Hand-in 2 FRTN30 Gustaf Sundell

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When completing this assignment I have discussed a few of the questions with Harald Österling and Fredrik Sidh.

a)

The shortest distance from node 1 to node 17 was found using the graphshortestpath function in Matlab. In order to do so, the sparse adjacency matrix was constructed using the node-link incidence matrix, B and the vector of lengths for all the links, l. The shortest distance was 0.533, and assuming this is in hours, it is roughly half an hour. The shortest path was

$$1 \to 2 \to 3 \to 9 \to 13 \to 17 \tag{1}$$

b)

The maximum flow between nodes 1 and 17 was found by using the Matlab-function graphmaxflow. Analogously to the previous task, a sparse square matrix needed to be constructed containing the capacities. The maximum flow was found to be 22448.

c)

In this task, the external inflow or outflow at each node was calculated. This is recognised as the exogenous flow-vector ν, which is given by $Bf = \nu$. The results are presented in Table 1.

d)

For this and the following tasks, the library cvx was used for optimization, and the delay function is

$$d_e(f_e) = \frac{l_e}{1 - \frac{f_e}{C_e}} \tag{2}$$
$$\text{where } 0 \leq f_e < C_e \ \forall e$$

whereas the cost function for each problem will vary. For the rest of the tasks, the exogenous flow will also be assumed to come from the origin node towards the destination node, with zero other inflow. To solve the social optimal flow f^*, the cost function used was:

$$\sum_{e \in \mathcal{E}} f_e d_e\left(f_e\right) = \sum_{e \in \mathcal{E}} \frac{f_e l_e}{1 - f_e/C_e} = \sum_{e \in \mathcal{E}} \left(\frac{l_e C_e}{1 - f_e/C_e} - l_e C_e \right) \tag{3}$$

The resulting optimal flow is presented in Table 2.

e)

The Wardrop equilibrium, $f^{(0)}$, was found by optimizing the problem with the following cost function:

$$\sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(s) ds = \sum_{e \in \mathcal{E}} \int_0^{f_e} \frac{l_e}{1 - \frac{s}{C_e}} ds = \sum_{e \in \mathcal{E}} -C_e l_e ln(1 - \frac{f_e}{C_e}) \tag{4}$$

Again, the optimization problem was solved by use of the cvx solver, and the results are presented Table 2.

f)

In this task, we introduce tolls. The tolls are derived from the delay function's derivative at the social optimal flow f_e. The optimal flow for this task, $f^{(\omega)}$, was found by minimizing a cost function that is a bit more complicated than the former.

First off, the tolls on each link e are given by:

$$\omega_e = f_e^* d_e'\left(f_e^*\right) = \frac{f_e^* l_e}{C_e} \frac{1}{(1 - \frac{f_e}{C_e})^2} \tag{5}$$

The tolls are not dependent of the flow that is being optimized, which can be seen when doing the integration in what finally is the cost function for optimizing $f^{(\omega)}$:

$$\sum_{e \in \mathcal{E}} \int_0^{f_e} \left(\frac{l_e}{1 - \frac{s}{C_e}} + \frac{f_e^* l_e}{C_e} \frac{1}{(1 - \frac{f_e}{C_e})^2} \right) ds \tag{6}$$

which simplifies to

$$\sum_{e \in \mathcal{E}} -l_e C_e ln(1 - \frac{f_e}{C_e}) + \frac{f_e l_e f_e^*}{C_e} \frac{1}{(1 - \frac{f_e}{C_e})^2} \tag{7}$$

The results were indeed equal to f^*, and are presented in Table 2.

g)

The first part of this task consists in finding an optimal flow, which I choose to call $f^{(g)}$, by minimizing the new cost function:

$$c_e(f_e) = f_e(d_e(f_e) - l_e) = \frac{l_e C_e}{1 - \frac{f_e}{C_e}} - l_e C_e - f_e l_e \tag{8}$$

subjected to the same constraints as before. Now, we want to find optimal tolls ω^*, such that solving the same problem as in 1.f), the Wardrop equilibrium flow $f^{(w^*)}$ coincides with $f^{(g)}$. Theorem 4.2 from the lecture notes provides a theoretical such toll vector. By setting

$$\omega^* = c_e'(f_e^{(g)}) - d_e(f_e^{(g)}) = \frac{f_e^{(g)} C_e l_e}{(C_e - f_e^{(g)})^2} - l_e \tag{9}$$

This should be the case. Finally, to test this, we find the new Wardrop equilibrium by minimizing the cost function:

$$\sum_{e \in \mathcal{E}} \int_0^{f_e^{(g)}} (d_e(s) + \omega_e^*) ds = \sum_{e \in \mathcal{E}} -l_e C_e ln(1 - \frac{f_e}{C_e}) + f_e \omega_e^* \tag{10}$$

By consulting the table, we can see that the two flows coincide. It is also noteworthy that ω_* contains negative values, which could be interpreted as reverse tolling, paying drivers to drive on these roads, which is somewhat counterintuitive.

1 Tables

Node	ν
1	16806
2	8570
3	19448
4	4957
5	-746
6	4768
7	413
8	-2
9	-5671
10	1169
11	-5
12	-7131
13	-380
14	-7412
15	-7810
16	-3430
17	-23544

Table 1: The exogenous flow vector, answering 1.c)

Link	f^*	$f^{(0)}$	$f^{(\omega)}$	$f^{(g)}$	$f^{(\omega^)}$
1	6642	6716	6642	6653	6653
2	6059	6716	6059	5775	5775
3	3132	2367	3132	3420	3420
4	3132	2367	3132	3420	3420
5	10164	10090	10164	10153	10153
6	4638	4645	4638	4643	4643
7	3006	2804	3006	3106	3106
8	2543	2284	2543	2662	2662
9	3132	3418	3132	3009	3009
10	583	0	583	878	878
11	0	177	0	0	0
12	2927	4171	2927	2355	2355
13	0	0	0	0	0
14	3132	2367	3132	3420	3420
15	5525	5445	5525	5510	5510
16	2854	2353	2854	3044	3044
17	4886	4933	4886	4882	4882
18	2215	1842	2215	2415	2415
19	464	697	464	444	444
20	2338	3036	2338	2008	2008
21	3318	3050	3318	3487	3487
22	5656	6087	5656	5495	5495
23	2373	2587	2373	2204	2204
24	0	0	0	0	0
25	6414	6919	6414	6301	6301
26	5505	4954	5505	5624	5624
27	4886	4933	4886	4882	4882
28	4886	4933	4886	4882	4882

Table 2: Table for the optimal flows, answering questions 1.d)-g).

Link	ω^*
1	1.8
2	0.1079
3	-0.0693
4	-0.0629
5	1.3004
6	0.393
7	0.0223
8	0.0059
9	0.1067
10	-0.0954
11	-0.1067
12	-0.0542
13	-0.1123
14	-0.0293
15	0.3548
16	0.0146
17	-0.0069
18	-0.0368
19	-0.0312
20	-0.024
21	0.0071
22	0.1277
23	-0.0207
24	-0.0542
25	0.2846
26	0.2409
27	0.0261
28	0.3771

Table 3: Values of ω^*, part of answer for question 1.g)