Waves and Fluids

Lecture notes for Spring 2023

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Welcome

These are the lecture notes for Waves and Fluids, part of the 2nd year Applied Maths module at the University of York in Spring 2023. Each chapter in these notes corresponds to one lecture.

These notes will be periodically revised. Whenever you spot something that is not quite right, please email me at gustav.delius@york.ac.uk or submit your correction in the correction form at https://forms.gle/w17c19vWnM7wpLpz7.

This module consists of two topics: Waves and Fluids. In the first half of the term, we will focus on Waves, while in the second half, we will delve into Fluids. The topics are interconnected by their use of partial differential equations to describe real-world phenomena in space and time.

An alternative title for this module could be "Introduction to Continuum Dynamics". It is designed to introduce you to the mathematical modeling and analysis of the behaviour of materials that are treated as continua, or continuous media. Unlike Newtonian Dynamics and Classical Dynamics, which deal with individual particles, Continuum Dynamics deals with emergent phenomena created by the interaction of large numbers of particles. It employs mathematical tools such as partial differential equations.

The concepts and methods presented in Continuum Dynamics form the basis of a large part of Applied Mathematics. You will encounter them again in future Applied Mathematics modules, and they are also used in fields such as biology, ecology, medicine, sociology, and economics. The examples presented in this module are less important than the way of thinking they introduce.

The goal of this module is to equip you with the skills and knowledge to apply mathematics to new phenomena in the real world. In your third and fourth year, you will have the opportunity to deepen this ability through various modules.

Throughout this module, we shall use SI units: length is measured in meters (m), time in seconds (s), mass in kilograms (kg).

Part I

Waves

Waves play a fundamental role in our universe, as they are the only means by which information can propagate through space and time. Some examples of waves that carry information are well-known, such as sound waves, light waves, radio waves, and the electrical waves that travel along our neurons. Other examples are less obvious, such as the waves that describe the motion of particles in quantum mechanics and the gravitational waves that convey the effects of gravity.

Because waves travel at a finite speed, information can only propagate at the speed of the wave. This has profound implications, as we see in the theory of special relativity. For instance, we can observe distant events in the past because some of the light waves that were emitted soon after the Big Bang, roughly 16 billion years ago, are only now reaching us. These waves are strongly red-shifted and detectable as microwaves.

The study of waves is not only important from a theoretical perspective, but also from a practical standpoint. Waves play a vital role in our modern technological world. Advances in our understanding of how to generate and control electromagnetic waves have led to the development of essential technologies such as radio, radar, and mobile phones.

Furthermore, understanding how waves propagate can also help us prepare for and prevent potential threats. For instance, studying how pests spread in the form of invasion waves into previously uninfected areas can help us develop effective interventions to prevent their spread. Similarly, understanding how density waves form in traffic flow and lead to traffic jams can help us design interventions to improve traffic flow and reduce congestion.

By studying waves, we can gain insights into the underlying mechanisms that govern a wide range of phenomena, from the behaviour of subatomic particles to the dynamics of traffic flow. These insights can be applied in various fields, including engineering, medicine, and environmental science. By deepening our understanding of waves, we can continue to make advancements in technology and address real-world challenges in a more effective manner.

1 Deriving the wave equation for a string

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 7
- Coulson and Jeffrey (1977) sections 17 and 19
- Baldock and Bridgeman (1983) section 1.10.3
- Simmons (1972) section 40

1.1 Why waves on a string?

The great diversity of waves in nature means that we need to choose some concrete wave phenomenon to concentrate on to start our investigation. In this module we will concentrate on the waves on a string (think of a guitar string) and generalise to waves on a membrane (think of the membrane of a drum). By studying this in detail you will develop the intuition and the skills that will allow you to understand other wave phenomena later. We'll come back to waves at the end of the part on Fluid Dynamics when we study waves on the surface of a fluid.

Personally I like studying vibrating strings because they are at the foundation of superstring theory. This is a "Theory of Everything" that posits that elementary particles are actually tiny strings, with different vibrational states corresponding to different elementary particles. As a Ph.D. student I showed how, if these strings move in certain higher-dimensional group supermanifolds, they behave like the elementary particles of our standard model of particle physics, including the chiral fermions. If we ignore the bit about group supermanifolds for the moment, the maths behind string theory is no more complicated than the maths we will discuss in this module and the partner module on quantum mechanics.

We consider a flexible, elastic string of linear density ρ (mass per unit length), which undergoes small *transverse* vibrations. (For example, it can be a guitar string.) The transverse vibrations mean that the displacements of each small element of the string is perpendicular to its length. We assume that the string does not move *longitudinally* (i.e. parallel to its length). Let $y(x,t)$ be its displacement from equilibrium position at time t and position x (see Fig. Figure 1.1).

The string is sufficiently simple so that we can understand it by pure thought. We will derive from first principles a PDE that describes its motion (the wave equation) and then solve it for various initial conditions. I find it amazing that this is possible.

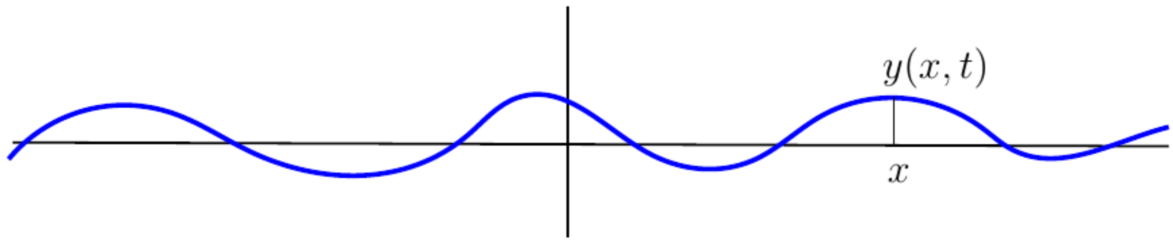

Figure 1.1: A string stretched in the x-direction and vibrating in the y-direction.

To achieve this mathematical understanding we will need to make several simplifications and approximations. For example, we neglect that the string is made up of lots of individual atoms and instead we will pretend that the mass is spread out continuously along the string. This is known as the **continuum approximation** and we will meet this again in the fluid dynamics part of this module.

To derive the equation of motion of the string we first need to discuss the force acting on it which we will do in the next section. Then in the section after that we can plug this into Newton's second law and out pops the wave equation. Along the way we will make approximations that allow us to linearise the equations.

1.2 Linearized tension force

We consider a small segment of the string between any two points x and at $x + \delta x$ as shown in Fig. Figure 1.2. We want to determine the force that is acting on this segment, so that we can later determine its motion using Newton's second law. We will concentrate on only the tension force of the string and ignore less important effects like gravity, friction, or stiffness.

We assume that the tension force $T(x)$ has constant magnitude throughout the string: $|T(x)| = T$. However its direction varies along the string, because it always acts in the tangential direction. At interior points the tension force pulling to one side will balance that pulling in the other direction. The net tension force on the segment will thus be determined by the tension forces at its ends. We have drawn these forces schematically in Figure Figure 1.2 where we have also split them into their x and y components.

The total force acting on the segment is

$$F = T(x + \delta x) - T(x). \tag{1.1}$$

We first consider the y component

$$T_y(x) = T \sin \theta(x), \tag{1.2}$$

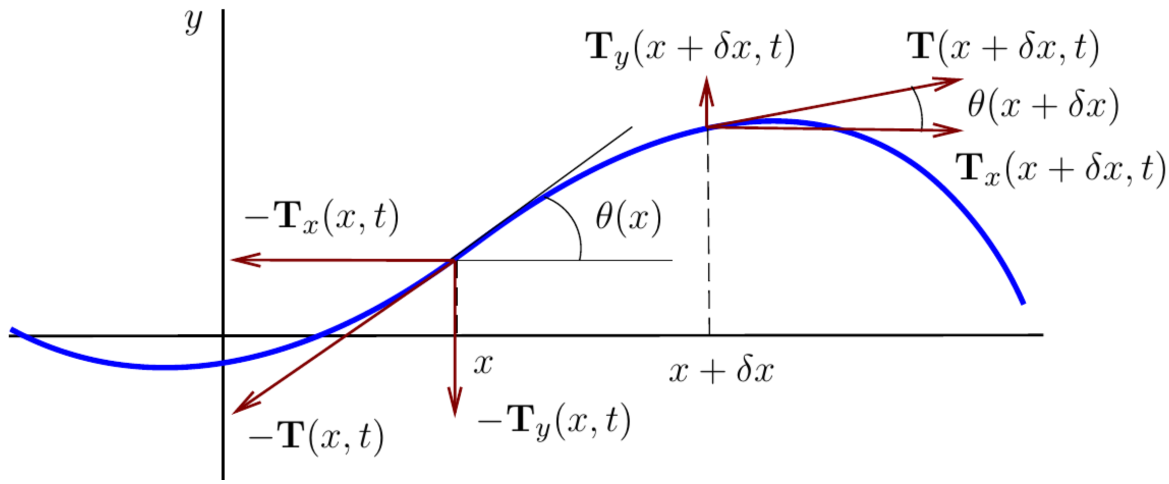

Figure 1.2: The tension forces acting on a segment of the string between x and $x + \delta x$.

where $\theta(x)$ is the angle that the string makes with the horizontal at x. The slope of the string at x is

$$\frac{\partial y}{\partial x} = \tan\theta(x). \tag{1.3}$$

We are now going to simplify the expressions by assuming that the slope and thus θ is small, $\theta << 1$. Then, by Taylor expansion,

$$\sin\theta = \theta + O(\theta^3), \quad \tan\theta = \theta + O(\theta^2). \tag{1.4}$$

We ignore all terms that are higher order in θ. This is known as the linear approximation. It is done very often, because it leads to linear equations that are so much easier to solve. So

$$\begin{aligned} F_y &= T_y(x+\delta x) - T_y(x) \\ &= T\sin\theta(x+\delta x) - T\sin\theta(x) \\ &\approx T\left(\theta(x+\delta x) - \theta(x)\right). \end{aligned} \tag{1.5}$$

We do another Taylor expansion and ignore higher-order terms in δx, which is fine because we want to look at only an infinitesimally small segment of string.

$$\begin{aligned} \theta(x+\delta x) &= \theta(x) + \delta x\frac{\partial\theta}{\partial x} + O(\delta x)^2 \\ &\approx \theta(x) + \delta x\frac{\partial\theta}{\partial x}. \end{aligned} \tag{1.6}$$

Substituting this into Eq. 1.5 gives

$$F_y \approx T\delta x\frac{\partial\theta}{\partial x}. \tag{1.7}$$

We would like to express this in terms of y instead of θ, which we can do by observing that

$$\theta \approx \tan\theta = \frac{\partial y}{\partial x}, \tag{1.8}$$

so we finally have

$$F_y \approx T\,\delta x\,\frac{\partial^2 y}{\partial x^2}. \tag{1.9}$$

We deal with the x component of the force similarly, using the Taylor expansion of $\cos\theta = 1 + O(\theta^2)$:

$$\begin{aligned} F_x &= T_x(x+\delta x) - T_x(x) \\ &= T\cos\theta(x+\delta x) - T\cos\theta(x) \\ &\approx T - T = 0. \end{aligned} \tag{1.10}$$

So in our approximation of small slope, there is no movement in the x direction. The string vibrates purely transversally.

1.3 Wave equation from Newton's 2nd law

To determine the motion in the y direction we use Newton's second law

$$ma_y = F_y, \tag{1.11}$$

where a_y is the acceleration in the y direction,

$$a_y = \frac{\partial^2 y}{\partial t^2} \tag{1.12}$$

and m is the mass of the infinitesimal segment which is obtained as the density times the length,

$$m = \rho\delta x. \tag{1.13}$$

We assume that density ρ is constant along the string. Plugging this together with our expression for F_y into Newton's second law gives

$$\rho\delta x\frac{\partial^2 y}{\partial t^2} = T\delta x\frac{\partial^2 y}{\partial x^2}. \tag{1.14}$$

We can cancel the δx and divide by ρ which finally gives us the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{1.15}$$

with **wave speed**

$$c = \sqrt{\frac{T}{\rho}}. \tag{1.16}$$

Why we call the constant c the wave speed will become clear in the next lecture.

1.4 Checking dimensions

After having derived an equation, it is always wise to check that its dimensions work out correctly.

We use square brackets to denote the dimension of a quantity. So $[y] = L$ says that y has dimension of length, $[m] = M$ says that m has dimension of mass, and $[t] = T$ says that t has dimension of time. [1] The dimension of both sides of an equation has to agree, so

$$\left[\frac{\partial^2 y}{\partial t^2}\right] = \frac{L}{T^2} = [c^2]\left[\frac{\partial^2 y}{\partial x^2}\right] = [c^2]\frac{1}{L}. \tag{1.17}$$

This shows that $[c] = L/T$, so it has the dimension of a velocity. Because T is a force we have $[T] = ML/T^2$. The density ρ has $[\rho] = M/L$. So

$$[c] = \left[\sqrt{\frac{T}{\rho}}\right] = \sqrt{\frac{ML/T^2}{M/L}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}. \tag{1.18}$$

This completes our check of the dimensions.

[1] Note the conflict of notation where we used T for the tension force while it is also the conventional symbol for the dimension of time. Such conflicts happen from time to time – the context determines the meaning of the symbol.

2 d'Alembert's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 8
- Coulson and Jeffrey (1977) sections 7 and 11
- Baldock and Bridgeman (1983) section 2.1

In this lecture, we consider an infinitely long string (this is physically justified if we consider waves propagating far away from any boundaries). Mathematically, this means that we are looking for solutions of the wave equation

$$\partial_t^2 y - c^2\,\partial_x^2 y = 0 \tag{2.1}$$

on the whole real axis $-\infty < x < +\infty$. Note that I have switched to the convenient notation using subscripts on derivatives to specify the variable with respect to which we are differentiating.

2.1 Characteristic coordinates

To solve the wave equation we use the *method of characteristics*, which involves a change of variables that makes the equation much simpler. We change from the variables x and t to the *characteristic coordinates*

$$\xi = x + ct, \quad \eta = x - ct. \tag{2.2}$$

By this we mean that for any function y that depends on the variables x and t we can introduce a function $\tilde{y}$ that depends on the variables ξ and η in such a way that it has the same values as y:

$$y(x,t) = \tilde{y}(\xi(x,t), \eta(x,t)) \text{ for all } x, t. \tag{2.3}$$

It is a conventional abuse of notation to drop the tilde and denote both functions by y. We will follow this abuse of notation.

We need to express the derivatives with respect to t and x via the derivatives with respect to ξ and η. This is done using the chain rule:

$$\begin{aligned}\partial_t\, y &= \frac{\partial y}{\partial \xi}\,\frac{\partial \xi}{\partial t} + \frac{\partial y}{\partial \eta}\,\frac{\partial \eta}{\partial t}\\ &= c\left(\partial_\xi - \partial_\eta\right) y\end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \partial_x y &= \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \left(\partial_\xi + \partial_\eta\right) y. \end{aligned} \tag{2.5}$$

Hence

$$\partial_t = c\left(\partial_\xi - \partial_\eta\right), \quad \partial_x = \partial_\xi + \partial_\eta. \tag{2.6}$$

Substituting these into the wave equation, we find that

$$c^2\left(\partial_\xi - \partial_\eta\right)^2 y - c^2\left(\partial_\xi + \partial_\eta\right)^2 y = 0. \tag{2.7}$$

Expanding the squares and cancelling terms gives

$$-4c^2\partial_\xi\partial_\eta y = 0. \tag{2.8}$$

We can divide both sides by the nonzero constant $-4c^2$. Thus the wave equation simplifies to

$$\partial_\xi\partial_\eta y = 0. \tag{2.9}$$

While here we have seen that the method of characteristics simplifies the wave equation, its applications extend far beyond this specific example. The method of characteristics can be used to solve a wide range of partial differential equations that arise in various areas of applied mathematics, including fluid dynamics. It can also be used to study the behaviour of complex systems, such as traffic flow, chemical reactions, and population dynamics. The key idea behind the method of characteristics is to transform a partial differential equation into a system of ordinary differential equations along characteristic curves, which can then be solved using standard techniques.

2.2 General solution of wave equation

The wave equation in characteristic coordinates is really easy to solve. First, we integrate Eq. 2.9 in the variable ξ:

$$\begin{aligned} &\int \partial_\xi\partial_\eta y(\xi,\eta)\, d\xi = 0 \\ \Leftrightarrow \quad &\partial_\eta y(\xi,\eta) = F(\eta) \end{aligned} \tag{2.10}$$

where F is an arbitrary function of one variable [1].

[1] You can verify that this is true by direct differentiation of Eq. 2.10 with respect to ξ.

Note

When we integrate a function of two variables in one of the two variable, we need to add to the result an arbitrary function of the other variable. This is similar to adding a constant of integration when we integrate a function of one variable.

Now we can integrate Eq. 2.10 in the variable η:

$$\begin{aligned} y(\xi,\eta) &= \int \partial_\eta y(\xi,\eta)\, d\eta \\ &= \int F(\eta) d\eta + g(\xi) \\ &= f(\eta) + g(\xi), \end{aligned} \tag{2.11}$$

where $g(\xi)$ is an arbitrary function of one variable and $f'(\eta) = F(\eta)$. Note that since F is arbitrary, so is f.

Returning to variables x and t, we can write the general solution of the wave equation as

$$y(x,t) = f(x-ct) + g(x+ct) \tag{2.12}$$

where f and g are arbitrary functions of one variable.

2.3 Travelling waves

We will now gain an initial understanding of this solution by visualising the two special cases where either f or g are zero.

If $g = 0$, then $y(x,t) = f(x-ct)$. At $t = 0$, the string has the shape described by the graph $y = f(x)$. At time $t > 0$, it will have the same shape relative to the variable $\eta = x - ct$: $y = f(\eta)$. Since $x = \eta + ct$, this means that the graph of y as a function of x for a fixed $t > 0$ is the graph of $f(x)$ shifted to the *right* (in the direction of positive x) by distance ct.

If $f = 0$, then $y(x,t) = g(x+ct)$. At $t = 0$, the string has the shape described by the graph $y = g(x)$. At time $t > 0$, it will have the same shape relative to the variable $\xi = x + ct$: $y = g(\xi)$. Since $x = \xi - ct$, this means that the graph of y as a function of x for a fixed $t > 0$ is the graph of $g(x)$ shifted to the *left* (in the direction of negative x) by distance ct.

Thus, $f(x-ct)$ and $g(x+ct)$ describe waves that propagate (without changing shape) to the right and to the left, respectively, and the general solution Eq. 2.12 represent the sum of such waves.

2.4 Initial value problem and d'Alembert's formula

The initial-value problem is to solve the wave equation

$$\partial_t^2 y - c^2 \partial_x^2 y = 0 \tag{2.13}$$

for $-\infty < x < +\infty$ and $0 < t < +\infty$ with the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \tag{2.14}$$

for $-\infty < x < +\infty$, where y_0 and v_0 are given functions of x. The first of the two initial conditions prescribes the initial displacement of the string, the second the initial velocity.

To solve an initial value one has to substitute the general solution into the initial conditions. We substitute the solution from Eq. 2.12 into the initial conditions in Eq. 2.14 and obtain

$$y_0(x) = f(x) + g(x), \tag{2.15}$$

$$v_0(x) = -cf'(x) + cg'(x). \tag{2.16}$$

So we have two equations for the two unknown functions f and g. To solve them, we first integrate Eq. 2.16:

$$-cf(x) + cg(x) = \int_0^x v_0(s)ds + a = V(x), \tag{2.17}$$

where a is an integration constant and $V(x)$ is just introduced to save writing below.

Next, we add and subtract Eq. 2.15 and Eq. 2.17 divided by c. This results in

$$\begin{aligned} y_0(x) - \frac{1}{c}\,V(x) &= 2\,f(x), \\ y_0(x) + \frac{1}{c}\,V(x) &= 2\,g(x), \end{aligned} \tag{2.18}$$

which implies that

$$\begin{aligned} f(x) &= \frac{1}{2}\,y_0(x) - \frac{1}{2c}\,V(x), \\ g(x) &= \frac{1}{2}\,y_0(x) + \frac{1}{2c}\,V(x). \end{aligned} \tag{2.19}$$

Substituting these into the formula for the general solution, we get

$$\begin{aligned} y(x,t) =& \frac{1}{2}\,y_0(x-ct) - \frac{1}{2c}\,V(x-ct) \\ &+ \frac{1}{2}\,y_0(x+ct) + \frac{1}{2c}\,V(x+ct) \end{aligned} \tag{2.20}$$

or

$$\begin{aligned} y(x,t) =& \frac{1}{2}\left[y_0(x-ct) + y_0(x+ct)\right] \\ &+ \frac{1}{2c}\left[V(x+ct) - V(x-ct)\right]. \end{aligned} \tag{2.21}$$

Note that only the difference $V(x+ct)-V(x-ct)$ appears, so the integration constant cancels and we can combine the two integrals into one because

$$\begin{aligned} V(x+ct)-V(x-ct) &= \int_0^{x+ct} v_0(s)\,ds - \int_0^{x-ct} v_0(s)\,ds \\ &= \int_{x-ct}^{x+ct} v_0(s)\,ds. \end{aligned} \tag{2.22}$$

Using this, we have

$$y(x,t) = \frac{1}{2}[y_0(x+ct)+y_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\,ds. \tag{2.23}$$

This is the solution formula for the initial-value problem (Eq. 2.13, Eq. 2.14) and it is called the **d'Alembert formula**.

Remark. Once we have the d'Alembert formula, we can consider solutions of the initial-value problem (Eq. 2.13, Eq. 2.14) corresponding to piecewise smooth (or even piecewise continuous) initial functions $y_0(x)$ and $v_0(x)$. This will result in *generalised solutions* of the wave equation which are defined everywhere in the upper half of the (x,t) plane except for a finite number of lines where values of $y(x,t)$ and/or its first derivatives are discontinuous.

3 Boundaries and Interfaces

You find content related to this lecture in the textbooks:

- Knobel (1999) chapter 9
- Baldock and Bridgeman (1983) section 2.1 and 2.5

We have seen that the solutions of the wave equation predict right- and left-moving waves that travel without changing their shapes. Eventually, in the real world at least, these waves are going to reach the end of the string. What will happen then? We know that the energy that is carried by the wave can not simply disappear. So we expect the wave to be reflected. But how is it reflected in detail?

3.1 Semi-infinite string with fixed end

Let us consider the case where a right-moving wave hits the right end of the string. We choose the x-coordinate so that the end is at $x = 0$. Thus we consider the wave equation on the left half-line $-\infty < x < 0$. In this section we consider the case where the end of the string is fixed, so we impose the boundary condition

$$y(0,t) = 0 \quad \text{for all } t \in \mathbf{R}. \tag{3.1}$$

This is known as a Dirichlet boundary condition.

We recall the general solution of the wave equation:

$$y(x,t) = f(x - ct) + g(x + ct). \tag{3.2}$$

As we know, the function f gives the shape of the right-moving wave. Imposing the boundary condition will tell us what g has to be, i.e., it will determine the shape of the left-moving reflected wave. Substituting the general solution into the boundary condition gives

$$y(0,t) = f(-ct) + g(ct) = 0. \tag{3.3}$$

This holds for any value of t, so

$$g(s) = -f(-s) \quad \text{for all } s \in \mathbf{R}. \tag{3.4}$$

This tells us that the reflected wave is the negative of the incoming wave and is flipped front-to-back. Thus the solution is

$$y(x,t) = f(x - ct) - f(-x - ct) \quad \text{for all } x \leq 0. \tag{3.5}$$

This is illustrated in Figure 3.1.

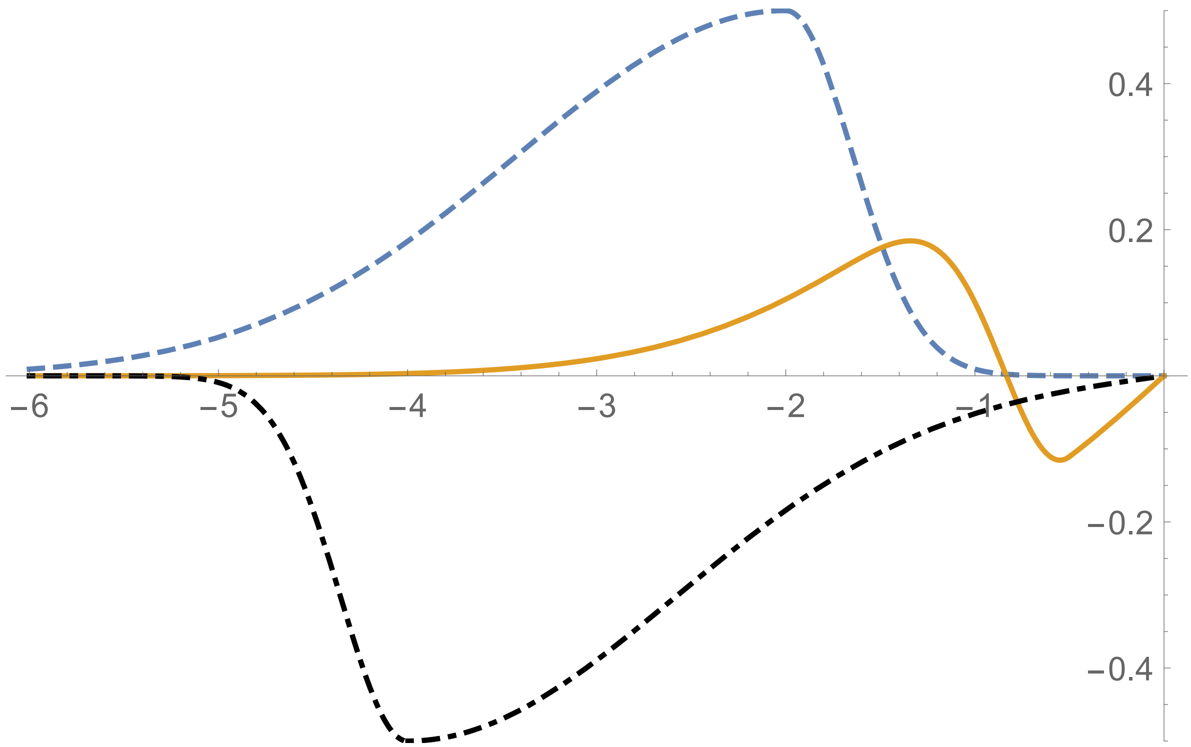

Figure 3.1: Reflection off a fixed end. Dashed line: incident right-moving wave. Solid line: wave interacting with the boundary. Dashdotted line: reflected left-moving wave. The reflected wave has the same shape as the incident wave but is flipped in both y and x.

3.2 Semi-infinite string with free end

Consider now a semi-infinite string ($0 < x < \infty$) with a free end at $x = 0$ (e.g. the end of the string can be attached to a small ring, which in turn can slide along a vertical rod without friction). This means that the vertical component of the tension force applied to the end of the string must be zero, which in turn means the string must be horizontal at x, i.e., we have the boundary condition

$$\partial_x y(0,t) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Conditions which specify the value of the normal derivative of the unknown function at the boundary are called *Neumann conditions*. So, here we have the homogeneous Neumann condition at $x = 0$. We now substitute the general solution. First we calculate its derivative

$$\partial_x y(x,t) = f'(x - ct) + g'(x + ct) \tag{3.7}$$

and thus the boundary condition says that

$$\partial_x y(0,t) = f'(-ct) + g'(ct) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{3.8}$$

Integrating this gives

$$\frac{-1}{c} f(-ct) + \frac{1}{c} g(ct) = \text{constant}. \tag{3.9}$$

Changing the constant only moves the string up or down on the y axis. We choose it to be zero. Because the boundary condition holds for all times, we have that

$$g(s) = f(-s) \quad \text{for all } s \in \mathbb{R}. \tag{3.10}$$

Thus the reflected wave has the same shape and the same sign as the incoming wave, but it is still flipped front-to-back.Thus the solution is

$$y(x,t) = f(x - ct) + f(-x - ct) \quad \text{for all } x \leq 0. \tag{3.11}$$

This is illustrated in Figure 3.2.

3.3 Reflection at a change of density

Consider two semi-infinite strings joined at the origin. The string on the left ($x < 0$) has constant density ρ_1 and the string on the right ($x > 0$) has constant density ρ_2. Let y_1 and y_2 be the displacements of the two strings. Since the strings have different densities, the wave speed in the two strings will be different:

$$c_1 = \sqrt{\frac{T}{\rho_1}} \quad \text{and} \quad c_2 = \sqrt{\frac{T}{\rho_2}}. \tag{3.12}$$

Suppose that we have a wave travelling to the right on the first string (an incident wave). When the wave meets the change in density, it will be partially reflected (back to the region $x < 0$) and partially transmitted (forward to the region $x > 0$). Waves travelling in the interval $x \in (-\infty, 0)$ are described by the wave equation with wave speed c_1; waves travelling in the interval $x \in (0, \infty)$ are described by the wave equation with wave speed c_2. This is illustrated in Figure Figure 3.3.

Therefore, we can write

$$y(x,t) = \begin{cases} y_1(x,t) = f_I(x - c_1 t) + f_R(x + c_1 t), & x < 0 \\ y_2(x,t) = f_T(x - c_2 t), & x > 0 \end{cases} \tag{3.13}$$

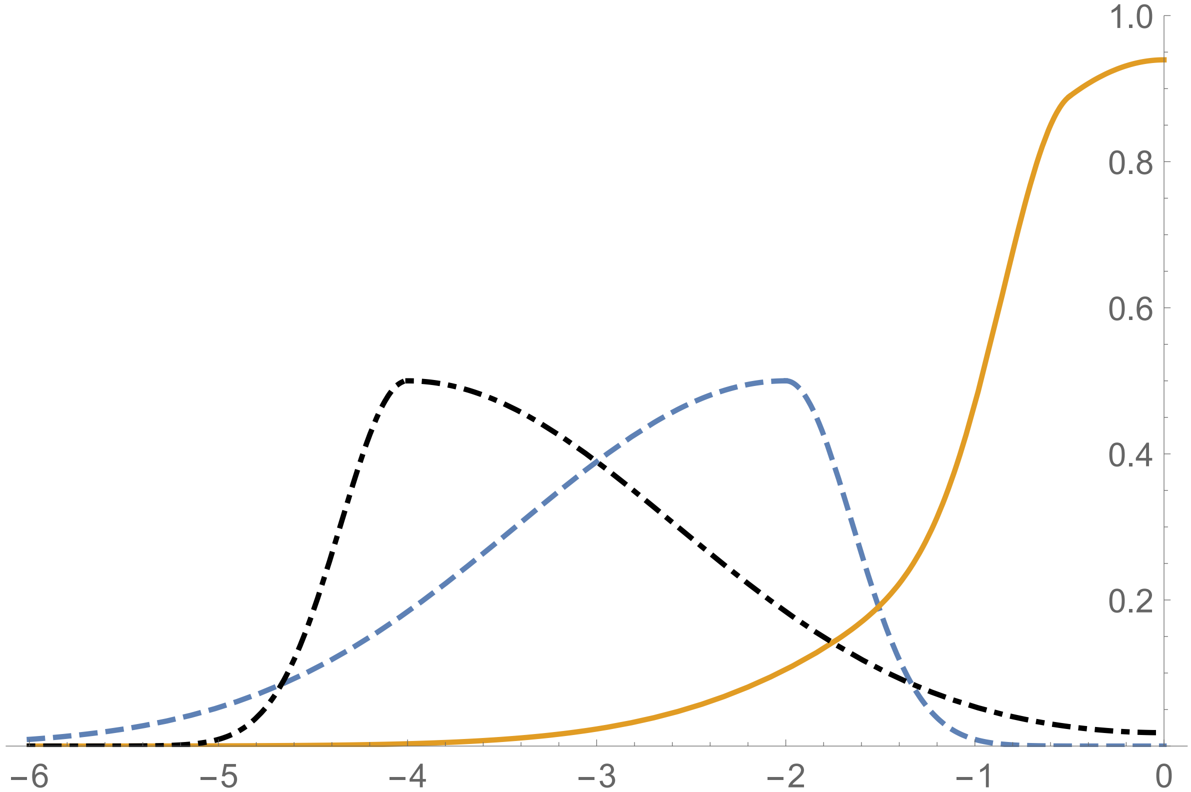

Figure 3.2: Reflection off a free end. Dashed line: incident right-moving wave. Solid line: wave interacting with the boundary. Dashdotted line: reflected left-moving wave. The reflected wave has the same shape as the incident wave but is flipped in x.

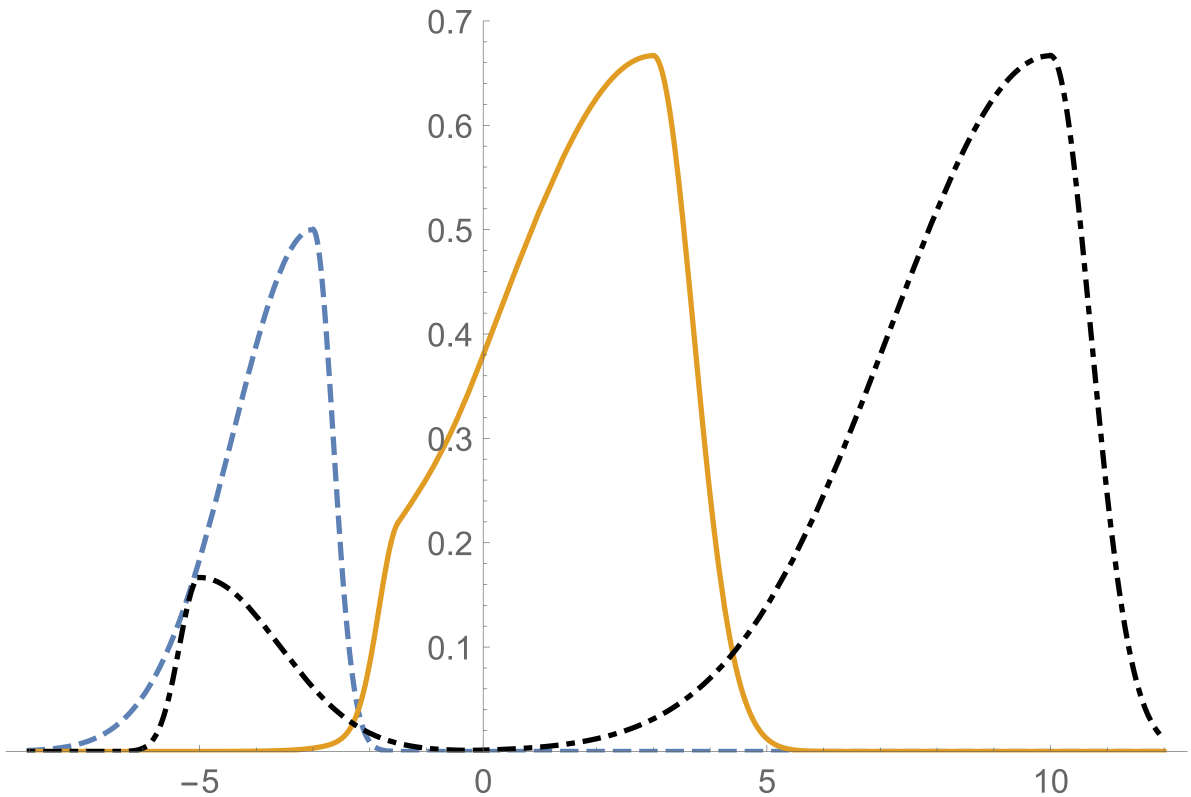

Figure 3.3: A right-moving wave being partially reflected and partially transmitted at the interface between two strings with wave velocities $c_1 = 1$ (on the left half-line) and $c_2 = 2$ (on the right half-line). Dashed line: incident right-moving wave. Solid line: wave interacting with the interface. Dashdotted line: partially reflected left-moving and partially transmitted right-moving wave.

where f_I, f_R and f_T represent the incident, reflected and transmitted waves, respectively. At the point of contact of the two strings ($x = 0$), we impose the following two conditions:

$$y_1(0,t) = y_2(0,t) \quad \text{for all } t, \tag{3.14}$$

$$\partial_x y_1(0,t) = \partial_x y_2(0,t) \quad \text{for all } t. \tag{3.15}$$

Condition 3.14 says that the solution for the combined string should be continuous at $x = 0$ (because the strings are attached to each other at the point). Condition 3.15 states that the slopes of the strings at $x = 0$ should be the same (if this is not so, there will be a finite force applied to an infinitesimal part of the combined string at $x = 0$, producing unphysical infinite acceleration).

Suppose that the incident wave is given, i.e. the function of one variable f_I is known. Can we find f_R and f_T?

Substitution of Eq. 3.13 into condition Eq. 3.14 yields

$$f_I(-c_1 t) + f_R(c_1 t) = f_T(-c_2 t) \quad \text{for all } t \tag{3.16}$$

or equivalently (writing $s = c_1 t$)

$$f_I(-s) + f_R(s) = f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.17}$$

Similarly, on substituting Eq. 3.13 into condition Eq. 3.15, we find that

$$f_I'(-c_1 t) + f_R'(c_1 t) = f_T'(-c_2 t) \quad \text{for all } t \tag{3.18}$$

or

$$f_I'(-s) + f_R'(s) = f_T'\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.19}$$

Integrating this equation in s, we get

$$-f_I(-s) + f_R(s) = -\frac{c_1}{c_2}\, f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.20}$$

Eliminating $f_R(s)$ from Eq. 3.17 and Eq. 3.20, we obtain

$$2 f_I(-s) = \left(1 + \frac{c_1}{c_2}\right) f_T\left(-\frac{c_2}{c_1}\, s\right) \quad \text{for all } s. \tag{3.21}$$

or, equivalently,

$$f_T(\tilde{s}) = \frac{2c_2}{c_2 + c_1}\, f_I\left(\frac{c_1}{c_2}\, \tilde{s}\right) \quad \text{for all } \tilde{s} \tag{3.22}$$

where $\tilde{s} = -(c_2/c_1)s$. Eq. 3.22 defines the function (of one variable) f_T in terms of known function f_I.

To find f_R, we substitute Eq. 3.22 into Eq. 3.17. This yields the formula

$$f_R(s) = \frac{c_2 - c_1}{c_2 + c_1} f_I(-s) \quad \text{for all } s. \tag{3.23}$$

Finally, on substituting Eq. 3.22 and Eq. 3.23 into Eq. 3.13, we get the solution formula

$$y(x,t) = \begin{cases} f_I(x - c_1 t) + A_R f_I(-x - c_1 t), & x < 0 \\ A_T f_I\left(\frac{c_1}{c_2}(x - c_2 t)\right), & x > 0 \end{cases} \tag{3.24}$$

where A_R (the ratio of the amplitude of the reflected wave to that of the incident wave) and A_T (the ratio of the amplitude of the transmitted wave to that of the incident wave) are given by

$$A_R = \frac{c_2 - c_1}{c_2 + c_1}, \quad A_T = \frac{2c_2}{c_2 + c_1}. \tag{3.25}$$

To check whether our answer makes sense, it is useful to consider a few limit cases.

i. If $\rho_1 = \rho_2$, then $c_1 = c_2$, and we have $A_R = 0$ and $A_T = 1$ (as one would expect for an infinite homogeneous string).

ii. If $\rho_1 \ll \rho_2$ (the first string is much lighter than the second one), then $c_1 \gg c_2$, and we have $A_R \approx -1$ and $A_T = 0$ (so that the heavy string effectively arrests the displacement at $x = 0$ as if the right end of the first string was fixed).

iii. If $\rho_1 \gg \rho_2$ (the first string is much heavier than the second one), then $c_1 \ll c_2$, and we have $A_R \approx 1$ and $A_T = 2$ (so that the light second string does not effect the displacement at $x = 0$ as if the right end of the first string was free).

4 Bernoulli's solution

You find content related to this lecture in the textbooks:

- Knobel (1999) sections 11.2 and 11.3, chapters 13 and 14
- Baldock and Bridgeman (1983) sections 3.1, 3.2, 3.3
- Simmons (1972) sections 39 and 40

We will now solve the wave equation again in a totally different way from d'Alembert, following instead Bernoulli. This is a method you have seen before, namely the method of separation of variables.

4.1 Separation of variables

We make the Ansatz that the solution factorises into one function of only x and one function of only t:

$$y(x,t) = X(x)T(t). \tag{4.1}$$

Substituting this into the wave equation

$$\partial_x^2 y(x,t) = \frac{1}{c^2}\partial_t^2 y(x,t) \tag{4.2}$$

we get

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t). \tag{4.3}$$

We divide both sides by $X(x)T(t)$, which will separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}. \tag{4.4}$$

It the last equation, we have a function of x only on the left hand side and a function of t only on the right hand side. The equation must hold for all x and t. This is only possible if both functions are equal to a constant. It will turn out to be convenient to write this constant as $-k^2$ for some new constant k. Thus we set

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = -k^2. \tag{4.5}$$

This means that $X(x)$ and $T(t)$ must be solutions of the ODEs

$$X'' = -k^2 X \quad \text{and} \quad T'' = -k^2 c^2 T. \tag{4.6}$$

The general solutions of these ODEs are

$$\begin{aligned} X(x) &= A \sin(kx) + B \cos(kx), \\ T(t) &= F \sin(kct) + G \cos(kct), \end{aligned} \tag{4.7}$$

where A, B, F, G and k are arbitrary constants.

4.2 Finite string with fixed ends

Bernoulli was most interested in the finite string. Let us put the ends of the string at $x = 0$ and $x = \pi$. Now let the ends of the string be fixed. This means that we need to impose homogeneous Dirichlet boundary conditions

$$y(0,t) = 0 \quad \text{and} \quad y(\pi,t) = 0 \quad \text{for all} \quad t. \tag{4.8}$$

These conditions will be satisfied if $X(0) = 0$ and $X(\pi) = 0$. Imposing the condition $X(0) = 0$ on the general solution for $X(x)$, we find that we need $B = 0$. We then have from $X(\pi) = 0$ that

$$A \sin(k\pi) = 0. \tag{4.9}$$

The last equation implies that either $A = 0$ or $\sin(k\pi) = 0$. We reject the first option because it results in a zero solution. The second option yields

$$\sin(k\pi) = 0 \quad \Rightarrow k \in \mathbb{Z}. \tag{4.10}$$

Thus, for each integer k we have a solution of the wave equation of the form

$$y_k(x,t) = \sin(kx)\left(F_k \sin(kct) + G_k \cos(kct)\right) \tag{4.11}$$

4.3 Standing waves and superposition

The solutions we obtained in the previous section are standing waves. They don't change their shape and don't move, only their amplitude oscillates with time. In the lecture videos I make various sketches and animations to illustrate this.

So we now have a stark contrast between what d'Alembert found and what Bernoulli found. According to d'Alembert, the solutions of the wave equation are travelling waves of arbitrary shape. According to Bernoulli, the solutions are standing waves that have sine shape. It is always wonderful when one has two very different ways of looking at the same phenomenon. That is where deep understanding comes from.

To resolve the apparent conflict between d'Alembert's solution and Bernoulli's solution we have to use the **superposition principle**: For any set of linear homogeneous equations, any sum of solutions is also a solution.

Since the wave equation is linear and also the boundary conditions we imposed were linear, a linear combination of any number of harmonic standing waves is also a solution. So, we can construct a general solution of the wave equation by summing up all [1] the harmonic standing waves from Eq. 4.11:

$$y(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left(F_k \sin(kct) + G_k \cos(kct)\right). \tag{4.12}$$

Here the G_k and the F_k are undetermined constants, to be fixed from the initial conditions.

The resolution of the apparent paradox is that a superposition of standing waves can give a travelling wave, and that a superposition of sine waves can give any shape.

4.4 Initial value problem

In Section 2.4 we imposed initial conditions on d'Alembert's general solution and obtained d'Alembert's formula in Eq. 2.23. We now similarly impose initial conditions on Bernoulli's solution. Suppose now that we are given the initial conditions

$$y(x,0) = y_0(x), \quad \partial_t y(x,0) = v_0(x) \quad \text{for } x \in [0,\pi]. \tag{4.13}$$

We want to use these initial conditions to determine the unknown coefficients F_k and G_k for all $k \in \mathbb{N}$. As always, the procedure is to substitute the general solution into the initial conditions. When we substitute Eq. 4.12 into Eq. 4.13 we get

$$y_0(x) = \sum_{k=1}^{\infty} G_k \sin(kx), \quad v_0(x) = \sum_{k=1}^{\infty} F_k\, k\, c \sin(kx). \tag{4.14}$$

These formulae look like Fourier series for the functions y_0 and v_0. So, to find the coefficients F_k and G_k we have to perform the inverse Fourier transform. This uses the identity

$$\int_0^{\pi} \sin(kx)\sin(lx)dx = \frac{\pi}{2}\delta_{kl},$$

[1] We do not need negative k because they will produce the same solutions, and we do not need $k=0$ because it yields zero solution. Also, we have absorbed the constant A into the constants F_k and G_k.

where δ_{kl} is the Kronecker delta:

$$\delta_{kl} = \begin{cases} 1 & \text{if} \quad k = l \\ 0 & \text{if} \quad k \neq l \end{cases}$$

We multiply each of Eq. 4.14 by $2/\pi \ \sin(lx)$ and integrate over the interval $[0, \pi]$:

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi} y_0(x)\sin(lx)dx &= \frac{2}{\pi}\sum_{k=1}^{\infty} G_k \int_0^{\pi} \sin(kx)\sin(lx)dx \\ &= \sum_{k=1}^{\infty} G_k \delta_{kl} = G_l \end{aligned} \tag{4.15}$$

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi} v_0(x)\sin(lx)dx &= \frac{2}{\pi}\sum_{k=1}^{\infty} F_k\, kc \int_0^{\pi} \sin(kx)\sin(lx)dx \\ &= \sum_{k=1}^{\infty} F_k\, kc\, \delta_{kl} = F_l\, lc \end{aligned} \tag{4.16}$$

Thus we know how to determine the constants F_k and G_k in the general solution Eq. 4.12 so as to satisfy given initial conditions.

The method of separation of variables that we have used in this lecture to solve the wave equation is a powerful technique that can be used to solve a wide range of partial differential equations beyond the wave equation. The idea behind this method is to assume that the solution of a partial differential equation can be expressed as a product of two functions, one depending only on the spatial variables and the other only on the temporal variables. This reduces the partial differential equation to a set of ordinary differential equations, which can be solved independently. You will encounter this method repeatedly, in fluid mechanics, quantum mechanics, mathematical ecology and epidemiology and many other fields.

5 Harmonic waves

You can find related material in the textbooks:

- Coulson and Jeffrey (1977) section 3
- Baldock and Bridgeman (1983) and Bridgeman section 1.7

5.1 Harmonic waves

Harmonic waves are waves that are described by sines and cosines. A travelling harmonic wave can be written as

$$y(x,t) = a\cos\left(kx - \omega t + \phi\right) \tag{5.1}$$

or

$$y(x,t) = a\cos\left(2\pi(\hat{k}x - \nu t) + \phi\right) \tag{5.2}$$

where a is the amplitude, $\hat{k}$ the wave number, ν the frequency, ϕ the phase, and where

$$k = 2\pi\hat{k} \quad \text{and} \quad \omega = 2\pi\nu \tag{5.3}$$

are the angular wave number and the angular frequency, respectively. Note that k and ω are used more often by mathematicians than $\hat{k}$ and ν and that the prefix 'angular' is often discarded.

Terminology

- *Frequency* = number of cycles (oscillations) per unit time.
- *Wave number* = number of cycles (oscillations) per unit length.
- *Period* = time P needed to complete one cycle (oscillation):

$$P = \frac{1}{\nu} = \frac{2\pi}{\omega}. \tag{5.4}$$

- *Wave length* = distance between two consecutive wave crests (peaks):

$$\lambda = \frac{1}{\hat{k}} = \frac{2\pi}{k}. \tag{5.5}$$

- *Wave speed* = speed at which the wave is travelling:

$$c = \frac{\lambda}{P} = \frac{\nu}{\tilde{k}} = \frac{\omega}{k}. \tag{5.6}$$

(Sometimes, the wave speed is also called the *phase speed*.)

The harmonic wave Eq. 5.1 can be written as

$$\begin{aligned} y(x,t) &= a\cos\left(kx - \omega t + \phi\right) \\ &= \mathrm{Re}\left(a e^{i\phi} e^{i(kx-\omega t)}\right) \\ &= \mathrm{Re}\left(A e^{i(kx-\omega t)}\right), \end{aligned} \tag{5.7}$$

where we have included the phase factor $e^{i\phi}$ into the complex amplitude: $A = ae^{i\phi}$.

Consider now the complex function

$$y(x,t) = A\, e^{i(kx-\omega(k)t)} \tag{5.8}$$

for any $A \in \mathbb{C}$, any $k \in \mathbb{R}$ and some $\omega(k)$. Substituting this into the wave equation gives us an equation for $\omega(k)$:

$$\partial_t^2 y = c^2 \partial_x^2 y \quad \Rightarrow \quad -\omega^2 A = -c^2 k^2 A. \tag{5.9}$$

Therefore, if

$$\omega(k) = \pm ck \tag{5.10}$$

then the complex function in Eq. 5.8 is a solution of the wave equation. We will refer to these complex solutions as *complex harmonic waves*. They are often more convenient to work with than their real counterpart in Eq. 5.1.

Eq. 5.10 is an example of a *dispersion relation*. It states that for the wave equation, ω is proportional to k. But complex harmonic waves can also solve other PDEs, as we will see in the next subsection, and that will lead to more complicated dispersion relations.

5.2 Solving PDEs with harmonic waves

! Important

Any linear homogeneous PDE (in variables x and t) with constant coefficients has complex harmonic wave solutions Eq. 5.8 for some $\omega(k)$.

Example 5.1. Consider the damped string (with friction force proportional to velocity):

$$\partial_t^2 y = c^2 \partial_x^2 y - p\,\partial_t y \quad \text{where} \quad p > 0. \tag{5.11}$$

Substituting the complex harmonic wave from Eq. 5.8 into this equation, we obtain the dispersion relation

$$-\omega^2 y = -c^2 k^2 y + ip\,\omega y. \tag{5.12}$$

Cancelling the y we obtain a quadratic equation for ω^2 which has the complex solution

$$\omega = -\frac{ip}{2} \pm \sqrt{c^2 k^2 - \frac{p^2}{4}}. \tag{5.13}$$

Thus we have the following solution for the damped string:

$$\begin{aligned} y(x,t) &= A\, e^{i\left(kx + \frac{ip}{2} t \pm \sqrt{c^2k^2 - \frac{p^2}{4}}\,t\right)} \\ &= A\, e^{-\frac{pt}{2}} e^{ik\left(x \pm \sqrt{c^2 - \frac{p^2}{4k^2}}\,t\right)} \end{aligned} \tag{5.14}$$

The factor $e^{-pt/2}$ shows that we have a wave with exponentially decreasing amplitude. This is a consequence of the damping. The wave speed is now dependent on the wave number k:

$$c(k) = \sqrt{c^2 - p^2/(4k^2)}. \tag{5.15}$$

If we want to, we can get a real solution by taking the real part of the complex solution:

$$\mathrm{Re}(y(x,t)) = a\, e^{-pt/2} \cos\left[k\left(x \pm t\,\sqrt{c^2 - p^2/(4k^2)}\right) + \phi\right] \tag{5.16}$$

where $a = |A|$ and $\phi = \arg(A)$.

Note that the imaginary part of ω produces the damping exponential and the real part of ω determines the wave speed.

6 Energy

Besides Information, waves transmit another practically important quantity: Energy. Note that waves do not transport matter. Matter may oscillate up and down or forth and back as a wave passes, but it is not swept away with the wave. But energy is. In this lecture we are going to first introduce the expression for the energy in a wave on a string as an integral over the energy density. The energy density in turn is made up out of kinetic and potential energy density. We will then calculate the energy in a few example waves, and then discuss the conservation of energy.

You can find related material in the textbooks:

- Baldock and Bridgeman (1983) section 2.2
- Coulson and Jeffrey (1977) sections 18, 25, 30
- Knobel (1999) chapter 15

6.1 Energy density

Consider an infinitesimal bit of string between x and $x + \delta x$. Its kinetic energy is

$$\delta K = \frac{1}{2} m\, v^2 = \frac{1}{2} \rho\, \delta x\, \left(\partial_t y\right)^2 . \tag{6.1}$$

The kinetic energy of the entire string is then obtained by integrating over its infinitesimal parts:

$$K = \int \frac{\rho}{2} (\partial_t y)^2\, dx = \int \mathcal{E}_K\, dx. \tag{6.2}$$

The quantity $\mathcal{E}_K$ is the *kinetic energy density.*

To derive the formula for the potential energy, we again look first at an infinitesimal segment of the string. It has been stretched from a length of δx to the longer length δs. The work done to change the length from δx to δs is $T(\delta s - \delta x)$. This gives the potential energy (we neglect the potential energy coming from gravity). We have

$$\delta s = \sqrt{1 + \left(\partial_x y\right)^2}\, \delta x \approx \delta x \left(1 + \frac{\left(\partial_x y\right)^2}{2} + \cdots\right), \tag{6.3}$$

where we have only kept the first two terms in the Taylor expansion because, as we did when we derived the wave equation, we assume that the slope of the string is small and thus the higher order terms in $\partial_x y$ are negligible. Thus the potential energy in the infinitesimal segment of the string is

$$\delta V = T\left(\delta s - \delta x\right) = \frac{T}{2}\left(\partial_x y\right)^2 \, \delta x. \tag{6.4}$$

Summing up contributions from all small elements of the string (i.e. integrating over the whole string), we find the potential energy

$$V = \int T\, \frac{(\partial_x y)^2}{2}\, dx = \int \mathcal{E}_V \, dx. \tag{6.5}$$

The quantity $\mathcal{E}_V$ is the *potential energy density.*

The total energy E is the sum of the kinetic and potential energy:

$$\begin{aligned} E &= K + V \\ &= \int \left(\frac{\rho}{2}(\partial_t y)^2 + \frac{T}{2}(\partial_x y)^2\right) dx \\ &= \int \mathcal{E}\, dx, \end{aligned} \tag{6.6}$$

where $\mathcal{E} = \mathcal{E}_K + \mathcal{E}_V$ is the *total energy density.*

6.2 Energy density of example waves

Example 6.1. Consider a localised wave $y(x,t) = f(x - ct)$ travelling to the right with speed c. Substituting this into the general expression for the energy density

$$\mathcal{E} = \frac{\rho}{2}(\partial_t y)^2 + \frac{T}{2}(\partial_x y)^2 \tag{6.7}$$

gives

$$\mathcal{E}(x,t) = \frac{\rho}{2}(-c)^2(f'(x-ct))^2 + \frac{T}{2}(f'(x-ct))^2. \tag{6.8}$$

Because $c^2\rho = T$, we see that the kinetic and the potential energy densities are equal. This phenomenon is referred to as “equipartition” of the energy. Together we have

$$\mathcal{E}(x,t) = T\,(f'(x-ct))^2. \tag{6.9}$$

Note how the energy density is travelling along with the wave profile.

Example 6.2 (Standing harmonic wave)**.** Now we consider solutions of the form

$$\begin{aligned} y(x,t) &= \sin(kx)(F\sin(kct) + G\cos(kct)) \\ &= \alpha\cos(kct+\phi). \end{aligned} \tag{6.10}$$

We calculate the energy densities

$$\begin{aligned} \mathcal{E}_K &= \frac{\rho}{2}(\partial_t y(x,t))^2 \\ &= \frac{\rho}{2}\alpha^2 k^2 c^2(-\sin(kct+\phi))^2\sin^2(kx) \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} \mathcal{E}_V &= \frac{T}{2}(\partial_x y(x,t))^2 \\ &= \frac{T}{2}\alpha^2 k^2(\cos(kct+\phi))^2\cos^2(kx) \end{aligned} \tag{6.12}$$

Again we notice that the prefactors are the same because $c^2\rho = T$. For the energies we find

$$\begin{aligned} K &= \frac{T}{2}\alpha^2 k^2\sin^2(kct+\phi)\int_0^\pi \sin^2(kx)dx \\ &= \frac{T\alpha^2 k^2\pi}{4}\sin^2(kct+\phi) \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} T &= \frac{T}{2}\alpha^2 k^2\cos^2(kct+\phi)\int_0^\pi \cos^2(kx)dx \\ &= \frac{T\alpha^2 k^2\pi}{4}\cos^2(kct+\phi). \end{aligned} \tag{6.14}$$

For standing waves, both the kinetic energy and the potential energy depend on time and are not equal. However, their averages, averaged over a period in t, are equal. The total energy is constant

$$E = K + T = \frac{T\alpha^2 k^2\pi}{4}(\sin^2(kct+\phi) + \cos^2(kct+\phi)) = \frac{T\alpha^2 k^2\pi}{4} \tag{6.15}$$

Example 6.3 (Sum of two standing harmonic waves)**.** Consider two harmonic waves

$$\begin{aligned} y_k &= \alpha_k\sin(kx)\cos(kct+\phi_k) \quad \text{and} \\ y_l &= \alpha_l\sin(lx)\cos(lct+\phi_l) \end{aligned} \tag{6.16}$$

with $k \neq l$ and let us calculate the energy of $y = y_k + y_l$. We have

$$
\begin{aligned}
K &= \frac{\rho}{2} \int_0^\pi (\partial_t y)^2 dx = \frac{\rho}{2} \int_0^\pi (\partial_t y_k + \partial_t y_l)^2 dx \\
&= K_k + K_l + \rho\, \alpha_k\, \alpha_l\, k\, l\, c^2 \cos(kct + \phi_k) \cos(lct + \phi_l) \cdot \\
&\qquad\qquad \int_0^\pi \sin(kx) \sin(lx) dx \\
&= K_k + K_l
\end{aligned}
\tag{6.17}
$$

where K_k and K_l are the kinetic energies of the individual harmonic waves. A similar calculation shows that also the potential energy of the sum is the sum of the potential energies and so this is also true of the total energy:

$$
E[y_k + y_l] = E[y_k] + E[y_l]. \tag{6.18}
$$

This is one of the nice properties of harmonic waves.

Example 6.4 (Complex exponential wave)**.** We calculate the energy density of the complex solution

$$
y(x,t) = A\, e^{i(kx - \omega t)}. \tag{6.19}
$$

The expression for the energy density of complex solutions involves the absolute value squared:

$$
\mathcal{E}[y] = \frac{\rho}{2} |\partial_t y|^2 + \frac{T}{2} |\partial_x y|^2. \tag{6.20}
$$

This has the effect that the energy density is the sum of the energy density of the real part of the solution and the energy density of the imaginary part of the solution. We find

$$
\begin{aligned}
\mathcal{E}[A\, e^{i(kx-\omega t)}] &= \frac{\rho}{2} \left| -i\omega\, A\, e^{i(kx-\omega t)} \right|^2 + \frac{T}{2} \left| ik\, A\, e^{i(kx-\omega t)} \right|^2 \\
&= \left(\frac{\rho}{2} \omega^2 + \frac{T}{2} k^2 \right) |A|^2
\end{aligned}
\tag{6.21}
$$

So another miracle of these complex exponential solutions is that their energy density is constant.

6.3 Conservation equation

Let $y(x,t)$ be a solution of the wave equation for the string and $\mathcal{E}$ its energy density

$$
\mathcal{E} = \frac{\rho}{2} (\partial_t y)^2 + \frac{T}{2} (\partial_x y)^2. \tag{6.22}
$$

For the time derivative of the energy density we find

$$\begin{aligned}\partial_t \mathcal{E} &= \rho\, \partial_t y\, \partial_t^2 y + T\, \partial_x y\, \partial_t \partial_x y \\ &= \partial_t y\, T\, \partial_x^2 y + T\, \partial_x y\, \partial_t \partial_x y \quad (\text{using} \quad \rho\, \partial_t^2 y = T\, \partial_x^2 y\ ) \\ &= -\partial_x \left(-T\, \partial_t y\, \partial_x y\right).\end{aligned} \tag{6.23}$$

In terms of the quantity $\mathcal{F} = -T\, \partial_t y\, \partial_x y$ this equation takes the form

$$\partial_t \mathcal{E} = -\partial_x \mathcal{F}. \tag{6.24}$$

The quantity $\mathcal{F}$ is called the *energy flux*.

Eq. 6.24 implies the law of conservation of energy (and therefore is called a *conservation equation*}). Indeed, we have

$$\frac{dE}{dt} = \int_{x_1}^{x_2} \partial_t \mathcal{E}\, dx = -\int_{x_1}^{x_2} \partial_x \mathcal{F}\, dx = \mathcal{F}(x_1) - \mathcal{F}(x_2). \tag{6.25}$$

So if we interpret the energy flux $\mathcal{F}(x)$ as the rate at which energy flows through a point x from left to right, then the conservation equation expresses that the rate at which energy in a region changes is equal to the difference between the rate at which energy flows in and the rate at which energy flows out of the region.

We already showed conservation of energy of a finite string with fixed boundary conditions in the previous section. We will now use the above machinery to show the conservation of the energy of a finite string with free boundary conditions. We consider a string between $x = 0$ and $x = \pi$ satisfying the free boundary conditions

$$\partial_x y(0,t) = 0 = \partial_x y(\pi, t) \text{ for all } t. \tag{6.26}$$

The energy flux at the left end of the string at $x = 0$ is

$$\mathcal{F}(0,t) = -T\, \partial_t y(0,t)\, \partial_x y(0,t). \tag{6.27}$$

This flux is zero due to the boundary condition. Similarly the flux at the right boundary is zero:

$$\mathcal{F}(\pi,t) = -T\, \partial_t y(\pi,t)\, \partial_x y(\pi,t) = 0. \tag{6.28}$$

So by the conservation equation it follows that the energy is conserved.

7 Two-dimensional waves

So far we have been discussing the one-dimensional wave equation. We have motivated the equation in terms of a string stretched along the x dimension and oscillating in the y dimension. We now want to extend this to higher dimensions.

One way to introduce another dimension would be to let the string vibrate in both the y and z dimension. However that would not give anything new. In the small-angle approximation to which we have been working, the oscillations in the y direction and the oscillations in the z direction would be independent and we would just end up with two independent one-dimensional wave equations,

$$\partial_t^2 y(x,t) = c^2 \partial_x^2 y(x,t) \quad \text{and} \quad \partial_t^2 z(x,t) = c^2 \partial_x^2 z(x,t).$$

Instead we will introduce the second dimension by going from the string stretched in the x dimension to a membrane stretched in both the x and y dimensions. It will vibrate in the z direction. This toy example of a vibrating membrane will give us the two-dimensional wave equation, which of course appears in many other applications as well.

You find content related to this lecture in the textbooks:

- Baldock and Bridgeman (1983) section 7.3
- Coulson and Jeffrey (1977) sections 32 and start of 33

7.1 Two-dimensional wave equation

Consider an infinite two-dimensional membrane of homogeneous density ρ (mass per unit area, measured in kg/m^2). In the equilibrium state, it is flat and coincides with the (x, y) plane in $\mathbb{R}^3$. We assume that it is stretched to a tension T (force per unit length, measured in N/m). This means that for any line on the surface of the membrane, the part of the membrane on one side of the line exerts a force T (per unit length of the line) on the other part of the membrane (on the other side of the line), and the direction of the force is perpendicular to the line.

The perturbed membrane may be described as a time-dependent surface $z = z(x, y, t)$ in $\mathbb{R}^3$, where $z(x, y, t)$ is the vertical (in the z direction) displacement of the membrane at point (x, y) and time t. To derive the equation of motion, we consider a small element of the membrane of size δx and δy in x and y, as shown in Figure 7.1.

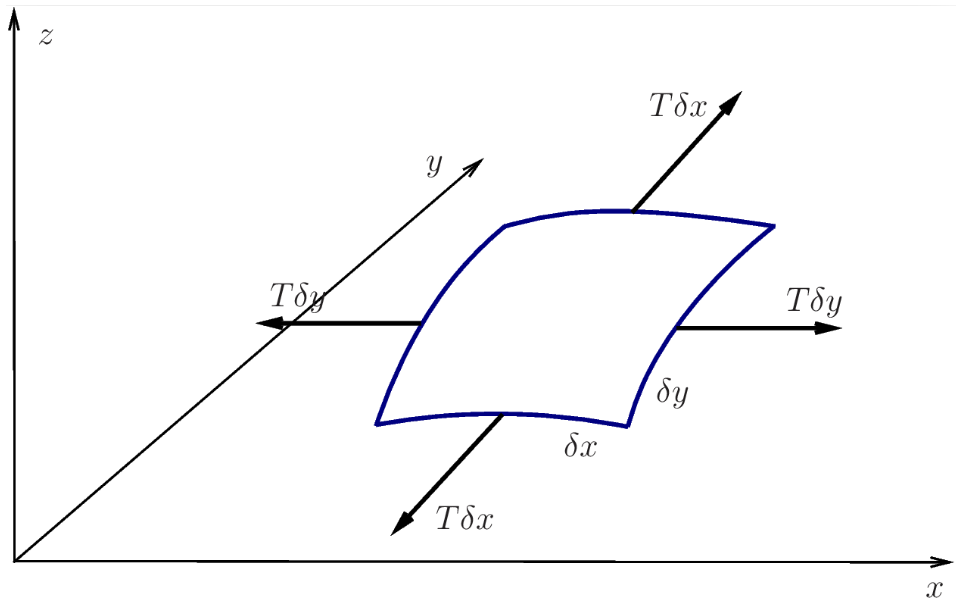

Figure 7.1: A membrane stretched in the x–y plane and vibrating in the z direction.

We assume that there is only transverse motion of the membrane and that the partial derivatives $\partial_x z$ and $\partial_y z$ are small: $|\partial_x z| \ll 1$ and $|\partial_y z| \ll 1$. Almost the same arguments as those in Section 1.2 lead to the conclusion that the vertical component of the force is a sum of two terms: $T\delta y\, \partial_x^2 z\, \delta x$ (coming from the tension forces that are nearly parallel to the x-axis in Figure 7.1) and $T\delta x\, \partial_y^2 z\, \delta y$ (coming from the tension forces that are nearly parallel to the y-axis in Figure 7.1). So, Newton's equation of motion ($ma = F$) yields

$$(\rho\, \delta x\, \delta y)\partial_t^2 z = T\delta y\, \partial_x^2 z\, \delta x + T\delta x\, \partial_y^2 z\, \delta y. \tag{7.1}$$

Dividing this by $\delta x\, \delta y$ and ρ, we get the governing equation for the membrane:

$$\partial_t^2 z = \frac{T}{\rho}\left(\partial_x^2 z + \partial_y^2 z\right) \tag{7.2}$$

or, equivalently,

$$\partial_t^2 z - c^2 \nabla^2 z = 0, \quad c = \sqrt{\frac{T}{\rho}}. \tag{7.3}$$

Here c in the wave speed and ∇^2 is the Laplace operator:

$$\nabla^2 = \partial_x^2 + \partial_y^2. \tag{7.4}$$

Sometimes, the Laplace operator is also denoted by Δ (i.e. $\Delta = \nabla^2 = \partial_x^2 + \partial_y^2$). Eq. 7.3 is the two-dimensional wave equation. Written in terms of the Laplace operator it easily generalises to any higher dimension.

7.2 Energy of membrane

The energy density of the membrane is

$$\mathcal{E}(x,y,t) = \frac{\rho}{2}(\partial_t z)^2 + \frac{T}{2}\left((\partial_x z)^2 + (\partial_y z)^2\right), \tag{7.5}$$

where the term involving the density ρ is the kinetic energy density $\mathcal{E}_K$ and the term involving the tension T is the potential energy density $\mathcal{E}_V$. The latter we could write in vector notation as $\mathcal{E}_v = T|\underline{\nabla} z|^2/2$ where $\underline{\nabla} z$ is the gradient of z.

To check that Eq. 7.5 is a good expression for the energy density, we check that $\mathcal{E}$ satisfies a conservation equation. So we calculate

$$\begin{aligned} \partial_t \mathcal{E} &= \rho\,(\partial_t z)(\partial_t^2 z) + T\,(\partial_t\partial_x z)(\partial_x z) + T\,(\partial_t\partial_y z)(\partial_y z) \\ &= T\,(\partial_t z)(\partial_x^2 z + \partial_y^2 z) + T\,\left((\partial_t\partial_x z)(\partial_x z) + (\partial_t\partial_y z)(\partial_y z)\right) \\ &= -\partial_x\left(-T(\partial_x z)(\partial_t z)\right) - \partial_y\left(-T(\partial_x z)(\partial_t z)\right), \end{aligned} \tag{7.6}$$

where for the second equality we have used the wave equation. We introduce the two-dimensional energy flux density vector $\underline{\mathcal{F}} = (\mathcal{F}_x, \mathcal{F}_y)$ with

$$\mathcal{F}_x = -T(\partial_x z)(\partial_t z), \qquad \mathcal{F}_y = -T(\partial_y z)(\partial_t z). \tag{7.7}$$

In terms of this we have derived the conservation equation

$$\partial_t \mathcal{E} = -\underline{\nabla}\cdot\underline{\mathcal{F}}. \tag{7.8}$$

To understand how this two-dimensional conservation equation leads to energy conservation let us look at the energy in a region R in the (x,y) plane. The energy in this region is given by

$$E = \iint_R \mathcal{E}\,dA. \tag{7.9}$$

Here $dA = dx\,dy$ is the area element. The rate of change in the energy is then

$$\begin{aligned} \frac{dE}{dt} &= \iint_R \partial_t\mathcal{E} dA = -\iint_R \underline{\nabla}\cdot\underline{\mathcal{F}}\,dA \\ &= -\int_{\partial R} \mathcal{F}\cdot\underline{n}\,ds. \end{aligned} \tag{7.10}$$

For the last equality we used the divergence theorem. $\underline{n}$ is the outward unit normal to the boundary ∂R of the region R and ds is the line element on ∂R. So again we see that the change of energy in a region is equal to the net flow of energy into the region.

7.3 Travelling plane waves

A 2D wave is a plane wave if $z(x,y,t)$ varies only in one spatial direction, say, parallel to a constant unit vector $\underline{n} = (n_x, n_y)$ (and $z(x,y,t)$ is constant in the direction perpendicular to $\underline{n}$). This means that

$$z(x,y,t) = f(\underline{n} \cdot \underline{x} - ct) = f(n_x x + n_y y - ct) \tag{7.11}$$

describes a plane wave travelling with wave speed c in the direction of vector $\underline{n}$.

Let's verify that Eq. 7.11 is a solution of the 2D wave equation. We have

$$\begin{aligned}
\partial_t^2 z &= \partial_t^2 f(\underline{n} \cdot \underline{x} - ct) = f''(\underline{n} \cdot \underline{x} - ct)\, c^2, \\
\partial_x^2 z &= \partial_x^2 f(\underline{n} \cdot \underline{x} - ct) = f''(\underline{n} \cdot \underline{x} - ct)\, n_x^2, \\
\partial_y^2 z &= \partial_y^2 f(\underline{n} \cdot \underline{x} - ct) = f''(\underline{n} \cdot \underline{x} - ct)\, n_y^2.
\end{aligned} \tag{7.12}$$

Hence,

$$\begin{aligned}
&\partial_t^2 z - c^2 \left(\partial_x^2 z + \partial_y^2 z\right) \\
=&c^2 f''(\underline{n} \cdot \underline{x} - ct)\left(1 - n_x^2 - n_y^2\right) = 0
\end{aligned} \tag{7.13}$$

because $\underline{n}$ is a unit vector, i.e., $n_x^2 + n_y^2 = 1$.

Because the wave equation is linear, any superposition of plane wave solutions is also a solution.

The *harmonic plane wave* corresponds to the choice $f(s) = e^{iks}$, so that

$$z(x,y,t) = e^{ik(\underline{n}\cdot\underline{x} - ct)} = e^{i(\underline{k}\cdot\underline{x} - \omega(\underline{k})t)}, \tag{7.14}$$

where $\underline{k} = k\,\underline{n}$ is called the *wave vector*. We have the dispersion relation $\omega(\underline{k}) = c\,|\underline{k}|$.

7.4 Higher dimensions

By writing our equations in vector notation, we can see that they work in any dimension. So the n-dimensional wave equation for a real-valued function $z : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ or a complex-valued function $z : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is

$$\partial_t^2 z(\underline{x}, t) = c^2\, \nabla^2 z(\underline{x}, t). \tag{7.15}$$

Its energy density is

$$\mathcal{E} = \frac{T}{2}\left(\frac{1}{c^2}|\partial_t z|^2 + |\underline{\nabla} z|^2\right). \tag{7.16}$$

It satisfies the conservation equation

$$\partial_t \mathcal{E} = -\underline{\nabla} \cdot \underline{\mathcal{F}} \tag{7.17}$$

where the energy flux is

$$\underline{\mathcal{F}} = -T\, Re\,(\partial_t z\, \underline{\nabla} z) \tag{7.18}$$

According to the n-dimensional divergence theorem, the rate of change of the energy in an n-dimensional region $R \subset \mathbb{R}^n$ with $n-1$-dimensional boundary ∂R is

$$\begin{aligned} \frac{dE}{dt} &= \int_R \partial_t \mathcal{E}\, dV = -\int_R \underline{\nabla} \cdot \underline{\mathcal{F}}\, dV \\ &= \int_{\partial R} \underline{\mathcal{F}} \cdot \underline{n}\, dS. \end{aligned} \tag{7.19}$$

The wave equation has plane wave solutions

$$z(\underline{x}, t) = f(\underline{n} \cdot \underline{x} - ct) \tag{7.20}$$

for any choice of $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{C}$.

When you study Electromagnetism you will meet an even nicer way of writing these equations in terms of $n+1$-dimensional space-time vectors and tensors.

8 Waves on rectangular domain

You find content related to this lecture in the textbooks:

- Baldock and Bridgeman (1983) section 7.4
- Coulson and Jeffrey (1977) section 33

Consider a rectangular membrane: $D = \{(x, y) \in \mathbb{R}^2 | 0 < x < a,\ 0 < y < b\}$. Let's solve the wave equation

$$\partial_t^2 z - c^2 \left(\partial_x^2 z + \partial_y^2 z\right) = 0 \quad \text{in} \quad D \tag{8.1}$$

subject to the fixed (Dirichlet) boundary condition

$$z(x, y, t) = 0 \quad \text{on} \quad \partial D \tag{8.2}$$

or, equivalently,

$$z(0, y, t) = 0, \quad z(a, y, t) = 0 \tag{8.3}$$

and

$$z(x, 0, t) = 0, \quad z(x, b, t) = 0. \tag{8.4}$$

To find a solution, we use the method of separation of variables (see Section 4.1), i.e., we make the Ansatz

$$z(x, y, t) = X(x)Y(y)T(t). \tag{8.5}$$

Substituting this Ansatz into the wave equation, we get

$$XYT'' = c^2 \left(X''YT + XY''T\right), \tag{8.6}$$

which, after dividing by XYT gives

$$\frac{T''(t)}{T(t)} = c^2 \left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right). \tag{8.7}$$

In the last equation, we have a function of one variable, t, on the left side and a function of two different variables, x and y, on the right side. The equation can be satisfied for all x, y and t only if both sides are equal to a constant. As in Section 4.1, we choose this constant to be negative and (for convenience) equal to $-k^2c^2$ (for arbitrary real k). Again, the possibility of a positive constant is excluded, because with a positive constant it is impossible to find solutions satisfying the boundary conditions. Thus, we have

$$\frac{T''(t)}{T(t)} = c^2 \left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right) = -k^2c^2. \tag{8.8}$$

This leads to the ODE

$$T''(t) = -c^2 k^2\, T(t) \tag{8.9}$$

and to the equation

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -k^2. \tag{8.10}$$

The general solution of Eq. 8.9 is

$$T(t) = F\,\sin(kct) + G\,\cos(kct), \tag{8.11}$$

where F and G are arbitrary constants.

Rewriting Eq. 8.10 as

$$\frac{X''(x)}{X(x)} = -k^2 - \frac{Y''(y)}{Y(y)}, \tag{8.12}$$

we conclude that for this equation to hold for all x and y, both sides must be equal to a constant, which we choose to write as $-\nu^2$ (for some real ν). Also, introducing the constant μ so that $\nu^2 + \mu^2 = k^2$ gives us the equations

$$X''(x) = -\nu^2 X(x), \quad Y''(y) = -\mu^2 Y(y). \tag{8.13}$$

The general solutions are

$$\begin{aligned} X(x) &= A\sin(\nu x) + B\cos(\nu x), \\ Y(y) &= C\sin(\mu y) + D\cos(\mu y), \end{aligned} \tag{8.14}$$

for arbitrary constants A, B, C, D.

Now we are ready to impose the boundary conditions.

The condition $z(0, y, t) = 0$ for all y, t requires that $X(0) = 0$ and, because $X(0) = B$, this implies that $B = 0$. Similarly the condition $z(a, y, t) = 0$ requires that $X(a) = 0$ and because $X(a) = A\sin(\nu\, a)$ (because we already know that $B = 0$), this implies that either $A = 0$, which is not an interesting case because it makes the entire solution vanish, or that $\nu = n\pi/a$ with $n \in \mathbb{Z}$. Without loss of generality we can take $n \in \mathbb{N}$ because negative n just give the same solution up to a sign that can be absorbed into the arbitrary constant A, and $n = 0$ gives the zero solution.

The conditions $z(x, 0, t) = 0 = z(x, b, t)$ similarly require that $Y(0) = 0 = Y(\pi)$ and thus $D = 0$ and $\mu = m\pi/a$ with $m \in \mathbb{N}$.

Thus we have found the following solutions satisfying the boundary conditions:

$$\begin{aligned} z_{nm}(x, y, t) = &\sin\left(\frac{\pi}{a} nx\right) \sin\left(\frac{\pi}{b} my\right) \\ &\cdot \left(F_{nm} \sin(k_{nm} ct) + G_{nm} \cos(k_{nm} ct)\right) \end{aligned} \tag{8.15}$$

with

$$k_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \tag{8.16}$$

for any choice of $n, m \in \mathbb{N}$ and $F_{nm}, G_{nm} \in \mathbb{R}$.

Note that solutions Eq. 8.15 already satisfy the boundary conditions Eq. 8.3 and Eq. 8.4. Such solutions are called *normal modes* of the membrane. Snapshots of normal modes with $(n, m) = (1, 1)$, $(n, m) = (1, 2)$, $(n, m) = (2, 2)$ and $(n, m) = (3, 2)$ are shown in Figure 8.1.

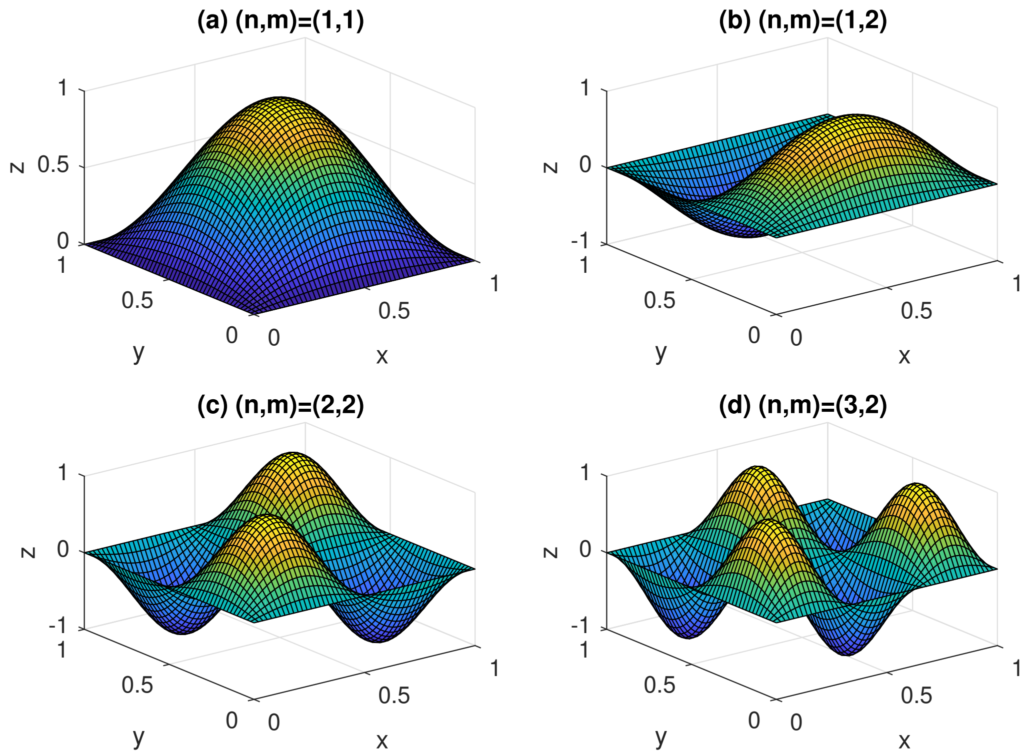

Figure 8.1: Some normal modes of a rectangular membrane

The amplitudes of the normal modes oscillate over time. The following video shows an animation of various normal modes:

https://youtu.be/yDZsCZn3lSk

Any solution of the wave equation Eq. 8.1 satisfying the boundary conditions Eq. 8.3 and Eq. 8.4 can be presented as a linear combination of normal modes Eq. 8.15:

$$z(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z_{nm}. \tag{8.17}$$

The free constants F_{nm} and G_{nm} are then determined by the initial conditions, using the Fourier transform technique as in the one-dimensional case.

To construct a solution satisfying the initial conditions

$$z(x,y,0) = z_0(x,y), \quad \partial_t z(x,y,0) = v_0(x,y) \tag{8.18}$$

we evaluate our general solution and its t derivative at the boundary. This gives

$$z(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{\pi n x}{a}\,\sin\frac{\pi m y}{b} G_{nm} = z_0(x,y), \tag{8.19}$$

$$\partial_t z(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_{nm} c\, B_{nm}\,\sin\frac{\pi n x}{a}\,\sin\frac{\pi m y}{b} = v_0(x,y). \tag{8.20}$$

Comparing these with the (double) Fourier series for functions $z_0(x,y)$ and $v_0(x,y)$, one can deduce that

$$\begin{aligned} G_{nm} &= \frac{4}{ab}\int_0^a dx \int_0^b dy\, z_0(x,y)\sin\frac{\pi n x}{a}\,\sin\frac{\pi m y}{b}, \\ F_{nm} &= \frac{1}{k_{nm}c}\,\frac{4}{ab}\int_0^a dx \int_0^b dy\, v_0(x,y)\sin\frac{\pi n x}{a}\,\sin\frac{\pi m y}{b}. \end{aligned} \tag{8.21}$$

9 Waves on circular domain

You find content related to this lecture in the textbooks:

- Baldock and Bridgeman (1983) section 7.5
- Coulson and Jeffrey (1977) sections 7 (eqs.(31) - (35)) and section 35

Let's find normal mode solution for a circular membrane of radius a with fixed boundary (see Figure 9.1). Mathematically, we need to solve the wave equation (Eq. 8.1) in the disc of radius a, i.e. in region $D = \{(x, y) \in \mathbb{R}^2 | r^2 = x^2 + y^2 < a^2\}$. It is convenient to use polar coordinates (r, θ), defined as (see Figure 9.1)

$$x = r \cos\theta, \quad y = r \sin\theta. \tag{9.1}$$

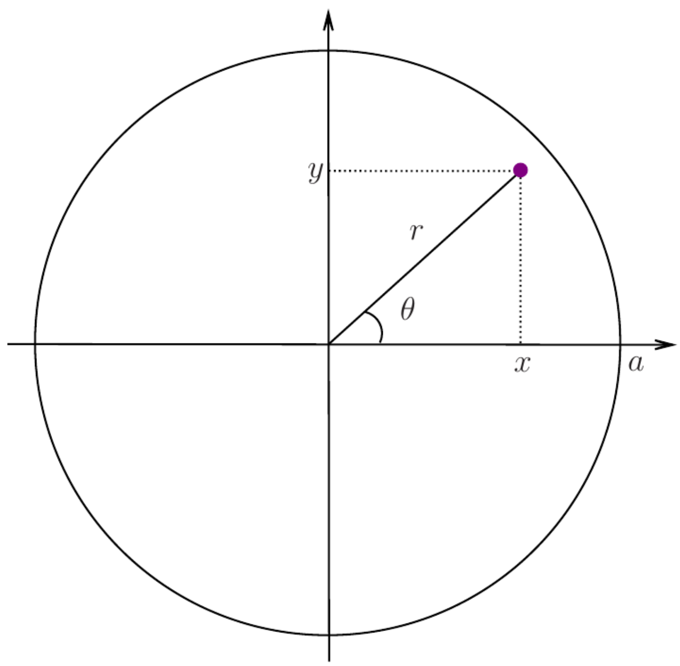

Figure 9.1: Circular membrane with radius a and polar coordinates.

Let

$$Z(r, \theta, t) = z\left(x(r, \theta), y(r, \theta), t\right), \tag{9.2}$$

i.e., $Z(r,\theta,t)$ is a solution of the wave equation in D, expressed in terms of polar coordinates.

We know that the wave equation can be written in vector notation using the Laplace operator ∇^2 as

$$\partial_t^2 z - c^2 \nabla^2 z = 0. \tag{9.3}$$

It is well-known (see, e.g., Theorem 3.43 in the Spring 2020 Calculus notes or Theorem 4.6 in the Spring 2021 Calculus notes) that the Laplacian in polar coordinates is given by

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\partial_\theta^2. \tag{9.4}$$

Therefore, the 2D wave equation in polar coordinates has the form

$$\partial_t^2 Z - c^2 \left(\partial_r^2 Z + \frac{1}{r}\,\partial_r Z + \frac{1}{r^2}\,\partial_\theta^2 Z \right) = 0. \tag{9.5}$$

Our aim is to solve it subject to the boundary condition Eq. 9.20. As in the preceding section, we employ the method of separation of variables, i.e. we assume that

$$Z(r,\theta,t) = R(r)\Theta(\theta)T(t). \tag{9.6}$$

Substituting this into Eq. 9.5, we find that

$$R\Theta T'' = c^2 \left(R''\Theta T + \frac{1}{r}\,R'\Theta T + \frac{1}{r^2}\,R\Theta'' T \right) \tag{9.7}$$

Dividing by $R\Theta T$ gives

$$\frac{T''(t)}{T(t)} = c^2 \left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} \right). \tag{9.8}$$

Employing the same arguments as in the preceding section, we conclude that

$$\frac{T''(t)}{T(t)} = c^2 \left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} \right) = -k^2c^2 \tag{9.9}$$

for some constant k. This means that we have the ODE

$$T''(t) = -k^2c^2\,T(t) \tag{9.10}$$

and the equation

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = -k^2 \tag{9.11}$$

The general solution of Eq. 8.3 is (cf. Eq. 8.11)

$$T(t) = F\,\sin(kct) + G\,\cos(kct) \tag{9.12}$$

where F and G are arbitrary constants.

Rewriting Eq. 9.12 as

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} + k^2 r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}, \tag{9.13}$$

we conclude that for this equation to hold for all r and θ, both sides must be equal to a constant, which we choose to be n^2 (for some constant n), i.e.

$$r^2\left(\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)}\right) + k^2 r^2 = n^2, \quad -\frac{\Theta''(\theta)}{\Theta(\theta)} = n^2 \tag{9.14}$$

(the constant, n^2, cannot be negative because, with a negative constant, it will be impossible to obtain a solution). Thus, we we have obtained the following two ODEs:

$$R''(r) + \frac{1}{r}R'(r) + \left(k^2 - \frac{\lambda^2}{r^2}\right)R(r) = 0 \tag{9.15}$$

and

$$\Theta''(\theta) = -n^2\Theta(\theta). \tag{9.16}$$

The general solution of Eq. 9.16 is

$$\Theta(\theta) = A\sin(n\theta) + B\cos(n\theta) \tag{9.17}$$

for arbitrary constants A and B. We impose the *periodicity* condition

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad \text{for all} \quad \theta.$$

This is a natural condition because (r, θ) and $(r, \theta + 2\pi)$ represent a single point in domain D. Thus we require $n \in \mathbb{Z}$. In fact we only need to consider $n \in \mathbb{N} \cup \{0\}$ because of the symmetry of sin and cos.

Eq. 9.15 has the form

$$r^2 R''(r) + r R'(r) + (r^2 k^2 - n^2) R(r) = 0. \tag{9.18}$$

It is a well-known equation, called the Bessel differential equation. Its solutions are not elementary functions. We are only interested in solutions that are finite at $r = 0$. There is a whole family of such solutions,

$$R(r) = J_n(kr), \tag{9.19}$$

where J_n is the nth Bessel functions of the first type. We have already determined above that we are only interested in the cases where $n \in \mathbb{N} \cup \{0\}$.

Next we need to impose the zero Dirichlet boundary condition (the edge of the membrane is fixed)

$$Z(a, \theta, t) = 0. \tag{9.20}$$

To do this, we only need to know that each Bessel function has an infinite number of zeros. A plot of the first few Bessel functions of the 1st kind is shown in Figure 9.2.

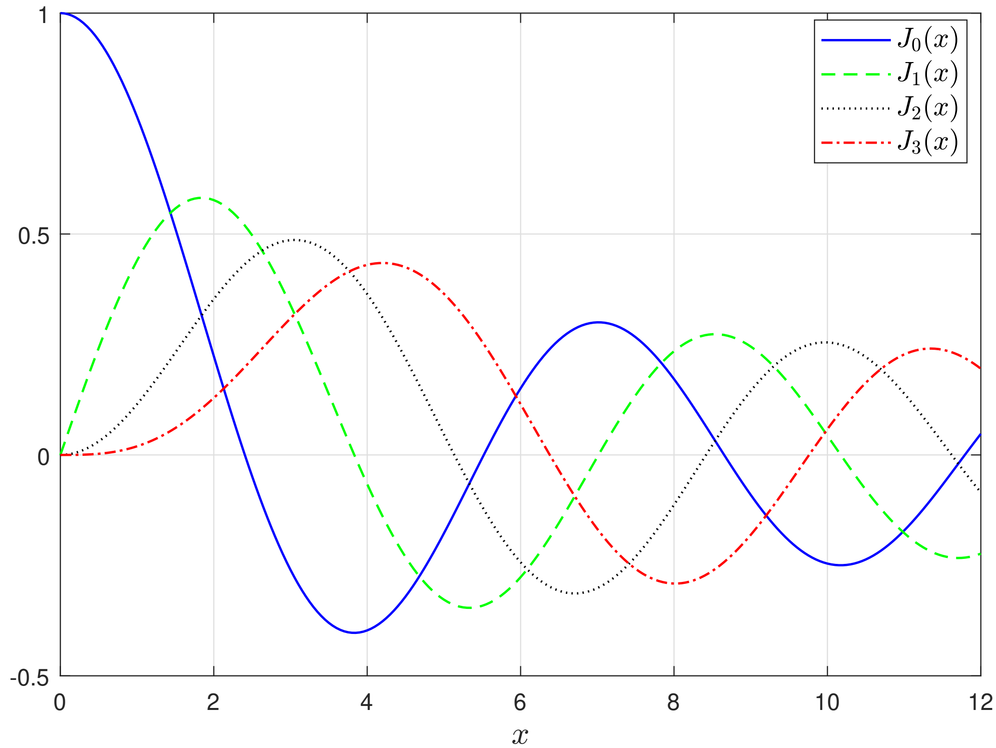

Figure 9.2: Plot of the first four Bessel functions of the 1st kind.

We denote the mth zero of the nth Bessel function of the 1st kind by k_{nm}. We have

$$\begin{aligned} R(a) = 0 \quad &\Rightarrow \quad J_n(k\,a) = 0 \\ &\Rightarrow \quad k\,a = k_{nm} \\ &\Rightarrow \quad k = \frac{k_{nm}}{a} \end{aligned}$$

for $m \in \mathbb{N}$.

Finally, combining the above with Eq. 9.12 and Eq. 9.17, we obtain the following solutions (that satisfy the required boundary condition):

$$\begin{aligned} Z_{nm}(r,\theta,t) =& J_m\left(k_{nm}\frac{r}{a}\right) \\ &\cdot\Big[\sin(n\theta)\left(F_{nm}\,\sin(k_{nm}ct) + G_{nm}\,\cos(k_{nm}ct)\right) \\ &+ \cos(n\theta)\left(\tilde{F}_{nm}\,\sin(k_{nm}ct) + \tilde{G}_{nm}\,\cos(k_{nm}ct)\right)\Big] \end{aligned} \tag{9.21}$$

where $n \in \mathbb{N}\cup\{0\}$, $m \in \mathbb{N}$ and F_{nm}, G_{nm}, $\tilde{F}_{nm}$ and $\tilde{G}_{nm}$ are arbitrary real constants. Solutions in the form Eq. 9.21 are the normal modes of vibrations of the circular membrane. Once normal modes are known, we can find a solution satisfying some initial conditions by using a linear combination of the normal modes

$$Z(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} Z_{nm}(r,\theta,t). \tag{9.22}$$

Note that constants F_{nm}, G_{nm}, $\tilde{F}_{nm}$ and $\tilde{G}_{nm}$ (in the formula for Z_{nm} for each n,m) are still arbitrary. To determine them, we need to substitute Eq. 9.22 into initial conditions. This will lead to the so-called Fourier-Bessel series. This is based on the identities

$$\int_0^a J_n\left(k_{nm}\frac{r}{a}\right) J_n\left(k_{nl}\frac{r}{a}\right) r\,dr = \delta_{ml}\frac{a^2}{2}\left(J_{n+1}(k_{nm})\right)^2. \tag{9.23}$$

Part II

Fluids

Fluid dynamics studies the motion of fluids (gases and liquids, e.g. air or water).

For us fluid dynamics provides a rich and fascinating area of study that draws upon many of the mathematical skills from your other modules, including calculus, vector calculus and functions of a complex variable. This part of the module will expose you to new important methods in mathematical modelling.

The study of fluid dynamics also has a wide range of practical applications, including in the design of aircraft, ships, and other vehicles that must move through fluids. It is also important in understanding the behaviour of natural phenomena such as weather patterns, ocean currents, and the flow of blood in the human body.

Fluid dynamics is a huge subject. In this introduction to the subject we will concentrate on ideal fluids, which we will define later. In spite of this restriction we will be able to understand some important phenomena, like the lift on an airplane wing or the surface waves on water.

We have a strong research group in this department using fluid dynamics in mathematical biology. In your third and fourth year you will be able to take further module that go more deeply into the subject, and you could choose to use fluid dynamics in your final-year project. Fluid dynamics is an active area of research with many open questions and opportunities for new discoveries. If you are interested in pursuing graduate studies in mathematics or related fields you will find that fluid dynamics provides a rich and exciting area for research, often in collaboration across disciplinary boundaries.

10 How to describe fluids

You find content related to this lecture in the textbooks:

- Batchelor (2000) section 2.1
- Acheson (1990) section 1.2
- Paterson (1983) chapter 3 and sections 4.1, 4.2
- Bernard (2015) chapters 1, 2, 10

10.1 Fluid flow

Fluids are considered as continuous media. Because any small volume element of a fluid contains a huge number of molecules, instead of describing the location and velocity of all the individual molecules, we work with quantities which should be thought of as averages over very small neighbourhoods.

So we describe the distribution of matter via a smooth density function $\rho(x, y, z, t)$ (mass/volume) and its motion via a smooth velocity field $\underline{u}(x, y, z, t)$.

The fluid velocity is a vector field, which means that it is a vector whose direction and magnitude may be different at different points in space:

$$\begin{aligned}\underline{u}(x, y, z, t) &= (u_x(x, y, z, t), u_y(x, y, z, t), u_z(x, y, z, t)) \\ &= u_x(x, y, z, t)\underline{e}_x + u_y(x, y, z, t)\underline{e}_y + u_z(x, y, z, t)\underline{e}_z.\end{aligned} \tag{10.1}$$

We will also use the shorthand $\underline{u}(\underline{x}, t)$ where $\underline{x} = (x, y, z)$ is the position vector.

We will sometimes be looking at particular simple kinds of flows:

Definition 10.1. A **steady flow** is a flow whose velocity field at every point is independent of time:

$$\partial_t \underline{u} = 0, \quad \text{i.e.,} \quad \underline{u}(\underline{x}, t) = \underline{u}(\underline{x}).$$

Definition 10.2. A **two-dimensional (2D) flow** is a flow whose velocity field is independent of the third coordinate (which we will always choose to be the z coordinate) and has no component in that third direction:

$$\partial_z \underline{u} = 0 \text{ and } u_z = 0,$$

i.e.,

$$\underline{u}(\underline{x}, t) = u_x(x, y, t)\underline{e}_x + u_y(x, y, t)\underline{e}_y = (u_x(x, y, t), u_y(x, y, t), 0).$$

A vector field can be visualised by drawing vectors attached to different points in space. In the video lecture we work through the sketch of the velocity field for the 2D flow $\underline{u} = (-y, x, 0)$.

10.2 Pathlines and streamlines

10.2.1 Pathlines

We want to answer the question: given the velocity field $\underline{u}(\underline{x}, t)$, how does a fluid particle (i.e. a very small volume element of the fluid or a small particle embedded in the fluid) move?

Let $\underline{x}(t)$ be the position of a fluid particle in the flow with the velocity $\underline{u}(\underline{x}, t)$ at time t. Then it must satisfy the following vector ODE:

! Pathline equation

$$\frac{d\underline{x}(t)}{dt} = \underline{u}(\underline{x}(t), t). \tag{10.2}$$

Equivalently, we can write this vector ODE as a system of three scalar ODEs:

$$\begin{aligned} \frac{dx(t)}{dt} &= u_x(x(t), y(t), z(t), t), \\ \frac{dy(t)}{dt} &= u_y(x(t), y(t), z(t), t), \\ \frac{dz(t)}{dt} &= u_z(x(t), y(t), z(t), t). \end{aligned} \tag{10.3}$$

Example 10.1. Find the pathlines for the 2D flow $\underline{u} = (-y, x, 0)$.

Solution. We solve the ODEs

$$\frac{dx(t)}{dt} = -y(t), \quad \frac{dy(t)}{dt} = x(t), \quad \frac{dz(t)}{dt} = 0. \tag{10.4}$$

The general solution of this system is

$$x(t) = A\sin t + B\cos t, \quad y(t) = -A\cos t + B\sin t, \quad z(t) = C \tag{10.5}$$

for some constants A, B, C. If we have initial conditions $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$, we find that $B = x_0$, $A = y_0$ and $C = z_0$. Thus, we have

$$\begin{aligned} x(t) &= -y_0 \sin t + x_0 \cos t, \\ y(t) &= y_0 \cos t + x_0 \sin t, \\ z(t) &= z_0. \end{aligned} \tag{10.6}$$

It is clear now that the pathline starting at (x_0, y_0) is a circle of radius $\sqrt{x_0^2 + y_0^2}$.

Next we look at an example of a non-steady flow.

Example 10.2. We consider the 2D flow $\underline{u} = (x, t, 0)$. In the video lecture we sketch this velocity field at $t = 0$ and $t = 1$.

To find the pathlines we solve the ODEs

$$\frac{dx(t)}{dt} = x(t), \quad \frac{dy(t)}{dt} = t, \quad \frac{dz(t)}{dt} = 0. \tag{10.7}$$

The solution of this system with initial values $\underline{x}(0) = (x_0, y_0, z_0)$ is

$$x(t) = x_0 e^t, \quad y(t) = \frac{1}{2}t^2 y_0, \quad z(t) = z_0. \tag{10.8}$$

10.2.2 Streamlines

A *streamline* of a flow is a curve such that, at each point of the curve, the tangent line to it is parallel to the velocity at the same point. If the velocity field depends on time, then streamlines may be different at different moments of time. Let $\underline{u}(\underline{x}, t)$ be a velocity field. It follows from this definition that if $\underline{x}(s) = (x(s), y(s), z(s))$ is a parametrisation of a streamline (with some parameter s along the streamline), then the tangent vector to the curve, $d\underline{x}(s)/ds$, must be parallel to $\underline{u}(\underline{x}(s), t)$. This means

$$\frac{d\underline{x}(s)}{ds} = \lambda\, \underline{u}(\underline{x}(s), t) \tag{10.9}$$

where λ is a nonzero scalar which may depend on both $\underline{x}$ and t. Clearly, there is a freedom in choosing λ. The simplest choice that we shall use from now on is $\lambda = 1$. So, the streamlines are solutions of the following vector ODE:

! **Streamline equation**

$$\frac{d\underline{x}(s)}{ds} = \underline{u}(\underline{x}(s), t) \tag{10.10}$$

In components this equation becomes

$$\begin{aligned}
\frac{dx(s)}{ds} &= u_x(x(s), y(s), z(s), t), \\
\frac{dy(s)}{ds} &= u_y(x(s), y(s), z(s), t), \\
\frac{dz(s)}{ds} &= u_z(x(s), y(s), z(s), t).
\end{aligned} \tag{10.11}$$

So in Example 10.2 the ODEs for the streamlines are

$$\frac{dx(s)}{ds} = x(s), \quad \frac{dy(s)}{ds} = t, \quad \frac{dz(s)}{ds} = 0. \tag{10.12}$$

The solution of this system for the streamline starting at $\underline{x}(0) = (x_0, y_0, z_0)$ is

$$x(s) = x_0 e^s, \quad y(t) = \frac{1}{2}st + y_0, \quad z(t) = z_0. \tag{10.13}$$

Note that this is different from the pathlines that we determined for Example 10.2

For steady flows ($\underline{u} = \underline{u}(\underline{x})$) streamlines coincide with pathlines, because Eq. 10.11 will be exactly the same as Eq. 10.3, with $s = t$. But in general pathlines and streamlines represent different quantities and should not be confused with each other. To summarise:

Pathlines are the paths traced out by individual fluid particles as they move through a flow field over a period of time. In other words, a pathline is the trajectory followed by a single fluid particle in the flow field, and it can be used to visualize the history of the particle's motion. Pathlines can be curved, twisted, or convoluted, and they can cross and intersect with each other.

Streamlines, on the other hand, are imaginary lines that are everywhere tangent to the velocity vector of the flow at a given instant in time. In other words, they represent the instantaneous direction of fluid motion at every point in the flow field. Streamlines do not represent the paths of individual fluid particles, but rather they represent the direction of the flow at any given instant. Streamlines are useful to visualize the flow pattern of a fluid.

10.3 Material derivative

Let $f(\underline{x}, t)$ be some quantity of interest (e.g. a component of the velocity $\underline{u}$ or the density of the fluid ρ). The rate of change of f at a fixed point in space (i.e. at fixed $\underline{x}$) is $\partial_t f(\underline{x}, t)$. What is the rate of change of f following the fluid (i.e. the rate of change of f at a given fluid particle as it moves with the fluid)? We denote this as Df/Dt and will refer to it as the *material derivative* of f. We have

$$\begin{aligned}\frac{Df}{Dt} &= \frac{d}{dt}\, f(x(t), y(t), z(t), t) \\ &= \partial_t f(x, y, z, t) + \partial_x f(x, y, z, t)\, \frac{dx}{dt} \\ &\qquad + \partial_y f(x, y, z, t)\, \frac{dy}{dt} + \partial_z f(x, y, z, t)\, \frac{dz}{dt} \\ &= \partial_t f(x, y, z, t) + \partial_x f(x, y, z, t)\, u_x \\ &\qquad + \partial_y f(x, y, z, t)\, u_y + \partial_z f(x(t), y, z, t)\, u_z \\ &= \partial_t f(\underline{x}, t) + \underline{u} \cdot \underline{\nabla} f(\underline{x}, t).\end{aligned} \tag{10.14}$$

Thus, we have

!

Definition 10.3. The **material derivative** of a function $f(x, y, z, t)$ is

$$\frac{Df}{Dt} = \partial_t f + \underline{u} \cdot \underline{\nabla} f. \tag{10.15}$$

In particular, $Df/Dt = 0$ means that quantity f is a constant for each fluid particle.

Example 10.3. Assume the temperature of the fluid is given by $T(x, z, y, t) = -y$ and the velocity field is $\underline{u} = (-y, x, 0)$. So there is no time dependence in either the temperature nor in the velocity field. Nevertheless a fluid particle is going to experience a time-dependent temperature because

$$\begin{aligned}\frac{DT}{Dt} &= \partial_t T + u_x \partial_x T + u_y \partial_y T + u_z \partial_z T \\ &= u_y \partial_y T = -x = y_0 \sin t - x_0 \cos t.\end{aligned} \tag{10.16}$$

We used the expression for $x(t)$ from Eq. 10.6.

10.3.1 Acceleration of a fluid particle

The velocity of a fluid particle (whose position at time t is $\underline{x}(t)$) is $\underline{u}(\underline{x}(t), t)$. To find the acceleration $\underline{a}(\underline{x}(t), t)$ of this fluid particle we need to differentiate the velocity with respect to time. Thus

!

Definition 10.4. The **acceleration** experienced by a fluid particle is

$$\underline{a}(\underline{x}, t) = \frac{D\underline{u}}{Dt} = \partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla})\underline{u}. \tag{10.17}$$

If we calculate the acceleration in Example Example 10.2 where $\underline{u} = (x, t, 0)$ we get the acceleration

$$\begin{aligned} \underline{a}(\underline{x}, t) &= \frac{D\underline{u}}{Dt} = \partial_t \underline{u} + x\, \partial_x \underline{u} + t\, \partial_y \underline{u} + 0\, \partial_z \underline{u} \\ &= (0, 1, 0) + x(1, 0, 0) = (x, 1, 0). \end{aligned} \tag{10.18}$$

10.4 Conservation of mass

Let $\rho(\underline{x}, t)$ be the density of the fluid, i.e. the mass of the fluid per unit volume. The density is strictly positive: $\rho(\underline{x}, t) > 0$ for all $\underline{x}$ and t. We shall deduce an equation governing the evolution of ρ from the law of conservation of mass.

Consider an arbitrary fixed (in space) volume V. The mass of the fluid in this volume is

$$\int\limits_V \rho(\underline{x}, t)\, dV. \tag{10.19}$$

Here dV is the volume element (i.e. $dV = dx\, dy\, dz$). Let ∂V be the boundary of V and $\underline{n}$ be the outward unit normal on ∂V (see Figure 10.1). The rate at which mass of the fluid is flowing through a surface element dS on ∂V is $\rho \underline{u} \cdot \underline{n}\, dS$ (i.e. the normal component of the *mass flux density* $\rho \underline{u}$ multiplied by the area dS of the surface element). If $\underline{u} \cdot \underline{n}$ is positive, then the fluid flows out of volume V through the surface element dS. If $\underline{u} \cdot \underline{n}$ is negative, then the fluid flows into V. So, the net rate at which mass is flowing out of V though the boundary of V is

$$\oint\limits_{\partial V} \rho \underline{u} \cdot \underline{n}\, dS. \tag{10.20}$$

If this integral is positive, this means that more fluid flows out than in; if it is negative, then more fluid flows in than out.

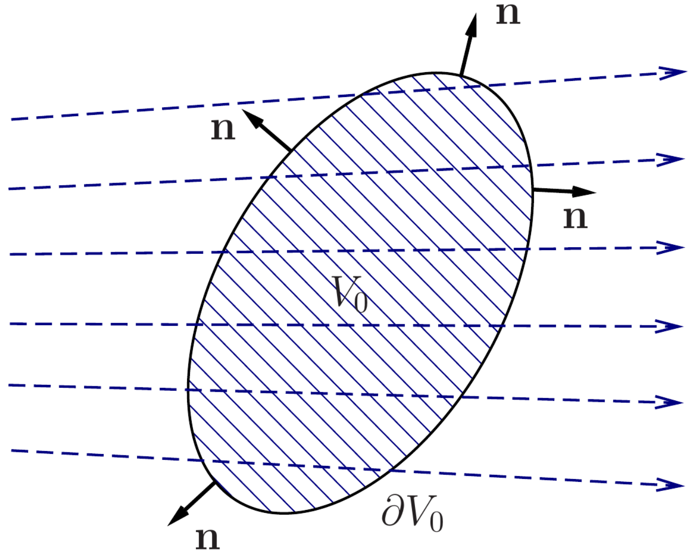

Figure 10.1: Flow through a fixed volume.

The rate of change of mass in volume V is the derivative with respect to time of the integral in Eq. 10.19. In view of to the law of conservation of mass, it must be equal to the change of mass (per unit time) due to the inflow and outflow of the fluid. Therefore, we obtain

$$\frac{d}{dt}\int\limits_V \rho\, dV = \int\limits_V \partial_t \rho\, dV = -\oint\limits_{\partial V} \rho \underline{u} \cdot \underline{n}\, dS. \tag{10.21}$$

Here the minus before the surface integral appears because $\underline{n}$ is chosen to be the outward normal (so that the change of mass is negative if the fluid flows out of V).

Now, we need to recall the divergence theorem (also known as the Gauss-Ostrogradsky theorem) of Vector Calculus which we already used in Section 7.2 when we discussed energy conservation in the example of the two-dimensional wave equation.

Theorem 10.1 (Divergence theorem)**.** *If V is a region in $\mathbb{R}^3$ with boundary ∂V and $\underline{F}(\underline{x})$ is a vector field defined in V, then*

$$\int\limits_V \underline{\nabla} \cdot \underline{F}\, dV = \oint\limits_{\partial V} \underline{F} \cdot \underline{n}\, dS. \tag{10.22}$$

Applying the divergence theorem to the surface integral on the right hand side of Eq. 10.21, we obtain

$$\int\limits_V \partial_t \rho\, dV = -\int\limits_V \underline{\nabla} \cdot (\rho\, \underline{u})\, dV, \tag{10.23}$$

which we can rewrite as

$$\int_V \left[\partial_t \rho + \underline{\nabla} \cdot (\rho\, \underline{u})\right]\, dV = 0. \tag{10.24}$$

The equation must hold for any volume V. This is possible only if the integrand is zero. Thus we have derived the conservation equation for the mass, which is also known as the continuity equation.

! Mass conservation equation

$$\partial_t \rho + \underline{\nabla} \cdot (\rho\, \underline{u}) = 0. \tag{10.25}$$

Note that since $\underline{\nabla} \cdot (\rho\, \underline{u}) = \underline{u} \cdot \underline{\nabla} \rho + \rho\, \underline{\nabla} \cdot \underline{u}$, the equation of continuity can be written as

$$\partial_t \rho + \underline{u} \cdot \underline{\nabla} \rho + \rho\, \underline{\nabla} \cdot \underline{u} = 0 \tag{10.26}$$

or

$$\frac{D\rho}{Dt} + \rho\, \underline{\nabla} \cdot \underline{u} = 0. \tag{10.27}$$

If the fluid is incompressible then the density of each fluid element must remain constant as the fluid moves, i.e.

$$\frac{D\rho}{Dt} = 0. \tag{10.28}$$

It then follows from Eq. 10.27 that

$$\underline{\nabla} \cdot \underline{u} = 0. \tag{10.29}$$

!

Definition 10.5. A fluid whose flow everywhere satisfies the *incompressibility condition*

$$\underline{\nabla} \cdot \underline{u} = 0. \tag{10.30}$$

is called **incompressible**.

A vector field $\underline{u}$ satisfying $\underline{\nabla} \cdot \underline{u}$ is known as a *solenoidal vector field* or a *divergence-free vector field*.

11 Ideal fluid

You find content related to this lecture in the textbooks:

- Batchelor (2000) section 2.2
- Acheson (1990) start of section 1.3
- Paterson (1983) section 10.1

11.1 The Euler equations for an ideal fluid

Euler wrote down the partial differential equations describing an ideal fluid already in 1755. This was one of the first partial differential equations systems to be studied.

An ideal fluid has two simplifying properties:

1. **Constant density**: The density is constant throughout the fluid at all times, so the density that in a general fluid could be a function of space and time is a constant: $\rho(\underline{x}, t) = \rho$, i.e.,
$$\partial_t \rho = 0 = \underline{\nabla} \rho. \tag{11.1}$$
This implies that also the material derivative is zero:
$$\frac{D\rho}{Dt} = \partial_t \rho + \underline{u} \cdot \underline{\nabla} \rho = 0. \tag{11.2}$$
We had seen already that this implies that the fluid is incompressible: $\underline{\nabla} \cdot \underline{u} = 0$. But the condition on an ideal fluid is stronger than incompressibility, because it demands that the density is not only constant as experienced by each fluid particle, but that it is also the same across different fluid particles.

2. **No viscosity**: The force that the fluid exerts on any infinitesimal surface element is solely in the direction of the normal vector to the surface element. Thus this surface force contribution can be written as
$$d\underline{F}^{(s)} = p\, \underline{n}\, dS \tag{11.3}$$
where $p = p(\underline{x}, t)$ is called the **pressure**, $\underline{n}$ is the normal vector to the surface, and the factor of dS expresses that the force is proportional to the area of the surface element.

> Thus in an ideal fluid there are no surface forces acting tangentially to the surface, also called shear forces. Such a fluid is also called **inviscid**. The opposite would be a **viscous** fluid. In reality every fluid has at least a little bit of viscosity, but in circumstances where these viscous forces are negligible compared to the other forces we can neglect them and treat the fluid as inviscid.

There are phenomena that we will not be able to model correctly when we neglect viscosity. For example, when we will study the flow of air over an aerofoil, a thin layer of fluid near the surface experiences a high level of shear stress due to the no-slip condition at the surface. This results in the formation of a boundary layer, which is a region of fluid where the velocity gradients are large and the viscosity of the fluid plays a critical role in determining the behaviour of the flow. Luckily the details of the flow in this boundary layer does not affect the lift force that we want to calculate.

Another phenomenon we can not describe with inviscid flows is turbulence. Turbulent flows arise for example at the rear of real-world aerofoils when the boundary layer detaches from the aerofoil. There the viscosity of the fluid plays a critical role in dissipating the kinetic energy of the turbulent motion into heat and this produces the drag on the airplane. But again this does not affect the lift force, just the drag force.

The equation of motion for the fluid is simply derived from Newton's equation $m\mathbf{a} = \underline{F}$. We consider a small volume V inside the fluid. We sum up the mass multiplied by the acceleration for each fluid particle inside the volume, which due to our continuity approximation becomes an integral of the density multiplied by the acceleration vector field $D\underline{u}/Dt$ over the volume V:

$$m\underline{a} = \int_V \rho \, \frac{D\underline{u}}{Dt} \, dV. \tag{11.4}$$

As discussed above, for an ideal fluid the only force acting on the surface of the volume V is the force due to the pressure in the surrounding fluid. The total force exerted on the volume V is obtained by integrating the contributions from all infinitesimal surface elements:

$$\underline{F}^{(s)} = -\oint_{\partial V} p \, \underline{n} \, dS. \tag{11.5}$$

The reason for the minus sign is that by convention $\underline{n}$ denotes the outwards normal, but we are interested in the force acting on the volume from the outside.

We now use a form of the divergence theorem that you may not have met yet:

Theorem 11.1. *For any scalar function $f(\underline{x})$*

$$\oint_{\partial V} f \, \underline{n} \, dS = \int_V \underline{\nabla} f \, dV. \tag{11.6}$$

Proof. The ith component of the above vector equation can be obtained by setting $\underline{F} = f\underline{e}_i$ in the divergence theorem Theorem 10.1.

□

Applying this to the expression for the surface force $\underline{F}^{(s)}$ by setting $f = p$ allows us to express the force as a volume integral:

$$\underline{F}^{(s)} = -\int\limits_V \underline{\nabla} p \, dV. \tag{11.7}$$

We added the superscript (s) to the pressure force to indicate that this is a force acting on a surface – a so-called *surface force*. An ideal fluid can also be subject to other forces, called *body forces*, that act at every point in the fluid, like the gravitational force or possibly electromagnetic forces. We will here consider the gravitational force. We obtain the gravitational force acting on our volume V by summing the gravitational force (mass times gravitational acceleration $\underline{g}$) on all particles, which in our continuum approximation becomes an integral:

$$\underline{F}^{(g)} = \int_V \rho \, \underline{g} \, dV. \tag{11.8}$$

So, Newton's equation $m\underline{a} = \underline{F}$ applied to our volume V of fluid takes the form

$$\int\limits_V \rho \, \frac{D\underline{u}}{Dt} \, dV = -\int\limits_V \underline{\nabla} p \, dV + \int_V \rho \, \underline{g} \, dV. \tag{11.9}$$

This must hold for any volume V. Therefore, we conclude that

$$\rho \, \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \, \underline{g}. \tag{11.10}$$

This equation, together with the incompressibility condition Eq. 10.30 are called *Euler's equations* for an ideal fluid.

! Euler's equations for an ideal fluid

$$\begin{aligned} \frac{D\underline{u}}{Dt} &= -\frac{1}{\rho}\underline{\nabla} p + \underline{g} \\ \underline{\nabla} \cdot \underline{u} &= 0 \end{aligned} \tag{11.11}$$

11.2 Water in a rotating bucket

Example 11.1. We will now use the steady flow from Example Example 10.1:

$$\underline{u} = (-y, x, 0) \tag{11.12}$$

to illustrate the Euler equations. We have seen that this is a circular flow. This describes for example the flow in a rotating bucket of water. [1]

We now want to show that $\underline{u}$ satisfies the Euler equations and find the pressure p. We will then see that Euler's equations can tell us the shape of the surface of the water in the rotating bucket. You probably know from personal observation that you expect the surface to be higher towards the rim of the bucket than in the middle because the centrifugal force presses water outwards. We will easily find the exact shape.

We have written Euler's equations in vector notation. Let's expand them in components. Let the z axis be directed vertically up, then $\underline{g} = (0, 0, -g)$. The equations for the material derivatives of the components of $\underline{u}$ are.

$$\begin{aligned}
\partial_t u_x + \left(u_x\partial_x + u_y\partial_y + u_z\partial_z\right) u_x &= -\frac{1}{\rho}\,\partial_x p,\\
\partial_t u_y + \left(u_x\partial_x + u_y\partial_y + u_z\partial_z\right) u_y &= -\frac{1}{\rho}\,\partial_y p,\\
\partial_t u_z + \left(u_x\partial_x + u_y\partial_y + u_z\partial_z\right) u_z &= -\frac{1}{\rho}\,\partial_z p - g.
\end{aligned} \tag{11.13}$$

Also, we have the incompressibility condition

$$\partial_x u_x + \partial_y u_y + \partial_z u_z = 0. \tag{11.14}$$

Thus, we have 4 equations for 4 unknowns: three components of the velocity, u_x, u_y and u_z, and pressure, p.

> **Note**
>
> The pressure cannot be uniquely determined from Eq. 11.13, because if $\underline{u}(\underline{x}, t)$ and $p(\underline{x}, t)$ represent a solution, then $\underline{u}(\underline{x}, t)$ and $p(\underline{x}, t)+f(t)$ for arbitrary function f is also a solution. To determine the pressure uniquely, we heed to impose some additional condition. For example, if we consider a flow in the whole space, we may require that the pressure at

[1] The reason why the water in a rotating bucket rotates along with the bucket is that shear forces act until the steady rotating flow is achieved. By neglecting viscosity we can not model this initial phase of the flow but we can model the steady flow that is achieved eventually.

infinity is a given constant: $p(\underline{x}, t) \to p_0 = const$ as $|\underline{x}| \to \infty$.

First we check that the flow with $\underline{u} = (-y, x, 0)$ is incompressible, i.e., that Eq. 11.14 is satisfied. We find

$$\partial_x u_x + \partial_y u_y + \partial_z u_z = \partial_x(-y) + \partial_y x + \partial_z 0 = 0. \tag{11.15}$$

So the flow is incompressible and thus can be the flow of an ideal fluid.

Substituting $\underline{u} = (-y, x, 0)$ into Eq. 11.13, most terms are zero and we get

$$\begin{cases} x\partial_y(-y) = -\frac{1}{\rho}\,\partial_x p, \\ -y\partial_x\, x = -\frac{1}{\rho}\,\partial_y p, \\ 0 = -\frac{1}{\rho}\,\partial_z p - g \end{cases} \quad \Rightarrow \quad \begin{cases} \partial_x p = \rho\, x, \\ \partial_y p = \rho\, y, \\ \partial_z p = -\rho\, g \end{cases}$$

We see that these equations for the pressure do indeed have a solution:

$$p = \rho\left(\frac{1}{2}\,(x^2 + y^2) - gz\right) + \text{ constant.} \tag{11.16}$$

This shows that this velocity field satisfies Euler's equations and thus describes an ideal fluid with the given pressure.

At the surface of the water in the bucket the pressure is constant – equal to the atmospheric pressure. From that we can deduce that

$$z = \frac{1}{2g}(x^2 + y^2) + \text{ constant.} \tag{11.17}$$

Thus the surface of the water is a perfect paraboloid.

12 Bernoulli's principle and vorticity

You find content related to this lecture in the textbooks:

- Batchelor (2000) sections 1.3, 1.4, and 1.5
- Braithwaite (2017) section 3.1
- Paterson (1983) section 10.4 and 10.2

Solving Euler's equations is hard. It is therefore worthwhile to first try to derive consequences of the equations without actually solving them. So in this lecture we will use our vector calculus skills to manipulate Euler's equations a bit until they reveal to us Bernoulli's principle and the vorticity equation.

12.1 Bernoulli's principle

In this section we will derive two theorems that make precise the principle discovered by Daniel Bernoulli in the 18th century that as the speed of a fluid increases, the pressure of the fluid decreases. This will later allow us to understand the origin of the lift force on an aerofoil, but of course has many other important practical applications.

To facilitate the derivation of Bernoulli's theorems, we will make two observations that allow us to rewrite Euler's equations in new ways.

1. Because gravity is a conservative force, it can be written as minus the gradient of a potential: $\underline{g} = -\underline{\nabla}\chi$. Hence, Eq. 11.11 can also be written as

$$\frac{D\underline{u}}{Dt} = -\underline{\nabla}\left(\frac{p}{\rho} + \chi\right). \tag{12.1}$$

   In the simple case where gravitational force is constant and in the negative z direction, $\chi = g\,z$ so that $\underline{g} = -g\,\underline{e}_z$.

2. We will use the vector calculus identity

$$(\underline{u} \cdot \underline{\nabla})\underline{u} = (\underline{\nabla} \times \underline{u}) \times \underline{u} + \underline{\nabla}\left(\frac{u^2}{2}\right), \tag{12.2}$$

which holds for any differentiable vector field $\underline{u}$ and thus in particular for the velocity field. Using this, together with the definition of the material derivative, in the left-hand side of Eq. 12.1 gives

$$\begin{aligned}\frac{D\underline{u}}{Dt} &= \partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla})\underline{u} \\ &= \partial_t \underline{u} + (\underline{\nabla} \times \underline{u}) \times \underline{u} + \underline{\nabla}\left(\frac{u^2}{2}\right) \\ &= -\underline{\nabla}\left(\frac{p}{\rho} + \chi\right).\end{aligned} \tag{12.3}$$

This we can rewrite as

$$\partial_t \underline{u} + (\underline{\nabla} \times \underline{u}) \times \underline{u} = -\underline{\nabla}\left(\frac{p}{\rho} + \frac{u^2}{2} + \chi\right). \tag{12.4}$$

In this form the Euler equation involves the curl of $\underline{u}$ which is known as the *vorticity*

$$\underline{\omega} = \operatorname{curl} \underline{u} = \underline{\nabla} \times \underline{u}, \tag{12.5}$$

which we will study in more detail in the next section. If we also introduce the function

$$H = \frac{p}{\rho} + \frac{u^2}{2} + \chi, \tag{12.6}$$

sometimes known as *Bernoulli's integral*, we can write Euler's equation simply as

$$\partial_t \underline{u} + \underline{\omega} \times \underline{u} = -\underline{\nabla} H. \tag{12.7}$$

This simplifies even further if we consider a steady steady flow, so that $\partial_t \underline{u} = 0$. Then Eq. 12.7 simplifies to

$$\underline{\omega} \times \underline{u} = -\underline{\nabla} H. \tag{12.8}$$

To get rid of the vector product in this equation, we take the dot product with $\underline{u}$ on both sides:

$$\underline{u} \cdot (\underline{\omega} \times \underline{u}) = -\underline{u} \cdot \underline{\nabla} H. \tag{12.9}$$

On the left hand side the cross product $\underline{\omega} \times \underline{u}$ is orthogonal to $\underline{u}$ so that the dot product with $\underline{u}$ is zero. So the left-hand side of the above equation is zero and we have

$$\underline{u} \cdot \underline{\nabla} H = 0. \tag{12.10}$$

In words this says that the derivative of H in the direction of the flow, i.e., along streamlines, is zero, which gives:

!

Theorem 12.1 (Bernoulli's streamline theorem)**.** *If an ideal fluid is in steady flow then*

$$H = \frac{p}{\rho} + \frac{u^2}{2} + \chi \tag{12.11}$$

is constant along streamlines.

To keep H constant, higher velocity must correspond to lower pressure, and vice versa. So we notice Bernoulli's principle. But notice that this is only true along each individual streamline. This theorem says nothing about how H varies between streamlines.

Next we restrict ourselves to flows where there is no vorticity, $\underline{\omega} = 0$. Such flows are called **irrotational** flows. In this case Eq. 12.8 implies that $\underline{\nabla} H = 0$, which together with $\partial_t H = 0$ gives us

!

Theorem 12.2 (Bernoulli's theorem for irrotational flow)**.** *If an ideal fluid is in steady* irrotational *flow then*

$$H = \frac{p}{\rho} + \frac{u^2}{2} + \chi \tag{12.12}$$

is constant.

This theorem determines the pressure in a steady, irrotational ideal fluid once the velocity is known.

We can use this theorem to explain the lift created by the airflow around an aerofoil: because the air flows faster along the top of the aerofoil than along the bottom, there is a lower pressure above the aerofoil than below. However this is not a complete explanation until we have answered two questions:

1. Why is the flow irrotational?
2. Why is the velocity higher along the top?

We'll address the first question in the next section and the second question in the following lectures.

12.2 Vorticity

Definition 12.1. The **vorticity** $\underline{\omega}$ of a flow with the velocity field $\underline{u}$ is defined as the *curl* of $\underline{u}$:

$$\underline{\omega} = \operatorname{curl} \underline{u} = \underline{\nabla} \times \underline{u}. \tag{12.13}$$

An example will make it easier to understand how to best think of vorticity:

Example 12.1. We consider the very simple flow $\underline{u} = (\alpha y, 0, 0)$ for some constant $\alpha > 0$. This flow is parallel to the x axis, with the magnitude of the velocity increasing with increasing y. The streamlines are simply straight horizontal lines parallel to the x axis. These are also the pathlines because this is a steady flow. So nothing flows in circles, so one might be tempted to guess that this is an irrotational flow. However the calculation shows that

$$\begin{aligned}\underline{\omega} = \underline{\nabla} \times \underline{u} &= (\partial_y \underline{u}_z - \partial_z \underline{u}_y, \partial_z \underline{u}_x - \partial_x \underline{u}_z, \partial_x \underline{u}_y - \partial_y \underline{u}_x) \\ &= (0, 0, -\partial_y \underline{u}_x) = (0, 0, -\alpha) \neq \underline{0}.\end{aligned} \tag{12.14}$$

So there is a non-zero vorticity – the flow is not irrotational. To see this intuitively, imagine placing an extended object into the flow. If you want to do this in your bathtub, I recommend gluing two matches together to form a cross and place this on the water surface. If you now create a flow, this object may start to rotate. It will do so in the flow in this example because the upper end of the object finds itself in an area of faster flow while the lower end is in an area of slower flow, so the object will rotate clockwise.

12.2.1 The evolution of vorticity

To derive an equation which governs the evolution of vorticity, we take the *curl* of Euler's equation in the form given in Eq. 12.7.

$$\underline{\nabla} \times \partial_t \underline{u} + \underline{\nabla} \times (\underline{\omega} \times \underline{u}) = -\underline{\nabla} \times \underline{\nabla} H. \tag{12.15}$$

Using the fact that *curl* of *grad* is zero, we obtain

$$\partial_t \underline{\omega} + \underline{\nabla} \times (\underline{\omega} \times \underline{u}) = \underline{0}. \tag{12.16}$$

We will now use the general vector calculus identity

$$\underline{\nabla} \times (\underline{a} \times \underline{b}) = (\underline{\nabla} \cdot \underline{b})\underline{a} + (\underline{b} \cdot \underline{\nabla})\underline{a} - (\underline{\nabla} \cdot \underline{a})\underline{b} - (\underline{a} \cdot \underline{\nabla})\underline{b} \tag{12.17}$$

which holds for any two differentiable vector fields $\underline{a}$ and $\underline{b}$. Applying this with $\underline{a} = \underline{\omega}$ and $\underline{b} = \underline{u}$ gives

$$\partial_t \underline{\omega} + (\underline{\nabla} \cdot \underline{u})\underline{\omega} + (\underline{u} \cdot \underline{\nabla})\underline{\omega} - (\underline{\nabla} \cdot \underline{\omega})\underline{u} - (\underline{\omega} \cdot \underline{\nabla})\underline{u} = \underline{0}. \tag{12.18}$$

Using the incompressibility condition $\underline{\nabla} \cdot \underline{u} = 0$ and the fact that the *div* of a *curl* is zero and thus $\underline{\nabla} \cdot \underline{\omega} = 0$, the above simplifies to

$$\partial_t \underline{\omega} + (\underline{u} \cdot \underline{\nabla})\underline{\omega} = (\underline{\omega} \cdot \underline{\nabla})\underline{u}. \tag{12.19}$$

Using the definition of the material derivative we obtain the

! Vorticity equation

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega} \cdot \underline{\nabla})\underline{u}. \tag{12.20}$$

Note that the pressure p does not appear in the vorticity equation.

This simplifies for a 2D flow $\underline{u} = (u_x(x, y, t), u_y(x, y, t), 0)$. In the calculation of the vorticity most terms vanish leaving only

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = (0, 0, \omega) \quad \text{with } \omega = \partial_x u_y - \partial_y u_x. \tag{12.21}$$

Then

$$(\underline{\omega} \cdot \underline{\nabla})\underline{u} = \omega\, \partial_z \underline{u} = \underline{0} \tag{12.22}$$

because there is no z dependence in the 2D flow. As a result, the vorticity equation Eq. 12.20 reduces to

$$\frac{D\omega}{Dt} = 0. \tag{12.23}$$

Thus:

!

Theorem 12.3 (Conservation of vorticity)**.** *In a 2D flow, the vorticity of each fluid particle is conserved, i.e., the vorticity is constant along each pathline.*

This now allows us to answer our first question on the way to understanding the lift on an aerofoil. If we assume that far away in front of the aerofoil the air is still and thus described by a flow with no vorticity, then the vorticity stays zero along the paths of all the air molecules also as the aerofoil passes. Thus the entire flow is irrotational and we can apply Bernoulli's theorem for irrotational flows to argue that the aerofoil experiences a lift force. All we still need to establish is that the velocity is higher above the aerofoil than below. That is going to occupy us for the next three lectures.

Vorticity is a fundamental concept in fluid dynamics that plays a critical role in determining the behaviour of a fluid flow. It is closely related to the formation of vortices, the conservation of angular momentum, and the formation of turbulence. We'll leave all of this for later modules.

13 Velocity potential, stream function, and complex potential

In this section we will develop a powerful method that provides us an unlimited supply of two dimensional incompressible and irrotational flows and hence solutions of the Euler equations. This will make clever use of complex analysis.

You find content related to this lecture in the textbooks:

- Acheson (1990) sections 4.2 and 4.3
- Batchelor (2000) sections 2.7, 6.2
- Paterson (1983) sections 11.1 and 16.1

In the discussions in this lecture the time variable does not play a role, so we neglect to write it in all equations. But everything is allowed to have a time dependence.

13.1 Velocity potential

Consider an **irrotational** flow, so that $\underline{\nabla} \times \underline{u} = \underline{0}$. Then you know from Vector Calculus that there is a scalar function ϕ so that $\underline{u} = \underline{\nabla}\phi$. This ϕ is called the velocity potential.

!

Definition 13.1. The **velocity potential** of an irrotational flow is a function ϕ such that

$$\underline{u} = \underline{\nabla}\phi. \tag{13.1}$$

i As you know from Vector Calculus, the velocity potential can be obtained from a line integral:

$$\phi(\underline{x}) = \int_{\underline{0}}^{\underline{x}} \underline{u}(\underline{\tilde{x}}) \cdot d\underline{\tilde{x}}. \tag{13.2}$$

Here we chose the origin $\underline{0}$ of our coordinate system as the starting point of the line integral. But we could have chosen any other starting point. A change in the starting point only leads to a change in ϕ by a constant and such a constant does not contribute to the gradient of ϕ.

If the domain in which the velocity field $\underline{u}$ is defined is not simply connected, then ϕ may be multi-valued. We will see an example of that later in Section 14.2, but first we will look at a simple example to make sure we understand the line integral.

Example 13.1. Show that the velocity field, given by

$$\underline{u} = (ax + by, bx + cy, -(a+c)z) \tag{13.3}$$

(where a, b and c are constants), represents an irrotational flow and find the velocity potential.

Solution. First we check whether the flow is irrotational by calculating the vorticity

$$\begin{aligned} \underline{\nabla} \times \underline{u} &= (\partial_y u_z - \partial_z u_y, \partial_z u_x - \partial_x u_z, \partial_x u_y - \partial_y u_x) \\ &= (0, 0, b - b) = \underline{0}. \end{aligned} \tag{13.4}$$

Then, to calculate

$$\phi(\underline{x}) = \int_{\underline{0}}^{\underline{x}} \underline{u}(\underline{\tilde{x}}) \cdot d\underline{\tilde{x}}. \tag{13.5}$$

we choose the contour that consists of three straight lines: it starts from $(0,0,0)$ and goes along the x axis to $(x,0,0)$, then goes straight to $(x,y,0)$ and then from there straight to (x,y,z). This splits the integral into three bits:

$$\begin{aligned} \phi(\underline{x}) &= \int_0^x u_x(\tilde{x}, 0, 0)\, d\tilde{x} + \int_0^y u_y(x, \tilde{y}, 0)\, d\tilde{y} + \int_0^z u_y(x, y, \tilde{z})\, d\tilde{z} \\ &= \int_0^x a\tilde{x}\, d\tilde{x} + \int_0^y (bx + c\tilde{y})\, d\tilde{y} - \int_0^z (a+c)\tilde{z}\, d\tilde{z} \\ &= \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 - \frac{a+c}{2}z^2. \end{aligned} \tag{13.6}$$

Alternatively, and equivalently, we can find ϕ by solving the component differential equations contained in $\underline{u} = \underline{\nabla}\phi$:

$$\left\{ \begin{array}{l} ax + by = \partial_x \phi, \\ bx + cy = \partial_y \phi, \\ -(a+c)z = \partial_z \phi, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \phi = \frac{a}{2}x^2 + bxy + f(y,z), \\ \phi = bxy + \frac{c}{2}y^2 + g(x,z), \\ \phi = -\frac{a+c}{2}z^2 + h(x,y) \end{array} \right. \tag{13.7}$$

The last three equations must represent the same function. Therefore,

$$\phi = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 - \frac{a+c}{2}z^2 + \text{ constant}. \tag{13.8}$$

13.2 Stream function

Consider an **incompressible, two-dimensional** flow, so $\underline{\nabla} \cdot \underline{u} = \underline{0}$ and $\underline{u}(\underline{x}) = (u_x(x,y), u_y(x,y), 0)$. Then we can write this in terms of a scalar function $\psi(x,y)$ as follows:

$$\underline{u} = \underline{\nabla} \times (\psi\, \underline{e}_z) = (\partial_y \psi, -\partial_x \psi, 0). \tag{13.9}$$

It is easy to check that this always gives an incompressible flow:

$$\underline{\nabla} \cdot \underline{u} = \partial_x u_x + \partial_y u_y = \partial_x \partial_y \psi - \partial_y \partial_x \psi = 0. \tag{13.10}$$

The function ψ is known as the stream function.

!

Definition 13.2. The **stream function** of an incompressible two-dimensional flow is a function ψ such that

$$\underline{u} = \underline{\nabla} \times (\psi\, \underline{e}_z). \tag{13.11}$$

The reason for the name lies in the following useful fact:

i A stream function is constant along streamlines.

To check this fact we calculate the derivative of ψ in the direction of a streamline, i.e., in the direction of the velocity field (recall that the velocity field is tangent to the streamlines).

$$\underline{u} \cdot \underline{\nabla} \psi = u_x \partial_x \psi + u_y \partial_y \psi = u_x(-u_y) + u_y u_x = 0. \tag{13.12}$$

Thus the stream function gives an easy way to obtain streamlines: they are the lines along which the stream function is constant.

i We can find ψ from the following two-dimensional line integral:

$$\psi(x,y) = \int_{(0,0)}^{(x,y)} (-u_y, u_x) \cdot d\underline{x}. \tag{13.13}$$

Again it does not matter where we start the line integral because that only changes the integral by a constant that does not affect the velocity field.

Example 13.2. Consider the velocity field $\underline{u} = (ax, -ay, 0)$ that you have met before in your homework and that we know is irrotational. We can obtain its stream function from the line integral in Eq. 13.13 :

$$\begin{aligned} \psi(x,y) &= \int_0^x (-u_y(\tilde{x},0))d\tilde{x} + \int_0^y u_x(x,\tilde{y})d\tilde{y} \\ &= \int_0^x 0\, d\tilde{x} + \int_0^y ax\, d\tilde{y} = axy. \end{aligned} \tag{13.14}$$

The streamlines are the lines along which $\psi(x,y) = c$ for some constant c. Here this gives $axy = c$ and hence

$$y = \frac{c}{a}\frac{1}{x}. \tag{13.15}$$

So the stream lines are hyperbolas, as you already determined in a different way in the homework.

13.3 Complex potential

Now we consider flows $\underline{u}$ that are irrotational and incompressible and two-dimensional, so they can be described by both a velocity potential ϕ as $\underline{u} = \underline{\nabla}\phi$ and by a stream function ψ as $\underline{u} = \underline{\nabla} \times (\psi \underline{e}_x)$. In components this gives

$$\begin{aligned} u_x &= \partial_x \phi = \partial_y \psi, \\ u_y &= \partial_y \phi = -\partial_x \psi. \end{aligned} \tag{13.16}$$

You will recognise these as the Cauchy-Riemann equations. They tell us that the function $w = \phi + i\psi$ is a holomorphic function of $z = x + iy$. This function is known as the complex potential.

!

Definition 13.3. The **complex potential** of an irrotational, incompressible two-dimensional flow with velocity potential ϕ and stream function ψ is the complex-valued function

$$w(\underline{x},t) = \phi(\underline{x},t) + i\psi(\underline{x},t). \tag{13.17}$$

Note that here we are identifying the $x - y$ plane with the complex plane and use z to denote the complex number in that plane. This has nothing to do with the z coordinate that we used when discussing 3-dimensional flows. While in general it is a bad idea to use the same letter in the same module for different things, it is just so conventional to use the letter z both for the third Cartesian coordinate as well as for complex numbers that it is excusable in this case.

To extract the velocity field from the complex potential, we just need to differentiate:

$$\begin{aligned}\frac{dw}{dz} &= \partial_x w = \partial_x(\phi + i\psi) \\ &= \partial_x\phi + i\partial_x\psi = u_x - iu_y.\end{aligned} \tag{13.18}$$

If you are wondering why $dw/dz = \partial_x w$, remind yourself of what is special about holomorphic functions: The derivative at any point does not depend on the direction from which we approach the point.

So we can extract the velocity components from the real and imaginary parts of the derivative of the complex potential:

$$u_x = \operatorname{Re}\frac{dw}{dz}, \quad u_y = -\operatorname{Im}\frac{dw}{dz}. \tag{13.19}$$

It is now easy to come up with examples of fluid flows, because we can simply choose any homomorphic function for our complex potential.

Any holomorphic function gives us a solution of Euler's equations, because any holomorphic function can be used as a complex potential describing an incompressible irrotational two-dimensional flow. Such a flow solves Euler's equations.

To see this we start with Euler's equation in the form given in Eq. 12.7 :

$$\partial_t\underline{u} + (\underline{\nabla}\times\underline{u})\times\underline{u} = -\underline{\nabla}\left(\frac{p}{\rho} + \frac{u^2}{2} + \chi\right), \tag{13.20}$$

where χ is the gravitational potential. For our irrotational flow the vorticity $\underline{\nabla}\times\underline{u}$ vanishes. Substituting $\underline{u} = \underline{\nabla}\phi$ on the left-hand side and moving the time derivative through the gradient we obtain

$$\underline{\nabla}(\partial_t\phi) = -\underline{\nabla}\left(\frac{p}{\rho} + \frac{u^2}{2} + \chi\right), \tag{13.21}$$

This equation is satisfied if the pressure is given by

$$p = -\rho\left(\partial_t\phi + \frac{u^2}{2} + \chi\right) + \text{ constant}. \tag{13.22}$$

Example 13.3. Consider the complex potential

$$w(z) = \frac{a}{2}z^2 = \frac{a}{2}(x+iy)^2 = \frac{a}{2}(x^2-y^2) + iaxy = \phi + i\psi. \tag{13.23}$$

We recognise the stream function $\psi = axy$ to be the one for the stagnation flow in Example 13.2. We obtain the velocity field from the derivative of the complex potential:

$$\frac{dw}{dz} = az = ax + iay, \tag{13.24}$$

so $u_x = ax$ and $u_y = -ay$, which again agrees with Example 13.2.

Example 13.4. Consider the complex potential

$$w(z) = ae^{i\alpha}z, \tag{13.25}$$

so that the derivative is just

$$\frac{dw}{dz} = ae^{i\alpha} = a(\cos\alpha + i\sin\alpha) \tag{13.26}$$

from which we can read off that $u_x = a\cos\alpha$ and $u_y = -a\sin\alpha$. This is a constant flow at an angle $-\alpha$ to the horizontal.

14 Flow around a cylinder, circulation, and the Magnus effect

In this lecture we will explain a fascinating phenomenon: a rotating cylinder flying through air experiences a sideways force perpendicular to the direction of motion. The effect is named after the German physicist Heinrich Gustav Magnus, who first described it in 1852.

The Magnus effect has important applications in sports and engineering. In sports, the effect is used to control the flight of balls, such as in soccer, baseball, and cricket. By spinning the ball, players can create a curved trajectory that is difficult for the opposing team to predict. In engineering, the effect is used to design a range of devices, such as helicopter rotors, wind turbines, and rotating cylinders used in certain industrial processes.

The Magnus effect arises due to the interaction between the spinning object and the surrounding fluid. As the object moves through the fluid, it creates a region of high pressure on one side and a region of low pressure on the other side. The direction of the pressure difference is perpendicular to the direction of motion, and the resulting force causes the object to move sideways.

For us the calculation of the force acting on the rotating cylinder will be a good warm-up for our later task of calculating the lift force acting on an aerofoil. We will meet the concept of *circulation*, which is the quantity that determines the strength of the lift force.

You find content related to this lecture in the textbooks:

- Acheson (1990) sections 1.6 and 4.5
- Batchelor (2000) section 6.6

14.1 Flow around a cylinder

We are on the way to describing the flow around an aerofoil. We want the flow to have the following three properties:

1. The flow is two-dimensional, incompressible and irrotational.
2. Far away from the aerofoil, the flow should be uniform and straight.
3. On the surface of the aerofoil, the flow follows the surface.

We will now argue that the flow described by the complex potential

$$w(z) = a\left(z + \frac{R^2}{z}\right) \tag{14.1}$$

has these properties, albeit for a cylindrical aerofoil of radius R centred on the origin, as depicted in the first picture in Figure 14.1. We'll see in the next lecture how we can deform the cylinder to a more aerofoil-like shape. Let us verify the three properties.

1. Any flow described by a complex potential is two-dimensional, incompressible and irrotational.
2. Far away from the origin, $|z| \ll 1$ and the term in w that is proportional to $1/z$ becomes irrelevant and $w(z) \approx az$. We have already seen in Example 13.4 that this corresponds to a uniform horizontal flow.
3. The flow is a steady flow, so it follows the streamlines. The streamlines are lines where the stream function is constant

   So we need to show that the stream function is constant on the circle of radius R. The stream function is the imaginary part of the complex potential, so we evaluate the complex potential at all points of the form $z = Re^{i\theta}$:

   $$\begin{aligned} w\left(Re^{i\theta}\right) &= a\left(Re^{i\theta} + \frac{R^2}{Re^{i\theta}}\right) \\ &= aR\left(e^{i\theta} + e^{-i\theta}\right) = aR\cos\theta. \end{aligned} \tag{14.2}$$

   Thus, on the circle of radius R the complex potential has no imaginary part, i.e., the stream function is zero there, hence constant.

14.2 Circulation

We now look at a flow around a rotating cylinder, in which the air close to the cylinder is rotating along with it, as depicted in the second flow in Figure 14.1. [1] We start with the complex potential

$$w(z) = ia\log z. \tag{14.3}$$

This is holomorphic outside the cylinder.

[1] Of course in a perfectly inviscid fluid the cylinder would not be able to drag the fluid along with it due to the absence of shear forces, so we can not describe how this circulating flow is created, but once it is established we can describe it.

We find the velocity field from the derivative

$$\begin{aligned} \frac{dw}{dz} &= \frac{ia}{z} = \frac{ia}{x+iy} = \frac{ia(x-iy)}{x^2+y^2} \\ &= \frac{ay}{x^2+y^2} - i\frac{-ax}{x^2+y^2}. \end{aligned} \tag{14.4}$$

Introducing $r^2 = x^2 + y^2$ we read off the velocity components

$$u_x = \frac{a}{r^2}y \quad \text{and} \quad u_y = -\frac{a}{r^2}x. \tag{14.5}$$

This is similar to the circular flow from Example 10.1 just with an extra factor of $-a/r^2$, so this flow too is a circular flow, but with its angular velocity Ω decreasing with increasing distance from the centre, $\Omega = -a/r$. It is easy to look at this in polar coordinates: we simply need to write $z = re^{i\theta}$. We then have

$$\begin{aligned} w(z) &= ia \log z = ia \log\left(re^{i\theta}\right) \\ &= ia\left(\log r + i\theta\right) = -a\theta + ia\log r. \end{aligned} \tag{14.6}$$

So the velocity potential is $\phi = -a\theta$ and the stream function is $\psi = a\log r$. The streamlines are lines of constant ψ, so lines of constant r, hence circles centred on the origin.

Remember that any flow described by a complex potential is irrotational. Yes, the fluid rotates around the origin, but an extended particle placed in the fluid does not rotate around its own axis. Now, if you do not believe this, in spite of the maths, you can also do the experiment. You can create something close to this flow by filling your bathroom sink and then pulling out the stopper. You will observe that the angular velocity of the water decreases with the distance from the plughole. If you glue together two matches to form a cross and place them on the water, you will see the cross circling the plughole, but without rotating around its own axis. Instead its orientation will stay constant. The vorticity of the flow is zero. To make the situation even more perverse, this flow is called the **line vortex flow**, in spite of its zero vorticity.

While the line vortex flow is irrotational, it has non-zero **circulation**.

!

Definition 14.1. The **circulation** Γ of a velocity field along a closed curve C is the line integral of the velocity field along that closed curve,

$$\Gamma = \oint_C \underline{u} \cdot d\underline{x}. \tag{14.7}$$

Thus the circulation is a measure of the net flow of fluid around the curve, and it contains information about the strength and direction of the fluid motion. If as the closed curve we choose the surface of an obstacle to the flow, like the cylinder in our example, we get the circulation around that obstacle. We will see that the lift force experienced by the object is proportional to that circulation.

The circulation is a scalar quantity that can be positive, negative, or zero, depending on the direction and strength of the fluid flow. If the fluid flows around the curve in a clockwise direction, the circulation is negative, and if it flows in a counterclockwise direction, the circulation is positive. If there is no net flow around the curve, the circulation is zero.

According to Stoke's theorem,

$$\oint_C \underline{u} \cdot d\underline{x} = \int_S (\underline{\nabla} \times \underline{u}) \cdot dS, \tag{14.8}$$

where S is a surface enclosed by C. If the flow is irrotational, so that $\underline{\nabla} \times \underline{u} = \underline{0}$, this seems to indicate that the circulation will always be zero. However this argument only works if the vector field is defined everywhere on the surface S. This will not be the case if the curve C encircles a line along which the vector field is not defined. This is the case for the line vortex flow, which is undefined along the entire z axis.

For a two-dimensional flow with a complex potential w, the line integral for the circulation Γ in Eq. 14.7 can be written as a contour integral in the complex plane. To derive this we start with the fact that $dw/dz = u_x - iu_y$. Hence

$$\begin{aligned} \oint_C \frac{dw}{dz} dz &= \oint_C (u_x - iu_y)(dx + idy) \\ &= \oint_C (u_x\, dx + u_y\, dy) - i \oint_C (u_y\, dx - u_x\, dy) \end{aligned} \tag{14.9}$$

Thus

$$\Gamma = \oint_C \underline{u} \cdot d\underline{x} = \oint_C (u_x\, dx + u_y\, dy) = \mathrm{Re}\left(\oint_C \frac{dw}{dz} dz\right). \tag{14.10}$$

In the example of the vortex flow with the complex potential $w = ia \log z$ this gives

$$\Gamma = \mathrm{Re}\left(\oint_C \frac{ia}{z} dz\right) = -2\pi\, a. \tag{14.11}$$

The contour integral was easy to perform thanks to Cauchy's residue theorem. It is nice to see how vector calculus and complex analysis merge to be helpful in fluid dynamics.

14.3 Magnus effect

We now explain the Magnus effect, whereby a spinning cylinder in an air flow will experience a force perpendicular to the airflow. This is similar to the phenomenon of curve balls, except that the flow around the cylinder can be described by a two-dimensional flow unlike the flow around a spinning ball.

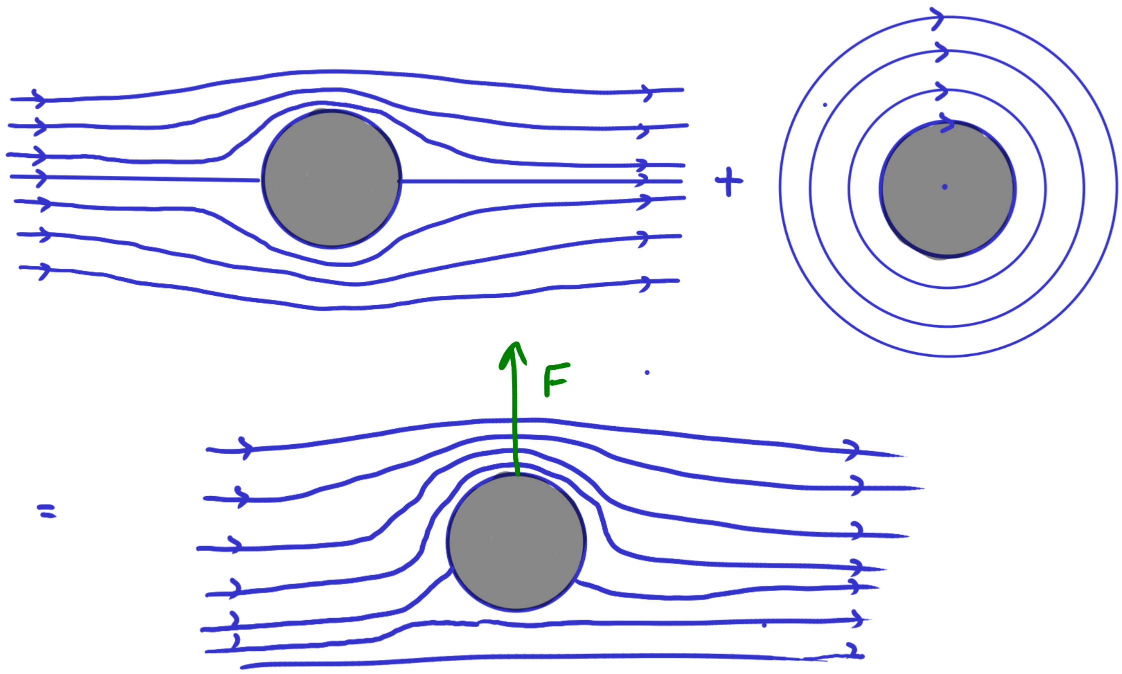

Figure 14.1: The flow around a rotating cylinder moving through still air is obtained as the sum of the flows from the previous sections, which individually did not create a lift force.

The idea is that near the rotating cylinder the air is dragged along, leading to a flow as depicted in Figure 14.1, with a faster flow above the cylinder than below the cylinder. This flow can be obtained by adding a line vertex flow (see Section 14.2) to the flow around the non-rotating cylinder (see Section 14.1. If we add two complex potentials, we again obtain a valid complex potential (because the sum of two holomorphic functions is again holomorphic). So we use the complex potential

$$w(z) = a\left(z + \frac{R^2}{z}\right) + ib\log z. \tag{14.12}$$

The stagnation points – the points at which the flow velocity is zero – are where the velocity of the line-vortex flow exactly cancels the velocity of the flow around the non-rotating cylinder. They are moved downwards by the clockwise rotation of the cylinder.

Bernoulli's theorem for irrotational flow gives us the pressure as

$$p = -\frac{\rho}{2} u^2 + \text{ constant.} \tag{14.13}$$

We do not care about the value of the constant because the force is caused only by pressure differences. In order to calculate the force acting on the cylinder we need to calculate the pressure everywhere on the surface of the cylinder, so we need the velocities there. We calculate these by taking the derivative of the complex potential and evaluating it at $z = Re^{i\theta}$,

$$\begin{aligned} \frac{dw}{dz} &= a\left(1 - \frac{R^2}{z^2}\right) + \frac{ib}{z} = a\left(1 - e^{-2i\theta}\right) + \frac{ib}{R} e^{-i\theta} \\ &= e^{-i\theta}\left(2ia\sin(\theta) + \frac{ib}{R}\right). \end{aligned} \tag{14.14}$$

From this we get the square of the velocity as

$$\begin{aligned} u^2 &= u_x^2 + u_y^2 = (u_x - iu_y)(u_x + iu_y) = \left|\frac{dw}{dz}\right|^2 \\ &= \left(2a\sin\theta - \frac{b}{R}\right)^2 \end{aligned} \tag{14.15}$$

and thus the pressure at the surface of the cylinder is

$$p(\theta) = -2\rho\left(a^2\sin^2\theta - a\frac{b}{R}\sin\theta\right) + \text{ constant.} \tag{14.16}$$

This now allows us to calculate the force dF on an infinitesimal angular segment of the cylinder between θ and $\theta + d\theta$. The force is constant along the length of the cylinder, so we calculate just the force per unit length. According to the definition of the pressure this force is

$$d\underline{F}(\theta) = -p(\theta)\,\underline{n}\,dl = -p(\theta)\,\underline{n}R\,d\theta. \tag{14.17}$$

where $\underline{n}$ is the outwards normal. We are interested in the y component of this:

$$dF_y(\theta) = -p(\theta)\,R\sin\theta\,d\theta. \tag{14.18}$$

Integrating this over the entire circle of radius R gives

$$\begin{aligned} F_y &= -\int_0^{2\pi} p(\theta)\,R\sin\theta\,d\theta \\ &= \int_0^{2\pi} 2\rho\left(a^2\sin^3\theta - a\frac{b}{R}\sin^2\theta\right) R\,d\theta \\ &= 2\pi\rho\,a\,b. \end{aligned} \tag{14.19}$$

So we have derived that the cylinder experiences a non-zero lift force that is proportional to the air density, the velocity of the air flow, and the angular velocity of the cylinder.

I encourage you to also calculate the x component of the force and to discuss the result.

14.4 A comment on units and dimensions

When describing the real world, quantities have units and dimensions. For example, the radius of a cylinder has dimension of length and can be measured in either meters or feet or centimetres or ... There are three ways to deal with this:

1. Agree on a system of units and measure everything in terms of these units. The most common system is the SI system, where length is measured in meters (m), time - in seconds (s), mass - in kilograms (kg). So we would write

   $$\text{radius } = R \text{ meters}$$

   and then work with R in our formulas. This seems natural to a mathematician because it means that the mathematician never has to think about units and dimensions because there is an agreed way to express all quantities as just numbers. But one looses some benefits.

2. Choose units that are natural for the problem at hand. For example if the radius of the cylinder is the most important length in the problem, then it is natural to measure all other lengths as multiples of this radius. This has the nice effect that

   $$\text{radius } = 1 \text{ radius.}$$

   None of the formulas will ever have to involve a parameter R. By similarly choosing a natural time and a natural mass, we can get rid of 3 parameters. This is why applied mathematicians sometimes like this approach because it leads to simpler expressions. It goes by the name of “non-dimensionalisation”. Theoretical physicists use it to set $c = \hbar = G = 1$.

3. Don't specify units and keep all quantities as dimensionful quantities. So

   $$\text{radius } = R.$$

   This has the disadvantage over the previous approach that it does not reduce the number of parameters. But it has the advantage that we have an easy way to check whether the equations we derive make sense. If in any equation the different terms do not all have the same dimension, then we know we have made a mistake. For people like me who make mistakes very easily, this is very valuable.

It is clear that the first approach above has no advantages, because it neither decreases the number of parameters in the equations nor gives us a way to use the dimensions of the terms in the equation as a check. Nevertheless, unfortunately, it is often adopted by mathematicians and we did so in this lecture. Take a look for example in Eq. 14.12 :

$$w(z) = a\left(z + \frac{R^2}{z}\right) + ib\log z.$$

The fact that it contains $\log z$ shows that z is dimensionless (because you can't take the log of a dimensionful number). Thus we did not take approach 3. The fact that it still contains R means that we did not take approach 2, i.e., z is not measured in multiples of R.

We will switch to using approach 3 for the rest of the module to always have a check on the results.

15 Why airplanes fly

Our calculation of the lift on the rotating cylinder in the previous section was very specific to the particular example. In this section we will see that, by using complex analysis techniques, we can get an expression for the lift force that works more generally. The result turns out to be independent of the details of the flow and only involves the flow velocity far from the aerofoil and the circulation around the aerofoil.

You will not be examined on the material in this section. But I hope that it shows you a nice example of how powerful advanced mathematics (complex analysis in this case) can be when brought to bear on an applied problem (the lift on an airplane wing in this case).

You find content related to this lecture in the textbooks:

- Acheson (1990) sections 4.6 to 4.11
- Batchelor (2000) section 6.6
- Paterson (1983) sections 16.5 and chapter 17

15.1 Blasius theorem

Theorem 15.1 (Blasius theorem)**.** *The force $\underline{F}$ per unit length on an aerofoil in an incompressible, irrotational flow described by a complex potential $w(z)$ is given by*

$$F_x - iF_y = i\frac{\rho}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz, \tag{15.1}$$

where C is any contour encircling the aerofoil.

In this subsection we will derive this expression. For this purpose we first consider the force $d\underline{F}$ on an infinitesimal segment of length dl of the cross section of the aerofoil. In terms of the angle θ that is indicated in Figure 15.1 we have

$$dF_x = p\cos\theta\, dl, \quad dF_y = -p\sin\theta\, dl, \tag{15.2}$$

and thus

$$dF_x - i\, dF_y = p\,(\cos\theta + i\sin\theta)\, dl = p\, e^{i\theta} dl. \tag{15.3}$$

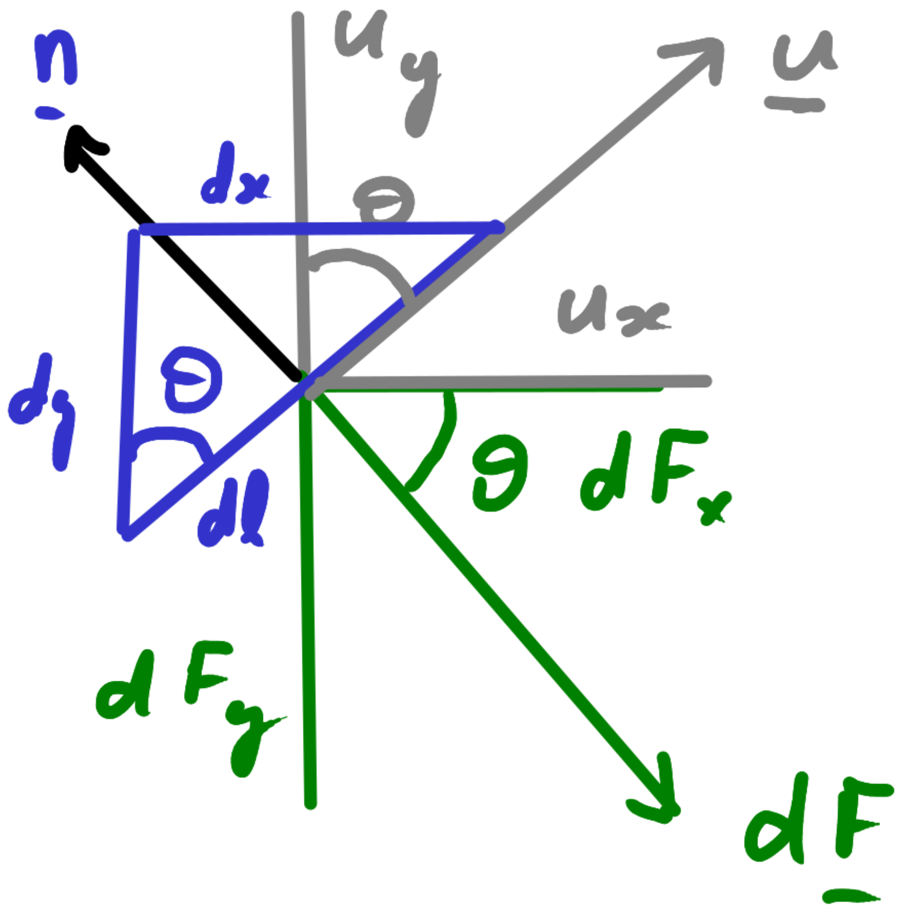

Figure 15.1: Force $d\underline{F}$ acting on an infinitesimal segment dl.

Notice how by reinterpreting the x-y plane as the complex plane we can avoid having to deal with sines and cosines.

The flow is tangential to the aerofoil, so the velocity components are

$$u_x = |u| \sin\theta, \quad u_y = |u| \cos\theta. \tag{15.4}$$

We can relate that to the derivative of the complex potential,

$$\frac{dw}{dz} = u_x - iu_y = |u| (\sin\theta - i\cos\theta) = -i|u|e^{i\theta}, \tag{15.5}$$

and thus

$$|u| = ie^{-i\theta}\frac{dw}{dz}. \tag{15.6}$$

This makes it easy to calculate

$$u^2 = -e^{-2i\theta}\left(\frac{dw}{dz}\right)^2. \tag{15.7}$$

We can use this in Bernoulli's theorem for irrotational flow (in the absence of body forces) to find the pressure

$$p = -\frac{\rho}{2}u^2 + constant = \frac{\rho}{2}e^{-2i\theta}\left(\frac{dw}{dz}\right)^2 + constant. \tag{15.8}$$

The value of the constant is irrelevant because only pressure differences create forces. Substituting this expression

$$\begin{aligned} dF_x - i\,dF_y &= p(\cos\theta + i\sin\theta)dl = pe^{i\theta}\,dl \\ &= \frac{\rho}{2}\left(\frac{dw}{dz}\right)^2 e^{-i\theta}dl \\ &= \frac{\rho}{2}\left(\frac{dw}{dz}\right)^2 (\cos\theta\,dl - i\sin\theta\,dl) \\ &= i\,\frac{\rho}{2}\left(\frac{dw}{dz}\right)^2 dz. \end{aligned} \tag{15.9}$$

where we used that

$$dx = -\sin\theta\,dl, \quad dy = -\cos\theta\,dl \quad \text{and} \quad dz = dx + i\,dy. \tag{15.10}$$

We can now integrate this along the aerofoil to get the total force on the aerofoil as

$$F_x - iF_y = i\,\frac{\rho}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz. \tag{15.11}$$

Because we are integrating a holomorphic function, the result will be the same no matter which contour C we are using, as long as it encircles the aerofoil once in a counterclockwise direction. We have thus derived Blasius' theorem.

15.2 Kutta-Joukowski lift theorem

We now simplify the expression for the force acting on an aerofoil that we derived in the previous subsection even further by considering the Laurent expansion

$$\frac{dw}{dz} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots. \tag{15.12}$$

(You can find a discussion of Laurent expansions in your lecture notes from 'Functions of a complex argument'.) We know that the velocity far away from the aerofoil (which we will denote as $\underline{u}(\infty)$) has to be finite, and therefore the Laurent expansion of dw/dz can not have any terms with positive powers of z. Far away from the aerofoil we have

$$\lim_{|z|\to\infty} \frac{dw}{dz} = a_0 = u_x(\infty) - i\, u_y(\infty). \tag{15.13}$$

We also get the Laurent expansion

$$\left(\frac{dw}{dz}\right)^2 dz = a_0^2 + \frac{2a_0 a_1}{z} + \frac{a_1^2 + 2a_0 a_2}{z^2} + \dots. \tag{15.14}$$

Using this in Blasius theorem and using Cauchy's residue theorem gives us

$$F_x - iF_y = i\,\frac{\rho}{2} \oint_C \left(\frac{dw}{dz}\right)^2 dz = -2\pi\rho\, a_0\, a_1. \tag{15.15}$$

We get a_1 from the contour integral of dw/dz:

$$\begin{aligned} 2\pi i a_1 &= \oint_C \frac{dw}{dz} dz = \oint_C (u_x - i\, u_y)(dx + i\, dy) \\ &= \oint (u_x dx + u_y dy) + \oint_C (iu_x dy - iu_y dx) \\ &= \oint_C \underline{u} \cdot d\underline{x} + i \oint_C \underline{\nabla}\psi \cdot d\underline{x}. \end{aligned} \tag{15.16}$$

The second term vanishes. To see this consider integrating along the aerofoil. Because the flow follows the surface of the aerofoil, this means that that surface is a streamline. Thus the streamfunction ψ is constant along the contour, hence $\underline{\nabla}\psi \cdot d\underline{x} = 0$. The remaining first term is just the circulation Γ. Hence we have that

$$a_1 = \frac{\Gamma}{2\pi i}. \tag{15.17}$$

Substituting our results for a_0 and a_1 into Eq. 15.15 gives

$$F_x - iF_y = i\rho(u_x(\infty) - iu_y(\infty))\Gamma, \tag{15.18}$$

from which we can extract the real and imaginary parts to get our final result:

$$F_x = \rho\,\Gamma u_y(\infty), \quad F_y = -\rho\,\Gamma u_x(\infty). \tag{15.19}$$

As promised, the forces depend only on the circulation and the velocity far away from the aerofoil.

Let us check that this reproduces the results from Section 14.3 on the Magnus effect. There we had $u_y(\infty) = 0$ and thus $F_x = 0$. Thus there is no drag force. (In the real world of course there is drag, which will be created by the turbulent wake behind the cylinder, but we are not modelling that.) We also had $u_x(\infty) = a$ and $\Gamma = -2\pi b$ so that $F_y = 2\pi\rho ab$, which reproduces the old result in Eq. 14.19 .

15.3 Conformal mappings

From the previous lecture we have a flow around a cylindrical aerofoil. We want the flow around a less symmetrical shape that is rounded at the front but has a tip at the end. This is illustrated in Figure 15.2. We are going to use a conformal mapping that maps from one to the other.

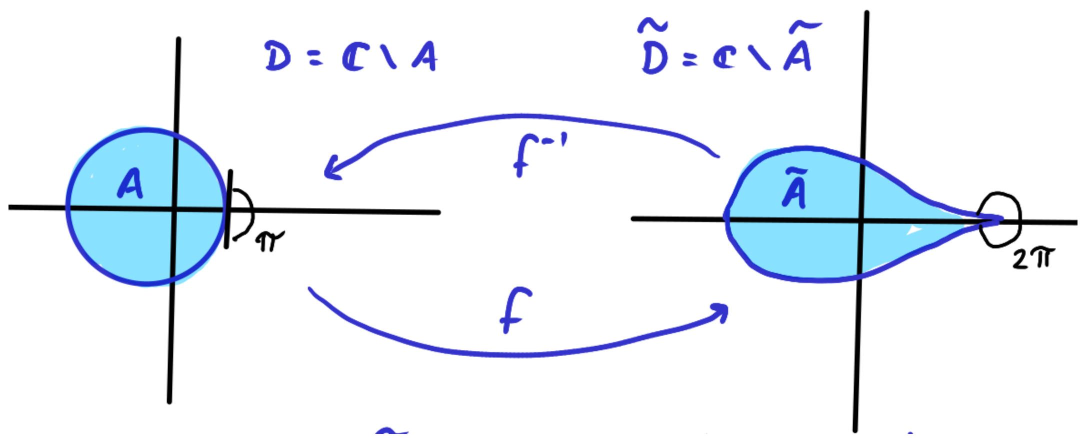

Figure 15.2: A conformal mapping f from the region D outside a disk A to the region $\tilde{D}$ outside the cross section $\tilde{A}$ of an aerofoil.

Let A be the cross section of a cylindrical aerofoil, i.e., a closed disc, with radius R, centred at $z = -\lambda$. Let $\tilde{A}$ be the desired cross section of the aerofoil. Then $D = \mathbb{C} \quad A$ and $\tilde{D} = \mathbb{C} \quad \tilde{A}$ are the regions outside the aerofoils.

Let $f : D \to \tilde{D}$ be a holomorphic invertible function. View this as a map from the region in one complex plane to another, which we will refer to as the z plane and the $\tilde{z}$ plane respectively, where $\tilde{z} = f(z) = \tilde{x} + i\tilde{y}$. We want to know what happens to the fluid flow under this map.

We introduce the complex potential on $\tilde{D}$ to be everywhere identical to the complex potential at the corresponding point in D, i.e.,

$$\tilde{w}(\tilde{z}) = w(z) = w(f^{-1}(\tilde{z})). \tag{15.20}$$

This is a holomorphic function on $\tilde{D}$. It gives us the vector potential $\tilde{\phi}$ and the stream function $\tilde{\psi}$ on $\tilde{D}$ as usual:

$$\tilde{w}(\tilde{z}) = \tilde{\phi}(\tilde{x}, \tilde{y}) + i\tilde{\psi}(\tilde{x}, \tilde{y}). \tag{15.21}$$

Because the stream function $\tilde{\psi}(\tilde{z})$ on $\tilde{D}$ takes the same values as the original stream function ψ on the corresponding point on D, the stream lines on D get mapped by f to streamlines on $\tilde{D}$. We can find the mapped velocity field on $\tilde{D}$ from the complex potential $\tilde{w}$ as

$$\begin{aligned} \frac{d\tilde{w}}{d\tilde{z}} &= \tilde{u}_x - i\tilde{u}_y = \frac{d}{d\tilde{z}} w\left(f^{-1}(\tilde{z})\right) \\ &= \frac{dw}{dz}\frac{df^{-1}(\tilde{z})}{d\tilde{z}} = \frac{dw/dz}{f'}, \end{aligned} \tag{15.22}$$

where we used that the derivative of the inverse function f^{-1} is the reciprocal of the derivative of the function f. Thus we find that the components of the velocity field are simply rescaled by f' at every point.

We now consider how angles behave under the mapping f. See Figure 15.3 for a drawing illustrating the following discussion. We consider an infinitesimal line from some point z_0 to point $z_0 + \delta z$ for some infinitesimally small δz. This gets mapped into a line from $\tilde{z}_0 = f(z_0)$ to

$$\begin{aligned} \tilde{z}_0 + \delta\tilde{z} &= f(z_0 + \delta z) \\ &= f(z_0) + \delta z\, f'(z_0) + \frac{(\delta z)^2}{2} f''(z_0) + \\ &\qquad \cdots + \frac{(\delta z)^n}{n!} f^{(n)}(z_0) + \cdots \end{aligned} \tag{15.23}$$

In the case where $f^{(n)}(z_0) = 0$ for all $n < m$ and $f^{(m)}(z_0) \neq 0$ this gives

$$\delta\tilde{z} = f(z_0 + \delta z) = \frac{(\delta z)^m}{m!} f^{(m)}(z_0) + \text{ higher-order terms.} \tag{15.24}$$

The higher-order terms can be neglected for infinitesimal δz. For the angle that the line segments make with the horizontal we find

$$\arg(\delta\tilde{z}) = m \arg(\delta z) + \arg(f^{(m)}(z_0)). \tag{15.25}$$

Let us now consider a second infinitesimal line element from z_0 to $z_0 + \delta z_2$, which gets mapped into a line from $\tilde{z}_0$ to $\tilde{z}_0 + \delta\tilde{z}_2$ and consider the angle

$$\theta = \arg(\delta z_2) - \arg(\delta z) \tag{15.26}$$

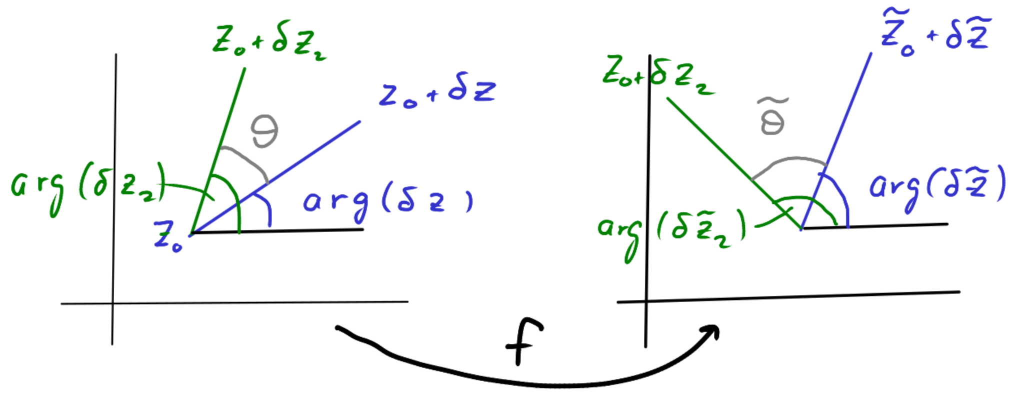

Figure 15.3: A figure illustrating the change in angle under the holomorphic mapping f.

between them. This gets mapped to

$$\begin{aligned}\tilde{\theta} &= \arg(\delta\tilde{z}_2) - \arg(\delta\tilde{z}) \\ &= m\,\arg(\delta z_2) - m\,\arg(\delta z) = m\,\theta.\end{aligned} \tag{15.27}$$

Thus the angle is multiplied by the integer m. But $f'(z) \neq 0$ for all z in D because otherwise f would not be invertible. Hence $m = 1$ and all angles are preserved. This is why such mappings by holomorphic invertible functions are called **conformal mappings**.

15.4 Joukouwski mapping

We want the aerofoil to have a cusp at the trailing edge, see Figure Figure 15.2. We will see soon why that is important. At the cusp the angle between the upper and lower surfaces of the aerofoil is 2π. In the original cylindrical aerofoil the angle is everywhere equal to π. So we need to choose a conformal map f that, when extended to the surface of the aerofoil, has a vanishing derivative at the point $c = R - \lambda$ but a non-vanishing second derivative,

$$f'(c) = 0 \quad \text{and} \quad f''(c) \neq 0, \tag{15.28}$$

because then, according to Eq. 15.27 , the angles at $z = c$ are doubled.

We also want the conformal mapping to not affect the flow far away from the aerofoil. We know from Eq. 15.22 that the velocity is scaled by a factor of f', so we need

$$\lim_{|z|\to\infty} f'(z) = 1. \tag{15.29}$$

A function f that satisfies the conditions in Eq. 15.28 and Eq. 15.29 is

$$f(z) = z - \frac{c^2}{z}. \tag{15.30}$$

It is holomorphic on D because the poles at $z = \pm c$ are not in D. The point $z = -c$ lies inside the aerofoil and the point $z = c$ on its surface.

We now start with the flow around the cylinder discussed in the section on the Magnus effect, described by the complex potential

$$w(z) = a\left(z + \frac{R^2}{z}\right) + ib\log z. \tag{15.31}$$

We rotate everything by an angle α counterclockwise and shift the centre of the cylinder to the left by λ . This is achieved by replacing z by $(z+\lambda)e^{-i\alpha}$. We also make the argument of the logarithm dimensionless by dividing by R, see discussion in section Section 14.4. This thus gives the flow on the left in Figure 15.4.

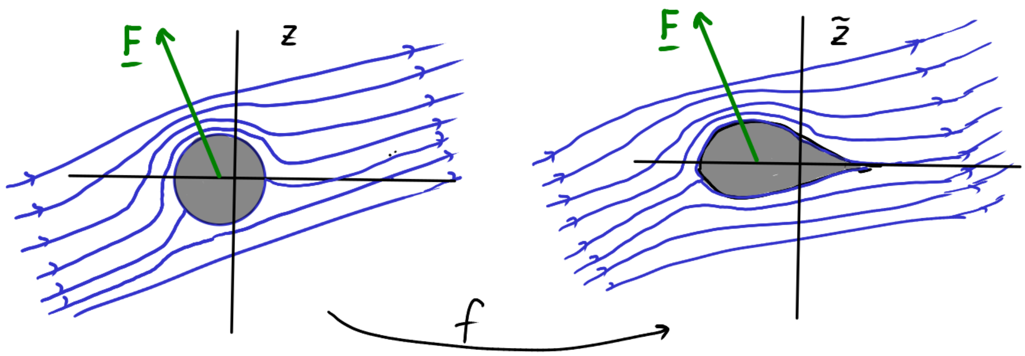

Figure 15.4: The Joukowski mapping f maps the flow around the cylinder to the flow around the aerofoil.

We set $b = -\Gamma/(2\pi R)$. So we use

$$w(z) = a\left((z+\lambda)e^{-i\alpha} + \frac{R^2}{z+\lambda}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi R}\log\frac{z+\lambda}{R}. \tag{15.32}$$

From the derivative of this complex potential

$$\frac{dw}{dz} = a\left(e^{-i\alpha} - \frac{R^2}{(z+\lambda)^2}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\frac{1}{z+\lambda} \tag{15.33}$$

we see that far away we have constant flow at an angle α to the horizontal:

$$\begin{aligned}\lim_{|z|\to\infty}\frac{dw}{dz} &= a\,e^{-i\alpha} = a\cos\alpha - ia\sin\alpha \\ &= u_x(\infty) - iu_y(\infty).\end{aligned} \tag{15.34}$$

By calculating the contour integral along a contour surrounding the aerofoil using Cauchy's residue theorem we find that the circulation is

$$\oint_C \frac{dw}{dz} dz = 2\pi i \left(-\frac{i\Gamma}{2\pi} \right) = \Gamma. \tag{15.35}$$

We can now use Eq. 15.22 to find the velocity field after applying the conformal mapping f:

$$\begin{aligned} \frac{d\tilde{w}}{d\tilde{z}} &= \frac{dw/dz}{f'} \\ &= \frac{a\left(e^{-i\alpha} - \dfrac{R^2}{(z+\lambda)^2} e^{i\alpha}\right) - \dfrac{i\Gamma}{2\pi} \dfrac{1}{z+\lambda}}{1 - \dfrac{(R-\lambda)^2}{z^2}}. \end{aligned} \tag{15.36}$$

We note that due to the zero of the denominator at $z = R - \lambda$ the velocity is infinite at that point unless the numerator vanishes at that point, i.e., unless

$$\Gamma = -4\pi a R \sin \alpha. \tag{15.37}$$

The cusp at the rear of the aerofoil forces the airflow to have this particular amount of circulation.

We can therefore now use the Kutta-Joukowski theorem to determine the lift force:

$$\begin{aligned} F_x &= \rho \Gamma u_y(\infty) = 4\pi\rho\, a^2\, R \sin \alpha(-\sin \alpha), \\ F_y &= -\rho \Gamma u_y(\infty) = 4\pi\rho\, a^2\, R \sin \alpha(\cos \alpha). \end{aligned} \tag{15.38}$$

We used the fact that the mapping did not change the velocity far away and did not change the circulation because $\tilde{w}(\tilde{z}) = w(z)$ and thus

$$\oint_C \frac{d\tilde{w}}{d\tilde{z}} d\tilde{z} = \oint_{f^{-1}(C)} \frac{dw}{dz} dz = \Gamma, \tag{15.39}$$

due to the fact that the contour $f^{-1}(C)$ also surrounds the cylinder in the counterclockwise direction.

We have thus found that the lift force is acting at an angle α to the vertical and has a magnitude of

$$4\pi\rho\, a^2\, R \sin \alpha. \tag{15.40}$$

16 Surface waves 1

We will now look at an example that combines the two topics of this module: waves and fluids. We will look at the waves on the surface of water. As waves move over the surface, of course the fluid below the surface needs to rearrange, so the shape of the wave and the flow of the fluid are coupled.

You find content related to this lecture in the textbooks:

- Acheson (1990) section 3.2
- Baldock and Bridgeman (1983) sections 10.1, 10.2 and 10.3.1
- Paterson (1983) sections 13.1 - 13.3
- Braithwaite (2017) section 5.1

If we use the y coordinate as the vertical coordinate we can describe the surface by the equation $y = \eta(x, z, t)$ for some function η. We will simplify our calculations by looking for plane waves. We choose the z axis to point in the direction along which the plane wave is constant. Then we can describe the surface by a function $\eta(x, t)$ that is independent of z. This is depicted in Figure 16.1. This will lead to a two-dimensional flow in the fluid that we can describe by a velocity field $\underline{u} = (u_x(x, y, t), u_y(x, y, t), 0)$.

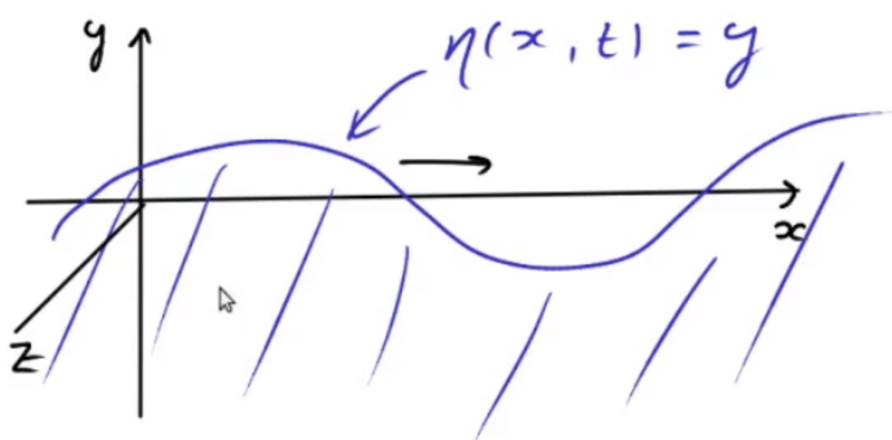

Figure 16.1: Plane wave on the surface of water described by $y = \eta(x, t)$.

We further simplify the problem by assuming that the water is infinitely deep and infinitely extended in the x and z directions, so that we do not have to impose any boundary conditions except at the surface. This is akin to how in the waves section of this module we first looked at the infinite string.

We will assume that the flow of the water is irrotational. This is certainly justified if we think of surface waves created for example by wind blowing over the surface of water at rest. This means that we can describe the flow by a velocity potential $\phi(x, y, t)$ so that $\underline{u} = \underline{\nabla}\phi$, or, in components, $u_x = \partial_x\phi$ and $u_y = \partial_y\phi$.

We describe the water as an ideal fluid. In particular we treat water as incompressible, which implies that the flow is divergence free:

$$\underline{\nabla} \cdot \underline{u} = \underline{\nabla} \cdot \underline{\nabla}\phi = \underline{\nabla}^2\phi = \partial_x^2\phi + \partial_y^2\phi = 0. \tag{16.1}$$

This is the Laplace equation. The Laplace equation has many other applications in both pure and applied mathematics. Solutions of the Laplace equation are known as *harmonic functions*. You will meet them again in many other modules.

We have seen that the real part of any holomorphic function satisfies the Laplace equation. But we are looking not for an arbitrary incompressible flow but one that fits the wave on its surface. This means that next we need to write down equations that couple the shape of the surface described by η to the flow of the water described by $\underline{u}$ or ϕ. These are the surface conditions.

16.1 Surface conditions

We consider the interface between the water and the air. We use the fact that the density of the air is negligible compared to that of water and thus treat the surface as a free surface. At such a surface there are two conditions: the kinematic surface condition and the dynamic surface condition, which we will introduce in the next two subsections.

16.1.1 Kinematic condition

The kinematic condition states that fluid particles on the surface stay on the surface.

If we introduce the distance d of a point from the surface

$$d(x, y, t) = y - \eta(x, t) \tag{16.2}$$

then the particles at the surface have $d = 0$ and this will stay constant as the particle moves around, i.e.,

$$\begin{aligned} \frac{Dd}{Dt} &= \partial_t d + \underline{u} \cdot \underline{\nabla} d = \partial_t d + u_x\partial_x d + u_y\partial_y d \\ &= -\partial_t\eta - u_x\partial_x\eta + u_y = 0. \end{aligned} \tag{16.3}$$

This is the kinematic surface condition and holds at all points at the surface, i.e., all points with $y = \eta(x, t)$.

16.1.2 Dynamic condition

The dynamic condition states that the pressure at the surface equals the atmospheric pressure p_0.

According to Euler's equation

$$\partial_t \underline{u} + (\underline{\nabla} \times \underline{u}) \times \underline{u} = -\underline{\nabla} \left(\frac{p}{\rho} + \frac{1}{2} u^2 + \chi \right), \tag{16.4}$$

where p is the pressure, ρ is the density and χ is the gravitational potential. The left-hand side simplifies for an irrotational flow described by a velocity potential ϕ,

$$\underline{\nabla}(\partial_t \phi) = -\underline{\nabla} \left(\frac{p}{\rho} + \frac{1}{2} u^2 + \chi \right). \tag{16.5}$$

We can integrate this to give

$$\partial_t \phi + \frac{p}{\rho} + \frac{1}{2} u^2 + \chi = G(t) \tag{16.6}$$

for some integration constant $G(t)$. (Note that by 'constant' in this constant we mean independent of the spatial variables. A time dependence does not affect the gradient.) The gravitational potential is acting in the negative y direction, so $\chi = g\,y$. At the surface $y = \eta$ we have $p = p_0$, so

$$\partial_t \phi + \frac{1}{2} u^2 + g\eta = G(t) - \frac{p_0}{\rho}, \tag{16.7}$$

where we have collected all the constant terms on the right-hand side. We now use that shifting a potential by a constant does not make a difference because it is only the gradient of the potential that is relevant. We can thus simplify the equation by shifting $\phi \to \phi + s(t)$ where we choose $s(t)$ such that $s'(t) = G(t) - p_0/\rho$. This new potential satisfies

$$\partial_t \phi + \frac{1}{2} u^2 + g\eta = 0 \tag{16.8}$$

This is the dynamical surface condition and holds at all points at the surface, i.e., the points with $y = \eta(x, t)$.

16.2 Linear approximation

The surface conditions contain non-linear terms. That makes them impossible to solve analytically. So we linearise the equations. For the kinematic surface condition Eq. 16.3 this means

$$-\partial_t \eta - u_x \partial_x \eta + u_y = 0 \longrightarrow -\partial_t \eta + u_y = 0. \tag{16.9}$$

We assume that the term that we are dropping is small, something we will need to verify later. Even after dropping the quadratic term, there is still a complicated dependence on η which we need to drop as well:

$$\partial_t\eta(x,t) = u_y(x,\eta(x,t),t) \longrightarrow \partial_t\eta(x,t) = u_y(x,0,t). \tag{16.10}$$

This is the linearised kinematic surface condition that we will use.

Similarly we linearise the dynamic surface condition to

$$\partial_t\phi(x,0,t) = -g\eta(x,t). \tag{16.11}$$

16.3 Harmonic travelling wave solution

Our task now is to solve the equations 16.1, 16.10 and 16.11. We will do this with the harmonic travelling wave Ansatz that we discussed in lecture 5[1]:

$$\eta(x,t) = A\cos(kx-\omega t). \tag{16.12}$$

Then in order to have a chance of satisfying Eq. 16.11 we need to make the following Ansatz for ϕ:

$$\phi(x,y,t) = f(y)\sin(kx-\omega t), \tag{16.13}$$

where $f(y)$ is an as yet undetermined function. To determine it we substitute this Ansatz into the Laplace equation Eq. 16.1. This gives

$$-k^2 f(y)\sin(kx-\omega t) + f''(y)\sin(kx-\omega t) = 0. \tag{16.14}$$

Dividing by $\sin(kx-\omega t)$ leaves us with a very simple ODE for f:

$$f''(y) = k^2 f(y). \tag{16.15}$$

The general solution is

$$f(y) = De^{k\,y} + Ee^{-k\,y} \tag{16.16}$$

for some undetermined constants D and E. However $E = 0$ in the case of infinite deep water because otherwise the function would go to ∞ as $y \to -\infty$.

Substituting into the dynamic surface condition Eq. 16.11 gives

$$-\omega f(0)\cos(kx-\omega t) = -g\,A\cos(kx-\omega t) \tag{16.17}$$

and thus

$$D = \frac{g\,A}{\omega}. \tag{16.18}$$

[1]We could include a phase as in lecture 5 but it would just come along for the ride and not add anything interesting.

Finally, substituting into the kinematic surface condition Eq. 16.10 gives

$$\omega\, A \sin(kx - \omega t) = f'(0) \sin(kx - \omega t). \tag{16.19}$$

Because $f'(0) = kD = k\, g\, A/\omega$, this gives

$$\omega^2 = k\, g \quad \text{or} \quad \omega = \pm\sqrt{k\, g}. \tag{16.20}$$

This is the dispersion relation.

So we have found a harmonic wave solution

$$\eta = A \cos(kx - \omega t) \tag{16.21}$$

for any wave number k and amplitude A, where $\omega = \pm\sqrt{k\, g}$. Thus the wave moves with a velocity of $c = \omega/k = \pm\sqrt{g/k}$. The flow underneath the wave is described by the velocity potential

$$\phi = \frac{A\omega}{k} e^{ky} \sin(kx - \omega t). \tag{16.22}$$

17 Surface waves 2

In this lecture we want to study the wave solution that we derived in the previous lecture. We will first look at the paths that the particles in the fluid follow. Then we will investigate under what condition the linear approximation that we made to simplify the surface conditions is valid. Finally we look at the dispersion in the surface waves.

You find content related to this lecture in the textbooks:

- Acheson (1990) sections 3.2 and 3.3
- Baldock and Bridgeman (1983) section 10.3
- Paterson (1983) section 13.3

17.1 Pathlines

From the velocity potential ϕ in Eq. 16.22 we can read off the components of the velocity field as

$$\begin{aligned} u_x &= \partial_x \phi = A\omega e^{k\,y} \cos(kx - \omega t), \\ u_y &= \partial_y \phi = A\omega e^{k\,y} \sin(kx - \omega t). \end{aligned} \tag{17.1}$$

The factor of $e^{k\,y}$ shows us that the velocity decreases exponentially with depth.

We want to understand how the fluid particles move, i.e., we want to determine the pathlines. These satisfy

$$\frac{\underline{x}(t)}{dt} = \underline{u}(\underline{x}(t), t). \tag{17.2}$$

Unfortunately these nonlinear equations are too complicated to solve analytically. So we linearise. We know from experience that, while the waves on the surface move over long distances, the particles in the fluid are not swept away with the wave but only move up and down and forth and back a little bit. So we write the position of the particle as

$$\underline{x}(t) = \underline{\bar{x}} + \underline{\hat{x}}(t), \tag{17.3}$$

where $\underline{\bar{x}}$ is the mean position and the $\underline{\hat{x}}$ are assumed to be small. Substituting this into Eq. 17.2 and looking at the components gives

$$\begin{aligned} \frac{d\hat{x}}{dt} &\approx A\omega e^{k\,\bar{y}} \cos(k\bar{x} - \omega t), \\ \frac{d\hat{y}}{dt} &\approx A\omega e^{k\,\bar{y}} \sin(k\bar{x} - \omega t). \end{aligned} \tag{17.4}$$

This is easy to integrate to

$$\begin{aligned}\hat{x} &\approx -Ae^{k\,\bar{y}}\sin(k\bar{x}-\omega t),\\ \hat{y} &\approx Ae^{k\,\bar{y}}\cos(k\bar{x}-\omega t).\end{aligned} \tag{17.5}$$

This describes motion counterclockwise around a circle of radius $Ae^{k\,\bar{y}}$ around the mean position. When that radius is small, the pathlines are close to perfect circles. Closer to the surface where the radius gets bigger the circle will get deformed a bit.

17.2 Validity of linear approximation

Now that we have the solutions to the linearised surface conditions we can check under what circumstances the linearisation was a good approximation. For example in Eq. 16.9 we dropped the term $u_x\partial_x\eta$. This is a good approximation only if the dropped term is much smaller in magnitude than the terms that remain. So we need

$$|u_x\partial_x\eta| \ll |u_y|. \tag{17.6}$$

Substituting our solutions from Eq. 17.1 and Eq. 16.21 gives

$$|A\omega e^{k\,\eta}\cos(kx-\omega t)A\,k\sin(kx-\omega t)| \ll |A\omega e^{k\,\eta}\sin(kx-\omega t)|. \tag{17.7}$$

We can cancel a sin from both sides. The factors of cosine can be dropped because it only takes values between -1 and 1. So, after cancellations, the condition becomes

$$|A\,k| \ll 1. \tag{17.8}$$

Written in terms of the wavelength λ this becomes

$$A \ll \frac{\lambda}{2\pi}. \tag{17.9}$$

So the amplitude of the wave must be small compared to its wavelength. This turns out to be exactly the same condition that we also needed in our linearisation while deriving the equation for waves on a string in lecture 1.

We made other approximations while we linearised the surface conditions and we need to check whether they too are valid under the same circumstances. For example we replaced $u_y(x,\eta,t)$ by $u_y(x,0,t)$. This is a good approximation if

$$|u_y(x,\eta,t)-u_y(x,0,t)| \ll |u_y(x,0,t)|. \tag{17.10}$$

Substituting our solution gives

$$|A\omega\,(e^{k\,\eta}-1)\sin(kx-\omega t)| \ll |A\omega\sin(kx-\omega t)|, \tag{17.11}$$

which simplifies to

$$\left|e^{k\,\eta}-1\right| \ll 1. \tag{17.12}$$

This holds if $|k\,\eta| \ll 1$ which in turn holds if $|k\,A| \ll 1$.

I leave it up to you to go through the approximations that we made in Section 16.1.2 to verify that they too are good when $|k\,A| \ll 1$.

17.3 Circular waves

You will have observed that when you drop a pebble into a pond, this creates waves that travel outwards from the disturbance with circular wave crests. We can easily find the solutions describing such circular waves by switching to polar coordinates r, θ for the x-z plane such that $x = r\cos\theta$ and $z = r\sin\theta$.

The circular waves we are looking for look the same in all directions, so we will be looking for solutions for the velocity potential ϕ and the surface function η that are independent of the θ coordinate:

$$\phi = \phi(r, y, t), \quad \eta = \eta(r, t).$$

The Laplace equation for the velocity potential ϕ then becomes

$$0 = \nabla^2\phi = \frac{1}{r}\partial_r^2(r\phi) + \partial_y^2\phi. \tag{17.13}$$

Introducing again the distance d from the surface, $d(r, y, t) = y - \eta(r, t)$, the kinematic surface condition in these coordinates is

$$\begin{aligned}\frac{Dd}{Dt} &= \partial_t d + \underline{u}\cdot\underline{\nabla}d = \partial_t d + u_r\partial_r d + u_y\partial_y d \\ &= -\partial_t\eta - u_r\partial_r\eta + u_y = 0.\end{aligned} \tag{17.14}$$

After linearisation this reduces to

$$\partial_t\eta(r, t) = u_y(r, 0, t). \tag{17.15}$$

This is the same as in the case of Carthesian coordinates, just with r taking the place of x. Similarly the linearised dynamic surface condition becomes

$$\partial_t\phi(r, 0, t) = -g\eta(r, t). \tag{17.16}$$

Thus the only difference between the problem of circular waves compared to the problem of plane waves we studied earlier is that the $\partial_x^2\phi$ term in the Laplace equation has been replaced by $1/r\partial_r^2(r\phi)$. This motivates us to make the modified Ansatz

$$\phi(r, y, t) = f(y)\frac{1}{r}\sin(kr - \omega t).$$

Substituting this into the Laplace equation -Eq. 17.13 again gives the equation $f'' - k^2 f$ and imposing that the solution must stay finite as $y \to -\infty$ singles out the solution $f(y) = D\,e^{ky}$ for some constant D. Thus

$$\phi(r, y, t) = De^{ky}\frac{1}{r}\sin(kx - \omega t)$$

Substituting this velocity potential into the dynamical surface condition -Eq. 17.16 gives

$$\eta(r,t) = \frac{\omega\, D}{g}\frac{1}{r}\cos(kr - \omega t)$$

and then the kinematic surface condition -Eq. 17.15 gives the dispersion relation

$$\omega = \sqrt{k\, g}.$$

This is identical to the dispersion relation for the plane wave solution.

18 Wave packets

You find content related to this lecture in the textbooks:

- Coulson and Jeffrey (1977) section 92

Harmonic waves are not localised in x. Instead they are periodic in x and and have an infinite number of crests and troughs. However, if you have ever thrown a pebble into a pond, you will know that real-world waves are not like that. Instead you will see packets with a finite number of crests and troughs moving away from the place where you dropped the pebble.

We will use the principle of superposition that we already discussed in lecture 4 that tells us that the sum of waves of a linear wave equation is also a solution. We have already found an uncountable family of harmonic plane wave solutions to the wave equation, parametrized by the wave number k. We will use the harmonic plane waves in the complex exponential notation that we introduced in lecture 5:

$$y(x,t) = Ae^{i(kx-\omega(k)t)}.$$

The superposition principle allows us not only to add a finite or countable number of solutions to obtain a new solution but it also allows us to integrate together an uncountable number of solutions by integrating rather than summing. It turns out that this makes it possible to construct a localised *wave packet*.

So we look at solutions where we give an amplitude $A(k)$ to the harmonic wave with wave number k and then integrate over all wave numbers from $-\infty$ to ∞:

$$y(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}dk. \tag{18.1}$$

This is again a solution. There is of course no guarantee that such an integral gives a finite result unless the coefficient function $A(k)$ is chosen carefully to fall off sufficiently quickly towards positive and negative infinity. Rather than discussing the general theory, which you may meet in a future module on functional analysis, we are going to look at the most important example: the Gaussian wave packet.

18.1 Gaussian wave packet

Let the amplitude function $A(k)$ be given by the Gaussian function

$$A(k) = e^{-\frac{(k-k_0)^2}{2\sigma^2}} \tag{18.2}$$

for some real k_0 and positive σ. This function has a single peak at $k = k_0$ and rapidly decays to zero away from it. The parameter σ is approximately the 'half width' of the peak. The sketch of this function is shown in Figure 18.1 a).

To understand what this wave packet looks like in space, we first look at $t = 0$, where we have

$$y(x,0) = \int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\sigma^2}}\, e^{ikx} dk. \tag{18.3}$$

To perform the integral we use a trick called "completing the square" whereby we move all k dependence into the square in the exponential. So we rewrite the exponent as follows:

$$-\frac{(k-k_0)^2}{2\sigma^2} + ikx = -\frac{(k-k_0-ix\sigma^2)^2}{2\sigma^2} + ik_0x - \frac{x^2\sigma^2}{2}.$$

Using this in Eq. 18.3 gives

$$y(x,0) = e^{-\frac{x^2\sigma^2}{2}} e^{ik_0x} \int_{-\infty}^{\infty} \exp\left(-\frac{(k-k_0-ix\sigma^2)^2}{2\sigma^2}\right) dk. \tag{18.4}$$

The integral over the Gaussian simply gives $\sqrt{2\pi}\sigma$, so that we have the solution

$$y(x,0) = \sqrt{2\pi}\,\sigma \exp\left(-\frac{\sigma^2x^2}{2}\right) e^{ik_0x}. \tag{18.5}$$

The factor e^{ik_0x} represents a harmonic wave, the other factors represent a varying amplitude (the envelope of the wave). The graph of the real part of Eq. 18.5 is shown in Figure 18.1 b.

Note that the half width of the wave packet is $1/\sigma$, i.e. it is inversely proportional to the width of $A(k)$. This means that that a more localised wave corresponds to a broader range in k and vice versa. This property has an important consequence in quantum mechanics – the uncertainty principle.

18.2 Moving wave packets and group velocity

If we have the standard wave equation, so that $\omega(k) = c\,k$ with a constant c (no dispersion), then, according to Eq. 18.1, we have $y(x,t) = f(x-ct)$ where $f(x) = y(x,0)$ is given by

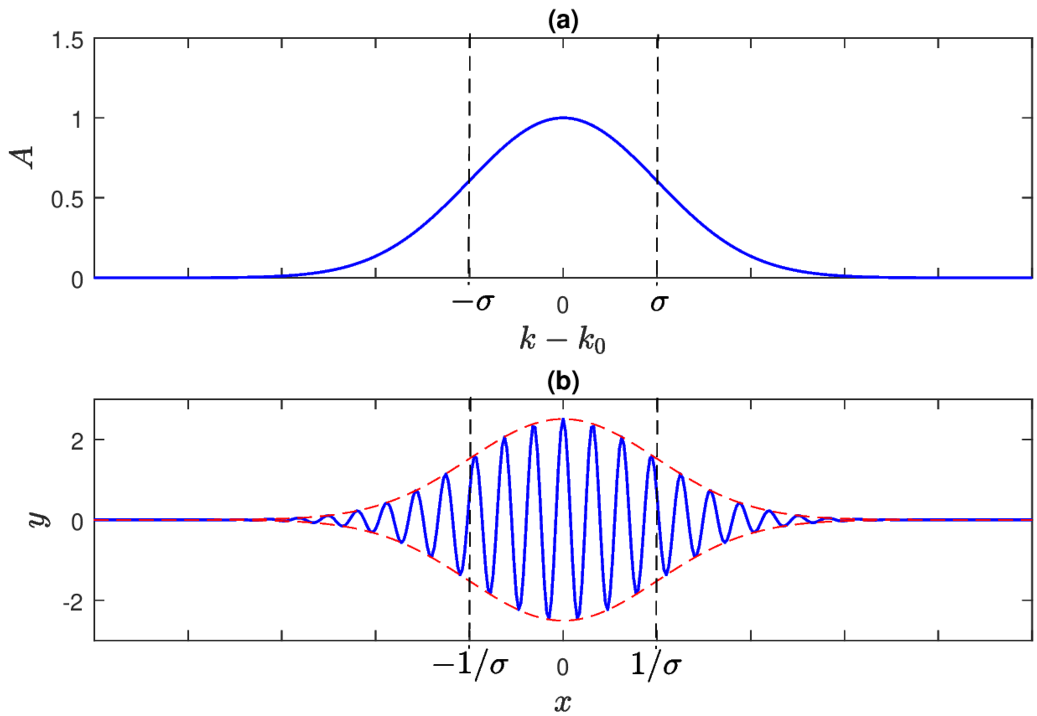

Figure 18.1: Plot of the Gaussian amplitude function A(k) and the resulting Gaussian wave packet.

Eq. 18.5. Thus, if there is no dispersion, the wave packet propagates with the (constant) wave speed c without changing its initial form.

The situation becomes more interesting if there is dispersion, i.e., if $\omega(k)/k$ depends on k. If we assume that $A(k)$ is well localised, i.e. σ is sufficiently small, then only values of k in a small neighbourhood of k_0 will contribute to the integral given by Eq. 18.1. With this in mind, we replace $\omega(k)$ with its Taylor expansion about $k = k_0$, keeping only two nonzero terms, i.e.

$$\omega(k) = \omega_0 + \alpha(k - k_0) + O(|k - k_0|^2),$$

where

$$\omega_0 = \omega(k_0), \quad \alpha = \frac{d\omega}{dk}(k_0).$$

Then Eq. 18.1 with $A(k)$ given by Eq. 18.1 yields

$$\begin{aligned} y(x,t) &\approx \int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\sigma^2} + i(kx - \omega_0 t - \alpha(k-k_0)t)}\, dk \\ &= e^{-i(\omega_0 - \alpha k_0)t} \int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\sigma^2}}\, e^{ik(x-\alpha t)}\, dk. \end{aligned}$$

We notice that the integral is similar to the one in Eq. 18.3 , just with x replaced by $x - \alpha t$, so we can reuse the result from the previous section. In view of Eq. 18.5, we obtain

$$y(x,t) \approx \sqrt{2\pi}\,\sigma\, e^{-\frac{\sigma^2(x-\alpha t)^2}{2}} e^{i(k_0 x - \omega_0 t)}. \tag{18.6}$$

The factor $e^{i(k_0 x - \omega_0 t)}$ represents a harmonic wave travelling with the wave speed $c = \omega_0/k_0$, while the other factors give us the envelope of the wave. Now, however, that envelope propagates not with the wave speed c but with the **group velocity** c_g where

$$c_g = \alpha = \frac{d\omega}{dk}(k_0).$$

In general, the group velocity is different from the wave speed. It is equal to the wave speed only if there is no dispersion.

In the wave packet in Eq. 18.6 the shape of the envelope does not change over time as the wave packet travels through space. However, if we keep the quadratic term in the Taylor expansion of $\omega(k)$, we see that there are some changes in the shape of the wave packet over time. The width of the Gaussian envelope will increase with time.

18.3 Strange experiences of a water strider

Imagine a water strider sitting on the surface of a pond (a water strider is an insect that is light enough to be able to sit on water). It sees a wave packet approaching, say with 6 noticeable wave crests. How often do you think the water strider will move up and down as that packet with its 6 wave crests moves past? The answer is 12! We'll now derive this.

We have derived in Chapter 16 that the harmonic wave $\eta = A\cos(kx - \omega t)$ is a solution describing a surface wave if $\omega = \pm\sqrt{g\,k}$. The wave moves with the wave speed

$$c = \frac{\omega}{k} = \pm\sqrt{\frac{g}{k}} = \pm\sqrt{\frac{g\lambda}{2\pi}}.$$

We see that longer waves move faster than shorter waves.

We observed in the previous subsection that a wave packet with wave numbers localised around wave number k moves with the group velocity. We can now calculate this group velocity for our surface waves:

$$c_g = \frac{d\omega}{dk} = \pm\frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c.$$

So for water waves the group velocity is half the phase velocity. This explains why the water strider moves up an down twice as often as one would have expected by looking at a snapshot of the wave packet.

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