planktr: A multispecies plankton size-spectrum model in R

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[1]This document is included as a vignette (a LaTeX document created using the R package `knitr`) of the package `planktr`. It is automatically downloaded together with the package and can be accessed through R typing `vignette("planktr_vignette")`.

1 Phytoplankton

1.1 Population balance equation

We specialise the population balance equation (**?**, eq.(2.10)) in the paper to the case of a discrete set of species, i.e.,

$$p(w, w_*) = \sum_{i=1}^{M} \delta(w_* - w_i)\, w_i^{-\xi} p_i(w/w_i). \tag{1.1}$$

Substituting this into eq.(2.10), using the scaling properties of the rates

$$\begin{aligned} G(w, w_*) &= w_*^{1-\xi} g(w/w_*), & K(w, w_*) &= w_*^{-\xi} k(w/w_*), \\ M(w, w_*) &= w_*^{-\xi} m(w/w_*), & Q(w, w') &= q(w/w')/w', \end{aligned}$$

and performing a change of variables to $\omega = w/w_i$, we obtain the equation

$$\begin{aligned} w_i^{\xi} \partial_t p_i(\omega) = &-\partial_\omega \left[g(\omega) p_i(\omega)\right] + 2 \int q(\omega/\omega') \left[k(\omega') p_i(\omega')\right] \omega'^{-1} d\omega' \\ &- \left[k(\omega) + m(\omega)\right] p_i(\omega) \end{aligned}$$

for all $\omega \in [\omega_{\min}, 1]$, were $\omega_{\min}$ is the smallest size possible for any cell.

To see that the integral is a convolution integral we write $\omega = e^x$ and $\omega' = e^y$ and observe that then $\omega'^{-1} d\omega' = dy$ so that the integral takes the form

$$\begin{aligned} \int q(\omega/\omega') \left[k(\omega') p_i(\omega')\right] \omega'^{-1} d\omega' &= \int q(e^{x-y}) k(e^y) p_i(e^y) dy \\ &= \int_{x_{\min}}^{0} f_1(x-y) g_1(y) dy = C_1(x). \end{aligned}$$

The subscript 1 is to distinguish this convolution integral from two others arising when we start including predation in the model. We need to calculate the convolution integral for all $x \in [x_{\min}, 0]$, where $x_{\min} = \log \omega_{\min}$ is the smallest possible cell size. The reason we can restrict the integral to run only from x_{th} to 0, where x_{th} is the log of the threshold size below which not division is taking place, is that the factor $g(y)$ is zero outside that interval anyway. With y ranging only over the interval $[x_{\text{th}}, 0]$, the largest region in which the function f gets evaluated is the interval $[x_{\min}, -x_{\text{th}}]$. Thus we can replace f by a periodic function $\bar{f}$ with period $L = -x_{\min}$, without changing the value of the integral, because $\bar{f}$ is zero on the interval from 0 to $-xt$ and agrees with f on the interval $[x_{\min}, 0]$. We can then also replace g by a periodic function $\bar{g}$ with the same period L, which again does not change the integral because $\bar{g}$ agrees with g on the interval $[x_{\min}, 0]$ and is not used outside that interval. This gives us

$$C_1(x) = \int_{x_{\min}}^{0} \bar{f}(x-y) \bar{g}(y) dy.$$

Because this is an integral over a complete period of the two periodic functions $\bar{f}$ and $\bar{g}$ it can be calculated by Fast Fourier Transform.

We also express the derivative in the birth term in terms of the variable $x = \log \omega$ so that it too can be calculated with spectral methods: $\partial_w = w^{-1} \partial_x$.

1.2 Nutrient

Substituting the discrete set of species expression (**??**) into the expression (**?**, (2.13)) for the rate of nutrient consumption, and choosing units for the nutrient concentration so that $\theta = 1$, we get

$$\sigma(N,p) = a(N)\sum_{i=1}^{M} w_i^{2-\xi-\gamma}\int \omega^{\alpha} p_i(\omega) d\omega. \tag{1.2}$$

Again performing the change of variable $x = \log(\omega)$ in the integral gives

$$\int \omega^{\alpha} p_i(\omega) d\omega = \int \omega^{\alpha+1} p_i(\omega) dx.$$

The reason we prefer the integral in the form on the right-hand side is that in our code we will work with logarithmically spaced intervals in ω which translates to equally-spaced intervals in x, which then simplifies the calculation of the integral with respect to dx by the Riemann sum. For equally-spaced interpolation points and a periodic integrand the second-order trapezoidal rule simplifies to the Riemann sum.

2 Predation

We now allow cells to eat other cells. We assume that the rate at which a given cell of size w eats another givem cell of size w' is given by $w^{\nu} S(w/w')$. This will add two additional terms to the right-hand side of the population balance equation (**??**): one additional death term and one additional growth term. The death term takes the form

$$-p_i(\omega)\sum_{j=1}^{M}(w_i - w_j)^{\xi}\int S(\omega'/\omega_j)\omega'^{\nu} p_j(\omega') dw',$$

where $\omega_j = w/w_j$.

References

Cuesta, J.A., Delius, G.W., Law, R., 2016. Sheldon Spectrum and the Plankton Paradox: Two Sides of the Same Coin. A trait-based plankton size-spectrum model. arXiv preprint arXiv:1607.04158.