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A Generalized-Alpha–Beta-Skew Normal Distribution with Applications

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Abstract
Recently there is a lot of research related to skewed distributions and their growing relevance in data analytics. In the present work we introduce a new generalized version of alpha beta skew normal distribution and some of its basic properties are investigated. Some extensions of the proposed distribution have also been studied. A simulation study has been conducted to see the performance of the obtained estimators of the parameters using Metropolis–Hastings (MH) algorithm. The appropriateness of the proposed distribution has been tested by comparing it with twelve closely related and nested distributions using Akaike Information Criterion. The Likelihood Ratio test has been employed for testing the relevance of the induction of the additional parameters in the proposed model.

Keywords Skew distribution · Alpha–beta-skew distribution · Bimodal distribution · AIC

Mathematics Subject Classification 60E05 · 62H10 · 62H12

1 Introduction

The role of probability distribution in the realm of data science is in studying various characteristics of underlying random variable for modelling uncertainty. While dealing with the large scale data arising in data science it's extremely important to investigate descriptive characteristics from the given data, to get hold of useable summary information. Probability distribution allows one to carry out model based data analytics and many statistical methods' used there are based on strong

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distributional assumptions about the underlying population from which data is supposedly sampled. Asymptotic is of paramount importance while dealing with large data also needs distributional assumptions (see [1–3] for example). Recently skew distributions are being extensively investigated and have found many applications in Data science (see [4–8] among others).

In many real life situations the data exhibit many modes as well as asymmetry, e.g., in the fields of demography, insurance, medical sciences, physics, etc. (for details see [9–16] and among others). Azzalini [17] first introduced the skew normal distribution, denoted by $SN(\lambda)$ and the density function of this distribution is given by

$$f_Z(z;\lambda) = 2\varphi(z)\Phi(\lambda z); \quad -\infty < z < \infty,\ -\infty < \lambda < \infty \tag{1}$$

where $\varphi(.)$ is the density function of standard normal distribution, $\Phi(.)$ is the cumulative distribution function (cdf) of standard normal distribution and λ is the asymmetry parameter.

Thereafter, many generalizations have been proposed to serve the same purpose. Some of the important among these are the alpha skew normal distribution of Elal-Olivero [18] and the alpha–beta skew normal distribution of Shafiei et al. [19] which are defined as below.

Elal-Olivero [18] introduced a new form of skew distribution which has both unimodal as well as bimodal behavior and is known as alpha skew normal distribution, denoted by $ASN(\alpha)$ and its density function is given by

$$f(z;\alpha) = \left(\frac{(1-\alpha z)^2+1}{2+\alpha^2}\right)\varphi(z); \quad z,\alpha \in R \tag{2}$$

Along the same line of the Eq. (2), Venegas et al. [20] studied the logarithmic form of alpha-skew normal distribution and used for modeling chemical data. Sharafi et al. [21] introduced a generalization of $ASN(\alpha)$ distribution.

Shafiei et al. [19] introduced a new family of skew distributions with more flexibility than the Azzalini [17] and the Elal-Olivero [18] distributions and is defined as follows: A random variable Z is said to be an alpha–beta skew normal distribution, denoted by $ABSN(\alpha,\beta)$ if its density function is given by

$$f(z;\alpha,\beta) = \left(\frac{(1-\alpha\, z-\beta\, z^3)^2+1}{2+\alpha^2+15\,\beta^2+6\,\alpha\,\beta}\right)\varphi(z); \quad z,\alpha,\beta \in R \tag{3}$$

The main motivation of this work is to propose a flexible generalization of $ABSN(\alpha,\beta)$ distribution to include $ABSN(\alpha,\beta)$ distribution, $ASN(\alpha)$ distribution, $SN(\lambda)$ distribution, and normal distribution and suitable enough to deal with multimodal data up to four modes. Such a situation may arise due to many applications, for example, (1) too many outliers in the data set. (2) a small sample may not exhibit normality always and in fact will more often look multimodal (For detail see Chakraborty et al. [13]).

In order to check the advantages of the proposed generalization in real life data modelling we have compared it with some of its sub models and a few recently introduced distributions by considering two widely used datasets the first of which comprises N latitude degree samples from world lakes, and the other is about white cells count (WCC) among athletes. Our finding clearly shows the superiority of the proposed model over the rest.

The article is summarized as follows: In Sect. 2 we define the proposed distribution, identify its special case and provide a useful results regarding stochastic representation. In Sect. 3 we study some more of its important distributional properties and a location—scale extension of this distribution. The logarithmic form of this distribution is discussed in Sect. 4. The parameter estimation, simulation, real life data modelling and the likelihood ratio test among the nested models of the proposed distribution are provided in Sect. 5. Conclusions are given in Sect. 6 followed by references. The articles end with an appendix.

2 A Generalized Alpha–Beta Skew Normal Distribution

In this section we introduce a new generalized form of alpha–beta skew normal distribution and investigate some of its basic properties.

Definition 1 If a random variable Z has a density function

$$f(z;\alpha,\beta,\lambda) = \frac{[(1-\alpha z-\beta z^3)^2+1]}{C(\alpha,\beta,\lambda)}\varphi(z)\,\Phi(\lambda z)\,;\; z\in R \tag{4}$$

where $C(\alpha,\beta,\lambda) = 1+3\,\alpha\,\beta-\alpha\,b\delta-\beta\,b\delta\frac{3+2\lambda^2}{1+\lambda^2}+\frac{\alpha^2}{2}+\frac{15\beta^2}{2}$, $b=\sqrt{\frac{2}{\pi}}$, $\delta=\frac{\lambda}{\sqrt{1+\lambda^2}}$, then it is said to follow *generalized alpha–beta skew normal distribution* with skewness parameters $(\alpha,\beta,\lambda)^T\in R^3$. We denote it as $GABSN(\alpha,\beta,\lambda)$. See "Appendix" for computation of the normalizing constant $C(\alpha,\beta,\lambda)$.

2.1 Particular Cases of $GABSN(\alpha,\beta,\lambda)$

- If $\alpha=0$, then we get $f(z;\beta,\lambda)=\frac{[(1-\beta z^3)^2+1]}{1-\beta\,b\delta(3+2\lambda^2)/(1+\lambda^2)+15\beta^2/2}\varphi(z)\,\Phi(\lambda z)$.
  This is known as generalized beta skew normal $GBSN(\beta,\lambda)$ distribution.
- If $\beta=0$, then we get the generalized $ASN(\alpha)$ distribution of Sharafi et al. [21] given by $f(z;\alpha,\lambda,\delta)=\frac{[(1-\alpha z)^2+1]}{1-\alpha\,b\delta+\alpha^2/2}\varphi(z)\,\Phi(\lambda z)$.
- If $\lambda=0$, then we get the $ABSN(\alpha,\beta)$ distribution of Shafiei et al. [19] given by $f(z;\alpha,\beta)=\frac{[(1-\alpha z-\beta z^3)^2+1]}{2+\alpha^2+6\alpha\beta+15\beta^2}\varphi(z)$.
- If $\alpha=\beta=0$, then we get the $SN(\lambda)$ distribution of Azzalini [17] and is given by $f(z;\lambda)=2\varphi(z)\,\Phi(\lambda z)$.
- If $\alpha=\beta=\lambda=0$, then we get the standard normal distribution and is given by $f(z)=\varphi(z)$.
- If $Z\sim GABSN(\alpha,\beta,\lambda)$, then $-Z\sim -GABSN(\alpha,\beta,\lambda)$.

2.2 Limiting Cases

- If $\alpha \to \pm\infty$, then $f(z;\lambda) = 2\,z^2\varphi(z)\,\Phi(\lambda\,z)$, where, $2\,z^2\varphi(z)$ is the pdf of Bimodal Normal $BN(2)$ (for details see [22]). Therefore, as $\alpha \to \pm\infty$, $GABSN(\alpha, \beta, \lambda) \to GBN(2)$, where GBN is the Generalized Bimodal Normal.
- If $\beta \to \pm\infty$, then $GABSN(\alpha, \beta, \lambda) \to GBN(6)$, and the pdf of $GBN(6)$ is $f(z;\lambda) = (2\,z^6/15)\,\varphi(z)\,\Phi(\lambda\,z)$.
- For fixed α and β, if $\lambda \to +\infty$ then

$$f(z;\alpha, \beta) = \frac{[(1-\alpha\,z-\beta\,z^3)^2+1]}{1-\alpha\,b-2\beta\,b+3\alpha\beta+\alpha^2/2+15\beta^2/2}\varphi(z)\,I(z>0)$$

and if $\lambda \to -\infty$ then
$f(z;\alpha, \beta) = \frac{[(1-\alpha\,z-\beta\,z^3)^2+1]}{1+\alpha\,b+2\beta\,b+3\alpha\beta+\alpha^2/2+15\beta^2/2}\varphi(z)\,I(z<0)$.

2.3 Plots of the Density Function

The density functions of $GABSN(\alpha, \beta, \lambda)$ distribution for different choices of the parameters α, β and λ are plotted in Fig. 1.

2.4 A Stochastic Representation for $GABSN(\alpha, \beta, \lambda)$ Distribution

Theorem 1 *The conditional distribution of* $W|\{\lambda\,W > X\}$ *follows* $GABSN(\alpha, \beta, \lambda)$ *distribution, if* $W \sim ABSN(\alpha,\ \beta)$ *and* $X \sim N(0\,,\ 1)$, *and are independent.*

Proof Assume $Z = W|\{\lambda\,W > X\}$. Then, we can have

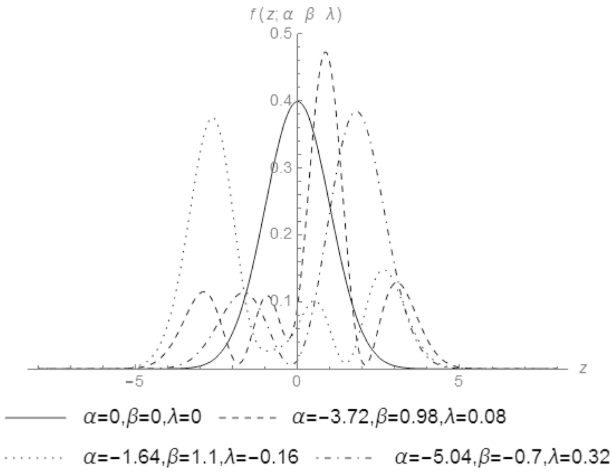


Fig. 1 Plots of the probability density function of $GABSN(\alpha, \beta, \lambda)$

$$P(Z \leq z) = P(W \leq z | \lambda\, W > X) = \frac{P(W \leq z,\ \lambda\, W > X)}{P(\lambda\, W > X)}$$

Using Eq. (3), we get, $P(W \leq z, \lambda\, W > X) = \int\limits_{-\infty}^{z} \frac{(1-\alpha\, u-\beta\, u^3)^2+1}{2+\alpha^2+15\,\beta^2+6\,\alpha\,\beta}\varphi(u)\,\Phi(\lambda\, u)du$ and $P(\lambda\, W > X) = \int\limits_{-\infty}^{\infty} \frac{(1-\alpha\, u-\beta\, u^3)^2+1}{2+\alpha^2+15\,\beta^2+6\,\alpha\,\beta}\varphi(u)\,\Phi(\lambda\, u)du \ = \frac{1}{2+\alpha^2+15\,\beta^2+6\,\alpha\,\beta}C(\alpha, \beta, \lambda).$

Therefore,

$$P(Z \leq z) = \frac{2+\alpha^2+15\,\beta^2+6\,\alpha\,\beta}{C(\alpha, \beta, \lambda)} \int\limits_{-\infty}^{z} \frac{(1-\alpha\, u-\beta\, u^3)^2+1}{2+\alpha^2+15\,\beta^2+6\,\alpha\,\beta}\varphi(u)\,\Phi(\lambda\, u)du$$

$$= \int\limits_{-\infty}^{z} \frac{(1-\alpha\, u-\beta\, u^3)^2+1}{C(\alpha, \beta, \lambda)}\varphi(u)\,\Phi(\lambda\, u)du$$

and the density function of the conditional distribution of $Z = W|\{\lambda\, W > X\}$ is

$$f_Z(z) = \frac{(1-\alpha\, z-\beta\, z^3)^2+1}{C(\alpha, \beta, \lambda)}\varphi(z)\,\Phi(\lambda\, z) \sim GABSN(\alpha, \beta, \lambda)$$

Or, we write $W|\{\lambda\, W > X\} \sim GABSN(\alpha, \beta, \lambda)$.□

3 Distributional Properties

In this section we investigate various important distributional characteristics of the proposed distribution.

Theorem 2 *The cdf of $GABSN(\alpha, \beta, \lambda)$ distribution is given by*

$$\begin{aligned} F_Z(z) = {} & \frac{1}{2(1+\lambda^2)^{5/2}}(b\delta(\alpha^2(1+\lambda^2)^2 - 2\alpha\beta(1+\lambda^2)(5+z^2+(3+z^2)\lambda^2) - \beta(-2z(1+\lambda^2)^2 \\ & + z^4\beta(1+\lambda^2)^2 + z^2\beta(1+\lambda^2)(9+5\lambda^2) \\ & + \beta(33+40\lambda^2+15\lambda^4)))\varphi(z\sqrt{1+\lambda^2}) \\ & - \frac{1}{\sqrt{\pi}}(1+\lambda^2)^2(\sqrt{2}(\alpha+2\beta)\lambda\, Erf(z\sqrt{1+\lambda^2}/\sqrt{2}) \\ & + \sqrt{\pi}(2(\alpha^2 z + 2\alpha(-1+z(3+\lambda^2)\beta) + \beta(-2(2+\lambda^2)+z(15+5z^2+z^4)\beta)) \\ & \sqrt{1+\lambda^2}\varphi(z)\,\Phi(\lambda z) + 2b\beta\delta\,\Phi(z\sqrt{1+\lambda^2}) - 3(1+2\alpha\beta+5\beta^2)\sqrt{1+\lambda^2}\,\Phi(z;\lambda)))) \end{aligned} \tag{5}$$

Proof see "Appendix". □

Theorem 3 *The k^{th} order moment of $GABSN(\alpha, \beta, \lambda)$ distribution is given by*

$$E(Z^k) = \frac{1}{C(\alpha,\beta,\lambda)}\left[E(Z_\lambda^k) - \alpha\, E(Z_\lambda^{k+1}) + \frac{\alpha^2}{2}E(Z_\lambda^{k+2}) \right.$$
$$\left. -\beta\, E(Z_\lambda^{k+3}) + \alpha\,\beta\, E(Z_\lambda^{k+4}) + \frac{\beta^2}{2}E(Z_\lambda^{k+6})\right] \tag{6}$$

where $E(Z_\lambda^k)$ is the k^{th} moment of $Z_\lambda \sim SN(\lambda)$.

Proof See “Appendix”.

Thus the moments of $Z \sim GABSN(\alpha,\beta,\lambda)$ can be obtained by applying the moments of $Z_\lambda \sim SN(\lambda)$ (Henze [23]). We obtained the following results for $k = 1, 2, 3, 4$:

$$E(Z) = \frac{1}{C(\alpha,\beta,\lambda)}\left[-\alpha - 3\beta + b\,\delta + \frac{b\,\delta\,\alpha^2(3+2\lambda^2)}{2(1+\lambda^2)} + \frac{b\,\delta\,\alpha\,\beta(c_1)}{(1+\lambda^2)^2} + \frac{3\,b\,\delta\,\beta^2(c_2)}{2(1+\lambda^2)^3}\right]$$
$$E(Z^2) = \frac{1}{C(\alpha,\beta,\lambda)}\left[1 + 15\alpha\beta + \frac{3\alpha^2}{2} + \frac{105\beta^2}{2} - \frac{b\,\delta\,\alpha(3+2\lambda^2)}{(1+\lambda^2)} - \frac{b\,\delta\,\beta(c_1)}{(1+\lambda^2)^2}\right]$$
$$E(Z^3) = \frac{1}{C(\alpha,\beta,\lambda)}\left[-3\alpha - 15\beta + \frac{b\,\delta\,(3+2\lambda^2)}{(1+\lambda^2)} + \frac{b\,\delta\,\alpha^2(c_1)}{2\,(1+\lambda^2)^2} + \frac{3b\,\delta\,\alpha\,\beta\,(c_2)}{(1+\lambda^2)^3} + \frac{3\,b\,\delta\,\beta^2(c_3)}{2\,(1+\lambda^2)^4}\right]$$
$$E(Z^4) = \frac{1}{C(\alpha,\beta,\lambda)}\left[3 + 105\alpha\beta + \frac{15\alpha^2}{2} + \frac{945\beta^2}{2} - \frac{b\,\delta\,\alpha(c_1)}{(1+\lambda^2)^2} - \frac{3b\delta\beta(c_2)}{(1+\lambda^2)^3}\right]$$
$$Var(Z) = \frac{1}{[C(\alpha,\beta,\lambda)]^2}\left[C(\alpha,\beta,\lambda)\left(1 + 15\alpha\beta + \frac{3\alpha^2}{2} + \frac{105\beta^2}{2} - \frac{b\,\delta\,\alpha(3+2\lambda^2)}{(1+\lambda^2)} - \frac{b\,\delta\,\beta(c_1)}{(1+\lambda^2)^2}\right) - \right.$$
$$\left. \left(-\alpha - 3\beta + b\,\delta + \frac{b\,\delta\,\alpha^2(3+2\lambda^2)}{2(1+\lambda^2)} + \frac{b\,\delta\,\alpha\,\beta(c_1)}{(1+\lambda^2)^2} + \frac{3\,b\,\delta\,\beta^2(c_2)}{2(1+\lambda^2)^3}\right)^2\right]$$

where $c_1 = 15 + 20\lambda^2 + 8\lambda^4$, $c_2 = 35 + 70\lambda^2 + 56\lambda^4 + 16\lambda^6$, $c_3 = 315 + 8\lambda^2(105 + 126\lambda^2 + 72\lambda^4 + 16\lambda^6$. □

Remark 1 Using numerical optimization of $E(Z)$ and $Var(Z)$ with respect to α, β and λ, the following bounds for mean and variance can be obtained as $-2.7739 \leq E(Z) \leq 2.7739$ and $0.43658 \leq Var(Z) \leq 8.16228$. The same can be observed in Fig. 2.

Remark 2 By taking limit $\alpha \to \pm\infty$ in the moments of $GABSN(\alpha,\beta,\lambda)$ distribution, the moments of $GBN(2)$ distribution can be obtained as

$$E(Z) \to \frac{b\,\delta\,(3+2\lambda^2)}{(1+\lambda^2)}; \quad Var(Z) \to \frac{3\pi(1+\lambda^2)^3 - 2\lambda^2(3+2\lambda^2)^2}{\pi(1+\lambda^2)^3}.$$

Remark 3 By taking limit $\lambda \to +\infty$ *or* $-\ \infty$ in the moments of $GABSN(\alpha,\beta,\lambda)$ distribution, we get when $\lambda \to +\infty$.

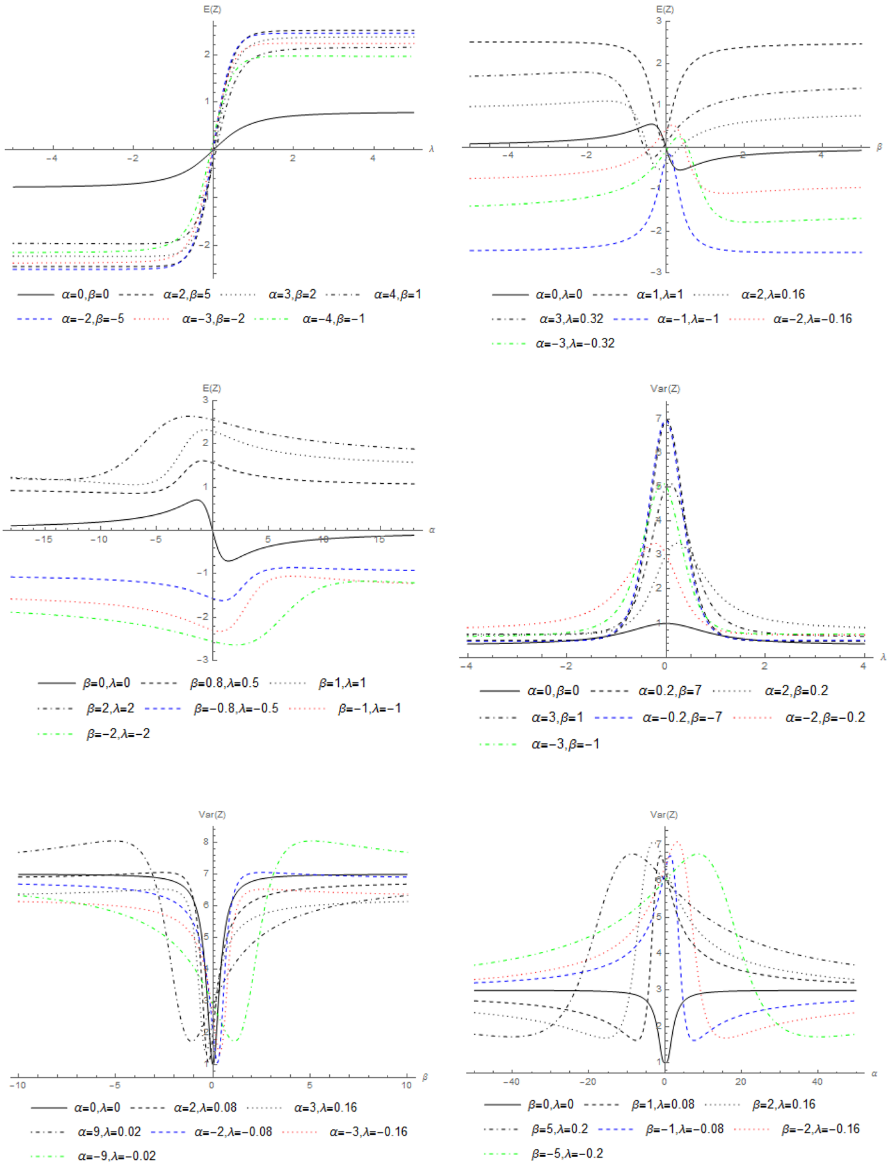


Fig. 2 Plots of the mean and variance of $GABSN(\alpha, \beta, \lambda)$ distribution for different values of the parameters

$$E(Z) = \frac{b(-2 - 2\alpha^2 + \alpha(\sqrt{2\pi} - 16\beta) + 3\sqrt{2\pi}\beta - 48\beta^2)}{-2 + 2b\alpha - \alpha^2 + 4b\beta - 6\alpha\beta - 15\beta^2};$$
$$Var(Z) = -\frac{2(2 - \sqrt{2\pi}\alpha + 2\alpha^2 - 3\sqrt{2\pi}\beta + 16\alpha\beta + 48\beta^2)^2}{\pi(2 - 2b\alpha + \alpha^2 - 4b\beta + 6\alpha\beta + 15\beta^2)^2} + \frac{2 - 4b\alpha + 3\alpha^2 - 16b\beta + 30\alpha\beta + 105\beta^2}{2 - 2b\alpha + \alpha^2 - 4b\beta + 6\alpha\beta + 15\beta^2}$$

and when $\lambda \to -\infty$ then

$$E(Z) = \frac{b(2 + 2\alpha^2 + \alpha(\sqrt{2\pi} + 16\beta) + 3\sqrt{2\pi}\beta + 48\beta^2)}{2 + 2b\alpha + \alpha^2 + 4b\beta + 6\alpha\beta + 15\beta^2},$$
$$Var(Z) = -\frac{2(2 + \sqrt{2\pi}\alpha + 2\alpha^2 + 3\sqrt{2\pi}\beta + 16\alpha\beta + 48\beta^2)^2}{\pi(2 + 2b\alpha + \alpha^2 + 4b\beta + 6\alpha\beta + 15\beta^2)^2} + \frac{2 + 4b\alpha + 3\alpha^2 + 16b\beta + 30\alpha\beta + 105\beta^2}{2 + 2b\alpha + \alpha^2 + 4b\beta + 6\alpha\beta + 15\beta^2}$$

Remark 4 By taking limit $\beta \to \pm\infty$ in the moments of $GABSN(\alpha, \beta, \lambda)$ distribution, the moments of $GBN(6)$ distribution can be obtained easily as

$$E(Z) \to \frac{b\,\delta(35 + 70\lambda^2 + 56\lambda^4 + 16\lambda^6)}{5(1 + \lambda^2)^3}; \quad Var(Z) = \frac{175\pi(1 + \lambda^2)^7 - 2\lambda^2(35 + 70\lambda^2 + 56\lambda^4 + 16\lambda^6)^2}{25\pi(1 + \lambda^2)^7}.$$

Remark 5 The skewness and kurtosis of $GABSN(\alpha, \beta, \lambda)$ distribution is obtained respectively, by using the formulae

$$\beta_1 = \frac{\left(E(Z^3) - 3E(Z^2)E(Z) + 2[E(Z)]^3\right)^2}{\left(E(Z^2) - [E(Z)]^2\right)^3}, \beta_2 = \frac{E(Z^4) - 4E(Z^3)E(Z) + 6E(Z^2)[E(Z)]^2 - 3[E(Z)]^4}{\left(E(Z^2) - [E(Z)]^2\right)^2}$$

where $E(Z)$, $E(Z^2)$, $E(Z^3)$ and $E(Z^4)$ are provided in Sect. 3 above. These cannot be expressed conveniently as the expressions are very vast.

Remark 6 Bounds for skewness and kurtosis are calculated by numerically optimizing β_1 and β_2 with respect to α, β and λ as $0 \le \beta_1 \le 6.70451$ and $1.22732 \le \beta_2 \le 14.1965$. The same can be observed in Fig. 3.

Remark 7 The skewness and kurtosis of $GBN(2)$ distribution can be obtained easily by taking limit $\alpha \to \pm\infty$ in the results of $GABSN(\alpha, \beta, \lambda)$ distribution as

$$\beta_1 = \frac{2\left(-4\lambda^3(3 + 2\lambda^2)^3 + \pi\lambda(1 + \lambda^2)^2(12 + 25\lambda^2 + 10\lambda^4)\right)^2}{\left(3\pi(1 + \lambda^2)^3 - 2\lambda^2(3 + 2\lambda^2)^2\right)^3}$$
$$\beta_2 = \frac{15\pi^2(1 + \lambda^2)^6 - 12\lambda^4(3 + 2\lambda^2)^4 + 4\pi(\lambda + \lambda^3)^2(-9 + 9\lambda^2 + 16\lambda^4 + 4\lambda^6)}{\left(3\pi(1 + \lambda^2)^3 - 2\lambda^2(3 + 2\lambda^2)^2\right)^2}.$$

Remark 8 The skewness and kurtosis of $GBN(6)$ distribution can be obtained easily by taking limit $\beta \to \pm\infty$ in the results of $GABSN(\alpha, \beta, \lambda)$ distribution as

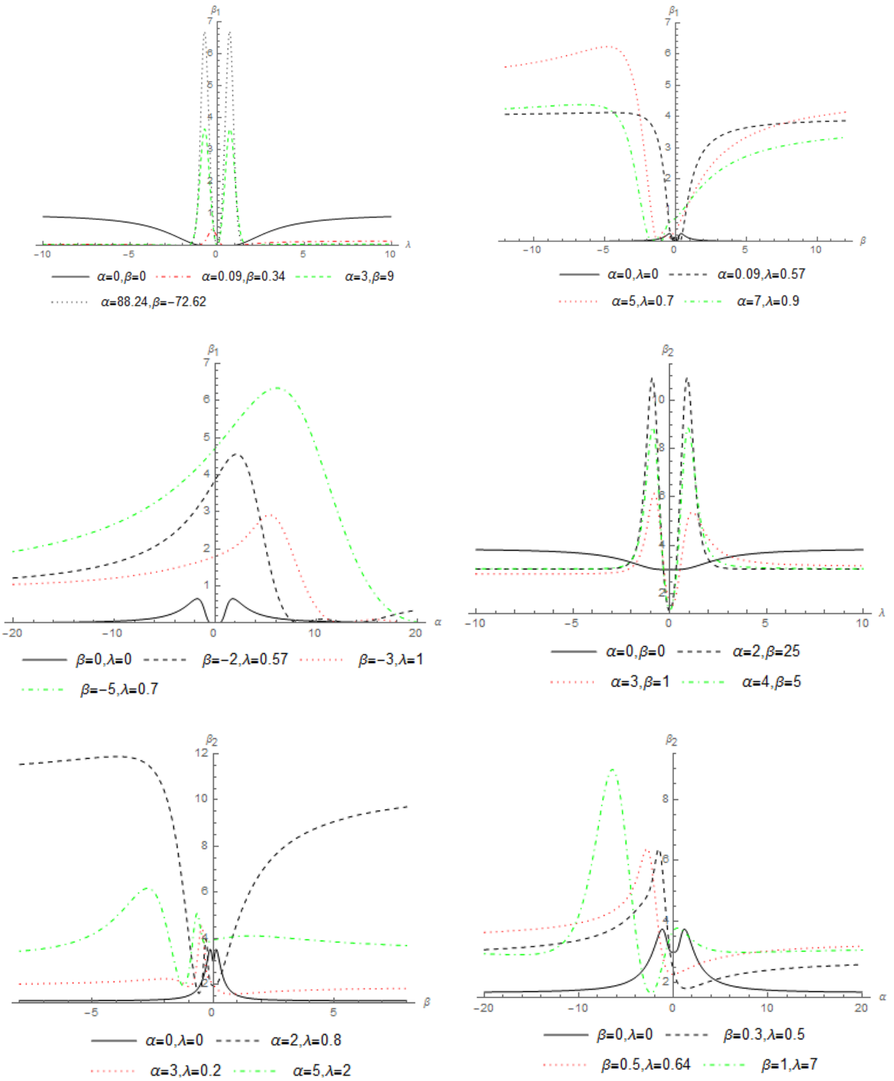


Fig. 3 Plots of the skewness and kurtosis of $GABSN(\alpha, \beta, \lambda)$ distribution for different values of the parameters

$$\beta_1 = \frac{2\lambda^2(4\lambda^2(c_2)^3 - 25\pi(1+\lambda^2)^6(c_4))^2}{15625\pi^3(1+\lambda^2)^{21}\left(c_6\right)^3},$$

$$\beta_2 = \frac{39375\pi^2(1+\lambda^2)^{14} - 12\lambda^4(c_2)^4 + 500\pi\lambda^2(1+\lambda^2)^6 c_2 c_5}{625\pi^2(1+\lambda^2)^{14}\left(c_6\right)^2}.$$

where $c_4 = 420 + 1365\lambda^2 + 1638\lambda^4 + 936\lambda^6 + 208\lambda^8$, $c_5 = 21 + 105\lambda^2 + 126\lambda^4 + 72\lambda^6 + 16\lambda^8$, and $c_6 = 7 - \frac{2\lambda^2(35+70\lambda^2+56\lambda^4+16\lambda^6)^2}{25\pi(1+\lambda^2)^7}$.

Note By taking the limit $\lambda \to +\infty$ *or* $-\infty$ in the results of $GABSN(\alpha, \beta, \lambda)$ distribution, we obtained the results but we are unable to express them conveniently as the expression is very messy.

Theorem 4 *The $GABSN(\alpha, \beta, \lambda)$ distribution has at most four modes.*

Proof Let $Z \sim GABSN(\alpha, \beta, \lambda)$ distribution then by Eqs. (3) and (4) we have,

$$f(z;\alpha, \beta, \lambda) = \frac{2 + \alpha^2 + 15\,\beta^2 + 6\,\alpha\,\beta}{C(\alpha, \beta, \lambda)} f(z;\alpha, \beta)\,\Phi(\lambda\,z). \tag{7}$$

Then by differentiating we get,

$$f'(z;\alpha, \beta, \lambda) = \frac{2 + \alpha^2 + 15\,\beta^2 + 6\,\alpha\,\beta}{C(\alpha, \beta, \lambda)} \left[f'(z;\alpha, \beta)\,\Phi(\lambda\,z) + \lambda f(z;\alpha, \beta)\,\varphi(\lambda\,z)\right]. \tag{8}$$

To prove that the distribution (7) has at most four modes, we have to show that the Eq. (8) has one or seven roots. For this, we apply a graphical approach. We write,

$$f'(z;\alpha, \beta, \lambda) = G_1(z) - G_2(z)$$

where

$$G_1(z) = \frac{2 + \alpha^2 + 15\,\beta^2 + 6\,\alpha\,\beta}{C(\alpha, \beta, \lambda)} f'(z;\alpha, \beta)\,\Phi(\lambda\,z)$$
$$G_2(z) = -\frac{2 + \alpha^2 + 15\,\beta^2 + 6\,\alpha\,\beta}{C(\alpha, \beta, \lambda)} \lambda f(z;\alpha, \beta)\,\varphi(\lambda\,z)$$

Then,

$$f'(z;\alpha, \beta, \lambda) = 0 \Rightarrow G_1(z) = G_2(z). \tag{9}$$

Since the pdf in Eq. (4) vanishes outside $(-5,\ 5)$ (see Fig. 1), we plot the curves of $C1 : y = G_1(z)$ for $\alpha = -3.72, \beta = 1.08, \lambda = 0.3$ and $C2 : y = G_2(z)$ for $\alpha = -1.53, \beta = 0.1, \lambda = 3$ in Figs. 4 and 5 respectively in the range $(-5{,}5)$.

From Figs. 4 and 5, it is shown that the two curves have at least one and at most seven intersection points and the values of z of these points are the roots of Eq. (9).

Since $\lim_{z\to\pm\infty} f(z;\alpha, \beta, \lambda) = 0$, then if Eq. (9) has one root it should be the mode of the pdf in Eq. (7) and if Eq. (9) has seven roots, then pdf in Eq. (7) should have four modes and hence the $GABSN(\alpha, \beta, \lambda)$ distribution has at least one and at most four modes. □

Remark 9. A Location Scale Extension If $Z \sim GABSN(\alpha, \beta, \lambda)$ then $Y = \mu + \sigma\,Z$ is said to be the location (μ) and scale (σ) extension of Z and has the density function is given by

Fig. 4 The Plot of C1 and C2 for $\alpha = -3.72, \beta = 1.08, \lambda = 0.3$

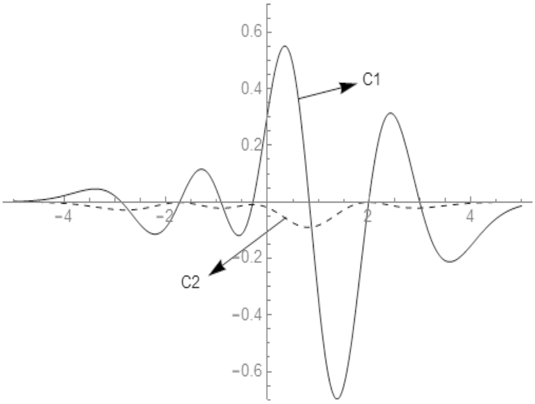
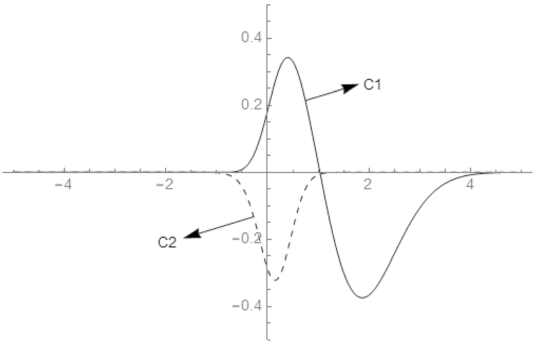


Fig. 5 The Plot of C1 and C2 for $\alpha = -1.53, \beta = 0.1, \lambda = 3$



$$f_Z(z;\alpha,\beta,\lambda,\mu,\sigma) = \frac{1}{C(\alpha,\beta,\lambda)}\left(\left[1-\alpha\left(\frac{z-\mu}{\sigma}\right)-\beta\left(\frac{z-\mu}{\sigma}\right)^3\right]^2+1\right)\varphi\left(\frac{z-\mu}{\sigma}\right)\Phi\left(\lambda\left(\frac{z-\mu}{\sigma}\right)\right). \tag{10}$$

where $z \in R, \alpha, \beta, \lambda, \mu \in R$, and $\sigma > 0$. We denote it by $Y \sim GABSN\ (\mu, \sigma, \alpha, \beta, \lambda)$.

$GABSN\ (\mu, \sigma, \alpha, \beta, \lambda)$ distribution will be used for data modelling.

4 Log-Generalized Alpha Beta Skew Normal Distribution

In this section, using the idea of Venegas et al. [20], we present the definition and some simple properties of log-generalized alpha beta skew normal distribution. Let $Z = e^Y$, then $Y = Log(Z)$, therefore, the density function of Z is defined as follows:

Definition 2 If the random variable Z has the density function given by

$$f(z;\alpha,\beta,\lambda)=\frac{[(1-\alpha\,y-\beta\,y^3)^2+1]}{C(\alpha,\beta,\lambda)\,z}\varphi(y)\,\Phi(\lambda\,y)\,;\ \ z>0, \tag{11}$$

then we say that Z is distributed according to the log-generalized alpha beta skew normal distribution with parameters $(\alpha,\beta,\lambda)^T\in R^3$ where $y=Log(z)$ and $\varphi(z)$ is the density function of the standard log-normal distribution. We denote it by $LGABSN(\alpha,\beta,\lambda)$. This distribution may have wide applications in all the fields where the log normal distribution has been applied.

Properties of $LGABSN(\alpha,\beta,\lambda)$

- If $\beta=0$, then we get $f(z;\alpha,\lambda)=\frac{[(1-\alpha\,y)^2+1]}{(1-\alpha\,b\,\delta+\alpha^2/2)\,z}\varphi(y)\,\Phi(\lambda\,y)$.
  This is known as log-generalized $ASN(\alpha)$ distribution.
- If $\alpha=0$, then we get $f(z;\beta,\lambda)=\frac{[(1-\beta\,y^3)^2+1]}{\left(1-\beta\,b\,\delta(3+2\lambda^2)/(1+\lambda^2)+15\beta^2/2\right)z}\varphi(y)\,\Phi(\lambda\,y)$.
  This is known as log-generalized beta skew normal $LGBSN(\beta,\lambda)$ distribution.
- If $\lambda=0$, then we get $f(z;\alpha,\beta)=\frac{[(1-\alpha\,y-\beta\,y^3)^2+1]}{(2+\alpha^2+6\alpha\beta+15\beta^2)z}\varphi(y)$.
  This is known as log-generalized $ABSN(\alpha,\beta)$ distribution.

- If $\alpha=\beta=0$, then we get $f(z;\lambda)=2\,\varphi(y)\,\Phi(\lambda\,y)/z$.
  This is known as log-$SN(\lambda)$ distribution.
- If $\alpha=\beta=\lambda=0$, then we get the standard log-normal distribution and is given by $f(z)=\varphi(y)/z$.
- If $\alpha\to\pm\infty$, then we get the log-generalized bimodal normal $LGBN(2)$ distribution given by $f(z;\lambda)=\ (y^2/z)\varphi(y)\,\Phi(\lambda\,y)$.
- If $\beta\to\pm\infty$, then we get the log-generalized bimodal normal $LGBN(6)$ distribution given by $f(z;\lambda)=(y^6/15z)\varphi(y)\,\Phi(\lambda\,y)$.
- If $Z\sim LGABSN(\alpha,\beta,\lambda)$, then $-Z\sim -LGABSN(\alpha,\beta,\lambda)$.

5 Maximum Likelihood Estimation

Let $y_1,\ y_2,\dots,y_n$ be a random sample from the distribution of the random variable $Y\sim GABSN\ (\mu,\ \sigma,\alpha,\beta,\lambda)$ so that the log-likelihood function for the parameters $\theta=(\mu,\ \sigma,\alpha,\beta,\lambda)$ is given by

$$l(\theta)=\sum_{i=1}^{n}\log\left[\left\{1-\alpha\Big(\frac{y_i-\mu}{\sigma}\Big)-\beta\Big(\frac{y_i-\mu}{\sigma}\Big)^3\right\}^2+1\right]-n\log C(\alpha,\beta,\lambda)-n\,\log(\sigma)\,-\frac{n}{2}\log\,(2\pi)$$
$$-\sum_{i=1}^{n}\frac{1}{2}\Big(\frac{y_i-\mu}{\sigma}\Big)^2+\sum_{i=1}^{n}\log\left[\Phi\Big(\frac{\lambda(y_i-\mu)}{\sigma}\Big)\right] \tag{12}$$

Taking Partial derivatives of Eq. (12) w.r.t. the parameters, the following normal equations are obtained:

$$\frac{\partial l(\theta)}{\partial \mu} = \sum_{i=1}^{n} \frac{(y_i-\mu)}{\sigma^2} + \sum_{i=1}^{n} \frac{2D}{1+D^2}\left(\frac{\alpha}{\sigma} + \frac{3\beta(y_i-\mu)^2}{\sigma^3}\right) - \frac{\lambda}{\sigma}\sum_{i=1}^{n} W\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)$$

$$\frac{\partial l(\theta)}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n}\left(\frac{(y_i-\mu)^2}{\sigma^3}\right) + \sum_{i=1}^{n} \frac{2D}{1+D^2}\left(\frac{\alpha(y_i-\mu)}{\sigma^2} + \frac{3\beta(y_i-\mu)^3}{\sigma^4}\right) - \frac{\lambda}{\sigma^2}\sum_{i=1}^{n} (y_i-\mu)\, W\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)$$

$$\frac{\partial l(\theta)}{\partial \alpha} = -\frac{n(\alpha+3\beta-b\delta)}{C(\alpha,\beta,\lambda)} - \sum_{i=1}^{n}\frac{2(y_i-\mu)D}{\sigma(1+D^2)}$$

$$\frac{\partial l(\theta)}{\partial \beta} = -\frac{n}{C(\alpha,\beta,\lambda)}\left(3\alpha+15\beta-\frac{b\delta(3+2\lambda^2)}{(1+\lambda^2)}\right) - \sum_{i=1}^{n}\frac{2(y_i-\mu)^3 D}{\sigma^3(1+D^2)}$$

$$\frac{\partial l(\theta)}{\partial \lambda} = -\frac{n}{C(\alpha,\beta,\lambda)}\left(\frac{b(\alpha+3\beta+\alpha\lambda^2)}{(1+\lambda^2)^{5/2}}\right) + \frac{1}{\sigma}\sum_{i=1}^{n}(y_i-\mu)\, W\left(\frac{\lambda(y_i-\mu)}{\sigma}\right)$$

where $D = 1 - \frac{\alpha(y_i-\mu)}{\sigma} - \frac{\beta(y_i-\mu)^3}{\sigma^3}$ and $W(.) = \frac{\varphi(.)}{\Phi(.)}$.

Solving simultaneously of the above equations is not mathematically tractable so one should apply some numerical optimization routine to get the solutions.

5.1 Simulation

In this section a simulation study has been conducted for taking sample sizes $n = 100$, 300 and 500 with different combinations of the true values of the parameters α, β and λ for fixed values $\mu = 0$ and $\sigma = 1$. Here we have applied Metropolis–Hastings (MH) algorithm of Chib and Greenberg [24] to generate the random samples and taking r (number of replications) = 1000. For each sample size we computed MLEs using GenSA package in R and then Bias and MSE. The formula for Bias and MSE are given as follows: $Bias(\widehat{\theta}) = E(\widehat{\theta}) - \theta$ and $MSE(\widehat{\theta}) = E[(\widehat{\theta} - \theta)^2]$ where θ is the true value of the parameter.

Table 1(a), (b) and (c) (see "Appendix") show that the average estimated Bias and MSE for different choices of values of the parameters gradually decreases to zero are as expected.

5.2 Real Life Applications

Here we have considered two datasets which are related to N latitude degrees in 69 samples from world lakes, which appear in Column 5 of the Diversity data set in website: http://users.stat.umn.edu/sandy/courses/8061/datasets/lakes.lsp and the white cells count (WCC) of 202 Australian athletes, given in Cook and Weisberg [25] for the purpose of data fitting. The two datasets are respectively as follows:

Data set-I: 47.5, 44, 62, 42, 52, 39.1, 33.8, 43.2, 39, 45.1, 47.6, 42.9, 43.1, 46, 42.4, 28, 68.6, 43.1, 46, 71.3, 74.7, 46, 33.8, 49.7, 41.4, 49.3, 46, 40.1, 43.9, 49.3, 49.3, 44, 41.3, 42.3, 42.4, 41.4, 46.2, 50.3, 43, 42.4, 38.8, 40.6, 46.2, 40, 39, 43.6, 41.4, 41.6, 39, 42.2, 42.5, 42.5, 71.3, 44.1, 32.8, 38.7, 71.3, 71.3, 38.6, 39, 43, 45.3, 37.2, 32.8, 38.6, 38.6, 43, 52.8, 37.1.

Data set-II: 7.5, 8.3, 5, 5.3, 6.8, 4.4, 5.3, 5.7, 8.9, 4.4, 5.3, 7.3, 7.8, 6.2, 6, 5.8, 7.3, 8.3, 8.1, 6.9, 5.7, 3.3, 9.5, 6.4, 5.8, 5.6, 5.8, 7.6, 7.5, 6.6, 6.4, 10.1, 6.6, 5.9, 7.3, 13.3, 6, 7.6, 6.4, 5.8, 6.1, 5, 6.6, 5.5, 9.7, 10.6, 6.3, 9.1, 9.6, 5.1, 10.7, 10.9, 9.3, 8.4,

Table 1 (a–c) Results of simulation

(a)

			$\mu = 0$		$\sigma = 1$		$\alpha = 1$		β		λ	
β	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
−1	−2	100	0.0110	0.0216	0.0400	0.0319	0.0089	0.0149	0.0312	0.0170	−0.0222	0.0107
		300	−0.0091	0.0094	−0.0129	0.0101	0.0090	0.0120	−0.0106	0.0130	−0.0140	0.0091
		500	−0.0090	0.0076	−0.0110	0.0196	0.0074	0.0099	0.0096	0.0087	0.0101	0.0090
	−1	100	−0.0416	0.0314	0.0110	0.0090	0.0190	0.0146	−0.0111	0.0256	0.1265	0.0956
		300	−0.0094	0.0109	−0.0040	0.0090	−0.0097	0.0093	0.123	0.0162	0.0314	0.0420
		500	0.0089	0.0097	−0.0089	0.0076	−0.0097	0.0084	−0.0099	0.0100	0.0186	0.0131
	0	100	0.0512	0.0617	0.0099	0.0111	0.0923	0.1206	−0.0740	0.0516	0.0110	0.0123
		300	−0.0100	0.0107	0.0093	0.0103	−0.0642	0.0459	0.0231	0.0234	−0.0120	0.0090
		500	0.0099	0.0081	0.0081	0.0076	0.0513	0.0761	0.0110	0.0090	0.0099	0.0090
	1	100	0.0106	0.0177	0.1341	0.1141	−0.0104	0.0177	−0.0149	0.0163	0.2162	0.1273
		300	0.0091	0.0109	−0.1213	0.0916	−0.0097	0.0163	0.0130	0.0143	0.1064	0.0940
		500	0.0090	0.0075	−0.0941	0.0430	0.0090	0.0105	−0.0099	0.0089	0.0995	0.0753
	2	100	0.0167	0.0171	0.0310	0.0413	−0.0315	0.0262	−0.0666	0.0321	0.2167	0.1010
		300	−0.0113	0.0095	−0.0126	0.0110	−0.0267	0.0110	0.0326	0.0284	−0.1207	0.0232
		500	−0.0010	0.0097	−0.0092	0.0106	0.0112	0.0111	0.00099	0.0110	0.0910	0.0230

Table 1 (continued)

(a)

β	λ	n	$\mu = 0$		$\sigma = 1$		$\alpha = 1$		β		λ	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0	−2	100	0.1566	0.1461	0.0810	0.0518	−0.0110	0.0214	−0.1496	0.1223	0.0634	0.0552
		300	0.0706	0.0752	−0.0243	0.0310	−0.0099	0.0146	0.0112	0.0568	−0.0511	0.0403
		500	−0.0152	0.0462	−0.0099	0.0153	−0.0098	0.0121	−0.0099	0.0158	0.0099	0.0099
	−1	100	−0.0090	0.0095	0.0358	0.0432	0.0089	0.0126	−0.0099	0.0146	−0.2333	0.1291
		300	0.0091	0.0086	−0.0250	0.0210	0.0089	0.0112	0.0098	0.0121	−0.0660	0.0556
		500	−0.0081	0.0082	−0.0110	0.0112	0.0086	0.0095	0.0081	0.0098	−0.0501	0.0321
	0	100	−0.0096	0.0152	0.0100	0.0213	0.0112	0.0251	0.0148	0.0222	0.0141	0.0211
		300	0.0081	0.0111	−0.0095	0.0106	−0.0106	0.0129	−0.120	0.164	−0.0120	0.0111
		500	0.0078	0.0097	0.0095	0.0096	−0.0100	0.0126	−0.0096	0.0111	0.0099	0.0098
	1	100	0.0193	0.0456	0.0240	0.0210	0.1209	0.0967	0.0210	0.0218	0.0333	0.0405
		300	−0.0145	0.0216	−0.0112	0.0096	−0.0607	0.0345	0.0158	0.0100	−0.0264	0.0100
		500	0.0110	0.0119	0.0090	0.0095	−0.0110	0.0098	−0.0108	0.0101	−0.0110	0.0098
	2	100	−0.0446	0.0512	−0.0113	0.0253	0.0110	0.0222	0.1109	0.0993	0.1080	0.0957
		300	0.0112	0.0171	−0.0127	0.0126	−0.0091	0.0151	−0.0101	0.0420	−0.0990	0.0826
		500	−0.0094	0.0143	−0.0096	0.0081	−0.0090	0.0091	−0.0093	0.0134	−0.0136	0.0412

Table 1 (continued)

(a)

β	λ	n	$\mu = 0$ Bias	MSE	$\sigma = 1$ Bias	MSE	$\alpha = 1$ Bias	MSE	β Bias	MSE	λ Bias	MSE
1	−2	100	0.0614	0.0714	−0.0130	0.0321	0.0156	0.0210	0.0091	0.0191	0.0660	0.0707
		300	−0.0454	0.0545	−0.0096	0.0167	0.0129	0.0200	0.0106	0.0133	−0.0512	0.0653
		500	0.0111	0.0099	0.0091	0.0093	−0.0100	0.0096	0.0083	0.0110	−0.0110	0.0430
	−1	100	−0.0113	0.0142	0.1000	0.0906	−0.0090	0.0090	0.0356	0.0426	0.0411	0.0323
		300	0.0091	0.0136	−0.0213	0.0219	−0.0089	0.0087	0.0111	0.0127	−0.0106	0.0219
		500	0.0068	0.0047	0.0099	0.0103	−0.0081	0.0081	0.0101	0.0099	0.0099	0.0141
	0	100	−0.0094	0.0161	0.0089	0.0100	0.0086	0.0091	−0.0101	0.0096	0.0112	0.0213
		300	−0.0093	0.0109	0.0089	0.0101	0.0081	0.0082	−0.0099	0.0096	0.0111	0.0142
		500	0.0063	0.0091	0.0073	0.0090	−0.0009	0.0073	0.0089	0.0090	0.0090	0.0099
	1	100	0.0606	0.0914	−0.0310	0.0421	0.0600	0.0145	0.2064	0.1142	0.0213	0.0123
		300	−0.0413	0.0446	0.0104	0.0270	0.0526	0.0131	0.0123	0.0090	0.0089	0.0112
		500	0.0131	0.0112	−0.0100	0.0093	−0.0093	0.0100	−0.0093	0.0081	0.0087	0.0099
	2	100	−0.0911	0.0934	0.1160	0.0994	−0.0117	0.0214	−0.1904	0.0970	0.2163	0.1906
		300	−0.0451	0.0550	−0.0949	0.0163	−0.0193	0.0153	0.1768	0.0852	0.0566	0.0703
		500	−0.0197	0.0232	0.0321	0.0405	−0.0063	0.0140	0.0914	0.0755	0.0111	0.0091

Table 1 (continued)

(b)

β	λ	n	$\mu = 0$		$\sigma = 1$		$\alpha = 0$		β		λ	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
−1	−2	100	−0.0203	0.0146	0.0150	0.0216	0.0090	0.0129	−0.1010	0.0906	0.0099	0.0147
		300	0.0106	0.0152	0.0117	0.0101	0.0091	0.0111	−0.0603	0.0523	0.0099	0.0099
		500	0.0096	0.0090	−0.0101	0.0093	0.0086	0.0096	−0.0101	0.0093	0.0067	0.0099
	−1	100	0.0098	0.0152	0.0094	0.0190	0.0106	0.0277	−0.1112	0.0676	−0.3216	0.1010
		300	−0.0097	0.0111	0.0093	0.0094	−0.0107	0.0109	0.0410	0.0214	−0.0103	0.0606
		500	−0.0084	0.0099	−0.0089	0.0094	−0.0090	0.0096	0.0090	0.0111	−0.0099	0.0193
	0	100	0.0097	0.0106	0.0444	0.0312	0.0091	0.0106	0.0113	0.0213	0.0310	0.0101
		300	−0.0089	0.0096	−0.0421	0.0111	−0.0092	0.099	−0.0093	0.0101	−0.0099	0.0107
		500	−0.0084	0.0095	0.0120	0.0090	−0.0004	0.0067	0.0090	0.0090	−0.0093	0.0091
	1	100	−0.0921	0.0526	−0.0103	0.0126	0.0096	0.0110	0.1112	0.0951	0.1112	0.1229
		300	0.0321	0.0426	−0.0090	0.0120	−0.0097	0.0101	−0.0703	0.0334	−0.0665	0.0456
		500	0.0120	0.0121	0.0090	0.0093	0.0089	0.0096	−0.0445	0.0116	−0.0632	0.0513
	2	100	0.5216	0.3261	0.0161	0.0210	0.0122	0.0321	−0.0127	0.0219	−0.1001	0.0955
		300	−0.0145	0.0643	−0.0167	0.0174	−0.0123	0.0214	0.0112	0.0124	−0.0214	0.0211
		500	−0.0099	0.0152	0.0091	0.0138	−0.0093	0.0168	0.0099	0.0111	−0.0099	0.0106

Table 1 (continued)

(b)

β	λ	n	$\mu = 0$ Bias	$\mu = 0$ MSE	$\sigma = 1$ Bias	$\sigma = 1$ MSE	$\alpha = 0$ Bias	$\alpha = 0$ MSE	β Bias	β MSE	λ Bias	λ MSE
0	−2	100	0.0210	0.0320	0.0999	0.0510	0.0445	0.0364	−0.0119	0.0236	0.0112	0.0167
		300	−0.0129	0.0310	−0.0301	0.0321	−0.0249	0.0296	−0.0106	0.0146	−0.0099	0.0102
		500	−0.0101	0.0099	0.0113	0.0193	0.0123	0.0111	−0.0094	0.0110	−0.0096	0.0094
	−1	100	0.0111	0.0214	0.2103	0.1111	0.0091	0.0123	0.0099	0.0100	−0.0476	0.0346
		300	−0.0123	0.0126	−0.0193	0.0214	−0.0087	0.0099	0.0084	0.0096	0.0143	0.0212
		500	0.0090	0.0100	0.0130	0.0090	−0.0082	0.0093	0.0083	0.0096	0.0119	0.0106
	0	100	−0.0090	0.0152	0.0093	0.0111	0.1060	0.0906	0.0190	0.0312	0.1001	0.0942
		300	−0.0081	0.0101	0.0090	0.0093	−0.0094	0.0089	0.0102	0.0094	−0.0312	0.0453
		500	−0.0050	0.0093	−0.0084	0.0087	0.0091	0.0089	−0.0090	0.0091	0.0108	0.0093
	1	100	0.0706	0.0619	0.0210	0.0346	−0.0111	0.0096	−0.0932	0.0646	−0.0103	0.0230
		300	−0.0119	0.0096	0.0131	0.0093	−0.0093	0.0096	0.0120	0.0090	−0.0100	0.0076
		500	−0.0093	0.0095	−0.0103	0.0090	0.0090	0.0073	0.0099	0.0090	0.0076	0.0076
	2	100	0.0821	0.0912	−0.0242	0.0180	−0.0090	0.0091	0.0210	0.0110	0.1209	0.1100
		300	−0.0644	0.0506	0.0249	0.0178	−0.0086	0.0096	−0.0196	0.0143	−0.0890	0.0706
		500	0.0120	0.0340	0.0100	0.0160	0.0071	0.0080	−0.0093	0.0120	0.0190	0.0507

Table 1 (continued)

(b)

β	λ	n	$\mu = 0$ Bias	$\mu = 0$ MSE	$\sigma = 1$ Bias	$\sigma = 1$ MSE	$\alpha = 0$ Bias	$\alpha = 0$ MSE	β Bias	β MSE	λ Bias	λ MSE
1	−2	100	−0.1106	0.0946	−0.1001	0.0914	0.0096	0.0120	0.1106	0.0929	0.0090	0.0151
		300	0.0320	0.0411	0.0521	0.0621	−0.0097	0.0095	0.0226	0.0112	0.0090	0.0122
		500	0.0090	0.0103	0.0099	0.0210	0.0081	0.0076	0.0090	0.0091	0.0081	0.0096
	−1	100	−0.0620	0.0523	0.0112	0.0412	−0.0142	0.0426	−0.1231	0.0946	0.0111	0.0125
		300	0.0162	0.0491	−0.0099	0.0312	−0.0101	0.0292	−0.0426	0.0312	−0.0090	0.0122
		500	0.0131	0.0222	−0.0093	0.0099	−0.0099	0.0123	−0.0109	0.0099	−0.0090	0.0094
	0	100	0.0090	0.0126	0.0200	0.0312	0.0090	0.0093	0.0120	0.0202	−0.1001	0.0931
		300	−0.0091	0.0090	−0.0191	0.0096	−0.0090	0.0094	0.0090	0.0191	0.0210	0.0343
		500	−0.0081	0.0089	−0.0009	0.0080	0.0081	0.0091	0.0089	0.0090	−0.0139	0.0111
	1	100	−0.0119	0.0210	0.0341	0.0445	0.0090	0.0110	0.1001	0.0656	0.0112	0.0321
		300	−0.0106	0.0105	−0.0210	0.0301	0.0089	0.0093	−0.0106	0.0231	−0.0106	0.0216
		500	−0.0100	0.0090	−0.0110	0.0210	−0.0009	0.0081	0.0090	0.0078	−0.0099	0.0110
	2	100	−0.0921	0.0464	0.1131	0.0949	0.0094	0.0099	0.0813	0.0513	0.1100	0.0906
		300	0.0149	0.0139	0.0493	0.0314	−0.0099	0.0149	−0.0707	0.0512	0.0312	0.0450
		500	−0.0091	0.0143	−0.0166	0.0099	−0.0086	0.0127	−0.0327	0.0150	0.0096	0.0121

Table 1 (continued)

(c)

β	λ	n	$\mu = 0$ Bias	$\mu = 0$ MSE	$\sigma = 1$ Bias	$\sigma = 1$ MSE	$\alpha = -1$ Bias	$\alpha = -1$ MSE	β Bias	β MSE	λ Bias	λ MSE
−1	−2	100	−0.0726	0.0601	−0.0091	0.0085	0.0512	0.0710	−0.0091	0.0087	−0.0201	0.0210
		300	−0.0652	0.0531	0.0083	0.0081	0.0091	0.0134	0.0080	0.0075	0.0097	0.0089
		500	0.0010	0.0065	0.0071	0.0079	0.0090	0.0083	−0.0080	0.0085	0.0058	0.0079
	−1	100	0.0098	0.0092	0.0273	0.0196	0.0301	0.0411	−0.0090	0.0117	−0.3102	0.1345
		300	−0.0091	0.0086	−0.0097	0.0085	−0.0094	0.0081	−0.0083	0.0091	0.0101	0.0091
		500	0.0086	0.0057	0.0385	0.0126	0.0087	0.0078	0.0076	0.0084	0.0099	0.0081
	0	100	0.0088	0.0081	−0.1276	0.0976	0.0079	0.0091	−0.0090	0.0081	0.0087	0.0099
		300	0.0090	0.0071	−0.0432	0.0126	0.0060	0.0077	−0.0089	0.0083	−0.0081	0.0086
		500	0.0064	0.0068	−0.0099	0.0091	−0.0053	0.0066	−0.0071	0.0066	−0.0070	0.0080
	1	100	0.1013	0.1113	−0.0126	0.0811	0.0191	0.0098	0.1003	0.0806	0.0089	0.0081
		300	0.0181	0.0202	0.0110	0.0201	−0.0096	0.0091	−0.0135	0.0090	−0.0088	0.0080
		500	−0.0210	0.0193	−0.0091	0.0100	−0.0063	0.0073	0.0093	0.0073	−0.0076	0.0069
	2	100	0.0112	0.0079	−0.0881	0.0911	−0.0144	0.0099	−0.0091	0.0101	−0.0091	0.0093
		300	0.0100	0.0127	−0.0631	0.0726	−0.0096	0.0081	0.0091	0.0079	0.0083	0.0089
		500	−0.0090	0.0086	0.0129	0.0113	0.0083	0.0081	0.0073	0.0078	−0.0073	0.0081

Table 1 (continued)

(c)

β	λ	n	$\mu = 0$ Bias	MSE	$\sigma = 1$ Bias	MSE	$\alpha = -1$ Bias	MSE	β Bias	MSE	λ Bias	MSE
0	−2	100	0.1281	0.0976	0.1111	0.1310	0.0081	0.0093	0.0143	0.0099	−0.0643	0.0556
		300	−0.0199	0.0202	0.0929	0.1003	0.0081	0.0086	0.0093	0.0092	−0.0103	0.0214
		500	0.0131	0.0213	−0.0322	0.0291	−0.0067	0.0070	−0.0090	0.0084	−0.0094	0.0101
	−1	100	0.0432	0.0512	0.0331	0.0412	−0.2101	0.1131	0.0090	0.0091	0.0193	0.0211
		300	0.0192	0.0261	−0.0301	0.0210	−0.0099	0.0089	−0.0090	0.0076	−0.0108	0.0119
		500	−0.0091	0.0097	0.0101	0.0091	0.0081	0.0084	0.0071	0.0077	0.0101	0.0210
	0	100	0.0093	0.0081	0.0092	0.0091	0.0081	0.0091	0.0080	0.0090	−0.0901	0.0450
		300	0.0073	0.0069	0.0081	0.0076	−0.0063	0.0086	−0.0077	0.0086	−0.0849	0.0379
		500	−0.0071	0.0074	0.0062	0.0071	−0.0009	0.0013	−0.0007	0.0049	0.0099	0.0156
	1	100	0.0145	0.0146	−0.0101	0.0121	−0.0091	0.0089	0.0301	0.0130	−0.0159	0.0233
		300	−0.0123	0.0222	0.0091	0.0102	0.0085	0.0081	−0.0100	0.0097	0.0099	0.0155
		500	−0.0096	0.0103	0.0089	0.0091	0.0055	0.0067	0.0083	0.0094	0.0091	0.0109
	2	100	0.1931	0.1212	0.2100	0.2346	0.0091	0.0081	0.0089	0.0091	−0.0345	0.0100
		300	0.0726	0.0910	0.1249	0.1945	−0.0089	0.0076	0.0081	0.0090	0.0091	0.0095
		500	−0.0176	0.0261	−0.0781	0.1006	−0.0071	0.0071	0.0071	0.0083	0.0089	0.0071

Table 1 (continued)

(c)

β	λ	n	$\mu = 0$ Bias	MSE	$\sigma = 1$ Bias	MSE	$\alpha = -1$ Bias	MSE	β Bias	MSE	λ Bias	MSE
1	−2	100	−0.0927	0.1000	0.3331	0.1276	0.0081	0.0100	0.0090	0.0091	−0.1111	0.1023
		300	−0.0871	0.0891	0.2107	0.1097	0.0080	0.0091	0.0088	0.0093	0.0176	0.0465
		500	0.0656	0.0721	0.0310	0.0421	0.0071	0.0076	0.0072	0.0081	0.0091	0.0097
	−1	100	0.1932	0.1076	0.0177	0.0131	0.0113	0.0110	0.0141	0.0091	−0.0088	0.0079
		300	0.1222	0.0990	0.0093	0.0101	0.0076	0.0079	0.0096	0.0081	−0.0090	0.0077
		500	−0.0954	0.0949	0.0081	0.0092	0.0070	0.0078	0.0081	0.0074	−0.0062	0.0066
	0	100	0.0091	0.0092	0.0097	0.0098	0.0079	0.0091	0.0081	0.0095	−0.0071	0.0081
		300	−0.0079	0.0081	0.0091	0.0101	−0.0071	0.0084	−0.0076	0.0083	−0.0060	0.0076
		500	0.0070	0.0063	−0.0087	0.0097	−0.0061	0.0088	0.0009	0.0046	0.0055	0.0064
	1	100	0.3126	0.2921	0.1073	0.0931	−0.0097	0.0081	0.0099	0.0098	−0.0101	0.0121
		300	0.1201	0.1220	−0.1000	0.0911	0.0091	0.0089	−0.0087	0.0069	0.0090	0.0111
		500	0.0901	0.1009	−0.0347	0.0217	−0.0080	0.0051	−0.0060	0.0067	−0.0081	0.0091
	2	100	0.1001	0.0921	0.1211	0.1018	0.0167	0.0312	0.1246	0.0912	0.2131	0.1110
		300	−0.0712	0.0767	−0.0801	0.0612	−0.0093	0.0109	−0.0127	0.0212	0.0156	0.0210
		500	−0.0701	0.0810	−0.0315	0.0442	0.0070	0.0089	−0.0091	0.0067	0.0097	0.0081

Table 2 MLE's, log− likelihood and AIC for N latitude degrees in 69 samples from world lakes.

Distribution	Parameters						
	μ	σ	λ	α	β	log L	AIC
$N(\mu, \sigma^2)$	45.165	9.549	–	–	–	−253.60	511.198
$LG(\mu, \beta)$	43.639	–	–	–	4.493	−246.65	497.290
$SN(\mu, \sigma, \lambda)$	35.344	13.7	3.687	–	–	−243.04	492.072
$BSN(\mu, \sigma, \beta)$	54.47	5.52	–	–	0.74	−242.53	491.060
$SLG(\mu, \beta, \lambda)$	36.787	–	2.828	–	6.417	−239.05	490.808
$La(\mu, \beta)$	43.0	–	–	–	5.895	−239.25	482.496
$ASLG(\mu, \beta, \alpha)$	49.087	–	–	0.861	3.449	−237.35	480.702
$SLa(\mu, \beta, \lambda)$	42.3	–	0.255	–	5.943	−236.90	479.799
$ASLa(\mu, \beta, \alpha)$	42.3	–	–	−0.22	5.44	−236.08	478.159
$ASN(\mu, \sigma, \alpha)$	52.147	7.714	–	2.042	–	−235.37	476.739
$ABSN(\alpha, \beta, \mu, \sigma)$	53.28	9.772	–	2.943	−0.292	−234.36	476.719
$GASN(\mu, \sigma, \alpha, \lambda)$	56.319	8.544	−0.672	12.052	–	−230.53	469.062
$GABSN(\mu, \sigma, \alpha, \beta, \lambda)$	58.333	6.489	−0.616	−6.144	−5.870	**−225.37**	**460.738**

6.9, 8.4, 6.6, 8.5, 5.5, 5.9, 4.9, 8.1, 8.3, 5.8, 5.3, 5.1, 7, 9.5, 9.5, 5.8, 6.8, 9, 7.1, 9.3, 7.5, 7.3, 7.6, 6.9, 6.1, 6.5, 6.9, 6.4, 6.6, 6, 7.6, 6.8, 7.2, 8.2, 7.8, 4.2, 4, 7.9, 6.6, 6.4, 7.2, 6.4, 9, 5, 4.9, 6.4, 7.1, 7.6, 4.7, 4.1, 6.7, 7.1, 6, 8.6, 6.6, 4.8, 5.2, 6.2, 4.3, 8.2, 7.1, 5.3, 5.9, 9.3, 6.8, 8.4, 6.5, 6.8, 5.4, 7.5, 10.1, 5, 6, 8, 7.2, 5.9, 5.8, 6.7, 8, 7.5, 9.2, 8.3, 8.9, 7.4, 6.4, 6.7, 5.55, 7.2, 7.3, 7.5, 8.9, 9.6, 6.3, 6.3, 4.5, 3.9, 9, 7.3, 4.5, 6.1, 6.1, 5.8, 4, 4.3, 8.2, 4.6, 6.4, 8.9, 6.2, 8.4, 9, 7.1, 6.6, 7.6, 4.6, 4.8, 5.2, 7.2, 5.9, 7.9, 6.6, 6.4, 9.3, 8.3, 8.9, 8.7, 10.8, 9.1, 10.2, 7.5, 10, 12.9, 12.7, 6.1, 9.8, 7.5, 7.4, 8.5, 6, 14.3, 7, 6.2, 8.9, 7.6, 8.3, 6.4, 8.8, 6.3.

Using GenSA package in R (See GenSA package version-1.0.3, Xiang et al. [26]), we have fitted our proposed distribution, i.e., $GABSN(\mu, \sigma, \alpha, \beta, \lambda)$ along with the normal $N(\mu, \sigma^2)$ distribution, the logistic $LG(\mu, \beta)$ distribution, the Laplace $La(\mu, \beta)$ distribution, the skew-normal $SN(\mu, \sigma, \lambda)$ distribution of Azzalini [17], the skew-logistic $SLG(\mu, \beta, \lambda)$ distribution of Wahed and Ali [27], the skew-Laplace $SLa(\mu, \beta, \lambda)$ distribution of Nekoukhou and Alamatsaz [28], the alpha-skew-normal $ASN(\mu, \sigma, \alpha)$ distribution of Elal-Olivero [18], the alpha-skew-Laplace $ASLa(\mu, \beta, \alpha)$ distribution of Harandi and Alamatsaz [29], the alpha-skew-logistic $ASLG(\mu, \beta, \alpha)$ distribution of Hazarika and Chakraborty [30], the alpha–beta-skew-normal $ABSN(\mu, \sigma, \alpha, \beta)$ distribution and beta-skew-normal $BSN(\mu, \sigma, \beta)$ distribution of Shafiei et al. [19], and the generalized alpha-skew-normal $GASN(\mu, \sigma, \alpha, \lambda)$ distribution of Sharafi et al. [21] for comparison purpose.

The values of the MLE's of the parameters for different distributions along with log-likelihood and AIC are given in Tables 2 and 3.

From Tables 2 and 3, it is seen that the proposed generalized alpha beta skew normal $GABSN(\mu, \sigma, \alpha, \beta, \lambda)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood and the AIC. The plots

Table 3 MLE's, log-likelihood and AIC for white cells count (WCC) of 202 Australian athletes

Distribution	Parameters						
	μ	σ	λ	α	β	log L	AIC
$N(\mu,\sigma^2)$	7.109	1.796	–	–	–	−404.919	813.838
$La\,(\mu,\beta)$	6.844	–	–	–	1.38	−407.142	818.284
$ASLa(\mu,\beta,\alpha)$	6.4	–	–	−0.265	1.263	−400.992	807.984
$LG(\mu,\beta)$	6.996	–	–	–	0.991	−401.612	807.224
$SLa(\mu,\beta,\lambda)$	6.4	–	0.762	–	1.287	−400.366	806.732
$BSN(\mu,\sigma,\beta)$	6.813	1.687	–	–	−0.061	−399.638	805.276
$ASLG(\mu,\beta,\alpha)$	6.413	–	–	−0.21	0.949	−398.943	803.886
$ASN(\mu,\sigma,\alpha)$	8.195	1.684	–	0.874	–	−398.393	802.786
$GASN(\mu,\sigma,\alpha,\lambda)$	5.569	2.689	2.242	0.5115	–	−395.545	799.090
$SN(\mu,\sigma,\lambda)$	5.105	2.691	2.729	–	–	−396.161	798.322
$SLG(\mu,\beta,\lambda)$	5.512	–	1.736	–	1.331	−395.897	797.794
$ABSN(\mu,\sigma,\alpha,\beta)$	7.758	1.796	–	0.813	−0.128	−394.833	797.666
$GABSN(\mu,\sigma,\alpha,\beta,\lambda)$	9.729	1.508	−0.669	0.411	0.428	**−392.788**	**795.576**

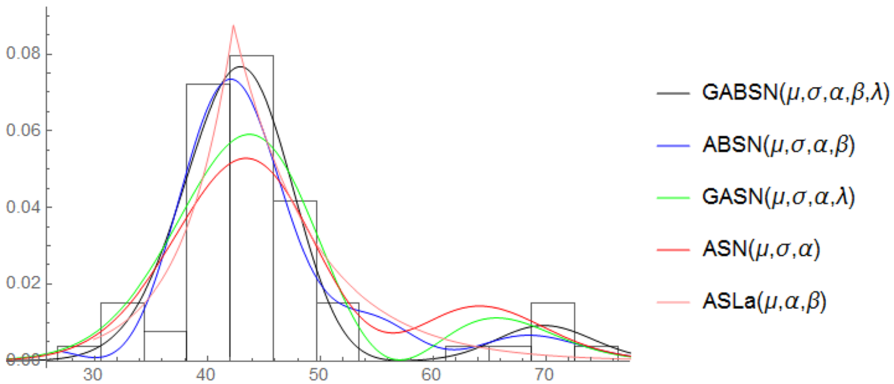


Fig. 6 Plots of observed and expected densities of some distributions for N latitude degrees in 69 samples from world lakes

of observed (in histogram) and expected densities (lines) presented in Figs. 6 and 7 also confirms our finding.

5.3 Likelihood Ratio Test

Since $N(\mu,\sigma^2)$, $SN(\mu,\sigma,\lambda)$, $ABSN(\mu,\sigma,\alpha,\beta)$, $GASN(\mu,\sigma,\alpha,\lambda)$, $GBSN(\mu,\sigma,\beta,\lambda)$ and $GABSN(\mu,\sigma,\alpha,\beta,\lambda)$ distributions are nested models, the likelihood ratio (LR) test is used to discriminate between them. The LR test is carried out to test the following hypothesis as shown in Table 4.

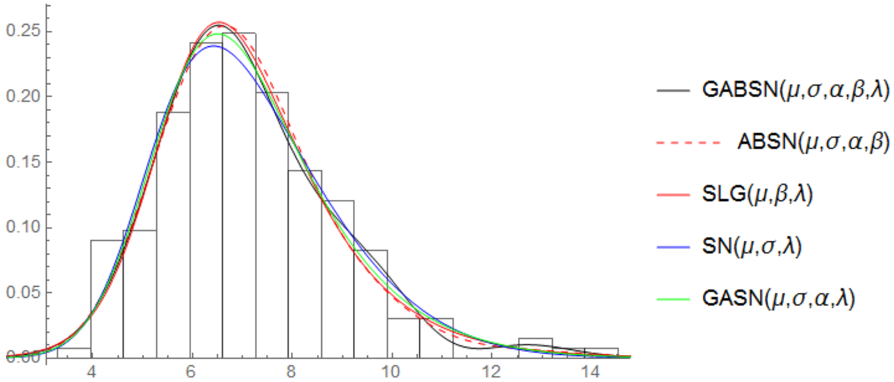


Fig. 7 Plots of observed and expected densities of some distributions for white cells count (WCC) of 202 Australian athletes

Table 4 The values of LR test statistic for different hypothesis

Hypothesis	LR test statistic values		Degrees of Freedom	Critical values at 5%
	Dataset 1	Dataset 2		
$H_0 : \beta = 0$ versus $H_1 : \beta \neq 0$	10.324	5.514	1	3.841
$H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$	17.982	4.09	1	3.841
$H_0 : \alpha = 0, \beta = 0$ versus $H_1 : \alpha \neq 0, \ \beta \neq 0$	35.342	6.746	2	5.991
$H_0 : \alpha = 0, \beta = 0, \lambda = 0$ versus $H_1 : \alpha \neq 0, \ \beta \neq 0, \ \lambda \neq 0$	56.460	24.262	3	7.815

Since all the values of LR test statistics for different hypothesis exceed the critical values at 5% level of significance. Thus we accept the alternative hypothesis. Therefore, we may conclude that the sampled data comes from $GABSN(\mu, \sigma, \alpha, \beta, \lambda)$ distribution.

6 Conclusions

A generalized-alpha–beta-skew-normal distribution which can have up to four modes is introduced and some of its basic structural properties are investigated. A few useful extensions of the proposed distribution along with their basic characteristics are discussed.

Parameter estimation with maximum likelihood method is implemented. A simulation study is conducted to see the performance of the maximum likelihood estimators by generating random sample using the Metropolis–Hastings algorithm revealed expected outcome. The importance of the proposed distribution is established by comparing it with as many as twelve closely related and nested distributions by considering two data sets from literature. Likelihood ratio test is employed for testing the relevance of the additional parameters inducted in proposing the model.

Furthermore, there is scope of extending the present work by considering the Logistic and the Laplace distributions. Moreover, logarithmic forms and bivariate generalizations can also be considered as future work.

Appendix

Derivation of normalizing constant $C(\alpha, \beta, \lambda)$

$$
\begin{aligned}
C(\alpha,\beta,\lambda) &= \int_{-\infty}^{\infty} [(1-\alpha z-\beta z^3)^2+1]\varphi(z)\,\Phi(\lambda z)\,dz \\
&= \int_{-\infty}^{\infty} (2-2\alpha z+\alpha^2 z^2-2\beta z^3+2\alpha\beta z^4+\beta^2 z^6)\varphi(z)\,\Phi(\lambda z)\,dz \\
&= \int_{-\infty}^{\infty} \varphi(z;\lambda)\ dz-\alpha E(Z_\lambda)+\frac{\alpha^2}{2}E(Z_\lambda^2)-\beta E(Z_\lambda^3)+\alpha\beta E(Z_\lambda^4)+\frac{\beta^2}{2}E(Z_\lambda^6) \\
&= 1+\frac{\alpha^2}{2}+3\alpha\beta+\frac{15\beta^2}{2}-\alpha\sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}-\beta\sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\frac{3+2\lambda^2}{1+\lambda^2} \\
&= 1+3\alpha\beta-\alpha b\delta-\beta b\delta\frac{3+2\lambda^2}{1+\lambda^2}+\frac{\alpha^2}{2}+\frac{15\beta^2}{2},
\end{aligned}
$$

where $b=\sqrt{\frac{2}{\pi}},\ \delta=\frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\varphi(z;\lambda)$ is the density function of $Z_\lambda \sim SN(\lambda)$.

Derivation of the cdf

$$
\begin{aligned}
F_Z(z) = P(Z\le z) &= \int_{-\infty}^{z} \frac{(1-\alpha t-\beta t^3)^2+1}{C(\alpha,\beta,\lambda)}\varphi(t)\,\Phi(\lambda t)dt \\
&= \frac{1}{C(\alpha,\beta,\lambda)}\int_{-\infty}^{z} (2-2\alpha t+\alpha^2 t^2-2\beta t^3+2\alpha\beta t^4+\beta^2 t^6)\varphi(t)\,\Phi(\lambda t)dt \\
&= \frac{1}{C(\alpha,\beta,\lambda)}\Bigg[\int_{-\infty}^{z} 2\varphi(t)\,\Phi(\lambda t)dt-2\alpha\int_{-\infty}^{z} t\varphi(t)\,\Phi(\lambda t)dt+\alpha^2\int_{-\infty}^{z} t^2\varphi(t)\,\Phi(\lambda t)dt-2\beta\int_{-\infty}^{z} t^3\varphi(t)\,\Phi(\lambda t)dt+ \\
&\quad 2\alpha\beta\int_{-\infty}^{z} t^4\varphi(t)\,\Phi(\lambda t)dt+\beta^2\int_{-\infty}^{z} t^6\varphi(t)\,\Phi(\lambda t)dt\Bigg]
\end{aligned}
$$

Putting the above following results we get the desired result.

$$\int_{-\infty}^{z} 2\varphi(t)\,\Phi(\lambda t)dt = \Phi(z;\lambda),\ \int_{-\infty}^{z} t\varphi(t)\,\Phi(\lambda t)dt = -\varphi(z)\,\Phi(\lambda z) + \frac{\lambda\,Erf\left((z\sqrt{1+\lambda^2})/\sqrt{2}\right)}{2\sqrt{2\pi}\sqrt{1+\lambda^2}}$$

$$\int_{-\infty}^{z} t^2\varphi(t)\,\Phi(\lambda t)dt = -z\varphi(z)\,\Phi(\lambda z) + \frac{b\,\delta\,\varphi\left(z\sqrt{1+\lambda^2}\right)}{2\sqrt{1+\lambda^2}},$$

$$\int_{-\infty}^{z} t^3\varphi(t)\,\Phi(\lambda t)dt = -(2+z^2)\varphi(z)\,\Phi(\lambda z) + \frac{\lambda\,Erf\left((z\sqrt{1+\lambda^2})/\sqrt{2}\right)}{\sqrt{2\pi}\sqrt{1+\lambda^2}} - z\frac{b\,\delta\,\varphi\left(z\sqrt{1+\lambda^2}\right)}{2\sqrt{1+\lambda^2}} + \frac{b\,\delta\,\Phi\left(z\sqrt{1+\lambda^2}\right)}{2\sqrt{1+\lambda^2}},$$

$$\int_{-\infty}^{z} t^4\varphi(t)\,\Phi(\lambda t)dt = -z(3+z^2)\varphi(z)\,\Phi(\lambda z) - \frac{b\,\delta\,\varphi\left(z\sqrt{1+\lambda^2}\right)\left(5+z^2+\lambda^2(3+z^2)\right)}{2\left(\sqrt{1+\lambda^2}\right)^3} + \frac{3}{2}\Phi(z;\lambda),$$

$$\int_{-\infty}^{z} t^6\varphi(t)\,\Phi(\lambda t)dt = -z(15+5z^2+z^4)\varphi(z)\,\Phi(\lambda z) + \frac{15}{2}\Phi(z;\lambda) +$$
$$\frac{b\,\delta\,\varphi\left(z\sqrt{1+\lambda^2}\right)\left(-33-40\lambda^2-15\lambda^4-z^4(1+\lambda^2)^2-z^2(9+14\lambda^2+5\lambda^4)\right)}{2\left(\sqrt{1+\lambda^2}\right)^5}.$$

Derivation of the moment

$$\begin{aligned}
E(Z^k) &= \int_{-\infty}^{\infty} z^k \frac{(1-\alpha z-\beta z^3)^2+1}{C(\alpha,\beta,\lambda)}\varphi(z)\,\Phi(\lambda z)\,dz \\
&= \frac{1}{C(\alpha,\beta,\lambda)}\Bigg[\int_{-\infty}^{\infty} 2\,z^k\,\varphi(z)\,\Phi(\lambda z)dz - 2\alpha\int_{-\infty}^{\infty} z^{k+1}\varphi(z)\,\Phi(\lambda z)\,dz + \alpha^2\int_{-\infty}^{\infty} z^{k+2}\varphi(z)\,\Phi(\lambda z)\,dz - \\
&\quad 2\beta\int_{-\infty}^{\infty} z^{k+3}\varphi(z)\,\Phi(\lambda z)\,dz + 2\alpha\beta\int_{-\infty}^{\infty} z^{k+4}\varphi(z)\,\Phi(\lambda z)\,dz + \beta^2\int_{-\infty}^{\infty} z^{k+6}\varphi(z)\,\Phi(\lambda z)\,dz\Bigg] \\
&= \frac{1}{C(\alpha,\beta,\lambda)}\left[E(Z_\lambda^k) - 2\alpha\,E(Z_\lambda^{k+1}) + \alpha^2\,E(Z_\lambda^{k+2}) - 2\beta\,E(Z_\lambda^{k+3}) + 2\alpha\beta\,E(Z_\lambda^{k+4}) + \beta^2\,E(Z_\lambda^{k+5})\right]
\end{aligned}$$

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Declarations

Conflicts of interest Not Applicable.

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