Exercícios Estatística Matemática
Capítulo 4

Sumário

0.1 Questão 1

a) Podemos encontrar a distribuição de W = X+Y através do método Jacobiano utilizando a distribuição conjuntar de X e Y. $\begin{cases} W = X - Y \\ Z = Y \end{cases}$

$$f_{W,Z}(w,z) = (1-p)^{w+2z}p^2$$

$$f_W(w) = \begin{cases} \sum\limits_{z=-w}^{\infty} (1-p)^{w+2z}p^2, & w < 0 \\ \sum\limits_{z=0}^{\infty} (1-p)^{w+2z}p^2, & w \geq 0 \end{cases}$$

$$f_W(w) = \begin{cases} \frac{p(1-p)^{-w}}{2-p} & w < 0 \\ \frac{p(1-p)^{w}}{2-p}, & w \geq 0 \end{cases}$$

$$P(W=0) = \frac{p}{2-p}$$

$$P(W<0) = 1 - P(W \geq 0) = 1 - \sum_{w=0}^{\infty} \frac{p(1-p)^w}{2-p} = \frac{1-p}{2-p}$$

b) Primeiro podemos encontrar a Distribuição de T = X+Y da mesma forma em que encontramos a distribuição de W.
$\begin{cases} T = X + Y \\ Z = Y \end{cases}$

$$f_{T,Z}(t,z) = p^2(t-1)(1-p)^t, \qquad t > z > 0$$

$$f_T(t) = \sum_{z=0}^{t} f_{T,Z}(t,z)$$

$$f_T(t) = (1-p)^t p^2(1+t), \qquad t > 0$$

$$P(X = x | X+Y = w) = \frac{P(X=x, Y=y)}{P(T=t)} = \frac{(1-p)^{(x+y)}p^2}{(1-p)^{(t)}p^2(t+1)}$$

$$\frac{(1-p)^{(x+y)}p^2}{(1-p)^{(x+y)}p^2(t+1)} = \frac{1}{t+1}$$

0.2 Questão 7

a)

$$P(\boldsymbol{X} = x_1, \ldots, x_n) = \frac{n!}{x_1!x_2!\cdots x_n!}p_1^{x_1}p_2^{x_2}\cdots p_n^{x_n}$$

$$P(X_1 = x_1) = \frac{n!}{x_1!}p_1^{x_1}\sum_{x_2,\ldots,x_n}^{n-x_1}\frac{1}{x_2!\cdots x_n!}p_2^{x_2}\cdots p_n^{x_n}$$

Sabemos que $\sum_i^n x_i = n$ e $\sum_i^n p_i = 1$, então:

$$= \frac{n!}{x_1!(n-x_1)!}p_1^{x_1}\sum_{x_2,\ldots,x_n}^{n-x_1}\frac{(n-x_1)!}{x_2!\cdots x_n!}p_2^{x_2}\cdots p_n^{x_n}$$

Pelo teorema multinomial, temos que:

$$\sum_{x_2,\ldots,x_n}^{n-x_1}\frac{(n-x_1)!}{x_2!\cdots x_n!}p_2^{x_2}\cdots p_n^{x_n} = (p_2 + \cdots + p_n)^{n-x_1} = (1-p_1)^{n-x_1}$$

$$\therefore P(X_i = x_i) = \frac{n!}{x_i!(n-x_i)!}p_i^{x_i}(1-p_i)^{n-x_i}\forall i \in \{1, \ldots, n\}$$

0.3 Questão 10

$X \sim Bin(n,p)$ $Y \sim Bin(m,p)$

$$P(X|X+Y) = \frac{P(X=x)P(Y=z-x)}{P(X+Y=s)} = \frac{P(X=x)P(Y=z-x)}{P(X+Y=s)}$$

$$P(X+Y=t) = \sum_x \binom{n}{x}\binom{m}{z-x} p^x(1-p)^{n-x}p^{z-x}(1-p)^{m-z+x}$$

$$= \sum_x \binom{n}{x}\binom{m}{z-x} p^z(1-p)^{(n+m)-z} = \binom{n+m}{z} p^z(1-p)^{(n+m)-z}$$

$$\therefore P(X|X+Y) = \frac{\binom{n}{x}\binom{m}{z-x}\cancel{p^x(1-p)^{n-x}p^{z-x}(1-p)^{m-z+x}}}{\binom{n+m}{z}\cancel{p^z(1-p)^{(n+m)-z}}} = \frac{\binom{n}{x}\binom{m}{z-x}}{\binom{n+m}{z}} \sim HiperGeo(m+n,m)$$

0.4 Questão 15

$X \sim U[0,1]$ e $U[-x,x]$

$$f(x,y) = f(y)\cdot f(x|y) =$$

$$f(y) = \int_{-|y|}^{1} \frac{1}{2x}dx = \frac{1}{2}log(x)\Big|_{|y|}^{1} = -\frac{1}{2}log(|y|)$$

$$f(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2x}}{-\frac{1}{2}log(|y|)} = -\frac{1}{xlog(|y|)}$$

0.5 Questão 27

$$Cov(X, E(Y|X)) = E(X \cdot E(Y|X))) - E(X) \cdot E(E(Y|X))$$
$$E(E(Y|X)) = E(Y) \tag{1}$$
$$E(X \cdot E(Y|X))) = E(E(X \cdot E(Y|X)))|X) \overbrace{=}^{P5} E(XY) \tag{2}$$

Então, por (1) e (2)

$$Cov(X, Y) = Cov(X, E(Y|X))$$

0.6 Questão 28

$$f(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0, & c.c \end{cases}$$

a)

$$f(y|x) = \frac{f(x, y)}{f(x)}$$
$$f(x) = \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-x^2}}{\pi}$$
$$\therefore f(y|x) = \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-x^2}}{\pi}} = \frac{1}{2\sqrt{1-x^2}}$$
$$E(X|Y) = \frac{1}{\pi} \int \frac{x}{2\sqrt{1-x^2}} dx = 0$$

b)

$$f(y) = \frac{2\sqrt{1-y^2}}{\pi}$$

$$f(x) \cdot f(y) = \frac{2\sqrt{1-x^2}}{\pi} \cdot \frac{2\sqrt{1-y^2}}{\pi} \neq f(x, y)$$

c)

$$Cov(X, Y) = Cov(X, E(Y|X)) = Cov(X, 0) = 0$$

0.7 Questão 34

A desigualdade de Cauchy-Schwarz nos diz que:

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

Podemos definir:

$$\mathbf{I}_1 = \begin{cases} 1, & X \leq x \\ 0, & c.c \end{cases}$$

$$\mathbf{I}_2 = \begin{cases} 1, & Y \leq y \\ 0, & c.c \end{cases}$$

Aplicando a desigualdade de Cauchy-Schwarz, temos:

$$[E(\mathbf{I}_1 \cdot \mathbf{I}_2)]^2 \leq E(\mathbf{I}_1^2)E(\mathbf{I}_2^2)$$

Note que a função indicadora é idempotente, então:

$$[E(I_{\{X \leq x, Y \leq y\}})]^2 \leq E(\mathbf{I}_1)E(\mathbf{I}_2)$$

Também é conhecido que a $E(\mathbf{I}_A) = P(A)$, logo:

$$[P(X \leq x, Y \leq y)]^2 \leq P(X \leq x)P(Y \leq y)$$

$$\therefore P(X \leq x, Y \leq y) \leq \sqrt{P(X \leq x)P(Y \leq y)}$$

0.8 Questão 35

$X \sim Exp(\frac{1}{2})$ e $Y|X \sim U[0, x^2]$

a) $V = Y|X^2$
   Podemos considerar a região $A : A \in (t - \varepsilon, t + \varepsilon)$, Então:
   $P(X^2 \in A) = P(t - \varepsilon < X^2 < t + \varepsilon) = P(\sqrt{t - \varepsilon} < X < \sqrt{t + \varepsilon}) = P(X \in \sqrt{A})$

   Podemos calcular
$$\lim_{\varepsilon \to 0} P(Y|X \in \sqrt{A}) = P(Y|X^2 = t)$$
$$\therefore Y|X^2 \sim U[0, t]$$

b) $E(X) = \frac{1}{\frac{1}{2}} = 2$
   $E(Y) = E(E(Y|X)) = E\left(\frac{x^2}{2}\right) = \frac{Var(X)+E(X)^2}{2} = \frac{4+4}{2} = 2$
   $cov(X, Y) = Cov(X, E(Y|X)) = E(X \cdot E(Y|X)) - E(X)E(E(Y|X))$
   $= \frac{1}{2}6 \cdot 8 - 2 \cdot 4 = 16$

0.9 Questão 40

$f(x, y) = \frac{1}{2x}$, $-\frac{1}{2}log(|y|)$, $X \sim U[0, 1]$, $Y \sim U[-x, x]$
$f(x|y) = \frac{-1}{xlog(|y|)}$
$E(x|y) = \int_{|y|}^{1} xf(x|y)dx = x\frac{-1}{xlog(|y|)}\Big|_{|y|}^{1} = \frac{|y|-1}{log(y)}$
$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{f(x|y)f(y)}{f(x)}$
$f(x) = \int_{-x}^{x} \frac{1}{2x}dy = 1$
$\therefore f(y|x) = \frac{f(x|y)f(y)}{1} = \frac{-1}{xlog(|y|)} - \frac{1}{2}log(|y|) = \frac{1}{2x}$
$E[y|x] = \int_{-x}^{x} y\frac{1}{2x}dy = 0$
$Cov(X, Y) = Cov(X, E(Y|X)) = E(X, E(Y|X)) - E(X)E(Y|X)$
$= E(X \cdot 0) - E(X) \cdot 0 = 0$

0.10 Questão 41

a) Variável aleatória discreta.

b) $E(Y) = E(E(Y|X)) = E(nX) = nE(X) = n\frac{1}{2}$
   $Var(Y) = E(Y^2) - E(Y)^2$
   $E(Y^2) = E(E(Y^2|X)) = E(n(n-1)x^2 + xn) = n(n-1)E(x^2) + E(X)n = n(n-1)\frac{1}{3} + n\frac{1}{2}$

$$\therefore Var(Y) = \frac{n^2}{3} - \frac{n}{3} + \frac{n}{2} - \frac{n^2}{4} = \frac{n(n+2)}{12}$$