Time-Varying Density Control for Large-Scale Multi-Agent Systems Using Heat and Continuity Equation

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Abstract—**The presented work examines the feasibility of coordinating a large-scale multi-agent system according to a time-varying probability density function using nature-inspired control techniques. The proposed method works in a decentralized manner where only information about neighboring agents positions are required. The information is then used to estimate the local density which further is used to generate a velocity command. Building upon previous work, a feedback law based on heat equation is extended by a feed-forward term based on the continuity equation. This accommodates a velocity field driving the agents towards the desired time-varying spatial density. We show how the proposed technique can be used to track both simple and complex time-varying densities. The proposed control law is tested in a simulation to verify its performance.**

Index Terms—**Swarm Robotics Control, Density Based Control, Multi-agent System, Heat Equation, Continuity Equation**

I. INTRODUCTION

This paper explores the compatibility of swarm control with time-varying formations. The concept of swarm control refers to the control of the coordination and movement of multiple agents, and has in recent years gained increased attention and research efforts [1]. Large-scale multi-agent systems emerge in various settings such as autonomous vehicle fleets, opinion dynamics or herd analysis. Applications of large-scale multi-agent control has a growing potential due to the continuing development of robotics and autonomous systems. Deploying many agents in a coordinated manner enables advantages in terms of redundancy, reconfigurability and scalability. These properties are useful for numerous tasks, such as search-and-rescue missions, distributed transportation coordination or autonomous driving for vehicle fleets.

Moreover, swarm control separates the control objectives to a local and global level, facilitating specialization of robots focusing on local tasks (such as sensing or motion) while the more complex global tasks (such as consensus or formation) are solved in a distributed manner overcoming economic and computational constraints.

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The research of swarm control has utilized physical principles of natural systems and flocking phenomena to achieve similar abilities [1], as well as modelling the discrete placement of agents as a spatial density – so-called *mean field approximation* [3].

Previous work include modelling the configuration space as a grid of cells, with the density of agents described as a arkov chain and their movement controlled using a Markovian matrix [2], [6]. [7] instead achieves the decentralized control with a continuous vector field, applying the heat equation to mimic an appropriate swarm motion. This approach is further explored in [5], by extending it with feed-forward control and agent localization in situations of constrained sensing between agents, and in [8] by analyzing the guarantee for collision-free control in one dimensional space.

The work presented in this paper builds on the work of [7]. The swarm of agents are controlled by a deterministic but time-varying velocity field which mimics the behaviour of heat transfer, resulting in a velocity field driving agents in a locally uniform manner to the desired spatial density. Each agent is given the information of the desired spatial density formation, and locally estimates the current density formation as a kernel density estimation based on the positions of other agents. The difference between the desired and estimated density at each agent is then used to generate the velocity field, yielding a distributed control input for the motion of each agent.

Contribution:

Extending the setting from [7] to include a time-varying desired density formation, we propose a feed forward term in order to retain the same asymptotic stability properties. Given the nominal control architecture in [7], the feed forward gain can be modelled as the inverse solution of the continuity equation. We further show that there exists an analytical solution for Gaussian densities, which can be extended to complex distributions using Gaussian Mixture Models and optimal transport theory.

The rest of the paper is organized as follow. Section II presents the control problem and estimation method of local density information. Section III presents the added feed-forward term and provides analysis on its properties, stability and convergence. Section IV describes implementation of the problem and provides simulated results of the added feed-forward term. Finally, Section V states the conclusion and outlook of the work.

II. PROBLEM STATEMENT

Let $\mathcal{S} = \{\boldsymbol{r}_1(t), \boldsymbol{r}_2(t), ..., \boldsymbol{r}_N(t)\}$ be a homogeneous swarm of N agents with point mass dynamics moving freely in a planar configuration space $\mathcal{R}$, i.e $\boldsymbol{r}_i(t) \in \mathcal{R}, \mathcal{R} \subseteq \mathbb{R}^2\ \forall i$. Let $f_{\mathcal{R}}(t, \boldsymbol{x})$ denote the swarm density of the configuration space dependent on time and coordinate $\boldsymbol{x} \in \mathcal{R}$, and $\boldsymbol{v}(t, \boldsymbol{r}_i(t))$ the velocity field acting on agent i at time t. The continuous time dynamics can then be expressed as (1a) and (1b). We refer to [7] for a detailed derivation.

$$\dot{\boldsymbol{r}}_i(t) = \boldsymbol{v}(t, \boldsymbol{r}_i(t)) \tag{1a}$$

$$\dot{f}_{\mathcal{R}}(t, \boldsymbol{x}) = -\nabla \cdot [\boldsymbol{v}(t, \boldsymbol{x}) f_{\mathcal{R}}(t, \boldsymbol{x})] \tag{1b}$$

The swarm density control objective is to design a velocity field control law $\boldsymbol{v}(t, \boldsymbol{x})$ given a desired time-varying density $f^d_{\mathcal{R}}(t, \boldsymbol{x}) \in \mathcal{C}^1$ such that:

$$f_{\mathcal{R}}(t, \boldsymbol{x}) = f^d_{\mathcal{R}}(t, \boldsymbol{x}) \quad \forall t. \tag{2}$$

We tackle the problem by applying the control architecture proposed in [7]. Figure 1 and 2 describe the high level control architecture and the control algorithm respectively. The main idea, which will be derived in Section III, is to find a control law $\dot{\boldsymbol{r}}(t, \boldsymbol{x}) = \boldsymbol{v}(t, \boldsymbol{x})$ consisting of the feedback law derived in [7] and expanding it with a feed forward term to enable tracking of time-varying target densities while retaining the asymptotic stability properties of the nominal feedback law.

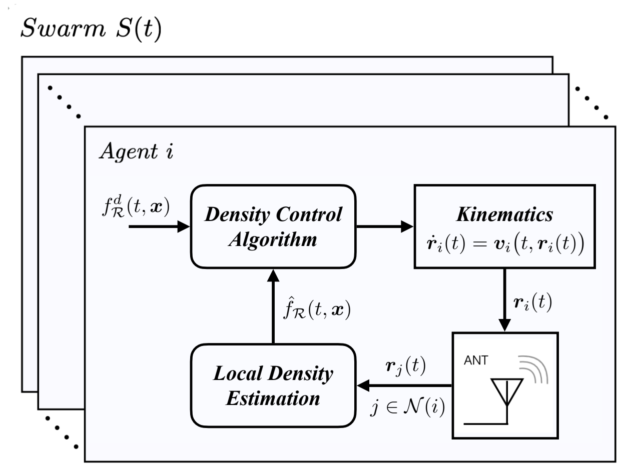


Fig. 1: The figure shows the high level control architecture for a swarm of agents. Figure is taken from [7] and modified.

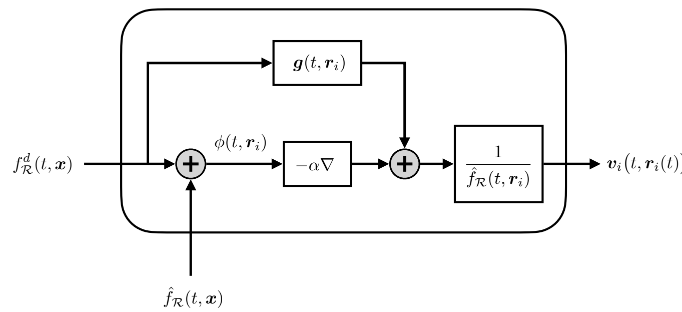


Fig. 2: The inner block diagram of the Density Control Algorithm shown in figure 1. $\boldsymbol{g}(t, \boldsymbol{r}_i)$ is the feed forward gain.

As evident from Figure 2, the velocity field $\boldsymbol{v}(t, \boldsymbol{x})$ is a function of both estimated and target swarm densities. The target density which is constructed analytically is trivial to evaluate. We do not however have direct access to the swarm density. Consequently $f_{\mathcal{R}}(t, \boldsymbol{x})$ must be estimated using local density estimation techniques. Similarly to [7] we employ Gaussian kernel density estimation:

$$\hat{f}_{\mathcal{R}}(t, \boldsymbol{r}_i(t)) = \frac{1}{Nh^d} \sum_{j=1}^{N} [K(\boldsymbol{r}_{\boldsymbol{i}}(t) - \boldsymbol{r}_{\boldsymbol{j}}(t))] \tag{3a}$$

$$K(\boldsymbol{x}) = -\frac{2}{\pi} \exp\left(-\frac{\boldsymbol{x}\boldsymbol{x}^\top}{h^2}\right). \tag{3b}$$

where N is the number of agents, h is the smoothing parameter and d is the dimension of configuration space.

III. CONTROL DESIGN

Given a static desired swarm density function $f^d_{\mathcal{R}}(\boldsymbol{x})$ the previous approach in [7] applies a local density error estimate to compute a velocity field. In the derivation of this field, the heat equation plays a crucial role. Their approach however relies upon the assumption that the desired density is constant in time. To overcome this assumption we introduce a feed forward gain from the reference signal $f^d_{\mathcal{R}}(t, \boldsymbol{x})$. The derivation of the feed forward gain results in solving the continuity equation, and thus the heat equation is accompanied by a fellow PDE.

Solving PDEs like the continuity equation can be notoriously hard, yet we will show that for the special case of a desired density modelled as Gaussian Mixture Model with a time-varying mean, analytical solutions exist. Furthermore, we will extend this result to complex distributions by using aspects of optimal transport theory.

A. Velocity field generation using heat and continuity equation

Let the difference between the current and desired swarm density be

$$\Phi(t, \boldsymbol{x}) = f_{\mathcal{R}}(t, \boldsymbol{x}) - f^d_{\mathcal{R}}(t, \boldsymbol{x}). \tag{4}$$

In [7] the following feedback law for controlling each agents velocity was proposed as

$$\boldsymbol{v}_{fb}(t, \boldsymbol{x}) = -\alpha \frac{\nabla \Phi(t, \boldsymbol{x})}{f_{\mathcal{R}}(t, \boldsymbol{x})}. \tag{5}$$

They further showed, assuming that the desired density is constant in time, that the proposed feed back law in (5) transforms equation (1b) to the heat equation (6). They also showed that this results yields some desired properties to the closed loop behavior of the swarm, in terms of asymptotic stability.

$$\dot{\Phi}(t, \boldsymbol{x}) = \alpha \nabla^2 \Phi(t, \boldsymbol{x}) \tag{6}$$

When accounting for the time varying desired densities, the closed loop behavior of $\Phi(t, \boldsymbol{x})$ gets slightly perturbed by an extra term $\frac{\partial}{\partial t} f_{\mathcal{R}}^d$ which is no longer zero:

$$\dot{\Phi}(t, \boldsymbol{x}) = \dot{f}_{\mathcal{R}} - \dot{f}_{\mathcal{R}}^d = \alpha \nabla^2 \Phi(t, \boldsymbol{x}) - \frac{\partial f_{\mathcal{R}}^d}{\partial t}(t, \boldsymbol{x}). \quad (7)$$

To compensate for this we take inspiration from control theory and introduce a feed forward term

$$\boldsymbol{v}_{ff}(t, \boldsymbol{x}) = \frac{f_{\mathcal{R}}^d(t, \boldsymbol{x})}{f_{\mathcal{R}}(t, \boldsymbol{x})} \boldsymbol{g}(t, \boldsymbol{x}), \quad (8)$$

where $\boldsymbol{g}(t, \boldsymbol{x})$ is the feed forward gain field.

Accounting for the feed forward term (8) we get the following agents dynamics:

$$\dot{\mathbf{r}}(t) = \boldsymbol{v}_{fb}(t, \boldsymbol{x}) + \boldsymbol{v}_{ff}(t, \boldsymbol{x}) = \frac{-\alpha \nabla \Phi(t, \boldsymbol{x}) + f_{\mathcal{R}}^d(t, \boldsymbol{x}) \boldsymbol{g}(t, \boldsymbol{x})}{f_{\mathcal{R}}(t, \boldsymbol{x})}, \quad (9)$$

such that

$$\begin{aligned} \dot{\Phi}(t, \boldsymbol{x}) &= -\nabla \cdot [-\alpha \nabla \Phi(t, \boldsymbol{x}) + f_{\mathcal{R}}^d(t, \boldsymbol{x}) \boldsymbol{g}(t, \boldsymbol{x})] - \frac{\partial f_{\mathcal{R}}^d}{\partial t}(t, \boldsymbol{x}) \\ &= \alpha \nabla^2 \Phi(t, \boldsymbol{x}) - \nabla \cdot [f_{\mathcal{R}}^d(t, \boldsymbol{x}) \boldsymbol{g}(t, \boldsymbol{x})] - \frac{\partial f_{\mathcal{R}}^d}{\partial t}(t, \boldsymbol{x}). \end{aligned} \quad (10)$$

To retrieve the same closed loop properties as in the nominal case of static desired density, we get the following condition on the feed forward gain $\boldsymbol{g}(t, \boldsymbol{x})$:

$$\nabla \cdot (f_{\mathcal{R}}^d(t, \boldsymbol{x}) \boldsymbol{g}(t, \boldsymbol{x})) + \frac{\partial}{\partial t} f_{\mathcal{R}}^d(t, \boldsymbol{x}) = 0. \quad (11)$$

Looking closer at equation (11) one recognizes the continuity equation, governing mass conserving flow of a compressible fluid.

Our main result is that by introducing the feed forward term in (8) that satisfies the continuity equation in (11) with respect to the desired swarm density $f_{\mathcal{R}}^d(t, \boldsymbol{x})$ in addition to the feedback law (5), we retain similar controller performance as in the nominal case with static desired density. That is, the desired feed forward gain $\boldsymbol{g}(t, \boldsymbol{x})$ is the flow field of the exact density we are trying to track. Which along with its mathematical beauty also is a very logical result.

Statement 1. Consider a swarm $\mathcal{S}(t)$ commanded with continuous desired density function $f_{\mathcal{R}}^d(t, \boldsymbol{x})$ where the motion of each agent is controlled as in equation (9). Choosing the feed forward gain $\boldsymbol{g}(t, \boldsymbol{x})$ such that it satisfies the continuity equation (11) with respect to the desired density function, yields the same asymptotic stability properties as proved in Theorem 6 in [7].

Proof. To conduct the same type of stability proof as in [7] we introduce a relative agent velocity

$$\tilde{\boldsymbol{u}}_i = \dot{\boldsymbol{r}}_i - \frac{f_{\mathcal{R}}^d(t, \boldsymbol{r}_i)}{f_{\mathcal{R}}(t, \boldsymbol{r}_i)} \boldsymbol{g}(t, \boldsymbol{r}_i). \quad (12)$$

and define the following positive definite function as the Lyapunov function for the swarm:

$$\begin{aligned} V &= \frac{1}{2} \sum_{i=1}^{N} \left(\frac{f_{\mathcal{R}}(t, \boldsymbol{r}_i)}{D} \right)^2 (\tilde{\boldsymbol{u}}_i)^T (\tilde{\boldsymbol{u}}_i) \\ &= \frac{1}{2} \sum_{i=1}^{N} \nabla \Phi(t, \boldsymbol{r}_i)^T \nabla \Phi(t, \boldsymbol{r}_i) \end{aligned} \quad (13)$$

Note that this function is defined at agent locations to ensure that V is positive definite, because it is guaranteed that $f_{\mathcal{R}}(t, \boldsymbol{r}_i) > 0 \forall i$. Taking the time derivative yields,

$$\dot{V} = \sum_{i=1}^{N} \nabla \Phi(t, \boldsymbol{r}_i)^T \nabla \dot{\Phi}(t, \boldsymbol{r}_i). \quad (14)$$

Recalling that $\boldsymbol{g}(t, \boldsymbol{r}_i)$ satisfies the continuity equation, we can use the heat equation $\dot{\Phi}(t, \boldsymbol{r}_i) = \alpha \nabla^2 \Phi(t, \boldsymbol{r}_i)$. The continuation of the proof is conducted exactly as in [7], and we hereby conclude that asymptotic stability is achieved. For a more detailed analysis and conditions we refer to [7].

B. Solving the continuity equation for time-varying Gaussian densities

Statement 2. Consider a time-varying density function parameterized by a Gaussian distribution with a time-varying mean $\boldsymbol{\mu}(t)$, such that $f_{\mathcal{R}}^d(t, \boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}(t), \Sigma)$ as in (15). Then the flow field candidate $\boldsymbol{g}(t, \boldsymbol{x}) = \dot{\boldsymbol{\mu}}(t)$ satisfies the continuity equation (11) with respect to $f_{\mathcal{R}}^d(t, \boldsymbol{x})$.

$$f_{\mathcal{R}}^d(t, \boldsymbol{x}) = |2\pi\Sigma|^{-1/2} \exp\Big[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}(t))^\top \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu}(t)) \Big] \quad (15)$$

Proof. Taking derivative with respect to time yields

$$\frac{\partial f_{\mathcal{R}}^d}{\partial t} = \frac{\partial f_{\mathcal{R}}^d}{\partial \boldsymbol{\mu}} \dot{\boldsymbol{\mu}}(t). \quad (16)$$

Further using that $\boldsymbol{\mu}(t)$ is only a function of time we get

$$\nabla \cdot (f_{\mathcal{R}}^d \dot{\boldsymbol{\mu}}) = \frac{\partial f_{\mathcal{R}}^d}{\partial \boldsymbol{x}} \dot{\boldsymbol{\mu}} \quad (17)$$

From the Gaussian distribution with time-varying mean (15) we have that $\frac{\partial}{\partial \boldsymbol{x}} f_{\mathcal{R}}^d = -\frac{\partial}{\partial \boldsymbol{\mu}} f_{\mathcal{R}}^d$. Using this together with equations (16) and (17) we get

$$\nabla \cdot (f_{\mathcal{R}}^d \dot{\boldsymbol{\mu}}) = \frac{\partial f_{\mathcal{R}}^d}{\partial \boldsymbol{x}} \dot{\boldsymbol{\mu}} = -\frac{\partial f_{\mathcal{R}}^d}{\partial \boldsymbol{\mu}} \dot{\boldsymbol{\mu}} = -\frac{\partial f_{\mathcal{R}}^d}{\partial t} \quad (18)$$

which concludes the proof.

C. Extension to complex time-varying densities

A Gaussian mixture model (GMM) defined in (19) can be used to parameterise complex distributions.

$$P(\boldsymbol{x}; \boldsymbol{\theta}) = \sum_{i=1}^{K} p_i \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i, \Sigma_i) \quad (19a)$$

$$\sum_{i=1}^{K} p_i = 1 \quad (19b)$$

$$p_i \geq 0 \quad \forall\, i \in [1, K] \quad (19c)$$

where K is the number of mixture components.

In [4] an interpolation technique between Gaussian mixtures was introduced in an optimal transport theory setting. The interpolation builds upon an auxiliary GMM where the Gaussian components have-time varying means and optionally time-varying covariances. This yields a framework to parameterize complex time-varying density functions. By adding one feed forward gain for each Gaussian component in the GMM, our results in Section III-A and Section III-B are directly extended to the case with GMMs with time-varying means.

Statement 3. Consider a swarm $\mathcal{S}(t)$ commanded with a continuous desired density function parameterized as Gaussian Mixture Model (GMM), where the means are time-varying such that $\boldsymbol{\mu}_i = \boldsymbol{\mu}_i(t)\ \forall\, i \in [1, K]$. Choosing the feed forward term

$$V_{ff} = \frac{1}{f_{\mathcal{R}}(t, \boldsymbol{x})} \sum_{i=1}^{K} p_i \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \dot{\boldsymbol{\mu}}_i(t) \tag{20}$$

will convert the dynamics in (1b) into the heat equation in (6) and hence the same stability properties proved in Statement 1 are retained.

Proof. Leveraging the linearity of the divergence operator we have that

$$\nabla \cdot \sum_{i=1}^{K} p_i \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \dot{\boldsymbol{\mu}}_i(t) \tag{21a}$$

$$= \sum_{i=1}^{K} p_i \nabla \cdot [\mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \dot{\boldsymbol{\mu}}_i(t)] \tag{21b}$$

Further recalling from Statement 2 that

$$\nabla \cdot [\mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \dot{\boldsymbol{\mu}}_i(t)] = -\frac{\partial}{\partial t} \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \tag{22}$$

and using the linearity of the differentiation operator we get

$$\nabla \cdot \sum_{i=1}^{K} p_i \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \dot{\boldsymbol{\mu}}_i(t) \tag{23a}$$

$$= -\sum_{i=1}^{K} p_i \frac{\partial}{\partial t} \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \tag{23b}$$

$$= -\frac{\partial}{\partial t} \sum_{i=1}^{K} p_i \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \Sigma_i) \tag{23c}$$

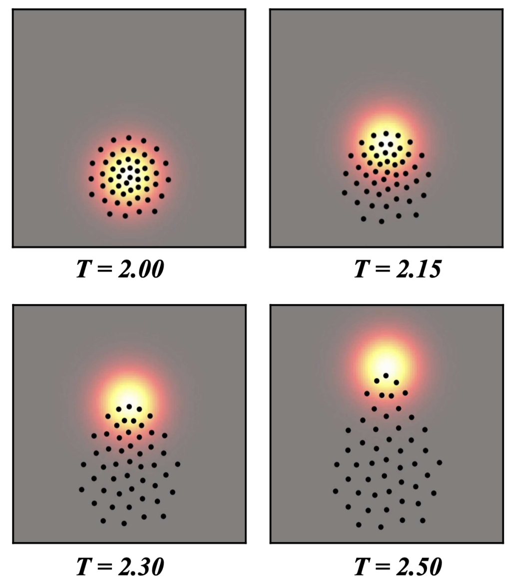
which we recognize as the continuity equation (11) for a time-varying GMM.

Since the feed forward term satisfies the continuity equation the closed loop dynamics of the density error $\Phi(t, \boldsymbol{x})$ will satisfy the heat equation (6), and hence asymptotic stability is obtained.

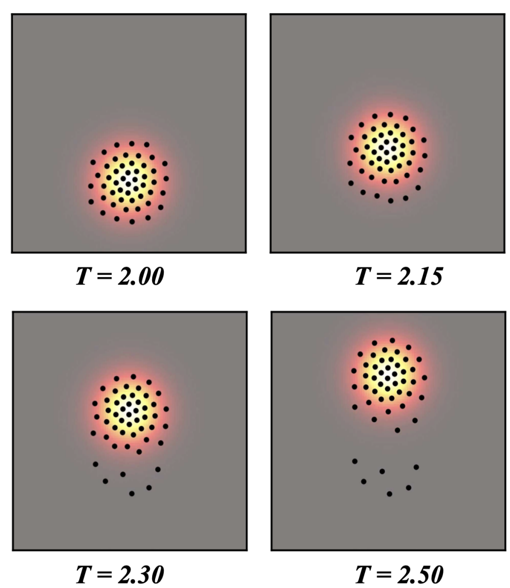
IV. SIMULATION RESULTS

In order to evaluate the performance of (9) we conduct a series of simulations where we compare the tracking ability of time varying densities of our control law to it's predecessor introduced in [7]. Figures 3 and 4 illustrates the quantitative performance gain achieved by the added feed forward term. All simulations were conducted with a smoothing parameter $h = L/20$ for the local density estimator (3a), where L is the size of the square domain, and diffusion constant of $\alpha = 5$ in (6).

The complex distribution in (4) was generated by sampling frames from a video of a walking man, generating Gaussian mixtures for the individual frames and then interpolating



(a) Without feed forward control

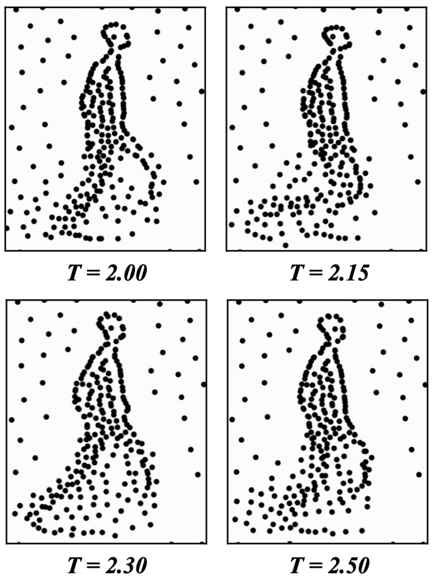


(b) With feed forward control

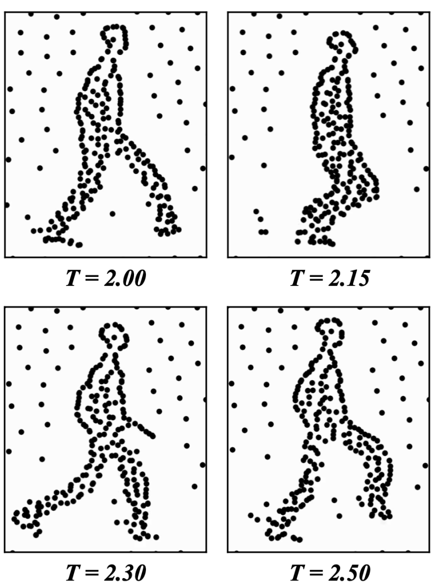
Fig. 3: Tracking of a Gaussian distribution with a time varying mean without (a) and with (b) feed forward control. The simulation was conducted with N = 50 agents. The heat map of the desired density is plotted in the background.

between them with an auxiliary Gaussian mixture model as described in [4].

For the frames, the GMMs were generated by relating each pixel to a component of (19) where the mixture weight coefficient p_i is directly proportional to the pixel intensity. This paper is also accompanied by supplementary videos for a more comprehensive understanding and visual comparison, which can be accessed through the link *https://gustafzach.github.io*.



(a) Without feed forward control



(b) With feed forward control

Fig. 4: Tracking of a complex multivariate Gaussian distribution with time varying means without (a) and with feed forward control (b). The simulation was conducted with N = 300 agents.

V. CONCLUSION & OUTLOOK

By continuing the exploration of control applications of the heat equation, the purpose and feasibility of time-varying density control of large-scale multi-agent systems are further documented. The analytical results demonstrate the simple nature of nature-inspired control techniques. The simulation results demonstrates the tractability of the objective, and the generality of the solution (the system input was a generic mp4-video). The simulation showed, however, the limitations of the distributed density estimation method resulting in oscillating behaviour where agents are concentrated and agents getting stuck when spread out. This could be because the density estimation method is based on local information, such that high-density areas yield high gradients and low-density areas yield close-to-zero gradients.

A logical next step in the exploration would be to extend the time-varying velocity fields into a finite-horizon control task, realizing a Model Predictive Control technique for large-scale multi-agent systems.

Moreover, future works are encouraged to investigate the relation and potential synergies with Optimal Mass Transport theory. Implementation of a PDE-solver enabling more advanced density distributions and feedback functions can also be looked into, as well as continuing the work in [8] of analyzing the control properties in terms of collision-avoidance and robustness.

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