Solutions for assignment 1

Mathematical Ecology

Immigration

Exercise 1.1 Let us try to model the population of a country with currently $N_0 = 70$ million inhabitants. Let us assume that the per-capita death rate is $d = 0.015$ deaths per year and the per-capita birth rate is $b = 0.01$ births per year. In addition there is a constant rate of immigration of $a = 300,000$ individuals per year.

i) Write down the ODE for the population number $N(t)$. At this point, do not use the numerical values yet but the symbols.

ii) Solve the ODE for $N(t)$ with the given initial condition. You may not have solved an ODE for some while so may need to look back at your Calculus notes. But don't panic: the equation from part (i) should be a linear, non-homogeneous, first-order ODE with constant coefficients, so you definitely know how to solve it. The easiest way to go about it is to first convert it into a homogeneous ODE by shifting the dependent variable.

iii) Substitute the numerical values to obtain the projected population after 10 years.

Solutions

i) The ODE for the population number $N(t)$ is

$$\frac{dN}{dt} = rN + a$$

where $r = b - d$ is the net growth rate.

ii) To solve the ODE we first convert it into a homogeneous ODE by shifting the dependent variable. We set $M(t) = N(t) + a/r$ and obtain

$$\frac{dM}{dt} = rM.$$

The solution to this ODE is $M(t) = M(0)e^{rt}$. Substituting back we get

$$N(t) = \left(N(0) + \frac{a}{r}\right) e^{rt} - \frac{a}{r}.$$

iii) Substituting the numerical values we get

$$\begin{aligned} N(10) &= \left(70\,000\,000 + \frac{300\,000}{0.01 - 0.015}\right) e^{(0.01-0.015)\cdot 10} - \frac{300\,000}{0.01 - 0.015} \\ &= \left(70\,000\,000 + \frac{300\,000}{-0.005}\right) e^{-0.05} - \frac{300\,000}{-0.005} \\ &= (70\,000\,000 - 60\,000\,000)\, e^{-0.05} + 60\,000\,000 \\ &\approx 10\,000\,000 \cdot 0.9512294 + 60\,000\,000 \\ &\approx 69\,512\,294. \end{aligned}$$

Sketching solutions

Exercise 1.3 Consider the population model with carrying capacity and Allee effect given by the differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{K_0} - 1\right).$$

Here $r > 0$, $K > K_0 > 0$ are constants. Simply by considering the shape of the right hand side, sketch a graph of $N(t)$ against t for several solutions with different initial conditions. Include two initial conditions between 0 and K_0, two initial conditions between K_0 and K and one initial condition larger than K. Note that the graph only needs to be qualitatively correct, similar to the rough sketch for the solutions of the logistic model sketched in the first lecture.

Solutions

First we make a sketch of the right-hand side of the ODE. The right-hand side is a cubic polynomial with roots at $N = 0$, $N = K$ and $N = K_0$. The coefficient of the cubic term is positive, so the graph is a cubic with positive leading coefficient. This gives us a sketch that will qualitatively look like Figure 1.

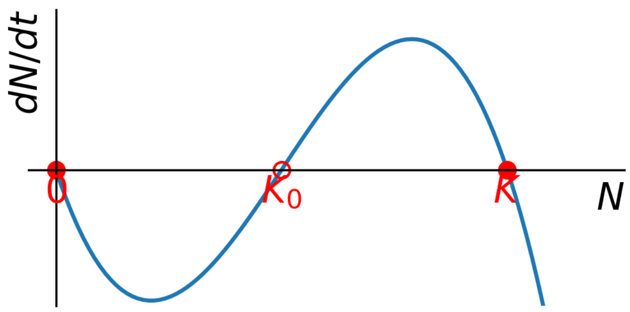

Figure 1: Plot of the rate of change.

To sketch solutions you have to draw lines that have a slope given by the rate of change which is the height of the plot in Figure 1. The slope of the solution is positive when the rate of change is positive and negative when the rate of change is negative. The slope is zero at the points $N = 0$, $N = K$ and $N = K_0$. The solution will be increasing when the slope is positive and decreasing when the slope is negative. The solution will be concave down when the slope is decreasing and concave up when the slope is increasing. This leads to a sketch qualitatively similar to the one in Figure 2.

Harvesting in Gompertz model

Exercise 1.5 Consider a population $N(t)$ that is described by the by the Gompertz model

$$\frac{dN}{dt} = \alpha N \log\frac{K}{N},$$

where α and K are positive constants. You want to harvest this population, for example by hunting or fishing, with some effort E. The rate at which you harvest individuals (which removes them from the population and hence results in an additional source of death) is proportional to the size of the population: $Y = EN$. This is called the yield. Write down the differential equation for $N(t)$ including this harvesting term. Determine the fixed points and their stability. Find the maximum sustainable yield, i.e., the maximum yield that can be sustained indefinitely.

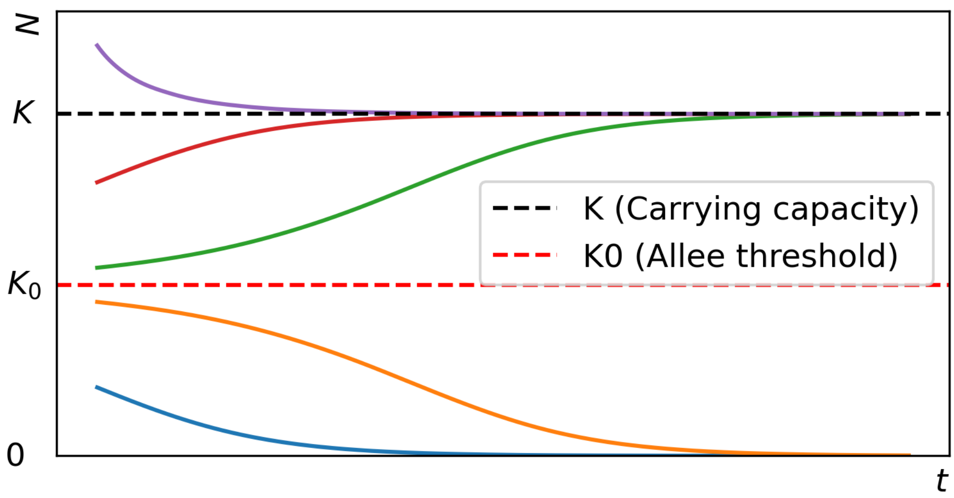

Figure 2: Plot of several solutions.

Solutions

If we harvest with a fixed effort E, the population is described by the equation

$$\frac{dN}{dt} = \alpha N \log \frac{K}{N} - EN = f(N).$$

To find the maximum sustainable yield we first need to find the non-zero fixed point of this equation by solving

$$0 = \alpha N^* \log \frac{K}{N^*} - EN^*.$$

There is a fixed point at zero, but this will not be relevant for the maximum sustainable yield. To find the non-zero fixed point we can divide out one factor of N^* to get

$$0 = \alpha \log \frac{K}{N^*} - E.$$

This we can solve for N^*:

$$N^* = K \exp\left(-\frac{E}{\alpha}\right).$$

To see that this fixed point is stable we just have to observe that the graph of the right-hand side of the equation crosses the x-axis from above to below at N^*. This is so because dN/dt becomes negative when N is larger than N^* and positive when N is smaller than N^*. We could also explicitly calculate $f'(N^*)$ but that is more work.

The yield at this fixed point is

$$Y^* = E\,K \exp\left(-\frac{E}{\alpha}\right).$$

To find the effort that maximises this yield we solve

$$0 = \frac{dY^*}{dE} = K\left(1 - \frac{E}{\alpha}\right) \exp\left(-\frac{E}{\alpha}\right).$$

This gives $E = \alpha$ and thus the maximum sustainable yield (MSY) is

$$MSY = \alpha\, K\, e^{-1}.$$

Discrete-time Ricker model

Exercise 2.4 Find the fixed points in the Ricker model

$$N_{t+1} = N_t \, e^{R_0\left(1-\frac{N_t}{K}\right)}.$$

and investigate their stability. Do this both analytically and by drawing cobweb diagrams. Allow also negative values of in your analysis, even though this is not ecologically realistic. Note that you will then need at least three cobweb diagrams because there are then two bifurcations.

Solutions

The fixed points are the solutions to

$$N^* = N^* \, e^{R_0\left(1-\frac{N^*}{K}\right)} =: f(N^*).$$

This equation has the obvious solutions $N^* = 0$. To find the non-zero solution we divide both sides by N^* to get

$$1 = e^{R_0\left(1-\frac{N^*}{K}\right)}.$$

Because the exponential can be equal to 1 only if the exponent is zero, we can solve for N^*:

$$N^* = K.$$

Next we determine the stability of the fixed points by calculating $f'(N^*)$. We have

$$f'(N^*) = R_0 e^{R_0\left(1-\frac{N^*}{K}\right)} \left(1 - \frac{N^* R_0}{K}\right).$$

For the zero fixed point we have $f'(0) = e^{R_0}$. So if $R_0 > 0$ we have $f'(0) > 1$ and the fixed point is unstable. If $R_0 < 0$ we have $f'(0) < 1$ and the fixed point is stable.

For the non-zero fixed point we have $f'(K) = 1 - R_0$. So if $R_0 \in (0, 2)$ we have $|f'(K)| < 1$ and the fixed point is stable. At $R_0 = 2$ the fixed point becomes unstable in a period-doubling bifurcation. This can also be seen from the cobweb diagram in Figure 3.

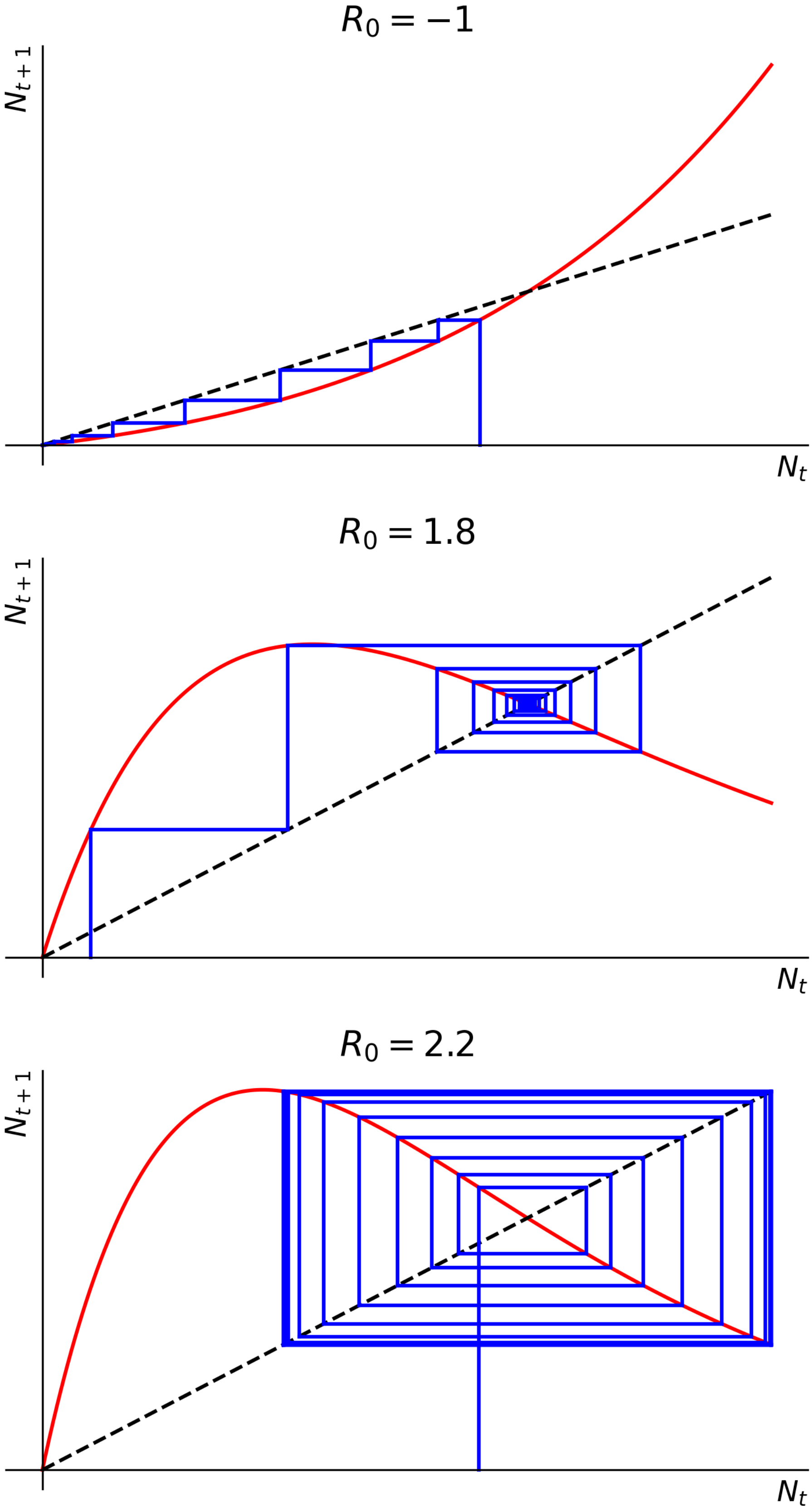

Figure 3: Cobweb diagrams for the Ricker model.