VALUATION OF DERIVATIVE ASSETS, FMSN25/MASM24
HOME ASSIGNMENT HT-21

The solutions should be handed in no later than Friday October 8 at 15.00. They shall be well written and properly motivated, so that no misunderstandings can occur. You may work together but *everyone should hand in their own solution.*

To pass you need correct solution to *most* exercises, but you may correct wrong solutions afterwards.

A: Pure Arbitrage relations for Basket call options. Here we will consider some pure arbitrage relations for Basket options. We will look at two types of Basket call options: weighted geometric and weighted arithmetic. We assume that the weights c_i are positive and sum to one. The pay-off at maturity for the geometric Basket call is defined as:

$$\left(\left(\prod_{i=1}^{n}(S_i(T))^{c_i}\right) - K\right)^{+} = \left(\exp\left(\sum_{i=1}^{n} c_i X_i(T)\right) - K\right)^{+},$$

where $X_i(T) = \log(S_i(T))$ and where $S_1, \cdots, S_n$ are n different stocks. The corresponding arithmetic Basket call has a pay-off defined as:

$$\left(\left(\sum_{i=1}^{n} c_i S_i(T)\right) - K\right)^{+},$$

where $S_1, S_2, \cdots, S_n$ are as above. In the following let $P_{E_i}^c(t, T, K)$, $P_{GB}^c(t, T, K)$ and $P_{AB}^c(t, T, K)$ be the price at time t of a European Call option with S_i as underlying asset, a geometric Basket call option and an arithmetic Basket call option all with strike price K and maturity at time T.

1. **Arbitrage bounds for the arithmetic Basket Call option.** We cannot find an analytical price for the arithmetic Basket Call. However, the price ($P_{AB}^c(0, T, K)$) of a call option must lie between the following bounds:

$$P_{GB}^c(0, T, K) \leqslant P_{AB}^c(0, T, K) \leqslant \sum_{i=1}^{n} c_i P_{E_i}^c(0, T, K_i),$$

where $\sum_{i=1}^{n} c_i K_i = K$.

Show this with a pure arbitrage argument. Hint: The upper bound is related to the convexity in x of $\max(x - K, 0)$. For the lower bound look at the convexity of the exponential function and the monotonicity of the payoff function.

B: Numerical calculation of prices Consider the multivariate Black-Scholes model for the stock prices $S_1, \cdots, S_n$. The volatilities of the stocks σ_i are all 0.4, the pairwise correlations $\rho_{ij}, i \neq j$ are all 0.6, the continuously compounded short interest rate is 0.15% per year and $S_1(0) = \cdots = S_n(0) = 100$ with $n = 12$. In B use $c_i = 1/12, i = 1 \cdots 12$.

1. Compute the arbitrage free price at $t = 0$ of the lower bound for the Arithmetic Basket call option , i.e. the Geometric Basket Call, on the stocks $S_1, \cdots, S_n$ with strike price $K = 80, 100, 120$, time to maturity five years.
   Hint: $\left(\prod_{i=1}^{n}(S_i(T))^{c_i}\right)$ has the same distribution as $\exp(X)$ where X is a Gaussian random variable with mean a and variance b. Start by calculating a and b. After that you can find the price in the same fashion as in the derivation of the Black Scholes formula.

2. Compute the arbitrage free price at $t = 0$ of the upper bound for the Arithmetic Basket call option, i.e. the weighted sum of European Calls, on the stocks $S_1, \cdots, S_n$ with strike price $K = 80, 100, 120$ and time to maturity as in B1.

3. Compute with Monte Carlo methods the arbitrage free price at $t = 0$ of the Arithmetic Basket call option on the stocks $S_1, \cdots, S_n$ with strike price $K = 80, 100, 120$ and time to maturity as in B1. Use $N = 1000, 10000, 100000$ where N is the number of replications used in the Monte Carlo calculation. This is essentially assignment 4.2 in computer exercise 2. Compare this with the price obtained from the upper and lower bounds. Use the Geometric Basket call as a control variate to reduce the variance of the estimate (see Åberg chapter 11). Part of this is in fact also one of the assignments on Computer exercise two.

Please turnover

4. Arithmetic Basket call options are common building blocks in one of the variations of the recently quite popular contract Equity linked notes (Aktieindexobligationer). Contracts of this type is available from most of the major Swedish banks as well as some other financial companies and they are mostly sold to private investors. There have also been a debate during recent years saying that some of these contracts contained hidden fees. We are now going to look at a contract with Boliden, Ericsson B, Getinge, HM B, MTG B, Sandvik, SCA B, Securitas B, SSAB B, Swedbank A, Telia and Volvo B as underlying assets (This resembles a contract that exists on the market but it is actually simpler than the original contract, which is an arithmetic Asian option on the basket of assets). The pay off at maturity $T = 5$ years is given by:

$$\Phi(T) = NA \left(1 + pr \left(\sum_{i=1}^{12} c_i \frac{S_i(T)}{S_i(0)} - 1 \right)^+ \right), \tag{1}$$

where $(c_i, i = 1 \cdots 12) = (0.07, 0.11, 0.11, 0.18, 0.06, 0.04, 0.01, 0.17, 0.04, 0.02, 0.03, 0.16)$, NA is the nominal amount, pr is the participation ratio (deltagandegrad), it describes how much of the risky asset that will contribute to the payoff and $\{S_i(0)\}_{i=1}^{12}$ are the initial stock prices. Note that we will always get at least the nominal amount back with this contract. The participation ratio is here chosen such that the price of the Equity linked noted at time zero is equal to 1.1NA.

(a) Your task is now to find pr, using Monte Carlo simulations, for this contract such that the fair price at time zero is equal to 1.1NA. Assume that S follows a multivariate Black-Scholes model with $(S_1(0), \cdots, S_{12}(0)) = (141.3, 80.9, 193.3, 316.7, 212.9, 75.6, 236.6, 108.8, 20.68, 192.9, 46.55, 88.95)$,

$$\rho = \begin{bmatrix}
1.00 & 0.30 & 0.28 & 0.32 & 0.36 & 0.52 & 0.29 & 0.37 & 0.44 & 0.41 & 0.38 & 0.41 \\
0.30 & 1.00 & 0.17 & 0.45 & 0.44 & 0.42 & 0.37 & 0.35 & 0.32 & 0.42 & 0.42 & 0.27 \\
0.28 & 0.17 & 1.00 & 0.20 & 0.17 & 0.31 & 0.23 & 0.24 & 0.20 & 0.25 & 0.27 & 0.21 \\
0.32 & 0.45 & 0.20 & 1.00 & 0.34 & 0.38 & 0.39 & 0.30 & 0.32 & 0.43 & 0.42 & 0.30 \\
0.36 & 0.44 & 0.17 & 0.34 & 1.00 & 0.47 & 0.22 & 0.33 & 0.39 & 0.38 & 0.31 & 0.31 \\
0.52 & 0.42 & 0.31 & 0.38 & 0.47 & 1.00 & 0.37 & 0.37 & 0.46 & 0.47 & 0.42 & 0.56 \\
0.29 & 0.37 & 0.23 & 0.39 & 0.22 & 0.37 & 1.00 & 0.25 & 0.26 & 0.42 & 0.29 & 0.22 \\
0.37 & 0.35 & 0.24 & 0.30 & 0.33 & 0.37 & 0.25 & 1.00 & 0.39 & 0.39 & 0.34 & 0.26 \\
0.44 & 0.32 & 0.20 & 0.32 & 0.39 & 0.46 & 0.26 & 0.39 & 1.00 & 0.43 & 0.38 & 0.29 \\
0.41 & 0.42 & 0.25 & 0.43 & 0.38 & 0.47 & 0.42 & 0.39 & 0.43 & 1.00 & 0.46 & 0.38 \\
0.38 & 0.42 & 0.27 & 0.42 & 0.31 & 0.42 & 0.29 & 0.34 & 0.38 & 0.46 & 1.00 & 0.39 \\
0.41 & 0.27 & 0.21 & 0.30 & 0.31 & 0.56 & 0.22 & 0.26 & 0.29 & 0.38 & 0.39 & 1.00
\end{bmatrix},$$

$\sigma = (0.32, 0.22, 0.31, 0.18, 0.27, 0.22, 0.21, 0.23, 0.34, 0.17, 0.18, 0.23)$, $NA = 100$ and $r = -0.1\%$ per year. First express the price of the Equity linked note at time zero as a function of the price of the Basket option and then use similar techniques to the ones that you have developed in B.3.

(b) Use pr from (a). A year has now passed and the values of the underlying assets are $(S_1(1), \cdots, S_{12}(1)) = (179.3, 57.45, 163.7, 260.6, 220.4, 89.85, 259.5, 143.8, 20.83, 194.7, 37.62, 97.5)$. The time left to maturity is now 4 years. Note that the pay-off is still the same as before (i.e. Eq. (1)) the differences are that we now condition on the values at time one and in the discounting factor. What is the fair price of the contract now?