Statistical analysis and experimental validation of data-driven dynamic measurement methods

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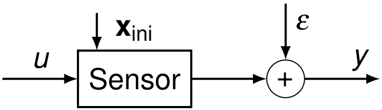
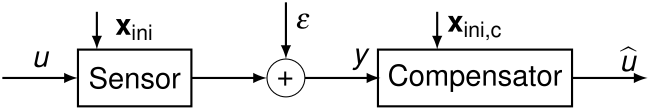
A measurement is a dynamic process

- The sensor interacts with its environment.
- The error is inevitably present in the sensor response.
- The aim is to estimate the input from the sensor response.



A compensator is typically used to estimate the input from the sensor response

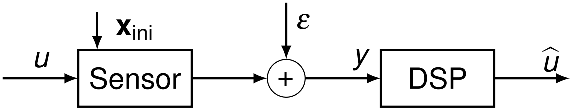
- The additional convolution inverts the sensor dynamics.



Digital signal processors can help to reduce the measurement time even more

Appropriate DSP programming can

- emulate dynamical systems, or
- implement data-driven methods.



In this work we focus on a data-driven step input estimation method

- The method is formulated as a structured and correlated errors-in-variables problem.
- The online solution is computed using recursive least squares.

The uncertainty of the data-driven step input estimation method was unknown

- The validation of the method was necessary to foster the method utilization in metrology applications.

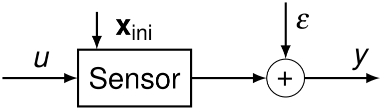
A statistical analysis was conducted to find the data-driven step input estimation method uncertainty

- This analysis unveils the step input estimation mean and variance, and thus, the method uncertainty.
- The estimation mean and variance knowledge permits to evaluate the method and its effectiveness.

The conducted research work covered

1. A statistical analysis of structured and correlated errors-in-variables problems.
2. An experimental validation based on the uncertainty of the step input estimation method.
3. The proposal of two affine input estimation methods.

To formulate the input estimation methods, we consider a measurement is a linear system problem



- With LTI sensor, the measurement dynamics can be described using state-space representation:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k), \quad \text{with} \quad \mathbf{x}_{\text{ini}} = \mathbf{x}(0)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + Du(k) + \varepsilon(k)$$

If the sensor model and initial conditions are known, and we have the exact data sensor response

- then, the input can be estimated from:

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^T \end{bmatrix}}_{\mathcal{O}} \mathbf{x}(0) + \underbrace{\begin{bmatrix} \mathbf{D} & & & & \\ \mathbf{CB} & \mathbf{D} & & & \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{D} & & \\ \vdots & \ddots & & & \\ \mathbf{CA}^{T-1}\mathbf{B} & \cdots & \mathbf{CAB} & \mathbf{CB} & \mathbf{D} \end{bmatrix}}_{\mathcal{T}} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T) \end{bmatrix}}_{\mathbf{u}}$$

A step input $u = \bar{u}s$ permits a representation with an augmented state space

- The step input adds a pole at (1,0) to the original system:

$$\mathbf{x}_\mathrm{a}(k+1) = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}_\mathrm{a}} \mathbf{x}_\mathrm{a}(k), \quad \text{where } \mathbf{x}_\mathrm{a}(k) = \begin{bmatrix} \mathbf{x}(k) \\ u(k) \end{bmatrix}, \ \mathbf{x}_\mathrm{ini} = \mathbf{x}(0)$$

$$y(k) = \underbrace{\begin{bmatrix} \mathbf{C} & D \end{bmatrix}}_{\mathbf{C}_\mathrm{a}} \mathbf{x}_\mathrm{a}(k)$$

Without sensor model, one alternative to estimate the step input starts by formulating

$$\underbrace{\begin{bmatrix} y(1) & y(2) & \cdots & y(n) \\ y(2) & y(3) & \cdots & y(n+1) \\ \iddots & \iddots & \iddots & \\ y(n) & y(n+1) & \cdots & y(2n-1) \end{bmatrix}}_{\mathcal{H}(y)} = \underbrace{\begin{bmatrix} \widehat{\mathbf{C}}_\mathrm{a} \\ \widehat{\mathbf{C}}_\mathrm{a}\widehat{\mathbf{A}}_\mathrm{a} \\ \vdots \\ \widehat{\mathbf{C}}_\mathrm{a}\widehat{\mathbf{A}}_\mathrm{a}^n \end{bmatrix}}_{\mathcal{O}_\mathrm{a}} \underbrace{\begin{bmatrix} \mathbf{x}_\mathrm{a}(0) & \mathbf{x}_\mathrm{a}(1) & \cdots & \mathbf{x}_\mathrm{a}(n) \end{bmatrix}}_{\mathbf{X}_\mathrm{ini}}$$

A singular value decomposition of $\mathcal{H}(y)$ permits the estimation of the observability matrix $\mathcal{O}_\mathrm{a}$ and the initial conditions $\mathbf{X}_\mathrm{ini}$

$$\widehat{\mathcal{O}}_\mathrm{a} = \mathbf{U}\sqrt{\boldsymbol{\Sigma}} \quad \textit{and} \quad \widehat{\mathbf{X}}_\mathrm{ini} = \sqrt{\boldsymbol{\Sigma}}\mathbf{V}n, \quad \textit{from} \quad \mathbf{U}\boldsymbol{\Sigma}\mathbf{V} = \mathcal{H}(y)$$

Once we have estimated the observability matrix $\mathcal{O}_\mathrm{a}$ and the initial conditions $\mathbf{X}_\mathrm{ini}$, we can write

$$y = G\,\bar{u} + \widehat{\mathcal{O}}_\mathrm{a}\,\mathbf{x}_\mathrm{a}(0),$$

- that is equivalent to:

$$\underbrace{\begin{bmatrix} y(0) \\ \vdots \\ y(T) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} G \otimes \mathbf{1}_{T+1} & \widehat{\mathcal{O}}_\mathrm{a} \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} \bar{u} \\ \mathbf{x}_\mathrm{a}(0) \end{bmatrix}$$

- and admits a least-squares solution

$$\begin{bmatrix} \widehat{u} \\ \widehat{\mathbf{x}}_\mathrm{ini} \end{bmatrix} = \left(\mathbf{K}^\top \mathbf{K}\right)^{-1} \mathbf{K}^\top \mathbf{y}$$

Another alternative to estimate the step input starts by differentiating the state-space representation

$$\Delta\mathbf{x}(k+1) = \mathbf{A}\Delta\mathbf{x}(k), \quad \Delta y(k) = \mathbf{C}\Delta\mathbf{x}(k), \quad \text{with} \quad \Delta\mathbf{x}_{\text{ini}} = \Delta\mathbf{x}(0)$$

where:
$\Delta = \sigma - 1$, and
$(\sigma^\tau y)(k) = y(k+\tau)$.

If Δy is persistently exciting of order L,

$$rank\,(\mathcal{H}_{L+1}(\Delta y)) \leq L$$

then, we can write the total response of the system as

$$\underbrace{\begin{bmatrix} y(n+1) \\ \vdots \\ y(T) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} G \otimes \mathbf{1}_{T+1} & \mathcal{H}(\Delta y) \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \bar{u} \\ \ell \end{bmatrix}}_{\boldsymbol{\theta}}$$

The data-driven step input estimation method is a structured and correlated errors-in-variables problem

considering the addition of the measurement noise

$$\widetilde{\mathbf{y}} = \mathbf{y} + \epsilon,$$
$$\widetilde{\mathbf{K}} = \mathbf{K} + \mathbf{E},$$

we have

$$\underbrace{\begin{bmatrix} \widetilde{y}(n+1) \\ \vdots \\ \widetilde{y}(T) \end{bmatrix}}_{\widetilde{\mathbf{y}}} = \underbrace{\begin{bmatrix} G \otimes \mathbf{1}_{T+1} & \mathcal{H}\left(\Delta \widetilde{y}\right) \end{bmatrix}}_{\widetilde{\mathbf{K}}} \underbrace{\begin{bmatrix} \bar{u} \\ \ell \end{bmatrix}}_{\theta}$$

What are the statistical moments of the least-squares solution of a structured and correlated errors-in-variables problem?

$$\widehat{\theta} = \widetilde{\mathbf{K}}^{\dagger}\widetilde{\mathbf{y}} = (\widetilde{\mathbf{K}}^{\top}\widetilde{\mathbf{K}})^{-1}\widetilde{\mathbf{K}}^{\top}\widetilde{\mathbf{y}}$$

Since the noise is assumed additive

$$
\begin{aligned}
\widehat{\boldsymbol{\theta}} &= \Big((\mathbf{K}+\mathbf{E})^\top (\mathbf{K}+\mathbf{E}) \Big)^{-1} (\mathbf{K}+\mathbf{E})^\top (\mathbf{y}+\epsilon) \\
&= (\mathbf{I}+\mathbf{M})^{-1} \mathbf{Q}^{-1} (\mathbf{K}+\mathbf{E})^\top (\mathbf{y}+\epsilon)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{Q} &= \mathbf{K}^\top \mathbf{K}, \quad \text{and} \\
\mathbf{M} &= \mathbf{Q}^{-1} (\mathbf{K}^\top \mathbf{E} + \mathbf{E}^\top \mathbf{K} + \mathbf{E}^\top \mathbf{E}).
\end{aligned}
$$

A second order Taylor series aproximation of the inverse matrix permits to study the LS solution

Using

$$(\mathbf{I}+\mathbf{M})^{-1} \approx \mathbf{I}-\mathbf{M}+\mathbf{M}^2,$$

the LS solution is approximated by

$$\widehat{\boldsymbol{\theta}} \approx \left(\mathbf{I}-\mathbf{M}+\mathbf{M}^2\right)\mathbf{Q}^{-1}(\mathbf{K}+\mathbf{E})^\top(\mathbf{y}+\boldsymbol{\epsilon})$$

To find the LS solution bias and covariance, we use

$$\boldsymbol{\mu}\left(\widehat{\boldsymbol{\theta}}\right) = \mathbb{E}\left\{\widehat{\boldsymbol{\theta}}\right\}, \quad \mathbf{b}\left(\widehat{\boldsymbol{\theta}}\right) = \boldsymbol{\mu}\left(\widehat{\boldsymbol{\theta}}\right) - \boldsymbol{\theta}$$

$$\mathbf{C}\left(\widehat{\boldsymbol{\theta}}\right) = \mathbb{E}\left\{\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\mu}\right)\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\mu}\right)^\top\right\}.$$

For an unstructured and uncorrelated EIV problem, the bias and covariance of the LS solution are

$$\mathbf{b}_p\left(\widehat{\theta}\right) \approx \sigma_{\mathbf{E}}^2 (2+2n-T)\,\mathbf{Q}^{-1}\theta$$

$$\begin{aligned}\mathbf{C}_p\left(\widehat{\theta}\right) \approx\ & \sigma_{\epsilon}^2\,\mathbf{Q}^{-1} + \sigma_{\mathbf{E}}^2\ \mathit{trace}\left(\theta\theta^\top\right)\mathbf{Q}^{-1} \\ & - \sigma_{\mathbf{E}}^4 (2+2n-T)^2\ \mathbf{Q}^{-1}\theta\theta^\top\mathbf{Q}^{-1}\end{aligned}$$

It is proposed to substitute the observed variables in the derived expressions.

The observed variables are $\widetilde{\mathbf{y}}$, $\widetilde{\mathbf{K}}$, and from them we compute $\widehat{\theta}$. The substitution gives an approximation of the estimation bias and covariance using the observed data.

$$\widetilde{\mathbf{b}}_p\left(\widehat{\theta}\right) \approx \sigma_{\mathbf{E}}^2 (2+2n-T)\,\widetilde{\mathbf{Q}}^{-1}\widehat{\theta}$$

$$\begin{aligned}\widetilde{\mathbf{C}}_p\left(\widehat{\theta}\right) \approx\ & \sigma_{\varepsilon}^2\,\widetilde{\mathbf{Q}}^{-1} + \sigma_{\mathbf{E}}^2\ \mathit{trace}\left(\widehat{\theta}\widehat{\theta}^{\top}\right)\widetilde{\mathbf{Q}}^{-1} \\ & - \sigma_{\mathbf{E}}^4 (2+2n-T)^2\,\widetilde{\mathbf{Q}}^{-1}\widehat{\theta}\widehat{\theta}^{\top}\widetilde{\mathbf{Q}}^{-1}\end{aligned}$$

For the structured and correlated EIV problem in the data-driven step input estimation method

the bias and covariance of the LS solution are

$$\mathbf{b}_p\left(\widehat{\theta}\right) \approx \mathbf{Q}^{-1}\left(\left(\mathbf{K}^\top\mathbf{B}_1 - \mathbf{B}_2\right)\mathbf{x} - \left(\mathbf{K}^\top\mathbf{B}_3 - \mathbf{B}_4\right)\right)$$

$$\mathbf{C}_p\left(\widehat{\theta}\right) \approx \mathbf{K}^\dagger\left(\sigma_\epsilon^2\mathbf{I}_{T-n} + \mathbf{C}_1 - \mathbf{C}_2 - \mathbf{C}_2^\top\right)\mathbf{K}^{\dagger\top} - \mathbf{b}_p\left(\widehat{\theta}\right)\mathbf{b}_p^\top\left(\widehat{\theta}\right)$$

where the matrices $\mathbf{B}_1$ to $\mathbf{B}_1$, and $\mathbf{C}_1$ to $\mathbf{C}_2$ can be obtained considering the structure and correlation of the problem.]

The substitution of the observed variables in the expressions also approximates the estimation bias and covariance

$$\mathbf{b}_{\widetilde{p}}\left(\widehat{\theta}\right) \approx \widetilde{\mathbf{Q}}^{-1}\left(\left(\widetilde{\mathbf{K}}^{\top}\widetilde{\mathbf{B}}_1 - \widetilde{\mathbf{B}}_2\right)\widehat{\theta} - \left(\widetilde{\mathbf{K}}^{\top}\widetilde{\mathbf{B}}_3 - \widetilde{\mathbf{B}}_4\right)\right)$$

$$\mathbf{C}_{\widetilde{p}}\left(\widehat{\theta}\right) \approx \widetilde{\mathbf{K}}^{\dagger}\left(\sigma_{\epsilon}^2\mathbf{I}_{T-n} + \widetilde{\mathbf{C}}_1 - \widetilde{\mathbf{C}}_2 - \widetilde{\mathbf{C}}_2^{\top}\right)\widetilde{\mathbf{K}}^{\dagger\top} - \mathbf{b}_{\widetilde{p}}\left(\widehat{\theta}\right)\mathbf{b}_{\widetilde{p}}^{\top}\left(\widehat{\theta}\right)$$

We can find the Cramér-Rao lower bound of the structured and correlated errors-in-variables problem

It is the lower limit on the estimation variance.

$$\mathrm{CRLB}(\boldsymbol{\theta}) = \left(\mathbf{I} + \frac{\partial \mathbf{b}(\widehat{\boldsymbol{\theta}})}{\partial(\widehat{\boldsymbol{\theta}})}\right)^{\top} \mathbf{Fi}^{-1}(\boldsymbol{\theta}) \left(\mathbf{I} + \frac{\partial \mathbf{b}(\widehat{\boldsymbol{\theta}})}{\partial(\widehat{\boldsymbol{\theta}})}\right)$$

Where the Fisher information matrix is

$$\mathbf{Fi}(x) = -\mathbb{E}\left\{\frac{\partial^2 l(\widehat{\boldsymbol{\theta}})}{\partial \widehat{\boldsymbol{\theta}}^2}\right\}$$

The structured EIV problem can be expressed as a linear in the measurements problem

$$e(\widehat{\theta},\widetilde{\mathbf{z}}) = \mathbf{M}_1(\widehat{\theta})\,\widetilde{\mathbf{z}} = \begin{bmatrix} \mathbf{I}_{T-n} & -\widehat{\theta}^T \otimes \mathbf{I}_{T-n} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{y}} \\ vec(\widetilde{\mathbf{K}}) \end{bmatrix} = 0.$$

where $\widetilde{\mathbf{z}} = \mathbf{z} + \varepsilon_{\mathbf{z}}$, and $\mathbf{C}_{\mathbf{z}}$ is the covariance matrix of $\varepsilon_{\mathbf{z}}$.
Then, the loglikelihood function is

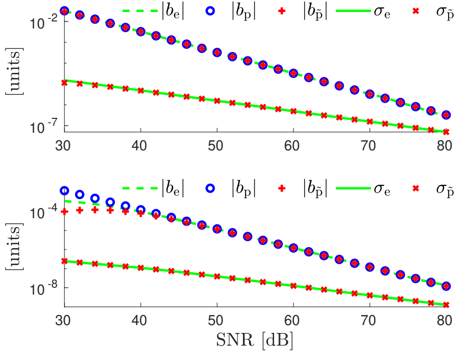
$$\ln l(\widetilde{\mathbf{z}},\widehat{\mathbf{z}},\widehat{\theta}) = -\frac{1}{2}\left(\widetilde{\mathbf{z}}-\widehat{\mathbf{z}}\right)^{\top}\mathbf{C}_{\mathbf{z}}^{-1}\left(\widetilde{\mathbf{z}}-\widehat{\mathbf{z}}\right) + constant,$$

and the Fisher information matrix is

$$\mathbf{Fi}(\theta) = \left(\frac{\partial e(\widehat{\theta},\mathbf{z})}{\partial\theta}\right)^{\top}\left(\mathbf{M}_1(\theta)\,\mathbf{C}_{\mathbf{z}}\,\mathbf{M}_1^{\top}(\theta)\right)^{-1}\left(\frac{\partial e(\widehat{\theta},\mathbf{z})}{\partial\theta}\right)$$

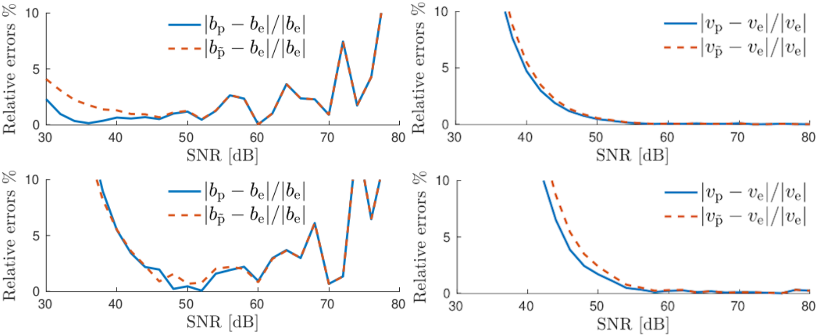
Simulation results for unstructured (top) and structured EIV problems (bottom)

- The absolute value of the bias and the standard error are proportional to the noise variance and standard deviation.
- The bias predictions coincide with the empirical bias, and
- the standard errors are smaller than the estimation bias.

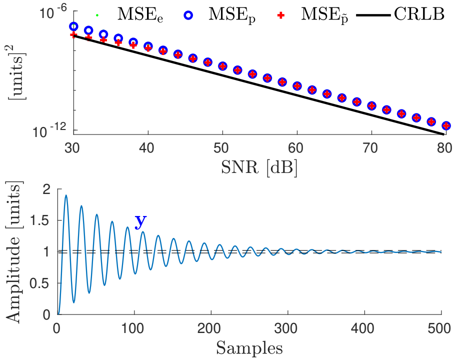


The relative errors of the bias and variance estimation show the accuracy of the derived expressions

- The empirical results are closely approximated



The MSE of the step input estimation is larger than its CRLB at most by one order of magnitude

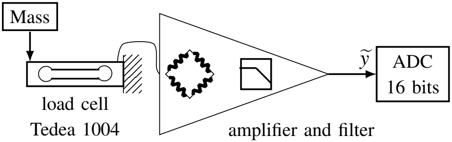


Observations of the statistical analysis

- The predictions in the structured case are more susceptible to perturbations.
- The bias and variance predictions accuracy depend on the Taylor series validity.
- This methodology can be applied to assess the uncertainty to other structured EIV problems solutions.
- The bias and variance expressions obtained depend on each specific structure.

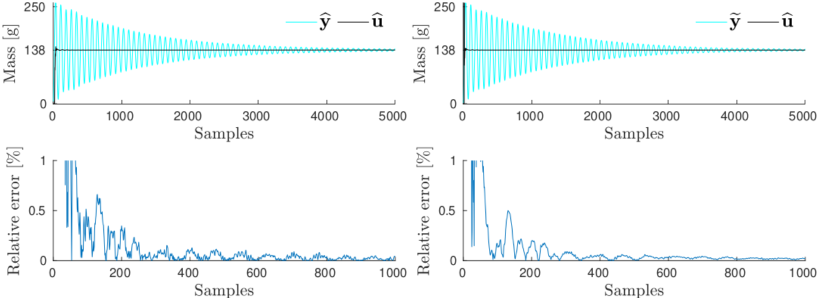
A description of the experimental validation of the step input estimation method

The diagram represents the setup used to measure mass.



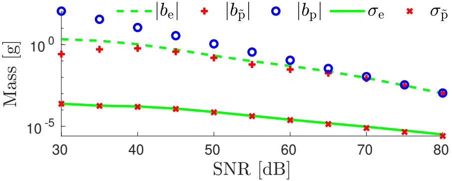
The step input estimation by processing a simulated (left) and an measured (right) sensor response

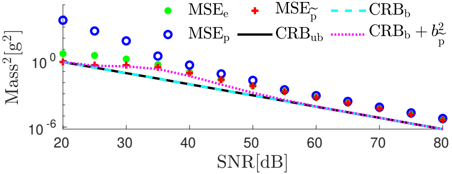
A model of the sensor was obtained using the system identification toolbox.



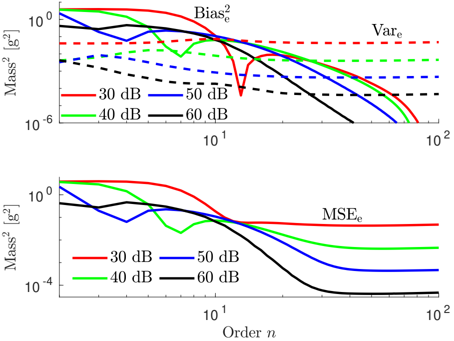
The bias and variance estimation estimation results allow to evaluate the effectiveness with respect to SNR

In simulation, with an order $n = 5$ the SNR region of validity is largar then 40 dB.



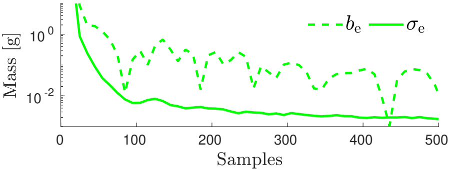


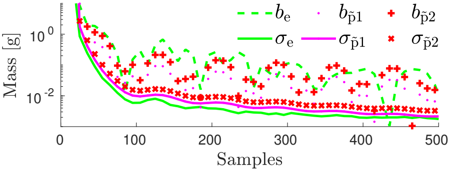
The uncertainty of the step input estimation does not decrease for larger order *n*



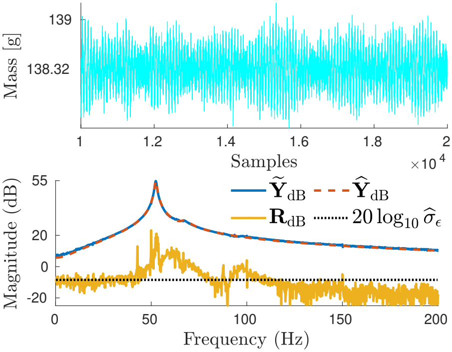
The empirical and the estimation of the bias and variance of the step input for different sample size

These results are the average from 100 experiments.

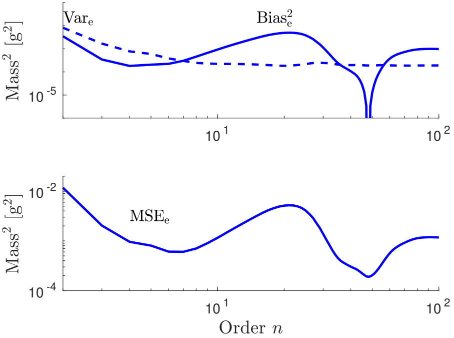




The measurement noise contains frequency components characteristic of mechanical systems



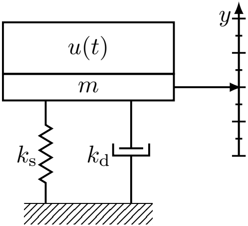
The uncertainty of the step input estimation does not depend on the order *n*



Observations of the experimental validation of the step input estimation method

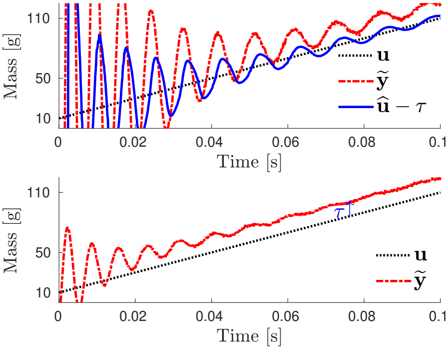
- In simulation, the input estimation MSE is close to the CRLB theoretical minimum for biased estimators.
- The step input estimation method is useful in practical applications where the whiteness assumption of the measurement noise is not fulfilled.
- The noise variance obtained from the sensor steady state response underestimates the measurement noise variance.
- Using the predictions, the uncertainty assessment is provided for given sample size and noise variance.

When an affine input $u = at + b$ is applied to weighing system, it becomes time varying



$$\frac{d}{dt}\left((at+b+m)\,\frac{dy}{dt}\right) + k_d\frac{dy}{dt} + k_s y = (at+b+m)\,g$$

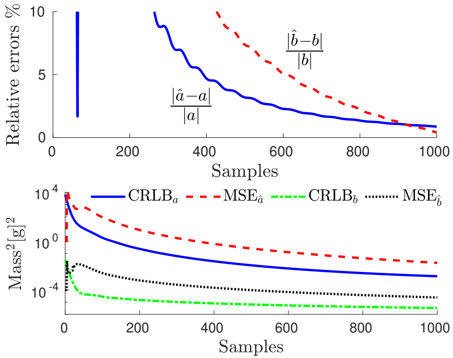
This is a typical sensor response to an affine input



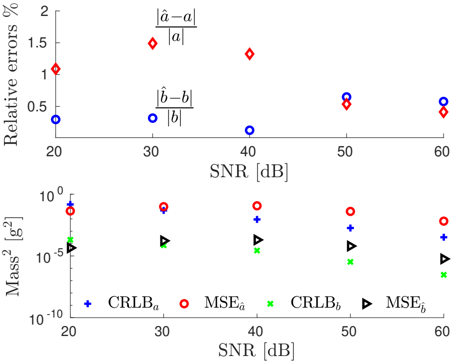
Data-driven affine input estimation method results with respect to sample size

In simulation, the input parameters converge towards the true value.

The MSEs come also near the CRLBs, by one order of magnitude.

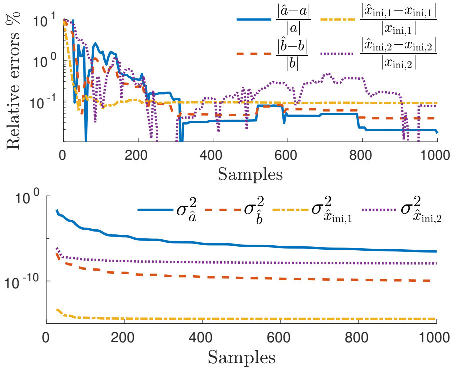


Data-driven affine input estimation method results with respect to SNR



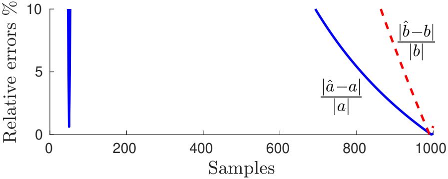
Maximum-likelihood affine input estimation method results with respect to sample size

The parameters converge after three iterations, but the required computational power is large.



Time varying affine input estimation method results

A conventional time varying filter, designed for weighing with affine inputs, has a poor performance in the same conditions.



Observations of the affine input estimation methods

- The subspace method estimates directly the affine input parameters, even when the sensor is time-varying.
- The subspace method is computationally cheap, simple and suitable for implementation on digital signal processor of low computational power.
- The maximum-likelihood method can estimate also model parameters or initial conditions, but needs a lot of computational resources.
- The proposed methods outperform a conventional time-varying filter.

Global conclusions of the conducted research 1

- The data-driven step input estimation method is valid for metrology applications.
- We have means to assess the estimation uncertainty by analysing the LS solution of a structured and correlated EIV problem.
- The derived expressions predict the input estimation bias and covariance for given sample size and perturbation level.

Global conclusions of the conducted research 2

- The input estimation MSE is considerably near to the minimal theoretical variance of the structured EIV problem.
- The implementation of the data-driven step input estimation method showed robustness under non Gaussian white noise from spurious mechanical vibrations.
- The adaptive data-driven affine input estimation method is also robust when processing time-varying sensor responses.

Future work

- Data-driven signal processing methods will continue to produce interesting results.
- Statistically efficient input estimation methods can be studied to reduce or eliminate bias.
- Efficient online optimization methods are required to simplify the implementation of measurement methods.