The Calculus of a Pivotal Voter Under a Simple Proxy Voting Regime

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Abstract

Advancements on communication technology and distributed ledgers have facilitated the implementation of voting mechanisms that were until recently deemed technically unfeasible or prohibitively expensive at scale. Proxy voting is one such mechanism that has enjoyed a renewed relevance. In this paper, we revisit the Condorcet Jury Theorem, a seminal result at the intersection of political science, law, and economics, through the lens of proxy voting. We introduce a nuanced yet significant modification to Condorcet's original thought exercise, and provide results that help elucidate the consequences of proxy voting on the calculus of a pivotal voter. In particular, we show how the fact that a voter is pivotal updates their beliefs about the competence of their peers and affects the trade-off they face when choosing between voting independently or delegating. We discuss how our findings can explain observations of "over-delegation" in recent proxy voting experiments and compare the accuracy of symmetric and asymmetric equilibria to a coordination problem that arises in this context.

1 Introduction

The search for an optimal voting system is a central theme in political science, economics, and legal scholarship. This debate reflects a deep-seated quest to aggregate individual preferences into collective decisions that are fair, representative, and efficient. The challenge of designing such a system is best demonstrated by Arrow's Impossibility Theorem (Arrow 1951), which suggests that no voting system can perfectly satisfy a set of *a priori* desirable criteria. This result, however, has not deterred scholars; instead, it has spurred a nuanced exploration of how information aggregation systems can approach these ideals, navigating the trade-offs that Arrow's Theorem illuminates.

This forces us to approach the study of voting systems through a multifaceted lens. One could consider whether a voting system is strategy-proof, averting manipulative voting behaviors (Gibbard 1973, Satterthwaite 1975); equitable, ensuring fairness in representation and influence among voters (May 1952); or whether the alternative they select has a majority support when put into a pairwise comparison with all other competing alternatives. This last criterion was first formulated by Marquis De Condorcet after noticing that majority rule systems can lead to a paradoxical situation in which different majorities of the same population can prefer distinct alternatives. Another of Condorcet's famous contributions to the debate on optimal voting is the Condorcet Jury Theorem (De Condorcet 1785). One of the oldest results in information aggregation literature, the Condorcet Jury Theorem (from here on referred to as CJT) provides a compelling statistical principle of how, under certain conditions, the chance that a majority selects the best alternative (when such alternative exists) converges to 1 as the size of the voter population grows. This theorem underscores the foundational perspective on the aggregation of individual judgments into a "wisdom of the crowd". A particularly interesting corollary of the CJT is that one can always find a number of voters for which the probability that a majority of them selects the correct alternative is greater than the probability that any single voter in that population selects the correct alternative (Austen-Smith and Banks 1996).

Proxy voting has recently entered the spotlight in this debate thanks to advancements on blockchain technology that allow for greater accountability of decisions made by decentralized authorities. We define a proxy voting mechanism to be one in which voters can choose between voting independently or delegating their voting power to another member of the population. The conceptual roots of proxy voting can be traced back to the writings of Charles Dodgson (know better by his pen name Lewis Carroll). In "Principles of Parliamentary Representation" (Carroll 1884), Carroll articulated a vision for societies where voters not only cast their ballots directly but also have the capacity to transfer their votes to others, essentially enabling a form of vote delegation similar to modern interpretations of proxy voting. The evolution of this idea through the 20^{th} century as detailed in Paulin 2014 underscores its enduring appeal and the challenges it presents. In 1912, William S. U'Ren's advocacy for interactive representation was reported by the New York Times, proposing a system where each elected proxy's influence is calibrated by the volume of votes they receive. By 1969, James C. Miller expanded on this, advocating for the option to vote on any question directly or appoint a representative—a sentiment echoed by Martin Shubik in 1970, who praised the idea while also voicing concerns about the practical challenges and the potential impact on the deliberative quality and timing of public debate. Shubik's apprehensions highlight a recurrent theme in the debate on proxy voting: the practical hurdles inherent to its implementation at scale. These historical perspectives reveal a consistent recognition of the potential proxy voting may have to democratize and decentralize decision-making. They also underscore a longstanding skepticism regarding its feasibility, particularly in the face of logistical and technological constraints.

In recent years, however, it has become increasingly clear that the dismissal of proxy voting on practical grounds may no longer hold. The advent of the internet and distributed ledgers has begun to dismantle the barriers to complex democratic processes, heralding a new era of potential for decentralization mechanisms. In the 21^{st} century this push towards less centralized decision-making is best represented by the development of Liquid Democracy (LD). LD is a system that combines instant referendums with proxy voting. In LD, a voter is always able to choose between the policy options available or choose a proxy representative to make the decision using their voting power. Therefore, instead of electing a subset of voters to represent the population on a set of topics every 4 years, LD allows for "modular congresses" to be composed for every new debate. Early applications of LD include the Google Votes experiment (Hardt and Lopes 2015), where Google employees were asked to review food trucks using LD. In that occasion, researchers observed a specialization effect where votes on vegetarian food trucks were delegated to vegetarian employees. The German Pirate Party used LD as their official internal decision-making tool from 2009 to 2015, abandoning it after concerns that LD had led the party to a situation where a few voters concentrated most of the voting power. In 2016, Democracy Earth, a political advocacy group, ran a mock LD referendum on the peace agreements between the Colombian government and the FARC guerrillas. The notable surge in LD's popularity has been closely followed by scholars in the computer science and economics circles. We will explore the papers around the concept of LD that are relevant to our study in the later half of the next section.

2 Literature Overview

The first idea that will be useful for us to review is the original thought exercise that led to the CJT. We will present this thought exercise in such a way that anticipates the corollary mentioned in the introduction. We choose to do this as it is the most relevant perspective of the CJT for the setup of our model. Consider a judge who is faced with the decision to convict or acquit a defendant. The only information available to the judge derives from the decision of a jury with an unspecified amount of jurors. The judge is given two different strategies to find the best decision. The first strategy is to copy the decision that the most competent (though not perfectly competent) juror makes. The second strategy is to collect signals from all jurors and follow the decision supported by a simple majority of them. If we make the assumption that the least competent jurors can perform better than random choice, then it follows from the Law of Large Numbers (LLN) that there will always exist some real number of jurors for which the decision made by their majority will outperform the most competent juror of that population. If individually all the jurors have competences higher than 50% then, by LLN, the number of jurors making the correct decision will converge to some proportion of the jury population above 50%. Consequently, the probability that more than 50% of the jury will make the correct decision converges to 100%. Therefore, as long as the most competent juror is not perfectly competent, we can always find a set of less competent jurors for which the majority of them will always outperform the single expert juror.

Expanding our theoretical backdrop, we refer to a variation of the CJT studied by T. Feddersen and Pesendorfer 1998. In that case, Feddersen and Pesendorfer were interested in investigating whether requiring juries to reach a unanimous verdict reduces the probability of convicting an innocent defendant while increasing the probability of acquitting a guilty defendant. Specifically, they were interested in testing whether this claim would hold true even if jurors were allowed to engage in strategic behavior. They reach the counter intuitive conclusion that imposing the unanimity restriction actually increases both the likelihood of convicting an innocent defendant and the likelihood of acquitting a guilty defendant. In fact, they find that the likelihood of committing both errors only increases as the number of jurors increases. They show how the majority rule performs much better when it comes to diminishing the likelihood of both of these errors. This study underscores the complexity of voting dynamics and informs the development of our own proxy voting model. Our analysis also draws inspiration from an earlier study by the same authors which introduces the calculus of a pivotal voter and develops the concept of the Swing Voter's Curse, providing information aggregation arguments for a voter's decision to abstain from an election (T. J. Feddersen and Pesendorfer 1996).

Recent works around the concept of proxy voting that also inform our model and our analysis include Revel *et al.* 2022. In it, the authors find that for voter competencies drawn from an arbitrary distribution whose support is bounded away from 1, a constant fraction of the total population is needed to maximize the probability that the representatives make the correct decision on behalf of the entire population. This results strengthens the notion that congresses should be much larger than they currently are in modern democracies, and that decentralized voting systems could improve over the current version of representative democracy many countries have.

Other papers offer a more cautionary perspective. Campbell *et al.* 2022 focuses on comparing the performance of Majority Rule with Abstention (MVA) *versus* LD. They study a model where voters share a common utility and are trying to maximize the chance that the majority will pick the ex-post correct alternative out of 2 options. Through two experiments, the authors show that even though LD could theoretically improve over MVA (as suggested by Revel *et* *al.* 2022), voters in LD exhibit a tendency to over-delegate, choosing to vote independently more than the optimal quantity. In applied settings, therefore, MVA has shown better results than decision mechanisms that allow for proxy-voting. Explanations as to why the "over-delegation" phenomenon occurs mostly center around the costs of voting. However, the results of our analysis suggest a new explanation based on the failure of a pivotal voter to account for how their pivotality condition affects the probability their peers are making the correct decision.

According to Bloembergen *et al.* 2019 a general NE existence theorem is the main open question in LD and proxy voting. In the final sections of our paper we discuss how different types of equilibria (symmetric and asymmetric) compare in a coordination problem that arises with proxy voting. Additional works that include similar models and motivations to the ones exposed here include Kahng *et al.* 2021, Felderhoff 2022, Green-Armytage 2015, and Cohensius *et al.* 2016.

In the next section we introduce our proxy voting model. We specifically focus on a jury model where jurors are able to either cast their verdict directly or transfer their voting weight to a more knowledgeable peer. This model remains faithful to the essence of the CJT while introducing a novel dimension of vote delegation to the most expert juror, thereby probing the strategic and outcome-based implications of proxy voting within a deliberative body. By threading these elements into our investigation, we aim to illuminate the place of proxy voting in the quest for improved aggregation of individual judgments into collective decisions, contributing to the broader discourse on the efficacy and integrity of democratic processes.

3 The Model

This section presents the formal model we are interested in studying. The model is constructed to explore the dynamics of the jury's decision-making under ex-ante uncertainty and the trade-off between the quality and quantity of information suggested by the CJT.

- **Set of Jurors:** Consider a jury of size $n \in \mathbb{N}$, with $n > 2$ and n odd, that deliberates to reach a verdict on whether to convict or acquit a defendant. We index jurors by $I = \{1, 2, \dots, i, \dots, n\}$.

- **Set of States and Alternatives:** There exists a set of possible states of nature denoted by $Z = \{0, 1\}$, corresponding to the defendant being innocent or guilty. Similarly, there exists a set of alternatives, representing the jury's possible decisions, denoted by $X = \{0, 1\}$, corresponding to the jury acquitting or convicting the defendant.

- **Set of Actions:** Jurors can take one of two actions, represented by the set $A = \{\upsilon, \delta\}$, reflecting their decision to vote independently (υ) or vote by proxy (δ).

- **Nature's Role:** The true state of nature, $z \in Z$, is determined by a uniformly random draw by nature, reflecting the inherent ex-ante uncertainty in the trial's outcome.
- **Juror Types:** We distinguish jurors into two types within the set $T = \{e, \neg e\}$, where $\neg e$ denotes non-experts with a common accuracy $p \in (0.5, 1)$ and e denotes experts with a strictly higher competence level $q \in (p, 1)$. Note that even though $\neg e$ jurors share the accuracy p, each p is independent from each other so that the probability 2 $\neg e$ are simultaneously correct is equal to p^2.
- **Utility Functions:** All jurors, regardless of type, share a common utility function $u(x, z)$, which is 1 if their decision matches the true state ($x = z$) and 0 otherwise.
- **Decision Rule:** The jury's verdict is determined by a weighted majority rule, where each juror i has a weight ω_i.

Throughout all the analysis contained in this paper it is assumed for simplicity that there exists only 1 expert juror and $n - 1$ non-expert jurors. By focusing on the trade-off between adding one more weight to the most expert juror *versus* adding a new non-expert juror, this assumption makes us ready to study the trade-off between the quality of a source and quantity of independent sources.

4 Analysis

In our model, any juror can only distinguish action υ from δ if their vote can change the jury's verdict. A juror that finds themselves in such position is referred to as the pivotal juror. In this section we turn to the analysis of the problem faced by this pivotal juror and their equilibrium solution. An important idea is that, in equilibrium, no jurors will choose a strictly dominated strategy. From the description of our model we observe that if the pivotal juror is the expert juror then, in equilibrium, they will always choose action υ, since voting independently strictly dominates delegating to any non-expert juror. If the pivotal juror is a non-expert juror then, in equilibrium, they will never choose to delegate to another non-expert juror as this strategy is strictly dominated by both delegating to the expert juror and voting independently. Therefore, for the remainder of our analysis we will focus on studying the decision of a non-expert pivotal juror between voting independently or using the expert juror as their proxy. As detailed in the description of our model, non-expert jurors have an inherit competence p and cannot change the probability they will vote correctly. Nonetheless, they still have a relevant decision to make. They must consider how voting independently or voting by proxy affects the probability that the majority of the jury will be correct. Recall that all jurors have the same utility function. Therefore, a pivotal juror i is always interested in maximizing the probability that the majority makes the correct decision. So we will use the probability that the majority is correct as their objective function. The first part of this section is dedicated the construction of this function.

4.1 Constructing the Pivotal Juror's Objective Function

When the expert juror has the majority of votes it is easy to calculate the probability that the majority is correct. Either the expert juror is correct and consequently the majority is correct, or the expert is wrong and consequently the majority is incorrect. Therefore, this case can be summarized by

$$\mathbb{P}\left(\text{Majority is Correct}|w_e \geq \frac{n+1}{2}\right) = q. \tag{1}$$

When the expert juror does not have the majority of votes, then the scenarios in which the majority is correct become a bit more involved. If the expert is correct, then we need another $\frac{n+1}{2} - \omega_e$ non-expert jurors to be correct in order to form a correct majority. This case can be modeled as

$$\mathbb{P}\left(\text{Majority is Correct}|\omega_e < \frac{n+1}{2}, e \text{ is correct}\right) =$$
$$q\left[\sum_{j=\frac{n+1}{2}-\omega_e}^{n-\omega_e} \binom{n-\omega_e}{j} p^j (1-p)^{n-\omega_e-j}\right]. \tag{2}$$

Now, if the expert juror does not have the majority of votes and is incorrect, we need $\frac{n+1}{2}$ non-expert jurors to be correct in order to form a correct majority. This case is defined by the expression below

$$\mathbb{P}\left(\text{Majority is Correct}|\omega_e < \frac{n+1}{2}, e \text{ is incorrect}\right) =$$
$$(1-q)\left[\sum_{j=\frac{n+1}{2}}^{n-\omega_e} \binom{n-\omega_e}{j} p^j (1-p)^{n-\omega_e-j}\right]. \tag{3}$$

Given the probabilities described by equations 1, 2, and 3, the probability that the majority is correct is given by their sum

$$\mathbb{P}(\text{Majority is correct}) =$$

$$q\left[\sum_{j=\max\left\{\frac{n+1}{2}-\omega_e, 0\right\}}^{n-\omega_e} \binom{n-\omega_e}{j} p^j (1-p)^{n-\omega_e-j}\right]$$
$$+ (1-q)\left[\sum_{j=\frac{n+1}{2}}^{n-\omega_e} \binom{n-\omega_e}{j} p^j (1-p)^{n-\omega_e-j}\right]. \tag{4}$$

Equation 4 completely describes the objective function of the pivotal juror in our model. Note that we added the term $\max\left\{\frac{n+1}{2} - \omega_e, 0\right\}$ to account for the case where $\omega_e > \frac{n+1}{2}$ and e is correct. In the next section we use equation 4 to setup the problem faced by the pivotal juror.

4.2 The Pivotal Juror's Optimization Problem

We begin the setup of the pivotal juror optimization problem by noting that once one is aware of the CJT, it is intuitive to propose that, for distinct sizes of juries, there exists different optimal amounts of weight that should be allocated to the expert juror. In fact, it seems reasonable to suppose that this optimal weight lies somewhere between absolute power to the expert and a majority rule where all jurors have the same weight. The setup of our model allows the jury to make a decision informed only by the juror with the best information by setting the weight of the expert $w_e \geq \frac{n+1}{2}$. It also allows the jury to make a decision informed by a majority rule between all available sources by distributing 1 vote to each juror. But more importantly, the jury is able to reach strategies in between these two extremes, where the most informed voter may have a higher weight than others yet not hold absolute power.

Given equation 4 and the constraints of our model, we can characterize the pivotal juror's problem as the following maximization problem:

$$\max_{\{\omega_e^*\}} \{\mathbb{P}(\text{Majority is Correct})\}$$

Subject to:

$$\sum_{i=1}^{n} \omega_i = n,$$

$$\omega_e \geq 1,$$

$$\omega_i = \{0, 1\} \ \forall \ i \neq e.$$

Assuming a uniform distribution of parameters p and q, we can use expected values to generate the utility of jurors as a function of the weight of the expert juror. These curves are plotted in Figure 1 for different values of n. These curves suggest that for any n the expected relationship between the probability that the majority is correct and the optimal weight of the expert juror follows a single-peaked pattern. We observe that, for $n > 3$, the peak of the curve seems to always fall in between the two extreme solutions. For the values of n selected in Figure 1, it looks like the optimal value of ω_e changes very slowly with respect to n.

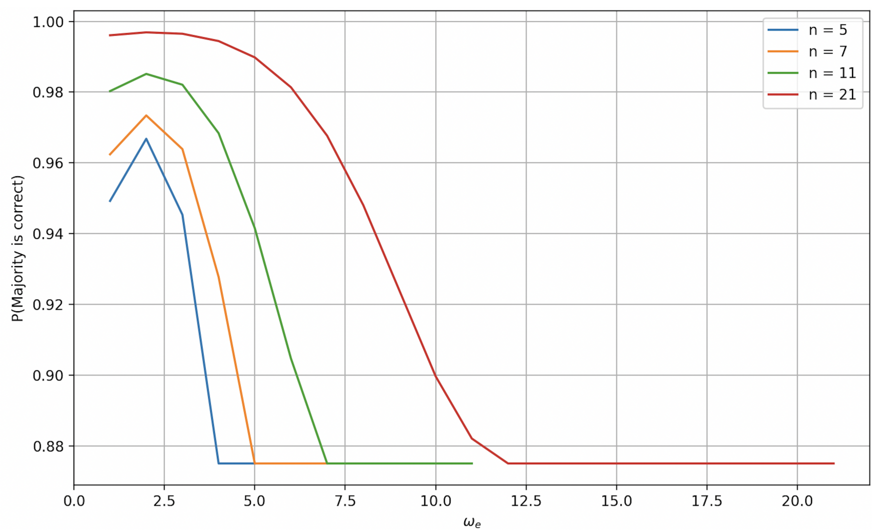

Figure 1: $\mathbb{E}[\mathbb{P}(\text{Majority is Correct})]$ as a Function of ω_e for Select Values of n.

4.3 The Decision of the Pivotal Juror

In this section we are interested in determining when the non-expert pivotal juror i prefers action υ over action δ. This decision is informed by the difference of the utilities derived from each action. Therefore, i's optimal choice of action is represented by

$$A^*(i) = \begin{cases} \upsilon \text{ if } u_i(\upsilon) > u_i(\delta), \\ \delta \text{ if } u_i(\upsilon) < u_i(\delta), \\ \text{Indifferent otherwise.} \end{cases} \tag{5}$$

We can start analyzing the decision of pivotal juror i by determining the probability that i finds themselves to be the pivotal juror. In our model, i can only be pivotal when there are exactly $\frac{n-1}{2}$ jurors voting for convicting the defendant and $\frac{n-1}{2}$ voting for acquitting the defendant. Evidently, this can only happen when $\omega_e < \frac{n+1}{2}$, so we do not need to consider any scenarios involving $\omega_e \geq \frac{n+1}{2}$. Then, there are two distinct occasions in which the non-expert juror is pivotal: one in which the expert juror is correct, and another in which the expert juror is incorrect. In the case when the expert juror is correct we require that exactly $\frac{n-1}{2} - \omega_e$ non-expert jurors be correct and, consequently, $\frac{n-1}{2}$ non-expert jurors be incorrect. We can represent this scenario as

$$\mathbb{P}(i \text{ is a pivotal juror}|e \text{ is correct}) = \binom{n-\omega_e-1}{\frac{n-1}{2}} p^{\frac{n-1}{2}-\omega_e} + (1-p)^{\frac{n-1}{2}}. \tag{6}$$

Similarly, in the case when the expert juror is incorrect, we require that exactly $\frac{n-1}{2} - \omega_e$ non-expert jurors are also incorrect and, consequently, $\frac{n-1}{2}$ non-expert jurors are correct.

$$\mathbb{P}(i \text{ is a pivotal juror}|e \text{ is incorrect}) = \binom{n-\omega_e-1}{\frac{n-1}{2}} p^{\frac{n-1}{2}} (1-p)^{\frac{n-1}{2}-\omega_e}. \quad (7)$$

By summing equations 6 and 7 using the probabilities that the expert juror is correct and incorrect as weights for each of the corresponding terms, we get the total probability that a non-expert juror i is pivotal. This is represented by equation 8 below

$$\mathbb{P}(i \text{ is pivotal}) = q \left[\binom{n-\omega_e-1}{\frac{n-1}{2}} p^{\frac{n-1}{2}-\omega_e} (1-p)^{\frac{n-1}{2}} \right] + (1-q) \left[\binom{n-\omega_e-1}{\frac{n-1}{2}} p^{\frac{n-1}{2}} (1-p)^{\frac{n-1}{2}-\omega_e} \right]. \quad (8)$$

Equation 8 fully captures the probability that i is a pivotal juror. Note that $\mathbb{P}(i \text{ is pivotal})$ is not influenced by the $\mathbb{P}(i \text{ is correct}) = p$. The next step is to find the utility of i associated with choosing action υ *versus* choosing δ given that i is pivotal. Let's start by calculating the utility of voting.

$$u_i(\upsilon) = \mathbb{P}(\text{Majority is Correct}|A(i) = \upsilon). \quad (9)$$

And

$$\mathbb{P}(\text{Majority is Correct}|A(i) = \upsilon) = \mathbb{P}(i \text{ is correct}|i \text{ is pivotal}). \quad (10)$$

Since the probability that i is pivotal is independent of the probability i is correct, then we have that the utility of i from choosing action υ is

$$u_i(\upsilon) = p. \quad (11)$$

Now we calculate $u_i(\delta)$. Naturally, the utility derived from delegating varies depending on whether the expert is correct or incorrect. We account for that in the equations below

$$u_i(\delta) = \mathbb{P}(\text{Majority is Correct}|A(i) = \delta). \quad (12)$$

$$\mathbb{P}(\text{Majority is Correct}|A(i) = \delta) = \mathbb{P}(e \text{ is correct}|i \text{ is pivotal}). \quad (13)$$

From Bayes' Rule we have that

$$\mathbb{P}(e \text{ is correct}|i \text{ is pivotal}) = \frac{\mathbb{P}(i \text{ is pivotal}|e \text{ is correct})\mathbb{P}(e \text{ is correct})}{\mathbb{P}(i \text{ is pivotal})}. \quad (14)$$

From equation 8 and from the definitions of our model we have

$$\mathbb{P}(e \text{ is correct}|i \text{ is pivotal}) = \frac{q\left[\binom{n-\omega_e-1}{\frac{n-1}{2}}p^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}\right]}{q\left[\binom{n-\omega_e-1}{\frac{n-1}{2}}p^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}\right]+(1-q)\left[\binom{n-\omega_e-1}{\frac{n-1}{2}}p^{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}-\omega_e}\right]}$$

$$= \frac{\binom{n-\omega_e-1}{\frac{n-1}{2}}\left[qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}\right]}{\binom{n-\omega_e-1}{\frac{n-1}{2}}\left(\left[qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}\right]+(1-q)\left[p^{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}-\omega_e}\right]\right)}$$

$$= \frac{qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}}{qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}+(1-q)\left[p^{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}-\omega_e}\right]}.$$

So i's utility from delegating is equal to

$$u_i(\delta) = \frac{qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}}{qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}+(1-q)\left[p^{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}-\omega_e}\right]}. \tag{15}$$

The pivotal juror would be indifferent between voting independently or voting by proxy as long as $u_i(\upsilon) = u_i(\delta)$. This means they will be indifferent whenever $p = \mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$ or, from equation 15, whenever

$$p = \frac{qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}}{qp^{\frac{n-1}{2}-\omega_e}(1-p)^{\frac{n-1}{2}}+(1-q)\left[p^{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}-\omega_e}\right]}. \tag{16}$$

If $p > \mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$, the pivotal juror will prefer to vote independently. If $p < \mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$, the pivotal juror will prefer to vote by proxy. Having established the relationship that orients the decision of the non-expert pivotal juror, we conclude the analysis section of our paper and move on to an exploration of the results of our analysis.

5 Results

The main result of our analysis is equation 16 and how it relates to the action choice of a non-expert pivotal voter in a proxy voting context. Figure 2 plots both p and $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$ as a function of the share of votes to the expert, $\frac{\omega_e}{n}$. From figure 2 we can dissect the many moving parts that make up the decision of the pivotal juror. Let's start by identifying the point where the horizontal line p crosses the blue curve $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$. It is here that we find the share of votes $\frac{\omega_e^*}{n}$ that makes i indifferent between voting independently or voting by proxy, that is, the optimal weight for the expert.

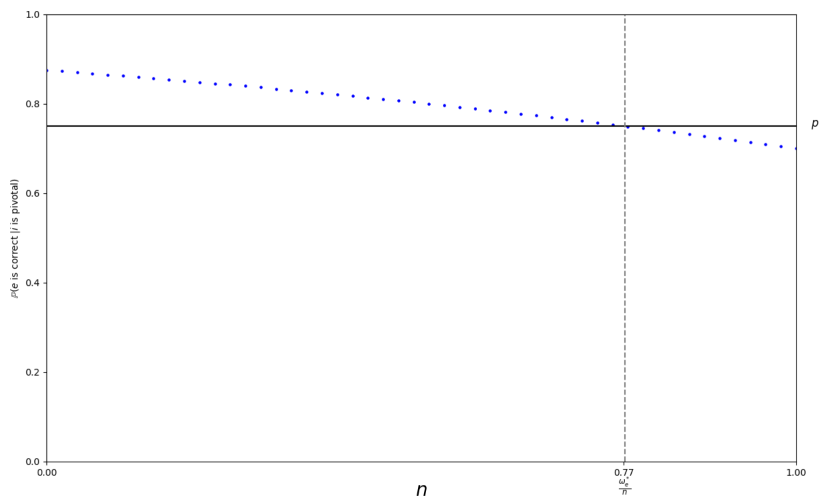

Figure 2: Dynamics of the Trade-Off Faced by a Pivotal Juror ($p = .75$, $q = .875$)

As $q \to 1$, the curve that represents $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$ approaches a constant $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal}) = 1$. As $q \to 0.5$, the optimal share of votes to the expert goes to 0.

As we move p, two effects are observed. The first is a change in $\frac{\omega_e^*}{n}$ due to the vertical shift of p. That is, all else constant, if p shifts down, $\frac{\omega_e^*}{n}$ increases. If p shifts up, $\frac{\omega_e^*}{n}$ decreases. The second effect is on the shape of $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$. As p increases, we observe a linearization of $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$. Therefore, as p increases, there are two combined effects that make $\frac{\omega_e}{n}$ attain its optimal at lower values. As p increases, we observe a curvilinearization of $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$. So as p decreases, there also exists two simultaneous effects that make $\frac{\omega_e}{n}$ attain its optimal at higher values.

At $\frac{\omega_e}{n} = 0$, the $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$ approaches q. As we change n, we observe no effects on p or $\mathbb{P}(e \text{ is correct}|i \text{ is pivotal})$. This means that the value of ω_e^* does not change with respect to n. Therefore, holding all other parameters constant,

$$\lim_{n\to\infty} \frac{\omega_e^*}{n} = 0. \quad (17)$$

Figure 3 provides a visualization of how the optimal allocation of share of votes to the expert varies as n grows for $p = 0.75$ and $q = 0.875$. It illustrates perhaps the most important result of this paper: the probability of the expert being correct given that i is pivotal drops very quickly as the population size

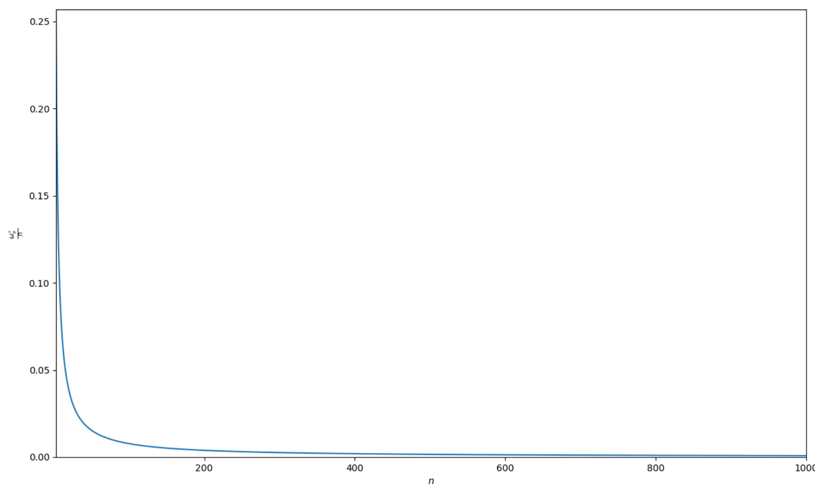

Figure 3: $\frac{\omega_e^*}{n}$ as a Function of n

increases. Here's an intuition for why this phenomenon happens: First, consider what happens to the share of jurors that receive the correct signal as the population of the jury increases. As n goes to infinity, the share of jurors that receive the correct signal converges to p as demonstrated by

$$\lim_{n\to\infty} \frac{p(n-1)+q}{n} \implies \lim_{n\to\infty} \frac{np-p+q}{n} = p. \tag{18}$$

From the construction of our model, we know that $p > 0.5$. So as the population increases, the population of correct jurors should be strictly greater than 50% of the jury population. Now, in a situation where i is pivotal, it must be that at least $\frac{n-1}{2}$ jurors received a signal opposite to the one the expert received. As the population grows and i remains pivotal, the proportion of signals that are opposite to the expert's signal converges to 50% of all realized signals. We can see that in the following equation

$$\lim_{n\to\infty} \frac{\left(\frac{n-1}{2}\right)}{n} = \frac{1}{2}. \tag{19}$$

So from the fact that $n \to \infty$ we have at the same time that the proportion of the population that is correct is greater than 50%, and that the proportion of the population that received signals opposite to the expert juror is at least 50%. Therefore, as $n \to \infty$, the expert juror must belong to the incorrect minority. This result suggests a severe limitation to the ability of proxy voting to improve on the aggregation of information distributed among members of a jury, specially if that jury consists of a large population.

Another way to visualize the decision of the pivotal juror is by plotting i's indifference curves between delegating and voting for every value of ω_e over the pq-plane. In Figure 4 we plot the indifference curves for $\omega_e = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ to illustrate the movement of i's indifference curve as ω_e grows.

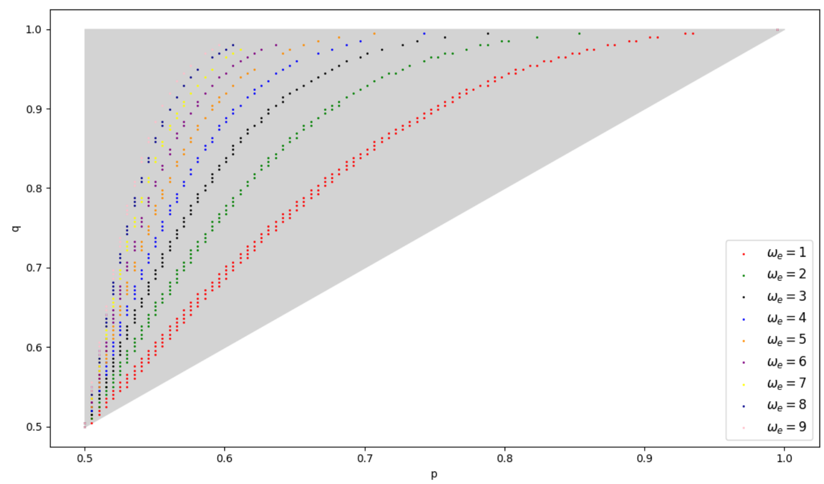

Figure 4: Pivotal Juror's Indifference Curves for $0 < \omega_e \leq 9$

These curves hold for any value of n. Each of them plots the combinations of (p, q) for which the pivotal juror i is indifferent between ω_e and $\omega_e + 1$. If the specific curve represents a value of $\omega_e > \frac{n-1}{2}$, then it has no meaning since this condition violates the concept that i is a pivotal juror. In that sense, these curves represent the combinations of (p, q) for which ω_e is a sufficient and necessary for the non-expert pivotal juror to be indifferent between voting or delegating. Any (p, q) above this specific curve would lead i to delegate their vote, and any (p, q) below it would lead the i for the pivotal juror to prefer to delegate. We observe that, as ω_e grows, i's indifference curve moves closer to the lines $p = 0.5$ and $q = 1$, the limits of the region of feasible (p, q) to the left and to the top. This shows again how the higher the number of jurors delegating gets, the lower the probability that the pivotal juror will prefer to delegate.

6 Discussion

In this section we explore how our findings contribute to recent debates around the concept on proxy voting.

6.1 An Explanation to Over-Delegation

A main finding from the two experiments conducted by Campbell *et al.* 2022 is that participants in LD tend to delegate their votes more frequently than the optimal amount of delegation. The authors use this fact to explain why outcomes from MV and MVA outperform decisions made through LD. However, they do not provide a compelling explanation of why this phenomenon occurs. Our work may offer such explanation.

The insight that the pivotal juror's calculus between voting and delegating depends on the probability that the expert juror is correct conditional on the fact that they are pivotal suggests a reason why this bias towards delegating may be happening. Humans are limited in their abilities to recognize and use probability appropriately in their every day life. It is very plausible that, when deciding whether to vote or to delegate, participants in the LD experiment made their decisions based only on their beliefs about the difference between the face value competence of other jurors and the face value of their own competence. By failing to consider for the relationship of dependence between the probability that their proxies were correct and the probability that they were the pivotal voter, these participants would surely delegate more often than the optimal amount.

6.2 Coordination Problems

From figure 4 we notice that a coordination problem arises given a proxy-voting setting. Suppose all jurors are aware of the distribution of competence levels among the population. Each of them can use figure 4 to identify the optimal ω_e that the expert should have. All jurors are commonly interested in giving the expert that exact amount of votes, so what's the problem? There still exists uncertainty about which jurors should delegate and which should vote. If they are not allowed to communicate, then some coordination strategy must be adopted to reach the optimal at equilibrium.

A possible symmetric equilibrium would be for all jurors to choose to delegate with probability $\frac{\omega_e}{n}$. No jurors would be interested in deviating from this strategy and, for a large enough population, the amount of individuals delegating would approach the optimal value very closely.

Another solution that would reach the optimal at equilibrium would for ω_e jurors to always delegate and $n - \omega_e$ to always vote. Again, no jurors would have the incentive to move away from their own strategy. This asymmetric equilibrium describes a solution to the coordination problems through the specialization of voters. Naturally, the asymmetric equilibrium would always outperform the symmetric equilibrium since the former attains the optimal ω_e at every n, whereas the latter approaches the optimal value of ω_e, but never attains it exactly.

7 Conclusion

In this paper we explore the nuanced dynamics of proxy voting through the adaptation of the Condorcet Jury Theorem, offering a fresh perspective on the strategic calculus of pivotal voters. We delve into the intricacies of proxy voting mechanisms, where voters face the choice between voting independently or delegating their vote to a more informed peer. This decision-making process is scrutinized within a common interest model that incorporates the varying competencies of jurors, the influence of an expert juror, and the collective goal of reaching an accurate majority decision.

Our analysis reveals that the decision of a pivotal voter to delegate or vote directly hinges on the interplay between their own competence, the competence of the expert juror, and the distribution of delegated votes. The findings suggest that as the size of the voting body increases, the optimal share of votes for the expert juror diminishes, highlighting a potential limitation in the scalability of proxy voting systems for large electorates. Furthermore, we address the phenomenon of over-delegation observed in empirical studies of proxy voting, proposing that a failure to account for conditional probabilities may lead voters to delegate more often than is theoretically optimal.

Our discussion extends to the coordination challenges inherent in proxy voting systems, contrasting symmetric and asymmetric equilibria as potential solutions. While a symmetric strategy, where voters delegate with a certain probability, may approach optimality in large populations, an asymmetric strategy, predicated on a fixed division of voters into delegates and direct voters, could more reliably achieve optimal outcomes.

By elucidating the strategic considerations of pivotal voters and the coordination dilemmas posed by proxy voting, we provide valuable insights into the potential and limitations of this voting system. As technology continues to evolve, enabling more sophisticated forms of democratic participation, the insights garnered from this study will inform future explorations into the optimal structuring of voting mechanisms to harness the collective wisdom of diverse electorates.

References

[1] K. J. Arrow, *Social choice and individual values.* Yale University Press, 1951, vol. 12.

[2] A. Gibbard, "Manipulation of voting schemes: A general result," *Econometrica: journal of the Econometric Society*, pp. 587–601, 1973.

[3] M. A. Satterthwaite, "Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of economic theory*, vol. 10, no. 2, pp. 187–217, 1975.

[4] K. O. May, "A set of independent necessary and sufficient conditions for simple majority decision," *Econometrica: Journal of the Econometric Society*, pp. 680–684, 1952.

[5] N. De Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix.* Cambridge University Press, 1785.

[6] D. Austen-Smith and J. S. Banks, "Information aggregation, rationality, and the condorcet jury theorem," *American Political Science Review*, vol. 90, no. 1, pp. 34–45, 1996. DOI: 10.2307/2082796.

[7] L. Carroll, *The Principles of Parliamentary Representation.* Harrison and Sons, 1884.

[8] A. Paulin, "Through liquid democracy to sustainable non-bureaucratic government," in *Proc. Int. Conf. for E-democracy and open government*, 2014, pp. 205–217.

[9] S. Hardt and L. C. Lopes, "Google votes: A liquid democracy experiment on a corporate social network," 2015.

[10] T. Feddersen and W. Pesendorfer, "Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting," *American Political Science Review*, vol. 92, no. 1, pp. 23–35, 1998.

[11] T. J. Feddersen and W. Pesendorfer, "The swing voter's curse," *The American Economic Review*, pp. 408–424, 1996.

[12] M. Revel, T. Lin, and D. Halpern, "How many representatives do we need? the optimal size of a congress voting on binary issues," in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 36, 2022, pp. 9431–9438.

[13] J. Campbell, A. Casella, L. de Lara, V. R. Mooers, and D. Ravindran, "Liquid democracy: Two experiments on delegation in voting," National Bureau of Economic Research, Tech. Rep. w30794, 2022.

[14] D. Bloembergen, D. Grossi, and M. Lackner, "On rational delegations in liquid democracy," in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 33, 2019, pp. 1796–1803.

[15] A. Kahng, S. Mackenzie, and A. Procaccia, "Liquid democracy: An algorithmic perspective," *Journal of Artificial Intelligence Research*, vol. 70, pp. 1223–1252, 2021.

[16] L. Felderhoff, "Single-peaked electorates in liquid democracy," 2022.

[17] J. Green-Armytage, "Direct voting and proxy voting," *Constitutional Political Economy*, vol. 26, pp. 190–220, 2015.

[18] G. Cohensius, S. Manor, R. Meir, E. Meirom, and A. Orda, "Proxy voting for better outcomes," *arXiv preprint arXiv:1611.08308*, 2016.