

# Definitions used in Cohn's *Measure Theory* (second edition)

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## 1 Measures

**Definition 1.1** (Algebra). Let  $X$  be a set, an arbitrary collection of subsets  $\mathcal{A}$  of  $X$  is an *algebra* on  $X$  if

- (a)  $X \in \mathcal{A}$ ,
- (b) if  $A \in \mathcal{A}$  then  $A^c \in \mathcal{A}$ ,
- (c) for each finite sequence  $\{A_n\}_{n=1}^N$  of sets in  $\mathcal{A}$ , the set  $\bigcup_{n=1}^N A_n$  belongs to  $\mathcal{A}$ , and
- (d) for each finite sequence  $\{A_n\}_{n=1}^N$  of sets in  $\mathcal{A}$ , the set  $\bigcap_{n=1}^N A_n$  belongs to  $\mathcal{A}$ .

**Definition 1.2** ( $\sigma$ -Algebra). Let  $X$  be a set, an arbitrary collection of subsets  $\mathcal{A}$  of  $X$  is a  *$\sigma$ -algebra* on  $X$  if

- (a)  $X \in \mathcal{A}$ ,
- (b) if  $A \in \mathcal{A}$  then  $A^c \in \mathcal{A}$ ,
- (c) for each infinite sequence  $\{A_n\}_{n=1}^\infty$  of sets in  $\mathcal{A}$ , the set  $\bigcup_{n=1}^\infty A_n$  belongs to  $\mathcal{A}$ , and
- (d) for each infinite sequence  $\{A_n\}_{n=1}^\infty$  of sets in  $\mathcal{A}$ , the set  $\bigcap_{n=1}^\infty A_n$  belongs to  $\mathcal{A}$ .

**Definition 1.3** (Borel  $\sigma$ -algebra on  $\mathbb{R}^d$ ). The *Borel  $\sigma$ -algebra on  $\mathbb{R}^d$* , denoted  $\mathcal{B}(\mathbb{R}^d)$ , is generated by the collection of open subsets of  $\mathbb{R}^d$ . **Proposition 1.1.5** states that  $\mathcal{B}(\mathbb{R}^d)$  is generated by each of the collections of sets

- (a) the collection of all closed subsets of  $\mathbb{R}^d$ ;
- (b) the collection of all closed half-spaces in  $\mathbb{R}^d$  that have the form  $\{(x_1, \dots, x_d) : x_i \leq b\}$  for some  $b \in \mathbb{R}$ ;
- (c) the collection of all rectangles in  $\mathbb{R}^d$  that have the form

$$\{(x_1, \dots, x_d) : a_i < x_i \leq b_i \text{ for } i = 1, \dots, d\}.$$

**Definition 1.4** (Measure). Let  $\mathcal{A}$  be a  $\sigma$ -algebra. A function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is called *countably additive* if

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n),$$

for each infinite sequence of disjoint sets  $\{A_n\}_{n=1}^\infty$  in  $\mathcal{A}$ . If  $\mu$  in addition to being countably additive also satisfies  $\mu(\emptyset) = 0$ ,  $\mu$  is said to be a *measure* on  $\mathcal{A}$ .

**Definition 1.5** (Measure space). Let  $X$  be a set,  $\mathcal{A}$  a  $\sigma$ -algebra on  $X$  and  $\mu$  a measure on  $\mathcal{A}$ . The triplet  $(X, \mathcal{A}, \mu)$  is then called a *measure space*, the pair  $(X, \mathcal{A})$  is often called a *measurable space*.

**Definition 1.6** (Outer measure). Let  $X$  be a set, and let  $\mathcal{P}(X)$  be the power set of  $X$ . An *outer measure* on  $X$  is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  such that

- (a)  $\mu^*(\emptyset) = 0$ ,
- (b)  $A \subseteq B \subseteq X$  implies  $\mu^*(A) \leq \mu^*(B)$ , and
- (c) if  $\{A_n\}$  is an infinite sequence of sets in  $\mathcal{P}(X)$ , then  $\mu^*(\bigcup A_n) \leq \sum \mu^*(A_n)$ .

**Definition 1.7** (Lebesgue outer measure). *Lebesgue outer measure* on  $\mathbb{R}^d$  which we denote by  $\lambda^*$  is defined as follows. For each set  $A \subseteq \mathbb{R}^d$  define the set  $\mathcal{C}_A$  of all sequences  $\{R_n\}$  of bounded and open  $d$ -cells  $R_n$  such that  $A \subseteq \bigcup_{n=1}^{\infty} R_n$ . Then

$$\lambda^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \text{vol}(R_n) : \{R_n\} \in \mathcal{C}_A \right\}.$$

**Definition 1.8** ( $\mu^*$ -measurable set). Let  $X$  be a set, and let  $\mu^*$  be an outer measure on  $X$ . A subset  $B$  of  $X$  is  $\mu^*$ -*measurable* if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c),$$

for all  $A \subseteq X$ .

**Definition 1.9** (Complete measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space. The measure  $\mu$ , or the measure space  $(X, \mathcal{A}, \mu)$ , is called *complete* if  $A \in \mathcal{A}$ ,  $\mu(A) = 0$ , and  $B \subseteq A$  implies  $B \in \mathcal{A}$ .

**Definition 1.10** ( $\mu$ -negligible set). A subset  $B$  of  $X$  is called  $\mu$ -*negligible* or  $\mu$ -*null* if there exists  $A \in \mathcal{A}$  such that  $\mu(A) = 0$  and  $B \subseteq A$ . Thus  $(X, \mathcal{A}, \mu)$  is complete if and only if every  $\mu$ -negligible set belongs to  $\mathcal{A}$ .

**Definition 1.11** (Completion of a  $\sigma$ -algebra under a measure). Let  $(X, \mathcal{A})$  be a measurable space. The *completion* of  $\mathcal{A}$  under  $\mu$  is the collection  $\mathcal{A}_\mu$  of  $A \subseteq X$  for which there exists  $E, F \in \mathcal{A}$  such that

$$E \subseteq A \subseteq F,$$

and

$$\mu(F \setminus E) = 0.$$

A set that belongs to  $\mathcal{A}_\mu$  is sometimes said to be  $\mu$ -*measurable*.

**Definition 1.12** (Completion of a measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space. The *completion* of  $\mu$  is defined as  $\bar{\mu} : \mathcal{A}_\mu \rightarrow [0, \infty]$  by letting  $\bar{\mu}(A)$  be the common value of  $\mu(E), \mu(F)$ , defined in the above definition.

**Definition 1.13** (Inner and outer measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $A$  be an arbitrary subset of  $X$ . The *inner measure*  $\mu_*$  of  $A$  is defined by

$$\mu_*(A) = \sup \{ \mu(B) : B \subseteq A \text{ and } B \in \mathcal{A} \}.$$

The *outer measure*  $\mu^*$  of  $A$  meanwhile, is defined by

$$\mu^*(A) = \inf \{ \mu(B) : A \subseteq B \text{ and } B \in \mathcal{A} \}.$$

**Remark.** According to **Proposition 1.5.4**, the outer measure defined in **Definition 1.13** satisfies the conditions placed on an outer measure in **Definition 1.6**.

**Definition 1.14** (Regular measure). Let  $\mathcal{A}$  be a  $\sigma$ -algebra on  $\mathbb{R}^d$  that includes  $\mathcal{B}(\mathbb{R}^d)$ . A measure  $\mu$  on  $\mathcal{A}$  is regular if

- (a) each compact subset  $K$  of  $\mathbb{R}^d$  satisfies  $\mu(K) < \infty$ ,
- (b) each set  $A$  in  $\mathcal{A}$  satisfies

$$\mu(A) = \inf \{ \mu(U) : U \text{ is open and } A \subseteq U \}, \text{ and}$$

- (c) each open subset  $U$  of  $\mathbb{R}^d$  satisfies

$$\mu(U) = \sup \{ \mu(K) : K \text{ is compact and } K \subseteq U \}.$$

**Definition 1.15** (Dynkin class). Let  $X$  be a set. A collection  $\mathcal{D}$  is a *d-system*, or *Dynkin class*, on  $X$  if

- (a)  $X \in \mathcal{D}$ ,
- (b)  $A \setminus B \in \mathcal{D}$  whenever  $A, B \in \mathcal{D}$  and  $A \supseteq B$ , and
- (c)  $\bigcup A_n \in \mathcal{D}$  whenever  $\{A_n\}$  is an increasing sequence of sets in  $\mathcal{D}$ .

**Definition 1.16** ( $\pi$ -system). A collection of subsets of  $X$  is a  $\pi$ -system if it is closed under the formation of finite unions.

## 2 Functions and Integrals

**Definition 2.1** ( $\mathcal{A}$ -measurable function). Let  $(X, \mathcal{A})$  be a measurable space, and let  $A \in \mathcal{A}$ . A function  $f : A \rightarrow [-\infty, \infty]$  is *measurable with respect to  $\mathcal{A}$*  if it satisfies any of the conditions, and thus all, of the conditions in **Proposition 2.1.1**. That is any of

- (a)  $\forall t \in \mathbb{R} \quad \{x \in A : f(x) \leq t\} \in \mathcal{A},$
- (b)  $\forall t \in \mathbb{R} \quad \{x \in A : f(x) < t\} \in \mathcal{A},$
- (c)  $\forall t \in \mathbb{R} \quad \{x \in A : f(x) \geq t\} \in \mathcal{A},$
- (d)  $\forall t \in \mathbb{R} \quad \{x \in A : f(x) > t\} \in \mathcal{A}.$

A function that is measurable with respect to  $\mathcal{A}$  may be called  *$\mathcal{A}$ -measurable* or if what  $\sigma$ -algebra is meant is obvious from context, simply *measurable*. In the case  $X = \mathbb{R}^d$  functions measurable with respect to  $\mathcal{B}(\mathbb{R}^d)$  are called *Borel measurable* or *Borel functions*. A function measurable with respect to  $\mathcal{M}_{\lambda^*}$  is called *Lebesgue measurable*.

**Definition 2.2** (Almost everywhere). Let  $(X, \mathcal{A}, \mu)$  be a measure space. A property of points on  $X$  is said to hold  *$\mu$ -almost everywhere* if the set of points in  $X$  where it fails to hold is  $\mu$ -negligible. The expression  $\mu$ -almost everywhere is often abbreviated  $\mu$ -a.e. or to a.e. $[\mu]$ . If the measure is clear from context one may simply say *almost everywhere*.

### 2.1 Construction of the integral

**Definition 2.3** (Integral of a simple non-negative function). Let  $\mu$  be a measure on  $(X, \mathcal{A})$ . If  $f$  is a real-valued, simple,  $\mathcal{A}$ -measurable function given by  $f = \sum_{i=1}^m a_i \chi_{A_i}$ , where each  $a_i \geq 0$  and  $A_i \in \mathcal{A}$  are disjoint. Then the *integral of  $f$  with respect to  $\mu$*  is then defined to be

$$\int f d\mu = \sum_{i=1}^m a_i \mu(A_i).$$

**Definition 2.4** (Integral of arbitrary  $\mathcal{A}$ -measurable, non-negative function). Let  $f$  be an arbitrary  $\mathcal{A}$ -measurable function, with image in  $[0, \infty]$ . The integral of  $f$  is then defined as

$$\int f d\mu = \sup \left\{ \int g d\mu : g \in \mathcal{S}_+ \text{ and } g \leq f \right\}.$$

**Definition 2.5** (Integral of arbitrary measurable function). Let  $f : X \rightarrow [-\infty, \infty]$  be a measurable function on  $(X, \mathcal{A}, \mu)$ . If  $\int f^+ d\mu$  and  $\int f^- d\mu$  are both finite, then  $f$  is called *integrable* and its *integral* is defined by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

The integral of  $f$  is said to *exist* if at least one of  $\int f^+ d\mu$  and  $\int f^- d\mu$  is finite, in this case the integral is defined  $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$ .

**Definition 2.6** (Integral over a subset). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $f : X \rightarrow [-\infty, \infty]$  be  $\mathcal{A}$ -measurable. The integral of  $f$  over a subset  $A \subseteq X$  is said to exist if the integral of  $f\chi_A$  exists. In that case the integral over  $A$  is defined to be

$$\int_A f d\mu = \int f\chi_A d\mu.$$

Likewise, if  $A \in \mathcal{A}$  and  $f : A \rightarrow [-\infty, \infty]$  is  $\mathcal{A}$ -measurable, then the integral of  $f$  over  $A$  is defined to be the integral of the function which agrees with  $f$  on  $A$  and vanishes on  $A^c$ .

**Definition 2.7** (Lebesgue integral). The case  $X = \mathbb{R}^d$  and  $\mu = \lambda$  we simply talk about *Lebesgue integrability* and the *Lebesgue integral*. We may use any of the following notations for the *Lebesgue integral* over an interval  $[a, b]$

$$\int_a^b f = \int_a^b f(x) dx = (L) \int_a^b f = (L) \int_a^b f(x) dx,$$

where the latter two are used to emphasise that we are talking about the Lebesgue integral.

**Definition 2.8** ( $\mathcal{L}^1$ ). We define  $\mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R})$ , or sometimes simply  $\mathcal{L}^1$ , as the set of all integrable functions  $f : X \rightarrow \mathbb{R}$ . (As opposed to  $[-\infty, \infty]$ -valued functions.)

## 2.2 Measurable functions again

**Definition 2.9** (Measurable function between sets). Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. A function  $f : X \rightarrow Y$  is *measurable with respect to  $\mathcal{A}$  and  $\mathcal{B}$*  if for each  $B \in \mathcal{B}$  the set  $f^{-1}(B)$  belongs to  $\mathcal{A}$ . In stead of saying measurable with respect to  $\mathcal{A}$  and  $\mathcal{B}$ , we may say that  $f$  is a *measurable function* from  $(X, \mathcal{A})$  to  $(Y, \mathcal{B})$ , or simply that  $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  is *measurable*.

**Definition 2.10** (Integral of complex-valued function). Let  $(X, \mathcal{A}, \mu)$  be a measure space. A complex-valued function  $f$  on  $X$  is *integrable* if its real and imaginary parts  $\Re(f)$  and  $\Im(f)$  are integrable; if  $f$  is integrable then its *integral* is defined by

$$\int f d\mu = \int \Re(f) d\mu + i \int \Im(f) d\mu.$$

**Definition 2.11** ( $\mu f^{-1}$ ). Let  $(X, \mathcal{A}, \mu)$  be a measure space, let  $(Y, \mathcal{B})$  be a measurable space, and let  $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  be measurable. Define  $\mu f^{-1} : \mathcal{B} \rightarrow [0, \infty]$  by  $\mu f^{-1}(B) = \mu(f^{-1}(B))$ . It is easy to show that  $\mu f^{-1}$  is a measure, this measure is sometimes called the *image of  $\mu$  under  $f$* .

### 3 Convergence

**Definition 3.1** (Convergence in measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $f$  and  $f_1, f_2, \dots$  be real valued  $\mathcal{A}$ -measurable functions on  $X$ . The sequence  $\{f_n\}$  converges to  $f$  *in measure* if

$$\lim_n \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0$$

for every  $\varepsilon > 0$ .

**Definition 3.2** (Almost uniform convergence). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $f$  and  $f_1, f_2, \dots$  be real valued  $\mathcal{A}$ -measurable functions on  $X$ . Then  $\{f_n\}$  converges to  $f$  *almost uniformly* if for all  $\varepsilon > 0$  there is  $B \in \mathcal{A}$  such that  $\{f_n\}$  converges to  $f$  on  $B$  and  $\mu(B^c) < \varepsilon$ .

**Definition 3.3** (Convergence in mean). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $f$  and  $f_1, f_2, \dots$  be real valued  $\mathcal{A}$ -measurable functions on  $X$ . Then  $\{f_n\}$  converges to  $f$  *in mean* if

$$\lim_n \int |f_n - f| d\mu = 0.$$

#### 3.1 Normed spaces

**Definition 3.4** (Norm & seminorm). Let  $V$  be a vector space over  $\mathbb{C}$ . A *norm* on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}$  that satisfies

- (a)  $\|v\| \geq 0$ ,
- (b)  $\|v\| = 0 \iff v = 0$ ,
- (c)  $\|\alpha v\| = |\alpha| \|v\|$ ,
- (d)  $\|u + v\| \leq \|u\| + \|v\|$

for each  $u, v \in V$  and  $\alpha \in \mathbb{C}$ . If condition (b) was replaced by " $\|v\| = 0 \iff v = 0$ "  $\|\cdot\|$  is a *seminorm*.

**Definition 3.5** (Metric & semimetric). A *metric* on a set  $S$  is a function  $d : S \times S \rightarrow \mathbb{R}$  that satisfies

- (a)  $d(s, t) \geq 0$ ,
- (b)  $d(s, t) = 0 \iff s = t$ ,
- (c)  $d(s, t) = d(t, s)$ ,
- (d)  $d(r, t) \leq d(r, s) + d(s, t)$

for all  $r, s, t \in S$ . If condition (b) is replaced by " $d(s, t) = 0 \iff s = t$ "  $d$  is a *semimetric*. A *metric space* is a set  $S$  together with a metric  $d$  on  $S$ . This may, if there is no risk for confusion with a measurable space, be written as  $(S, d)$ .

**Definition 3.6** (Converging sequence). Let  $(S, d)$  be a metric (or semimetric) space, a sequence  $\{s_n\}$  in  $S$  is said to *converge* to  $s \in S$  if for all  $\varepsilon > 0$  there exists  $N$  such that  $\forall n \geq N$   $d(s_n, s) \leq \varepsilon$ . The point  $s$  is then said to be the *limit point* of  $\{s_n\}$ . In particular,

if  $V$  is a normed linear space,  $v \in V$  and  $\{v_n\}$  is a sequence in  $V$ , then  $\{v_n\}$  converges to  $v$  (with respect to the metric induced by the norm on  $V$ ) if and only if  $\lim_n \|v_n - v\| = 0$ . Note that if  $d$  is a semimetric  $\{s_n\}$  may have several limit points.

**Definition 3.7** (Dense subset). Let  $(S, d)$  be a metric (or semimetric) space, a subset  $A \subseteq S$  is said to be *dense* in  $S$  if for all  $s \in S$  and  $\varepsilon > 0$  there exists  $a \in A$  such that  $d(s, a) < \varepsilon$ .

**Definition 3.8** (Separable space). Let  $(S, d)$  be a metric (or semimetric) space, if  $S$  has a countable dense subset,  $S$  is *separable*.

**Definition 3.9** (Cauchy sequences and completeness). Let  $(S, d)$  be a metric space, a *Cauchy sequence* is a sequence  $\{s_n\}$  in  $S$  such that for all  $\varepsilon > 0$  there exists  $N$  such that for all  $n, m \geq N$ ,  $d(s_n, s_m) < \varepsilon$ . A metric space  $(S, d)$  is said to be *complete* if all Cauchy sequences in  $(S, d)$  converge.

**Definition 3.10** (Banach space). If a normed linear space is complete, with respect to the metric induced by the norm on the space, then it is called a *Banach space*.

**Definition 3.11** (Inner product). Let  $V$  be a vector space over  $\mathbb{C}$ . A function  $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$  is an *inner product* on  $V$  if

- (a)  $(x, x) \geq 0$ ,
- (b)  $(x, x) = 0 \iff x = 0$ ,
- (c)  $(x, y) = \overline{(y, x)}$ , and
- (d)  $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

hold for all  $x, y, z \in V$  and  $\alpha, \beta \in \mathbb{C}$ . An *inner product space* is a vector space, together with an inner product. The *norm*  $\|\cdot\|$  associated to the inner product  $(\cdot, \cdot)$  is defined by  $\|x\| = \sqrt{(x, x)}$ .

**Definition 3.12** (Hilbert space). An inner product space that is complete under the norm associated with the inner product is called a *Hilbert space*.

## 3.2 $\mathcal{L}^p$ and $L^p$

**Definition 3.13** ( $\mathcal{L}^p$ ). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $p \in [1, \infty)$ . Then  $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{R})$  is the set of all  $\mathcal{A}$ -measurable functions  $f : X \rightarrow \mathbb{R}$  such that  $|f|^p$  is integrable, and  $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{C})$  is the set of  $\mathcal{A}$ -measurable functions  $f : X \rightarrow \mathbb{C}$  such that  $|f|^p$  is integrable.

**Definition 3.14** ( $\mathcal{L}^\infty$ ). Let  $(X, \mathcal{A}, \mu)$  be a measure space. We define  $\mathcal{L}^\infty(X, \mathcal{A}, \mu, \mathbb{R})$  to be the set of all<sup>1</sup> bounded real-valued  $\mathcal{A}$ -measurable functions, and  $\mathcal{L}^\infty(X, \mathcal{A}, \mu, \mathbb{C})$  as the set of all bounded complex-valued  $\mathcal{A}$ -measurable functions.

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<sup>1</sup>I think it's supposed to be almost everywhere bounded functions, otherwise exercise 3.3.7 fails with this definition (however not with the alternative definition).

**Remark.** Some authors<sup>2</sup> define  $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$  as the set of all *essentially bounded*  $\mathcal{A}$ -measurable functions on  $X$ . A function  $f : X \rightarrow \mathbb{C}$  is *essentially bounded* if there exists  $M > 0$  such that  $\{x \in X : |f(x)| > M\}$  is locally  $\mu$ -null. For most purposes, it does not matter which definition of  $\mathcal{L}^\infty$  one uses. However the study of liftings is convenient with **Definition 3.14**.

**Definition 3.15** (Locally  $\mu$ -null). Let  $(X, \mathcal{A}, \mu)$  be a measure space. A subset  $N \subseteq X$  is said to be *locally  $\mu$ -null* if for each  $A \in \mathcal{A}$  that satisfies  $\mu(A) < \infty$  the set  $A \cap N$  is  $\mu$ -null. A property is said to hold *locally almost everywhere* if the set on which the property doesn't hold is locally  $\mu$ -null.

**Definition 3.16** (Seminorm on  $\mathcal{L}^p$ ). In the case of  $p \in [1, \infty)$  we define a seminorm  $\|\cdot\|_p : \mathcal{L}^p(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$  by

$$\|f\|_p = \left( \int |f|^p d\mu \right)^{1/p}.$$

In the case  $p = \infty$  we define a seminorm  $\|\cdot\|_\infty : \mathcal{L}^\infty(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$  by

$$\|f\|_\infty = \inf \{M : \{x \in X : |f(x)| > M\} \text{ is locally } \mu\text{-null}\}.$$

**Definition 3.17** ( $\mathcal{N}^p$ ). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $\mathcal{N}^p(X, \mathcal{A}, \mu)$  be the subset of  $\mathcal{L}^p(X, \mathcal{A}, \mu)$  which consists of the functions  $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$  such that  $\|f\|_p = 0$ . That is, if  $p \in [1, \infty)$ , then  $\mathcal{N}^p(X, \mathcal{A}, \mu)$  is the set of functions in  $\mathcal{L}^p(X, \mathcal{A}, \mu)$  which vanish almost everywhere, and if  $p = \infty$  then  $\mathcal{N}^\infty(X, \mathcal{A}, \mu)$  is the set of functions in  $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$  which vanish locally almost everywhere.

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<sup>2</sup>Notably, the first edition of Cohn's *Measure Theory* uses this definition.



**Definition 3.18** ( $L^p$ ). Let  $(X, \mathcal{A}, \mu)$  be a measure space. We define  $L^p(X, \mathcal{A}, \mu)$  to be the quotient group  $\mathcal{L}^p(X, \mathcal{A}, \mu)/\mathcal{N}^p(X, \mathcal{A}, \mu)$ . That is  $L^p(X, \mathcal{A}, \mu)$  is the collection of cosets of  $\mathcal{N}^p(X, \mathcal{A}, \mu)$  in  $\mathcal{L}^p(X, \mathcal{A}, \mu)$ ; these cosets are by definition the equivalence classes induced by the equivalence relation  $\sim$ , where  $f \sim g$  holds if and only if  $f - g \in \mathcal{N}^p(X, \mathcal{A}, \mu)$ . Then if  $p \in [1, \infty)$ ,  $f \sim g \iff f = g$  almost everywhere.

**Definition 3.19** (Norm on  $L^p$ ). Let  $(X, \mathcal{A}, \mu)$  be a measure space. For each  $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$  let  $\langle f \rangle$  be the coset of  $\mathcal{N}^p(X, \mathcal{A}, \mu)$  in  $\mathcal{L}^p(X, \mathcal{A}, \mu)$  to which  $f$  belongs. Then  $L^p(X, \mathcal{A}, \mu)$  is a vector space and we can define a norm  $\|\cdot\|_p : L^p(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$  by  $\|\langle f \rangle\|_p = \|f\|_p$ , where on the right hand side  $\|\cdot\|_p : \mathcal{L}^p(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$  is given in **Definition 3.16**.

**Definition 3.20** (Convergence in  $p$ th mean). Let  $(X, \mathcal{A}, \mu)$  be a measure space, let  $p \in [1, \infty)$ , and let  $f, f_1, f_2, \dots \in \mathcal{L}^p(X, \mathcal{A}, \mu)$ . Then  $\{f_n\}$  converges to  $f$  in  $p$ th mean, or in  $L^p$  norm, if  $\lim_n \|f_n - f\|_p = 0$ .

### 3.3 Dual Spaces

**Definition 3.21** (Linear operator). Let  $V_1, V_2$  be normed vector spaces over  $\mathbb{C}$  (or over  $\mathbb{R}$ ), then a function  $T : V_1 \rightarrow V_2$  is a *linear operator* or *linear transformation* if  $T(\alpha v) = \alpha T(v)$  and  $T(u + v) = T(u) + T(v)$  hold for all  $\alpha \in \mathbb{C}$  (or  $\mathbb{R}$ ) and all  $u, v \in V_1$ .

**Definition 3.22** (Bounded linear operator). Let  $V_1, V_2$  be normed vector spaces, and let  $T : V_1 \rightarrow V_2$  be linear. Then a nonnegative number  $A$  such that  $\|T(v)\| \leq A \|v\|$  holds for every  $v \in V_1$  is called a *bound* for  $T$ , and the operator  $T$  is called *bounded* if there is a bound for it.

**Definition 3.23** (Norm of linear operator). Let  $T : V_1 \rightarrow V_2$  be a bounded linear operator, we define the *norm* of  $T$  by

$$\|T\| = \inf\{A : A \text{ is a bound for } T\}.$$

Then  $\|\cdot\|$  is a norm on the vector space of bounded linear operators from  $V_1$  to  $V_2$ .

**Definition 3.24** (Isometry). Let  $T : V_1 \rightarrow V_2$  be a linear operator between normed linear spaces. Then  $T$  is called an *isometry* if  $\|T(v)\| = \|v\|$  for every  $v \in V_1$ .

**Definition 3.25** (Isometric isomorphism). Let  $T : V_1 \rightarrow V_2$  be a linear operator between normed linear spaces. Then  $T$  is an *isometric isomorphism* if  $T$  is an isometry and is surjective. Because all isometries are injective,  $T$  is then bijective.

**Definition 3.26** (Linear functional). Let  $V$  be a normed linear space. A *linear functional* on  $V$  is a linear operator on  $V$  whose values lie in  $\mathbb{C}$ , if  $V$  is a vector space over  $\mathbb{C}$ , or in  $\mathbb{R}$ , if  $V$  is a vector space over  $\mathbb{R}$ .

**Definition 3.27** (Dual space). Let  $V$  be a normed linear space. The set of all bounded, and hence continuous, linear functionals on  $V$  then form a vector space. This vector space is called the *dual space* (or *conjugate space*) of  $V$ , and is denoted by  $V^*$ . Note that the function  $\|\cdot\| : V^* \rightarrow \mathbb{R}$  which assigns to each functional in  $V^*$  its norm, is in fact a norm on the vector space  $V^*$ .

## 4 Signed and Complex measures

**Definition 4.1** (Signed measure). Let  $(X, \mathcal{A})$  be a measurable space. A function  $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$  is called a *signed measure* if it is countably additive and satisfies  $\mu(\emptyset) = 0$ .

**Definition 4.2** (Positive & negative sets). Let  $\mu$  be a signed measure on a measurable space  $(X, \mathcal{A})$ . A set  $A \in \mathcal{A}$  is a *positive set* if every  $B \in \mathcal{A}$  such that  $B \subseteq A$  satisfies  $\mu(B) \geq 0$ . Likewise, a set  $A \in \mathcal{A}$  is a *negative set* if every  $B \in \mathcal{A}$  such that  $B \subseteq A$  satisfies  $\mu(B) \leq 0$ .

**Definition 4.3** (Hahn decomposition). A *Hahn decomposition* of a signed measure  $\mu$  on the measurable space  $(X, \mathcal{A})$  is a pair  $(P, N)$  of disjoint subsets in  $\mathcal{A}$  such that  $X = P \cup N$ , and  $P$  is a positive set and  $N$  is a negative set. Note that there may be several Hahn decomposition of the signed measure  $\mu$ .

**Definition 4.4** (Complex measure). Let  $(X, \mathcal{A})$  be a measurable space. A *complex measure* is a function  $\mu : \mathcal{A} \rightarrow \mathbb{C}$  that satisfies  $\mu(\emptyset) = 0$  and is countably additive. A complex measure  $\mu$  can be written as  $\mu = \mu' + i\mu''$  where  $\mu'$  and  $\mu''$  are finite signed measures.

**Definition 4.5** (Jordan decomposition). Let  $\mu$  be a signed measure on the measurable space  $(X, \mathcal{A})$ , and let  $(P, N)$  be a Hahn decomposition of  $\mu$ . Let  $\mu^+(A) = \mu(A \cap P)$  and  $\mu^-(A) = -\mu(A \cap N)$ , then  $\mu^+, \mu^-$  are measures on  $(X, \mathcal{A})$  and  $\mu = \mu^+ - \mu^-$ . The measures  $\mu^+$  and  $\mu^-$  are called the *positive part* and *negative part* of  $\mu$ , respectively. The representation  $\mu = \mu^+ - \mu^-$  is called the *Jordan decomposition* of the signed measure  $\mu$ . If  $\mu$  is a complex measure on  $(X, \mathcal{A})$  then the representation  $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$  is called the *Jordan decomposition* of  $\mu$ , if  $\mu' = \mu_1 - \mu_2$  and  $\mu'' = \mu_3 - \mu_4$  are the Jordan decompositions of the real and imaginary parts of  $\mu$ .

**Definition 4.6** (Variation). If  $\mu$  is a signed measure on the measurable space  $(X, \mathcal{A})$ , then the *variation* of  $\mu$  is defined to be  $|\mu| = \mu^+ + \mu^-$ , and the *total variation* of  $\mu$  is defined to be  $\|\mu\| = |\mu|(X)$ . If  $\mu$  is a complex measure on  $(X, \mathcal{A})$ , then the *variation* of  $\mu$  is defined by

$$|\mu|(A) = \sup \left\{ \sum_{j=1}^n |\mu(A_j)| : \{A_j\}_{j=1}^n \text{ are finite disjoint sequences in } \mathcal{A} \text{ such that } A = \bigcup_{j=1}^n A_j \right\}.$$

The total variation of  $\mu$  is defined to be  $\|\mu\| = |\mu|(X)$ .

**Definition 4.7.** Let  $(X, \mathcal{A})$  be a measurable space. Define  $M(X, \mathcal{A}, \mathbb{R})$  as the set of all finite signed measures on  $(X, \mathcal{A})$ , and  $M(X, \mathcal{A}, \mathbb{C})$  as the set of all complex measures on  $(X, \mathcal{A})$ . It is easy to see that  $M(X, \mathcal{A}, \mathbb{R})$  and  $M(X, \mathcal{A}, \mathbb{C})$  are vector spaces over  $\mathbb{R}$  and  $\mathbb{C}$  respectively, and that the total variation gives a norm on each of them.

**Definition 4.8** (Integration with signed measure). Let  $(X, \mathcal{A})$  be a measurable space. Denote by  $B(X, \mathcal{A}, \mathbb{R})$  the vector space of bounded real-valued  $\mathcal{A}$ -measurable functions on  $X$ . If  $\mu$  is a finite signed measure on  $(X, \mathcal{A})$ , and  $\mu = \mu^+ - \mu^-$  is the Jordan decomposition of  $\mu$ , and if  $f \in B(X, \mathcal{A}, \mathbb{R})$ , then the *integral of  $f$  with respect to  $\mu$*  is defined as

$$\int f d\mu = \int f d\mu^+ - \int f d\mu^-.$$

**Definition 4.9** (Integration with complex measure). Let  $(X, \mathcal{A})$  be a measurable space. Denote by  $B(X, \mathcal{A}, \mathbb{C})$  the vector space of bounded complex-valued  $\mathcal{A}$ -measurable functions on  $X$ . If  $\mu$  is a complex measure on  $(X, \mathcal{A})$ , and  $\mu_1, \mu_2$  are the real and imaginary parts of  $\mu$ , and if  $f \in B(X, \mathcal{A}, \mathbb{C})$ , then the *integral of  $f$  with respect to  $\mu$*  is defined by

$$\int f d\mu = \int f d\mu_1 + i \int f d\mu_2.$$

**Remark.** The formula  $f \mapsto \int f d\mu$  and  $\mu \mapsto \int f d\mu$  define a linear functionals on  $B(X, \mathcal{A})$  and  $M(X, \mathcal{A})$  respectively.

**Definition 4.10** (Absolute continuity). Let  $(X, \mathcal{A})$  be a measurable space, and let  $\mu$  and  $\nu$  be measures on  $(X, \mathcal{A})$ . We say that  $\nu$  is *absolutely continuous with respect to  $\mu$*  if every  $A \in \mathcal{A}$  such that  $\mu(A) = 0$  also satisfies  $\nu(A) = 0$ . This is sometimes indicated as  $\nu \ll \mu$ . A measure  $\nu$  on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  is called absolutely continuous if  $\nu \ll \lambda$ .

**Definition 4.11** (Absolute continuity of signed or complex measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space. A signed or complex measure  $\nu$  on  $(X, \mathcal{A})$  is *absolutely continuous with respect to  $\mu$* , written  $\nu \ll \mu$ , if the variation  $|\nu|$  is absolutely continuous with respect to  $\mu$ .

**Definition 4.12** (Radon-Nikodym derivative). Let  $(X, \mathcal{A})$  be a measurable space, let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{A})$  and let  $\nu$  be a, finite signed, complex, or  $\sigma$ -finite, measure on  $(X, \mathcal{A})$  such that  $\nu \ll \mu$ . A function  $g$  such that  $\nu(A) = \int_A g d\mu$  hold for every  $A \in \mathcal{A}$  is called a *Radon-Nikodym derivative* of  $\nu$  with respect to  $\mu$ , or in light of the  $\mu$ -almost uniqueness of such  $g$ , *the Radon-Nikodym derivative* of  $\nu$  with respect to  $\mu$ . The Radon-Nikodym derivative of  $\nu$  is sometimes denoted  $\frac{d\nu}{d\mu}$ .

**Definition 4.13** (Concentrated measure). Let  $(X, \mathcal{A})$  be a measurable space, a measure  $\mu$  is *concentrated on  $E \in \mathcal{A}$*  if  $\mu(E^c) = 0$ . A signed or complex measure  $\mu$  is said to be concentrated on  $E$  if  $|\mu|(E^c) = 0$ .

**Definition 4.14** (Singularity). Let  $(X, \mathcal{A})$  be a measurable space, let  $\mu$  and  $\nu$  be positive, signed, or complex measures on  $(X, \mathcal{A})$ . Then  $\mu$  and  $\nu$  are called *mutually singular* if there exists  $E \in \mathcal{A}$  such that  $\mu$  is concentrated on  $E$  and  $\nu$  is concentrated on  $E^c$ . That two measures are mutually singular is sometimes denoted  $\mu \perp \nu$ . Sometimes the statement  $\mu$  and  $\nu$  are mutually singular is said,  $\mu$  and  $\nu$  are singular,  $\mu$  is singular with respect to  $\nu$ , or that  $\nu$  is singular with respect to  $\mu$ . A positive, signed, or complex measure on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  is simply said to be *singular* if it is singular with respect to the  $d$ -dimensional Lebesgue measure on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ .

**Definition 4.15** (Lebesgue decomposition). Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $\nu$  be a finite signed, complex, or  $\sigma$ -finite positive measure on  $(X, \mathcal{A})$ . There are unique finite signed, complex, or  $\sigma$ -finite measures  $\nu_a$  and  $\nu_s$  on  $(X, \mathcal{A})$  that satisfy

- (a)  $\nu_a \ll \mu$ ,
- (b)  $\nu_s \perp \mu$ , and
- (c)  $\nu = \nu_a + \nu_s$ .

The decomposition  $\nu = \nu_a + \nu_s$  is called *the Lebesgue decomposition of  $\nu$* .

## 5 Product Measures

**Definition 5.1** (Product of  $\sigma$ -algebras). Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. A subset of  $X \times Y$  is called a *rectangle with measurable sides* if it has the form  $A \times B$  for some  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ . The  $\sigma$ -algebra on  $X \times Y$  generated by collection of rectangles with measurable sides is called the *product* of  $\mathcal{A}$  and  $\mathcal{B}$ , and is denoted by  $\mathcal{A} \times \mathcal{B}$ .

**Definition 5.2** (Product measure). Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces. The unique measure  $\mu \times \nu$  on  $\mathcal{A} \times \mathcal{B}$  which satisfies  $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ , for every  $A \in \mathcal{A}, B \in \mathcal{B}$ , is called the *product* of  $\mu$  and  $\nu$ .

## 10 Probability

**Definition 10.1** (Probability space). A *probability space* is a measure space  $(\Omega, \mathcal{A}, P)$  such that  $P(\Omega) = 1$ . The elements of  $\Omega$  are called the *elementary outcomes* or the *sample points* of our experiment, and the members of  $\mathcal{A}$  are called *events*. If  $A \in \mathcal{A}$ , then  $P(A)$  is the *probability* of the event  $A$ .

**Definition 10.2** (Random variable). A *real-valued random variable* on a probability space  $(\Omega, \mathcal{A}, P)$  is an  $\mathcal{A}$ -measurable function from  $\Omega$  to  $\mathbb{R}$ . Such a variable represents a numerical observation or measurement whose value depends on the outcome of the random experiment represented by  $(\Omega, \mathcal{A}, P)$ . More generally, a *random variable* with values in a measurable space  $(S, \mathcal{B})$  is a measurable function from  $(\Omega, \mathcal{A}, P)$  to  $(S, \mathcal{B})$ .

**Definition 10.3** (Distribution). Let  $X$  be a random variable with values in  $(S, \mathcal{B})$ . The *distribution* of  $X$  is the measure  $PX^{-1}$  (see **Definition 2.11**) defined on  $(S, \mathcal{B})$  by  $(PX^{-1})(A) = P(X^{-1}(A))$ . We will often write  $P_X$  for the distribution of a random variable  $X$ . If  $X_1, \dots, X_d$  are  $(S, \mathcal{B})$ -valued random variables on  $(\Omega, \mathcal{A}, P)$ , then the formula  $X(\omega) = (X_1(\omega), \dots, X_d(\omega))$  defines an  $S^d$ -valued random variable  $X$ ; the distribution of  $X$  is called the *joint distribution* of  $X_1, \dots, X_d$ .

**Definition 10.4** (Expected value). If a real-valued random variable on the probability space  $(\Omega, \mathcal{A}, P)$  is integrable, then the *expected value* of  $X$  is defined  $E(X) = \int X dP$ . The expected value of  $X$  is often denoted  $\mu_X$ .

**Definition 10.5** (Variance). If  $X$  is a real-valued random variable, then the *variance* of  $X$  is the expected value of the random variable  $(X - E(X))^2$ , often denoted  $\text{Var}(X)$  or  $\sigma_X^2$ . The numerical value  $\sqrt{\sigma_X^2} = \sigma_X$  is called the *standard deviation* of  $X$ .

**Definition 10.6.** If  $X$  is  $\mathbb{R}^d$  valued and  $P_X \ll \lambda$ , then the Radon-Nikodym derivative of  $P_X$   $f_X$ , is called the *density function* of  $X$ .

**Definition 10.7** (Independence). Let  $(\Omega, \mathcal{A}, P)$  be a probability space, and let  $\{A_i\}_{i \in I}$  be an indexed family of events in  $\mathcal{A}$ . The events  $A_i$  are called *independent* if for each finite subset  $I_0$  of  $I$  we have  $P(\cap_{i \in I_0} A_i) = \prod_{i \in I_0} P(A_i)$ . Let  $\{X_i\}_{i \in I}$  be an indexed family of random variables defined on  $(\Omega, \mathcal{A}, P)$  and with values in the measurable space  $(S, \mathcal{B})$ . The random variables  $X_i$  are called *independent* if for each choice of sets  $A_i \in \mathcal{B}$ ,  $i \in I$ , the events  $X_i^{-1}(A_i)$  are independent. Finally if  $\{\mathcal{A}_i\}_{i \in I}$  is an indexed family of sub- $\sigma$ -algebras of  $\mathcal{A}$ , then the  $\sigma$ -algebras  $\mathcal{A}_i$  are independent if for each choice  $A_i \in \mathcal{A}_i$  the events  $A_i$  are independent.