Hand-in, Derivative pricing.

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A:

The rightmost inequality simplifies to

$$e^{-r(T-t)} E\left(\left(\left(\sum_{i=1}^{n} c_i S_i(T)\right) - K\right)^+\right) \leq e^{-r(T-t)} E \sum_{i=1}^{n} c_i (S_i(T) - K_i)^+.$$

$\Rightarrow$ the discounting factor can be disgarded.

LHS expands to

$\left(\sum_{i=1}^{n} c_i (S_i(T) - K_i)\right)^+$, which is due to

that $\sum_{i=1}^{n} c_i K_i = K$.

Furthermore, as all $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$, this means that $c^T x$ for some $x \in \mathbb{R}^n$ and with $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a convex combination of the elements in x.

let $x_i = S_i(T) - K_i$.

so what we are comparing is

$E((c^T x)^+)$ to $E(c^T(x)^+)$, alt. $c^T(E((x))^+)$

(linearity of E $\Rightarrow$ where we put it doesn't seem to important here, but open for suggestions.

now, let $f(y) = (y)^+ = \max(y, 0)$

max is a convex function, And by the definition of convexity:

f convex $\Rightarrow$ $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$

where $\theta \in [0, 1]$

this generalises to bigger convex combinations, of which $c^T x$ is an example, so

$f(c^T x) \leq c^T f(x)$, proving the rightmost ineq.

Sorry, no time for the leftmost ineq!

B:

① For this, I simulate $\ln(S(T))$ (vector-valued).
and put $R = c^T \ln(S(T))$
and further obtain $\Phi(T) = (e^R - K)^+$
then obtain the price as
$e^{-r(T-t)} \cdot E^Q(\Phi(T))$, where the expectation is estimated as mean of simulations.

② This task was very similar, exept the payoff for each simulated asset of course was $(S_i(T) - K)^+$.
further the price was obtained as the mean of all $n = 12$ prices.

③ With the crude MC method, this was not so different from the 2 previous tasks, as well as the lab earlier this week.

However, to use the control variate, the following scheme was used.

↳

Re-using some of the previously mentioned notation, we have that the vector $S(T)$ follows a vectorvalued GBM as:

$$l = \ln(S(T)) = \ln(S(t)) + (r \cdot \mathbb{I}_n - \mathrm{diag}(\sigma\sigma^*)/2)(T-t) + \dots$$

$$\dots + \sigma\sqrt{T-t}\, G$$

where $G \sim N_n(0,1)$

and σ is the lower cholesky factorization of

$$\begin{bmatrix} \sigma_1 & 0 & 0 & \cdots \\ 0 & \sigma_2 & & \\ \vdots & & \ddots & \end{bmatrix} \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \\ \rho_{2,1} & 1 & & \\ \rho_{3,1} & & \ddots & \\ \vdots & & & \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots \\ 0 & \sigma_2 & & \\ \vdots & & \ddots & \end{bmatrix}$$

this means that

$$E(l) = \ln(S(t)) + (r\mathbb{I}_n - \mathrm{diag}(\sigma\sigma^*)/2)(T-t)$$

and

$$\mathrm{Var}(l) = \sigma\sigma^*(T-t)$$

and $R = c^T l \Rightarrow R$ is also Gaussian (since it's a linear comb. of Gaussians) with

$$E(R) = c^T E(l)$$

$$\mathrm{Var}(R) = c^T \mathrm{Var}(l) c.$$

$\hookrightarrow$

now: since $\prod_{i=1}^{n} (S_i(T))^{c_i} = e^R$

this simplifies to a log-normal random variable

if $Y = e^X$, $X \sim N(\mu, \sigma^2)$ then

$$E(Y) = e^{(\mu + \frac{1}{2}\sigma^2)}$$

So, using e^R as a control variate is easy, because we know the Expected value. The implementation in matlab is heavily borrowed from Lecture 10.

(4)

a) $\Pi_0 = 1.1 \cdot NA = e^{-r(T-t)} E^Q(\Phi(T)) =$

$$= e^{-r(T-t)} NA \left(1 + pr E\left(\left(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} - 1\right)^+\right)\right)$$

$$\Rightarrow pr = \frac{1.1 \cdot e^{r(T-t)} - 1}{E\left(\left(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(0)} - 1\right)^+\right)}$$

$\hookrightarrow$

note, as each S_i is assumed to follow a GBM, we have that if we have the vector l where $l_i = \ln\left(\frac{S_i(T)}{S_i(0)}\right)$ then

$$l = \left(r I_n - \mathrm{diag}(\sigma\sigma^*)/2\right)(T-t) + \sigma\sqrt{T-t}\cdot G,$$

with G & σ as before (note we now have a given ρ-matrix).

the simulation then becomes:

$$\widehat{pr} = \frac{1.1e^{r(T-t)} - 1}{\mathrm{mean}\left(\left(c^T e^{l} - 1\right)^+\right)} \approx 0.72.$$

b) Using the estimated $\widehat{pr}$, the price is simply calculated. I was a bit unsure if price at $t=1$ should still be compared to $S_i(0)$ or to $S_i(1)$. I opted for the latter:

$$\widehat{\Pi}_1 = e^{-r(T-t)} \cdot NA \cdot \left(1 + \widehat{pr} \cdot E\left(\left(\sum_{i=1}^{n} c_i \frac{S_i(T)}{S_i(1)} - 1\right)^+\right)\right) \approx 108.9.$$

where again $S_i(1)$ simplifies away.