The Calculus of a Pivotal Voter Under a Proxy Voting Regime

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Motivation

- Political economists have long debated what the optimal voting system should look like (Arrow 1951, May 1952, Downs 1957, Gibbard 1973).
- The celebrated Condorcet Jury Theorem (CJT) says that an increase in the number of voting committee members leads to a better information aggregation.
- In many modern setting, voters have an opportunity to delegate their vote to better informed members ("proxy voting").
- I revisit the CJT to analyze the consequences of proxy voting on the ability of voting rules to aggregate information.

Agenda

1. Related Literature
   - Condorcet Jury Theorem
   - Inspirations for the Model
   - Recent Literature on Proxy Voting
2. Description of the Proxy Voting Model
3. The Judge's Optimization Problem
4. The Pivotal Juror's Decision
5. Discussion
   - Contributions to normative and positive ideas around proxy voting

Related Literature

Condorcet Jury Theorem (CJT)

Given a binary decision where one of the options is true ex-post with some ex-ante uncertainty, one can always find a number of jurors which are more competent than random choice for which the outcome of a majority rule among those jurors will outperform the competence of any single juror.

The CJT holds true as a result of the following ideas:

- Jurors' competence level is bounded between 50% up to the accuracy of the most competent juror.
- Law of Large Numbers.

Take-Away 1: Intuition suggests that there exists a trade-off between quality and quantity of information.

Related Literature (cont'd)

- Feddersen and Pesendorfer (1995, 1998) study models of binary decision similar to the context of CJT. They introduce considerations on voter strategical behavior and the calculus of a pivotal voter.
- Revel et al. (2021) study the optimal size of congresses in an epistemic context. Optimal decision-making groups should have a size much larger than what we observe empirically.
- Campbell et al. (2022) run two experiments and show that, even though proxy voting could outperform majority vote (MV) in theory, in practice voters in a proxy-voting system tend to vote by proxy less than they should, to the point that MV consistently outperforms proxy voting.

Description of the Proxy Voting Model

Consider again a jury of size $n \in \mathbb{N}$ deciding whether to convict $(x = 1)$ or acquit $(x = 0)$ a defendant. Let $n > 2$, odd.

- Set of states $Z = \{0, 1\}$
- Set of alternatives $X = \{0, 1\}$
- Set of actions $A = \{\upsilon, \delta\}$
- Nature makes a uniformly random draw to determine state $z \in Z$
- Set of juror types $T = \{e, \neg e\}$, where
  - Non-experts $(\neg e)$: share a common accuracy of $p \in (0.5, 1)$.
  - Experts (e): with competence $q \in (p, 1)$.
- Jurors of all types share common utility $u(x, z) = \begin{cases} 1 \text{ if } x = z \\ 0 \text{ if } x \neq z \end{cases}$.
- The jury's decision is determined by a weighted majority rule, where the weight juror i is represented by ω_i

Description of the Proxy Voting Model (cont'd)

- For simplicity, we will assume that $\exists$ only 1 e juror and $(n-1)$ $\neg e$ jurors. We will also assume that only the single e juror can have weight $\omega_e > 1$.
- Given these assumptions we are ready to study the trade-off between quality and quantity of information suggested by CJT through the lenses of proxy voting.
- Note: The problem faced by common interested jurors is equivalent to the problem faced by a central planner who is endowed with a budget of n votes that must be allocated to the jurors in a way that maximizes the probability that the majority is correct. In the following slides we will refer to this central planner as the judge.

The Judge's Objective Function

The probability that a majority of jurors is correct when...
$\ldots \omega_e < \frac{n+1}{2}$ and e is correct:

$$\mathbb{P}\left(\text{Majority is Correct} \mid \omega_e < \tfrac{n+1}{2}, e \text{ is correct}\right) =$$

$$q\left[\sum_{j=\frac{n+1}{2}-\omega_e}^{n-\omega_e} \binom{n-\omega_e}{j} p^j (1-p)^{n-\omega_e-j}\right] \tag{1}$$

$\ldots \omega_e < \frac{n+1}{2}$ and e is incorrect:

$$\mathbb{P}(\text{Majority is Correct} \mid \omega_e < \tfrac{n+1}{2}, e \text{ is incorrect}) =$$

$$(1-q)\left[\sum_{j=\frac{n+1}{2}}^{n-\omega_e} \binom{n-\omega_e}{j} p^j (1-p)^{n-\omega_e-j}\right] \tag{2}$$

The Judge's Objective Function (cont'd)

From equations (1) and (2) we have that the probability the majority is correct is given by their sum:

$$\mathbb{P}(\text{Majority is Correct}) = q\left[\sum_{j=\max\left\{\frac{n+1}{2}-\omega_e,0\right\}}^{n-\omega_e}\binom{n-\omega_e}{j}p^j(1-p)^{n-\omega_e-j}\right] + (1-q)\left[\sum_{j=\frac{n+1}{2}}^{n-\omega_e}\binom{n-\omega_e}{j}p^j(1-p)^{n-\omega_e-j}\right] \tag{3}$$

Note that $j=\max\left\{\frac{n+1}{2}-\omega_e,0\right\}$ accounts for the special case where $\omega_e \geq \frac{n+1}{2}$ and e is correct.

The Judge's Optimization Problem

Next, we characterize the judge's optimization problem as the following maximization problem:

$$\max_{\{\omega_e^*\}} \{\mathbb{P}(\text{Majority is Correct})\}$$

Subject to:

$$\omega_e \geq 1$$

$$\sum_{i=1}^{n} \omega_i = n$$

$$\omega_i = \{0, 1\} \ \forall \ i \neq e$$

Numerical Solution to the Judge's Optimization Problem

The Single-Peak Property of the Judge's Preferences

For any $n \in (2, +\infty) \in \mathbb{N}$, the expected probability that the majority of the jury is correct as a function of the weight of the expert juror ω_e approximates a single-peaked function.

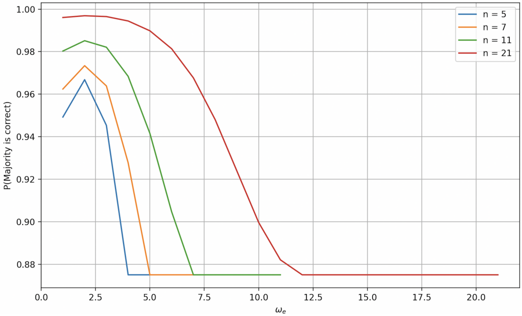

Figure: Expected Probability of Jury Voting Correctly as a Function of ω_e

Take-Away 2: Confirms our intuition on the quality *v.s.* quantity trade-off.

Numerical Solution (cont'd)

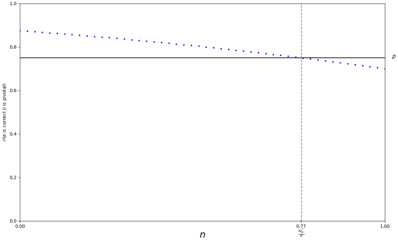

Figure: Optimal Share of Votes Allocated to the Expert as *n* increases

Take-Away 3: For a large enough jury size, the optimal share of votes the judge should allocate to their most informative source goes to 0. This is consistent with the CJT and results from Revel (2021).

Probability that a Juror is Pivotal

In our model, a juror i is pivotal when there are exactly $\frac{n-1}{2}$ votes on each alternative. The probability such a scenario occurs is given by:

$$\mathbb{P}(i \text{ is pivotal}) = q \cdot \left[\binom{n - \omega_e - 1}{\frac{n-1}{2}} \cdot p^{\frac{n-\omega_e-2}{2}} \cdot (1-p)^{\frac{n-1}{2}} \right]$$
$$+ (1-q) \cdot \left[\binom{n - \omega_e - 1}{\frac{n-1}{2}} \cdot p^{\frac{n-1}{2}} \cdot (1-p)^{\frac{n-\omega_e-2}{2}} \right] \quad (4)$$

The Decision of the Pivotal Juror (cont'd)

If i is pivotal, then the utility differential from choosing action υ versus δ is given by:

$$\mathbb{P}(\text{Majority is Correct})|e \text{ is incorrect} = p(1) + (1-p)(0) - 0 = p \tag{5}$$

$$\mathbb{P}(\text{Majority is Correct})|e \text{ is correct} =$$

$$p - \frac{q\left[\binom{n-\omega_e-1}{\frac{n-1}{2}} \cdot p^{\frac{n-\omega_e-2}{2}} \cdot (1-p)^{\frac{n-1}{2}}\right]}{q\left[\binom{n-\omega_e-1}{\frac{n-1}{2}} \cdot p^{\frac{n-\omega_e-2}{2}} \cdot (1-p)^{\frac{n-1}{2}}\right] + (1-q)\left[\binom{n-\omega_e-1}{\frac{n-1}{2}} \cdot p^{\frac{n-1}{2}} \cdot (1-p)^{\frac{n-\omega_e-2}{2}}\right]} \tag{6}$$

Combining equations (4) and (5) we have a representation of $[u_i(\upsilon) - u_i(\delta)]$ in terms of $n, p, q,$ and ω_e. Let's break down the expression:

$$[u_i(\upsilon) - u_i(\delta)] = qCA\left(p - \frac{qCA}{qCA + (1-q)CB}\right) - (1-q)CB(1-p) \quad (7)$$

where

- $A = p^{\frac{n-1}{2}}(1-p)^{\frac{n-\omega_e-2}{2}}$
- $B = p^{\frac{n-\omega_e-2}{2}}(1-p)^{\frac{n-1}{2}}$
- $C = \binom{n-\omega_e-1}{\frac{n-1}{2}}$

Combining equations (4) and (5) we have a representation of $\mathbb{E}[u_i(\upsilon) - u_i(\delta)]$ in terms of $n, p, q,$ and k:

$$\mathbb{P}(i \text{ is pivotal}) \times [\Delta\mathbb{P}(\text{Majority is Correct})|A(i) = \upsilon] =$$

$$q \cdot \left[\binom{n - \omega_e - 1}{\frac{n-1}{2}} \cdot p^{\frac{n-\omega_e-2}{2}} \cdot (1-p)^{\frac{n-1}{2}} \right] \times 2p$$

$$+(1-q) \cdot \left[\binom{n - \omega_e - 1}{\frac{n-1}{2}} \cdot p^{\frac{n-1}{2}} \cdot (1-p)^{\frac{n-\omega_e-2}{2}} \right] \times 2(p-1) \qquad (8)$$

Suppose $\mathbb{E}[u_i(\upsilon)] = \mathbb{E}[u_i(\delta)]$. Set equation (6) equal to zero. After taking the logs of terms and rearranging them so that k is expressed as a function of p and q, we find that competence levels describe a k sufficient for a pivotal juror to prefer action υ or be indifferent between and υ and δ.

The Decision of the Pivotal Juror (cont'd)

The k sufficient for $\mathbb{E}[u_i(\upsilon)] \geq \mathbb{E}[u_i(\delta)]$

Let $k \in [0,(n-2)] \in \mathbb{R}$ represent the number of jurors that choose to vote by proxy in a jury of size n. Then the pivotal juror will be indifferent between actions υ and δ if and only if

$$k = \frac{\ln(2p^2 - 2qp^2) - \ln(2q - 4qp + 2qp^2)}{\ln(1-p) - \ln p}. \tag{9}$$

.

- $\frac{\partial k}{\partial q} = \frac{1}{(q-1)q\ln\left(\frac{1}{p}-1\right)} > 0$, $\frac{\partial k}{\partial p} = \frac{-2\ln\left(\frac{1}{p}-1\right)+\ln\left(\frac{-p^2(q-1)}{(p-1)^2q}\right)}{(p-1)p\ln\left(\frac{1}{p}-1\right)^2} < 0.$
- Note that, given i is pivotal, n has no impact in their decision between $\{\upsilon, \delta\}$.

The Pivotal Juror's Decision at $k = 0$

From equation (7) we have that

$$q = \frac{p^{k+2}}{p^{k+2} + (1-p)^{k+2}} \tag{10}$$

At $k = 0$ the figure below gives us the graphical relationship between p and q that orients the pivotal juror. It holds true for any jury of size n as long as n is odd and > 2.

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Figure: Combinations of (p, q) and decision of pivotal juror at $k = 0$

Take-Away 4: Assuming uniform distribution of p and q, $k = 0 \implies \approx 40\%$ likelihood that $\upsilon \succ_i \delta$ and, consequently, $\approx 60\%$ likelihood that $\delta \succ_i \upsilon$.

The Pivotal Juror's Decision as $\frac{k}{n} \to \frac{1}{2}$

Plugging $k = \left(\frac{n-1}{2} - 1\right)$ into equation (8) we have

$$q = \frac{p^{\lceil \frac{n}{2} \rceil}}{p^{\lceil \frac{n}{2} \rceil} + (1-p)^{\lceil \frac{n}{2} \rceil}} \tag{11}$$

Then, the probability that a pivotal juror will choose to make the decision through a "dictatorship of the expert" is given by:

$$\mathbb{P}(i \text{ prefers } e\text{-dictatorship}) = \frac{\frac{1}{2} - \int_{\frac{1}{2}}^{1} \frac{p^{\frac{n+1}{2}}}{p^{\frac{n+1}{2}} + (1-p)^{\frac{n+1}{2}}}\, dp}{\frac{1}{8}} \tag{12}$$

The Pivotal Juror's Decision as $\frac{k}{n} \to \frac{1}{2}$ (cont'd)

graphs/graph2.png

Figure: $\mathbb{P}(i$ prefers e-dictatorship) as $\frac{k}{n} \to \frac{1}{2}$.

Take-Away 5: υ completely crowds out δ as $\frac{k}{n} \to \frac{1}{2}$, suggesting that jurors will choose υ more than the optimal level at every k (except at $k = 0$).

Discussion

- Information Aggregation Bias Explains Results from Campbell *et al* 2022
  - If we assume all voters behave according to a pivotal voter, then from Take-Away 5, for any $k > 0$, we would expect to observe more voters taking action υ when they should actually be taking action δ.
- Trustees versus Representatives
  - From Take-Away 4: If we assume all voters behave according to a pivotal voter, then when all voters have the same weight, a majority of them is expected to vote on trustees.
  - Still from Take-Away 4: Beliefs about the distribution of competence levels impacts the proportion of votes for trustees and representatives.
- Administrative State versus Court Rulings
  - From Take-Away 2: The single-peak property of the judge's preference suggests a naive solution to the debate over U.S. courts interpretations of the Chevron Doctrine.

Thank you!

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