\|f\|_{E^\star} = \|g\|_{G^\star}$.*

Corollary 1.3. *Let E be a normed real vector space. For every $x \in E$ there is $f \in E^\star$ such that $\|f\| = \|x\|$ and $\langle f, x\rangle = \|x\|^2$.*

Corollary 1.4. *For every x in the real normed vector space E we have*

$$\|x\| = \sup_{\substack{f \in E^\star \\ \|f\| \leq 1}} |\langle f, x\rangle| = \max_{\substack{f \in E^\star \\ \|f\| \leq 1}} |\langle f, x\rangle| .$$

Proposition 1.5. *Let E be a real normed vector space, and let $H = [f = \alpha] \subset E$ be an affine hyperplane. Then H is closed if and only f is continuous.*

Theorem 1.6 (Hahn-Banach, 1st geometric). *Let E be a normed vector space, and let $A, B \subset E$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume A is open, then there exists a closed hyperplane H that separates A and B.*

Lemma 1.2. *Let E be a normed vector space, and let $C \subset E$ be an open convex set with $0 \in C$. For every $x \in E$ set $p(x) = \inf\left\{\alpha : \alpha^{-1}x \in C\right\}$. Then p is a Minkowski functional. Furthermore, there is a constant M such that $0 \leq p(x) \leq M\|x\|$, and $C = \{x \in E : p(x) < 1\}$.*

Lemma 1.3. *Let E be a normed vector space, and let $C \subset E$ be an open convex set. Assume $x_0 \in E \setminus C$. Then there exists $f \in E^\star$ such that $f(x) < f(x_0)$ for all $x \in C$. In particular, $[f = f(x_0)]$ separates C and $\{x_0\}$.*

Theorem 1.7 (Hahn-Banach, 2nd geometric). *Let E be a normed vector space, and let $A, B \subset E$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume A is closed and B is compact, then there exists a closed hyperplane that strictly separates A and B.*

Corollary 1.8. *Let E be a normed vector space, and let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there is some $f \in E^\star \setminus \{0\}$ such that $\langle f, x\rangle = 0$ for all $x \in F$.*

Proposition 1.9. *Let E be a normed vector space, and suppose M is a linear subspace of E. Then $(M^\perp)^\perp = \overline{M}$. Furthermore, let N be a linear subset of $E^\star$, then $(N^\perp)^\perp \supset \overline{N}$.*