Waves and Fluids

Lecture notes for Spring 2023

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In this module we will explore the dynamics of continuous media, focusing on elementary fluid dynamics and the motion of waves. This lays the foundations for the full development of fluid dynamics in years 3 and 4, as well as for modules on electromagnetism and quantum mechanics. The mathematical techniques of vector calculus are employed and further developed, as are Fourier methods and methods from complex analysis.

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Welcome

These are the lecture notes for Waves and Fluids, part of the 2nd year Applied Maths module at the University of York in Spring 2023. Each chapter in these notes corresponds to one lecture.

These notes will be periodically revised. Whenever you spot something that is not quite right, please email me at gustav.delius@york.ac.uk or submit your correction in the correction form at https://forms.gle/w17c19vWnM7wpLpz7.

The material in this module consists of two topics: Waves and Fluids. We will spend the first half of the term on Waves and the second half on Fluids. The topics are linked by the fact that both use partial differential equations to describe real-world phenomena in space and time. Of course they share this feature with a large part of Applied Mathematics. So you will meet the ideas and methods introduced in this module again and again in future Applied Mathematics modules. In some sense, the actual subject matter of this module is less important than the way of thinking that it introduces.

The module is meant to prepare you for going out into the world and confronting new phenomena with the power of mathematics, not only in physics, but in biology, ecology, medicine, sociology, economics, and other areas. There will be many modules in the third and fourth year of you studies that will deepen that ability.

Throughout this module, we shall use SI units: length is measured in meters (m), time in seconds (s), mass in kilograms (kg).

Part I

Waves

Waves are so fundamentally important because waves are the only way information can propagate in this universe. Some waves that propagate information are obvious: sound waves, light and radio waves, electric waves travelling along our neurons. Others are less obvious: even if I communicate with you by shooting a particle at you, this is described by a wave, as you will learn in quantum mechanics. Gravitational effects are communicated via gravitational waves.

Because waves propagate at a finite speed, also information can only propagate at that wave speed. This has profound impacts, as you know from the theory of special relativity. For example, we can look far back into the past because some of the light waves emitted shortly after the big bang 16 billion years ago are only now arriving here, strongly red-shifted and thus detectable as microwaves.

Studying waves is also of great practical importance. They play an important part in our technological world. Improvements in our understanding of how to generate and control electromagnetic waves, for example, has led to radio, radar and mobile phones. Understanding how a pest propagates in the form of an invasion wave into so-far uninfected territory allows us to prepare adequate interventions. Understanding how density waves form in traffic flow, leading to traffic jams, allows us to design interventions that lead to smoother traffic flow.

1 Deriving the wave equation for a string

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 7
- Coulson and Jeffrey (1977) sections 17 and 19
- Baldock and Bridgeman (1983) section 1.10.3
- Simmons (1972) section 40

1.1 Why waves on a string?

The great diversity of waves in nature means that we need to choose some concrete wave phenomenon to concentrate on to start our investigation. In this module we will concentrate on the waves on a string (think of a guitar string) and generalise to waves on a membrane (think of the membrane of a drum). By studying this in detail you will develop the intuition and the skills that will allow you to understand other wave phenomena later. We'll come back to waves at the end of the part on Fluid Dynamics when we study waves on the surface of a fluid.

Personally I like studying vibrating strings because they are at the foundation of superstring theory. This is a "Theory of Everything" that posits that elementary particles are actually tiny strings, with different vibrational states corresponding to different elementary particles. As a Ph.D. student I showed how, if these strings move in certain higher-dimensional group supermanifolds, they behave like the elementary particles of our standard model of particle physics, including the chiral fermions. If we ignore the bit about group supermanifolds for the moment, the maths behind string theory is no more complicated than the maths we will discuss in this module and the partner module on quantum mechanics.

We consider a flexible, elastic string of linear density ρ (mass per unit length), which undergoes small *transverse* vibrations. (For example, it can be a guitar string.) The transverse vibrations mean that the displacements of each small element of the string is perpendicular to its length. We assume that the string does not move *longitudinally* (i.e. parallel to its length). Let $y(x, t)$ be its displacement from equilibrium position at time t and position x (see Fig. Figure 1.1).

The string is sufficiently simple, that we can understand it by pure thought. We will derive from first principles a PDE that describes its motion (the wave equation) and then solve it for various initial conditions. I find it amazing that this is possible.

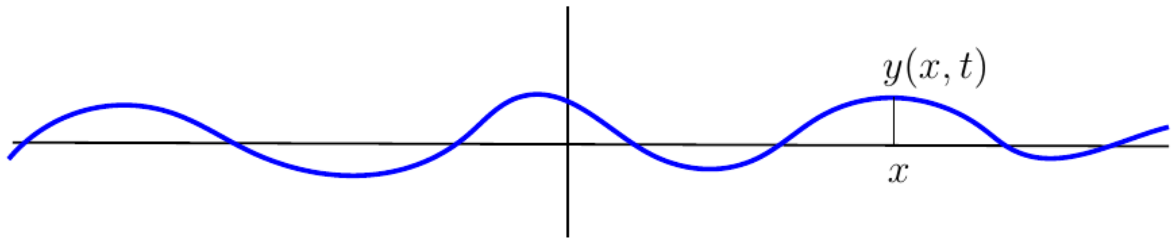

Figure 1.1: A string stretched in the x-direction and vibrating in the y-direction.

To derive the equation of motion of the string we first need to discuss the force acting on it which we will do in the next section. Then in the section after that we can plug this into Newton's second law and out pops the wave equation.

1.2 Linearized tension force

We consider a small segment of the string between any two points x and at $x + \delta x$ as shown in Fig. Figure 1.2. We want to determine the force that is acting on this segment, so that we can later determine its motion using Newton's second law. We will concentrate on only the tension force of the string and ignore less important effects like gravity, friction, or stiffness.

We assume that the tension force $T(x)$ has constant magnitude throughout the string: $|T(x)| = T$. However its direction varies along the string, because it always acts in the tangential direction. At interior points the tension force pulling to one side will balance that pulling in the other direction. The net tension force on the segment will thus be determined by the tension forces at its ends. We have drawn these forces schematically in Figure Figure 1.2 where we have also split them into their x and y components.

The total force acting on the segment is

$$F = T(x + \delta x) - T(x). \tag{1.1}$$

We first consider the y component

$$T_y(x) = T \sin\theta(x), \tag{1.2}$$

where $\theta(x)$ is the angle that the string makes with the horizontal at x. The slope of the string at x is

$$\frac{\partial y}{\partial x} = \tan\theta(x). \tag{1.3}$$

We are now going to simplify the expressions by assuming that the slope and thus θ is small, $\theta << 1$. Then, by Taylor expansion,

$$\sin\theta = \theta + O(\theta^3), \quad \tan\theta = \theta + O(\theta^2). \tag{1.4}$$

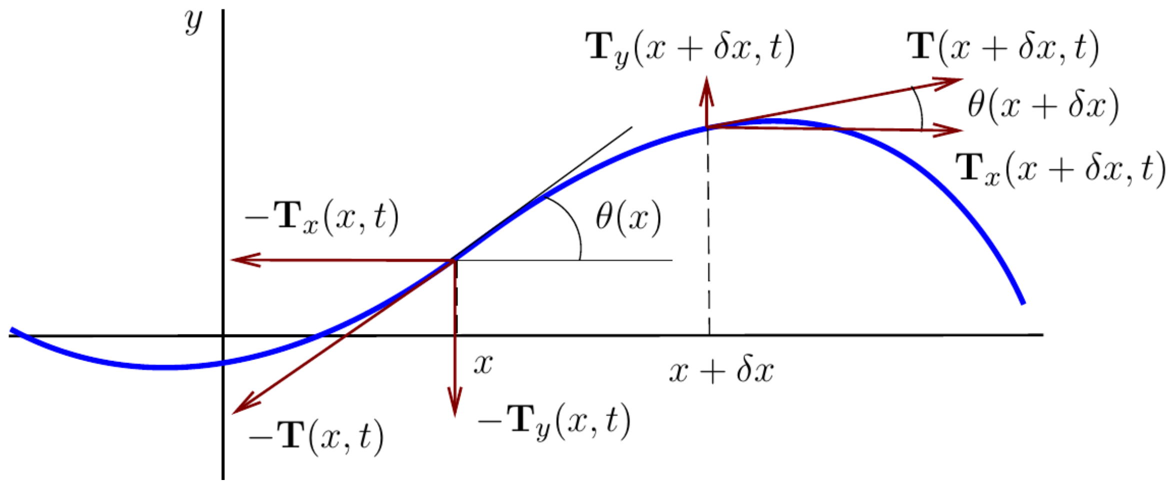

Figure 1.2: The tension forces acting on a segment of the string between x and $x + \delta x$.

We ignore all terms that are higher order in θ. This is known as the linear approximation. It is done very often, because it leads to linear equations that are so much easier to solve. So

$$\begin{aligned} F_y &= T_y(x+\delta x) - T_y(x) \\ &= T\sin\theta(x+\delta x) - T\sin\theta(x) \\ &\approx T\left(\theta(x+\delta x) - \theta(x)\right). \end{aligned} \tag{1.5}$$

We do another Taylor expansion and ignore higher-order terms in δx, which is fine because we want to look at only an infinitesimally small segment of string.

$$\begin{aligned} \theta(x+\delta x) &= \theta(x) + \delta x\frac{\partial\theta}{\partial x} + O(\delta x)^2 \\ &\approx \theta(x) + \delta x\frac{\partial\theta}{\partial x}. \end{aligned} \tag{1.6}$$

Substituting this into Eq. 1.5 gives

$$F_y \approx T\delta x\frac{\partial\theta}{\partial x}. \tag{1.7}$$

We would like to express this in terms of y instead of θ, which we can do by observing that

$$\theta \approx \tan\theta = \frac{\partial y}{\partial x}, \tag{1.8}$$

so we finally have

$$F_y \approx T\,\delta x\,\frac{\partial^2 y}{\partial x^2}. \tag{1.9}$$

We deal with the x component of the force similarly, using the Taylor expansion of $\cos\theta = 1 + O(\theta^2)$:

$$\begin{aligned} F_x &= T_x(x+\delta x) - T_x(x) \\ &= T\cos\theta(x+\delta_x) - T\cos\theta(x) \\ &\approx T - T = 0. \end{aligned} \tag{1.10}$$

So in our approximation of small slope, there is no movement in the x direction. The string vibrates purely transversally.

1.3 Wave equation from Newton's 2nd law

To determine the motion in the y direction we use Newton's second law

$$ma_y = F_y, \tag{1.11}$$

where a_y is the acceleration in the y direction,

$$a_y = \frac{\partial^2 y}{\partial t^2} \tag{1.12}$$

and m is the mass of the infinitesimal segment which is obtained as the density times the length,

$$m = \rho\delta x. \tag{1.13}$$

We assume that density ρ is constant along the string. Plugging this together with our expression for F_y into Newton's second law gives

$$\rho\delta x\frac{\partial^2 y}{\partial t^2} = T\delta x\frac{\partial^2 y}{\partial x^2}. \tag{1.14}$$

We can cancel the δx and divide by ρ which finally gives us the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{1.15}$$

with **wave speed**

$$c = \sqrt{\frac{T}{\rho}}. \tag{1.16}$$

Why we call the constant c the wave speed will become clear in the next lecture.

1.4 Checking dimensions

After having derived an equation, it is always wise to check that its dimensions work out correctly.

We use square brackets to denote the dimension of a quantity. So $[y] = L$ says that y has dimension of length, $[m] = M$ says that m has dimension of mass, and $[t] = T$ says that t has dimension of time. [1] The dimension of both sides of an equation has to agree, so

$$\left[\frac{\partial^2 y}{\partial t^2}\right] = \frac{L}{T^2} = [c^2]\left[\frac{\partial^2 y}{\partial x^2}\right] = [c^2]\frac{1}{L}. \tag{1.17}$$

This shows that $[c] = L/T$, so it has the dimension of a velocity. Because T is a force we have $[T] = ML/T^2$. The density ρ has $[\rho] = M/L$. So

$$[c] = \left[\sqrt{\frac{T}{\rho}}\right] = \sqrt{\frac{ML/T^2}{M/L}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}. \tag{1.18}$$

This completes our check of the dimensions.

[1] Note the conflict of notation where we used T for the tension force while it is also the conventional symbol for the dimension of time. Such conflicts happen from time to time – the context determines the meaning of the symbol.

2 d'Alembert's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 8
- Coulson and Jeffrey (1977) sections 7 and 11
- Baldock and Bridgeman (1983) section 2.1

First, we consider an infinitely long string (this is physically justified if we consider waves propagating far away from any boundaries). Mathematically, this means that we are looking for solutions of the wave equation on the whole real line $-\infty < x < +\infty$.

2.1 Wave equation in light-cone coordinates

We consider the wave equation

$$\partial_t^2 y - c^2\, \partial_x^2 y = 0 \tag{2.1}$$

for $-\infty < x < +\infty$. Note that I have switched to the convenient notation using subscripts on derivatives to specify the variable with respect to which we are differentiating.

Let's rewrite Eq. 2.1 using the *characteristic coordinates* (also known as light-cone coordinates)

$$\xi = x + ct, \quad \eta = x - ct. \tag{2.2}$$

By this we mean that for any function y that depends on the variables x and t we can introduce a function $\tilde{y}$ that depends on the variables ξ and η in such a way that it has the same values as y:

$$y(x,t) = \tilde{y}(\xi(x,t), \eta(x,t)) \text{ for all } x, t. \tag{2.3}$$

It is a conventional abuse of notation to drop the tilde and denote both functions by y. We will follow this abuse of notation.

We need to express the derivatives with respect to t and x via the derivatives with respect to ξ and η. This is done using the chain rule:

$$\begin{aligned} \partial_t\, y &= \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial t} \\ &= c\left(\partial_\xi - \partial_\eta\right) y \end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
\partial_x\, y &= \frac{\partial y}{\partial \xi}\,\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\,\frac{\partial \eta}{\partial x} \\
&= \left(\partial_\xi + \partial_\eta\right) y.
\end{aligned} \tag{2.5}
$$

Hence

$$
\partial_t = c\left(\partial_\xi - \partial_\eta\right), \quad \partial_x = \partial_\xi + \partial_\eta. \tag{2.6}
$$

Substituting these into the wave equation, we find that

$$
c^2\left(\partial_\xi - \partial_\eta\right)^2 y - c^2\left(\partial_\xi + \partial_\eta\right)^2 y = 0. \tag{2.7}
$$

Expanding the squares and cancelling terms gives

$$
-4c^2\partial_\xi\partial_\eta y = 0. \tag{2.8}
$$

We can divide both sides by the nonzero constant $-4c^2$. Thus the wave equation simplifies to

$$
\partial_\xi\partial_\eta y = 0. \tag{2.9}
$$

2.2 General solution of wave equation

The wave equation in light-cone variables is really easy to solve. First, we integrate Eq. 2.9 in the variable ξ [1]:

$$
\begin{aligned}
&\int \partial_\xi\partial_\eta y(\xi,\eta)\, d\xi = 0 \\
\Leftrightarrow \quad &\partial_\eta y(\xi,\eta) = F(\eta)
\end{aligned} \tag{2.10}
$$

where F is an arbitrary function of one variable [2]. Then we integrate Eq. 2.10 in the variable η:

$$
\begin{aligned}
y(\xi,\eta) &= \int \partial_\eta y(\xi,\eta)\, d\eta \\
&= \int F(\eta) d\eta + g(\xi) \\
&= f(\eta) + g(\xi),
\end{aligned} \tag{2.11}
$$

where $g(\xi)$ is an arbitrary function of one variable and $f'(\eta) = F(\eta)$. Note that since F is arbitrary, so is f.

Returning to variables x and t, we can write the general solution of the wave equation as

$$
y(x,t) = f(x-ct) + g(x+ct) \tag{2.12}
$$

where f and g are arbitrary functions of one variable.

[1] Note that when we integrate a function of two variables in one of the two variable, we need to add (to the result) an arbitrary function of the other variable. This is similar to adding a constant of integration when we integrate a function of one variable.

[2] You can verify that this is true by direct differentiation of Eq. 2.10 with respect to ξ.

2.3 Travelling waves

We will now gain an initial understanding of this solution by visualising the two special cases where either f or g are zero.

If $g = 0$, then $y(x,t) = f(x - ct)$. At $t = 0$, the string has the shape described by the graph $y = f(x)$. At time $t > 0$, it will have the same shape relative to the variable $\eta = x - ct$: $y = f(\eta)$. Since $x = \eta + ct$, this means that the graph of y as a function of x for a fixed $t > 0$ is the graph of $f(x)$ shifted to the *right* (in the direction of positive x) by distance ct.

If $f = 0$, then $y(x,t) = g(x + ct)$. At $t = 0$, the string has the shape described by the graph $y = g(x)$. At time $t > 0$, it will have the same shape relative to the variable $\xi = x + ct$: $y = g(\xi)$. Since $x = \xi - ct$, this means that the graph of y as a function of x for a fixed $t > 0$ is the graph of $g(x)$ shifted to the *left* (in the direction of negative x) by distance ct.

Thus, $f(x - ct)$ and $g(x + ct)$ describe waves that propagate (without changing shape) to the right and to the left, respectively, and the general solution Eq. 2.12 represent the sum of such waves.

2.4 Initial value problem and d'Alembert's formula

The initial-value problem is to solve the wave equation

$$\partial_t^2 y - c^2 \partial_x^2 y = 0 \tag{2.13}$$

for $-\infty < x < +\infty$ and $0 < t < +\infty$ with the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \tag{2.14}$$

for $-\infty < x < +\infty$, where y_0 and v_0 are given functions of x. The first of the two initial conditions prescribes the initial displacement of the string, the second the initial velocity.

The solution of this initial value problem can be found by substituting the general solution Eq. 2.12 into the initial conditions. This gives

$$y_0(x) = f(x) + g(x), \tag{2.15}$$

$$v_0(x) = -cf'(x) + cg'(x). \tag{2.16}$$

So we have two equations for the two unknown functions f and g. To solve them, we first integrate Eq. 2.16:

$$-cf(x) + cg(x) = \int_0^x v_0(s)ds + a = V(x) \tag{2.17}$$

where a is an integration constant and $V(x)$ is just introduced to save writing below.

Next, we add and subtract Eq. 2.15 and Eq. 2.17 divided by c. This results in

$$\begin{aligned} y_0(x) - \frac{1}{c}\,V(x) &= 2\,f(x) \\ y_0(x) + \frac{1}{c}\,V(x) &= 2\,g(x) \end{aligned} \tag{2.18}$$

which implies that

$$\begin{aligned} f(x) &= \frac{1}{2}\,y_0(x) - \frac{1}{2c}\,V(x) \\ g(x) &= \frac{1}{2}\,y_0(x) + \frac{1}{2c}\,V(x) \end{aligned} \tag{2.19}$$

Substituting these into the formula for the general solution, we get

$$\begin{aligned} y(x,t) =& \frac{1}{2}\,y_0(x-ct) - \frac{1}{2c}\,V(x-ct) \\ &+ \frac{1}{2}\,y_0(x+ct) + \frac{1}{2c}\,V(x+ct) \end{aligned} \tag{2.20}$$

or

$$\begin{aligned} y(x,t) =& \frac{1}{2}\,[y_0(x-ct) + y_0(x+ct)] \\ &+ \frac{1}{2c}\,[V(x+ct) - V(x-ct)] \end{aligned} \tag{2.21}$$

Note that only the difference $[V(x+ct) - V(x-ct)]$ appears so the integration constant cancels and also we can combine the two integrals into one because

$$\begin{aligned} V(x+ct) - V(x-ct) &= \int_0^{x+ct} v_0(s)\,ds - \int_0^{x-ct} v_0(s)\,ds \\ &= \int_{x-ct}^{x+ct} v_0(s)\,ds. \end{aligned} \tag{2.22}$$

Finally, we have

$$y(x,t) = \frac{1}{2}[y_0(x+ct) + y_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,ds. \tag{2.23}$$

This is the solution formula for the initial-value problem (Eq. 2.13, Eq. 2.14) and it is called the **d'Alembert formula**.

Remark. Once we have the d'Alembert formula, we can consider solutions of the initial-value problem (Eq. 2.13, Eq. 2.14) corresponding to piecewise smooth (or even piecewise continuous) initial functions $y_0(x)$ and $v_0(x)$. This will result in *generalised solutions* of the wave equation which are defined everywhere in the upper half of the (x,t) plane except for a finite number of lines where values of $y(x,t)$ and/or its first derivatives are discontinuous.

3 Boundaries and Interfaces

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 9
- Baldock and Bridgeman (1983) section 2.1 and 2.5

We have seen that the solutions of the wave equation predict right- and left-moving waves that travel without changing their shapes. Eventually, in the real world at least, these waves are going to reach the end of the string. What will happen then? We know that the energy that is carried by the wave can not simply disappear. So we expect the wave to be reflected. But how is it reflected in detail?

3.1 Semi-infinite string with fixed end

Let us consider the case where a right-moving wave hits the right end of the string. We choose the x-coordinate so that the end is at $x = 0$. Thus we consider the wave equation on the left half-line $-\infty < x < 0$. In this section we consider the case where the end of the string is fixed, so we impose the boundary condition

$$y(0,t) = 0 \quad \text{for all } t \in \mathbf{R}. \tag{3.1}$$

This is known as a Dirichlet boundary condition.

We recall the general solution of the wave equation:

$$y(x,t) = f(x - ct) + g(x + ct). \tag{3.2}$$

As we know, the function f gives the shape of the right-moving wave. Imposing the boundary condition will tell us what g has to be, i.e., it will determine the shape of the left-moving reflected wave. Substituting the general solution into the boundary condition gives

$$y(0,t) = f(-ct) + g(ct) = 0. \tag{3.3}$$

This holds for any value of t, so

$$g(s) = -f(-s) \quad \text{for all } s \in \mathbf{R}. \tag{3.4}$$

This tells us that the reflected wave is the negative of the incoming wave and is flipped front-to-back. Thus the solution is

$$y(x,t) = f(x-ct) - f(-x-ct) \quad \text{for all } x \le 0. \tag{3.5}$$

This is illustrated in Figure 3.1.

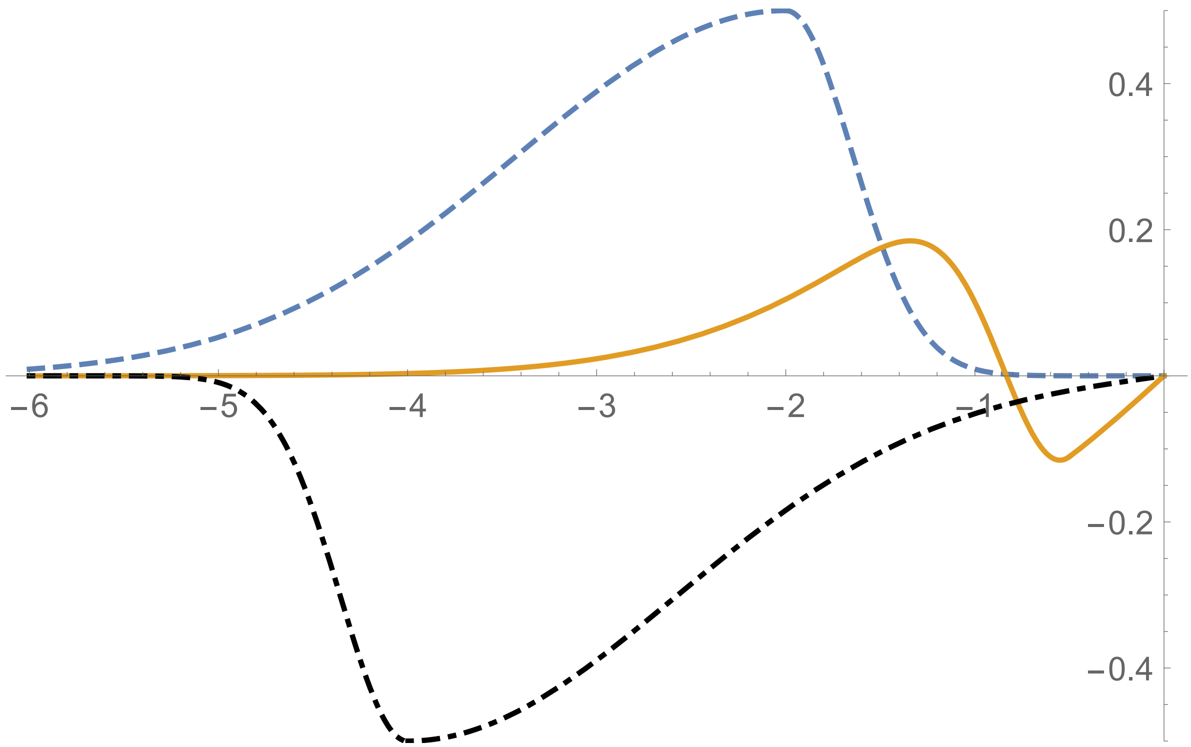

Figure 3.1: Reflection off a fixed end. Dashed line: incident right-moving wave. Solid line: wave interacting with the boundary. Dashdotted line: reflected left-moving wave. The reflected wave has the same shape as the incident wave but is flipped in both y and x.

3.2 Semi-infinite string with free end

Consider now a semi-infinite string ($0 < x < \infty$) with a free end at $x = 0$ (e.g. the end of the string can be attached to a small ring, which in turn can slide along a vertical rod without friction). This means that the vertical component of the tension force applied to the end of the string must be zero, which in turn means the string must be horizontal at x, i.e., we have the boundary condition

$$\partial_x y(0,t) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Conditions which specify the value of the normal derivative of the unknown function at the boundary are called *Neumann conditions.* So, here we have the homogeneous Neumann condition at $x = 0$. We now substitute the general solution. First we calculate its derivative

$$\partial_x y(x,t) = f'(x-ct) + g'(x+ct) \tag{3.7}$$

and thus the boundary condition says that

$$\partial_x y(0,t) = f'(-ct) + g'(ct) = 0 \quad \text{ for all } t \in \mathbb{R}. \tag{3.8}$$

Integrating this gives

$$\frac{-1}{c} f(-ct) + \frac{1}{c} g(ct) = \text{constant}. \tag{3.9}$$

Changing the constant only moves the string up or down on the y axis. We choose it to be zero. Because the boundary condition holds for all times, we have that

$$g(s) = f(-s) \quad \text{ for all } s \in \mathbb{R}. \tag{3.10}$$

Thus the reflected wave has the same shape and the same sign as the incoming wave, but it is still flipped front-to-back.Thus the solution is

$$y(x,t) = f(x-ct) + f(-x-ct) \quad \text{for all } x \leq 0. \tag{3.11}$$

This is illustrated in Figure 3.2.

3.3 Reflection at a change of density

Consider two semi-infinite strings joined at the origin. The string on the left ($x < 0$) has constant density ρ_1 and the string on the right ($x > 0$) has constant density ρ_2. Let y_1 and y_2 be the displacements of the two strings. Since the strings have different densities, the wave speed in the two strings will be different:

$$c_1 = \sqrt{\frac{T}{\rho_1}} \quad \text{and} \quad c_2 = \sqrt{\frac{T}{\rho_2}}. \tag{3.12}$$

Suppose that we have a wave travelling to the right on the first string (an incident wave). When the wave meets the change in density, it will be partially reflected (back to the region $x < 0$) and partially transmitted (forward to the region $x > 0$). Waves travelling in the interval $x \in (-\infty, 0)$ are described by the wave equation with wave speed c_1; waves travelling in the interval $x \in (0, \infty)$ are described by the wave equation with wave speed c_2. This is illustrated in Figure Figure 3.3.

Therefore, we can write

$$y(x,t) = \begin{cases} y_1(x,t) = f_I(x-c_1t) + f_R(x+c_1t), & x < 0 \\ y_2(x,t) = f_T(x-c_2t), & x > 0 \end{cases} \tag{3.13}$$

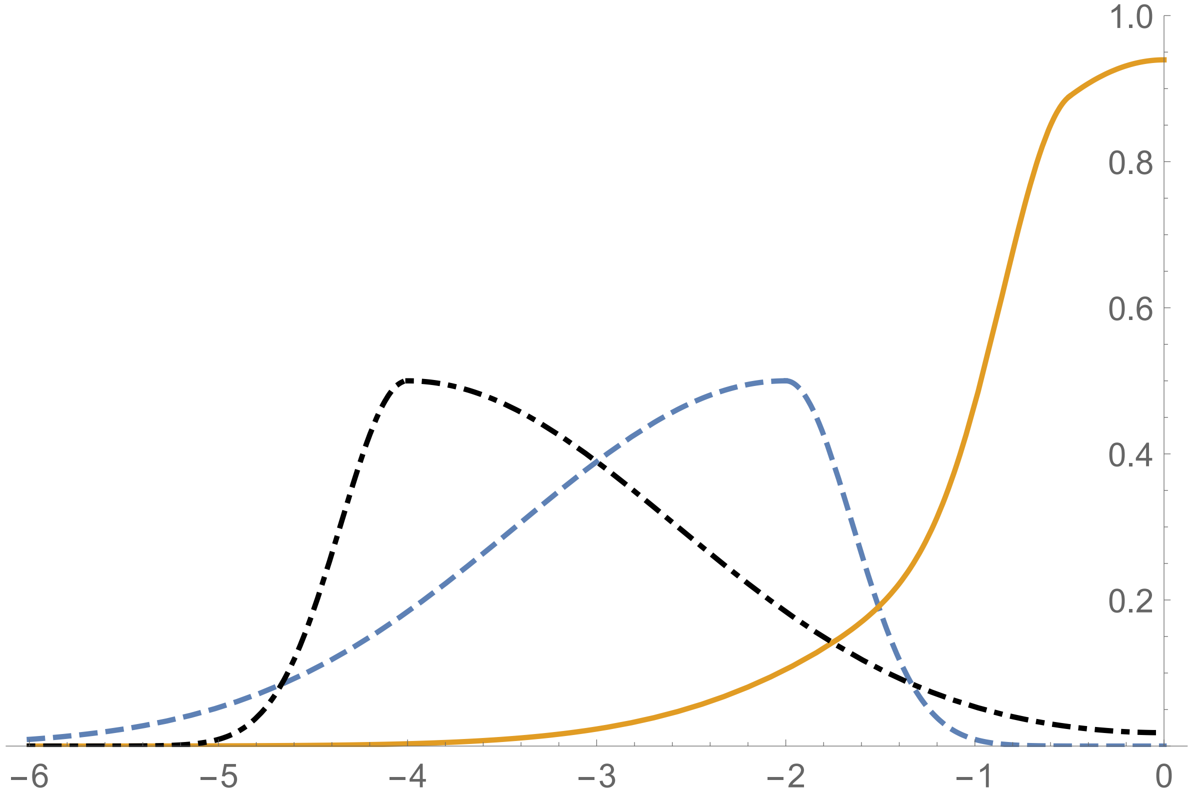

Figure 3.2: Reflection off a free end. Dashed line: incident right-moving wave. Solid line: wave interacting with the boundary. Dashdotted line: reflected left-moving wave. The reflected wave has the same shape as the incident wave but is flipped in x.

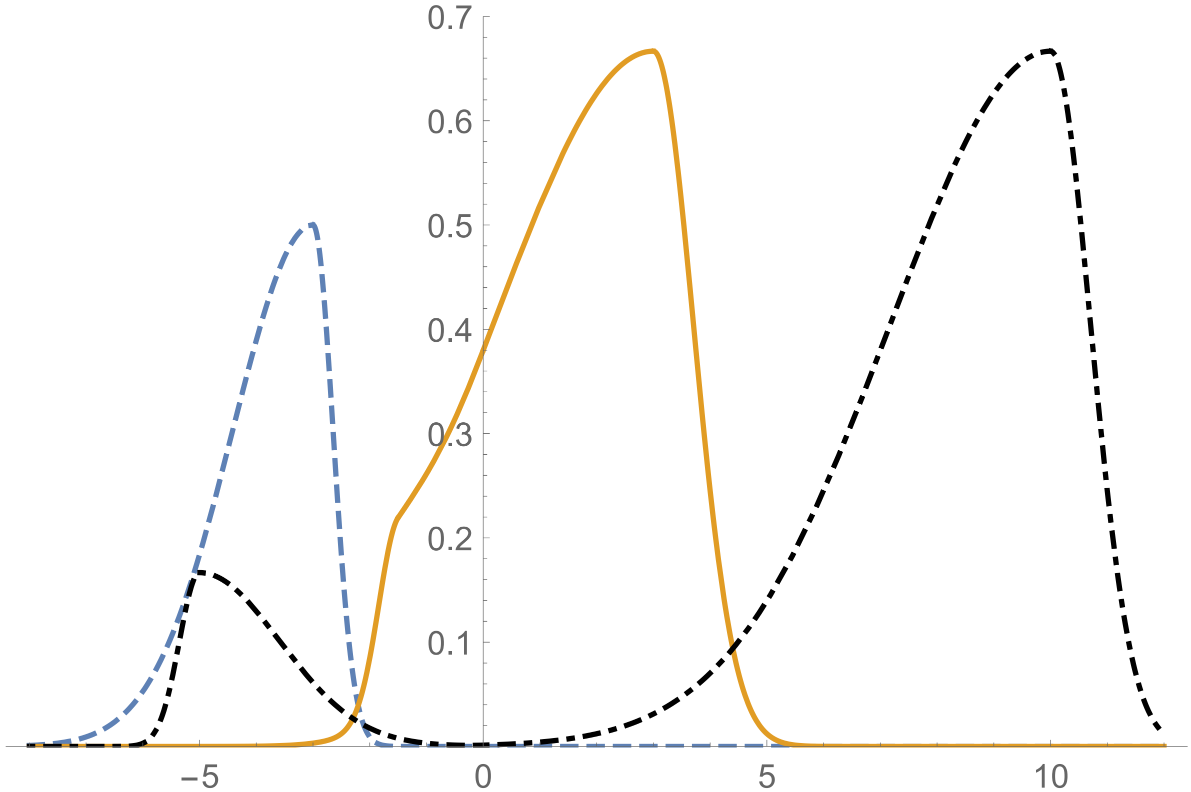

Figure 3.3: A right-moving wave being partially reflected and partially transmitted at the interface between two strings with wave velocities $c_1 = 1$ (on the left half-line) and $c_2 = 2$ (on the right half-line). Dashed line: incident right-moving wave. Solid line: wave interacting with the interface. Dashdotted line: partially reflected left-moving and partially transmitted right-moving wave.

where f_I, f_R and f_T represent the incident, reflected and transmitted waves, respectively. At the point of contact of the two strings ($x = 0$), we impose the following two conditions:

$$y_1(0,t) = y_2(0,t) \quad \text{for all } t, \tag{3.14}$$

$$\partial_x y_1(0,t) = \partial_x y_2(0,t) \quad \text{for all } t. \tag{3.15}$$

Condition 3.14 says that the solution for the combined string should be continuous at $x = 0$ (because the strings are attached to each other at the point). Condition 3.15 states that the slopes of the strings at $x = 0$ should be the same (if this is not so, there will be a finite force applied to an infinitesimal part of the combined string at $x = 0$, producing unphysical infinite acceleration).

Suppose that the incident wave is given, i.e. the function of one variable f_I is known. Can we find f_R and f_T?

Substitution of Eq. 3.13 into condition Eq. 3.14 yields

$$f_I(-c_1 t) + f_R(c_1 t) = f_T(-c_2 t) \quad \text{for all } t \tag{3.16}$$

or equivalently (writing $s = c_1 t$)

$$f_I(-s) + f_R(s) = f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.17}$$

Similarly, on substituting Eq. 3.13 into condition Eq. 3.15, we find that

$$f_I'(-c_1 t) + f_R'(c_1 t) = f_T'(-c_2 t) \quad \text{for all } t \tag{3.18}$$

or

$$f_I'(-s) + f_R'(s) = f_T'\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.19}$$

Integrating this equation in s, we get

$$-f_I(-s) + f_R(s) = -\frac{c_1}{c_2}\, f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.20}$$

Eliminating $f_R(s)$ from Eq. 3.17 and Eq. 3.20, we obtain

$$2 f_I(-s) = \left(1 + \frac{c_1}{c_2}\right) f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.21}$$

or, equivalently,

$$f_T(\tilde{s}) = \frac{2c_2}{c_2 + c_1}\, f_I\left(\frac{c_1}{c_2}\, \tilde{s}\right) \quad \text{for all } \tilde{s} \tag{3.22}$$

where $\tilde{s} = -(c_2/c_1)s$. Eq. 3.22 defines the function (of one variable) f_T in terms of known function f_I.

To find f_R, we substitute Eq. 3.22 into Eq. 3.17. This yields the formula

$$f_R(s) = \frac{c_2 - c_1}{c_2 + c_1} f_I(-s) \quad \text{for all } s. \tag{3.23}$$

Finally, on substituting Eq. 3.22 and Eq. 3.23 into Eq. 3.13, we get the solution formula

$$y(x,t) = \begin{cases} f_I(x - c_1 t) + A_R f_I(-x - c_1 t), & x < 0 \\ A_T f_I\left(\frac{c_1}{c_2}(x - c_2 t)\right), & x > 0 \end{cases} \tag{3.24}$$

where A_R (the ratio of the amplitude of the reflected wave to that of the incident wave) and A_T (the ratio of the amplitude of the transmitted wave to that of the incident wave) are given by

$$A_R = \frac{c_2 - c_1}{c_2 + c_1}, \quad A_T = \frac{2c_2}{c_2 + c_1}. \tag{3.25}$$

To check whether our answer makes sense, it is useful to consider a few limit cases.

i. If $\rho_1 = \rho_2$, then $c_1 = c_2$, and we have $A_R = 0$ and $A_T = 1$ (as one would expect for an infinite homogeneous string).

ii. If $\rho_1 \ll \rho_2$ (the first string is much lighter than the second one), then $c_1 \gg c_2$, and we have $A_R \approx -1$ and $A_T = 0$ (so that the heavy string effectively arrests the displacement at $x = 0$ as if the right end of the first string was fixed).

iii. If $\rho_1 \gg \rho_2$ (the first string is much heavier than the second one), then $c_1 \ll c_2$, and we have $A_R \approx 1$ and $A_T = 2$ (so that the light second string does not effect the displacement at $x = 0$ as if the right end of the first string was free).

4 Bernoulli's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) sections 11.2 and 11.3, chapters 13 and 14
- Baldock and Bridgeman (1983) sections 3.1, 3.2, 3.3
- Simmons (1972) sections 39 and 40

We will now solve the wave equation again in a totally different way from d'Alembert, following instead Bernoulli. This is a method you have seen before, namely the method of separation of variables.

4.1 Separation of variables

We make the Ansatz that the solution factorises into one function of only x and one function of only t:

$$y(x,t) = X(x)T(t). \tag{4.1}$$

Substituting this into the wave equation

$$\partial_x^2 y(x,t) = \frac{1}{c^2}\partial_t^2 y(x,t) \tag{4.2}$$

we get

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t). \tag{4.3}$$

We divide both sides by $X(x)T(t)$, which will separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}. \tag{4.4}$$

It the last equation, we have a function of x only on the left hand side and a function of t only on the right hand side. The equation must hold for all x and t. This is only possible if both functions are equal to a constant. It will turn out to be convenient to write this constant as $-k^2$ for some new constant k. Thus we set

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = -k^2. \tag{4.5}$$

This means that $X(x)$ and $T(t)$ must be solutions of the ODEs

$$X'' = -k^2 X \quad \text{and} \quad T'' = -k^2 c^2 T. \tag{4.6}$$

The general solutions of these ODEs are

$$\begin{aligned} X(x) &= A\,\sin(kx) + B\,\cos(kx), \\ T(t) &= F\,\sin(kct) + G\,\cos(kct), \end{aligned} \tag{4.7}$$

where A, B, F, G and k are arbitrary constants.

4.2 Finite string with fixed ends

Bernoulli was most interested in the finite string. Let us put the ends of the string at $x = 0$ and $x = \pi$. Now let the ends of the string be fixed. This means that we need to impose homogeneous Dirichlet boundary conditions

$$y(0,t) = 0 \quad \text{and} \quad y(\pi,t) = 0 \quad \text{for all} \quad t. \tag{4.8}$$

These conditions will be satisfied if $X(0) = 0$ and $X(\pi) = 0$. Imposing the condition $X(0) = 0$ on the general solution for $X(x)$, we find that we need $B = 0$. We then have from $X(\pi) = 0$ that

$$A \sin(k\pi) = 0. \tag{4.9}$$

The last equation implies that either $A = 0$ or $\sin(k\pi) = 0$. We reject the first option because it results in a zero solution. The second option yields

$$\sin(k\pi) = 0 \quad \Rightarrow k \in \mathbb{Z}. \tag{4.10}$$

Thus, for each positive integer k we have a solution of the wave equation of the form

$$y_k(x,t) = \sin(kx)\left(F_k\,\sin(kct) + G_k\,\cos(kct)\right) \tag{4.11}$$

[1]

4.3 Standing waves

The solutions we obtained in the previous section are standing waves. They don't change their shape and don't move, only their amplitude oscillates with time. In the lecture videos I make various sketches and animations to illustrate this.

[1] We do not need negative k because they will produce the same solutions, and we do not need $k = 0$ because it yields zero solution. Also, we have absorbed the constant A into the constants F_k and G_k.

So we now have a stark contrast between what d'Alembert found and what Bernoulli found. According to d'Alembert, the solutions of the wave equation are travelling waves of arbitrary shape. According to Bernoulli, the solutions are standing waves that have sine shape. It is always wonderful when one has two very different ways of looking at the same phenomenon. That is where deep understanding comes from.

The resolution of the apparent paradox is that a superposition of standing waves can give a travelling wave, and that a superposition of sine waves can give any shape.

4.4 Initial value problem

Since the wave equation is linear and also the boundary conditions we imposed were linear, a linear combination of any number of the solutions is also a solution (this is called the *superposition principle*. So, we can construct a general solution of the wave equation by summing up the harmonic standing waves from Eq. 4.11:

$$y(x,t) = \sum_{k=1}^{\infty} \sin(kx) \left(F_k \sin(kct) + G_k \cos(kct) \right). \tag{4.12}$$

Here the G_k and the F_k are undetermined constants, to be fixed from the initial conditions.

Suppose now that we are given the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \quad \text{for } x \in [0,\pi]. \tag{4.13}$$

How to choose constants F_k and G_k so as to satisfy the initial conditions? Let's substitute Eq. 4.12 into Eq. 4.13. We have

$$y_0(x) = \sum_{k=1}^{\infty} G_k \sin(kx), \quad v_0(x) = \sum_{k=1}^{\infty} F_k \, kc \sin(kx). \tag{4.14}$$

These formulae look like Fourier series for the functions y_0 and v_0. So, to find coefficients F_k and G_k, we use the identity

$$\int_0^{\pi} \sin(kx) \sin(lx) dx = \frac{\pi}{2} \delta_{kl},$$

where δ_{kl} is the Kronecker delta:

$$\delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

To make use of this, we multiply each of Eq. 4.14 by $2/\pi \ \sin(lx)$ and integrate over the interval $[0, \pi]$:

$$\begin{aligned}\frac{2}{\pi}\int\limits_0^\pi y_0(x)\sin(lx)dx &= \frac{2}{\pi}\sum_{k=1}^{\infty} G_k \int\limits_0^\pi \sin\left(kx\right)\sin(lx)dx \\ &= \sum_{k=1}^{\infty} G_k \delta_{kl} = G_l\end{aligned} \tag{4.15}$$

$$\begin{aligned}\frac{2}{\pi}\int\limits_0^\pi v_0(x)\sin(lx)dx &= \frac{2}{\pi}\sum_{k=1}^{\infty} F_k\, kc \int\limits_0^\pi \sin\left(kx\right)\sin(lx)dx \\ &= \sum_{k=1}^{\infty} F_k\, kc\, \delta_{kl} = F_l\, lc\end{aligned} \tag{4.16}$$

Thus we know how to determine the constants F_k and G_k in the general solution Eq. 4.12 such as to satisfy given initial conditions.

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