Definitions used in Romans's *Advanved Linear Algebra* (3rd ed)

Gustaf Bjurstam
bjurstam@kth.se

0 Selection of preliminaries

These are only a few of the definitions given in the Preliminaries chapter, those which were not immediately obvious to me, or those which I suspect will be used often. Zorn's lemma is also stated here.

Definition 0.1 (Multiset)**.** Let S be a nonempty set. A *multiset* M with *underlying set* S is a set of ordered pairs

$$M = \{(s_i, n_i) : s_i \in S, n_i \in \mathbb{Z}^+, s_i \neq s_j \text{ for } i \neq j\}.$$

The number n_i is referred to as the *multiplicity* of the elements s_i in M. The *size* of a multiset is the sum of the multiplicities of its elements.

Definition 0.2 (Invariant)**.** Let $\sim$ be an equivalence relation on a set S. A function $f : S \to T$, where T is any set, is called an *invariant* of $\sim$ if it is constant on the equivalence classes of $\sim$, that is,

$$a \sim b \Longrightarrow f(a) = f(b),$$

and a *complete invariant* if it is constant and distinct on the equivalence classes of $\sim$, that is,

$$a \sim b \Longleftrightarrow f(a) = f(b).$$

Definition 0.3 (Canonical forms)**.** Let $\sim$ be an equivalence relation on a set S. A subset $C \subseteq S$ is called a set of *canonical forms*, or just a *canonical form*, if each equivalence class under $\sim$ contains exactly one member of C.

Definition 0.4. (Partially ordered set) A *partially ordered set* is a pair $(P, \leq)$ where P is a nonempty set and $\leq$ is a binary relation called a *partial order* with the following properties,

1. **(Reflexivity)** For all $a \in P$, $a \leq a$,
2. **(Antisymmetry)** For all $a, b \in P$, $a \leq b \leq a$ implies $a = b$,
3. **(Transitivity)** for all $a, b, c \in P$, $a \leq b \leq c$ implies $a \leq c$.

Partially ordered sets are also called *posets*.

Definition 0.5. Let P be a partially ordered set.

1. The *maximum* (*largest, top*) element of P is, should it exist, an element $M \in P$ such that for all $p \in P$, $p \leq M$. Similarly, the *minimum* (*least, smallest, bottom*) element of P is, should it exist, an element $N \in P$ such that for all $p \in P$, $N \leq p$.
2. A *maximal element* is an element $m \in P$, such that there is no larger element in P, that is, $m \leq p \in P \implies p = m$. Similarly a *minimal element* is an element $n \in P$ such that there is no smaller element, that is, $p \leq n \implies p = n$.

3. Let $a, b \in P$. Then $u \in P$ is an *upper bound* for a and b, if $a \leq u$ and $b \leq u$. The unique smallest upper bound, if it exists, is called the *least upper bound* of a and b and is denoted by $\text{lub}\{(a, b\}$.

4. Let $a, b \in P$. Then $l \in P$ is an *lower bound* for a and b, if $l \leq a$ and $l \leq b$. The unique largest lower bound, if it exists, is called the *greatest lower bound* of a and b and is denoted by $\text{glb}\{(a, b\}$.

5. Let $S \subseteq P$. We say that $u \in P$ is an *upper bound* of S, if $s \leq u$ for all $s \in S$, lower bounds of S are similiarly defined.

Definition 0.6 (Totally ordered set). A partially ordered set is in which every pair of elements is compareable is called a *totally ordered set*, or a *linearly ordered set*. Any totally ordered subset of a partially ordered set P is called as *chain* in P.

Definition 0.7 (Well ordering). A *well ordering* on a set X, is a total order on X, with the property that every nonempty subset of X has a least element.

Lemma 0.1 (Zorn's lemma). *If P is a partially ordered set in which every chain has an upper bound, then P has a maximal element.*

Remark. Zorn's lemma is equivalent to the axiom of choice, and to the well ordering principle.

Theorem 0.1 (Well ordering principle). *Every nonempty set has a well ordering.*

Definition 0.8 (Algebra). An *algebra* over a field F is a nonempty set $\mathcal{A}$, together with three operations, *addition*, *multiplication*, and *scalar multiplication*, for which the following properties hold

1. $\mathcal{A}$ is a vector space over F under addition and scalar multiplication

2. A is a ring under addition and multiplication

3. If $r \in F$ and $a, b \in \mathcal{A}$, then $r(ab) = (ra)b = a(rb)$.

1 Vector spaces

Definition 1.1 (Vector space)**.** Let F be a field, whose elements are referred to as *scalars*. A *vector space* over F is a nonempty set V, whose elements are referred to as *vectors*, together with two operations. The first operation is called *addition*, and denoted by $+$, assigns each pair $(u, v) \in V^2$ a vector $u + v \in V$. The second operation, called *scalar multiplication*, assigns each pair $(r, u) \in F \times V$ a vector $ru \in V$. Furthermore, the following properties must be satisfied:

1. **(Associativity of addition)** For all vectors $u, v, w \in V$
$$u + (v + w) = (u + v) + w,$$
2. **(Commutativity of addition)** For all $u, v \in V$
$$u + v = v + u,$$
3. **(Existence of zero)** There is a vector $0 \in V$ such that for all $u \in V$
$$0 + u = u + 0 = u,$$
4. **(Existence of additive inverse)** For every $u \in V$ there exists a vector $-u$ such that
$$u + (-u) = (-u) + u = 0,$$
5. **(Properties of scalar multiplication)** For all scalars $a, b \in F$ and all $u, v \in V$
$$\begin{aligned} a(u + v) &= au + av \\ (a + b)u &= au + bu \\ (ab)u &= a(bu) \\ 1u &= u. \end{aligned}$$

Remark. Properties 1-4 can be summarised as V is an abelian group under addition.

Definition 1.2 (F-space)**.** A vector space over the field F is sometimes called an *F-space*. A vector space over $\mathbb{R}$ is called a *real vector space* and a vector space over $\mathbb{C}$ is called a *complex vector space*.

Definition 1.3 (Linear combination)**.** Let S be a nonempty subset of a vector space V over F. A *linear combination* in S is an expression of the form

$$a_1v_1 + \cdots + a_nv_n,$$

where $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in S$. The scalars a_i are called the *coefficients* of the linear combination. A linear combination is *trivial* if every coefficient is zero, otherwise it is *nontrivial.*

Definition 1.4 (Subspace)**.** A *subspace* of a vectors space V is a subset S of V that is a vector space in its own right under the operations obtained by restricting the operations of V to S. We use the notation $S \leq V$ to indicate that S is a subspace of V and $S < V$ to indicate that S is a *proper subspace* of V, that is, $S \leq V$ but $S \neq V$. The *zero subspace* of V is $\{0\}$.

Remark. The set $\mathcal{S}(V)$ of subspaces of V is partially ordered by set inclusion. If $A \subseteq \mathcal{S}(V)$, then $\text{glb}(A) = \bigcap_{S_i \in A} S_i$. Similarly $\text{lub}(A) = \sum_{S_i \in A} S_i$.

Definition 1.5. Let V be vector space and $A \subseteq \mathcal{S}(V)$. The *sum* $\sum_{S_i \in A} S_i$ is defined by

$$\sum_{S_i \in A} S_i = \left\{ s_1 + \cdots + s_n : s_j \in \bigcup_{S_i \in A} S_i \right\}.$$

Definition 1.6 (Lattice)**.** If P is a poset with the property that every pair of elements has a least upper bound and greatest lower bound, then P is called a *lattice.* If P has a smallest, and a largest, element, and has the property that every collection of elements has a least upper bound and greatest lower bound, then P is called a *complete lattice.* The glb of a collection is also called the *join* of the collection and the glb is called the *meet.*

Definition 1.7 (External direct sum)**.** Let $V_1, \ldots, V_n$ be vector spaces over a field F. The *external direct sum* of $V_1, \ldots, V_n$, denoted by $V = V_1 \boxplus \cdots \boxplus V_n$, is the vector space V whose elements are ordered n-tuples, $V = \{(v_1, \ldots, v_n) : v_i \in V_i, i = 1, 2, \ldots, n\}$, with component wise operations.

Definition 1.8 (Direct product)**.** Let $\mathcal{F} = \{V_i : i \in K\}$ be a family of vector spaces over a field F. The *direct product* of $\mathcal{F}$ is the vector space

$$\prod_{i \in K} V_i = \left\{ f : K \to \bigcup_{i \in K} V_i : f(i) \in V_i \right\}$$

thought of as a subspace of the vector space of all functions $f : K \to \bigcup V_i$.

Definition 1.9 (Support)**.** Let $\mathcal{F} = \{V_i : i \in K\}$ be a family of vector spaces over a field F. The *support* of a function $f : K \to \bigcup V_i$ is the set $\text{supp}(f) = \{i \in K : f(i) \neq 0\}$.

Definition 1.10 (External direct sum of family)**.** The *external direct sum* of the family $\mathcal{F} = \{V_i : i \in K\}$ of vector spaces, is the vector space

$$\bigoplus_{i \in K}^{\text{ext}} V_i = \left\{ f : K \to \bigcup V_i : f(i) \in V_i, f \text{ has finite support} \right\},$$

thought of as a subspace of the vector space of functions from K to $\bigcup V_i$.