Conversion between weight and length

In mizer we have the following PDE for a density $N(w)$ in a variable w:

$$\partial N/\partial t = -\partial J_w/\partial w - mw^(n-1)N$$

where

$$J_w = (Aw^n - Bw)N - Dw^{n+1}\partial N/\partial w.$$

We want to convert this into a PDE for a density $u(l,t)$ in a variable l related to w by $w = al^b$ where $n = 1 - 1/b$. Write the PDE in the form

$$\partial u/\partial t = -\partial J_l/\partial l - m/lu$$

where

$$J_l = k(L_\infty^0 - l)u - \alpha l \partial u/\partial l.$$

We would like to express the parameters k, L_∞^0 and α in terms of the parameters A, B and D.

Step 1: Relate the Densities using Conservation

The number of particles must be conserved under the change of variables. This means that the number of particles in an infinitesimal interval dw must be equal to the number of particles in the corresponding interval dl.

$$N(w,t)dw = u(l,t)dl$$

This gives the relationship between the two densities:

$$u(l,t) = N(w,t)\frac{dw}{dl}$$

Given the transformation $w = al^b$, we can calculate the derivative $\frac{dw}{dl}$:

$$\frac{dw}{dl} = abl^{b-1}$$

Substituting this into the relation between the densities, we get:

$$u(l,t) = N(w(l),t) \cdot (abl^{b-1})$$

From this, we can express $N(w,t)$ in terms of $u(l,t)$:

$$N(w,t) = \frac{u(l,t)}{abl^{b-1}}$$

Step 2: Transform the PDE

The original PDE is:

$$\frac{\partial N}{\partial t} = -\frac{\partial J_w}{\partial w} - mw^{n-1}N$$

We can write this in conservation form:

$$\frac{\partial N}{\partial t} + \frac{\partial J_w}{\partial w} = -mw^{n-1}N$$

To change variables from (w, t) to (l, t), we multiply the entire equation by $\frac{dw}{dl}$:

$$\frac{\partial N}{\partial t}\frac{dw}{dl} + \frac{\partial J_w}{\partial w}\frac{dw}{dl} = -mw^{n-1}N\frac{dw}{dl}$$

Let's transform each term:

- **Time derivative term:**

  $$\frac{\partial N}{\partial t}\frac{dw}{dl} = \frac{\partial}{\partial t}\left(N\frac{dw}{dl}\right) - N\frac{\partial}{\partial t}\left(\frac{dw}{dl}\right)$$

  Since $\frac{dw}{dl} = abl^{b-1}$ does not depend on time t, the second term is zero. Thus:

  $$\frac{\partial N}{\partial t}\frac{dw}{dl} = \frac{\partial}{\partial t}\left(N\frac{dw}{dl}\right) = \frac{\partial u}{\partial t}$$

- **Flux derivative term:** Using the chain rule:

  $$\frac{\partial J_w}{\partial w}\frac{dw}{dl} = \frac{\partial J_w}{\partial l}$$

- **Source term:**

  $$-mw^{n-1}N\frac{dw}{dl} = -mw^{n-1}u$$

  We use the given relations $w = al^b$ and $n = 1 - 1/b$, which implies $n - 1 = -1/b$.

  $$w^{n-1} = (al^b)^{n-1} = a^{n-1}l^{b(n-1)} = a^{-1/b}l^{-1}$$

  So the source term becomes:

  $$-ma^{-1/b}l^{-1}u = -\frac{ma^{-1/b}}{l}u$$

Substituting these transformed terms back into the PDE, we get:

$$\frac{\partial u}{\partial t} + \frac{\partial J_w}{\partial l} = -\frac{ma^{-1/b}}{l}u$$

Rearranging this gives:

$$\frac{\partial u}{\partial t} = -\frac{\partial J_w}{\partial l} - \frac{ma^{-1/b}}{l}u$$

This equation has the same form as the target equation if we identify $J_l = J_w$ and if the parameter m in the target equation is understood as a new parameter $m_{new} = m_{old}a^{-1/b}$.

Step 3: Transform the Flux J_w

Now, we express the flux J_w in terms of u and l. The expression for J_w is:

$$J_w = (Aw^n - Bw)N - Dw^{n+1}\frac{\partial N}{\partial w}$$

We need to substitute for w, N, and $\frac{\partial N}{\partial w}$. First, let's find the derivative $\frac{\partial N}{\partial w}$ in terms of u and l:

$$\frac{\partial N}{\partial w} = \frac{dl}{dw}\frac{\partial N}{\partial l} = \frac{1}{abl^{b-1}}\frac{\partial}{\partial l}\left(\frac{u}{abl^{b-1}}\right)$$

$$\frac{\partial N}{\partial w} = \frac{1}{abl^{b-1}}\left[\frac{1}{abl^{b-1}}\frac{\partial u}{\partial l} - u\frac{(b-1)}{abl^{b}}\right] = \frac{1}{(ab)^2 l^{2b-2}}\frac{\partial u}{\partial l} - \frac{u(b-1)}{(ab)^2 l^{2b-1}}$$

Now substitute this and the expressions for w and N into J_w:

- **First part of J_w (advection term):**

$$(Aw^n - Bw)N = (A(al^b)^n - B(al^b))\frac{u}{abl^{b-1}}$$

  Using $n = 1 - 1/b \Leftrightarrow bn = b-1$:

$$(Aa^n l^{b-1} - Bal^b)\frac{u}{abl^{b-1}} = \left(\frac{Aa^{n-1}}{b} - \frac{B}{b}l\right)u$$

- **Second part of J_w (diffusion term):**

$$-Dw^{n+1}\frac{\partial N}{\partial w} = -D(al^b)^{n+1}\left[\frac{1}{(ab)^2 l^{2b-2}}\frac{\partial u}{\partial l} - \frac{u(b-1)}{(ab)^2 l^{2b-1}}\right]$$

  The exponent of l is $b(n+1) = b(1 - 1/b + 1) = 2b - 1$.

$$-Da^{n+1}l^{2b-1}\left[\frac{1}{a^2b^2l^{2b-2}}\frac{\partial u}{\partial l} - \frac{u(b-1)}{a^2b^2l^{2b-1}}\right]$$

$$= -\frac{Da^{n-1}}{b^2}l\frac{\partial u}{\partial l} + \frac{Da^{n-1}(b-1)}{b^2}u$$

Combining all parts of J_w:

$$J_w = \left(\frac{Aa^{n-1}}{b} + \frac{Da^{n-1}(b-1)}{b^2}\right)u - \frac{B}{b}lu - \frac{Da^{n-1}}{b^2}l\frac{\partial u}{\partial l}$$

Step 4: Identify the Coefficients

We now have the transformed PDE:

$$\frac{\partial u}{\partial t} = -\frac{\partial J_w}{\partial l} - \dots$$

with J_w as derived above. We compare this flux with the target flux J_l:

$$J_l = k(L^0_\infty - l)u - \alpha l\frac{\partial u}{\partial l} = (kL^0_\infty)u - klu - \alpha l\frac{\partial u}{\partial l}$$

By setting $J_l = J_w$ and comparing the coefficients of the terms u, lu, and $l\frac{\partial u}{\partial l}$, we find the expressions for k, L^0_∞, and α.

- **Coefficient of $l\frac{\partial u}{\partial l}$:**

$$-\alpha = -\frac{Da^{n-1}}{b^2} \Leftrightarrow \alpha = \frac{Da^{n-1}}{b^2}$$

- **Coefficient of lu:**
$$-k = -\frac{B}{b} \Leftrightarrow k = \frac{B}{b}$$

- **Coefficient of u:**
$$\begin{aligned} & kL_\infty^0 = \frac{Aa^{n-1}}{b} + \frac{Da^{n-1}(b-1)}{b^2} \\ \Leftrightarrow \quad & L_\infty^0 = \frac{1}{k}\left(\frac{Aa^{n-1}}{b} + \frac{Da^{n-1}(b-1)}{b^2}\right) = \frac{b}{B}a^{n-1}\left(\frac{A}{b} + \frac{D(b-1)}{b^2}\right) \\ \Leftrightarrow \quad & L_\infty^0 = \frac{a^{n-1}}{B}\left(A + \frac{D(b-1)}{b}\right) \end{aligned}$$

Final Expressions

The expressions for k, L_∞^0, and α in terms of A, B, D, and the transformation parameters a and b are:

$$\begin{aligned} k &= \frac{B}{b}, \\ \alpha &= \frac{Da^{n-1}}{b^2}, \\ L_\infty^0 &= \frac{a^{n-1}}{B}\left(A + D\frac{b-1}{b}\right). \end{aligned}$$

These can also be expressed in terms of n by using the relations $n = 1 - 1/b$, which implies $b = \frac{1}{1-n}$ and $n - 1 = -1/b$. Also, the term $\frac{b-1}{b} = 1 - \frac{1}{b} = n$.

Substituting these into the expressions gives:

$$\begin{aligned} k &= B(1-n), \\ \alpha &= Da^{-(1-n)}(1-n)^2, \\ L_\infty^0 &= \frac{a^{-(1-n)}}{B}(A + Dn). \end{aligned}$$

Note: The parameter L_∞^0 is related to the asymptotic length L_∞ in the expression for the average growth rate
$$g(l) = k(L_\infty - l)$$
by
$$L_\infty = L_\infty^0 + \frac{\alpha}{k}.$$

Inverting the Relations

We will now solve express the parameters A, B, and D in terms of k, L_∞^0 and α.

1. Solving for B

From the first equation, we can directly isolate B:

$$\begin{aligned} & k = B(1-n) \\ \Leftrightarrow \quad & B = \frac{k}{1-n} \end{aligned}$$

2. Solving for D

From the second equation, we can directly isolate D:

$$\begin{aligned} &\alpha = Da^{-(1-n)}(1-n)^2 \\ \Leftrightarrow \qquad &D = \frac{\alpha}{a^{-(1-n)}(1-n)^2} \\ \Leftrightarrow \qquad &D = \frac{\alpha a^{1-n}}{(1-n)^2} \end{aligned}$$

3. Solving for A

From the third equation, we can isolate A:

$$L_\infty^0 = \frac{a^{-(1-n)}}{B}(A + Dn)$$

Rearranging the terms to solve for A:

$$\begin{aligned} &BL_\infty^0 = a^{-(1-n)}(A + Dn), \\ \Leftrightarrow \qquad &BL_\infty^0 a^{1-n} = A + Dn, \\ \Leftrightarrow \qquad &A = BL_\infty^0 a^{1-n} - Dn. \end{aligned}$$

Now, we substitute the expressions we found for B and D:

$$A = \left(\frac{k}{1-n}\right) L_\infty^0 a^{1-n} - \left(\frac{\alpha a^{1-n}}{(1-n)^2}\right) n$$

We can factor out common terms to simplify the expression for A:

$$A = \frac{a^{1-n}}{(1-n)} \left(kL_\infty^0 - \frac{n\alpha}{1-n}\right)$$

Final Inverted Relations

The final expressions for A, B, and D are:

$$\begin{aligned} A &= \frac{a^{1-n}}{1-n} \left(kL_\infty^0 - \frac{n\alpha}{1-n}\right), \\ B &= \frac{k}{1-n}, \\ D &= \frac{\alpha a^{1-n}}{(1-n)^2}. \end{aligned}$$