Statistical analysis and experimental validation of data-driven dynamic measurement methods

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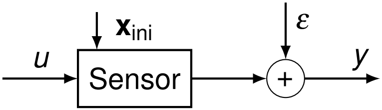
A measurement is a dynamic process

- The sensor interacts with its environment.
- The error is inevitably present in the sensor response.
- The aim is to estimate the input from the sensor response.



Statistical analysis and experimental validation of data-driven dynamic measurement methods

- Comparison of input estimation approaches.
- Formulation of a data-driven step input estimation method.
- Thesis contribution.
  - Statistical analysis, data-driven step input estimation method.
  - Experimental illustration.
  - Affine input estimation methods.

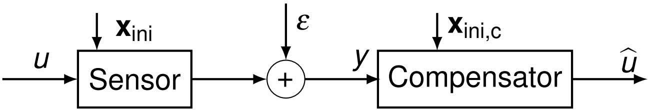
How to estimate the input from the sensor response?

Without sensor model

- System identification and input estimation.
  - Two stages, or
  - simultaneously.
- Adaptive filters.
- Optimization.

With given model

- Kalman filter.
- Compensator system.
- Deconvolution filter.



Overview of the classical approaches

- Kalman filter is optimal estimator under white noise.
- Compensator systems design is a established technique.
- The uncertainty propagation is well understood.

but

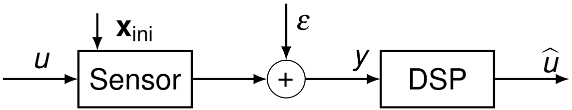
- Kalman filter and compensators need a model.
- Compensators also exhibit transient state.

- Model identification errors add uncertainty.
- sysId + estimation not suitable for real-time.
- Optimization uses large computational resources.

Digital signal processors are useful in metrology

Depending on its configuration, a DSP can

- emulate dynamical systems, or
- implement data-driven methods.



~~Overview of~~ DSP approaches

- DSP are versatile.
- Data-driven methods can bypass the model identification.
- Data-driven methods can speed-up measurements.

but

  - Estimation uncertainty is not always evident.
  - Stochastic properties are harder to find.

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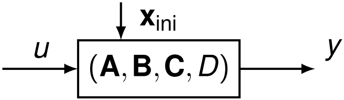
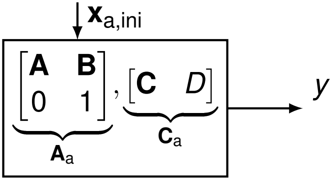
To formulate input estimation methods, consider a measurement as a linear system problem



- With model and exact data, we can solve

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^T \end{bmatrix}}_{\mathcal{O}} \mathbf{x}(0) + \underbrace{\begin{bmatrix} \mathbf{D} & & & & \\ \mathbf{CB} & \mathbf{D} & & & \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{D} & & \\ \vdots & \ddots & & & \\ \mathbf{CA}^{T-1}\mathbf{B} & \cdots & \mathbf{CAB} & \mathbf{CB} & \mathbf{D} \end{bmatrix}}_{\mathcal{T}} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T) \end{bmatrix}}_{\mathbf{u}}$$

With step input, the sensor permits an augmented autonomous state space representation



- without sensor model, we can estimate first $\mathcal{O}_\mathrm{a}$ and $\mathbf{X}_{\mathrm{ini}}$ in

$$\underbrace{\begin{bmatrix} y(1) & y(2) & \cdots & y(n)\\ y(2) & y(3) & \cdots & y(n+1)\\ \iddots & \iddots & \iddots & \\ y(n) & y(n+1) & \cdots & y(2n-1)\end{bmatrix}}_{\mathcal{H}(y)} = \underbrace{\begin{bmatrix}\mathbf{C}_\mathrm{a}\\ \mathbf{C}_\mathrm{a}\mathbf{A}_\mathrm{a}\\ \vdots\\ \mathbf{C}_\mathrm{a}\mathbf{A}_\mathrm{a}^n\end{bmatrix}}_{\mathcal{O}_\mathrm{a}} \underbrace{\begin{bmatrix}\mathbf{x}_\mathrm{a}(0) & \mathbf{x}_\mathrm{a}(1) & \cdots & \mathbf{x}_\mathrm{a}(n)\end{bmatrix}}_{\mathbf{X}_{\mathrm{ini}}}$$

Once we have estimated the observability matrix $\widehat{\mathcal{O}}_{\mathrm{a}}$ and the initial conditions matrix $\widehat{\mathbf{X}}_{\mathrm{ini}}$, we can write

$$y = G\,\bar{u} + \widehat{\mathcal{O}}_{\mathrm{a}}\,\mathbf{x}_{\mathrm{a}}(0),$$

- that is equivalent to

$$\underbrace{\begin{bmatrix} y(0) \\ \vdots \\ y(T) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} G \otimes \mathbf{1}_{T+1} & \widehat{\mathcal{O}}_{\mathrm{a}} \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} \bar{u} \\ \mathbf{x}_{\mathrm{a}}(0) \end{bmatrix}$$

To estimate directly the step input, first differentiate the state-space representation

$$\Delta \mathbf{x}_{\text{ini}} = \Delta \mathbf{x}(0)$$

$$\boxed{\begin{aligned} \Delta \mathbf{x}(k+1) &= \mathbf{A}\Delta \mathbf{x}(k) \\ \Delta y(k) &= \mathbf{C}\Delta \mathbf{x}(k) \end{aligned}} \longrightarrow \Delta y$$

where:
$\Delta = \sigma - 1$, and
$(\sigma^{\tau} y)(k) = y(k+\tau)$.

If Δy is persistently exciting of order L,
$$rank\,(\mathcal{H}_{L+1}(\Delta y)) \leq L$$

- then, the total response of this autonomous system is

$$\underbrace{\begin{bmatrix} y(n+1) \\ \vdots \\ y(T) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} G \otimes \mathbf{1}_{T-n} & \mathcal{H}(\Delta y) \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \bar{u} \\ \ell \end{bmatrix}}_{\boldsymbol{\theta}}$$

The data-driven step input estimation method formulates an errors-in-variables problem

$$\underbrace{\begin{bmatrix} \widetilde{y}(n+1) \\ \vdots \\ \widetilde{y}(T) \end{bmatrix}}_{\widetilde{\mathbf{y}}} = \underbrace{\begin{bmatrix} G \otimes \mathbf{1}_{T-n} & \mathcal{H}\left(\Delta \widetilde{y}\right) \end{bmatrix}}_{\widetilde{\mathbf{K}}} \underbrace{\begin{bmatrix} \bar{u} \\ \ell \end{bmatrix}}_{\boldsymbol{\theta}}$$

considering additive measurement noise

$$\widetilde{\mathbf{y}} = \mathbf{y} + \boldsymbol{\epsilon},$$
$$\widetilde{\mathbf{K}} = \mathbf{K} + \mathbf{E},$$

The output-error step input estimation is converted into errors-in-variables problem

- structured and correlated,
- no evident statistical properties,
- optimal or unbiased solution is harder to get,
- admits least-squares solution.

The recursive least squares solution is suboptimal, but it is suitable for real-time implementation.

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The data-driven step input estimation uncertainty assessment was pending

This requires to study

- the structured errors-in-variables problem
  - stochastic properties,
  - minimum variance, and
- the data-driven step input estimation
  - statistical moments,
  - mean-squared-error,
  - effectiveness.

To study the least-squares solution of an errors-in-variables problem

$$\widehat{\boldsymbol{\theta}} = (\widetilde{\mathbf{K}}^\top \widetilde{\mathbf{K}})^{-1} \widetilde{\mathbf{K}}^\top \widetilde{\mathbf{y}},$$

where the noise is assumed additive

$$\widehat{\boldsymbol{\theta}} = \Big((\mathbf{K}+\mathbf{E})^\top (\mathbf{K}+\mathbf{E}) \Big)^{-1} (\mathbf{K}+\mathbf{E})^\top (\mathbf{y}+\epsilon)$$

~~we write~~

$$\widehat{\boldsymbol{\theta}} = \Big(\mathbf{I} + \underbrace{\mathbf{Q}^{-1} (\mathbf{K}^\top \mathbf{E} + \mathbf{E}^\top \mathbf{K} + \mathbf{E}^\top \mathbf{E})}_{\mathbf{M}} \Big)^{-1} \big(\underbrace{\mathbf{K}^\top \mathbf{K}}_{\mathbf{Q}} \big)^{-1} (\mathbf{K}+\mathbf{E})^\top (\mathbf{y}+\epsilon)$$

The second order Taylor series aproximation

$$(\mathbf{I}+\mathbf{M})^{-1} \approx \mathbf{I}-\mathbf{M}+\mathbf{M}^2,$$

permits to express the solution as

$$\widehat{\boldsymbol{\theta}} \approx \left(\mathbf{I}-\mathbf{M}+\mathbf{M}^2\right)\mathbf{Q}^{-1}(\mathbf{K}+\mathbf{E})^{\top}(\mathbf{y}+\epsilon)$$

- The perturbation elements
  are no longer subject to inversion,
- the bias and covariance approximations
  - can be calculated,
  - have few terms,
  - depend on the structure and correlation.

~~The least-squares solution bias and covariance are~~

- ~~for an unstructured and uncorrelated EIV problem:~~

$$\mathbf{b}_p\left(\widehat{\theta}\right) \approx \sigma_{\mathbf{E}}^2 (2+2n-T)\,\mathbf{Q}^{-1}\theta$$

$$\mathbf{C}_p\left(\widehat{\theta}\right) \approx \sigma_{\epsilon}^2\,\mathbf{Q}^{-1} + \sigma_{\mathbf{E}}^2\,\mathit{trace}\left(\theta\theta^{\top}\right)\mathbf{Q}^{-1} - \sigma_{\mathbf{E}}^4 (2+2n-T)^2\,\mathbf{Q}^{-1}\theta\theta^{\top}\mathbf{Q}^{-1}$$

- ~~for a structured and correlated EIV problem:~~

$$\mathbf{b}_p\left(\widehat{\theta}\right) \approx \mathbf{Q}^{-1}\left(\left(\mathbf{K}^{\top}\mathbf{B}_1 - \mathbf{B}_2\right)\mathbf{x} - \left(\mathbf{K}^{\top}\mathbf{B}_3 - \mathbf{B}_4\right)\right)$$

$$\mathbf{C}_p\left(\widehat{\theta}\right) \approx \mathbf{K}^{\ddagger}\left(\sigma_{\epsilon}^2\mathbf{I}_{T-n} + \mathbf{C}_1 - \mathbf{C}_2 - \mathbf{C}_2^{\top}\right)\mathbf{K}^{\ddagger\top} - \mathbf{b}_p\left(\widehat{\theta}\right)\mathbf{b}_p^{\top}\left(\widehat{\theta}\right)$$

- They provide insight of the structure effect on the estimation.

The Cramér-Rao lower bound of the structured and correlated errors-in-variables problem is

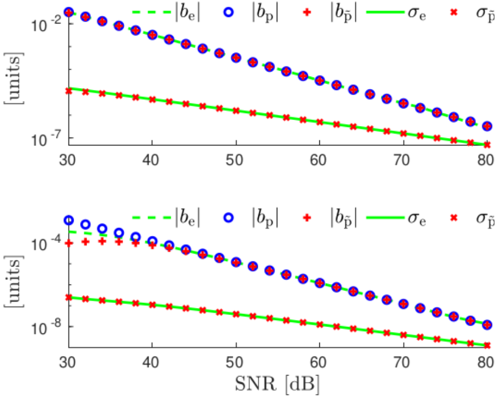
$$\mathrm{CRLB}(\boldsymbol{\theta}) = \left(\mathbf{I} + \frac{\partial \mathbf{b}(\widehat{\boldsymbol{\theta}})}{\partial(\widehat{\boldsymbol{\theta}})}\right)^{\top} \mathbf{Fi}^{-1}(\boldsymbol{\theta}) \left(\mathbf{I} + \frac{\partial \mathbf{b}(\widehat{\boldsymbol{\theta}})}{\partial(\widehat{\boldsymbol{\theta}})}\right)$$

where the Fisher information matrix is

$$\mathbf{Fi}(x) = -\mathbb{E}\left\{\frac{\partial^2 l(\widehat{\boldsymbol{\theta}})}{\partial \widehat{\boldsymbol{\theta}}^2}\right\}$$

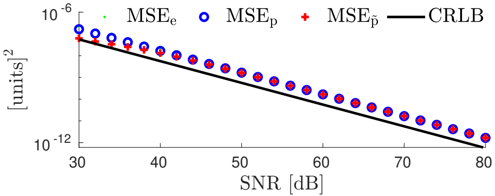
- Here we benefit from the obtained bias expression.

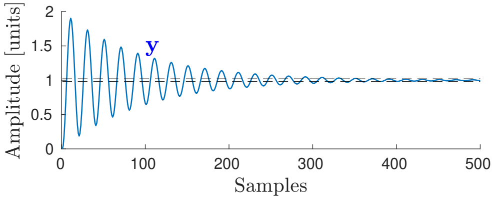
Simulation results for unstructured (top) and structured (bottom) EIV problems



- The structured case is more susceptible to perturbations.
- The predictions depend on the Taylor series validity.

The MSE of the step input estimation is larger than the CRLB at most by one order of magnitude





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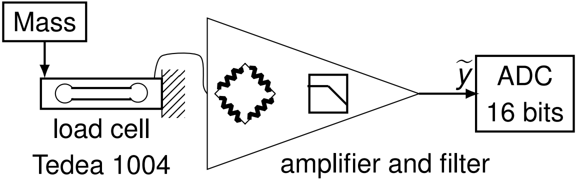
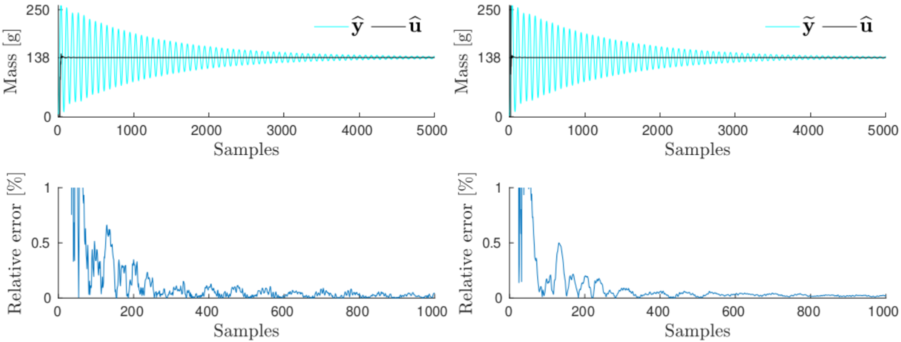
A weighing system is used to collect experimental data from step inputs

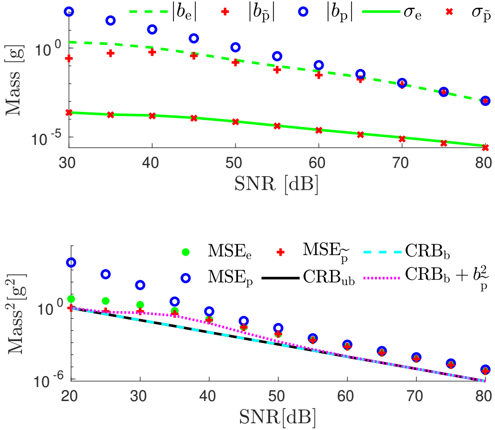


- A sensor model was obtained with the sysId toolbox.

The step input estimation by processing a simulated (left) and an measured (right) sensor response

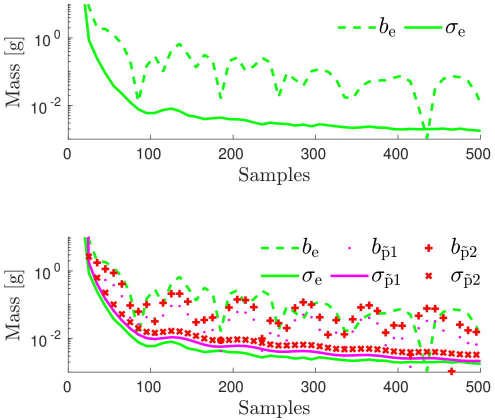


In simulation, for SNR > 40 dB the bias and variance are well predicted



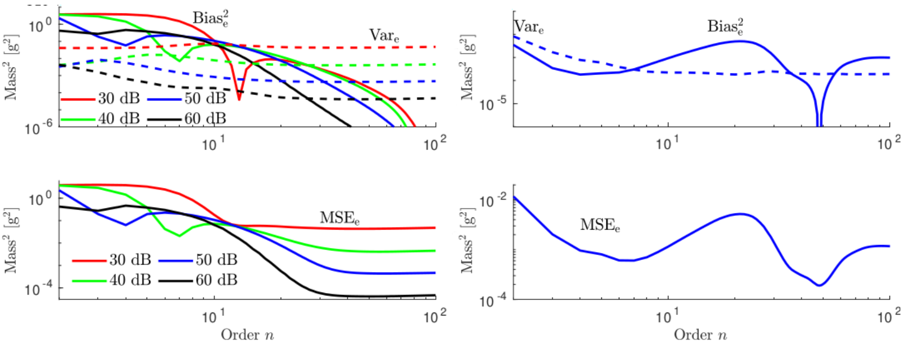
- The input estimation MSE is near the CRLB for biased estimators.

With experimental data,
the bias and variance decrease with sample size

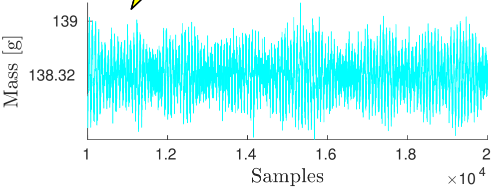


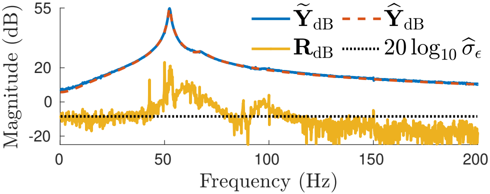
- The predictions are useful to assess the uncertainty for given sample size and noise variance.

In simulation (left) and in experiments (right), the estimation MSE does not decrease with order *n*



There are frequency components in measurement noise





- The step input estimation method is useful when the measurement noise is not white.

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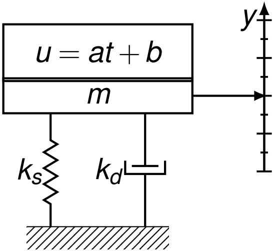
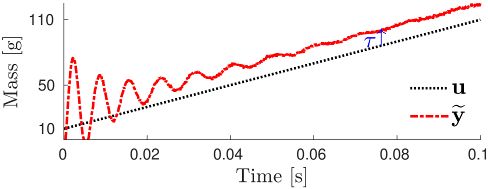
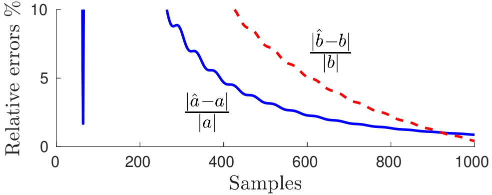
Can we estimate affine inputs with a data-driven method?

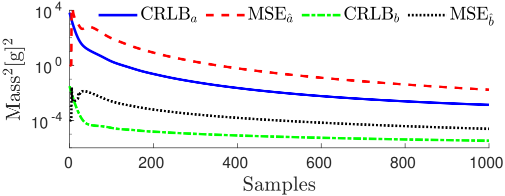


- An affine input turns the weighing system into time varying.



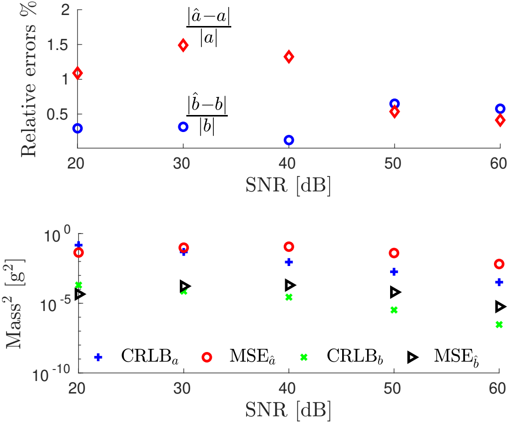
A data-driven affine input estimation method is adapted using exponentially weighted recursive LS





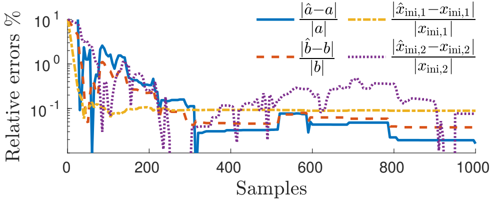
- Computationally cheap, and suitable for online implementation.
- MSEs become one order of magnitude larger than CRLBs.

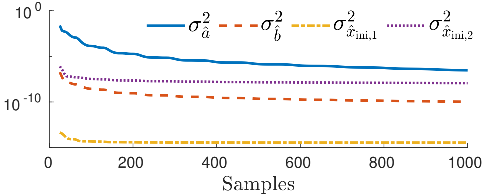
The data-driven affine input estimation method results with respect to SNR



- The benefit of having the MSEs near the CRLBs is inherited from the data-driven step input estimation method.

A maximum-likelihood affine input estimation method estimates also the initial conditions

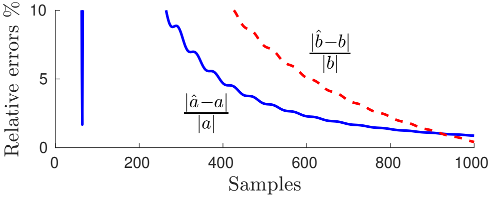




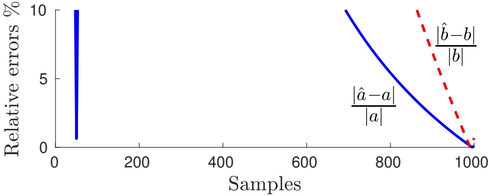
- The parameters converge after three iterations, but
- the required computational power is large.

The proposed methods outperform a conventional time varying filter

- Data-driven method results:



- Time-varying filter results:



- The methods process the same sensor response.

Conclusions

- A structured and correlated EIV problem
  - was analyzed to find its stochastic properties,
  - now we can predict the bias and variance of the LS solution,
  - for given sample size and perturbation level.
  - The solution MSE is considerably near to the CRLB.

Conclusions

- The data-driven input estimation methods
    - are valid for metrology applications,
    - are complete with the uncertainty assessment,
    - users can select sample size for desired MSE,
    - are robust under non Gaussian white noise,
    - can process responses from time-varying sensors.

The proposed future work is as follows

- Generalize the data-driven methods to estimate other input models.
- Design efficient solution methods for structured EIV problems to reduce bias.
- Design efficient online optimization methods to reduce its computational burden.