Waves and Fluids

Lecture notes for Spring 2023

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In this module we will explore the dynamics of continuous media, focusing on elementary fluid dynamics and the motion of waves. This lays the foundations for the full development of fluid dynamics in years 3 and 4, as well as for modules on electromagnetism and quantum mechanics. The mathematical techniques of vector calculus are employed and further developed, as are Fourier methods and methods from complex analysis.

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Welcome

These are the lecture notes for Waves and Fluids, part of the 2nd year Applied Maths module at the University of York in Spring 2023. Each chapter in these notes corresponds to one lecture.

These notes will be periodically revised. Whenever you spot something that is not quite right, please email me at gustav.delius@york.ac.uk or submit your correction in the correction form at https://forms.gle/w17c19vWnM7wpLpz7.

The material in this module consists of two topics: Waves and Fluids. We will spend the first half of the term on Waves and the second half on Fluids. The topics are linked by the fact that both use partial differential equations to describe real-world phenomena in space and time. Of course they share this feature with a large part of Applied Mathematics. So you will meet the ideas and methods introduced in this module again and again in future Applied Mathematics modules. In some sense, the actual subject matter of this module is less important than the way of thinking that it introduces.

The module is meant to prepare you for going out into the world and confronting new phenomena with the power of mathematics, not only in physics, but in biology, ecology, medicine, sociology, economics, and other areas. There will be many modules in the third and fourth year of you studies that will deepen that ability.

Throughout this module, we shall use SI units: length is measured in meters (m), time in seconds (s), mass in kilograms (kg).

Part I

Waves

Waves are so fundamentally important because waves are the only way information can propagate in this universe. Some waves that propagate information are obvious: sound waves, light and radio waves, electric waves travelling along our neurons. Others are less obvious: even if I communicate with you by shooting a particle at you, this is described by a wave, as you will learn in quantum mechanics. Gravitational effects are communicated via gravitational waves.

Because waves propagate at a finite speed, also information can only propagate at that wave speed. This has profound impacts, as you know from the theory of special relativity. For example, we can look far back into the past because some of the light waves emitted shortly after the big bang 16 billion years ago are only now arriving here, strongly red-shifted and thus detectable as microwaves.

Studying waves is also of great practical importance. They play an important part in our technological world. Improvements in our understanding of how to generate and control electromagnetic waves, for example, has led to radio, radar and mobile phones. Understanding how a pest propagates in the form of an invasion wave into so-far uninfected territory allows us to prepare adequate interventions. Understanding how density waves form in traffic flow, leading to traffic jams, allows us to design interventions that lead to smoother traffic flow.

1 Deriving the wave equation for a string

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 7
- Coulson and Jeffrey (1977) sections 17 and 19
- Baldock and Bridgeman (1983) section 1.10.3
- Simmons (1972) section 40

1.1 Why waves on a string?

The great diversity of waves in nature means that we need to choose some concrete wave phenomenon to concentrate on to start our investigation. In this module we will concentrate on the waves on a string (think of a guitar string) and generalise to waves on a membrane (think of the membrane of a drum). By studying this in detail you will develop the intuition and the skills that will allow you to understand other wave phenomena later. We'll come back to waves at the end of the part on Fluid Dynamics when we study waves on the surface of a fluid.

Personally I like studying vibrating strings because they are at the foundation of superstring theory. This is a "Theory of Everything" that posits that elementary particles are actually tiny strings, with different vibrational states corresponding to different elementary particles. As a Ph.D. student I showed how, if these strings move in certain higher-dimensional group supermanifolds, they behave like the elementary particles of our standard model of particle physics, including the chiral fermions. If we ignore the bit about group supermanifolds for the moment, the maths behind string theory is no more complicated than the maths we will discuss in this module and the partner module on quantum mechanics.

We consider a flexible, elastic string of linear density ρ (mass per unit length), which undergoes small *transverse* vibrations. (For example, it can be a guitar string.) The transverse vibrations mean that the displacements of each small element of the string is perpendicular to its length. We assume that the string does not move *longitudinally* (i.e. parallel to its length). Let $y(x,t)$ be its displacement from equilibrium position at time t and position x (see Fig. Figure 1.1).

The string is sufficiently simple, that we can understand it by pure thought. We will derive from first principles a PDE that describes its motion (the wave equation) and then solve it for various initial conditions. I find it amazing that this is possible.

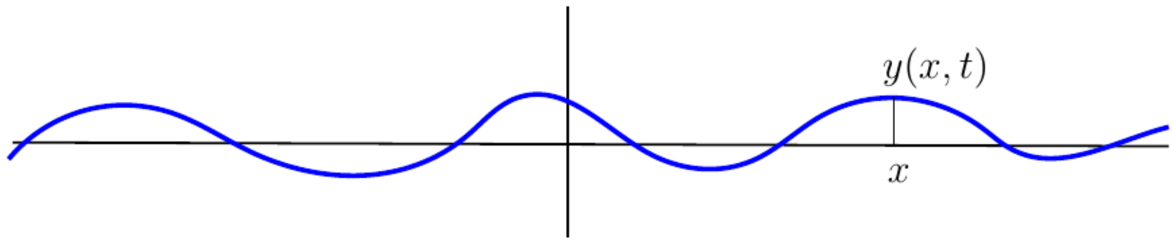

Figure 1.1: A string stretched in the x-direction and vibrating in the y-direction.

To derive the equation of motion of the string we first need to discuss the force acting on it which we will do in the next section. Then in the section after that we can plug this into Newton's second law and out pops the wave equation.

1.2 Linearized tension force

We consider a small segment of the string between any two points x and at $x + \delta x$ as shown in Fig. Figure 1.2. We want to determine the force that is acting on this segment, so that we can later determine its motion using Newton's second law. We will concentrate on only the tension force of the string and ignore less important effects like gravity, friction, or stiffness.

We assume that the tension force $T(x)$ has constant magnitude throughout the string: $|T(x)| = T$. However its direction varies along the string, because it always acts in the tangential direction. At interior points the tension force pulling to one side will balance that pulling in the other direction. The net tension force on the segment will thus be determined by the tension forces at its ends. We have drawn these forces schematically in Figure Figure 1.2 where we have also split them into their x and y components.

The total force acting on the segment is

$$F = T(x + \delta x) - T(x). \tag{1.1}$$

We first consider the y component

$$T_y(x) = T \sin\theta(x), \tag{1.2}$$

where $\theta(x)$ is the angle that the string makes with the horizontal at x. The slope of the string at x is

$$\frac{\partial y}{\partial x} = \tan\theta(x). \tag{1.3}$$

We are now going to simplify the expressions by assuming that the slope and thus θ is small, $\theta << 1$. Then, by Taylor expansion,

$$\sin\theta = \theta + O(\theta^3), \quad \tan\theta = \theta + O(\theta^2). \tag{1.4}$$

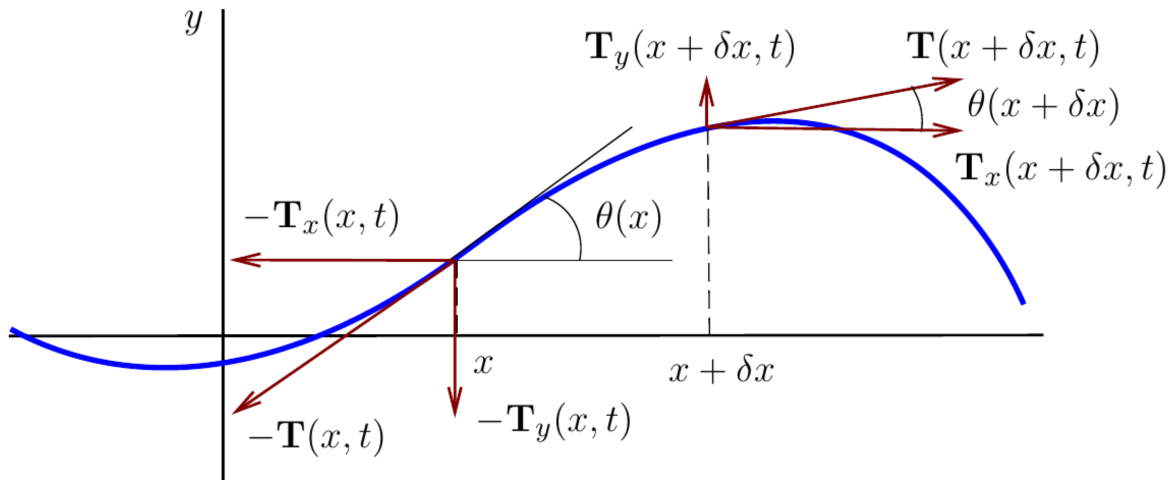

Figure 1.2: The tension forces acting on a segment of the string between x and $x + \delta x$.

We ignore all terms that are higher order in θ. This is known as the linear approximation. It is done very often, because it leads to linear equations that are so much easier to solve. So

$$\begin{aligned} F_y &= T_y(x+\delta x) - T_y(x) \\ &= T \sin\theta(x+\delta x) - T \sin\theta(x) \\ &\approx T\left(\theta(x+\delta x) - \theta(x)\right). \end{aligned} \tag{1.5}$$

We do another Taylor expansion and ignore higher-order terms in δx, which is fine because we want to look at only an infinitesimally small segment of string.

$$\begin{aligned} \theta(x+\delta x) &= \theta(x) + \delta x \frac{\partial\theta}{\partial x} + O(\delta x)^2 \\ &\approx \theta(x) + \delta x \frac{\partial\theta}{\partial x}. \end{aligned} \tag{1.6}$$

Substituting this into Eq. 1.5 gives

$$F_y \approx T\delta x \frac{\partial\theta}{\partial x}. \tag{1.7}$$

We would like to express this in terms of y instead of θ, which we can do by observing that

$$\theta \approx \tan\theta = \frac{\partial y}{\partial x}, \tag{1.8}$$

so we finally have

$$F_y \approx T\,\delta x\,\frac{\partial^2 y}{\partial x^2}. \tag{1.9}$$

We deal with the x component of the force similarly, using the Taylor expansion of $\cos\theta = 1 + O(\theta^2)$:

$$\begin{aligned} F_x &= T_x(x+\delta x) - T_x(x) \\ &= T\cos\theta(x+\delta_x) - T\cos\theta(x) \\ &\approx T - T = 0. \end{aligned} \tag{1.10}$$

So in our approximation of small slope, there is no movement in the x direction. The string vibrates purely transversally.

1.3 Wave equation from Newton's 2nd law

To determine the motion in the y direction we use Newton's second law

$$ma_y = F_y, \tag{1.11}$$

where a_y is the acceleration in the y direction,

$$a_y = \frac{\partial^2 y}{\partial t^2} \tag{1.12}$$

and m is the mass of the infinitesimal segment which is obtained as the density times the length,

$$m = \rho\delta x. \tag{1.13}$$

We assume that density ρ is constant along the string. Plugging this together with our expression for F_y into Newton's second law gives

$$\rho\delta x\frac{\partial^2 y}{\partial t^2} = T\delta x\frac{\partial^2 y}{\partial x^2}. \tag{1.14}$$

We can cancel the δx and divide by ρ which finally gives us the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{1.15}$$

with **wave speed**

$$c = \sqrt{\frac{T}{\rho}}. \tag{1.16}$$

Why we call the constant c the wave speed will become clear in the next lecture.

1.4 Checking dimensions

After having derived an equation, it is always wise to check that its dimensions work out correctly.

We use square brackets to denote the dimension of a quantity. So $[y] = L$ says that y has dimension of length, $[m] = M$ says that m has dimension of mass, and $[t] = T$ says that t has dimension of time. [1] The dimension of both sides of an equation has to agree, so

$$\left[\frac{\partial^2 y}{\partial t^2}\right] = \frac{L}{T^2} = [c^2]\left[\frac{\partial^2 y}{\partial x^2}\right] = [c^2]\frac{1}{L}. \tag{1.17}$$

This shows that $[c] = L/T$, so it has the dimension of a velocity. Because T is a force we have $[T] = ML/T^2$. The density ρ has $[\rho] = M/L$. So

$$[c] = \left[\sqrt{\frac{T}{\rho}}\right] = \sqrt{\frac{ML/T^2}{M/L}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}. \tag{1.18}$$

This completes our check of the dimensions.

[1] Note the conflict of notation where we used T for the tension force while it is also the conventional symbol for the dimension of time. Such conflicts happen from time to time – the context determines the meaning of the symbol.

2 d'Alembert's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 8
- Coulson and Jeffrey (1977) sections 7 and 11
- Baldock and Bridgeman (1983) section 2.1

First, we consider an infinitely long string (this is physically justified if we consider waves propagating far away from any boundaries). Mathematically, this means that we are looking for solutions of the wave equation on the whole real line $-\infty < x < +\infty$.

2.1 Wave equation in light-cone coordinates

We consider the wave equation

$$\partial_t^2 y - c^2\, \partial_x^2 y = 0 \tag{2.1}$$

for $-\infty < x < +\infty$. Note that I have switched to the convenient notation using subscripts on derivatives to specify the variable with respect to which we are differentiating.

Let's rewrite Eq. 2.1 using the *characteristic coordinates* (also known as light-cone coordinates)

$$\xi = x + ct, \quad \eta = x - ct. \tag{2.2}$$

By this we mean that for any function y that depends on the variables x and t we can introduce a function $\tilde{y}$ that depends on the variables ξ and η in such a way that it has the same values as y:

$$y(x,t) = \tilde{y}(\xi(x,t), \eta(x,t)) \text{ for all } x, t. \tag{2.3}$$

It is a conventional abuse of notation to drop the tilde and denote both functions by y. We will follow this abuse of notation.

We need to express the derivatives with respect to t and x via the derivatives with respect to ξ and η. This is done using the chain rule:

$$\begin{aligned}\partial_t\, y &= \frac{\partial y}{\partial \xi}\,\frac{\partial \xi}{\partial t} + \frac{\partial y}{\partial \eta}\,\frac{\partial \eta}{\partial t} \\ &= c\left(\partial_\xi - \partial_\eta\right) y\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\partial_x y &= \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \left(\partial_\xi + \partial_\eta\right) y.\end{aligned} \tag{2.5}$$

Hence

$$\partial_t = c\left(\partial_\xi - \partial_\eta\right), \quad \partial_x = \partial_\xi + \partial_\eta. \tag{2.6}$$

Substituting these into the wave equation, we find that

$$c^2\left(\partial_\xi - \partial_\eta\right)^2 y - c^2\left(\partial_\xi + \partial_\eta\right)^2 y = 0. \tag{2.7}$$

Expanding the squares and cancelling terms gives

$$-4c^2\partial_\xi\partial_\eta y = 0. \tag{2.8}$$

We can divide both sides by the nonzero constant $-4c^2$. Thus the wave equation simplifies to

$$\partial_\xi\partial_\eta y = 0. \tag{2.9}$$

2.2 General solution of wave equation

The wave equation in light-cone variables is really easy to solve. First, we integrate Eq. 2.9 in the variable ξ [1]:

$$\begin{aligned}&\int \partial_\xi\partial_\eta y(\xi,\eta)\, d\xi = 0 \\ \Leftrightarrow \quad &\partial_\eta y(\xi,\eta) = F(\eta)\end{aligned} \tag{2.10}$$

where F is an arbitrary function of one variable [2]. Then we integrate Eq. 2.10 in the variable η:

$$\begin{aligned}y(\xi,\eta) &= \int \partial_\eta y(\xi,\eta)\, d\eta \\ &= \int F(\eta) d\eta + g(\xi) \\ &= f(\eta) + g(\xi),\end{aligned} \tag{2.11}$$

where $g(\xi)$ is an arbitrary function of one variable and $f'(\eta) = F(\eta)$. Note that since F is arbitrary, so is f.

Returning to variables x and t, we can write the general solution of the wave equation as

$$y(x,t) = f(x-ct) + g(x+ct) \tag{2.12}$$

where f and g are arbitrary functions of one variable.

[1] Note that when we integrate a function of two variables in one of the two variable, we need to add (to the result) an arbitrary function of the other variable. This is similar to adding a constant of integration when we integrate a function of one variable.

[2] You can verify that this is true by direct differentiation of Eq. 2.10 with respect to ξ.

2.3 Travelling waves

We will now gain an initial understanding of this solution by visualising the two special cases where either f or g are zero.

If $g = 0$, then $y(x,t) = f(x - ct)$. At $t = 0$, the string has the shape described by the graph $y = f(x)$. At time $t > 0$, it will have the same shape relative to the variable $\eta = x - ct$: $y = f(\eta)$. Since $x = \eta + ct$, this means that the graph of y as a function of x for a fixed $t > 0$ is the graph of $f(x)$ shifted to the *right* (in the direction of positive x) by distance ct.

If $f = 0$, then $y(x,t) = g(x + ct)$. At $t = 0$, the string has the shape described by the graph $y = g(x)$. At time $t > 0$, it will have the same shape relative to the variable $\xi = x + ct$: $y = g(\xi)$. Since $x = \xi - ct$, this means that the graph of y as a function of x for a fixed $t > 0$ is the graph of $g(x)$ shifted to the *left* (in the direction of negative x) by distance ct.

Thus, $f(x - ct)$ and $g(x + ct)$ describe waves that propagate (without changing shape) to the right and to the left, respectively, and the general solution Eq. 2.12 represent the sum of such waves.

2.4 Initial value problem and d'Alembert's formula

The initial-value problem is to solve the wave equation

$$\partial_t^2 y - c^2 \partial_x^2 y = 0 \tag{2.13}$$

for $-\infty < x < +\infty$ and $0 < t < +\infty$ with the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \tag{2.14}$$

for $-\infty < x < +\infty$, where y_0 and v_0 are given functions of x. The first of the two initial conditions prescribes the initial displacement of the string, the second the initial velocity.

The solution of this initial value problem can be found by substituting the general solution Eq. 2.12 into the initial conditions. This gives

$$y_0(x) = f(x) + g(x), \tag{2.15}$$

$$v_0(x) = -cf'(x) + cg'(x). \tag{2.16}$$

So we have two equations for the two unknown functions f and g. To solve them, we first integrate Eq. 2.16:

$$-cf(x) + cg(x) = \int_0^x v_0(s)ds + a = V(x) \tag{2.17}$$

where a is an integration constant and $V(x)$ is just introduced to save writing below.

Next, we add and subtract Eq. 2.15 and Eq. 2.17 divided by c. This results in

$$\begin{aligned} y_0(x) - \frac{1}{c} V(x) &= 2 f(x) \\ y_0(x) + \frac{1}{c} V(x) &= 2 g(x) \end{aligned} \tag{2.18}$$

which implies that

$$\begin{aligned} f(x) &= \frac{1}{2} y_0(x) - \frac{1}{2c} V(x) \\ g(x) &= \frac{1}{2} y_0(x) + \frac{1}{2c} V(x) \end{aligned} \tag{2.19}$$

Substituting these into the formula for the general solution, we get

$$\begin{aligned} y(x,t) =& \frac{1}{2} y_0(x-ct) - \frac{1}{2c} V(x-ct) \\ &+ \frac{1}{2} y_0(x+ct) + \frac{1}{2c} V(x+ct) \end{aligned} \tag{2.20}$$

or

$$\begin{aligned} y(x,t) =& \frac{1}{2} \left[y_0(x-ct) + y_0(x+ct) \right] \\ &+ \frac{1}{2c} \left[V(x+ct) - V(x-ct) \right] \end{aligned} \tag{2.21}$$

Note that only the difference $[V(x+ct) - V(x-ct)]$ appears so the integration constant cancels and also we can combine the two integrals into one because

$$\begin{aligned} V(x+ct) - V(x-ct) &= \int_0^{x+ct} v_0(s)\, ds - \int_0^{x-ct} v_0(s)\, ds \\ &= \int_{x-ct}^{x+ct} v_0(s)\, ds. \end{aligned} \tag{2.22}$$

Finally, we have

$$y(x,t) = \frac{1}{2}[y_0(x+ct) + y_0(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(s)\, ds. \tag{2.23}$$

This is the solution formula for the initial-value problem (Eq. 2.13, Eq. 2.14) and it is called the **d'Alembert formula**.

Remark. Once we have the d'Alembert formula, we can consider solutions of the initial-value problem (Eq. 2.13, Eq. 2.14) corresponding to piecewise smooth (or even piecewise continuous) initial functions $y_0(x)$ and $v_0(x)$. This will result in *generalised solutions* of the wave equation which are defined everywhere in the upper half of the (x,t) plane except for a finite number of lines where values of $y(x,t)$ and/or its first derivatives are discontinuous.

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