# Waves and Fluids

**Lecture notes for Spring 2023**

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In this module we will explore the dynamics of continuous media, focusing on elementary fluid dynamics and the motion of waves. This lays the foundations for the full development of fluid dynamics in years 3 and 4, as well as for modules on electromagnetism and quantum mechanics. The mathematical techniques of vector calculus are employed and further developed, as are Fourier methods and methods from complex analysis.

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# Welcome

These are the lecture notes for Waves and Fluids, part of the 2nd year Applied Maths module at the University of York in Spring 2023. Each chapter in these notes corresponds to one lecture.

These notes will be periodically revised. Whenever you spot something that is not quite right, please email me at gustav.delius@york.ac.uk or submit your correction in the correction form at https://forms.gle/w17c19vWnM7wpLpz7.

The material in this module consists of two topics: Waves and Fluids. We will spend the first half of the term on Waves and the second half on Fluids. The topics are linked by the fact that both use partial differential equations to describe real-world phenomena in space and time. Of course they share this feature with a large part of Applied Mathematics. So you will meet the ideas and methods introduced in this module again and again in future Applied Mathematics modules. In some sense, the actual subject matter of this module is less important than the way of thinking that it introduces.

The module is meant to prepare you for going out into the world and confronting new phenomena with the power of mathematics, not only in physics, but in biology, ecology, medicine, sociology, economics, and other areas. There will be many modules in the third and fourth year of you studies that will deepen that ability.

Throughout this module, we shall use SI units: length is measured in meters (m), time in seconds (s), mass in kilograms (kg).

# Part I

# Waves

Waves are so fundamentally important because waves are the only way information can propagate in this universe. Some waves that propagate information are obvious: sound waves, light and radio waves, electric waves travelling along our neurons. Others are less obvious: even if I communicate with you by shooting a particle at you, this is described by a wave, as you will learn in quantum mechanics. Gravitational effects are communicated via gravitational waves.

Because waves propagate at a finite speed, also information can only propagate at that wave speed. This has profound impacts, as you know from the theory of special relativity. For example, we can look far back into the past because some of the light waves emitted shortly after the big bang 16 billion years ago are only now arriving here, strongly red-shifted and thus detectable as microwaves.

Studying waves is also of great practical importance. They play an important part in our technological world. Improvements in our understanding of how to generate and control electromagnetic waves, for example, has led to radio, radar and mobile phones. Understanding how a pest propagates in the form of an invasion wave into so-far uninfected territory allows us to prepare adequate interventions. Understanding how density waves form in traffic flow, leading to traffic jams, allows us to design interventions that lead to smoother traffic flow.

# 1 Deriving the wave equation for a string

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 7
- Coulson and Jeffrey (1977) sections 17 and 19
- Baldock and Bridgeman (1983) section 1.10.3
- Simmons (1972) section 40

## 1.1 Why waves on a string?

The great diversity of waves in nature means that we need to choose some concrete wave phenomenon to concentrate on to start our investigation. In this module we will concentrate on the waves on a string (think of a guitar string) and generalise to waves on a membrane (think of the membrane of a drum). By studying this in detail you will develop the intuition and the skills that will allow you to understand other wave phenomena later. We'll come back to waves at the end of the part on Fluid Dynamics when we study waves on the surface of a fluid.

Personally I like studying vibrating strings because they are at the foundation of superstring theory. This is a "Theory of Everything" that posits that elementary particles are actually tiny strings, with different vibrational states corresponding to different elementary particles. As a Ph.D. student I showed how, if these strings move in certain higher-dimensional group supermanifolds, they behave like the elementary particles of our standard model of particle physics, including the chiral fermions. If we ignore the bit about group supermanifolds for the moment, the maths behind string theory is no more complicated than the maths we will discuss in this module and the partner module on quantum mechanics.

We consider a flexible, elastic string of linear density $\rho$ (mass per unit length), which undergoes small *transverse* vibrations. (For example, it can be a guitar string.) The transverse vibrations mean that the displacements of each small element of the string is perpendicular to its length. We assume that the string does not move *longitudinally* (i.e. parallel to its length). Let $y(x, t)$ be its displacement from equilibrium position at time $t$ and position $x$ (see Fig. Figure 1.1).

The string is sufficiently simple, that we can understand it by pure thought. We will derive from first principles a PDE that describes its motion (the wave equation) and then solve it for various initial conditions. I find it amazing that this is possible.

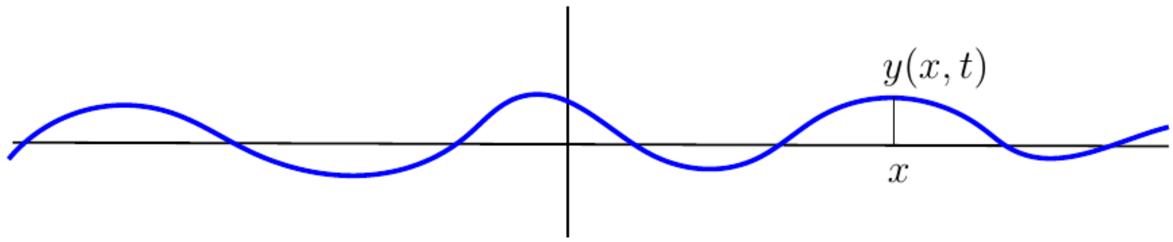


Figure 1.1: A string stretched in the $x$-direction and vibrating in the $y$-direction.

To achieve this mathematical understanding we will need to make several simplifications and approximations. For example, we neglect that the string is made up of lots of individual atoms and instead we will pretend that the mass is spread out continuously along the string. This is known as the **continuum approximation** and we will meet this again in the fluid dynamics.

To derive the equation of motion of the string we first need to discuss the force acting on it which we will do in the next section. Then in the section after that we can plug this into Newton's second law and out pops the wave equation. Along the way we will make approximations that allow us to linearise the equations.

## 1.2 Linearized tension force

We consider a small segment of the string between any two points $x$ and at $x + \delta x$ as shown in Fig. Figure 1.2. We want to determine the force that is acting on this segment, so that we can later determine its motion using Newton's second law. We will concentrate on only the tension force of the string and ignore less important effects like gravity, friction, or stiffness.

We assume that the tension force $T(x)$ has constant magnitude throughout the string: $|T(x)| = T$. However its direction varies along the string, because it always acts in the tangential direction. At interior points the tension force pulling to one side will balance that pulling in the other direction. The net tension force on the segment will thus be determined by the tension forces at its ends. We have drawn these forces schematically in Figure Figure 1.2 where we have also split them into their $x$ and $y$ components.

The total force acting on the segment is

$$F = T(x + \delta x) - T(x). \tag{1.1}$$

We first consider the $y$ component

$$T_y(x) = T \sin\theta(x), \tag{1.2}$$

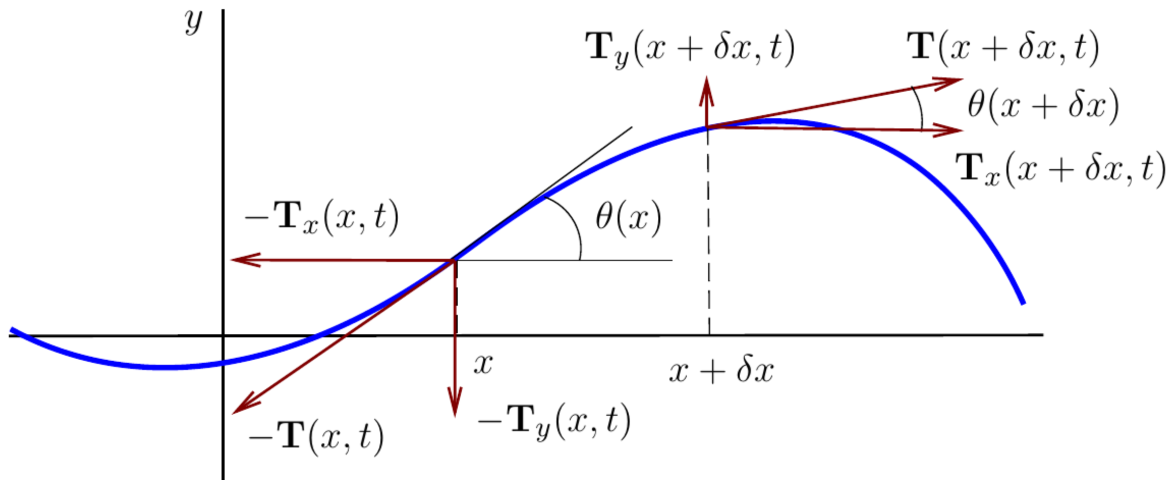


Figure 1.2: The tension forces acting on a segment of the string between $x$ and $x + \delta x$.

where $\theta(x)$ is the angle that the string makes with the horizontal at $x$. The slope of the string at $x$ is

$$\frac{\partial y}{\partial x} = \tan\theta(x). \tag{1.3}$$

We are now going to simplify the expressions by assuming that the slope and thus $\theta$ is small, $\theta << 1$. Then, by Taylor expansion,

$$\sin\theta = \theta + O(\theta^3), \quad \tan\theta = \theta + O(\theta^2). \tag{1.4}$$

We ignore all terms that are higher order in $\theta$. This is known as the linear approximation. It is done very often, because it leads to linear equations that are so much easier to solve. So

$$\begin{aligned} F_y &= T_y(x+\delta x) - T_y(x) \\ &= T\sin\theta(x+\delta x) - T\sin\theta(x) \\ &\approx T\left(\theta(x+\delta x) - \theta(x)\right). \end{aligned} \tag{1.5}$$

We do another Taylor expansion and ignore higher-order terms in $\delta x$, which is fine because we want to look at only an infinitesimally small segment of string.

$$\begin{aligned} \theta(x+\delta x) &= \theta(x) + \delta x \frac{\partial \theta}{\partial x} + O(\delta x)^2 \\ &\approx \theta(x) + \delta x \frac{\partial \theta}{\partial x}. \end{aligned} \tag{1.6}$$

Substituting this into Eq. 1.5 gives

$$F_y \approx T\delta x \frac{\partial \theta}{\partial x}. \tag{1.7}$$

We would like to express this in terms of $y$ instead of $\theta$, which we can do by observing that

$$\theta \approx \tan\theta = \frac{\partial y}{\partial x}, \tag{1.8}$$

so we finally have

$$F_y \approx T\,\delta x\,\frac{\partial^2 y}{\partial x^2}. \tag{1.9}$$

We deal with the $x$ component of the force similarly, using the Taylor expansion of $\cos\theta = 1 + O(\theta^2)$:

$$\begin{aligned} F_x &= T_x(x+\delta x) - T_x(x) \\ &= T\cos\theta(x+\delta x) - T\cos\theta(x) \\ &\approx T - T = 0. \end{aligned} \tag{1.10}$$

So in our approximation of small slope, there is no movement in the $x$ direction. The string vibrates purely transversally.

## 1.3 Wave equation from Newton's 2nd law

To determine the motion in the $y$ direction we use Newton's second law

$$ma_y = F_y, \tag{1.11}$$

where $a_y$ is the acceleration in the $y$ direction,

$$a_y = \frac{\partial^2 y}{\partial t^2} \tag{1.12}$$

and $m$ is the mass of the infinitesimal segment which is obtained as the density times the length,

$$m = \rho\delta x. \tag{1.13}$$

We assume that density $\rho$ is constant along the string. Plugging this together with our expression for $F_y$ into Newton's second law gives

$$\rho\delta x\frac{\partial^2 y}{\partial t^2} = T\delta x\frac{\partial^2 y}{\partial x^2}. \tag{1.14}$$

We can cancel the $\delta x$ and divide by $\rho$ which finally gives us the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{1.15}$$

with **wave speed**

$$c = \sqrt{\frac{T}{\rho}}. \tag{1.16}$$

Why we call the constant $c$ the wave speed will become clear in the next lecture.

## 1.4 Checking dimensions

After having derived an equation, it is always wise to check that its dimensions work out correctly.

We use square brackets to denote the dimension of a quantity. So $[y] = L$ says that $y$ has dimension of length, $[m] = M$ says that $m$ has dimension of mass, and $[t] = T$ says that $t$ has dimension of time. [1] The dimension of both sides of an equation has to agree, so

$$\left[\frac{\partial^2 y}{\partial t^2}\right] = \frac{L}{T^2} = [c^2]\left[\frac{\partial^2 y}{\partial x^2}\right] = [c^2]\frac{1}{L}. \tag{1.17}$$

This shows that $[c] = L/T$, so it has the dimension of a velocity. Because $T$ is a force we have $[T] = ML/T^2$. The density $\rho$ has $[\rho] = M/L$. So

$$[c] = \left[\sqrt{\frac{T}{\rho}}\right] = \sqrt{\frac{ML/T^2}{M/L}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}. \tag{1.18}$$

This completes our check of the dimensions.

[1] Note the conflict of notation where we used $T$ for the tension force while it is also the conventional symbol for the dimension of time. Such conflicts happen from time to time – the context determines the meaning of the symbol.

# 2 d'Alembert's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 8
- Coulson and Jeffrey (1977) sections 7 and 11
- Baldock and Bridgeman (1983) section 2.1

In this lecture, we consider an infinitely long string (this is physically justified if we consider waves propagating far away from any boundaries). Mathematically, this means that we are looking for solutions of the wave equation

$$\partial_t^2 y - c^2\, \partial_x^2 y = 0 \tag{2.1}$$

on the whole real axis $-\infty < x < +\infty$. Note that I have switched to the convenient notation using subscripts on derivatives to specify the variable with respect to which we are differentiating.

## 2.1 Characteristic coordinates

To solve the wave equation we use the *method of characteristics*, which involves a change of variables that makes the equation much simpler. We change from the variables $x$ and $t$ to the *characteristic coordinates*

$$\xi = x + ct, \quad \eta = x - ct. \tag{2.2}$$

By this we mean that for any function $y$ that depends on the variables $x$ and $t$ we can introduce a function $\tilde{y}$ that depends on the variables $\xi$ and $\eta$ in such a way that it has the same values as $y$:

$$y(x,t) = \tilde{y}(\xi(x,t), \eta(x,t)) \text{ for all } x, t. \tag{2.3}$$

It is a conventional abuse of notation to drop the tilde and denote both functions by $y$. We will follow this abuse of notation.

We need to express the derivatives with respect to $t$ and $x$ via the derivatives with respect to $\xi$ and $\eta$. This is done using the chain rule:

$$\begin{aligned} \partial_t\, y &= \frac{\partial y}{\partial \xi}\,\frac{\partial \xi}{\partial t} + \frac{\partial y}{\partial \eta}\,\frac{\partial \eta}{\partial t} \\ &= c\left(\partial_\xi - \partial_\eta\right) y \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \partial_x y &= \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \left(\partial_\xi + \partial_\eta\right) y. \end{aligned} \tag{2.5}$$

Hence

$$\partial_t = c\left(\partial_\xi - \partial_\eta\right), \quad \partial_x = \partial_\xi + \partial_\eta. \tag{2.6}$$

Substituting these into the wave equation, we find that

$$c^2\left(\partial_\xi - \partial_\eta\right)^2 y - c^2\left(\partial_\xi + \partial_\eta\right)^2 y = 0. \tag{2.7}$$

Expanding the squares and cancelling terms gives

$$-4c^2 \partial_\xi \partial_\eta y = 0. \tag{2.8}$$

We can divide both sides by the nonzero constant $-4c^2$. Thus the wave equation simplifies to

$$\partial_\xi \partial_\eta y = 0. \tag{2.9}$$

## 2.2 General solution of wave equation

The wave equation in light-cone variables is really easy to solve. First, we integrate Eq. 2.9 in the variable $\xi$:

$$\begin{aligned} &\int \partial_\xi \partial_\eta y(\xi, \eta)\, d\xi = 0 \\ \Leftrightarrow \quad &\partial_\eta y(\xi, \eta) = F(\eta) \end{aligned} \tag{2.10}$$

where $F$ is an arbitrary function of one variable [1].

> **ℹ Note**
>
> When we integrate a function of two variables in one of the two variable, we need to add to the result an arbitrary function of the other variable. This is similar to adding a constant of integration when we integrate a function of one variable.

Now we can integrate Eq. 2.10 in the variable $\eta$:

$$\begin{aligned} y(\xi, \eta) &= \int \partial_\eta y(\xi, \eta)\, d\eta \\ &= \int F(\eta) d\eta + g(\xi) \\ &= f(\eta) + g(\xi), \end{aligned} \tag{2.11}$$

[1] You can verify that this is true by direct differentiation of Eq. 2.10 with respect to $\xi$.

where $g(\xi)$ is an arbitrary function of one variable and $f'(\eta) = F(\eta)$. Note that since $F$ is arbitrary, so is $f$.

Returning to variables $x$ and $t$, we can write the general solution of the wave equation as

$$y(x,t) = f(x - ct) + g(x + ct) \tag{2.12}$$

where $f$ and $g$ are arbitrary functions of one variable.

## 2.3 Travelling waves

We will now gain an initial understanding of this solution by visualising the two special cases where either $f$ or $g$ are zero.

If $g = 0$, then $y(x,t) = f(x - ct)$. At $t = 0$, the string has the shape described by the graph $y = f(x)$. At time $t > 0$, it will have the same shape relative to the variable $\eta = x - ct$: $y = f(\eta)$. Since $x = \eta + ct$, this means that the graph of $y$ as a function of $x$ for a fixed $t > 0$ is the graph of $f(x)$ shifted to the *right* (in the direction of positive $x$) by distance $ct$.

If $f = 0$, then $y(x,t) = g(x + ct)$. At $t = 0$, the string has the shape described by the graph $y = g(x)$. At time $t > 0$, it will have the same shape relative to the variable $\xi = x + ct$: $y = g(\xi)$. Since $x = \xi - ct$, this means that the graph of $y$ as a function of $x$ for a fixed $t > 0$ is the graph of $g(x)$ shifted to the *left* (in the direction of negative $x$) by distance $ct$.

Thus, $f(x - ct)$ and $g(x + ct)$ describe waves that propagate (without changing shape) to the right and to the left, respectively, and the general solution Eq. 2.12 represent the sum of such waves.

## 2.4 Initial value problem and d'Alembert's formula

The initial-value problem is to solve the wave equation

$$\partial_t^2 y - c^2 \partial_x^2 y = 0 \tag{2.13}$$

for $-\infty < x < +\infty$ and $0 < t < +\infty$ with the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \tag{2.14}$$

for $-\infty < x < +\infty$, where $y_0$ and $v_0$ are given functions of $x$. The first of the two initial conditions prescribes the initial displacement of the string, the second the initial velocity.

To solve an initial value one has to substitute the general solution into the initial conditions. We substitute the solution from Eq. 2.12 into the initial conditions in Eq. 2.14 and obtain

$$y_0(x) = f(x) + g(x), \tag{2.15}$$

$$v_0(x) = -cf'(x) + cg'(x). \tag{2.16}$$

So we have two equations for the two unknown functions $f$ and $g$. To solve them, we first integrate Eq. 2.16:

$$-cf(x) + cg(x) = \int_0^x v_0(s)ds + a = V(x), \tag{2.17}$$

where $a$ is an integration constant and $V(x)$ is just introduced to save writing below.

Next, we add and subtract Eq. 2.15 and Eq. 2.17 divided by $c$. This results in

$$\begin{aligned} y_0(x) - \frac{1}{c}\,V(x) &= 2\,f(x), \\ y_0(x) + \frac{1}{c}\,V(x) &= 2\,g(x), \end{aligned} \tag{2.18}$$

which implies that

$$\begin{aligned} f(x) &= \frac{1}{2}\,y_0(x) - \frac{1}{2c}\,V(x), \\ g(x) &= \frac{1}{2}\,y_0(x) + \frac{1}{2c}\,V(x). \end{aligned} \tag{2.19}$$

Substituting these into the formula for the general solution, we get

$$\begin{aligned} y(x,t) =& \frac{1}{2}\,y_0(x-ct) - \frac{1}{2c}\,V(x-ct) \\ &+ \frac{1}{2}\,y_0(x+ct) + \frac{1}{2c}\,V(x+ct) \end{aligned} \tag{2.20}$$

or

$$\begin{aligned} y(x,t) =& \frac{1}{2}\left[y_0(x-ct) + y_0(x+ct)\right] \\ &+ \frac{1}{2c}\left[V(x+ct) - V(x-ct)\right]. \end{aligned} \tag{2.21}$$

Note that only the difference $V(x+ct)-V(x-ct)$ appears, so the integration constant cancels and we can combine the two integrals into one because

$$\begin{aligned} V(x+ct) - V(x-ct) &= \int\limits_0^{x+ct} v_0(s)\,ds - \int\limits_0^{x-ct} v_0(s)\,ds \\ &= \int\limits_{x-ct}^{x+ct} v_0(s)\,ds. \end{aligned} \tag{2.22}$$

Using this, we have

$$y(x,t) = \frac{1}{2}[y_0(x+ct) + y_0(x-ct)] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} v_0(s)\,ds. \tag{2.23}$$

This is the solution formula for the initial-value problem (Eq. 2.13, Eq. 2.14) and it is called the **d'Alembert formula**.

**Remark.** Once we have the d'Alembert formula, we can consider solutions of the initial-value problem (Eq. 2.13, Eq. 2.14) corresponding to piecewise smooth (or even piecewise continuous) initial functions $y_0(x)$ and $v_0(x)$. This will result in *generalised solutions* of the wave equation which are defined everywhere in the upper half of the $(x, t)$ plane except for a finite number of lines where values of $y(x, t)$ and/or its first derivatives are discontinuous.

# 3 Boundaries and Interfaces

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 9
- Baldock and Bridgeman (1983) section 2.1 and 2.5

We have seen that the solutions of the wave equation predict right- and left-moving waves that travel without changing their shapes. Eventually, in the real world at least, these waves are going to reach the end of the string. What will happen then? We know that the energy that is carried by the wave can not simply disappear. So we expect the wave to be reflected. But how is it reflected in detail?

## 3.1 Semi-infinite string with fixed end

Let us consider the case where a right-moving wave hits the right end of the string. We choose the $x$-coordinate so that the end is at $x = 0$. Thus we consider the wave equation on the left half-line $-\infty < x < 0$. In this section we consider the case where the end of the string is fixed, so we impose the boundary condition

$$y(0,t) = 0 \quad \text{for all } t \in \mathbf{R}. \tag{3.1}$$

This is known as a Dirichlet boundary condition.

We recall the general solution of the wave equation:

$$y(x,t) = f(x - ct) + g(x + ct). \tag{3.2}$$

As we know, the function $f$ gives the shape of the right-moving wave. Imposing the boundary condition will tell us what $g$ has to be, i.e., it will determine the shape of the left-moving reflected wave. Substituting the general solution into the boundary condition gives

$$y(0,t) = f(-ct) + g(ct) = 0. \tag{3.3}$$

This holds for any value of $t$, so

$$g(s) = -f(-s) \quad \text{for all } s \in \mathbf{R}. \tag{3.4}$$

This tells us that the reflected wave is the negative of the incoming wave and is flipped front-to-back. Thus the solution is

$$y(x,t) = f(x - ct) - f(-x - ct) \quad \text{for all } x \leq 0. \tag{3.5}$$

This is illustrated in Figure 3.1.

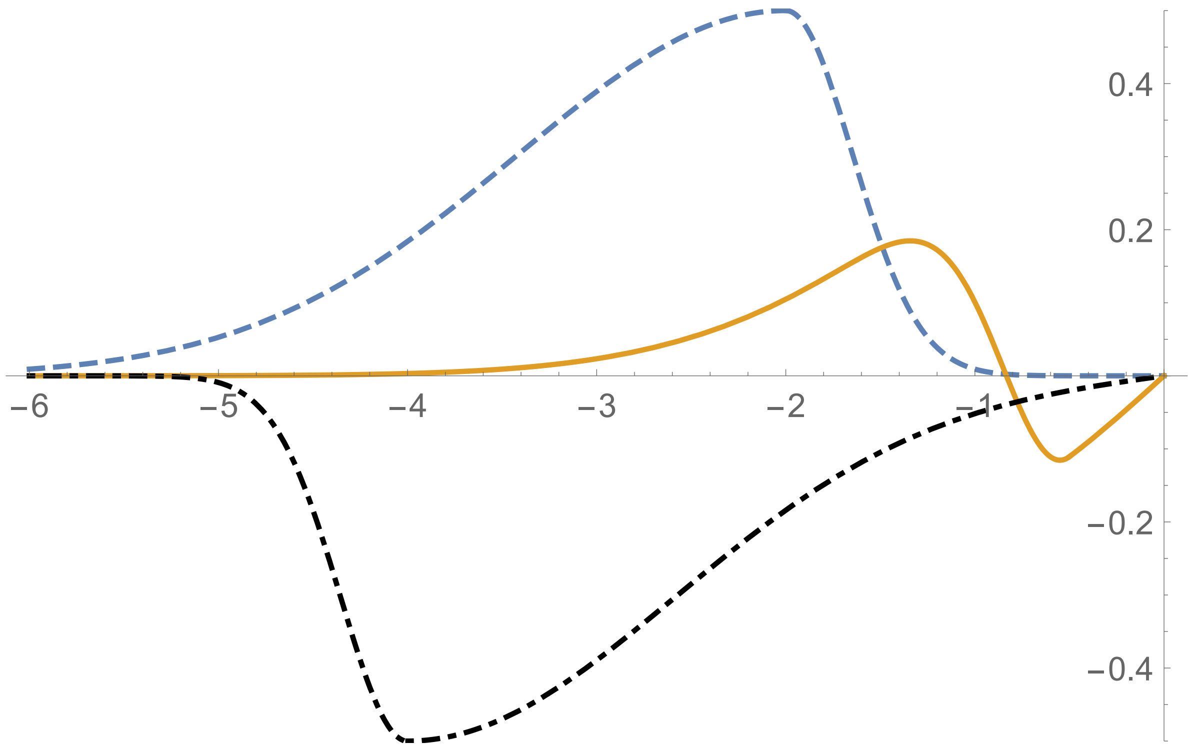


Figure 3.1: Reflection off a fixed end. Dashed line: incident right-moving wave. Solid line: wave interacting with the boundary. Dashdotted line: reflected left-moving wave. The reflected wave has the same shape as the incident wave but is flipped in both $y$ and $x$.

## 3.2 Semi-infinite string with free end

Consider now a semi-infinite string ($0 < x < \infty$) with a free end at $x = 0$ (e.g. the end of the string can be attached to a small ring, which in turn can slide along a vertical rod without friction). This means that the vertical component of the tension force applied to the end of the string must be zero, which in turn means the string must be horizontal at $x$, i.e., we have the boundary condition

$$\partial_x y(0,t) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Conditions which specify the value of the normal derivative of the unknown function at the boundary are called *Neumann conditions*. So, here we have the homogeneous Neumann condition at $x = 0$. We now substitute the general solution. First we calculate its derivative

$$\partial_x y(x,t) = f'(x - ct) + g'(x + ct) \tag{3.7}$$

and thus the boundary condition says that

$$\partial_x y(0,t) = f'(-ct) + g'(ct) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{3.8}$$

Integrating this gives

$$\frac{-1}{c} f(-ct) + \frac{1}{c} g(ct) = \text{constant}. \tag{3.9}$$

Changing the constant only moves the string up or down on the y axis. We choose it to be zero. Because the boundary condition holds for all times, we have that

$$g(s) = f(-s) \quad \text{for all } s \in \mathbb{R}. \tag{3.10}$$

Thus the reflected wave has the same shape and the same sign as the incoming wave, but it is still flipped front-to-back.Thus the solution is

$$y(x,t) = f(x - ct) + f(-x - ct) \quad \text{for all } x \leq 0. \tag{3.11}$$

This is illustrated in Figure 3.2.

## 3.3 Reflection at a change of density

Consider two semi-infinite strings joined at the origin. The string on the left ($x < 0$) has constant density $\rho_1$ and the string on the right ($x > 0$) has constant density $\rho_2$. Let $y_1$ and $y_2$ be the displacements of the two strings. Since the strings have different densities, the wave speed in the two strings will be different:

$$c_1 = \sqrt{\frac{T}{\rho_1}} \quad \text{and} \quad c_2 = \sqrt{\frac{T}{\rho_2}}. \tag{3.12}$$

Suppose that we have a wave travelling to the right on the first string (an incident wave). When the wave meets the change in density, it will be partially reflected (back to the region $x < 0$) and partially transmitted (forward to the region $x > 0$). Waves travelling in the interval $x \in (-\infty, 0)$ are described by the wave equation with wave speed $c_1$; waves travelling in the interval $x \in (0, \infty)$ are described by the wave equation with wave speed $c_2$. This is illustrated in Figure Figure 3.3.

Therefore, we can write

$$y(x,t) = \begin{cases} y_1(x,t) = f_I(x - c_1 t) + f_R(x + c_1 t), & x < 0 \\ y_2(x,t) = f_T(x - c_2 t), & x > 0 \end{cases} \tag{3.13}$$

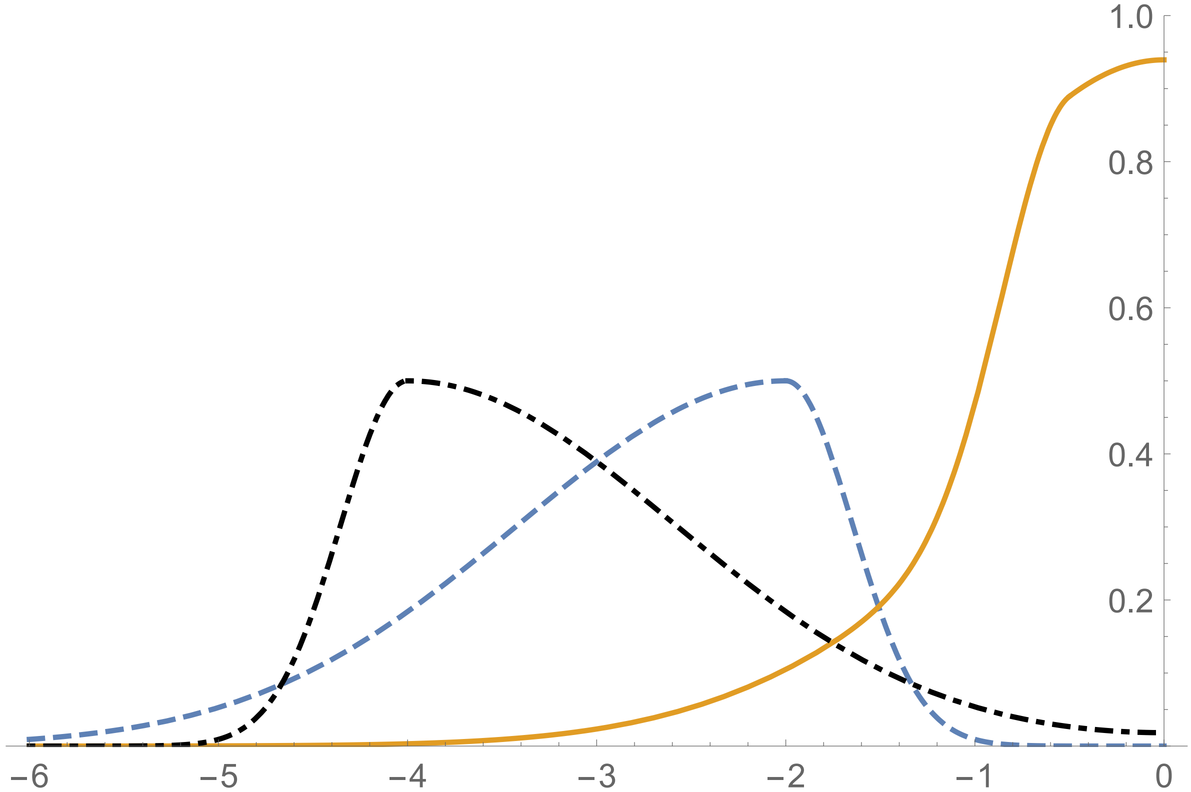


Figure 3.2: Reflection off a free end. Dashed line: incident right-moving wave. Solid line: wave interacting with the boundary. Dashdotted line: reflected left-moving wave. The reflected wave has the same shape as the incident wave but is flipped in $x$.

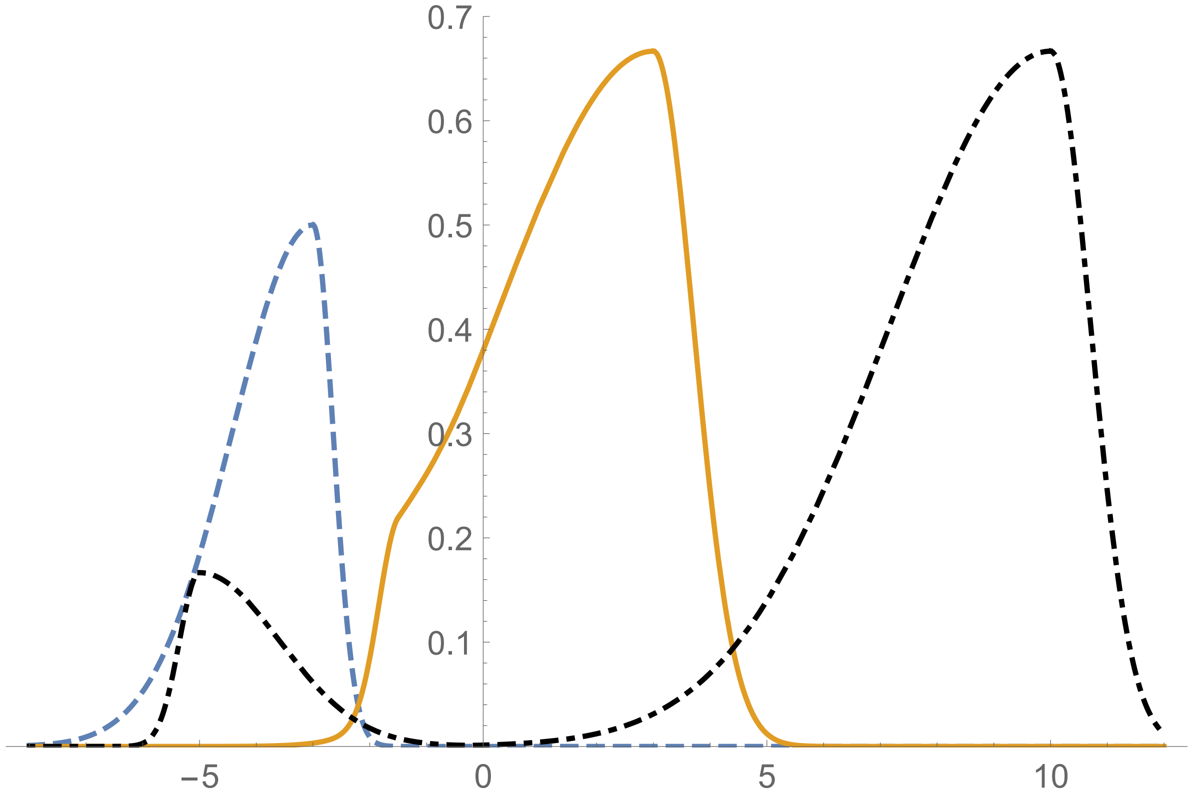


Figure 3.3: A right-moving wave being partially reflected and partially transmitted at the interface between two strings with wave velocities $c_1 = 1$ (on the left half-line) and $c_2 = 2$ (on the right half-line). Dashed line: incident right-moving wave. Solid line: wave interacting with the interface. Dashdotted line: partially reflected left-moving and partially transmitted right-moving wave.

where $f_I$, $f_R$ and $f_T$ represent the incident, reflected and transmitted waves, respectively. At the point of contact of the two strings ($x = 0$), we impose the following two conditions:

$$y_1(0,t) = y_2(0,t) \quad \text{for all } t, \tag{3.14}$$

$$\partial_x y_1(0,t) = \partial_x y_2(0,t) \quad \text{for all } t. \tag{3.15}$$

Condition 3.14 says that the solution for the combined string should be continuous at $x = 0$ (because the strings are attached to each other at the point). Condition 3.15 states that the slopes of the strings at $x = 0$ should be the same (if this is not so, there will be a finite force applied to an infinitesimal part of the combined string at $x = 0$, producing unphysical infinite acceleration).

Suppose that the incident wave is given, i.e. the function of one variable $f_I$ is known. Can we find $f_R$ and $f_T$?

Substitution of Eq. 3.13 into condition Eq. 3.14 yields

$$f_I(-c_1 t) + f_R(c_1 t) = f_T(-c_2 t) \quad \text{for all } t \tag{3.16}$$

or equivalently (writing $s = c_1 t$)

$$f_I(-s) + f_R(s) = f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.17}$$

Similarly, on substituting Eq. 3.13 into condition Eq. 3.15, we find that

$$f_I'(-c_1 t) + f_R'(c_1 t) = f_T'(-c_2 t) \quad \text{for all } t \tag{3.18}$$

or

$$f_I'(-s) + f_R'(s) = f_T'\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.19}$$

Integrating this equation in $s$, we get

$$-f_I(-s) + f_R(s) = -\frac{c_1}{c_2}\, f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.20}$$

Eliminating $f_R(s)$ from Eq. 3.17 and Eq. 3.20, we obtain

$$2 f_I(-s) = \left(1 + \frac{c_1}{c_2}\right) f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.21}$$

or, equivalently,

$$f_T(\tilde{s}) = \frac{2c_2}{c_2 + c_1}\, f_I\left(\frac{c_1}{c_2}\, \tilde{s}\right) \quad \text{for all } \tilde{s} \tag{3.22}$$

where $\tilde{s} = -(c_2/c_1)s$. Eq. 3.22 defines the function (of one variable) $f_T$ in terms of known function $f_I$.

To find $f_R$, we substitute Eq. 3.22 into Eq. 3.17. This yields the formula

$$f_R(s) = \frac{c_2 - c_1}{c_2 + c_1} f_I(-s) \quad \text{for all } s. \tag{3.23}$$

Finally, on substituting Eq. 3.22 and Eq. 3.23 into Eq. 3.13, we get the solution formula

$$y(x,t) = \begin{cases} f_I(x - c_1 t) + A_R f_I(-x - c_1 t), & x < 0 \\ A_T f_I\left(\frac{c_1}{c_2}(x - c_2 t)\right), & x > 0 \end{cases} \tag{3.24}$$

where $A_R$ (the ratio of the amplitude of the reflected wave to that of the incident wave) and $A_T$ (the ratio of the amplitude of the transmitted wave to that of the incident wave) are given by

$$A_R = \frac{c_2 - c_1}{c_2 + c_1}, \quad A_T = \frac{2c_2}{c_2 + c_1}. \tag{3.25}$$

To check whether our answer makes sense, it is useful to consider a few limit cases.

i. If $\rho_1 = \rho_2$, then $c_1 = c_2$, and we have $A_R = 0$ and $A_T = 1$ (as one would expect for an infinite homogeneous string).

ii. If $\rho_1 \ll \rho_2$ (the first string is much lighter than the second one), then $c_1 \gg c_2$, and we have $A_R \approx -1$ and $A_T = 0$ (so that the heavy string effectively arrests the displacement at $x = 0$ as if the right end of the first string was fixed).

iii. If $\rho_1 \gg \rho_2$ (the first string is much heavier than the second one), then $c_1 \ll c_2$, and we have $A_R \approx 1$ and $A_T = 2$ (so that the light second string does not effect the displacement at $x = 0$ as if the right end of the first string was free).

# 4 Bernoulli's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) sections 11.2 and 11.3, chapters 13 and 14
- Baldock and Bridgeman (1983) sections 3.1, 3.2, 3.3
- Simmons (1972) sections 39 and 40

We will now solve the wave equation again in a totally different way from d'Alembert, following instead Bernoulli. This is a method you have seen before, namely the method of separation of variables.

## 4.1 Separation of variables

We make the Ansatz that the solution factorises into one function of only $x$ and one function of only $t$:

$$y(x,t) = X(x)T(t). \tag{4.1}$$

Substituting this into the wave equation

$$\partial_x^2 y(x,t) = \frac{1}{c^2}\partial_t^2 y(x,t) \tag{4.2}$$

we get

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t). \tag{4.3}$$

We divide both sides by $X(x)T(t)$, which will separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}. \tag{4.4}$$

It the last equation, we have a function of $x$ only on the left hand side and a function of $t$ only on the right hand side. The equation must hold for all $x$ and $t$. This is only possible if both functions are equal to a constant. It will turn out to be convenient to write this constant as $-k^2$ for some new constant $k$. Thus we set

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = -k^2. \tag{4.5}$$

This means that $X(x)$ and $T(t)$ must be solutions of the ODEs

$$X'' = -k^2 X \quad \text{and} \quad T'' = -k^2 c^2 T. \tag{4.6}$$

The general solutions of these ODEs are

$$\begin{aligned} X(x) &= A\,\sin(kx) + B\,\cos(kx), \\ T(t) &= F\,\sin(kct) + G\,\cos(kct), \end{aligned} \tag{4.7}$$

where $A, B, F, G$ and $k$ are arbitrary constants.

## 4.2 Finite string with fixed ends

Bernoulli was most interested in the finite string. Let us put the ends of the string at $x = 0$ and $x = \pi$. Now let the ends of the string be fixed. This means that we need to impose homogeneous Dirichlet boundary conditions

$$y(0,t) = 0 \quad \text{and} \quad y(\pi,t) = 0 \quad \text{for all} \quad t. \tag{4.8}$$

These conditions will be satisfied if $X(0) = 0$ and $X(\pi) = 0$. Imposing the condition $X(0) = 0$ on the general solution for $X(x)$, we find that we need $B = 0$. We then have from $X(\pi) = 0$ that

$$A \sin(k\pi) = 0. \tag{4.9}$$

The last equation implies that either $A = 0$ or $\sin(k\pi) = 0$. We reject the first option because it results in a zero solution. The second option yields

$$\sin(k\pi) = 0 \quad \Rightarrow k \in \mathbb{Z}. \tag{4.10}$$

Thus, for each integer $k$ we have a solution of the wave equation of the form

$$y_k(x,t) = \sin(kx)\,(F_k\,\sin(kct) + G_k\,\cos(kct)) \tag{4.11}$$

## 4.3 Standing waves and superposition

The solutions we obtained in the previous section are standing waves. They don't change their shape and don't move, only their amplitude oscillates with time. In the lecture videos I make various sketches and animations to illustrate this.

So we now have a stark contrast between what d'Alembert found and what Bernoulli found. According to d'Alembert, the solutions of the wave equation are travelling waves of arbitrary shape. According to Bernoulli, the solutions are standing waves that have sine shape. It is always wonderful when one has two very different ways of looking at the same phenomenon. That is where deep understanding comes from.

To resolve the apparent conflict between d'Alembert's solution and Bernoulli's solution we have to use the **superposition principle**: For any set of linear homogeneous equations, any sum of solutions is also a solution.

Since the wave equation is linear and also the boundary conditions we imposed were linear, a linear combination of any number of harmonic standing waves is also a solution. So, we can construct a general solution of the wave equation by summing up all [1] the harmonic standing waves from Eq. 4.11:

$$y(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left(F_k \sin(kct) + G_k \cos(kct)\right). \tag{4.12}$$

Here the $G_k$ and the $F_k$ are undetermined constants, to be fixed from the initial conditions.

The resolution of the apparent paradox is that a superposition of standing waves can give a travelling wave, and that a superposition of sine waves can give any shape.

## 4.4 Initial value problem

In Section 2.4 we imposed initial conditions on d'Alembert's general solution and obtained d'Alembert's formula in Eq. 2.23. We now similarly impose initial conditions on Bernoulli's solution. Suppose now that we are given the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \quad \text{for } x \in [0,\pi]. \tag{4.13}$$

We want to use these initial conditions to determine the unknown coefficients $F_k$ and $G_k$ for all $k \in \mathbb{N}$. As always, the procedure is to substitute the general solution into the initial conditions. When we substitute Eq. 4.12 into Eq. 4.13 we get

$$y_0(x) = \sum_{k=1}^{\infty} G_k \sin(kx), \quad v_0(x) = \sum_{k=1}^{\infty} F_k\, k\, c \sin(kx). \tag{4.14}$$

These formulae look like Fourier series for the functions $y_0$ and $v_0$. So, to find the coefficients $F_k$ and $G_k$ we have to perform the inverse Fourier transform. This uses the identity

$$\int_0^{\pi} \sin(kx)\sin(lx)dx = \frac{\pi}{2}\delta_{kl},$$

[1] We do not need negative $k$ because they will produce the same solutions, and we do not need $k=0$ because it yields zero solution. Also, we have absorbed the constant $A$ into the constants $F_k$ and $G_k$.

where $\delta_{kl}$ is the Kronecker delta:

$$\delta_{kl} = \begin{cases} 1 & \text{if} \quad k = l \\ 0 & \text{if} \quad k \neq l \end{cases}$$

We multiply each of Eq. 4.14 by $2/\pi \, \sin(lx)$ and integrate over the interval $[0, \pi]$:

$$\begin{aligned} \frac{2}{\pi} \int_0^\pi y_0(x) \sin(lx) dx &= \frac{2}{\pi} \sum_{k=1}^{\infty} G_k \int_0^\pi \sin(kx) \sin(lx) dx \\ &= \sum_{k=1}^{\infty} G_k \delta_{kl} = G_l \end{aligned} \tag{4.15}$$

$$\begin{aligned} \frac{2}{\pi} \int_0^\pi v_0(x) \sin(lx) dx &= \frac{2}{\pi} \sum_{k=1}^{\infty} F_k \, kc \int_0^\pi \sin(kx) \sin(lx) dx \\ &= \sum_{k=1}^{\infty} F_k \, kc \, \delta_{kl} = F_l \, lc \end{aligned} \tag{4.16}$$

Thus we know how to determine the constants $F_k$ and $G_k$ in the general solution Eq. 4.12 so as to satisfy given initial conditions.

# 5 Harmonic waves

## 5.1 Harmonic waves

Harmonic waves are waves that are described by sines and cosines. A travelling harmonic wave can be written as

$$y(x,t) = a\cos\left(kx - \omega t + \phi\right) \tag{5.1}$$

or

$$y(x,t) = a\cos\Big(2\pi(\hat{k}x - \nu t) + \phi\Big) \tag{5.2}$$

where $a$ is the amplitude, $\hat{k}$ the wave number, $\nu$ the frequency, $\phi$ the phase, and where

$$k = 2\pi\hat{k} \quad \text{and} \quad \omega = 2\pi\nu \tag{5.3}$$

are the angular wave number and the angular frequency, respectively. Note that $k$ and $\omega$ are used more often by mathematicians than $\hat{k}$ and $\nu$ and that the prefix 'angular' is often discarded.

Terminology

- *Frequency* = number of cycles (oscillations) per unit time.
- *Wave number* = number of cycles (oscillations) per unit length.
- *Period* = time $P$ needed to complete one cycle (oscillation):

$$P = \frac{1}{\nu} = \frac{2\pi}{\omega}. \tag{5.4}$$

- *Wave length* = distance between two consecutive wave crests (peaks):

$$\lambda = \frac{1}{\hat{k}} = \frac{2\pi}{k}. \tag{5.5}$$

- *Wave speed* = speed at which the wave is travelling:

$$c = \frac{\lambda}{P} = \frac{\nu}{\hat{k}} = \frac{\omega}{k}. \tag{5.6}$$

(Sometimes, the wave speed is also called the *phase speed*.)

The harmonic wave Eq. 5.1 can be written as

$$\begin{aligned} y(x,t) &= a\cos\left(kx-\omega t+\phi\right) \\ &= \mathrm{Re}\left(ae^{i\phi}e^{i(kx-\omega t)}\right) \\ &= \mathrm{Re}\left(Ae^{i(kx-\omega t)}\right), \end{aligned} \tag{5.7}$$

where we have included the phase factor $e^{i\phi}$ into the complex amplitude: $A = ae^{i\phi}$.

Consider now the complex function

$$y(x,t) = A\,e^{i(kx-\omega(k)t)} \tag{5.8}$$

for any $A \in \mathbb{C}$, any $k \in \mathbb{R}$ and some $\omega(k)$. Substituting this into the wave equation gives us an equation for $\omega(k)$:

$$\partial_t^2 y = c^2\partial_x^2 y \quad \Rightarrow \quad -\omega^2 A = -c^2k^2A. \tag{5.9}$$

Therefore, if

$$\omega(k) = \pm ck \tag{5.10}$$

then the complex function in Eq. 5.8 is a solution of the wave equation. We will refer to these complex solutions as *complex harmonic waves*. They are often more convenient to work with than their real counterpart in Eq. 5.1.

Eq. 5.10 is an example of a *dispersion relation*. It states that for the wave equation, $\omega$ is proportional to $k$. But complex harmonic waves can also solve other PDEs, as we will see in the next subsection, and that will lead to more complicated dispersion relations.

## 5.2 Solving PDEs with harmonic waves

**!** Important

Any linear homogeneous PDE (in variables $x$ and $t$) with constant coefficients has complex harmonic wave solutions Eq. 5.8 for some $\omega(k)$.

**Example 5.1.** Consider the damped string (with friction force proportional to velocity):

$$\partial_t^2\, y = c^2\,\partial_x^2\, y - p\,\partial_t\, y \quad \text{where} \quad p > 0. \tag{5.11}$$

Substituting the complex harmonic wave from Eq. 5.8 into this equation, we obtain the dispersion relation

$$-\omega^2 y = -c^2k^2y + ip\,\omega y. \tag{5.12}$$

Cancelling the $y$ we obtain a quadratic equation for $\omega^2$ which has the complex solution

$$\omega = -\frac{ip}{2} \pm \sqrt{c^2k^2 - \frac{p^2}{4}}. \tag{5.13}$$

Thus we have the following solution for the damped string:

$$\begin{aligned} y(x,t) &= A\, e^{i\left(kx + \frac{ip}{2}t \pm \sqrt{c^2k^2 - \frac{p^2}{4}}\,t\right)} \\ &= A\, e^{-\frac{pt}{2}}\, e^{ik\left(x \pm \sqrt{c^2 - \frac{p^2}{4k^2}}\,t\right)} \end{aligned} \tag{5.14}$$

The factor $e^{-pt/2}$ shows that we have a wave with exponentially decreasing amplitude. This is a consequence of the damping. The wave speed is now dependent on the wave number $k$:

$$c(k) = \sqrt{c^2 - p^2/(4k^2)}. \tag{5.15}$$

If we want to, we can get a real solution by taking the real part of the complex solution:

$$\mathrm{Re}(y(x,t)) = a\, e^{-pt/2} \cos\left[k\left(x \pm t\,\sqrt{c^2 - p^2/(4k^2)}\right) + \phi\right] \tag{5.16}$$

where $a = |A|$ and $\phi = \arg(A)$.

Note that the imaginary part of $\omega$ produces the damping exponential and the real part of $\omega$ determines the wave speed.

# 6 Energy

Besides Information, waves transmit another practically important quantity: Energy. Note that waves do not transport matter. Matter may oscillate up and down or forth and back as a wave passes, but it is not swept away with the wave. But energy is. In this lecture we are going to first introduce the expression for the energy in a wave on a string as an integral over the energy density. The energy density in turn is made up out of kinetic and potential energy density. We will then calculate the energy in a few example waves, and then discuss the conservation of energy.

## 6.1 Energy density

Consider an infinitesimal bit of string between $x$ and $x + \delta x$. Its kinetic energy is

$$\delta K = \frac{1}{2} m\, v^2 = \frac{1}{2} \rho\, \delta x \left(\partial_t y\right)^2 . \tag{6.1}$$

The kinetic energy of the entire string is then obtained by integrating over its infinitesimal parts:

$$K = \int \frac{\rho}{2} (\partial_t y)^2 \, dx = \int \mathcal{E}_K \, dx. \tag{6.2}$$

The quantity $\mathcal{E}_K$ is the *kinetic energy density*.

To derive the formula for the potential energy, we again look first at an infinitesimal segment of the string. It has been stretched from a length of $\delta x$ to the longer length $\delta s$. The work done to change the length from $\delta x$ to $\delta s$ is $T(\delta s - \delta x)$. This gives the potential energy (we neglect the potential energy coming from gravity). We have

$$\delta s = \sqrt{1 + \left(\partial_x y\right)^2}\, \delta x \approx \delta x \left(1 + \frac{\left(\partial_x y\right)^2}{2} + \cdots \right), \tag{6.3}$$

where we have only kept the first two terms in the Taylor expansion because, as we did when we derived the wave equation, we assume that the slope of the string is small and thus the higher order terms in $\partial_x y$ are negligible. Thus the potential energy in the infinitesimal segment of the string is

$$\delta V = T\left(\delta s - \delta x\right) = \frac{T}{2} \left(\partial_x y\right)^2 \, \delta x. \tag{6.4}$$

Summing up contributions from all small elements of the string (i.e. integrating over the whole string), we find the potential energy

$$V = \int T \, \frac{(\partial_x y)^2}{2} \, dx = \int \mathcal{E}_V \, dx. \tag{6.5}$$

The quantity $\mathcal{E}_V$ is the *potential energy density.*

The total energy $E$ is the sum of the kinetic and potential energy:

$$\begin{aligned} E &= K + V \\ &= \int \left( \frac{\rho}{2} (\partial_t y)^2 + \frac{T}{2} (\partial_x y)^2 \right) dx \\ &= \int \mathcal{E} \, dx, \end{aligned} \tag{6.6}$$

where $\mathcal{E} = \mathcal{E}_K + \mathcal{E}_V$ is the *total energy density.*

## 6.2 Energy density of example waves

**Example 6.1.** Consider a localised wave $y(x,t) = f(x - ct)$ travelling to the right with speed $c$. Substituting this into the general expression for the energy density

$$\mathcal{E} = \frac{\rho}{2} (\partial_t y)^2 + \frac{T}{2} (\partial_x y)^2 \tag{6.7}$$

gives

$$\mathcal{E}(x,t) = \frac{\rho}{2} (-c)^2 (f'(x-ct))^2 + \frac{T}{2} (f'(x-ct))^2. \tag{6.8}$$

Because $c^2 \rho = T$, we see that the kinetic and the potential energy densities are equal. This phenomenon is referred to as "equipartition" of the energy. Together we have

$$\mathcal{E}(x,t) = T \, (f'(x-ct))^2. \tag{6.9}$$

Note how the energy density is travelling along with the wave profile.

**Example 6.2** (Standing harmonic wave)**.** Now we consider solutions of the form

$$\begin{aligned} y(x,t) &= \sin(kx)(F \sin(kct) + G \cos(kct)) \\ &= \alpha \cos(kct + \phi). \end{aligned} \tag{6.10}$$

We calculate the energy densities

$$\begin{aligned}\mathcal{E}_K &= \frac{\rho}{2}(\partial_t y(x,t))^2 \\ &= \frac{\rho}{2}\alpha^2 k^2 c^2(-\sin(kct+\phi))^2 \sin^2(kx)\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}\mathcal{E}_V &= \frac{T}{2}(\partial_x y(x,t))^2 \\ &= \frac{T}{2}\alpha^2 k^2 (\cos(kct+\phi))^2 \cos^2(kx)\end{aligned} \tag{6.12}$$

Again we notice that the prefactors are the same because $c^2\rho = T$. For the energies we find

$$\begin{aligned}K &= \frac{T}{2}\alpha^2 k^2 \sin^2(kct+\phi)\int_0^\pi \sin^2(kx)dx \\ &= \frac{T\alpha^2k^2\pi}{4}\sin^2(kct+\phi)\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}T &= \frac{T}{2}\alpha^2 k^2 \cos^2(kct+\phi)\int_0^\pi \cos^2(kx)dx \\ &= \frac{T\alpha^2k^2\pi}{4}\cos^2(kct+\phi).\end{aligned} \tag{6.14}$$

For standing waves, both the kinetic energy and the potential energy depend on time and are not equal. However, their averages, averaged over a period in $t$, are equal. The total energy is constant

$$E = K + T = \frac{T\alpha^2k^2\pi}{4}(\sin^2(kct+\phi)+\cos^2(kct+\phi)) = \frac{T\alpha^2k^2\pi}{4} \tag{6.15}$$

**Example 6.3** (Sum of two standing harmonic waves)**.** Consider two harmonic waves

$$\begin{aligned}y_k &= \alpha_k \sin(kx)\cos(kct+\phi_k) \quad \text{and} \\ y_l &= \alpha_l \sin(lx)\cos(lct+\phi_l)\end{aligned} \tag{6.16}$$

with $k \neq l$ and let us calculate the energy of $y = y_k + y_l$. We have

$$\begin{aligned}K &= \frac{\rho}{2}\int_0^\pi (\partial_t y)^2 dx = \frac{\rho}{2}\int_0^\pi (\partial_t y_k + \partial_t y_l)^2 dx \\ &= K_k + K_l + \rho\,\alpha_k\,\alpha_l\,k\,l\,c^2\cos(kct+\phi_k)\cos(lct+\phi_l)\cdot \\ &\qquad \int_0^\pi \sin(kx)\sin(lx)dx \\ &= K_k + K_l\end{aligned} \tag{6.17}$$

where $K_k$ and $K_l$ are the kinetic energies of the individual harmonic waves. A similar calculation shows that also the potential energy of the sum is the sum of the potential energies and so this is also true of the total energy:

$$E[y_k + y_l] = E[y_k] + E[y_l]. \tag{6.18}$$

This is one of the nice properties of harmonic waves.

**Example 6.4** (Complex exponential wave)**.** We calculate the energy density of the complex solution

$$y(x,t) = A\, e^{i(kx-\omega t)}. \tag{6.19}$$

The expression for the energy density of complex solutions involves the absolute value squared:

$$\mathcal{E}[y] = \frac{\rho}{2}|\partial_t y|^2 + \frac{T}{2}|\partial_x y|^2. \tag{6.20}$$

This has the effect that the energy density is the sum of the energy density of the real part of the solution and the energy density of the imaginary part of the solution. We find

$$\begin{aligned} \mathcal{E}[A\, e^{i(kx-\omega t)}] &= \frac{\rho}{2}\left|-i\omega\, A\, e^{i(kx-\omega t)}\right|^2 + \frac{T}{2}\left|ik\, A\, e^{i(kx-\omega t)}\right|^2 \\ &= \left(\frac{\rho}{2}\omega^2 + \frac{T}{2}k^2\right)|A|^2 \end{aligned} \tag{6.21}$$

So another miracle of these complex exponential solutions is that their energy density is constant.

## 6.3 Conservation equation

Let $y(x,t)$ be a solution of the wave equation for the string and $\mathcal{E}$ its energy density

$$\mathcal{E} = \frac{\rho}{2}(\partial_t y)^2 + \frac{T}{2}(\partial_x y)^2. \tag{6.22}$$

For the time derivative of the energy density we find

$$\begin{aligned} \partial_t \mathcal{E} &= \rho\, \partial_t y\, \partial_t^2 y + T\, \partial_x y\, \partial_t \partial_x y \\ &= \partial_t y\, T\, \partial_x^2 y + T\, \partial_x y\, \partial_t \partial_x y \quad (\text{using} \ \ \rho\, \partial_t^2 y = T\, \partial_x^2 y\ ) \\ &= -\partial_x \left(-T\, \partial_t y\, \partial_x y\right). \end{aligned} \tag{6.23}$$

In terms of the quantity $\mathcal{F} = -T\, \partial_t y\, \partial_x y$ this equation takes the form

$$\partial_t \mathcal{E} = -\partial_x \mathcal{F}. \tag{6.24}$$

The quantity $\mathcal{F}$ is called the *energy flux*.

Eq. 6.24 implies the law of conservation of energy (and therefore is called a *conservation equation*}). Indeed, we have

$$\frac{dE}{dt} = \int_{x_1}^{x_2} \partial_t \mathcal{E}\, dx = -\int_{x_1}^{x_2} \partial_x \mathcal{F}\, dx = \mathcal{F}(x_1) - \mathcal{F}(x_2). \tag{6.25}$$

So if we interpret the energy flux $\mathcal{F}(x)$ as the rate at which energy flows through a point $x$ from left to right, then the conservation equation expresses that the rate at which energy in a region changes is equal to the difference between the rate at which energy flows in and the rate at which energy flows out of the region.

We already showed conservation of energy of a finite string with fixed boundary conditions in the previous section. We will now use the above machinery to show the conservation of the energy of a finite string with free boundary conditions. We consider a string between $x = 0$ and $x = \pi$ satisfying the free boundary conditions

$$\partial_x y(0, t) = 0 = \partial_x y(\pi, t) \text{ for all } t. \tag{6.26}$$

The energy flux at the left end of the string at $x = 0$ is

$$\mathcal{F}(0, t) = -T\, \partial_t y(0, t)\, \partial_x y(0, t). \tag{6.27}$$

This flux is zero due to the boundary condition. Similarly the flux at the right boundary is zero:

$$\mathcal{F}(\pi, t) = -T\, \partial_t y(\pi, t)\, \partial_x y(\pi, t) = 0. \tag{6.28}$$

So by the conservation equation it follows that the energy is conserved.

# 7 Two-dimensional waves

So far we have been discussing the one-dimensional wave equation. We have motivated the equation in terms of a string stretched along the $x$ dimension and oscillating in the $y$ dimension. We now want to extend this to higher dimensions.

One way to introduce another dimension would be to let the string vibrate in both the $y$ and $z$ dimension. However that would not give anything new. In the small-angle approximation to which we have been working, the oscillations in the $y$ direction and the oscillations in the $z$ direction would be independent and we would just end up with two independent one-dimensional wave equations,

$$\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t) \quad \text{and} \quad \partial_t^2 z(x,t) = c^2 \partial_x^2 z(x,t).$$

Instead we will introduce the second dimension by going from the string stretched in the $x$ dimension to a membrane stretched in both the $x$ and $y$ dimensions. It will vibrate in the $z$ direction. This toy example of a vibrating membrane will give us the two-dimensional wave equation, which of course appears in many other applications as well.

## 7.1 Two-dimensional wave equation

Consider an infinite two-dimensional membrane of homogeneous density $\rho$ (mass per unit area, measured in $kg/m^2$). In the equilibrium state, it is flat and coincides with the $(x, y)$ plane in $\mathbb{R}^3$. We assume that it is stretched to a tension $T$ (force per unit length, measured in $N/m$). This means that for any line on the surface of the membrane, the part of the membrane on one side of the line exerts a force $T$ (per unit length of the line) on the other part of the membrane (on the other side of the line), and the direction of the force is perpendicular to the line.

The perturbed membrane may be described as a time-dependent surface $z = z(x, y, t)$ in $\mathbb{R}^3$, where $z(x, y, t)$ is the vertical (in the $z$ direction) displacement of the membrane at point $(x, y)$ and time $t$. To derive the equation of motion, we consider a small element of the membrane of size $\delta x$ and $\delta y$ in $x$ and $y$, as shown in Figure 7.1.

We assume that there is only transverse motion of the membrane and that the partial derivatives $\partial_x z$ and $\partial_y z$ are small: $|\partial_x z| \ll 1$ and $|\partial_y z| \ll 1$. Almost the same arguments as those in Section 1.2 lead to the conclusion that the vertical component of the force is a sum of two terms: $T\delta y\, \partial_x^2 z\, \delta x$ (coming from the tension forces that are nearly parallel to the $x$-axis in

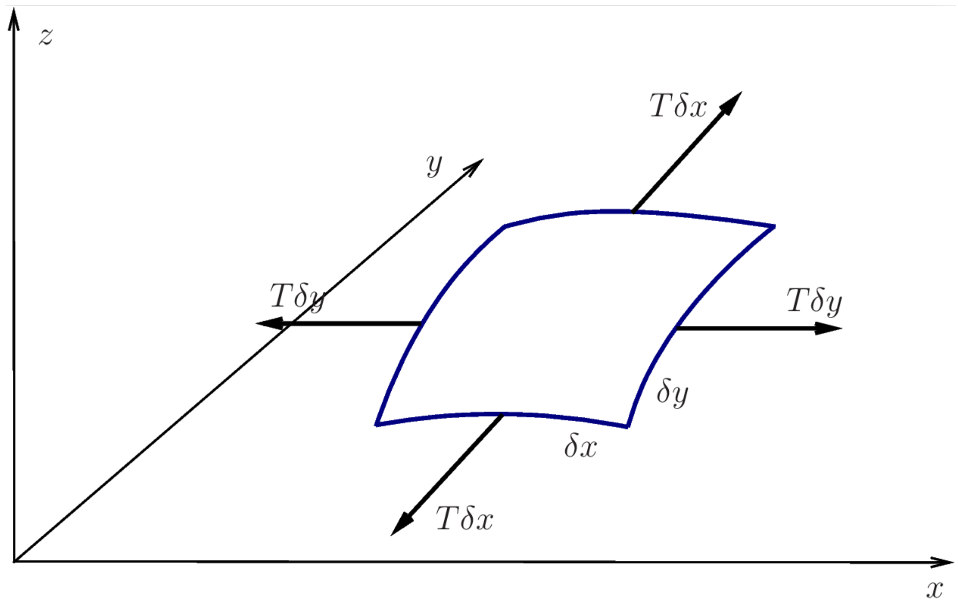


Figure 7.1: A membrane stretched in the $x$–$y$ plane and vibrating in the $z$ direction.

Figure 7.1) and $T\delta x\,\partial_y^2 z\,\delta y$ (coming from the tension forces that are nearly parallel to the $y$-axis in Figure 7.1). So, Newton's equation of motion ($ma = F$) yields

$$(\rho\,\delta x\,\delta y)\partial_t^2 z = T\delta y\,\partial_x^2 z\,\delta x + T\delta x\,\partial_y^2 z\,\delta y. \tag{7.1}$$

Dividing this by $\delta x\,\delta y$ and $\rho$, we get the governing equation for the membrane:

$$\partial_t^2 z = \frac{T}{\rho}\left(\partial_x^2\, z + \partial_y^2\, z\right) \tag{7.2}$$

or, equivalently,

$$\partial_t^2 z - c^2\nabla^2 z = 0, \quad c = \sqrt{\frac{T}{\rho}}. \tag{7.3}$$

Here $c$ in the wave speed and $\nabla^2$ is the Laplace operator:

$$\nabla^2 = \partial_x^2 + \partial_y^2. \tag{7.4}$$

Sometimes, the Laplace operator is also denoted by $\Delta$ (i.e. $\Delta = \nabla^2 = \partial_x^2 + \partial_y^2$). Eq. 7.3 is the two-dimensional wave equation. Written in terms of the Laplace operator it easily generalises to any higher dimension.

## 7.2 Energy of membrane

The energy density of the membrane is

$$\mathcal{E}(x,y,t) = \frac{\rho}{2}(\partial_t z)^2 + \frac{T}{2}\left((\partial_x z)^2 + (\partial_y z)^2\right), \tag{7.5}$$

where the term involving the density $\rho$ is the kinetic energy density $\mathcal{E}_K$ and the term involving the tension $T$ is the potential energy density $\mathcal{E}_V$. The latter we could write in vector notation as $\mathcal{E}_v = T|\underline{\nabla} z|^2/2$ where $\underline{\nabla} z$ is the gradient of $z$.

To check that Eq. 7.5 is a good expression for the energy density, we check that $\mathcal{E}$ satisfies a conservation equation. So we calculate

$$\begin{aligned}
\partial_t \mathcal{E} &= \rho\,(\partial_t z)(\partial_t^2 z) + T\,(\partial_t\partial_x z)(\partial_x z) + T\,(\partial_t\partial_y z)(\partial_y z) \\
&= T\,(\partial_t z)(\partial_x^2 z + \partial_y^2 z) + T\,\left((\partial_t\partial_x z)(\partial_x z) + (\partial_t\partial_y z)(\partial_y z)\right) \\
&= -\partial_x\left(-T(\partial_x z)(\partial_t z)\right) - \partial_y\left(-T(\partial_x z)(\partial_t z)\right),
\end{aligned} \tag{7.6}$$

where for the second equality we have used the wave equation. We introduce the two-dimensional energy flux density vector $\underline{\mathcal{F}} = (\mathcal{F}_x, \mathcal{F}_y)$ with

$$\mathcal{F}_x = -T(\partial_x z)(\partial_t z), \qquad \mathcal{F}_y = -T(\partial_y z)(\partial_t z). \tag{7.7}$$

In terms of this we have derived the conservation equation

$$\partial_t \mathcal{E} = -\underline{\nabla}\cdot\underline{\mathcal{F}}. \tag{7.8}$$

To understand how this two-dimensional conservation equation leads to energy conservation let us look at the energy in a region $R$ in the $(x,y)$ plane. The energy in this region is given by

$$E = \iint_R \mathcal{E}\,dA. \tag{7.9}$$

Here $dA = dx\,dy$ is the area element. The rate of change in the energy is then

$$\begin{aligned}
\frac{dE}{dt} &= \iint_R \partial_t \mathcal{E} dA = -\iint_R \underline{\nabla}\cdot\underline{\mathcal{F}}\,dA \\
&= -\int_{\partial R} \mathcal{F}\cdot\underline{n}\,ds.
\end{aligned} \tag{7.10}$$

For the last equality we used the divergence theorem. $\underline{n}$ is the outward unit normal to the boundary $\partial R$ of the region $R$ and $ds$ is the line element on $\partial R$. So again we see that the change of energy in a region is equal to the net flow of energy into the region.

## 7.3 Travelling plane waves

A 2D wave is a plane wave if $z(x, y, t)$ varies only in one spatial direction, say, parallel to a constant unit vector $\underline{n} = (n_x, n_y)$ (and $z(x, y, t)$ is constant in the direction perpendicular to $\underline{n}$). This means that

$$z(x, y, t) = f(\underline{n} \cdot \underline{x} - ct) = f(n_x x + n_y y - ct) \tag{7.11}$$

describes a plane wave travelling with wave speed $c$ in the direction of vector $\underline{n}$.

Let's verify that Eq. 7.11 is a solution of the 2D wave equation. We have

$$\begin{aligned} \partial_t^2 z &= \partial_t^2 f(\underline{n} \cdot \underline{x} - ct) = f''(\underline{n} \cdot \underline{x} - ct)\, c^2, \\ \partial_x^2 z &= \partial_x^2 f(\underline{n} \cdot \underline{x} - ct) = f''(\underline{n} \cdot \underline{x} - ct)\, n_x^2, \\ \partial_y^2 z &= \partial_y^2 f(\underline{n} \cdot \underline{x} - ct) = f''(\underline{n} \cdot \underline{x} - ct)\, n_y^2. \end{aligned} \tag{7.12}$$

Hence,

$$\begin{aligned} &\partial_t^2 z - c^2 \left(\partial_x^2 z + \partial_y^2 z\right) \\ =&c^2 f''(\underline{n} \cdot \underline{x} - ct) \left(1 - n_x^2 - n_y^2\right) = 0 \end{aligned} \tag{7.13}$$

because $\underline{n}$ is a unit vector, i.e., $n_x^2 + n_y^2 = 1$.

Because the wave equation is linear, any superposition of plane wave solutions is also a solution.

The *harmonic plane wave* corresponds to the choice $f(s) = e^{iks}$, so that

$$z(x, y, t) = e^{ik(\underline{n} \cdot \underline{x} - ct)} = e^{i(\underline{k} \cdot \underline{x} - \omega(\underline{k})t)}, \tag{7.14}$$

where $\underline{k} = k\, \underline{n}$ is called the *wave vector*. We have the dispersion relation $\omega(\underline{k}) = c\, |\underline{k}|$.

## 7.4 Higher dimensions

By writing our equations in vector notation, we can see that they work in any dimension. So the n-dimensional wave equation for a real-valued function $z : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ or a complex-valued function $z : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is

$$\partial_t^2 z(\underline{x}, t) = c^2\, \nabla^2 z(\underline{x}, t). \tag{7.15}$$

Its energy density is

$$\mathcal{E} = \frac{T}{2} \left( \frac{1}{c^2} |\partial_t z|^2 + |\underline{\nabla} z|^2 \right). \tag{7.16}$$

It satisfies the conservation equation

$$\partial_t \mathcal{E} = -\underline{\nabla} \cdot \underline{\mathcal{F}} \tag{7.17}$$

where the energy flux is

$$\underline{\mathcal{F}} = -T\, Re\,(\partial_t z\, \underline{\nabla} z) \tag{7.18}$$

According to the $n$-dimensional divergence theorem, the rate of change of the energy in an $n$-dimensional region $R \subset \mathbb{R}^n$ with $n-1$-dimensional boundary $\partial R$ is

$$\begin{aligned} \frac{dE}{dt} &= \int_R \partial_t \mathcal{E}\, dV = -\int_R \underline{\nabla} \cdot \underline{\mathcal{F}}\, dV \\ &= \int_{\partial R} \underline{\mathcal{F}} \cdot \underline{n}\, dS. \end{aligned} \tag{7.19}$$

The wave equation has plane wave solutions

$$z(\underline{x}, t) = f(\underline{n} \cdot \underline{x} - ct) \tag{7.20}$$

for any choice of $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{C}$.

When you study Electromagnetism you will meet an even nicer way of writing these equations in terms of $n+1$-dimensional space-time vectors and tensors.

# 8 Waves on rectangular domain

Consider a rectangular membrane: $D = \{(x, y) \in \mathbb{R}^2 | 0 < x < a,\ 0 < y < b\}$. Let's solve the wave equation

$$\partial_t^2 z - c^2 \left(\partial_x^2 z + \partial_y^2 z\right) = 0 \quad \text{in} \quad D \tag{8.1}$$

subject to the fixed (Dirichlet) boundary condition

$$z(x, y, t) = 0 \quad \text{on} \quad \partial D \tag{8.2}$$

or, equivalently,

$$z(0, y, t) = 0, \quad z(a, y, t) = 0 \tag{8.3}$$

and

$$z(x, 0, t) = 0, \quad z(x, b, t) = 0. \tag{8.4}$$

To find a solution, we use the method of separation of variables (see Section 4.1), i.e., we make the Ansatz

$$z(x, y, t) = X(x)Y(y)T(t). \tag{8.5}$$

Substituting this Ansatz into the wave equation, we get

$$XYT'' = c^2 \left(X''YT + XY''T\right), \tag{8.6}$$

which, after dividing by $XYT$ gives

$$\frac{T''(t)}{T(t)} = c^2 \left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right). \tag{8.7}$$

In the last equation, we have a function of one variable, $t$, on the left side and a function of two different variables, $x$ and $y$, on the right side. The equation can be satisfied for all $x$, $y$ and $t$ only if both sides are equal to a constant. As in Section 4.1, we choose this constant to be negative and (for convenience) equal to $-k^2c^2$ (for arbitrary real $k$). Again, the possibility of a positive constant is excluded, because with a positive constant it is impossible to find solutions satisfying the boundary conditions. Thus, we have

$$\frac{T''(t)}{T(t)} = c^2 \left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right) = -k^2c^2. \tag{8.8}$$

This leads to the ODE

$$T''(t) = -c^2k^2\, T(t) \tag{8.9}$$

and to the equation

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -k^2. \tag{8.10}$$

The general solution of Eq. 8.9 is

$$T(t) = F\,\sin(kct) + G\,\cos(kct), \tag{8.11}$$

where $F$ and $G$ are arbitrary constants.

Rewriting Eq. 8.10 as

$$\frac{X''(x)}{X(x)} = -k^2 - \frac{Y''(y)}{Y(y)}, \tag{8.12}$$

we conclude that for this equation to hold for all $x$ and $y$, both sides must be equal to a constant, which we choose to write as $-\nu^2$ (for some real $\nu$). Also, introducing the constant $\mu$ so that $\nu^2 + \mu^2 = k^2$ gives us the equations

$$X''(x) = -\nu^2 X(x), \quad Y''(y) = -\mu^2 Y(y). \tag{8.13}$$

The general solutions are

$$\begin{aligned} X(x) &= A\sin(\nu x) + B\cos(\nu x),\\ Y(y) &= C\sin(\mu y) + D\cos(\mu y), \end{aligned} \tag{8.14}$$

for arbitrary constants $A, B, C, D$.

Now we are ready to impose the boundary conditions.

The condition $z(0, y, t) = 0$ for all $y, t$ requires that $X(0) = 0$ and, because $X(0) = B$, this implies that $B = 0$. Similarly the condition $z(a, y, t) = 0$ requires that $X(a) = 0$ and because $X(a) = A\sin(\nu\, a)$ (because we already know that $B = 0$), this implies that either $A = 0$, which is not an interesting case because it makes the entire solution vanish, or that $\nu = n\pi/a$ with $n \in \mathbb{Z}$. Without loss of generality we can take $n \in \mathbb{N}$ because negative $n$ just give the same solution up to a sign that can be absorbed into the arbitrary constant $A$, and $n = 0$ gives the zero solution.

The conditions $z(x, 0, t) = 0 = z(x, b, t)$ similarly require that $Y(0) = 0 = Y(\pi)$ and thus $D = 0$ and $\mu = m\pi/a$ with $m \in \mathbb{N}$.

Thus we have found the following solutions satisfying the boundary conditions:

$$\begin{aligned} z_{nm}(x, y, t) = &\sin\left(\frac{\pi}{a}nx\right)\sin\left(\frac{\pi}{b}my\right)\\ &\cdot\left(F_{nm}\sin(k_{nm}ct) + G_{nm}\cos(k_{nm}ct)\right) \end{aligned} \tag{8.15}$$

with

$$k_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \tag{8.16}$$

for any choice of $n, m \in \mathbb{N}$ and $F_{nm}, G_{nm} \in \mathbb{R}$.

Note that solutions Eq. 8.15 already satisfy the boundary conditions Eq. 8.3 and Eq. 8.4. Such solutions are called *normal modes* of the membrane. Snapshots of normal modes with $(n, m) = (1, 1)$, $(n, m) = (1, 2)$, $(n, m) = (2, 2)$ and $(n, m) = (3, 2)$ are shown in Figure 8.1.

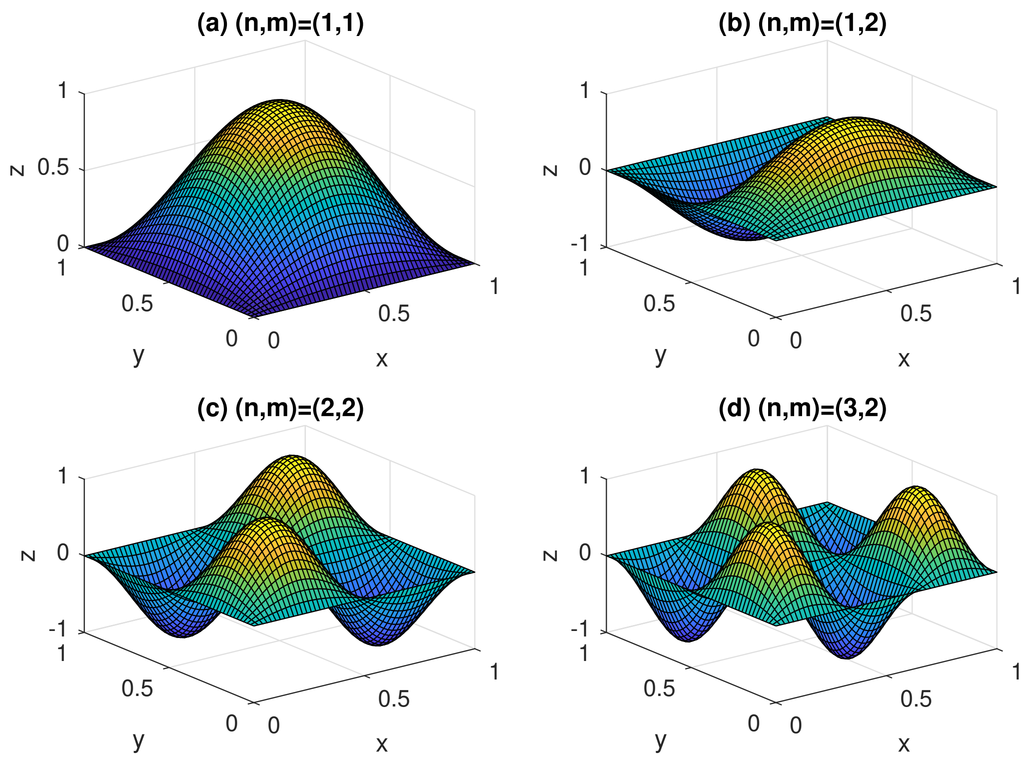


Figure 8.1: Some normal modes of a rectangular membrane

The amplitudes of the normal modes oscillate over time. The following video shows an animation of various normal modes:

https://youtu.be/yDZsCZn3lSk

Any solution of the wave equation Eq. 8.1 satisfying the boundary conditions Eq. 8.3 and Eq. 8.4 can be presented as a linear combination of normal modes Eq. 8.15:

$$z(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z_{nm}. \tag{8.17}$$

The free constants $F_{nm}$ and $G_{nm}$ are then determined by the initial conditions, using the Fourier transform technique as in the one-dimensional case.

To construct a solution satisfying the initial conditions

$$z(x,y,0) = z_0(x,y), \quad \partial_t z(x,y,0) = v_0(x,y) \tag{8.18}$$

we evaluate our general solution and its $t$ derivative at the boundary. This gives

$$z(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{\pi n x}{a}\, \sin\frac{\pi m y}{b} G_{nm} = z_0(x,y), \tag{8.19}$$

$$\partial_t z(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_{nm} c\, B_{nm}\, \sin\frac{\pi n x}{a}\, \sin\frac{\pi m y}{b} = v_0(x,y). \tag{8.20}$$

Comparing these with the (double) Fourier series for functions $z_0(x,y)$ and $v_0(x,y)$, one can deduce that

$$\begin{aligned} G_{nm} &= \frac{4}{ab}\int_0^a dx \int_0^b dy\, z_0(x,y) \sin\frac{\pi n x}{a}\, \sin\frac{\pi m y}{b}, \\ F_{nm} &= \frac{1}{k_{nm} c}\,\frac{4}{ab}\int_0^a dx \int_0^b dy\, v_0(x,y) \sin\frac{\pi n x}{a}\, \sin\frac{\pi m y}{b}. \end{aligned} \tag{8.21}$$

# 9 Waves on circular domain

Let's find normal mode solution for a circular membrane of radius $a$ with fixed boundary (see Figure 9.1). Mathematically, we need to solve the wave equation (Eq. 8.1) in the disc of radius $a$, i.e. in region $D = \{(x, y) \in \mathbb{R}^2 | r^2 = x^2 + y^2 < a^2\}$. It is convenient to use polar coordinates $(r, \theta)$, defined as (see Figure 9.1)

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{9.1}$$

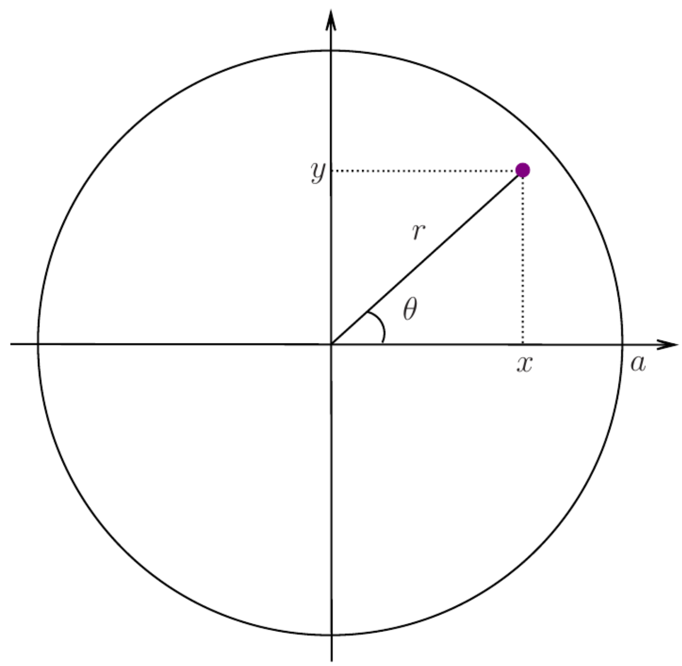


Figure 9.1: Circular membrane with radius $a$ and polar coordinates.

Let

$$Z(r, \theta, t) = z\left(x(r, \theta), y(r, \theta), t\right), \tag{9.2}$$

i.e., $Z(r, \theta, t)$ is a solution of the wave equation in $D$, expressed in terms of polar coordinates.

We know that the wave equation can be written in vector notation using the Laplace operator $\nabla^2$ as

$$\partial_t^2 z - c^2 \nabla^2 z = 0. \tag{9.3}$$

It is well-known (see, e.g., Theorem 3.43 in the Calculus notes) that the Laplacian in polar coordinates is given by

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\partial_\theta^2. \tag{9.4}$$

Therefore, the 2D wave equation in polar coordinates has the form

$$\partial_t^2 Z - c^2\left(\partial_r^2 Z + \frac{1}{r}\,\partial_r Z + \frac{1}{r^2}\,\partial_\theta^2 Z\right) = 0. \tag{9.5}$$

Our aim is to solve it subject to the boundary condition Eq. 9.20. As in the preceding section, we employ the method of separation of variables, i.e. we assume that

$$Z(r,\theta,t) = R(r)\Theta(\theta)T(t). \tag{9.6}$$

Substituting this into Eq. 9.5, we find that

$$R\Theta T'' = c^2\left(R''\Theta T + \frac{1}{r}\,R'\Theta T + \frac{1}{r^2}\,R\Theta'' T\right) \tag{9.7}$$

Dividing by $R\Theta T$ gives

$$\frac{T''(t)}{T(t)} = c^2\left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)}\right). \tag{9.8}$$

Employing the same arguments as in the preceding section, we conclude that

$$\frac{T''(t)}{T(t)} = c^2\left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)}\right) = -k^2c^2 \tag{9.9}$$

for some constant $k$. This means that we have the ODE

$$T''(t) = -k^2c^2\,T(t) \tag{9.10}$$

and the equation

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = -k^2 \tag{9.11}$$

The general solution of Eq. 8.3 is (cf. Eq. 8.11)

$$T(t) = F\,\sin(kct) + G\,\cos(kct) \tag{9.12}$$

where $F$ and $G$ are arbitrary constants.

Rewriting Eq. 9.12 as

$$\frac{r^2R''(r) + rR'(r)}{R(r)} + k^2r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}, \tag{9.13}$$

we conclude that for this equation to hold for all $r$ and $\theta$, both sides must be equal to a constant, which we choose to be $n^2$ (for some constant $n$), i.e.

$$r^2\left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)}\right) + k^2r^2 = n^2, \quad -\frac{\Theta''(\theta)}{\Theta(\theta)} = n^2 \tag{9.14}$$

(the constant, $n^2$, cannot be negative because, with a negative constant, it will be impossible to obtain a solution). Thus, we we have obtained the following two ODEs:

$$R''(r) + \frac{1}{r}\,R'(r) + \left(k^2 - \frac{\lambda^2}{r^2}\right) R(r) = 0 \tag{9.15}$$

and

$$\Theta''(\theta) = -n^2\Theta(\theta). \tag{9.16}$$

The general solution of Eq. 9.16 is

$$\Theta(\theta) = A\sin(n\theta) + B\cos(n\theta) \tag{9.17}$$

for arbitrary constants $A$ and $B$. We impose the *periodicity* condition

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad \text{for all} \quad \theta.$$

This is a natural condition because $(r, \theta)$ and $(r, \theta + 2\pi)$ represent a single point in domain $D$. Thus we require $n \in \mathbb{Z}$. In fact we only need to consider $n \in \mathbb{N} \cup \{0\}$ because of the symmetry of sin and cos.

Eq. 9.15 has the form

$$r^2\,R''(r) + r\,R'(r) + (r^2\,k^2 - n^2)\,R(r) = 0. \tag{9.18}$$

It is a well-known equation, called the Bessel differential equation. Its solutions are not elementary functions. We are only interested in solutions that are finite at $r = 0$. There is a whole family of such solutions,

$$R(r) = J_n(kr), \tag{9.19}$$

where $J_n$ is the $n$th Bessel functions of the first type. We have already determined above that we are only interested in the cases where $n \in \mathbb{N} \cup \{0\}$.

Next we need to impose the zero Dirichlet boundary condition (the edge of the membrane is fixed)

$$Z(a, \theta, t) = 0. \tag{9.20}$$

To do this, we only need to know that each Bessel function has an infinite number of zeros. A plot of the first few Bessel functions of the 1st kind is shown in Figure 9.2.

We denote the $m$th zero of the $n$th Bessel function of the 1st kind by $k_{nm}$. We have

$$R(a) = 0 \quad \Rightarrow \quad J_n(k\,a) = 0 \quad \Rightarrow \quad k\,a = k_{nm} \quad \Rightarrow \quad k = \frac{k_{nm}}{a}$$

for $m \in \mathbb{N}$.

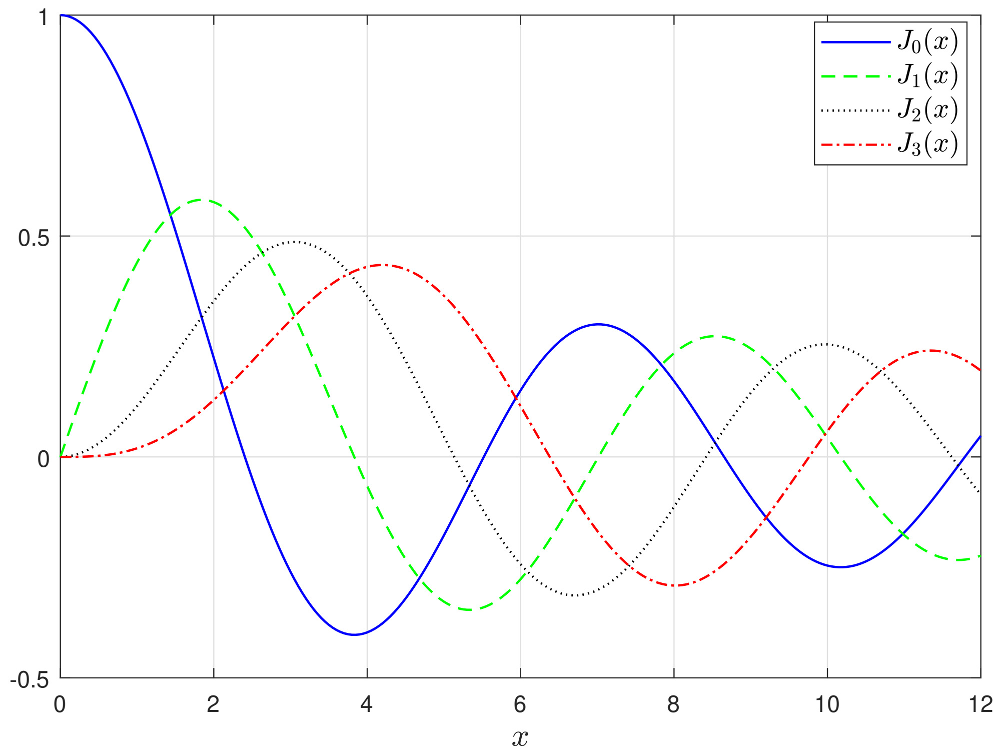


Figure 9.2: Plot of the first four Bessel functions of the 1st kind.

Finally, combining the above with Eq. 9.12 and Eq. 9.17, we obtain the following solutions (that satisfy the required boundary condition):

$$\begin{aligned} Z_{nm}(r,\theta,t) = & J_m\left(k_{nm}\frac{r}{a}\right)\left(A_{nm}\sin(n\theta)+\cos(n\theta)\right) \\ & \cdot\left(F_{nm}\,\sin(k_{nm}ct)+G_{nm}\,\cos(k_{nm}ct)\right) \end{aligned} \tag{9.21}$$

where $n \in \mathbb{N}\cup\{0\}$, $m \in \mathbb{N}$ and $A_{nm}$, $F_{nm}$ and $G_{nm}$ are arbitrary real constants. Solutions in the form Eq. 9.21 are the normal modes of vibrations of the circular membrane. Once normal modes are known, we can find a solution satisfying some initial conditions by using a linear combination of the normal modes

$$Z(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} Z_{nm}(r,\theta,t). \tag{9.22}$$

Note that constants $A_{nm}$, $F_{nm}$ and $G_{nm}$ (in the formula for $Z_{nm}$ for each $n,m$) are still arbitrary. To determine them, we need to substitute Eq. 9.22 into initial conditions. This will lead to the so-called Fourier-Bessel series. This is based on the identities

$$\int_0^a J_n\left(k_{nm}\frac{r}{a}\right) J_n\left(k_{nl}\frac{r}{a}\right) r\,dr = \delta_{ml}\frac{a^2}{2}\left(J_{n+1}(k_{nm})\right)^2. \tag{9.23}$$

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