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A Class of Bivariate Modified Weighted Distributions: Properties and Applications

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Abstract
In this paper, new bivariate weighted distributions are introduced based on Marshall and Olkin concept, different properties of these distributions are discussed. Moreover, the joint pdf, joint survival function, joint cdf, joint hazard function, product moments, marginal conditional density, and moment generating function are obtained explicitly in compact forms. Furthermore, it is shown that the new bivariate weighted distributions are obtained from the Marshall and Olkin survival copula, and a tail dependence measure is discussed. Explicit Bayesian estimators are obtained for the unknown parameters of these models and MLE are also discussed. Three data sets have been re-analyzed for illustrative purposes. Some simulations to see the performances of the estimators are performed. Absolutely continuous bivariate versions of these distributions are obtained and some of their properties are discussed.

Keywords Weighted distributions · Weibull distribution · Gumbel distribution · Maximum likelihood estimation · Bayesian estimation · Marginal distributions · Reversed hazard function

1 Introduction

The weighted distribution gives a technique for fitting models to the unknown weight functions when samples can be taken both from the parent distribution and the developed ones. It provides collective access to the problems of model specification and data interpretation. It has multiple applications in the studies related to reliability, analysis, Meta-analysis and analysis of intervention data, biomedicine,

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ecology, which used as a tool in the selection of suitable models for observed data depends on the choice of the weight function.

Fisher [1] and Rao [2] are the first who provided the concept of weighted distributions as follows: suppose X is a non-negative random variable with pdf $f_X(x)$. The pdf of the weighted random variable X_w is given by: $f_w(x) = \frac{f(x).w(x)}{E[w(x)]}$.Where $E[w(x)] = \int_{-\infty}^{\infty} w(x)f(x)dx$, and $w(x)$ be a non-negative weight function.

A new class of weighted distribution called modified weighted family of distributions is introduced by Aleem et al. [3]. The main aim of this paper is to introduce the bivariate extension of this family based on an idea similar to that of Theorem 3.2 proposed by Marshall and Olkin [4]. These authors introduced a multivariate exponential distribution whose marginals have exponential distributions and proposed a bivariate Weibull distribution.

The proposed bivariate weighted distributions are constructed from three independent distributions using a minimization process. These new distributions are singular, and they can be used quite conveniently if there are ties in the data. hence the new weighted models belong to the bivariate Marshal-Olkin family of distributions, which has great importance and has multiple applications such as; in Biological studies; Study of (blindness in the left and right eye, the age at death of parent and child in a genetic study, the relation between blood pressure and body weight for a patient and the failure time of the left and right kidney), in Economic studies; Study the relation between (years of education and personal income, personal income and expenditure and inflation and unemployment), in engineering studies; analyzing the lifetime of a twin-engine plane, also warranty policies based on failure time and warranty servicing time, as well as, different applications like shock model, competing risks model, stress model, maintenance model and longevity model.

The Weibull distribution and other related distribution like exponential, Rayleigh, and extreme value (Gumbel) distributions are very useful in survival analysis, reliability renewal theory, and branching processes. So these distributions will be used as parent distributions to produce bivariate weighted versions of them. Moreover, because of the wide applicability of the Weibull distribution, Dey et al. [5] introduced a new extension of it in both univariate and bivariate cases.

Bivariate statistical and data mining methods afford the opportunity to analyze two variables together, in order to understand how they function as a system, and how this system may change as the two variables changed together. By the way, Shi et al. [6] defining data mining algorithms as any analytic process of using artificial intelligence, statistics, optimization, and other mathematics algorithms to carry out more advanced data analysis than data pre-processing. The data mining tools are any commercial or non-commercial software packages performing data mining methods. Some researchers in the field of data mining have realized its importance of handling the massive rules or hidden patterns from data mining (Olson and Shi [7] and Tien [8]).And they deduced that finding suitable "relation measurement" to measure the interdependence between data, information and intelligent knowledge is a challenging task that research on appropriate classification and expression of intelligent knowledge may contribute to the establishment of the general theory of data mining, which has been a long-term unresolved problem.

The paper is organized as follows: In Sect. 2, the modified weighted Weibull distribution is introduced with its related distributions. The bivariate weighted Weibull distribution is presented in Sect. 3. The bivariate weighted Gumbel distribution is presented in Sect. 4. A numerical study is discussed in Sect. 5. Finally, conclude the paper in Sect. 6.

2 Modified Weighted Weibull Distribution and Related Distributions

Aleem et al. [3] defined the modified weighted Weibull (MWW) distribution based on the probability density function

$$f_W(x) = \frac{[1 - w(t(x))]^c f_X(x)}{E[1 - w(t(x))]^c}\,, \quad -\infty < x < \infty \tag{1}$$

where $0 < E[1 - w(t(x))] < \infty$. If $w(t(x)) = F_X(\theta x)$, then (1) becomes

$$f_W(x) = \frac{\left[1 - F_X(\theta x)\right]^c f_X(x)}{E\left[1 - F_X(\theta x)\right]^c}\,, \quad 0 < x < \infty \tag{2}$$

where θ and c are additional parameters. In the case of Weibull distribution which is defined by the following pdf and cdf respectively, as

$$f_X(x;\beta,\gamma) = \beta\gamma x^{\gamma-1}e^{-\beta x^\gamma} \text{ and } F_X(x;\beta,\gamma) = 1 - e^{-\beta x^\gamma} \tag{3}$$

and hence $F_X(\theta x) = 1 - e^{-\beta(\theta x)^\gamma}$.

$$\text{So, } E\left[1 - F_X(\theta x)\right]^c = \left[c\theta^\gamma + 1\right]^{-1} \tag{4}$$

Now, from (3) and (4) in (2), the MWW distribution is given as

$$f_{MWW}(x;\beta,\gamma,\theta,c) = \beta\gamma(c\theta^\gamma + 1)x^{\gamma-1}e^{-\beta x^\gamma(c\theta^\gamma+1)} \tag{5}$$

$$S_{MWW}(x;\beta,\gamma,\theta,c) = e^{-\beta x^\gamma(c\theta^\gamma+1)} \tag{6}$$

$$h_{MWW}(x;\beta,\gamma,\theta,c) = \beta\gamma(c\theta^\gamma + 1)x^{\gamma-1} \tag{7}$$

where S(.) and h(.) are the survival and hazard functions respectively.

MWW distribution is denoted by $MWW\ (\beta,\gamma,\theta,c)$. Moreover, the rth moment about zero for MWW distribution is given as $\acute{\mu}_r = \gamma(c\theta^\gamma + 1)\beta^{-\frac{r}{\gamma}}\Gamma\left(\frac{r}{\gamma+1}\right)(c\theta^\gamma + 1)^{-\frac{r}{\gamma}-1}$.

2.1 Special Cases and Related Distributions

(1) If $\gamma = 2$, then (5) reduced to the modified weighted Rayleigh (MWR)distribution

$$f_{MWR}(x;\beta,\theta,c) = 2\beta\left(c\theta^2 + 1\right)xe^{-\beta x^2\left(c\theta^2+1\right)} \tag{8}$$

(2) If $\gamma = 1$, then (5) reduced to the modified weighted exponential (MWE) distribution

$$f_{MWE}(x;\beta,\theta,c) = \beta(c\theta + 1)e^{-\beta x(c\theta+1)} \tag{9}$$

(3) If $Y = \log(\beta X^\gamma)$ and X is distributed as MWW distribution, hence Y is distributed as modified weighted extreme value distribution or modified weighted Gumbel (MWG) distribution with pdf and cdf respectively,

$$f_{MWG}(y;\gamma,\theta,c) = (c\theta^\gamma + 1)e^y e^{-e^y(c\theta^\gamma+1)}, \quad -\infty < y < \infty \tag{10}$$

$$F_{MWG}(y;\gamma,\theta,c) = 1 - e^{-e^y(c\theta^\gamma+1)}, \quad -\infty < y < \infty \tag{11}$$

$$S_{MWG}(y;\gamma,\theta,c) = e^{-e^y(c\theta^\gamma+1)}, \quad -\infty < y < \infty \tag{12}$$

$$h_{MWG}(y;\gamma,\theta,c) = (c\theta^\gamma + 1)e^y, \quad -\infty < y < \infty \tag{13}$$

The MWG can be denoted by $MWG(\gamma,\theta,c)$.The bivariate extensions for both MWW and MWG distributions will be discussed for the first time in detail in the next sections.

3 Bivariate Modified Weighted Weibull Distribution

In this section, the BMWW distribution will be introduced and its properties will be established. Suppose that $U_i \sim MWW\left(\beta_i,\gamma,\theta,c\right),\ i = 1,2,3$ such that $U_i's$ are mutually independent random variables and define $X_j = min\left(U_j, U_3\right),\ j = 1,2$. Such that; $X_j's$ are dependent random variables. Hence the joint survival function of the vector $\left(X_1,X_2\right)$ denoted by $S_{BMWW}\left(x_1,x_2\right)$ is given as

$$\begin{aligned} S_{BMWW}\left(x_1,x_2\right) =& S_{MWW}\left(x_1;\beta_1\right)S_{MWW}\left(x_2;\beta_2\right)S_{MWW}\left(x_3;\beta_3\right) \\ =& \exp\left\{-(c\theta^\gamma + 1)\left[\beta_1 x_1^\gamma + \beta_2 x_2^\gamma + \beta_3 x_3^\gamma\right]\right\} \end{aligned} \tag{14}$$

where $x_3 = \max\left(x_1,x_2\right)$.

The joint survival function of BMWW distribution can be extended in the following form

$$S_{BMWW}\left(x_1,x_2\right) = \begin{cases} S_{MWW}\left(x_1;\beta_1\right)S_{MWW}\left(x_2;\beta_{23}\right), & x_1 < x_2 \\ S_{MWW}\left(x_1;\beta_{13}\right)S_{MWW}\left(x_2;\beta_2\right), & x_1 > x_2 \\ S_{MWW}\left(x;\beta_{123}\right), & x_1 = x_2 = x \end{cases} \tag{15}$$

where $\beta_{13} = \beta_1 + \beta_3$, $\beta_{23} = \beta_2 + \beta_3$ and $\beta_{123} = \beta_1 + \beta_2 + \beta_3$.

Accordingly, the joint pdf of BMWW distribution can be obtained as

$$f_{BMWW}(x_1,x_2)=\begin{cases} f_{MWW}(x_1;\beta_1)f_{MWW}(x_2;\beta_{23}), & x_1<x_2\\ f_{MWW}(x_1;\beta_{13})f_{MWW}(x_2;\beta_2), & x_1>x_2\\ \frac{\beta_3}{\beta_{123}}f_{MWW}(x;\beta_{123}), & x_1=x_2\end{cases} \tag{16}$$

and its graph is shown in Fig. 1 for some parameters values.

The joint cdf of the BMWW distribution is given as

$$F_{BMWW}(x_1,x_2)=\begin{cases} F_{MWW}(x_1;\beta_{13})-F_{MWW}(x_1;\beta_1)\left[1-F_{MWW}(x_2;\beta_{23})\right], & x_1<x_2\\ F_{MWW}(x_2;\beta_{23})-F_{MWW}(x_2;\beta_2)\left[1-F_{MWW}(x_1;\beta_{13})\right], & x_2<x_1\\ 1-F_{MWW}(x;\beta_{123}), & x_1=x_2=x\end{cases} \tag{17}$$

The joint hazard function of the BMWW distribution is given as

$$h_{BMWW}(x_1,x_2)=\begin{cases} h_{MWW}(x_1;\beta_1)h_{MWW}(x_2;\beta_{23}), & x_1<x_2\\ h_{MWW}(x_1;\beta_{13})h_{MWW}(x_2;\beta_2), & x_1>x_2\\ \frac{\beta_3}{\beta_{123}}h_{MWW}(x;\beta_{123}), & x_1=x_2=x.\end{cases} \tag{18}$$

Moreover, its graph is shown in Fig. 2 for some parameters values.

The BMWW distribution has both an absolutely continuous part and a singular part. The joint survival function of the BMWW distribution can be factorized into the absolutely continuous part and singular part as follows in the following form

$$S_{\mathrm{BMWW}}(x_1,x_2)=\frac{\beta_{12}}{\beta_{123}}S_a(x_1,x_2)+\frac{\beta_3}{\beta_{123}}S_s(x_3) \tag{19}$$

where $x_3=\max(x_1,x_2)$, $S_s(x_3)=S_{MWW}(x;\beta_{123})$,$\beta_{123}=\beta_1+\beta_2+\beta_3$, and $S_a(x_1,x_2)=\frac{\beta_{123}}{\beta_{12}}S_{MWW}(x_1;\beta_1)S_{MWW}(x_2;\beta_2)S_{MWW}(x_3;\beta_3)-\frac{\beta_3}{\beta_{12}}S_{MWW}(x;\beta_{123})$.

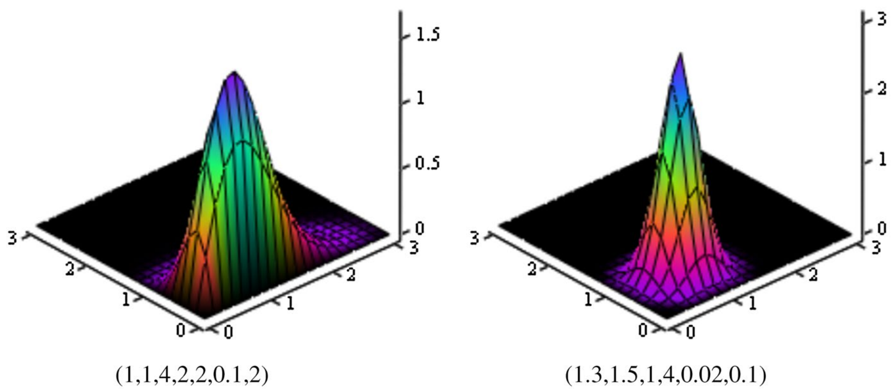

Fig. 1 Joint pdf for BMWW distribution for some value of $(\beta_1,\beta_2,\beta_3,\gamma,\theta,c)$

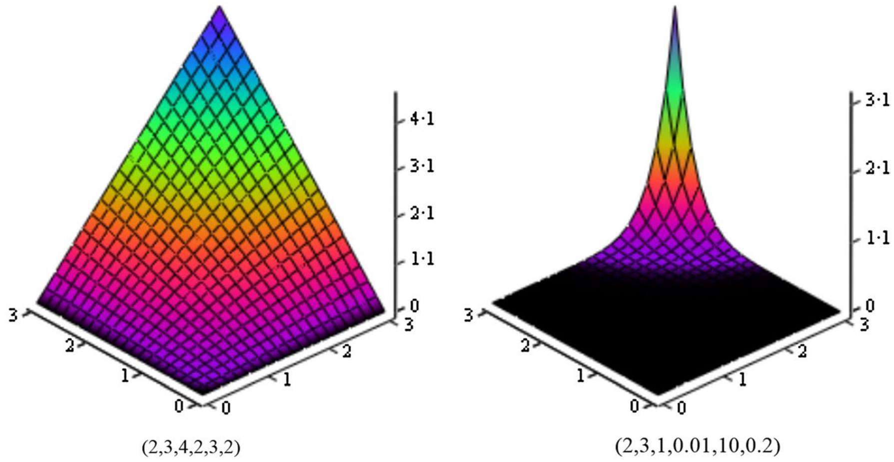

Fig. 2 Joint hazard function for BMWW distribution for some value of $(\beta_1, \beta_2, \beta_3, \gamma, \theta, c)$

Its cleat that, $S_s(\ldots)$ and $S_a(\ldots)$ are the singular and absolutely continuous parts respectively .

Accordingly, the pdf of the BMWW model can be factorized into an absolutely continuous part and singular part as follows

$$f_{\mathrm{BMWW}}(x_1, x_2) = \frac{\beta_{12}}{\beta_{123}} f_a(x_1, x_2) + \frac{\beta_3}{\beta_{123}} f_s(x_3) \tag{20}$$

where

$$f_a(x_1, x_2) = \frac{\beta_{123}}{\beta_{12}} \begin{cases} f_{MWW}(x_1;\beta_{13}) f_{MWW}(x_2;\beta_2), & x_1 < x_2 \\ f_{MWW}(x_1;\beta_1) f_{MWW}(x_2;\beta_{23}), & x_1 < x_2 \end{cases}$$

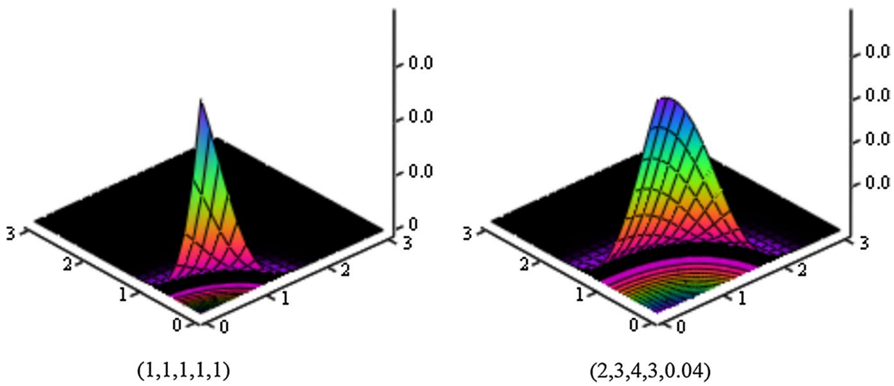

Fig. 3 Joint pdf for BMWG distribution for some value of $(c_1, c_2, c_3, \gamma, \theta)$

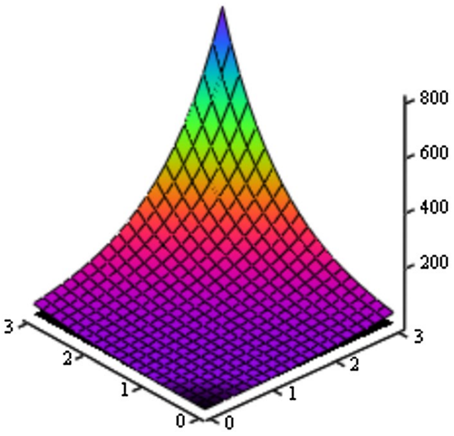

Fig. 4 Joint hazard function for BMWG distribution for some value of $\left(c_1, c_2, c_3, \gamma, \theta\right) = (2, 3, 4, 3, 0.04)$

and $f_s\left(x_3\right) = f_{MWW}\left(x; \beta_{123}\right)$.

Clearly, here $f_a(x_1, x_2)$ and $f_s(x_3)$ are the absolutely continuous and singular parts respectively.

The absolutely continuous part of the BMWW density may be unimodal depending on the values of β_3,θ and γ, that is $f_a(x_1, x_2)$ is unimodal and the respective modes are

$$\left\{ \left[\frac{\gamma - 1}{\beta_1 \gamma (c\theta^\gamma + 1)}\right]^{-\gamma}, \left[\frac{\gamma - 1}{\beta_{23} \gamma (c\theta^\gamma + 1)}\right]^{-\gamma} \right\} and \left\{ \left[\frac{\gamma - 1}{\beta_{13} \gamma (c\theta^\gamma + 1)}\right]^{-\gamma}, \left[\frac{\gamma - 1}{\beta_2 \gamma (c\theta^\gamma + 1)}\right]^{-\gamma} \right\}$$

The median for the absolutely continuous BMWW distribution is given as follows

$$\left[\frac{\ln 2}{\beta_{123}(c\theta^\gamma + 1)}\right]^{-\gamma}$$

It is important to mention that the marginal distributions of the BMWW distribution are univariate MWW with the following survival and density functions respectively,

$$S_{X_i}\left(x_i\right) = S_{MWW}\left(x_i; \beta_{i3}\right) = e^{-\beta_{i3}(c\theta^\gamma + 1)x_i^\gamma}, \quad i = 1, 2 \tag{21}$$

$$f_{X_i}\left(x_i\right) = f_{MWW}\left(x_i; \beta_{i3}\right) = \beta_{i3}\, \gamma\, (c\theta^\gamma + 1)\, x_i^{\gamma - 1}\, e^{-\beta_{i3}(c\theta^\gamma + 1)x_i^\gamma}, \quad i = 1, 2 \tag{22}$$

such that $\beta_{i3} = \beta_i + \beta_3, \quad i = 1, 2.$

Moreover, the distribution of the minimum of $\left(X_1, X_2\right) \sim BMWW\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ is also univariate MWW with shape parameter β_{123}. And the survival and density functions are given as follows

$$S_{\min(X_1, X_2)}(x) = S_{MWW}\left(x; \beta_{123}\right) = e^{-\beta_{123}(c\theta^\gamma + 1)x^\gamma} \tag{23}$$

and

$$f_{\min(X_1,X_2)}(x) = f_{MWW}\left(x;\beta_{123}\right) = \beta_{123}\,(c\theta^{\gamma}+1)x^{\gamma-1}e^{-\beta_{123}(c\theta^{\gamma}+1)x^{\gamma}} \tag{24}$$

where $x=\min\left(x_1,x_2\right)$ and $\beta_{123}=\beta_1+\beta_2+\beta_3$.

Based on the fact that marginal distributions of the vector $\left(X_1,X_2\right) \sim BMWW\left(\beta_1,\beta_2,\beta_3,\gamma,\theta,c\right)$ are univariate MWW distributions, then the conditional density of X_i given $X_j=x_j,\ i\neq j$ is calculated as follows

$$f_{i/j}\left(x_i/x_j\right)=\begin{cases} f_{i/j}^{(1)}\left(x_i/x_j\right), & x_i<x_j\\ f_{i/j}^{(2)}\left(x_i/x_j\right), & x_i>x_j\\ f_{i}^{(3)}\left(x_i\right), & x_i=x_j\end{cases} \tag{25}$$

where $f_{i/j}^{(1)}\left(x_i/x_j\right)=\beta_1\,\gamma(c\theta^{\gamma}+1)x_i^{\gamma-1}e^{-\beta_1(c\theta^{\gamma}+1)x_i^{\gamma}}$,

$$f_{i/j}^{(2)}\left(x_i/x_j\right)=\frac{\beta_2\beta_{13}}{\beta_{23}}\gamma(c\theta^{\gamma}+1)x_i^{\gamma-1}e^{-(c\theta^{\gamma}+1)\left[\beta_{13}x_i^{\gamma}-\beta_3x_j^{\gamma}\right]},$$

and $f_{i}^{(3)}\left(x_i\right)=\dfrac{\beta_3}{\beta_{23}}\left(\dfrac{x_i}{x_j}\right)^{\gamma-1}e^{-(c\theta^{\gamma}+1)\left[\beta_{123}x_i^{\gamma}-\beta_{23}x_j^{\gamma}\right]}$

According to the fact that the marginal distributions of the vector $\left(X_1,X_2\right)$ are univariate MWW distributions, then the moments of X_1 and X_2 can be obtained directly from the following marginals

$$E\left(X_1^k\right)=\ \gamma(c\theta^{\gamma}+1)\left(\beta_1+\beta_3\right)^{-\frac{k}{\gamma}}\Gamma\left(\frac{k}{\gamma+1}\right)(c\theta^{\gamma}+1)^{-\frac{k}{\gamma}-1}\text{ and}$$

$$E\left(X_2^k\right)=\ \gamma(c\theta^{\gamma}+1)\left(\beta_2+\beta_3\right)^{-\frac{k}{\gamma}}\Gamma\left(\frac{k}{\gamma+1}\right)(c\theta^{\gamma}+1)^{-\frac{k}{\gamma}-1}$$

Now, the rth and sth joint moments of $\left(X_1,X_2\right) \sim BMWW\left(\beta_1,\beta_2,\beta_3,\gamma,\theta,c\right)$, can be given by the following formula

$$
\begin{aligned}
E\left(X_1^r X_2^s\right) = & \frac{\beta_{23}(c\theta^\gamma+1)\Gamma\left(\frac{r+s}{\gamma}+2\right)}{\left[\beta_1(c\theta^\gamma+1)\right]^{\frac{r}{\gamma}}\left(\frac{r}{\gamma}+1\right)\left[1+\beta_{23}(c\theta^\gamma+1)\right]^{\frac{r+s}{\gamma}+2}} \\
& .F\left(1,\frac{r+s}{\gamma}+2;2+\frac{r}{\gamma};\left[1+\beta_{23}(c\theta^\gamma+1)\right]^{-1}\right) \\
& +\frac{\beta_{13}(c\theta^\gamma+1)\Gamma\left(\frac{r+s}{\gamma}+2\right)}{\left[\beta_2(c\theta^\gamma+1)\right]^{\frac{s}{\gamma}}\left(\frac{s}{\gamma}+1\right)\left[1+\beta_{13}(c\theta^\gamma+1)\right]^{\frac{r+s}{\gamma}+2}} \\
& .F\left(1,\frac{r+s}{\gamma}+2;2+\frac{s}{\gamma};\left[1+\beta_{13}(c\theta^\gamma+1)\right]^{-1}\right) \\
& +\frac{\beta_3(c\theta^\gamma+1)}{\left[\beta_{123}(c\theta^\gamma+1)\right]^{\frac{r+s}{\gamma}}}\Gamma\left(\frac{r+s}{\gamma}+1\right)
\end{aligned}
\tag{26}
$$

where $\Gamma(r)=\int\limits_0^\infty x^{r-1}e^{-x}dx$ is the gamma function and $F(a,b;c;z)=\sum_{s=0}^\infty \frac{(a)_s\,(b)_s}{\Gamma(c+s)s!}$. is a hypergeometric function.

3.1 Copula and Correlation Measures

In this section, how the BMWW distribution can be obtained through copulas is discussed. According to Sklar [9] let H be a bivariate distribution function with continuous marginals F_1 and F_2. moreover, let $\bar{F}_1$, $\bar{F}_2$ and $\bar{H}$ be the survival functions corresponding to F_1, F_2 and H respectively, its follows that there exist a unique function $C:[0,1]^2\rightarrow[0,1]$, which is named copula, such that $H\left(x_1,x_2\right)=C\left(F_1\left(x_1\right),\ F_2\left(x_2\right)\right)$. Analogously to Sklar's theorem, it follows there exists a unique function $\hat{C}:[0,1]^2\rightarrow[0,1]$, which is called survival copula such that $\bar{H}\left(x_1,x_2\right)=\hat{C}\left(\bar{F}_1\left(x_1\right),\ \bar{F}_2\left(x_2\right)\right)$.

The Marshall Olkin bivariate survival copula of $\left(X_1,X_2\right)$ is given by

$$\hat{C}\left(u_1,\ u_2\right)=u_1u_2\min\left(u_1^{-\vartheta_1},u_2^{-\vartheta_2}\right)$$

which is satisfies the following relation

$$S_{BMWW}\left(x_1,x_2\right)=\hat{C}\left(S_{MWW}\left(x_1;\beta_{13}\right),\ S_{MWW}\left(x_2;\beta_{23}\right)\right).$$

This construction leads to a copula family given by

$$
\begin{aligned}
C_{\vartheta_1,\vartheta_2}\left(u_1,\ u_2\right) & =\min\left(u_1^{1-\vartheta_1}u_2,\ u_1u_2^{1-\vartheta_2}\right) \\
& =\begin{cases} u_1^{1-\vartheta_1}\,u_2 & u_1^{\vartheta_1}>u_2^{\vartheta_2} \\ u_1u_2^{1-\vartheta_2} & u_1^{\vartheta_1}<u_2^{\vartheta_2}\end{cases}
\end{aligned}
$$

where $\vartheta_1 = \frac{\beta_3}{\beta_1 + \beta_3}$, $\vartheta_2 = \frac{\beta_3}{\beta_2 + \beta_3}$, $u_1 = F_1(x_1)$ and $u_2 = F_2(x_2)$.

It is noted that Marshall-Olkin copulas have both an absolutely continuous and a singular components since

$$\frac{\partial^2}{\partial u_1 \partial u_2} C_{\vartheta_1,\vartheta_2}(u_1, u_2) = \begin{cases} u_1^{-\vartheta_1} & u_1^{\vartheta_1} > u_2^{\vartheta_2} \\ u_2^{-\vartheta_2} & u_1^{\vartheta_1} < u_2^{\vartheta_2} \end{cases}$$

The mass of the singular component is concentrated on the curve $u_1^{\vartheta_1} = u_2^{\vartheta_2}$ in $[0, 1]^2$.

A copula provides a natural way to study and measure dependence between random variables. Spearman's rho $\rho S(C_{\vartheta_1,\vartheta_2})$ and Kendall's tau $\tau(C_{\vartheta_1,\vartheta_2})$ are quite easily evaluated respectively, for the Marshall-Olkin BMWW model as following

$$\begin{aligned} \rho S(C_{\vartheta_1,\vartheta_2}) &= 12 \int_0^1 \int_0^1 C_{\vartheta_1,\vartheta_2}(u, v) du dv - 3 \\ &= \frac{3\vartheta_1 \vartheta_2}{2\vartheta_1 + 2\vartheta_2 - \vartheta_1 \vartheta_2} = \frac{3\beta_3}{2\beta_1 + 2\beta_2 + 3\beta_3} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \tau(C_{\vartheta_1,\vartheta_2}) &= 4 \int_0^1 \int_0^1 C_{\vartheta_1,\vartheta_2}(u, v) dC_{\vartheta_1,\vartheta_2}(u, v) - 1 \\ &= \frac{\vartheta_1 \vartheta_2}{\vartheta_1 + \vartheta_2 - \vartheta_1 \vartheta_2} = \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3}. \end{aligned} \tag{28}$$

Moreover, Marshall-Olkin copulas have upper tail dependence. Without loss of generality assume that $\vartheta_1 > \vartheta_2$, then

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u} = \lim_{u \to 1} \frac{1 - 2u + u^2 \min\left(u^{-\vartheta_1}, u^{-\vartheta_2}\right)}{1 - u} = \vartheta_2 = \frac{\beta_3}{\beta_2 + \beta_3}$$

and hence $\lambda_U = \min\left(\frac{\beta_3}{\beta_1 + \beta_3}, \frac{\beta_3}{\beta_2 + \beta_3}\right)$ is the coefficient of upper tail dependence for the B MWW model.

3.2 Absolute Continuous BMWW Model

According to the idea of Block and Basu [10], an absolutely continuous BMWW (BMWW_{ac}) distribution will be introduced by removing the singular part from the Marshal-Olkin BMWW distribution and remaining only the absolutely continuous part.

A random vector (Y_1, Y_2) follows a BMWW_{ac} distribution if its pdf is given by

$$f_{BMWW}(y_1, y_2) = K. \begin{cases} f_{BMWW}(y_1; \beta_1) \cdot f_{BMWW}(y_2; \beta_2 + \beta_3) & if \quad y_1 < y_2 \\ f_{BMWW}(y_1; \beta_1 + \beta_3) \cdot f_{BMWW}(y_2; \beta_2) & if \quad y_1 > y_2 \end{cases}, \tag{29}$$

where $K = \frac{\beta_1+\beta_2}{\beta_1+\beta_2+\beta_3}$ is the normalizing constant.

It is denoted that $\left(Y_1, Y_2\right) \sim \mathrm{BMWW}_{ac}\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ if $\left(X_1, X_2\right)$ has a BMWW $\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ distribution, then $\left(X_1, X_2\right)$ given $X_1 \neq X_2$ has a BMWW_{ac} distribution.

The associated survival function of $\left(Y_1, Y_2\right) \sim \mathrm{BMWW}_{ac}\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ is given by

$$S_{\mathrm{BMWW}}\left(y_1, y_2\right) = \frac{\beta_{123}}{\beta_{12}} S_{MWW}\left(y_1;\beta_1\right) S_{MWW}\left(y_2;\beta_2\right) S_{MWW}\left(y;\beta_3\right) - \frac{\beta_3}{\beta_{12}} S_{MWW}\left(y;\beta_{123}\right). \tag{30}$$

where $y = \max\left(y_1, y_2\right)$ and $\beta_{123} = \beta_1 + \beta_2 + \beta_3$. Moreover, the marginal survival functions of Y_1 and Y_2 are given respectively, as

$$S_{Y_1}\left(y_1\right) = \frac{\beta_{123}}{\beta_{12}} S_{MWW}\left(y_1;\beta_{13}\right) - \frac{\beta_3}{\beta_{12}} S_{MWW}\left(y_1;\beta_{123}\right)$$

$$S_{Y_2}\left(y_2\right) = \frac{\beta_{123}}{\beta_{12}} S_{MWW}\left(y_1;\beta_{23}\right) - \frac{\beta_3}{\beta_{12}} S_{MWW}\left(y_2;\beta_{123}\right)$$

The corresponding marginal pdfs of Y_1 and Y_2 are given respectively as follows

$$f_{Y_1}\left(y_1\right) = K f_{MWW}\left(y_1;\beta_{13}\right) - K \frac{\beta_3}{\beta_{123}} f_{MWW}\left(y_1;\beta_{123}\right),$$

and $f_{Y_2}\left(y_2\right) = K f_{MWW}\left(y_2;\beta_{23}\right) - K \frac{\beta_3}{\beta_{123}} f_{MWW}\left(y_2;\beta_{123}\right)$.

Unlike those of the BMWW distribution, the marginals of the $\mathrm{BMWW}_{\mathrm{ac}}$ distribution are not MWW distributions. If $\beta_3 \to 0^+$, then Y_1 and Y_2 follow MWW distributions and in this case, Y_1 and Y_2 become independent.

The Stress- Strength parameter for $\left(Y_1, Y_2\right) \sim \mathrm{BMWW}_{ac}\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ has the following form; $R = P\left(Y_1 < Y_2\right) = \frac{\beta_1}{\beta_1+\beta_2}$.

Moreover,$\min\left(Y_1, Y_2\right) \sim MWW\left(\beta_{123}\right)$.

The product moments of $\left(Y_1, Y_2\right) \sim BMWW_{ac}\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ denoted by $\mu'_{r,s}$ are given as

$$\begin{aligned} E\left(Y_1^r Y_2^s\right) = & \frac{K\beta_{23}(c\theta^\gamma + 1)\Gamma\left(\frac{r+s}{\gamma} + 2\right)}{\left[\beta_1(c\theta^\gamma + 1)\right]^{\frac{r}{\gamma}}\left(\frac{r}{\gamma} + 1\right)\left[1 + \beta_{23}(c\theta^\gamma + 1)\right]^{\frac{r+s}{\gamma}+2}} \\ & \cdot F\left(1, \frac{r+s}{\gamma} + 2;2 + \frac{r}{\gamma}; \left[1 + \beta_{23}(c\theta^\gamma + 1)\right]^{-1}\right) \\ & + \frac{K\beta_{13}(c\theta^\gamma + 1)\Gamma\left(\frac{r+s}{\gamma} + 2\right)}{\left[\beta_2(c\theta^\gamma + 1)\right]^{\frac{s}{\gamma}}\left(\frac{s}{\gamma} + 1\right)\left[1 + \beta_{13}(c\theta^\gamma + 1)\right]^{\frac{r+s}{\gamma}+2}} \end{aligned}$$

$$\cdot F\left(1, \frac{r+s}{\gamma} + 2;2 + \frac{s}{\gamma}; \left[1 + \beta_{13}(c\theta^{\gamma} + 1)\right]^{-1}\right).$$

3.3 MLE Estimation for BMWW Distribution

In this section, the maximum likelihood estimators (MLEs) of the unknown parameters of the BMWW distribution will be considered. Suppose $\{(x_{11}, x_{21}), \dots, (x_{1n}, x_{2n})\}$ is a random sample from $BMWW\left(\beta_1, \beta_2, \beta_3, \gamma, \theta\right)$ distribution. Consider the following notation

$$I_1 = \{i;x_{1i} < x_{2i}\}, \quad I_2 = \{i;x_{1i} > x_{2i}\}, \quad I_3 = \{x_{1i} = x_{2i} = x_i\}, \quad I = I_1 \cup I_2 \cup I_3,$$

$$\left|I_1\right| = n_1, \quad \left|I_2\right| = n_2, \quad \left|I_3\right| = n_3, \quad \text{and} \quad n_1 + n_2 + n_3 = n.$$

The log-likelihood function $[l = \ln\ L(\Theta);\Theta = \left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)]$, of the sample of size n is given by

$$l = \ln L(\Theta) = \sum_{i\in I_1} \ln f_1(x_{1i}, x_{2i}) + \sum_{i\in I_2} \ln f_2(x_{1i}, x_{2i}) + \sum_{i\in I_3} \ln f_3(x_i)$$

$$\begin{aligned}
l =& n_1 \ln \beta_1 + n_2 \ln \beta_2 + n_3 \ln \beta_3 + n_1 \ln \beta_{23} + n_2 \ln \beta_{13} + \left(2n_1 + 2n_2 + n_3\right) \ln \gamma \\
&+ \left(2n_1 + 2n_2 + n_3\right) \ln (c\theta^{\gamma} + 1) - (c\theta^{\gamma} + 1)\beta_1 \sum_{i=1}^{n_1} x_{1i}^{\gamma} - (c\theta^{\gamma} + 1)\beta_{23} \sum_{i=1}^{n_1} x_{2i}^{\gamma} \\
&- (c\theta^{\gamma} + 1)\beta_2 \sum_{i=1}^{n_2} x_{2i}^{\gamma} - (c\theta^{\gamma} + 1)\beta_{13} \sum_{i=1}^{n_2} x_{1i}^{\gamma} - (c\theta^{\gamma} + 1)\beta_{123} \sum_{i=1}^{n_3} x_{i}^{\gamma} \\
&+ (\gamma - 1)\left[\sum_{i=1}^{n_1} \ln x_{1i} + \sum_{i=1}^{n_1} \ln x_{2i} + \sum_{i=1}^{n_2} \ln x_{1i} + \sum_{i=1}^{n_2} \ln x_{2i} + \sum_{i=1}^{n_3} \ln x_i\right] \\
&+ (\gamma - 1)\left[\sum_{i=1}^{n_1} \ln x_{1i} + \sum_{i=1}^{n_1} \ln x_{2i} + \sum_{i=1}^{n_2} \ln x_{1i} + \sum_{i=1}^{n_2} \ln x_{2i} + \sum_{i=1}^{n_3} \ln x_i\right]
\end{aligned} \tag{32}$$

Now, the first derivative of the log-likelihood function given in (32) concerning the six unknown parameters are given respectively as follows

$$\frac{\partial l}{\partial \beta_1} = \frac{n_1}{\beta_1} + \frac{n_2}{\beta_{13}} - (c\theta^{\gamma} + 1)\left[\sum_{i=1}^{n_1} x_{1i}^{\gamma} + \sum_{i=1}^{n_2} x_{1i}^{\gamma} + \sum_{i=1}^{n_3} x_{i}^{\gamma}\right],$$

$$\frac{\partial l}{\partial \beta_2} = \frac{n_2}{\beta_2} + \frac{n_1}{\beta_{23}} - (c\theta^{\gamma} + 1)\left[\sum_{i=1}^{n_1} x_{2i}^{\gamma} + \sum_{i=1}^{n_2} x_{2i}^{\gamma} + \sum_{i=1}^{n_3} x_{i}^{\gamma}\right],$$

$$\frac{\partial l}{\partial \beta_3} = \frac{n_3}{\beta_3} + \frac{n_1}{\beta_{23}} + \frac{n_2}{\beta_{13}} - (c\theta^\gamma + 1)\left[\sum_{i=1}^{n_1} x_{2i}^\gamma + \sum_{i=1}^{n_2} x_{1i}^\gamma + \sum_{i=1}^{n_3} x_i^\gamma\right],$$

$$\begin{aligned}\frac{\partial l}{\partial \gamma} =& \frac{2n_1 + 2n_2 + n_3}{\gamma} + \frac{c\theta^\gamma \ln\theta}{c\theta^\gamma + 1} - \beta_1 \sum_{i=1}^{n_1} A\left(x_{1i};\gamma,\theta,c\right) - \beta_{23}\sum_{i=1}^{n_1} A\left(x_{2i};\gamma,\theta,c\right) \\ & - \beta_2 \sum_{i=1}^{n_2} A\left(x_{2i};\gamma,\theta,c\right) - \beta_{13}\sum_{i=1}^{n_2} A\left(x_{1i};\gamma,\theta,c\right) - \beta_{123}\sum_{i=1}^{n_3} A\left(x_i;\gamma,\theta,c\right) + G\left(x_{1i}, x_{2i}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \theta} =& \frac{2n_1 + 2n_2 + n_3}{c\theta^\gamma + 1}\,\gamma\theta^{\gamma-1} - c\,\theta^{\gamma-1}\gamma\left[\beta_1 \sum_{i=1}^{n_1} x_{1i}^\gamma + \beta_{23}\sum_{i=1}^{n_1} x_{2i}^\gamma\right. \\ & \left. + \beta_2 \sum_{i=1}^{n_2} x_{2i}^\gamma + \beta_{13}\sum_{i=1}^{n_2} x_{1i}^\gamma + \beta_{123}\sum_{i=1}^{3} x_i^\gamma\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial c} =& \frac{2n_1 + 2n_2 + n_3}{c\theta^\gamma + 1}\,\theta^\gamma - \theta^\gamma\left[\beta_1 \sum_{i=1}^{n_1} x_{1i}^\gamma + \beta_{23}\sum_{i=1}^{n_1} x_{2i}^\gamma + \beta_2 \sum_{i=1}^{n_2} x_{2i}^\gamma\right. \\ & \left. + \beta_{13}\sum_{i=1}^{n_2} x_{1i}^\gamma + \beta_{123}\sum_{i=1}^{3} x_i^\gamma\right]\end{aligned}$$

where $G\left(x_{1i}, x_{2i}\right) = \sum_{i=1}^{n_1} \ln x_{1i} + \sum_{i=1}^{n_1} \ln x_{2i} + \sum_{i=1}^{n_2} \ln x_{1i} + \sum_{i=1}^{n_2} \ln x_{2i} + \sum_{i=1}^{n_3} \ln x_i$ and $A\left(x_i;\gamma,\theta,c\right) = x_i^\gamma\left[c\theta^\gamma \ln\theta + (c\theta^\gamma + 1)\ln x_i\right]$.

The numerical solutions for these equations will be considered to obtain $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\gamma}, \hat{\theta}$ and $\hat{c}$ as will be shown in Sect. 5.

For the BMWW distribution, the information matrix $I(\Theta)$ is a 6×6 symmetric matrix with elements

$$I_{ij}(\Theta) = -E\left[\frac{\partial^2 \log l\,(\Theta)}{\partial\Theta_i \partial\Theta_j}\right],\ \forall\ i,j$$

Now, elements of the Fisher information matrix for the $BMWW\left(\beta_1, \beta_2, \beta_3, \gamma, \theta, c\right)$ distribution are obtained to be

$$I_{11}\left(\hat{\Theta}\right) = \frac{n_1}{\hat{\beta}_1^2} + \frac{n_2}{\left(\hat{\beta}_1 + \hat{\beta}_3\right)^2}, \quad I_{22}\left(\hat{\Theta}\right) = \frac{n_2}{\hat{\beta}_2^2} + \frac{n_1}{\left(\hat{\beta}_2 + \hat{\beta}_3\right)^2},$$

$$I_{33}\left(\hat{\Theta}\right) = \frac{n_1}{\left(\hat{\beta}_2 + \hat{\beta}_3\right)^2} + \frac{n_2}{\left(\hat{\beta}_1 + \hat{\beta}_3\right)^2} + \frac{n_3}{\hat{\beta}_3^2},$$

$$I_{13}\left(\hat{\Theta}\right)=\frac{n_2}{\left(\hat{\beta}_1+\hat{\beta}_3\right)^2}\quad I_{23}\left(\hat{\Theta}\right)=\frac{n_1}{\left(\hat{\beta}_2+\hat{\beta}_3\right)^2},\ I_{12}\left(\hat{\Theta}\right)=I_{21}\left(\hat{\Theta}\right)=0,$$

$$I_{16}\left(\hat{\Theta}\right)=\hat{\theta}^{\hat{\gamma}}\left[\sum_{i=1}^{n_1}x_{1i}^{\hat{\gamma}}+\sum_{i=1}^{n_2}x_{1i}^{\hat{\gamma}}+\sum_{i=1}^{n_3}x_{i}^{\hat{\gamma}}\right],\ I_{26}\left(\hat{\Theta}\right)=\hat{\theta}^{\hat{\gamma}}\left[\sum_{i=1}^{n_1}x_{2i}^{\hat{\gamma}}+\sum_{i=1}^{n_2}x_{2i}^{\hat{\gamma}}+\sum_{i=1}^{n_3}x_{i}^{\hat{\gamma}}\right],$$

$$I_{36}\left(\hat{\Theta}\right)=\hat{\theta}^{\hat{\gamma}}\left[\sum_{i=1}^{n_1}x_{2i}^{\hat{\gamma}}+\sum_{i=1}^{n_2}x_{1i}^{\hat{\gamma}}+\sum_{i=1}^{n_3}x_{i}^{\hat{\gamma}}\right],\ I_{51}\left(\hat{\Theta}\right)=\hat{c}\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\left[\sum_{i=1}^{n_1}x_{1i}^{\hat{\gamma}}+\sum_{i=1}^{n_2}x_{1i}^{\hat{\gamma}}+\sum_{i=1}^{n_3}x_{i}^{\hat{\gamma}}\right],$$

$$I_{52}\left(\hat{\Theta}\right)=\hat{c}\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\left[\sum_{i=1}^{n_1}x_{2i}^{\hat{\gamma}}+\sum_{i=1}^{n_2}x_{2i}^{\hat{\gamma}}+\sum_{i=1}^{n_3}x_{i}^{\hat{\gamma}}\right],\ I_{53}\left(\hat{\Theta}\right)=\hat{c}\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\left[\sum_{i=1}^{n_1}x_{2i}^{\hat{\gamma}}+\sum_{i=1}^{n_2}x_{1i}^{\hat{\gamma}}+\sum_{i=1}^{n_3}x_{i}^{\hat{\gamma}}\right],$$

$$I_{41}\left(\hat{\Theta}\right)=\sum_{i=1}^{n_1}A\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\sum_{i=1}^{n_2}A\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\sum_{i=1}^{n_3}A\left(x_{i};\hat{\gamma},\hat{\theta},\hat{c}\right),$$

$$I_{42}\left(\hat{\Theta}\right)=\sum_{i=1}^{n_2}A\left(x_{2i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\sum_{i=1}^{n_1}A\left(x_{2i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\sum_{i=1}^{n_3}A\left(x_{i};\hat{\gamma},\hat{\theta},\hat{c}\right),$$

$$I_{43}\left(\hat{\Theta}\right)=\sum_{i=1}^{n_1}A\left(x_{2i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\sum_{i=1}^{n_2}A\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\sum_{i=1}^{n_3}A\left(x_{i};\hat{\gamma},\hat{\theta},\hat{c}\right),$$

$$I_{56}\left(\hat{\Theta}\right)=\frac{\hat{\theta}^{\hat{\gamma}}\left(2n_1+2n_2+n_3\right)}{(\hat{c}\hat{\theta}^{\hat{\gamma}}+1)^2}\left[\hat{\gamma}\theta^{\hat{\gamma}-1}+\hat{\beta}_1\sum_{i=1}^{n_1}x_{1i}^{\hat{\gamma}}+\hat{\beta}_{23}\sum_{i=1}^{n_1}x_{2i}^{\hat{\gamma}}+\hat{\beta}_2\sum_{i=1}^{n_2}x_{2i}^{\hat{\gamma}}+\hat{\beta}_{13}\sum_{i=1}^{n_2}x_{1i}^{\hat{\gamma}}+\hat{\beta}_{123}\sum_{i=1}^{3}x_{i}^{\hat{\gamma}}\right],$$
$$I_{66}\left(\hat{\Theta}\right)=\frac{\hat{\theta}^{2\hat{\gamma}}\left(2n_1+2n_2+n_3\right)}{(\hat{c}\hat{\theta}^{\hat{\gamma}}+1)^2},$$

$$\begin{aligned}I_{45}\left(\hat{\Theta}\right)=&-\frac{\hat{c}\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\ln\hat{\theta}}{(\hat{c}\hat{\theta}^{\hat{\gamma}}+1)^2}+\hat{\beta}_1\sum_{i=1}^{n_1}D\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\hat{\beta}_{23}\sum_{i=1}^{n_1}D\left(x_{2i};\hat{\gamma},\hat{\theta},\hat{c}\right)\\&+\hat{\beta}_{13}\sum_{i=1}^{n_2}D\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\hat{\beta}_{123}\sum_{i=1}^{n_3}D\left(x_{i};\hat{\gamma},\hat{\theta},\hat{c}\right),\end{aligned}$$

$$\begin{aligned}I_{46}\left(\hat{\Theta}\right)=&-\frac{\hat{\theta}^{\hat{\gamma}}\ln\hat{\theta}}{(\hat{c}\hat{\theta}^{\hat{\gamma}}+1)^2}+\hat{\beta}_1\sum_{i=1}^{n_1}E\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)\\&+\hat{\beta}_{23}\sum_{i=1}^{n_1}E\left(x_{2i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\hat{\beta}_{13}\sum_{i=1}^{n_2}E\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right)+\hat{\beta}_{123}\sum_{i=1}^{n_3}E\left(x_{i};\hat{\gamma},\hat{\theta},\hat{c}\right),\end{aligned}$$

$$I_{44}\left(\hat{\Theta}\right) = \frac{\left(2n_1 + 2n_2 + n_3\right)}{\hat{\gamma}^2} - \frac{\hat{c}\hat{\theta}^{\hat{\gamma}}\left(\ln\hat{\theta}\right))n^{\,2}}{(\hat{c}\hat{\theta}^{\hat{\gamma}} + 1)^2} + \hat{\beta}_1\sum_{i=1}^{n_1}B\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right) + \hat{\beta}_{23}\sum_{i=1}^{n_1}B\left(x_{2i};\hat{\gamma},\hat{\theta},\hat{c}\right)$$
$$+ \hat{\beta}_{13}\sum_{i=1}^{n_2}B\left(x_{1i};\hat{\gamma},\hat{\theta},\hat{c}\right) + \hat{\beta}_{123}\sum_{i=1}^{n_3}B\left(x_i;\hat{\gamma},\hat{\theta},\hat{c}\right),$$

$$I_{55}\left(\hat{\Theta}\right) = -\frac{\left(2n_1 + 2n_2 + n_3\right)}{(\hat{c}\hat{\theta}^{\hat{\gamma}} + 1)^2}\,\hat{\gamma}\theta^{\hat{\gamma}-2}\left(\hat{c}\hat{\theta}^{\hat{\gamma}} + \hat{\gamma} - 1\right)$$
$$+ \hat{c}\hat{\gamma}(\hat{\gamma} - 1)\theta^{\hat{\gamma}-2}\left[\hat{\beta}_1\sum_{i=1}^{n_1}x_{1i}^{\hat{\gamma}} + \hat{\beta}_{23}\sum_{i=1}^{n_1}x_{2i}^{\hat{\gamma}} + \hat{\beta}_2\sum_{i=1}^{n_2}x_{2i}^{\hat{\gamma}} + \hat{\beta}_{13}\sum_{i=1}^{n_2}x_{1i}^{\hat{\gamma}} + \hat{\beta}_{123}\sum_{i=1}^{3}x_i^{\hat{\gamma}}\right].$$ where

$B\left(x_i;\gamma,\theta,c\right) = x_i^{\gamma}\left[c\theta^{\gamma}(\ln\theta + \ln x_i)^2 + \left(\ln x_i\right)^2\right]$,

$D\left(x_i;\gamma,\theta,c\right) = x_i^{\gamma}c\theta^{\gamma-1}\left(\gamma\ln\theta x_i + 1\right)$, and $E\left(x_i;\gamma,\theta,c\right) = \left(x_i\theta\right)^{\gamma}\ln\theta x_i$.

Hence, the asymptotic confidence intervals for the parameters of the BMWW distribution will be presented. Let $\hat{\Theta} = \left(\widehat{\beta_1},\ \widehat{\beta_2},\ \widehat{\beta_3},\ \hat{\gamma},\hat{\theta},\hat{c}\right)$ be the MLEs of $\Theta = \left(\beta_1,\beta_2,\beta_3,\gamma,\theta,c\right)$ Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have:

$$\sqrt{n}\left(\hat{\Theta} - \Theta\right) \xrightarrow{d} N_6\left(\underset{\sim}{0}, I^{-1}(\Theta)\right) \tag{33}$$

where $I(\Theta)$ is the Fisher information matrix. The multivariate normal distribution with mean vector $\underset{\sim}{0} = (0,0,0,0,0,0)$ and variance–covariance matrix $I^{-1}(\Theta)$ can be used to construct confidence intervals for the model parameters. That is, a $(1-\alpha)\%$ two-sided confidence intervals for $\beta_1,\beta_2\ \beta_3\ \gamma,\theta,c$ can be introduced as

$$\begin{aligned}&\widehat{\beta_1} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta_1\beta_1}(\Theta)},\ \ \widehat{\beta_2} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta_2\beta_2}(\Theta)},\ \widehat{\beta_3} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta_3\beta_3}(\Theta)},\\ &\hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\gamma\gamma}(\Theta)},\ \hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\theta\theta}(\Theta)},\\ &\text{and,}\ \ \hat{c} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{cc}(\Theta)}\end{aligned} \tag{34}$$

Respectively, where $I^{-1}_{\beta_1\beta_1}(\Theta)$,$I^{-1}_{\beta_2\beta_2}(\Theta)$ $I^{-1}_{\beta_3\beta_3}(\Theta)$,$I^{-1}_{\gamma\gamma}(\Theta)$,$I^{-1}_{\theta\theta}(\Theta)$ and $I^{-1}_{cc}(\Theta)$ are diagonal elements of the variance–covariance matrix and $Z_{\frac{\alpha}{2}}$ is the $(\frac{\alpha}{2})^{th}$ percentile of a standard normal distribution.

3.4 Bayesian Estimation for BMWW Distribution

In this section, the Bayesian analysis for the BMWW distribution is considered. The explicit Bayes estimators under the squared error loss function are obtained. When the shape parameters γ,θ,c are known, we assume the same conjugate prior on β_1,β_2 and β_3 as considered by Kundu and Gupta [11] as follows:

Assume β_1,β_2 and β_3 are independent, and distributed as gamma as following

$$\pi_i(\beta_i) = \frac{b^{a_i}}{\Gamma(a_i)}\ \beta_i^{a_i-1} e^{-b_i\beta_i}\ ,\ i = 1,2,3\ ,\beta_i > 0$$

The joint prior density of β_1, β_2 and β_3 is given as follows

$$\pi_0(\beta_1, \beta_2, \beta_3) = \prod_{i=1}^{3} \frac{b^{a_i}}{\Gamma(a_i)}\ \beta_i^{a_i-1} e^{-b_i\ \beta_i}$$

Posterior Analysis and Bayesian Estimation

Suppose $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ is a random sample from $BMWW(\beta_1, \beta_2, \beta_3, \gamma, \theta, c)$ distribution. Consider the following notation.

$D = \{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$,$\Theta = (\beta_1, \beta_2, \beta_3)$ and $n = n_1 + n_2 + n_3$.

Then, the Likelihood function (32) can be written as

$$L(D\backslash\Theta) = Exp(\log L(D\backslash\Theta))$$

$$L(D\backslash\Theta) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\binom{n_1}{j}\binom{n_2}{k}\beta_1^{n_1+k}\ \beta_2^{n_2+j}\beta_3^{n-j-k}Exp(-\beta_1 T_1 - \beta_2 T_2 - \beta_3 T_3)$$

where $T_1 = Z_1(\gamma, \theta, c) + Z_4(\gamma, \theta, c) + Z_5(\gamma, \theta, c)$, $T_2 = Z_2(\gamma, \theta, c) + Z_3(\gamma, \theta, c) + Z_5(\gamma, \theta, c)$.

$T_3 = Z_2(\gamma, \theta, c) + Z_4(\gamma, \theta, c) + Z_5(\gamma, \theta, c)$, $Z_1(\gamma, \theta, c) = (c\theta^\gamma + 1)\sum_{i=1}^{n_1} x_{1i}^{\hat{\gamma}}$,,

$Z_2(\gamma, \theta, c) = (c\theta^\gamma + 1)\sum_{i=1}^{n_1} x_{2i}^{\hat{\gamma}}$, $Z_3(\gamma, \theta, c) = (c\theta^\gamma + 1)\sum_{i=1}^{n_2} x_{2i}^{\hat{\gamma}}$, since, $f(D, \Theta) = \pi_0(\Theta)L(D\backslash\Theta)$ and $f(D) = \int f(D\backslash\Theta)d\Theta = \int \pi_0(\Theta)L(D\backslash\Theta)d\Theta$.

Hence the joint posterior density function of $\Theta = (\beta_1, \beta_2, \beta_3)$ given the data D, denoted by $\pi_1(\ \Theta\backslash D)$ can be written as

$$\pi_1(\Theta\backslash D)\ = \frac{f(D, \Theta)}{f(D)}$$

$$\pi_1(\ \Theta\backslash D) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} w_{jk}\ Gamma\ [\beta_1; a_{1k},\ b_1 + T_1] \cdot Gamma\ [\beta_2; a_{2j},\ b_2 + T_2]$$
$$\cdot Gamma\ [\beta_3; a_{3jk},\ b_3 + T_3]$$

where $w_{ij}\ = \frac{C_{jk}}{\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} C_{jk}}$ and $C_{jk} = \binom{n_1}{j}\binom{n_2}{k}.\frac{\Gamma(a_{1k})}{[b_1+T_1]^{a_{1k}}}.\frac{\Gamma(a_{2j}\ )}{[b_2+T_2]^{a_{2j}}}.\frac{\Gamma(a_{3jk})}{[b_3+T_3]^{a_{3jk}}}$.

$$a_{1k} = a_1 + k + n_1,\ a_{2j} = a_2 + j + n_2\ and\ a_{3kj} = a_3 + n - j - k$$

Therefore, under the assumption of independence of β_1, β_2 and β_3 and γ, θ and c are assumed to be known. It is possible to get the Bayes estimators of β_1, β_2 and β_3 explicitly under the square error loss function as follows:

$\check{\beta}_1 = \frac{1}{b_1+T_1}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} w_{jk}a_{1k}$, $\check{\beta}_2 = \frac{1}{b_2+T_2}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} w_{jk}a_{2j}$ and $\check{\beta}_3 = \frac{1}{b_3+T_3}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} w_{jk}a_{3jk}$,

4 Bivariate Modified Weighted Gumbel Distribution

In this section, the BMWG distribution will be derived and its properties also will be established. Suppose that $U_i \sim MWG\left(c_i, \gamma, \theta\right),\ i = 1, 2, 3$ whish is defined as in Eqs. (10–12) such that $U_i's$ are mutually independent random variables and define $X_j = min\left(U_j, U_3\right),\ j = 1, 2$. Such that; $X_j's$ are dependent random variables. Hence the joint survival function of the vector $\left(X_1, X_2\right)$ denoted by $S_{BMWG}\left(x_1, x_2\right)$ is given as.

$$\begin{aligned} S_{BMWG}\left(x_1, x_2\right) &= S_{MWG}\left(x_1; c_1\theta^{\gamma} + 1\right)S_{MWG}\left(x_2; c_2\theta^{\gamma} + 1\right)S_{MWG}\left(x_3; c_3\theta^{\gamma} + 1\right) \\ &= \exp\left\{-\left(c_1\theta^{\gamma} + 1\right)e^{x_1} - \left(c_2\theta^{\gamma} + 1\right)e^{x_2} - \left(c_3\theta^{\gamma} + 1\right)e^{x_3}\right\} \end{aligned} \tag{35}$$

where $x_3 = \max\left(x_1, x_2\right)$ and $S_{MWG}(x;\gamma, \theta, c) = e^{-e^{x}(c\theta^{\gamma}+1)}$.

The joint survival function of BMWG distribution can be extended in the following form

$$S_{BMWW}\left(x_1, x_2\right) = \begin{cases} S_{MWG}\left(x_1; c_1\theta^{\gamma} + 1\right)S_{MWG}\left(x_2; c_{23}\theta^{\gamma} + 2\right), & x_1 < x_2 \\ S_{MWG}\left(x_1; c_{13}\theta^{\gamma} + 2\right)S_{MWG}\left(x_2; c_2\theta^{\gamma} + 1\right), & x_1 > x_2 \\ S_{MWG}\left(x; c_{123}\theta^{\gamma} + 3\right), & x_1 = x_2 = x \end{cases} \tag{36}$$

where $c_{13} = c_1 + c_3,\ c_{23} = c_2 + c_3$ and $c_{123} = c_1 + c_2 + c_3$.

Hence, the joint pdf of BMWG distribution can be obtained as

$$f_{BMWW}\left(x_1, x_2\right) = \begin{cases} f_{MWG}\left(x_1;c_1\theta^{\gamma} + 1\right)f_{MWG}\left(x_2;c_{23}\theta^{\gamma} + 2\right), & x_1 < x_2 \\ f_{MWG}\left(x_1;c_{13}\theta^{\gamma} + 2\right)f_{MWG}\left(x_2;c_2\theta^{\gamma} + 1\right), & x_1 > x_2 \\ \frac{c_3\theta^{\gamma}+1}{c_{123}\theta^{\gamma}+3}f_{MWG}\left(x;c_{123}\theta^{\gamma} + 3\right), & x_1 = x_2 \end{cases} \tag{37}$$

The joint cdf of the BMWG distribution is given by

$$F_{BMWG}\left(x_1, x_2\right) = \begin{cases} F_{MWG}\left(x_1;c_{13}\theta^{\gamma} + 2\right) - F_{MWG}\left(x_1;c_1\theta^{\gamma} + 1\right)\left[1 - F_{MWG}\left(x_2;c_{23}\theta^{\gamma} + 2\right)\right], & x_1 < x_2 \\ F_{MWG}\left(x_2;c_2\theta^{\gamma} + 1\right) - F_{MWG}\left(x_2;c_2\theta^{\gamma} + 1\right)\left[1 - F_{MWG}\left(x_1;c_{13}\theta^{\gamma} + 2\right)\right], & x_2 < x_1 \\ 1 - F_{MWG}\left(x;c_{123}\theta^{\gamma} + 3\right), & x_1 = x_2 = x. \end{cases}$$

The joint hazard function of the BMWG distribution is given as

$$h_{BMWG}\left(x_1, x_2\right) = \begin{cases} (c_1\theta^{\gamma} + 1)\left(c_{23}\theta^{\gamma} + 2\right)e^{x_1+x_2}, & x_1 < x_2 \\ (c_2\theta^{\gamma} + 1)\left(c_{13}\theta^{\gamma} + 2\right)e^{x_1+x_2}, & x_1 > x_2 \\ \left(c_3\theta^{\gamma} + 1\right)e^{x}, & x_1 = x_2 = x. \end{cases} \tag{38}$$

Similar to BMWW distribution the BMWG has both an absolutely continuous part and a singular part (Figs. 3, 4). The joint survival function of the BMWG distribution can be factorized into an absolutely continuous part and singular part as follows in the following form

$$S_{\mathrm{BMWG}}\left(x_1,x_2\right) = \frac{(c_{12}\theta^{\gamma}+2)}{\left(c_{123}\theta^{\gamma}+3\right)}\, S_a\left(x_1,x_2\right) + \frac{\left(c_3\theta^{\gamma}+1\right)}{\left(c_{123}\theta^{\gamma}+3\right)}\, S_s\left(x_3\right) \tag{39}$$

where $x_3 = \max\left(x_1,x_2\right), S_s\left(x_3\right) = S_{MWG}\left(x;c_{123}\theta^{\gamma}+3\right), c_{123} = c_1+c_2+c_3$.

a n d

$$S_a\left(x_1,x_2\right) = \frac{\left(c_{123}\theta^{\gamma}+3\right)}{(c_{12}\theta^{\gamma}+2)} S_{MWG}\left(x_1;c_1\theta^{\gamma}+1\right) S_{MWG}\left(x_2;c_2\theta^{\gamma}+1\right) S_{MWG}\left(x_3;c_3\theta^{\gamma}+1\right) - \frac{c_3\theta^{\gamma}+1}{c_{12}\theta^{\gamma}+2} S_{MWG}\left(x;c_{123}\theta^{\gamma}+3\right).$$

Its cleat that $S_s(\ldots)$ and $S_a(\ldots)$ are the singular and absolutely continuous parts respectively.

So, the pdf of the BMWG model can be factorized into an absolutely continuous part and singular part as follows

$$f_{\mathrm{BMWG}}\left(x_1,x_2\right) = \frac{c_{12}\theta^{\gamma}+2}{c_{123}\theta^{\gamma}+3} f_a\left(x_1,x_2\right) + \frac{c_3\theta^{\gamma}+1}{c_{123}\theta^{\gamma}+3} f_s\left(x_3\right) \tag{40}$$

where $f_a\left(x_1,x_2\right) = \frac{c_{123}\theta^{\gamma}+3}{c_{12}\theta^{\gamma}+2}\begin{cases} f_{MWG}\left(x_1;c_{13}\theta^{\gamma}+2\right) f_{MWG}\left(x_2;c_2\theta^{\gamma}+1\right), & x_1 < x_2 \\ f_{MWG}\left(x_1;c_1\theta^{\gamma}+1\right) f_{MWG}\left(x_2;c_{23}\theta^{\gamma}+2\right), & x_1 < x_2 \end{cases}$ and $f_s\left(x_3\right) = f_{MWG}\left(x;\, c_{123}\theta^{\gamma}+3\right)$.

Clearly, here $f_a(x_1,x_2)$ and $f_s(x_3)$ are the absolutely continuous and singular parts respectively.

The absolutely continuous part of the BMWG density may be unimodal depending on the values of c_1, c_2 and c_3 that is $f_a(x_1,x_2)$ is unimodal and the respective modes are.

$\left\{-\ln\left(c_1\theta^{\gamma}+1\right), -\ln\left(c_{23}\theta^{\gamma}+2\right)\right\}$ and $\left\{-\ln\left(c_{13}\theta^{\gamma}+2\right), -\ln\left(c_2\theta^{\gamma}+1\right)\right\}$.

The median for the absolutely continuous BMWG distribution is given as: $\ln\left(\frac{\ln 2}{c_{123}\theta^{\gamma}+3}\right)$.

The marginal distributions of the BMWG distribution are univariate MWG with the following survival and density functions respectively,

$$S_{X_i}\left(x_i\right) = S_{MWG}\left(x_i;\, c_{i3}\theta^{\gamma}+2\right) = e^{-(c_{i3}\theta^{\gamma}+2)e^{x_i}},\quad i = 1,2 \tag{41}$$

$$f_{X_i}\left(x_i\right) = f_{MWG}\left(x_i;\, c_{i3}\theta^{\gamma}+2\right) = \left(c_{i3}\theta^{\gamma}+2\right) e^{x_i} e^{-(c_{i3}\theta^{\gamma}+2)e^{x_i}},\quad i = 1,2 \tag{42}$$

such that $c_{i3} = c_i + c_3,\quad i = 1,2.$

Moreover, the distribution of the minimum of $\left(X_1,X_2\right) \sim BMWG\left(c_1,c_2,c_3,\gamma,\theta\right)$ is also univariate MWG. And the survival and density functions are given as follows

$$S_{\min(X_1,X_2)}(x) = S_{MWG}\left(x;\, c_{123}\theta^{\gamma}+3\right) = e^{-\left(c_{123}\theta^{\gamma}+3\right)e^{x}} \tag{43}$$

and

$$f_{\min(X_1,X_2)}(x) = f_{MWG}\left(x;\, c_{123}\theta^{\gamma} + 3\right) = \left(c_{123}\theta^{\gamma} + 3\right)e^{x}e^{-\left(c_{123}\theta^{\gamma}+3\right)e^{x}} \tag{44}$$

where $x = \min\left(x_1, x_2\right)$ and $c_{123} = c_1 + c_2 + c_3$.

Based on the fact that marginal distributions of the vector $\left(X_1, X_2\right) \sim BMWG\left(c_1, c_2, c_3, \gamma, \theta\right)$ are univariate MWG distributions, then the conditional density of X_i given $X_j = x_j,\ i \neq j$ is calculated as follows

$$f_{i/j}\left(x_i/x_j\right) = \begin{cases} f_{i/j}^{(1)}\left(x_i/x_j\right), & x_i < x_j \\ f_{i/j}^{(2)}\left(x_i/x_j\right), & x_i > x_j \\ f_i^{(3)}\left(x_i\right), & x_i = x_j \end{cases} \tag{45}$$

where $f_{i/j}^{(1)}\left(x_i/x_j\right) = \left(c_1\theta^{\gamma} + 1\right)e^{x_i}e^{-\left(c_1\theta^{\gamma}+2\right)e^{x_i}}$, $f_{i/j}^{(2)}\left(x_i/x_j\right) = \frac{\left(c_{13}\theta^{\gamma}+2\right)\left(c_2\theta^{\gamma}+1\right)}{\left(c_{23}\theta^{\gamma}+2\right)}e^{x_i}e^{-\left(c_{13}\theta^{\gamma}+2\right)e^{x_i}+\left(c_3\theta^{\gamma}+1\right)e^{x_j}}$, and $f_i^{(3)}\left(x_i\right) = \frac{c_3\theta^{\gamma}+1}{c_{23}\theta^{\gamma}+2}e^{x_i-x_j}e^{-\left(c_{123}\theta^{\gamma}+3\right)e^{x_i}+\left(c_{23}\theta^{\gamma}+2\right)e^{x_j}}$.

Now, the joint moment generating function of $\left(X_1, X_2\right) \sim BMWG\left(c_1, c_2, c_3, \gamma, \theta\right)$, can be given by the following formula

$$\begin{aligned} M_{MWG}\left(t_1, t_2\right) = & \frac{\left(t_1+1\right)^{-1}\left(c_{23}\theta^{\gamma}+2\right)\Gamma\left(t_1+t_2+2\right)}{\left(c_1\theta^{\gamma}+1\right)^{t_1}\left(c_{23}\theta^{\gamma}+3\right)^{t_1+t_2+2}} \\ & .F\left(1, t_1+t_2+2; t_1+2;\, \left(c_{23}\theta^{\gamma}+3\right)^{-1}\right) \\ & + \frac{\left(t_2+1\right)^{-1}\left(c_{13}\theta^{\gamma}+2\right)\Gamma\left(t_1+t_2+2\right)}{\left(c_2\theta^{\gamma}+1\right)^{t_1}\left(c_{13}\theta^{\gamma}+3\right)^{t_1+t_2+2}} \\ & .F\left(1, t_1+t_2+2; t_2+2;\, \left(c_{13}\theta^{\gamma}+3\right)^{-1}\right) \\ & + \frac{c_3\theta^{\gamma}+1}{\left(c_{123}\theta^{\gamma}+3\right)^{t_1+t_2+1}}\,\Gamma\left(t_1+t_2+1\right) \end{aligned} \tag{46}$$

where $\Gamma(r) = \int_0^{\infty} x^{r-1}e^{-x}dx$ is the gamma function and $F(a, b;c;z) = \sum_{s=0}^{\infty}\frac{(a)_s\ (b)_s}{\Gamma(c+s)s!}$. is a hypergeometric function.

4.1 Absolute Continuous BMWG Model

A random vector $\left(Y_1, Y_2\right)$ follows a BMWG_{ac} distribution if its pdf is given by

$$f_{BMWG}(y_1, y_2) = L.\begin{cases} f_{BMWG}(y_1;c_1\theta^{\gamma}+1)\cdot f_{BMWG}(y_2;c_{23}\theta^{\gamma}+2) & if \quad y_1 < y_2 \\ f_{BMWG}(y_1;c_{13}\theta^{\gamma}+2)\cdot f_{BMWG}(y_2;c_2\theta^{\gamma}+1) & if \quad y_1 > y_2 \end{cases}, \tag{47}$$

where $L = \frac{c_{12}\theta^{\gamma}+2}{c_{123}\theta^{\gamma}+3}$ is the normalizing constant.

It is denoted that $\left(Y_1, Y_2\right) \sim \mathrm{BMWG}_{ac}\left(c_1, c_2, c_3, \gamma, \theta\right)$ if $\left(X_1, X_2\right)$ has a BMWG $\left(c_1, c_2, c_3, \gamma, \theta\right)$ distribution, then $\left(X_1, X_2\right)$ given $X_1 \neq X_2$ has a BMWG_{ac} distribution.

The associated survival function of $\left(Y_1, Y_2\right) \sim \mathrm{BMWG}_{ac}\left(c_1, c_2, c_3, \gamma, \theta\right)$ is given by

$$\begin{aligned} S_{\mathrm{BMWG}}\left(y_1, y_2\right) =& \frac{c_{123}\theta^{\gamma} + 3}{c_{12}\theta^{\gamma} + 2} S_{MWG}\left(y_1; c_1\theta^{\gamma} + 1\right) S_{MWG}\left(y_2; c_2\theta^{\gamma} + 1\right) S_{MWG}\left(y; c_3\theta^{\gamma} + 1\right) \\ & - \frac{c_3\theta^{\gamma} + 1}{c_{12}\theta^{\gamma} + 2} S_{MWG}\left(y; c_{123}\theta^{\gamma} + 3\right). \end{aligned} \tag{48}$$

where $y = \max\left(y_1, y_2\right)$. Moreover, the marginal survival functions of Y_1 and Y_2 are given respectively, as.

$$S_{Y_1}\left(y_1\right) = \frac{c_{123}\theta^{\gamma} + 3}{c_{12}\theta^{\gamma} + 2} S_{MWG}\left(y_1; c_{13}\theta^{\gamma} + 2\right) - \frac{c_3\theta^{\gamma} + 1}{c_{12}\theta^{\gamma} + 2} S_{MWG}\left(y_1; c_{123}\theta^{\gamma} + 3\right),$$

$$S_{Y_2}\left(y_2\right) = \frac{c_{123}\theta^{\gamma} + 3}{c_{12}\theta^{\gamma} + 2} S_{MWG}\left(y_1; c_{23}\theta^{\gamma} + 2\right) - \frac{c_3\theta^{\gamma} + 1}{c_{12}\theta^{\gamma} + 2} S_{MWG}\left(y_2; c_{123}\theta^{\gamma} + 3\right).$$

The corresponding marginal pdfs of Y_1 and Y_2 are given respectively as follows

$$f_{Y_1}\left(y_1\right) = \frac{c_{123}\theta^{\gamma} + 3}{c_{12}\theta^{\gamma} + 2} f_{MWG}\left(y_1; c_{13}\theta^{\gamma} + 2\right) - \frac{c_3\theta^{\gamma} + 1}{c_{12}\theta^{\gamma} + 2} f_{MWG}\left(y_1; c_{123}\theta^{\gamma} + 3\right),$$

$$f_{Y_2}\left(y_2\right) = \frac{c_{123}\theta^{\gamma} + 3}{c_{12}\theta^{\gamma} + 2} f_{MWG}\left(y_2; c_{23}\theta^{\gamma} + 2\right) - \frac{c_3\theta^{\gamma} + 1}{c_{12}\theta^{\gamma} + 2} f_{MWG}\left(y_2; c_{123}\theta^{\gamma} + 3\right).$$

Again, the marginals of the $\mathrm{BMWG}_{\mathrm{ac}}$ distribution are not MWG distributions. Moreover,$\min\left(Y_1, Y_2\right) \sim MWG\left(c_{123}\theta^{\gamma} + 3\right)$.

4.2 MLE Estimation for BMWG Distribution

Suppose $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ is a random sample from $BMWG\left(c_1, c_2, c_3, \gamma, \theta\right)$ distribution. Using the likelihood (31) with the same notation for n_1, n_2 and n_3, and the pdf (4.4).The log-likelihood function in this case, $\left\{l(\Psi) = \ln L(\Psi); \Psi = \left(c_1, c_2, c_3, \gamma, \theta\right)\right\}$ is given as

$$\begin{aligned} l(\Psi) =& n_1 \ln\left(c_1\theta^{\gamma} + 1\right) + n_2 \ln\left(c_2\theta^{\gamma} + 1\right) + n_3 \ln\left(c_3\theta^{\gamma} + 1\right) + n_1 \ln\left(c_{23}\theta^{\gamma} + 2\right) \\ & + n_2 \ln\left(c_{13}\theta^{\gamma} + 1\right) + G\left(x_{1i}, x_{2i}\right) - \left(c_1\theta^{\gamma} + 1\right) \sum_{i=1}^{n_1} e^{x_{1i}} - \left(c_{23}\theta^{\gamma} + 2\right) \sum_{i=1}^{n_1} e^{x_{2i}} \\ & + \left(c_{13}\theta^{\gamma} + 2\right) \sum_{i=1}^{n_2} e^{x_{1i}} - \left(c_2\theta^{\gamma} + 1\right) \sum_{i=1}^{n_2} e^{x_{2i}} - \left(c_{123}\theta^{\gamma} + 3\right) \sum_{i=1}^{n_1} e^{x_i} \end{aligned}$$

$$\text{where } G\left(x_{1i},x_{2i}\right)=\sum_{i=1}^{n_1}\ln x_{1i}+\sum_{i=1}^{n_1}\ln x_{2i}+\sum_{i=1}^{n_2}\ln x_{1i}+\sum_{i=1}^{n_2}\ln x_{2i}+\sum_{i=1}^{n_3}\ln x_i \quad (49)$$

The first derivative of the log-likelihood function given in (41) concerning the five unknown parameters are given respectively as follows.

$$\frac{\partial l}{\partial c_1}=\frac{n_1\theta^{\gamma}}{c_1\theta^{\gamma}+1}+\frac{n_2\theta^{\gamma}}{c_{13}\theta^{\gamma}+2}-\theta^{\gamma}\left[\sum_{i=1}^{n_1}e^{x_{1i}}+\sum_{i=1}^{n_2}e^{x_{1i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$\frac{\partial l}{\partial c_2}=\frac{n_2\theta^{\gamma}}{c_2\theta^{\gamma}+1}+\frac{n_1\theta^{\gamma}}{c_{23}\theta^{\gamma}+2}-\theta^{\gamma}\left[\sum_{i=1}^{n_1}e^{x_{2i}}+\sum_{i=1}^{n_2}e^{x_{2i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$\frac{\partial l}{\partial c_3}=\frac{n_1\theta^{\gamma}}{c_{23}\theta^{\gamma}+2}+\frac{n_2\theta^{\gamma}}{c_{13}\theta^{\gamma}+2}+\frac{n_3\theta^{\gamma}}{c_3\theta^{\gamma}+1}-\theta^{\gamma}\left[\sum_{i=1}^{n_1}e^{x_{2i}}+\sum_{i=1}^{n_2}e^{x_{1i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$\begin{aligned}\frac{\partial l}{\partial \theta}=&\gamma\theta^{\gamma-1}[\frac{n_1}{c_1\theta^{\gamma}+1}+\frac{n_2}{c_2\theta^{\gamma}+1}+\frac{n_3}{c_3\theta^{\gamma}+1}+\frac{n_1}{c_{23}\theta^{\gamma}+2}+\frac{n_2}{c_{13}\theta^{\gamma}+2}-c_1\sum_{i=1}^{n_1}e^{x_{1i}}\\&-c_{23}\sum_{i=1}^{n_1}e^{x_{2i}}-c_{13}\sum_{i=1}^{n_2}e^{x_{1i}}-c_2\sum_{i=1}^{n_2}e^{x_{2i}}-c_{123}\sum_{i=1}^{n_3}e^{x_i}]\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \gamma}=&\theta^{\gamma}\ln\theta\left[\frac{n_1c_1}{c_1\theta^{\gamma}+1}+\frac{n_2c_2}{c_2\theta^{\gamma}+1}+\frac{n_3}{c_3\theta^{\gamma}+1}+\frac{n_1c_{23}}{c_{23}\theta^{\gamma}+2}+\frac{n_2c_{13}}{c_{13}\theta^{\gamma}+2}-c_1\sum_{i=1}^{n_1}e^{x_{1i}}\right.\\&\left.-c_{23}\sum_{i=1}^{n_1}e^{x_{2i}}-c_{13}\sum_{i=1}^{n_2}e^{x_{1i}}-c_2\sum_{i=1}^{n_2}e^{x_{2i}}-c_{123}\sum_{i=1}^{n_3}e^{x_i}\right].\end{aligned}$$

In the case of the BMWG distribution, the information matrix $I(\Psi)$ is a 5×5 symmetric matrix with elements

$$I_{ij}(\Psi)=-E\left[\frac{\partial^2\log l\,(\Psi)}{\partial\Psi_i\partial\Psi_j}\right],\ \forall\ i,j$$

Now, elements of the Fisher information matrix for the $BMWG\left(c_1,c_2,c_3,\gamma,\theta\right)$ distribution are obtained to be.

$I_{11}\left(\hat{\Psi}\right)=\frac{n_1\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_1\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\frac{n_2\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}$, $I_{22}\left(\hat{\Psi}\right)=\frac{n_2\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_2\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\frac{n_1\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}$,
$I_{13}\left(\hat{\Psi}\right)=\frac{n_2\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}$, $I_{23}\left(\hat{\Psi}\right)=\frac{n_1\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}$.
$I_{33}\left(\hat{\Psi}\right)=\frac{n_1\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}+\frac{n_2\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}+\frac{n_3\hat{\theta}^{2\hat{\gamma}}}{\left(\hat{c}_3\hat{\theta}^{\hat{\gamma}}+1\right)^2}$, $I_{12}\left(\hat{\Psi}\right)=I_{21}\left(\hat{\Psi}\right)=0$.
$I_{15}\left(\hat{\Psi}\right)=\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\left[\frac{-n_1}{\left(\hat{c}_1\hat{\theta}^{\hat{\gamma}}+1\right)^2}-\frac{n_2}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}+\sum_{i=1}^{n_1}e^{x_{1i}}+\sum_{i=1}^{n_2}e^{x_{1i}}+\sum_{i=1}^{n_3}e^{x_i}\right]$,

$$I_{25}\left(\hat{\Psi}\right)=\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\left[\frac{-n_1}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_2}{\left(\hat{c}_2\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\sum_{i=1}^{n_1}e^{x_{2i}}+\sum_{i=1}^{n_2}e^{x_{2i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$I_{35}\left(\hat{\Psi}\right)=\hat{\gamma}\hat{\theta}^{\hat{\gamma}-1}\left[\frac{-n_1}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_2}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_3}{\left(\hat{c}_3\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\sum_{i=1}^{n_1}e^{x_{2i}}+\sum_{i=1}^{n_2}e^{x_{1i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$I_{14}\left(\hat{\Psi}\right)=\hat{\theta}^{\hat{\gamma}}\ln\hat{\theta}\left[\frac{-n_1}{\left(\hat{c}_1\hat{\theta}^{\hat{\gamma}}+1\right)^2}-\frac{n_2}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}+\sum_{i=1}^{n_1}e^{x_{1i}}+\sum_{i=1}^{n_2}e^{x_{1i}}+\sum_{i=1}^{n_3}e^{x_i}\right]$$

$$I_{24}\left(\hat{\Psi}\right)=\hat{\theta}^{\hat{\gamma}}\ln\hat{\theta}\left[\frac{-n_1}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_2}{\left(\hat{c}_2\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\sum_{i=1}^{n_1}e^{x_{2i}}+\sum_{i=1}^{n_2}e^{x_{2i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$I_{34}\left(\hat{\Psi}\right)=\hat{\theta}^{\hat{\gamma}}\ln\hat{\theta}\left[\frac{-n_1}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_2}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_3}{\left(\hat{c}_3\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\sum_{i=1}^{n_1}e^{x_{2i}}+\sum_{i=1}^{n_2}e^{x_{1i}}+\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$I_{44}\left(\hat{\Psi}\right)=\hat{\theta}^{\hat{\gamma}}\left(\ln\hat{\theta}\right)^2\left[\frac{-n_1\hat{c}_1}{\left(\hat{c}_1\hat{\theta}^{\hat{\gamma}}+1\right)^2}-\frac{n_1\hat{c}_{23}}{\left(\hat{c}_{23}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_2\hat{c}_{13}}{\left(\hat{c}_{13}\hat{\theta}^{\hat{\gamma}}+2\right)^2}-\frac{n_2\hat{c}_2}{\left(\hat{c}_2\hat{\theta}^{\hat{\gamma}}+1\right)^2}+\hat{c}_1\sum_{i=1}^{n_1}e^{x_{1i}}\right.$$
$$\left.+\hat{c}_{23}\sum_{i=1}^{n_1}e^{x_{2i}}+\hat{c}_{13}\sum_{i=1}^{n_2}e^{x_{1i}}+\hat{c}_2\sum_{i=1}^{n_2}e^{x_{2i}}+\hat{c}_{123}\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$I_{45}\left(\hat{\Psi}\right)=-\xi\left(n_1,\hat{c}_1,\hat{\theta},\hat{\gamma}\right)-\xi\left(n_1,\hat{c}_{23},\hat{\theta},\hat{\gamma}\right)-\xi\left(n_2,\hat{c}_{13},\hat{\theta},\hat{\gamma}\right)-\xi\left(n_2,\hat{c}_2,\hat{\theta},\hat{\gamma}\right)-\xi\left(n_3,\hat{c}_3,\hat{\theta},\hat{\gamma}\right)$$
$$+\hat{\theta}^{\hat{\gamma}-1}\left(1+\hat{\gamma}\ln\hat{\theta}\right)\left[\hat{c}_1\sum_{i=1}^{n_1}e^{x_{1i}}+\hat{c}_{23}\sum_{i=1}^{n_1}e^{x_{2i}}+\hat{c}_{13}\sum_{i=1}^{n_2}e^{x_{1i}}+\hat{c}_2\sum_{i=1}^{n_2}e^{x_{2i}}+\hat{c}_{123}\sum_{i=1}^{n_3}e^{x_i}\right],$$

$$I_{55}\left(\hat{\Psi}\right)=-\epsilon\left(n_1,\hat{c}_1,\hat{\theta},\hat{\gamma}\right)-\epsilon\left(n_1,\hat{c}_{23},\hat{\theta},\hat{\gamma}\right)-\epsilon\left(n_2,\hat{c}_{13},\hat{\theta},\hat{\gamma}\right)-\epsilon\left(n_2,\hat{c}_2,\hat{\theta},\hat{\gamma}\right)-\epsilon\left(n_3,\hat{c}_3,\hat{\theta},\hat{\gamma}\right)$$
$$+\hat{\gamma}(\hat{\gamma}-1)\hat{\theta}^{\hat{\gamma}-2}\left[\hat{c}_1\sum_{i=1}^{n_1}e^{x_{1i}}+\hat{c}_{23}\sum_{i=1}^{n_1}e^{x_{2i}}+\hat{c}_{13}\sum_{i=1}^{n_2}e^{x_{1i}}+\hat{c}_2\sum_{i=1}^{n_2}e^{x_{2i}}+\hat{c}_{123}\sum_{i=1}^{n_3}e^{x_i}\right]$$

where $\xi(n,c,\theta,\gamma)=\frac{nc\theta^{\gamma-1}(1+\gamma\ln\theta)}{c\theta^{\gamma}+1}-\frac{c\gamma\theta^{2\gamma-1}\ln\theta}{(c\theta^{\gamma}+1)^2}$, and $\epsilon(n,c,\theta,\gamma)=\frac{nc\gamma\theta^{\gamma-2}[(\gamma-1)(c\theta^{\gamma}+1)-\gamma\theta^{\gamma}]}{(c\theta^{\gamma}+1)^2}$.

So, the asymptotic confidence intervals for the parameters of the BMWG distribution will be presented. Let $\left(\hat{\Psi}\right)=\left(\hat{c_1},\ \hat{c_2},\ \hat{c_3},\ \hat{\gamma},\hat{\theta},\right)$ be the MLEs of $\Psi=\left(c_1,c_2,c_3,\gamma,\theta\right)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have:

$$\sqrt{n}\left(\hat{\Psi}-\Psi\right)\xrightarrow{d}N_5\left(\underset{\sim}{0},I^{-1}(\Psi)\right)\tag{50}$$

where $I(\Psi)$ is the Fisher information matrix. The multivariate normal distribution with mean vector $\underset{\sim}{0}=(0,0,0,0,0)$ and variance–covariance matrix $I^{-1}(\Psi)$ can be used to construct confidence intervals for the model parameters. That is, a $(1-\alpha)\%$ two-sided confidence intervals for $c_1,c_2\,c_3\ \gamma,\theta$ can be introduced as.

$$\hat{c_1} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{c_1c_1}(\Psi)},\ \hat{c_2} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{c_2c_2}(\Psi)},\ \hat{c_3} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{c_3c_3}(\Psi)},$$
$$\hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\gamma\gamma}(\Psi)}\, and\ \hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\theta\theta}(\Psi)} \tag{51}$$

Respectively, where $I^{-1}_{c_1c_1}(\Psi)$,$I^{-1}_{c_2c_2}(\Psi)$ $I^{-1}_{c_3c_3}(\Psi)$,$I^{-1}_{\gamma\gamma}(\Psi) and\ I^{-1}_{\theta\theta}(\Psi)$ are diagonal elements of the variance–covariance matrix and $Z_{\frac{\alpha}{2}}$ is the $(\frac{\alpha}{2})^{th}$ percentile of a standard normal distribution.

4.3 Bayesian Estimation for BMWG Distribution

Assume c_1, c_2 and c_3 are independent, and distributed as gamma as following

$$\pi_i(c_i) = \frac{b^{a_i}}{\Gamma(a_i)}\, c_i^{a_i-1} e^{-b_i c_i}\ ,\ i = 1, 2, 3\ , c_i > 0$$

The joint prior density of c_1, c_2 and c_3 is given as follows

$$\pi_0(c_1, c_2, c_3) = \prod_{i=1}^{3} \frac{b^{a_i}}{\Gamma(a_i)}\, c_i^{a_i-1} e^{-b_i\ c_i}$$

Suppose $\{(x_{11}, x_{21}), \dots, (x_{1n}, x_{2n})\}$ is a random sample from $BMWG(c_1, c_2, c_3, \gamma, \theta)$ distribution. Consider the following notation.

$D = \{(x_{11}, x_{21}), \dots, (x_{1n}, x_{2n})\}$,$\Psi = (c_1, c_2, c_3)$ and $n = n_1 + n_2 + n_3$.

Then, the Likelihood function (4.18) can be written as

$$L(D\backslash\Psi) = \ Exp(\log L(D\backslash\Psi))$$

$$L(D\backslash\Psi) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{l=1}^{n_2}\sum_{m=1}^{n_1}\sum_{p=1}^{m}\sum_{r=}^{n_2}\sum_{q=1}^{r}\binom{n_1}{j}\binom{n_2}{k}\binom{n_3}{l}\binom{n_1}{m}\binom{m}{p}\binom{n_2}{r}\binom{r}{q}$$

$c_1^{q+j}\ c_2^{p+k} c_3^{l+m+r-p-q} Exp(-c_1 \mathrm{H}_1 - c_2 H_2 - c_3 H_3)$ where $H_1 = \theta^{\gamma}\left[\sum_{i=1}^{n_1} e^{x_{1i}} + \sum_{i=1}^{n_2} e^{x_{1i}} + \sum_{i=1}^{n_3} e^{x_i}\right]$, $H_2 = \theta^{\gamma}\left[\sum_{i=1}^{n_1} e^{x_{2i}} + \sum_{i=1}^{n_2} e^{x_{2i}} + \sum_{i=1}^{n_3} e^{x_i}\right]$. and $H_3 = \theta^{\gamma}\left[\sum_{i=1}^{n_1} e^{x_{2i}} + \sum_{i=1}^{n_2} e^{x_{1i}} + \sum_{i=1}^{n_3} e^{x_i}\right]$.

Hence the joint posterior density function of $\Psi = (c_1, c_2, c_3)$ given the data D, denoted by $\pi_1(\ \Psi\backslash D)$ can be written as.

$\pi_1(\ \Psi\backslash D)\ = \frac{f(D,\Psi)}{f(D)}$.

$$L(D\backslash\Psi) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{l=1}^{n_2}\sum_{m=1}^{n_1}\sum_{p=1}^{m}\sum_{r=}^{n_2}\sum_{q=1}^{r} \eta_{jr}\ Gamma\ \left[c_1; a_{1jq},\ b_1 + H_1\right]$$

$\cdot Gamma\left[c_2; a_{2kp}, b_2 + H_2\right] . Gamma\left[c_3; a_{3lr}, b_3 + H_3\right]$. where $\eta_{jr} = \frac{v_{jr}}{\sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{l=1}^{n_2}\sum_{m=1}^{n_1}\sum_{p=1}^{m}\sum_{r=}^{n_2}\sum_{q=1}^{r} v_{jr}}$ and $v_{jr} = \binom{n_1}{j}\binom{n_2}{k}\binom{n_3}{l}\binom{n_1}{m}\binom{m}{p}\binom{n_2}{r}\binom{r}{q} \cdot \frac{\Gamma(a_{1jq})}{\left[b_1+T_1\right]^{a_{1jq}}} \cdot \frac{\Gamma(a_{2kp})}{\left[b_2+T_2\right]^{a_{2kp}}} \cdot \frac{\Gamma(a_{3lr})}{\left[b_3+T_3\right]^{a_{3lr}}}$.

$a_{1jq} = a_1 + j + q$, $a_{2kp} = a_2 + k + p$ and $a_{3lr} = a_3 + l + m + r - q - p$.

Therefore, under the assumption of independence of c_1, c_2 and c_3 and γ, θ are assumed to be known. It is possible to get the Bayes estimators of c_1, c_2 and c_3 explicitly under the square error loss function as follows:

$$\check{c}_1 = \frac{1}{b_1 + H_1} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{l=1}^{n_2}\sum_{m=1}^{n_1}\sum_{p=1}^{m}\sum_{r=}^{n_2}\sum_{q=1}^{r} \eta_{jr}\, a_{1jq},$$

$$\check{c}_2 = \frac{1}{b_2 + H_2} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{l=1}^{n_2}\sum_{m=1}^{n_1}\sum_{p=1}^{m}\sum_{r=}^{n_2}\sum_{q=1}^{r} \eta_{jr}\, a_{2kp},$$

$$\text{and } \check{c}_3 = \frac{1}{b_3 + H_3} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{l=1}^{n_2}\sum_{m=1}^{n_1}\sum_{p=1}^{m}\sum_{r=}^{n_2}\sum_{q=1}^{r} \eta_{jr}\, a_{3lr}$$

5 Numerical Study

5.1 Data Analysis

To see how the BMWW and BMWG models work in practice, three data sets will be applied in this section (Tables 1, 2, 3). The marginal distributions of both models are fitted to each data set as shown in Table 4. For each of the data sets the Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike

Table 1 The football data

S. No.	X_1	X_2	S.N	X_1	X_2	S.N	X_1	X_2	S.N	X_1	X_2
1	26	20	11	72	72	21	53	39	31	49	49
2	63	18	12	66	62	22	54	7	32	24	24
3	19	19	13	25	9	23	51	28	33	44	30
4	66	85	14	41	3	24	76	64	34	42	3
5	4	4	15	16	75	25	64	15	35	27	47
6	49	49	16	18	18	26	26	48	36	28	28
7	8	8	17	22	14	27	16	16	37	2	2
8	69	71	18	42	42	28	44	6			
9	39	39	19	36	52	29	25	14			
10	82	48	20	34	34	30	55	11			

Table 2 Burr data

S. No.	X_1	X_2	S.N	X_1	X_2	S.N	X_1	X_2	S.N	X_1	X_2	S.N	X_1	X_2
1	0.04	0.06	11	0.24	0.16	21	0.24	0.12	31	0.24	0.14	41	0.02	0.16
2	0.02	0.12	12	0.04	0.12	22	0.22	0.24	32	0.16	0.06	42	0.18	0.32
3	0.06	0.14	13	0.14	0.24	23	0.12	0.06	33	0.32	0.04	43	0.22	0.18
4	0.12	0.04	14	0.16	0.06	24	0.18	0.02	34	0.18	0.14	44	0.14	0.24
5	0.14	0.14	15	0.08	0.02	25	0.24	0.18	35	0.24	0.22	45	0.06	0.22
6	0.08	0.16	16	0.26	0.18	26	0.32	0.22	36	0.22	0.14	46	0.04	0.04
7	0.22	0.08	17	0.32	0.22	27	0.16	0.14	37	0.16	0.06	47	0.14	0.14
8	0.12	0.26	18	0.28	0.14	28	0.14	0.02	38	0.12	0.04	48	0.26	0.26
9	0.08	0.32	19	0.14	0.22	29	0.08	0.18	39	0.24	0.16	49	0.18	0.18
10	0.26	0.22	20	0.16	0.16	30	0.16	0.22	40	0.06	0.24	50	0.16	0.16

Table 3 Cholesterol levels at 5 and 25 weeks after treatment in 30 patients

S. No.	X_1	X_2	S.N	X_1	X_2	S.N	X_1	X_2
1	325	246	11	217	252	21	316	283
2	278	245	12	248	305	22	243	245
3	257	212	13	225	225	23	305	272
4	192	192	14	287	208	24	197	197
5	276	325	15	233	217	25	243	247
6	262	294	16	198	198	26	315	283
7	309	232	17	229	179	27	205	205
8	287	287	18	310	352	28	315	255
9	304	245	19	214	274	29	263	215
10	215	261	20	253	209	30	210	271

information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are calculated for both BMWW and BMWG models and given in Table 5. Moreover, the MLE,the CI length and the variance covariance matrix is calculated for each data set to each model separately as shown in Tables 6 and 7.

5.1.1 Data Set 1

The data set has been obtained from Meintanis [12]. He explained that: the data represent the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. In this case, all possibilities exist, for example, $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

Table 4 Marginal fitting

	MODEL	Variables	$\hat{\beta}$	$\hat{C}$	k-S
Data set1	MWW	X1	2.667	…	1.009
		X2	3.595	…	1.024
		Min (X1, X2)	4.5222	…	1.024
	MWG	X1	…	4.438	1.026
		X2	…	4.584	0.579
		Min (X1, X2)	…	6.753	1.028
Data set2	MWW	X1	0.753	…	0.69
		X2	3.157	…	0.833
		Min (X1, X2)	1.551	…	0.911
	MWG	X1	…	1.05	0.671
		X2	…	1.6	0.572
		Min (X1, X2)	…	2.15	0.673
Data set3	MWW	X1	0.75	…	0.763
		X2	1.125	…	0.916
		Min (X1, X2)	1.5	…	0.916
	MWG	X1	…	0.8	0.677
		X2	…	1.6	0.6
		MIN	…	1.9	0.646

Table 5 Comparison between BMWW and BMWG Models

	MODEL	− 2lnl	AIC	BIC	AICC	HQIC
Data set1	BMWW	56.146	68.146	77.811	70.946	71.553
	BMWG	50.941	60.941	68.996	62.877	63.781
Data set2	BMWW	342.36	354.36	365.832	356.314	358.729
	BMWG	104.736	114.736	124.296	116.1	118.377
Data set3	BMWW	178.352	190.352	198.759	194.004	193.041
	BMWG	56.345	66.345	73.351	68.845	68.586

5.1.2 Data Set 2

The data set has been obtained from Shoaee [13]. This dataset contains 50 observations on the burr. In the first component, the hole diameter is 12 mm and the sheet thickness is 3.15 mm. In the second component, the hole diameter is 9 mm and the sheet thickness is 2 mm. These two datasets are derived from two different machines. Also, in this case, all possibilities exist, for example, $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

Table 6 The MLE, the CIL and the variance–covariance matrix for BMWW model

	Para	MLE	CIL	Var–cov					
Dataset1	$\hat{\beta}_1$	0.927	0.203	0.099	− 0.034	− 0.057	0.0088	− 0.0043	0.0033
	$\hat{\beta}_2$	1.855	0.208	− 0.034	0.104	− 0.055	0.019	− 0.0092	0.0072
	$\hat{\beta}_3$	1.74	0.228	− 0.057	− 0.055	0.125	0.0086	− 0.0057	0.0039
	$\hat{\gamma}$	1.543	0.171	0.0088	0.019	0.0086	0.07	0.03	− 0.023
	$\hat{\theta}$	1.758	0.091	− 0.0043	− 0.0092	− 0.0057	0.03	0.02	− 0.0048
	$\hat{c}$	0.433	0.051	0.0033	0.0072	0.0039	− 0.023	− 0.0048	0.0062
Dataset 2	$\hat{\beta}_1$	0.413	0.046	0.0069	− 0.0024	− 0.0039	0.0048	− 0.0001	0.0019
	$\hat{\beta}_2$	0.798	0.053	− 0.0024	0.0092	− 0.0049	0.015	− 0.0005	0.0062
	$\hat{\beta}_3$	0.34	0.053	− 0.0039	− 0.0049	0.0092	0.0038	− 0.0001	0.0015
	$\hat{\gamma}$	0.596	0.062	0.0048	0.015	0.0038	0.012	0.0045	− 0.044
	$\hat{\theta}$	0.301	0.035	− 0.0001	− 0.0005	− 0.0001	0.0045	0.0039	− 0.0025
	$\hat{c}$	0.502	0.096	0.002	0.0062	0.0015	− 0.044	− 0.0025	0.03
Dataset 3	$\hat{\beta}_1$	0.385	0.072	0.0099	− 0.0038	− 0.0051	0.0088	0.00015	0.0032
	$\hat{\beta}_2$	0.756	0.089	− 0.0038	0.016	− 0.0076	0.029	0.00072	0.011
	$\hat{\beta}_3$	0.251	0.084	− 0.0051	− 0.0076	0.014	0.0076	0.00028	0.0026
	$\hat{\gamma}$	0.57	0.066	0.0088	0.029	0.0076	0.0086	0.0066	− 0.078
	$\hat{\theta}$	0.306	0.058	0.00015	0.00072	0.00028	0.0066	0.00662	− 0.0041
	$\hat{c}$	0.016	0.158	0.0032	0.011	0.0026	− 0.078	− 0.0041	0.049

Table 7 The MLE,the CIL and the Variance–covariance matrix for BMWG model

	Para	MLE	CIL	Var−cov				
Dataset1	$\hat{c}_1$	2.169	0.7	1.18	− 0.999	2.75	− 0.084	− 0.076
	$\hat{c}_2$	2.315	0.552	− 0.996	0.733	2.167	− 0.065	− 0.058
	$\hat{c}_3$	2.269	1.637	2.75	2.167	6.45	0.215	0.194
	$\hat{\gamma}$	1.418	0.088	− 0.084	− 0.065	0.215	0.019	− 0.0024
	$\hat{\theta}$	3	0.059	− 0.076	− 0.058	0.194	− 0.0024	0.0084
Dataset2	$\hat{c}_1$	0.55	0.571	1.063	− 1.182	− 0.118	− 0.438	0.014
	$\hat{c}_2$	1.1	0.539	− 1.182	0.947	− 0.083	− 0.428	0.013
	$\hat{c}_3$	0.5	0.329	− 0.118	− 0.083	0.351	0.011	− 0.0003
	$\hat{\gamma}$	0.6	0.221	− 0.438	− 0.428	0.011	0.159	− 0.0067
	$\hat{\theta}$	0.375	0.076	0.014	0.013	− 0.0003	− 0.0067	0.019
Dataset3	$\hat{c}_1$	0.3	0.379	0.28	0.166	− 0.122	0.068	− 0.074
	$\hat{c}_2$	1.1	0.477	0.166	0.443	− 0.119	0.066	− 0.071
	$\hat{c}_3$	0.5	0.425	− 0.122	− 0.119	0.353	− 0.0014	0.0015
	$\hat{\gamma}$	0.6	0.113	0.068	0.066	− 0.0014	0.025	− 0.059
	$\hat{\theta}$	0.625	0.024	− 0.074	− 0.071	0.0015	− 0.059	0.0011

5.1.3 Data Set 3

This data set contains cholesterol levels at 5 and 25 weeks after treatment in 30 patients. Before analyzing this data, the transformation (X − 150)/100 is applied to all data, this transformation will not effect on the analysis and are for computational reasons only. This data set was used by Shoaee [13]. Again, in this case all possibilities exist, i.e., $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

5.2 Simulation Study

In this section, the results of a Monte Carlo simulation study testing the performance of MLE of the model Parameters will be introduced. The evaluation of the MLEs was performed based on the following quantities for each sample size: the Average Estimates (AE), the Mean Squared Error, (*MSE*) Relative Absolute Bias (RAB) and Confidence Interval Length (CIL) are estimated from $R = 1000$ replications for $\widehat{\beta}_1$, $\widehat{\beta}_2$ $\widehat{\beta}_3, \hat{\gamma}$, $\hat{\theta}$ and $\hat{c}$ the sample size has been considered at $n = 35, 50, 70, 100$ and 150, and some values for β_1, β_2 β_3, γ, θ and c have been considered in the following sets:

Set1 (0.2, 0.2, 0.9, 0.9, 1, 1).
Set2 (0.5, 0.5, 1, 1.5, 1.5, 1.5).
Set 3 (0.03, 0.02, 0.01, 0.9, 0.9, 0.9).

Algorithm to generate from BMWW distribution

Step 1 Generate U_1, U_2 and U_3 from U(0, 1).
Step Compute $Z_1 = \frac{-\ln U_1}{\beta_1(c\theta^{\gamma}+1)}$, $Z_2 = \frac{-\ln U_2}{\beta_2(c\theta^{\gamma}+1)}$ and $Z_3 = \frac{-\ln U_3}{\beta_3(c\theta^{\gamma}+1)}$.
Step3 Obtain $X_1 = \min\left(Z_1, Z_3\right)$ and $X_2 = \min\left(Z_2, Z_3\right)$.
Step4 Define the indicator functions as.
$\delta_{1i} = \begin{cases} 1\,; & x_{1i} < x_{1i} \\ 0; & otherwise \end{cases}$, $\delta_{2i} = \begin{cases} 1\,; & x_{1i} > x_{1i} \\ 0; & otherwise \end{cases}$ and $\delta_{3i} = \begin{cases} 1\,; & x_{1i} = x_{1i} \\ 0; & otherwise \end{cases}$.
Step5 The corresponding sample size n must satisfy $n = n_1 + n_2 + n_3$.
Such that $n_1 = \sum_{i=1}^{n} \delta_{1i}$, $\quad n_2 = \sum_{i=1}^{n} \delta_{2i}$ and $\quad n_3 = \sum_{i=1}^{n} \delta_{3i}$.

For different choices of sample sizes a 1000, data set is generated using the MATHCAD program that is employed to solve the nonlinear likelihood equations. It can be noted from Table 8 through Table 9 that the estimates are work well and MSE and RAB decrease as the sample size increases (Table, 10).

Table 8 The AE, MSE, RAB and CL for BMWW Model for Set 1

Sample size	Parameters	AE	MSE	RAB	CL (lower, upper)
$n = 35$	$\hat{\beta}_1$	0.171	0.00083	0.144	0.059 (0.142,0.201)
	$\hat{\beta}_2$	0.239	0.0020	0.195	0.074 (0.202, 0.276)
	$\hat{\beta}_3$	0.964	0.0041	0.071	0.194(0.867, 1.061)
	$\hat{\gamma}$	0.973	0.0054	0.081	0.096 (0.925, 1.021)
	$\hat{\theta}$	0.925	0.0057	0.075	0.00045 (0.922, 0.927)
	$\hat{c}$	0.903	0.0094	0.097	0.228 (0.789, 1.017)
$n = 50$	$\hat{\beta}_1$	0.369	0.029	0.847	0.07 (0.334,0.405)
	$\hat{\beta}_2$	0.352	0.023	0.759	0.07 (0.317, 0.387)
	$\hat{\beta}_3$	0.667	0.054	0.259	0.089 (0.622, 0.711)
	$\hat{\gamma}$	1.029	0.017	0.143	0.106 (0.976, 1.082)
	$\hat{\theta}$	0.579	0.177	0.421	0.0093 (0.574, 0.583)
	$\hat{c}$	0.559	0.195	0.441	0.456 (0.331, 0.787)
$n = 70$	$\hat{\beta}_1$	0.362	0.026	0.812	0.049 (0.338, 0.387)
	$\hat{\beta}_2$	0.346	0.021	0.728	0.045 (0.323, 0.368)
	$\hat{\beta}_3$	0.628	0.074	0.303	0.059 (0.598,0.657)
	$\hat{\gamma}$	0.949	0.0024	0.054	0.063 (0.917,0.98)
	$\hat{\theta}$	0.595	0.164	0.405	0.00681 (0.591, 0.598)
	$\hat{c}$	0.56	0.194	0.44	0.238 (0.441,0.679)
$n = 100$	$\hat{\beta}_1$	0.215	0.00023	0.076	0.024 (0.203, 0.227)
	$\hat{\beta}_2$	0.26	0.00360	0.302	0.027 (0.247,0.274)
	$\hat{\beta}_3$	1.023	0.015	0.137	0.07 (0.988, 1.058)
	$\hat{\gamma}$	0.988	0.0077	0.097	0.046 (0.965, 1.01)
	$\hat{\theta}$	0.934	0.0044	0.066	0.00381 (0.932,0.936)
	$\hat{c}$	0.963	0.0014	0.037	0.174 (0.875,1.05)
$n = 150$	$\hat{\beta}_1$	0.301	0.01	0.507	0.02 (0.291, 0.311)
	$\hat{\beta}_2$	0.311	0.012	0.555	0.02 (0.301, 0.321)
	$\hat{\beta}_3$	0.892	0.000071	0.00934	0.041 (0.871, 0.912)
	$\hat{\gamma}$	0.988	0.0077	0.097	0.03 (0.973, 1.003)
	$\hat{\theta}$	0.696	0.092	0.304	0.00204 (0.695, 0.697)
	$\hat{c}$	0.762	0.057	0.238	0.187 (0.668, 0.855)

6 Conclusion

In this paper, the BMWW and BMWG distributions are introduced. Marginals of these bivariate distributions are also MWW and MWG distributions respectively. It is observed that the new distributions are singular and they have an absolutely continuous part and a singular part. Since the joint distribution function and the joint density function are in closed forms, therefore these distributions can be used in

Table 9 The AE, MSE, RAB and CL for BMWW Model for Set 3

Sample size	Parameters	AE	MSE	RAB	CL (lower, upper)
$n = 35$	$\hat{\beta}_1$	0.044	0.00021	0.481	0.0041 (0.042, 0.046)
	$\hat{\beta}_2$	0.024	0.00001	0.181	0.0033 (0.022, 0.025)
	$\hat{\beta}_3$	0.031	0.00046	2.134	0.005 (0.029, 0.034)
	$\hat{\gamma}$	0.92	0.00039	0.022	0.071 (0.884, 0.998)
	$\hat{\theta}$	0.873	0.00074	0.03	0.0033 (0.871, 0.874)
	$\hat{c}$	1	0.01	0.111	0.076 (0.962, 1.038)
$n = 50$	$\hat{\beta}_1$	0.041	0.00012	0.367	0.0041 (0.039, 0.043)
	$\hat{\beta}_2$	0.022	0.000003	0.086	0.0016 (0.021, 0.023)
	$\hat{\beta}_3$	0.027	0.00028	1.666	0.0028 (0.025, 0.028)
	$\hat{\gamma}$	0.936	0.0013	0.04	0.051 (0.91, 0.961)
	$\hat{\theta}$	0.918	0.00031	0.02	0.013 (0.911, 0.924)
	$\hat{c}$	0.968	0.0046	0.075	0.128 (0.903, 1.032)
$n = 70$	$\hat{\beta}_1$	0.035	0.000024	0.164	0.00413 (0.034, 0.036)
	$\hat{\beta}_2$	0.026	0.000036	0.299	0.00137 (0.025, 0.027)
	$\hat{\beta}_3$	0.026	0.00027	1.628	0.0019 (0.025, 0.027)
	$\hat{\gamma}$	0.919	0.00036	0.021	0.031 (0.903, 0.935)
	$\hat{\theta}$	0.916	0.00025	0.018	0.0077 (0.912, 0.92)
	$\hat{c}$	0.982	0.0067	0.091	0.087 (0.938, 1.025)
$n = 100$	$\hat{\beta}_1$	0.031	0.0000006	0.027	0.0041 (0.03, 0.031)
	$\hat{\beta}_2$	0.021	0.000008	0.044	0.0011 (0.02, 0.023)
	$\hat{\beta}_3$	0.022	0.00016	1.247	0.00126 (0.022, 0.023)
	$\hat{\gamma}$	0.962	0.0038	0.068	0.021 (0.951, 0.972)
	$\hat{\theta}$	1.003	0.011	0.114	0.0065 (1, 1.006)
	$\hat{c}$	0.941	0.001699	0.046	0.064 (0.909, 0.973)
$n = 150$	$\hat{\beta}_1$	0.037	0.00005	0.235	0.0041 (0.037, 0.038)
	$\hat{\beta}_2$	0.023	0.0000065	0.128	0.0007 (0.022, 0.023)
	$\hat{\beta}_3$	0.025	0.00021	1.451	0.0091 (0.024, 0.025)
	$\hat{\gamma}$	0.918	0.00032	0.02	0.014 (0.911, 0.925)
	$\hat{\theta}$	0.864	0.0013	0.04	0.0031 (0.863, 0.866)
	$\hat{c}$	1	0.01	0.111	0.04 (0.98, 1.02)

practice for non-negative and positively correlated random variables. Several properties of the BMWW such as conditional distributions, product moments, joint survival function and joint reversed hazard function have been discussed. Furthermore, it is showed that the BMWW distribution is obtained from the Marshall and Olkin survival copula and a tail dependence measure is discussed. Moreover, the joint moment generating function for BMWG is obtained. Explicit Bayesian estimators are obtained for the unknown parameters of these models and MLE are also discussed. Three data sets have been re-analyzed for illustrative purposes. Along the

Table 10 The AE, MSE, RAB and CL for BMWW Model for Set 2

Sample size	Parameters	AE	MSE	RAB	CL (lower, upper)
$n = 35$	$\hat{\beta}_1$	1.019	0.269	1.038	0.192 (0.923, 1.115)
	$\hat{\beta}_2$	0.927	0.182	0.854	0.17 (0.842,1.012)
	$\hat{\beta}_3$	2.757	3.087	1.757	0.409 (2.553, 2.961)
	$\hat{\gamma}$	0.987	0.263	0.342	0.131 (0.921, 1.052)
	$\hat{\theta}$	1.409	0.0083	0.061	0.021 (1.398, 1.42)
	$\hat{c}$	0.762	0.454	0.492	0.106 (0.709, 0.815)
$n = 50$	$\hat{\beta}_1$	1.043	0.295	1.087	0.168 (0.96, 1.127)
	$\hat{\beta}_2$	0.967	0.218	0.935	0.142 (0.897, 1.038)
	$\hat{\beta}_3$	2.385	1.848	1.26	0.442 (2.164,2.606)
	$\hat{\gamma}$	0.961	0.29	0.359	0.122 (0.9, 1.022)
	$\hat{\theta}$	1.114	0.149	0.257	0.01 (1.109, 1.12)
	$\hat{c}$	0.89	0.372	0.407	0.087 (0.847, 0.934)
$n = 70$	$\hat{\beta}_1$	0.801	0.091	0.602	0.117 (0.742, 0.86)
	$\hat{\beta}_2$	0.831	0.11	0.663	0.129 (0.767, 0.896)
	$\hat{\beta}_3$	2.27	1.613	1.27	0.344 (2.098, 2.442)
	$\hat{\gamma}$	0.966	0.285	0.356	0.097 (0.917, 1.014)
	$\hat{\theta}$	1.134	0.134	0.244	0.00885 (1.13, 1.138)
	$\hat{c}$	0.896	0.364	0.402	0.087 (0.853, 0.94)
$n = 100$	$\hat{\beta}_1$	1.024	0.275	1.049	0.111 (0.969, 1.08)
	$\hat{\beta}_2$	0.919	0.176	0.838	0.113 (0.862, 0.976)
	$\hat{\beta}_3$	2.517	2.301	1.517	0.288 (2.373, 2.661)
	$\hat{\gamma}$	0.975	0.275	0.35	0.072 (0.939, 1.014)
	$\hat{\theta}$	1.149	0.123	0.234	0.0073 (1.145, 1.153)
	$\hat{c}$	0.864	0.404	0.424	0.098 (0.815, 0.913)
$n = 150$	$\hat{\beta}_1$	0.869	0.136	0.738	0.111 (0.83, 0.908)
	$\hat{\beta}_2$	0.909	0.167	0.818	0.079 (0.869, 0.948)
	$\hat{\beta}_3$	2.429	2.042	1.429	0.235 (2.312, 2.547)
	$\hat{\gamma}$	0.965	0.286	0.357	0.064 (0.933, 0.997)
	$\hat{\theta}$	1.109	0.153	0.26	0.0038 (1.108, 1.111)
	$\hat{c}$	0.893	0.369	0.405	0.054 (0.866, 0.92)

same line as Block and Basu (1974) bivariate exponential model, absolutely continuous version of the BMWW and BMWG models also obtained and several of their properties are presented.

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Code Availability Mathcad code is provided as per need for the reviewers.

Declarations

Conflict of interest The author declare that she have no conflict of interest.

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