PL and QL

I Translation and Real World Grounding With (Nohra)

I.1 Anschauung 1

1. TODO: Add the figure from the notebook.

2. A very natural starting point is the real world. Within it one man, within the group of humans, and one dog within the group of dogs.

3. The word, written on paper by one man 'dog' is linked to a structure in his brain. This happens for multiple men, this turns into the human abstraction 'dog' which has components both inside the human culture and outside of it. Even if the human culture did not exist, the dogs might still. This is simply the human culture sensing something outside of it.

4. Within each man's brain, there is the black box of a language, which encodes all of man's experience, and reflections on then. These include compressed versions of abstractions. These abstractions include processes such as physical ones.

5. One of these processes is that of the simplest potentially infinite dynamic process of repeating the same thing over and over. With a bit of added structure, this is formally and informally translated into 'induction' and the 'natural numbers'.

6. Since each man's brain is being a black box, most probably (and also provably) different than the other's. It has been useful for matters of discussion settling (and shocking by 'proof' using the trick of first inventing the system of proof without telling the shocked about it), that is for matters of power over the other man, at least intellectual, but also practical by prediction of real world processes, to formalize discussion settling such as to make it possible to categorically settle questions given a set of initial fixities (the individual interpretation of which having no effect of the settling). At least one basic component of such a discussion-settling system, which is then directly a tiny part of the human culture hence of the world, the two valued dichotomy T(rue) and F(alse).

7. However, this must be assigned to some 'things'. The structure of these things must also be formalized and this turns out to be not a small challenge.

8. This gives us a 'domain' of discussion-settling (the 'things') and a codomain (True, False). But the allowed relation between the two has also to be settled. And this gives us the full range of allowed 'logics'.

9. One can allow the relation to be a function or not, injective or not, etc. Per example one could allow it to be a relation, and one thing could then be both true and false. In systems where this leads to (by some deductive structure) every 'thing' being both true and false, one could say that the discussion settling has then failed, since nothing is settled. This is what is adopted in classical logic and the answer to Wittgenstein's "Der Widerspruch. Warum gerade dieses ein Gespenst". But this is merely a fact that must be described and not an argument for the making of 'classical logic' the right choice of allowed 'logic'. Each logic will probably be found to model some useful aspect of the whole world and even when not, has its place by pluralism, unless some choice is found to be natural by some very strong and convincing criterium.

10. The challenge is obviously overwhelming to be tackled 'directly'. Instead, we gamify it. We invent games which we hope capture amount to more or less expressive disucssion-settling systems. This is an indirect way. At the end of the day, we look at all of our games and see how far we have come in formally settling discussions.

11. Calling this gamification immediately allows us to drop all the long winded and confused justifications of why such or such concept in such of such formal language's or theory's syntax or semantics is this or that way. This is confused and confusing. Knowing the above, makes us really understand that this is just an attempt at a game. Take or or create your own.

12. The domain is not easy to capture. Many times, a model of 'induction' is brought in because as we model a discussion, one can always in a dynamical process kind of way, create and combine 'things' and this is something one wishes to cover in the settling system. One commonly used way to capture this is 'syntax'. When this is so, we call the syntax game a 'wff game' viz. a 'well formed formula game'.

I.2 Anschauung 2

13. Figure from our notebook at page 36. LDL: draw circle world labelled "world (0)", draw circle "discussion domain" alias dd inside world, label dd with arrow to lower left of world, draw circle "discussion bootstrap" alias db inside dd, label db with arrow at 3 o'clock of world, draw labelled circles A,B,C inside db, give db 1/10 density

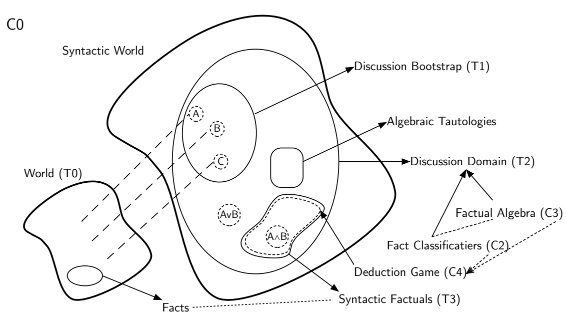

Figure 1: Translation Choices and Things

14. We first make a choice (C0) about the way we picture our (game of) discussion settling deductive system. In effect, we fix what we mean by a 'deductive system' although there may be many choices for what this means. Everything that follows assumes this choice.

15. The first 'thing' (T0) that we have a is world.

16. Note that there is nothing wrong with calling things as simply 'things' without further qualification, since the qualification is implicit in the structure of the whole system (think Hilbert's points and lines). As long as we identify the thing (e.g T0), we are fine.

17. From this world, by translation, observation, sensing, etc. we have our second thing which is a syntactic world (T1). This world contains syntactical atoms (a bit like propositions). T1 can be seen as a kind of discussion bootstrap as well as a 'reasoning intuition' removal and 'expression ambiguity' filtering device. Note that there is nothing against the world T0 being syntactic itself.

18. We make a second choice (C2) of how we classify 'facts'. We could take {T,F}, {T,F,neither}, {T,F,T and F, neither ,...}, {1,2,3,4,...}, etc.

19. We make a third choice (C3) for a 'factual algebra'. This can per example be boolean algebra which if taken, must be of course compatible with C2. A boolean algebra per example forces an excluded middle, implies tautologies such as 'a or not a', etc. Another way to see this is that we choose the kind of function that maps syntactic atoms to factuality. If this is a relation, a fact can have any of {}, {T}, {F}, {T,F} factualities. A surjective function yields the classical kind of mapping.

20. In the case boolean algebra, one could see it as directly given some connectives like $\wedge$ but there is a more essential way to see it. The boolean algebra can be described, given the finiteness of its elements, by its unary and binary operators, everything else being composed of the former. The complete set of unary operators is thence:

a	f_0	f_1	$\overbrace{f_2}^{\neg}$	f_3
1	0	1	0	1
0	0	0	1	1

and of binary operators:

a	b	g_0	g_1	g_i	g_n
1	1	0	1	·	1
1	0	0	0	·	1
0	1	0	0	·	1
0	0	0	0	·	1

1. In an ad-hok manner (usually by (so called 'structural') induction on the complexity (length) of terms), or using (modern) algebra, one can show that there is a small set of operators that can be used to produce all the functions. These are then given names such as '$\neg$', '$\wedge$' and '$\Rightarrow$'. One such set of adequate operators can then be chosen for the syntax. We note here that the binary operators can be shown to form a poset [@c3, pp.30]. For boolean algebra, such forms can be methodically studied and proved to be adequate using normal forms (in a sense, canonical representations of all boolean functions). When this is not possible (e.g because of infinitary considerations, is the syntactic translation doomed?)

2. A methodical treatment of the matter of adequacy is to be found in [**?**]. Specifically, there are 26 minimally adequate sets of connectives and 4 maximally inadequate sets. Also, for a set of connectives to be adequate it is necessary and sufficient that for it to have

   - F and T (or formulas with these values).
   - An odd connective (a connective with arity larger than 1 is called odd if it has odd number of Ts in its truth table).
   - A non-monotone connective (a connective that turning an F to a T will make its value change from T to F).

3. In fact it seems that the first serious and exhaustive treatment of this was done in [**?**], resulting in the iconic Post lattice[†].

21. T1, by a kind of algebraic completion related to C2 and C3, gives rise to T2, the syntactic discussion domain, that includes all of T1, or better said, all of T1's contents, which are all 'atoms'.

22. The world's translation also includes delineating a subset of the discussion domain as expressing facts in the world. It is required that for any selection of facts in the real world, the counterparts in T2 are 'T' but not only that, it is required that the algebraic consequences of these being 'T' hold in the real world, or at least do not break the hoped discussion settling model. Per example them not being T but actually relating to 'nothing of value' might be acceptable.

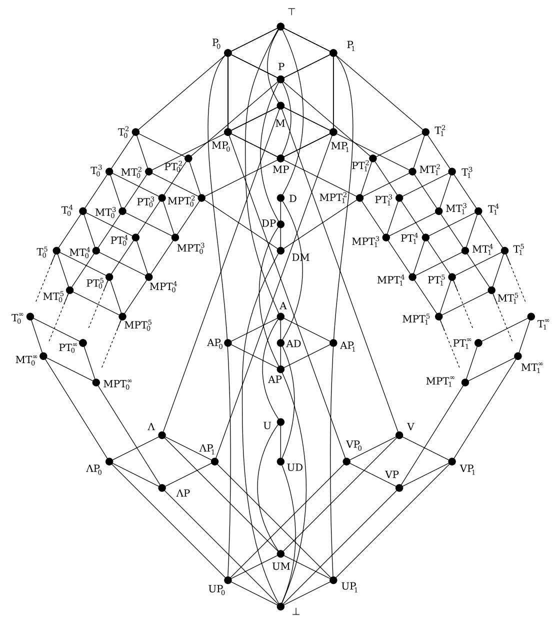

Figure 2: Post's lattice [1]

23. The part of T2 that translates to the world's factuals is called T3.

24. Per example, consider the '$a \vee b$' world, where this is the only fact. We note the following:

1. Any fact 'c' other than 'a' and 'b' has no business being in the syntactic world, since it does not correspond in any relation to any other fact, even if it represents 'something in the world'. In general it is required that any atom in the syntactic world be part in some factual relation to one or a group of other atoms, or be assigned a factual itself. Maybe better said, every syntactic atom has to have some factual content. Per example in our example world, 'a' has no factual content in relation to itself (is not T or F), however it enters into a factual relation together with 'b'.

[1]"Post-lattice" by EmilJ - Own work. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Post-lattice.svg#/media/File:Post-lattice.svg

2. In this example, T or F cannot be assigned to any of 'a' or 'b'. Nevertheless, it is a 'constraint' or 'knowledge' about this world.

25. There might be algebraic tautologies in T2, but they are in this context, not very interesting since (again only in this context), they highlight a certain structure in the algebra itself which is more often than not 'uselessly and trivially obvious and un-enlighting' when translated to the real world (e.g 'a or not a'). This is so because the choice of the algebra already contains 'what we have observed to be a pattern in the world and can be used as a completion algebra' and that is why we chose it. What is 'interesting', what makes the world differentiable from other worlds with the same number of atoms and the same choices, is the delineated (non-logical) facts. Let us call this the world's contingent.

26. Having done this, we proceed to find a game that models discussion-settling if-then-ism. We decide that this game happens entirely inside T3 and follows a slight generalization of human if-then-ism, the technique the humans are naturally good at, as opposed to an implicit equilibrium of facts without specific 'root', which actually is closer to the truth about factual structures. We call this choice of game C4. Pictorially C4 looks like what follows in its most general (and not very human) form of multiple 'outputs':

$$\frac{\begin{array}{c}\text{—}\\ \text{—}\\ \text{—}\end{array}}{\begin{array}{c}\text{—}\\ \text{—}\end{array}}$$

27. C4 is hence a game of 'facts to facts'. Depending on the previous choices, the algebra dictates how this game is extracted from it. Since the game C4 is syntactic, it is completely mechanical (formal). It is mechanical in the sense that we can check, given the rules and a set of moves, whether the moves were legal. At the same time, for most interesting systems, the search for moves that get from some choice of facts to some other seems to necessitate a hopeless raw search. This is due to the 'non-invertability' of deduction one could say.

28. It is important that our choice of C4 allows to recover all of T3 using the game C4 alone and not only a part of T3.

29. The tautologies of the algebra find their use exactly when extracting C4 since then, we decided to work only in 'T', that is, with tautologies. All lines in the deductive game must be tautologies. An example of the extraction of C4 is then the following:

	a	b	$a \vee b$
1	T	F	T
2	F	T	T
3	T	T	T
4	F	F	F

The tautological part of '$a \vee b$' consisting of lines (1,2,3) has no tautological counterpart for 'a' {T,F,T} or for 'b' {F,T,T}, hence we cannot extract any of these two rules for C4: $\frac{a\vee b}{a}$, $\frac{a\vee b}{b}$. This is in contrast with '$a \wedge b$' from which we can extract both $\frac{a\wedge b}{a}$ and $\frac{a\wedge b}{b}$. It is also in contrast with looking at the table from the tautological perspective of 'a' viz. lines (1,3), which actually allows the extraction of $\frac{a}{a\vee b}$. Similarly, 'b' has a different set of tautological lines (2,3) from which we nevertheless can extract $\frac{b}{a\vee b}$.

30. Using this 'choice exposing methodology' our understanding of at least some paradoxes become clearer.

1. We have established elsewhere that with such a paradox, one way to see the problem is that truth is 'dynamic' and not 'static' or that the structure of deduction is a graph and not a tree.

2. Informally, we can speak of anything, and we can also choose almost anything for our syntactic and semantic systems.

3. What we now see is that some choices are incompatible with the {T,F} choice and it is as simple as that, but these are then called inconsistent. They are in fact simply an incompatibility between choices.

4. Imagine a relay switch set itself to turn itself off when it is on and vice versa (our experiment as a child leading to a buzzer). some kind of a PL with 'switch is on' is simply incompatible with the choice {T,F} unless we add time somewhere (modal logic?). This is a sloppy example of incompatibility between choices.

5. A famous example is probably naive set theory, which is incompatible with {T,F}.

6. Para-consistent logic probably offers a set of choices that can accommodate such informal factual structures.

7. We could say that one goal of logic is the search for compatible choices.

8. Elsewhere we wrote as topic: 'the possibility of a self-referential formal theory' as a precursor to studying Goedel's result.

    - A precursor of the possibility of such a theory is the possibility of an informal one.
    - The barber paradox is a prime example. TODO: can we formalize it minimally?.
    - In fact, the barber paradox could have been the Kuhnian 'exemplar' for Goedel! He knew it from naive set theory, and then 'simply' tried to replicate if for a theory of his interest. This helps demystify his whole enterprise. Our observation turns out to be confirmed in [@c13]: "Godel's original proof relied on a syntactic version of the liar's paradox, the latter being the paradoxical (semantic) utterance "I am lying".39 Godel formulated within Peano arithmetic the self-referential statement, "I am not a theorem", and showed it to be an undecidable sentence."

I.3 Anschauung 3

31. A discussion settling system is of no use if, to achieve its purpose it must defy the laws of physics as known at the time of the discussion. Per example, it cannot rely on a result that needs an infinity of steps in the process of its computation. Because of this, 'formal' must be 'computable'. As an example, syntax is required to be comprised of finite propositions.

32. Let us note here that by the above, we find the right way to describe 'fin-ifinity' in mathematics, per example in uncomputable reals (not describable in any finite way, incompressible): What is needed for mathematics to work is that the discussion-settling system is 'computable' but this does not say anything about about the computability of any mathematical object. This also confirms the necessity of introduction of any kind of 'infinity' through axioms, about which one can 'formally' (hence computably) reason as games of fin-infinity.

33. Let us note that the utility of a theory, which is actually the final arbiter for it, does not require formalization per se, neither does convincing others with arguments about its validity (which is different from discussion settling).

34. The difference between convincing and the stricter 'discussion settling' is that in the latter, we require a total specification of the theory, where no term escapes (implicitly) and no term is unnecessary.

35. The impossibility of syntax-only formalization tells us that we require another structure on which a formal proof is to act, since informal proofs about any single instance of formalization (e.g. syntax only), will necessarily either be informal or self-justifying (which might not be a bad thing).

36. The Formalization Dilemma, or, the impossibility of single-sided formalization.

- We wish to formalize a theory. by formalize we mean 'remove intuition', 'remove subjective burden', allow 'machine representation and manipulation'.
- We model our 'informal' theory into a formal syntactic one.
- To prove that this is what we mean, what we wanted (soundness, completeness, etc.) we have to do it informally.
- We wish to formalize the proof that our formal theory is 'what we mean'.
- We are presented with yet another theory to formalize because of this.
- Ad infinitum, except if we find a self-justifying theory, but that, again does not allow us to show

that it is what we meant since it is hermetically closed.

- In fact, this leads us to **Infinite Discussion Guarantee**. Except for the (provenly not very 'expressive') self justifying theories, there is ground for infinite discussion. In fact, we find an unexpected example of the problem in [@c13]. Of course, the problem is not seen in this clarity by the author and he says: "We wanted to be user-friendly (which often means "sloppy") in the first instance, and said that the above defines a "concept": "Occurs' vs. 'does not occur". In reality, all such "concepts" that we may define by recursion on formulae are just functions. For example, this "concept" can be captured by the function "occurs(p, A)", where occurs(p, A) = 0 means "p occurs in A", and occurs(p, A) = 1 means "p does not occur in A"."

Figure 3: The **Infinite Discussion Guarantee**, illustrated by stacked pots (dicussion settling formalisms), that hopelessly expose a concave volume (yet another 'proof' to formalize) unless a top is used: an agreement outside formalism, per example the acceptance of the consistency of a system in equilibrium as in ZFC.

- For the above we note that even for self justifying theory, their use by humans will necessitate proving that they 'fit exactly' what they are being used for, so even there, the 'Infinite Discussion Guarantee' might hold.

Researching this leads us to:

- 'Proof and Other Dilemmas: Mathematics and Philosophy'
- 'Issues in Mathematical Linguistics: Workshop on Mathematical Linguistics'
- 'Representation and Productive Ambiguity in Mathematics and the Sciences'
- 'A Logical Approach to Discrete Math'

37. Figure from our notebook p. 42.

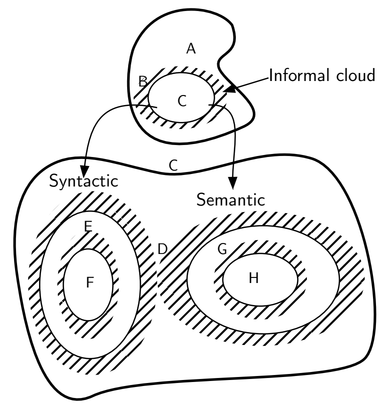

Figure 4: Impossibility of Single-Sided Formalization

38. In the figure, what exactly is in each of A...H?

1. In F, arguably the simplest, we make rules (implicit of explicit) for concrete world building (games).

39. It seems that proving 'soundness' and completeness of 'PL' in the book is merely informally establishing that truth tables and propositional calculus are effectively the same 'thing' drawn in two ways on paper, nothing more nothing less. In other words, truth tables and propositional calculus are two forms of the same 'thing' and here, one could even say the same 'syntax', and this explains our early troubles with seeing the separation into 'semantics'.

40. TODO:As we shall see in the diagram, there appears to arise again the necessity of a choice. We must choose what we mean 'semantically'. ...

41. TODO: http://math.stackexchange.com/questions/173735/how-to-avoid-perceived-circularity-when-defining-a-formal-language

42. TODO: You can't close all your pots if all you have is pots. You need at least one top. of pots and lids: what is the 'structure' of an informal theory. Closing the mathematical lid. The structure of informal mathematical theories. https://books.google.de/books?id=HFa03eq-9LQC&pg=PA43&lpg=PA43&dq=the+nature+of+informal+mathematical+theories&source=bl&ots=BU-6p6DQ5P&sig=OKde-KNq_bPh8sWOsR3-b8jWSvY&hl=en&sa=X&ved=0ahUKEwjyyc7JgLjJAhWD1iwKHSE8CO8Q6AEINzAE#v=onepage&q=the%20nature%20of%20informal%20mathematical%20theories&f=false

43. TODO: "From Frege To Gödel: A Source Book in Mathematical Logic" Contains some nice historic content.

44. TODO: 'Certainty' is a ridiculous goal. One can see that even before doing one line of mathematics.

45. TODO: "The Search for Certainty: a philosophical account of foundations of mathematics"

46. TODO: It's less like a circle and more like a spiral. You use logic #1 to define and study, say, set theory #2. You use set theory #2 to develop a theory of formal logic #3. Formal logic #3 can be used to define set theory #4. And so forth. Logic #5 and logic #3 are not the same thing. There are strong similarities, of course, and this fact can be expressed, e.g., by set theory #2. To some extent, if you're "within" the spiral, it doesn't really matter how far along you are, things will still "look" the same. We then invoke a meta-mathematical assumption: that "real world" mathematics is described by some point on the spiral. Recognizing the spiral isn't just for philosophical issues. If we're using set theory #2, then the actual construction and main applications of formal logic are those of formal logic #3. It is, for example, formal logic #3's notion of "the theory of a group" – not formal logic #1's notion – whose models (in set theory #2) are precisely the groups (in set theory #2). Skolem's paradox is something that can happen when you neglect the spiral: it arises, for example, when you fail to distinguish between the word "countable" from set theory #2 and the word "countable" from set theory #4. But if you are interested in philosophical, foundational issues, the IMO winding twice around the spiral is the right way to go, I think. e.g. maybe you start with meta-logic #1, from which you build an "ambient" set theory #2. Set theory #2 is used to construct "ambient" logic #3, which can discuss set theory #4. Now, you do the rest of mathematics within set theory #4, except for a few special occasions, such as when you need set theory #2 to express the similarity between logic #3 and logic #5 so that you can take statements about what logic #5 says about set theory #6 and try to infer something about set theory #4. In this way, we've insulated ourselves somewhat from dealing with the "real world", and are working entirely within the 'mathematical universe'. But it does require the discipline to be content working in set theory #4 and not inquire too strongly about set theory #2 or meta-logic #1.

47. TODO: Utility as the only judge, but utility is seldom treated in mathematics books, it is scattered all over the world, while lack of utility is seldom explicitly mentioned due to the inexistence of the journal of negative results, let alone of not very fruitful results.

48. TODO: "A Concise Introduction to Mathematical Logic" has a better chapter intro to QL than does @c7. These notes also look good: http://www.math.ucla.edu/~dam/135.12s/135notes3.pdf

49. TODO: "Knowledge Representation and Reasoning" (Brachman, Levesque) has an extremely pragmatic view on the matter, in line with our attitudes, we should consult it for FOL.

50. TODO: "THE ROAD TO MODERN LOGIC—AN INTERPRETATION" seems to fill many missing pieces in our 'mother structure' for logic.

51. TODO: Finally we are able to deal with clarity with [@c8 p. 109, Theorem 8]. It is a perfect example of the usefulness of the 'harmless' completion by appeal to 'the' 'complete ordered field', and we now understand the subtlety of this theorem and its relation to LEM without it blinding us.

\begin{note} QL

1. Why do we need a counterpart at all for syntax in order to find tautologies, can it be done without?

2. There are found unary operators on {T,F}, can negation be interpreted as identity, zero or one? why not?

3. We impose rules on syntax, and we impose an uncertain cloud of more subtle rules by choosing the idea of interpretation based on sets[161].

4. What are the constraints on these interpretations? viz. why cannot they be anything? Given that we can analyze their relation to syntax.

5. This split is probably but one solution. Is it the simplest one?

6. Are 'informal' sets the simplest most abstract structure? Maybe yes for finite sets, are they simpler than any algebraic construct?

7. Why not use 'finite' sets? Because we want to syntaxify mathematics which already includes completed infinities and hence is retrospectively richer than syntax!

The answer to all of this can actually be quite easy if we:

- Do not search for depth where these is none.
- Do not grant any of syntax or semantics a bias of importance or dominance.

To create our machinistic deductive system

1. We choose a language for it (syntax and grammar).

2. We choose what that language is intended to formalize. There may very well be 'other' ways to use the language, but what we intend is the kind of set-theory based modeling and interpretation (as given in [@c7 p.54]). All other 'ways' to use the language do not concern us.

3. We look at the relation between our two tentative choices.

We said that 'at the end of the day' we take a look at our system and see what we achieved. If we look at PL, we see that we have only scratched the surface of mechanization. We have not mechanized the proof that 'Socrates is Mortal'. About mechanization: one could say that a goal of a machinistic system is the ability to create a machine that spits out all possible proofs if let run forever (proven possible: 103}. Of course this is useless in practice, but the ability to create such a machine is definitely one way to summarize the goal of machinization, of formal proof.

- We literally did 'scratch the surface' because our propositions are too hermetic.
- We have not penetrated propositions and formalized 'is', as in 'Socrates is a man'. We could say we have not introduced 'types' since Socrates is of 'type'[2] man.
- To mechanize an even larger part of informal discourse, we penetrate into the propositions of PL, or at least into one very fruitful kind, with QL, and that allows us to formalize actually most of mathematics.
- This explains why the mention of 'expressivity' has always confused us. In fact, expressivity is really about a mechanization of a larger part of the discourse, and that is in the case of the passage from PL to QL, a breaking a certain kind of monolithic propositions into pieces that are amenable to mechanization, in a way no more or less deep that what we outlined above.
- An example of such confusing mention is: "Thus we see that propositional logic is not strong enough to verify the validity of certain arguments and that additional concepts are needed. In this chapter, we will study first-order languages. These languages are more expressive than the language of propositional logic, and their additional expressive power will enable us to..."

\end{note}

[2]

52. We used to think there should be 'some method' of defining what we require from 'interpretations', but this cannot be. Since each really distinct formal language is essentially different, 'what we mean by it' must also be essentially distinct, with no 'template' like what we sought. This should not be confused with intended interpretation. There is an intendended interpretation, and 'other' interepretation, but there is only one 'what we mean'/definition of what makes an interpretation (even if it is a choice, it is a not so arbitrary one).

53. But wait, there are alternative semantics ("Herbrand Semantics" by Genesereth and Kao) (http://logic.stanford.edu/herbrand/herbrand.html): "The traditional semantics for First Order Logic (sometimes called Tarskian semantics) is based on the notion of interpretations of constants. Herbrand semantics is an alternative semantics based directly on truth assignments for ground sentences rather than interpretations of constants. Herbrand semantics is simpler and more intuitive than Tarskian semantics; and, consequently, it is easier to teach and learn. Moreover, it is more expressive. For example, while it is not possible to finitely axiomatize integer arithmetic with Tarskian semantics, this can be done easily with Herbrand Semantics. The downside is a loss of some common logical properties, such as compactness and completeness. However, there is no loss of inferential power. Anything that can be proved according to Tarskian semantics can also be proved according to Herbrand semantics. In this article, we define Herbrand semantics and report in detail on its properties." and "Model theory in Herbrand logic is much simpler than it is in first-order logic.".

54. Maybe this is related? https://www.yumpu.com/en/document/view/21957330/the-herbrand-topos

55. TODO: this has simple pictures: [3] (The Herbrand Manifesto)

56. Tautology in QL: continue here ("A First Course in Logic, An introduction to model theory, proof theory, computability, and complexity" p.65) "In these examples we are able to show that certain formulas are satisfiable by exhibiting structures in which they hold. Using this same idea, we can show that a given formula is not a tautology, that one formula is not a consequence of another, and that two given formulas are not equivalent. How- ever, we have no way at present to show that a formula is unsatisfiable, or a tautology, or that one formula is a consequence of another. This is the topic of Chapter 3 where we define both formal proofs and resolution for first-order logic."

57. Why can't we reinterpret the formal language in any way we like? The definition of what an interpretation is does not bar reinterpretation of the logical symbols?

1. The question is not moot, a quite relevant book dealing with Kripke and skepticism is: "The Taming of the True" (Neil Tennant). This reminds us of the concept of 'private language' and Wittgenstein's 'meaning is use'.
2. Interesting is Dummet's Harmony theory and Belnap's Tonk, Plonk and Plink https://en.wikipedia.org/wiki/Logical_harmony. It betrays hidden constraints and choices.
3. Coming back to alternative Semantics, we find that it really IS a choice, we find here a list https://en.wikipedia.org/wiki/Truth-value_semantics:
   - Game semantics
   - Kripke semantics
   - Model-theoretic semantics
   - Proof-theoretic semantics
   - Truth-value semantics
   - Truth-conditional semantics
   - Tarskian semantics
4. In the above article, we also read the common point of view that the lack of compactness is a 'disadvantage' (this is biased). "Truth-value semantics is not without its problems. First, the strong completeness theorem and compactness fail. To see this consider the set F(1), F(2),.... Clearly the formula □xF(x) is a logical consequence of the set,

[3] https://conference.imp.fu-berlin.de/cade-25/uploads/2015_CADE_ruleml_genesere.pdf

but it is not a consequence of any finite subset of it (and hence it is not deducible from it). It follows immediately that both compactness and the strong completeness theorem fail for truth-value semantics. This is rectified by a modified definition of logical consequence as given in Dunn and Belnap 1968."

5. Relevant resources here are:

   - "The Adventure of Reason: Interplay Between Philosophy of Mathematics and Mathematical Logic"
   - "Radical Interpretation and Indeterminacy" (maybe not too relevant).
   - These two books talk about (the nontrivial) topic of nonstrandard intepretations of FOL quantifiers, and the probable indeterminacy of that, and the proven indeterminacy of that in the case of SOL. Here, reinterpretation is not as 'free' as we were thinking, so this is too advanced for the moment. At the same time, the first book gives us too excellent words to use 'orthodoxy' and 'heretic'. What we have been having problems with is in fact implicit 'orthodoxy'. The books are

     (a) "Quantifiers, Quantifiers, and Quantifiers: Themes in Logic, Metaphysics, and Language" (p.66,173). (This could turn out to be a real gem in answering our questions of 'free interpretation').

     (b) "ON THE GENERAL INTERPRETATION OF FIRST-ORDER QUANTIFIERS" http://www.aldo-antonelli.org/Papers/Genint-RSL.pdf. We quote:

       – "It appears that the idea of providing nonstandard interpretations for the first-order quan- tifiers, even in a generalized setting, first made its appearance with Thomason & Johnson (1969). Arguably, Thomason & Johnson were the first to combine the generalized view- point (consideration of quantifiers other than $\forall$ and $\exists$) with the nonstandard approach (consideration of models that in some way or other "liberalize" the interpretation of the quantifiers). Their approach is both more and less general than the present one. It is more general in that they consider ordinary first-order logic augmented with quantifier variables (while the present approach deals with quantifier constants): accordingly they consider what formulas $Qx\varphi(x)$ are valid under all interpretations of the quantifier variable Q. As it turns out, doing so introduces intrinsically higher-order features in the semantics, as shown by Väänänen (1978) (improving upon previous results of Yasuhara (1969)). But the approach of Thomason & Johnson is also less general than the present one in that they restrict themselves to quantifiers that satisfy a form of permutation invariance, and moreover (as mentioned) only in the context of logics extending ordinary first-order logic. The ordinary $\forall$ and $\exists$ are required to express invariance under permutation, but in their absence the logic of quantifier variables becomes much weaker—decidable, in fact (see Anapolitanos & Väänänen, 1981). The notion of permutation invariance is also used by Thomason & Johnson to identify a class of nonstandard models, by allowing models in which the interpretations of the quantifier variables are not required to be invariant under all permutations, but just those in a given collection."

  - The mentioned paper above gives us the following gem, a definition of ’quantifier’: “Since a genuine quantifier should have to do with quantity,it should not depend on properties of the various individuals that happen to belong to a given set. Following Mostowski in [4], we make this precise as follows. A quantifier on a domain D is a function q from $\mathcal{P}(D)$, the power set of D, into $2 = \{0, 1\}$ which is invariant under permutations of D.”

- Quite intriguing is: ”Logic and Games on Automatic Structures, Playing with Quantifiers and Decompositions”
- ”Universal Logic: An Anthology”, which promises “ investigate the domain of validity and application of fundamental results such as compactness and completeness”.
- One review of the above book confirms our interest since it seems the book directly targets pluralism and explicit but not arbitrary choice. It says: “The quest for common ground in the expression of logical statementsregardless of the given syntax, semantics, or grammarhas taken the best efforts of notable logicians throughout the 20th century. This book gathers a collection of some of the most notable of their works. Its structure is chronological, but it is not merely a book on the history of universal logic. Each major work is discussed by eminent contemporary researchers, providing context on why a particular idea in logic came into existence. For example, Rougier’s 1941 work on logical pluralism is discussed by Marion, who traces the context of origin of this work to Lewis’ alternative systems of logic, or Carnap’s logical syntax of language, around the mid1930s. This leads to the question of whether it is possible to lead away from logical monism, and instead embrace the possibility of different logical systems, which, though free, are not arbitrary, since they must comply with the domain of facts to which the research is relevant. Dana Scott’s discussion on completeness and axiomatizability in manyvalued logics is tackled by Humberstone, who argues that it is actually a “finitized” version of the multipleconclusion logics, inspired by the logics of Łukasiewicz. Other major works either discussed or mentioned include those of Russell and Whitehead (from Principia), Gödel, and Dov Gabbay (his 1996 work seeking to combine logical systems, which is relevant for software engineering). I highly recommend this book to logicians, mathematicians working on provability theory, and software engineers.”
- Maybe Troulakis comes to the rescue here with a chapter that starts with “This chapter is on naive semantics. That is, we see here what these abstract, ”meaningless”, strings—the first-order formulae—actually say. Specifically, we will see what it means, and how, to compute truth values of such formulae. 1 would like to emphasize the qualifier naive. In the more advanced literature one defines semantics rigorously: either defining the truth or falsehood of formulae within informal mathematics—that is, in the metatheory—a process originated by Tarski and nicknamed Tarski semantics, or defining the truth or falsehood of formulae of the original language within some otherformal theory, T, possibly over a different language, in which case one speaks of formal semantics. In formal semantics, formulae of the original language are first translated into formulae over the language of T. Then one defines that a formula in the original language is ”true” iff its translation is provable in T (cf. [45, 53, 54]). Here we do neither, but instead imitate Tarski informal semantics, albeit in a very simple and purposely sloppy (8.1.2) manner. But this will do for our purpose, which is to learn to easily build counterexamples that expose

fallacious statements in predicate logic."

6. It could simply be that we were confused, and that interpretations and the freedom therein exclude connectives, which are not 'interpreted' but must mean 'what we mean' and not something else. This is again a kind of an informal constraint since how do we 'prove' that within a certain interpretation, 'what we mean' by the quantifiers per example in the interpretation, is 'what we mean' in general.

58. One answer to an old question of ours, which resolves 'interpretations' such as q-addition (we retain the good keyword 'deviant logic', 'deviant interpretation'), and allowing the kinds of constructions that make any 'compression' possible at all (we do not have to write out S(S(...) 100 times to 'prove' that it actually does match '100') our new rule: **"The No Abstract Leprechaun Assumption"**. Of course, this is only allowed when we are delineating a system (formal or not), and the whole problem with 'certainty in general'[44] being a ridiculous concept is exactly that in the real world, there can never be a proof of that assumption. This would be the **"No Real Leprechaun Disproof"**. In fact, this rules can serve as the cornerstone of a proof that induction works (proofs of this always beg the question).

Figure 5: The Leperchaun: can be assumed away (think Euclid's common notions) but never really disproved.

I.3.1 Resolution of Our Confusion Regarding Interpretation

59. Indeed, Tourlakis confirms our finding that, unlike what we thought, the quantifiers in FOL are not to be 'reinterpreted'.

60. As part of the translation from the formal to the interpreted: "(iv) We replace each occurrence of $(\forall x)$ in A by $(\forall x \in D)$, which means "for all values of x in D"."

61. We finally notice the 'subtlety' with set theory where membership is not a logical symbol, and hence can be 'freely reinterpreted' "(viii) We emphasize once more what was left unsaid in the transformations (i)-(vii) above: Every Boolean connective, $=$, and brackets are translated as themselves."

62. The above is affirmed by where we see an example of a 'free interpretation' of membership: " As a frivolous example, we may let the domain of discourse be the set Z of integers,and interpret $x \in y$ as $x < y$. This is a legitimate interpretation for the language of set theory, even though the sentence $\forall x \exists y (y \in x)$ is true under this interpretation but refutable from ZFC. Of course, not all the axioms of ZFC are true under this interpretation. In the intended interpretation, under which the axioms of ZFC are presumed true, $x \in y$ is interpreted to mean that x is a member of y, but the domain of discourse is somewhat harder to describe."

63. We see here that our confusion was justified. It is 'chosen' that 'nonlogical' symbols are up for interpretation, while 'logical' ones are not. And it is 'chosen' what is to be called 'logical' and what not. All these hidden choices totally confuse any mind that is fully unbiased.

64. Why Don't We Talk About Models in PL? Again here, we have been confused. When we were 'designing' PL using truth tables, to create a formal deductive system, we did not then go on to see which other semantics could 'in some way' fit our formal system, where the logical connectives are 'freely reinterpreted'. Now we understand this. But, why don't we speak of 'models' and 'interpretations' in PL? Simply because in PL, we have not 'looked inside' propositions. Hence, interpretation plays no role whatsoever, the propositions being fully opaque. But this does not answer everything. In other words: 'We mostly don't reinterpret the logical symbols'. We have to be very careful with the preceding sentence, by reinterpret, we mean that very informally, and not in the sense of QL 'interpretation'. 'Mostly', that is, with the exception of studies such as 5.

65. In PL, we extracted inference rules from semantic tautologies. Can't we do the same for QL? Is this exactly Tarski's truth? I suppose that we cannot easily capture the tautologies due to infinity, and hence try to capture them more elegantly with Tarski's definition? It may very well be, and it probably is formulated to be more general than the needs of QL alone, is that why it is the way it is? The answer to this is quite simple. Tarski's definition is not restricted to the universal quantifier, but its purpose is to tickle out all tautologies [@c13, p.202), and from that point of view, it is the right definition.

66. Despite the above, it remains to be said that a heretic feels there is some 'orthodoxy glue' at play, since we have not defined exactly which interpretations are allowed and which are disallowed. Probably, interpretations which 'reinterpret' quantifiers such as $\forall$ (in the case of FOL), are barred from being called 'interpretations'. This 'orthodoxy glue' is explained (without calling that) in [@c13, p.202-204]. Per example, we see how in 'Valid Axioms 2', the quantifiers are not to be reinterpreted to anything else than then mechanism of substitution, all while this is nowhere explicitly required of an 'interpretation'. In this sense, the 'orthodoxy glue' is simply the fixation of quantifier being exactly substitution and nothing else. The heretic must abide to a rule that is nowhere a definition or an axiom; or is it? The next step in this note is to search for it. We find our 'orthodoxy glue' as part of two definitions: 8.1.1 and 8.1.2 these definitions go under the title 'Interpreting a language, translating the alphabet and the formulas'. In Hodel [@c14], such definitions occur (in not as much clarity for our puprose here) in section 4.4.

67. (We could, as an exercise, try to force fit Taski's definition into PL.)

68. At the end of this Anschauung, it seems that all our current confusions are resolved, and Tourlakis emerges again as the clearest and most resonating, while Hodel looks very good for support with explicit practical examples.

II Anschauung 4

69. See the accompanying documents 'PL Flavors' and 'PL Short'.

70. Exactly during trying to put in terse or formal form what a syntactic deductive system is, we were struck by the difference of this undertaking as opposed to describing grammars or semantics simply using sets or BNF notation. Researching this difficulty leads us to finally ask the right questions that unlock many doors on many levels, even outside formal logic.

71. This problem is actually solved by 'abstract algebra', and it is sad that only now we understand what 'abstract' is. Abstract is the contrary of concrete (we shall call this the abstractness misconception). Very importantly, this means that the analysis of abstract things does not pass through analysing their (concrete) elements, but through analysing the properties of the abstract things, that is by analysing their axioms. The properties (axioms) are the concrete currency of abstract algebra.

72. The above finally opens the door to usefully understanding 'interpretation', lattices (and order) and their importance: despite us passing over this ground multiple times before, without the critical mental change above, it was all useless and confusing.

73. Accessible now become Algebraic logic. Even though not exactly to the point, a great introduction is

74. Here are some examples that contribute to a clear removal of the misconception:

- a

75. The 'abstractness misconception' is such that one thinks one 'understands' abstract algebra (or mathematics), per example by thinking that 'there is nothing abstract about them', while this is totally not the point. The attribute abstract is fully justified and precise.

76. We now know that this 'abstractness misconception' is all but human, the mathematicians who contributed here are: .Of course as usual, the theories are then studied for their own sake, in a totally anti-pedagogical manner.

77. What is the mental representation of set theory for somebody who has not understood the 'abstractness misconception'? It is this:

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$$

. But how is this any different from

$$0, 1, 2, \ldots$$

? It is not. The concrete currency of 'abstract' mathematics are the axioms. They are what needs to be essentially considered, at least as essentially as 'represe-ntiations'.

78. Note that in 'Algebra', the treatment of boolean algebra is very short. Why? Because we have the misconception that 'algebras' can be surveyed, while they mostly can not. It is better to focus on the cases at hand, and have a map of properties and theorems of the more basic objects, those in 'Algebra'.

79. As an example of the 'abstractness misconception', we can finally read https://en.wikipedia.org/wiki/Lattice_(order) properly, differentiating between the two sections as properly indicated by the remark: "In mathematics, a lattice is a partially ordered set in which every two elements ... Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order theory and universal algebra." This would have only triggered a 'OK' acknowledgement from the misconpeting mind. Now we understand how important this is.

80. TODO copy the tables from https://en.wikipedia.org/wiki/Lattice_(order) and https://en.wikipedia.org/wiki/Binary_relation

81. We note a vague parallelism that we thought of some time ago: The 'coordinates' are the concrete things of 'coordinate free' things.

82. Additional research on this topic leads to this scatter-map:

1. Initially, we find this shocking statement [@c20]: "Alfred Tarski in 1953 formalized set theory in the equational theory of relation algebras [37, 38]. Why did he do so? Because the ***equational theory of relation algebras (RA) corresponds to a logic without individual variables, in other words, to a propositional logic***. This is why the title of the book [39] is "Formalizing set theory without variables". Tarski got the ***surprising result that a propositional logic can be strong enough to "express all of mathematics"***, to be the arena for mathematics. The classical view before this result was that propositional logics in general were weak in expressive power, decidable, uninteresting in a sense. By using the fact that set theory can be built up in it, Tarski proved that the equational theory of RA is undecidable. This was ***the first propositional logic shown to be undecidable***."

2. So are there 'propositional logics' (plural)? What makes a logic 'propositional'? Although not directly to the point, the survey [@c21] is basic, but very useful: " It is possible to investigate a logic as an algebraic structure, the properties of that structure giving insight in to the logic itself. We will discuss that ***connection between Boolean algebras and the system of classical propositional logic***. We will also introduce the idea of different logical systems, and investigate their algebraic equivalents as well."

3. We finally find a survey [@c22] that treats the ***Lindenbaum-Tarski algebras*** in detail. Here we read

    - "In contrast to ***Boolean, cylindric, polyadic, and Wajsberg algebras*** which were defined before the Lindenbaum-Tarski method was first applied to generate them from the appropriate assertional systems, Heyting algebras seem to be ***the first algebras of logic that were identified by applying the Lindenbaum-Tarski method to a known as-***

sertional system, namely the intuitionistic propositional calculus."

- "***Traditionally algebraic logic has focused on the algebraic investigation of particular classes of algebras of logic***, whether or not they could be con- nected to some known assertional system by means of the Lindenbaum- Tarski method. However, when such a connection could be established, there was interest in investigating the relationhip between various metalogical properties of the logistic system and the algebraic properties of the associated class of algebras (obtaining what are sometimes called ***"bridge theorems"***). For example, ***it was discovered that there is a natural relation between the interpolation theorems of classical, intuitionistic, and intermediate propositional calculi, and the amalgamation properties of varieties of Heyting algebras***. Similar connections were investigated between interpola- tion theorems in the predicate calculus and amalgamation results in varieties of cylindric and polyadic algebras."
- "Although interest in the traditional areas of algebraic logic has not diminished, the field has evolved considerably in other directions. The ad hoc methods by which a class of algebras is associated to a given logic have given way to a ***systematic investigation of broad classes of logics in an algebraic context***. The focus has shifted to the process by which a class of algebras is associated with an arbitrary logic and away from the particular classes of algebras that are obtained when the process is applied to specific logics. The general theory of the algebraization of logical systems that has developed is called Abstract Algebraic Logic."
- "One of the goals of AAL is to discover general criteria for a class of algebras (or for a class of mathematical objects closely related to algebras such as, for instance, logical matrices or generalized matrices) to be the algebraic counterpart of a logic, and to develop the methods for obtaining this algebraic counterpart. In this connection an ***abstraction of the Lindenbaum-Tarski method plays a major role***."
- "Bridge theorems relating metalogical properties of a logic to algebraic properties of its algebraic counterpart take on added interest in the con- text of AAL. For example, it was known for some time that there is a close connection between the deduction theorem and the property of a class of algebras that its members have uniformly equationally definable principal congruences, but it is only in the more general context of AAL that the connection can be made precise. Indeed, the desire to find the proper context in which this connection could be made precise partly motivated the development of AAL."

4. 'Systems of Formal Logic' (Hackstaff) seems to be a very soft introduction into the topic and might be worthwhile.

5. 'The Mathematical Structure of Logical Syntax' (Jean-Yves Béziau) finally treats ***zero-order forumlas*** and ***absolutely free algebra*** in some detail, this could be our ***ground-zero***. We study this in II.1

83. The continuation of this Anschauung consists on the elaboration of this scatter-map before going back to the original goals.

II.1 Logical Wisdom with (Béziau)

84. TODO: tabulate the explained confusing choices of words 'word', 'sentence', etc.

85. The author hits it on the nail with the difficulties, for an 'ordinary man', of a recursive definition, and he is the only one who explains that, as we expect, the need for a ***limitation clause*** was only realized when problems occured. He says: "It is not obvious that this definition, as simple as it is, can be understood by an ordinary man, with no mathematical practice. Such an ordinary man may have some difficulty to understand the recursive process involved in (L2) and he will perhaps not see the necessity of (L3) (Hisiorically (L3) was at the beginning not mentioned, see section 7). There is a more intuitive definition (DST) which is a construction by stages."

86. He also correctly acknowledges the non-obviousness of a bottom-up (by levels, the way a human untrained constructive mind works) and top-down (which needs 'minimal set'-like clauses) by saying "Following Enderton's terminolory (cf. [END], we can say that the first defini- tion (DLI) is from the top d'oun and the second (DST) from the bottom up. To see that these two defintions are equivalent is not at â.ll obvious without entering into some mathematical considerations."

87. "We analyse the various ways of constructing the set of zero-order formulas (i.e. propositionâl or sentetrtial formulas) : from the intuitive definition, based on a simple linguistic notion (combination of signs), to the abstract definition of absolutely free algebrâ. Col1necting this last concept vÿith Peâno axithmetic, we show why ***the set of zero-order formulas cannot be axiomatized in first-order logic*** and explain how this can be used against the formalist approach to logic and mathematics. Finally we try to invetigate the ***historical development of the conception of the set of zero-order formulas***"

88. "A first course of logic usually begins with what is viewed as the mo6t elementaxy paft of logic, namely classical propositional logic. And such a course begins with the definitiou of the set of formulas. Most people, even philosophers v/ithout âny mathematical background, can understand this definitiou, mainly because it is based on a linguistic intuition : formulas are presented as well-formed expressions built over a,n alphabet, and the set of formulas is commonly called the "lânguage" , and considered as paxt of "logicâl syntax". However there is a very big gap between this intuitive definitior and the math- ematical concept which is behind it, the concept of absolutely free algebra. In farct, though this coucept is used by Polish logicans since at least 40 years (cf. [LOS]), most logicians (except those fa.rniliar with Polish logic or working in algebraic logic) outside Poland don't kno r it and don't use it. Such notion is, for example, never mentioned by philosophers of logic. They still work only v/ith the intuitive linguistic approach in particular when they deal with the ontological problem "sentence versus proposition". Here, to avoid ontological commitment, we will used the neutral PoLish terminology "zeroorder formulas" and "zero-order logic Nevertheless we will try to explain the philosophical import of a more sophisticated view on the set of zero-order formulas. 'We will show that the set of zero-order formulas cannot be axiomatized in first-order logic a.nd explain how this result may dismiss the formalist approach to logic and mathematics."

89. ""'Many contemporaxy logicians adopt a formalist view because they do not want to be bothered with philosophical questions""

90. This is golden, proper wisdom, and hints at 'induction', which by now we now is the 'one-and-only' mathematical abstraction of constructive algorithms (that is, human-constructive-mind-like). He explains "These definitions are a ***strange mixture of mathematical and non-mathematical material***. They is already a strong idealization of informal linguistic intuition. Notions such as infinite, recursion, set, union, intersection, etc., appear. Of course these notions can be understood at the intuitive level, but very soon we need more precise mathematicâl concepts. Even for example if we want to prove a simple result saying that any formula has the same number of right and left parentheses. Someone with no idea of what is mathematical induction would not be able to rigorously prove this property."

91. Here we must add our old realization that mathematical proofs that 'induction itself works', as we know, 'prove nothing', and can only be salvaged by mixing a meta-level with a bit of glue, that is, as the author says, a 'strange mixture'.

92. The death of the WFF is beautifully announced with a fair accusation of pleonastism and a bit of history: "This is the English translation of the French terminology 'Expressions bien formées', which is abbreviated by EBFs. In English people use the terminology 'Well-formed-formulas', abbreviated by WFFs. However this terminology is quite absurd because formulas are generally not considered as any combinations of signs (i.e. words or expressions). Therefore this terminology is ***pleonastic***."

93. Intriguing are the author's books 'The Square of Opposition: A General Framework for Cognition' and 'Around and Beyond the Square of Opposition', along with the paper 'The new rising of the square of opposition', which explains: "my interest for paraconsistent logic was based on questions regarding the foundations of logic rather than by a childish attraction to paradoxes. If the ***principle of (non) contradiction is not a fundamental principle, what are the fundamental principles of logic, if any?*** How is it possible to reason without the principle of contradiction? Those were the questions I was exploring." From our primitive point of view, 'logic' is also just a part of what is being 'translated' (that is, modelled), if that what is being translated functions using a certain 'logic' then this logic is the logic of what is being modelled. Everything else is simply human 'coining'.

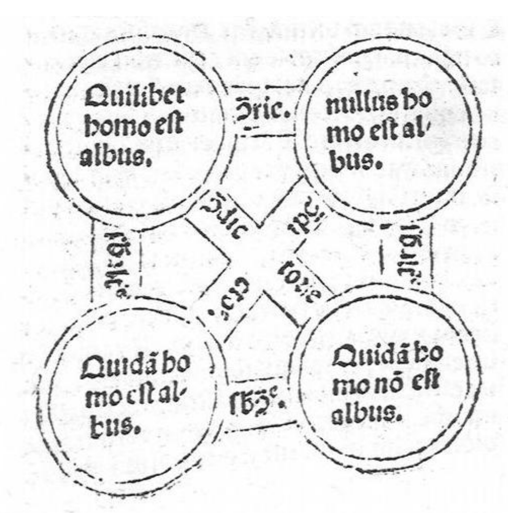

Figure 6: Artistotle's square of opposition [4]

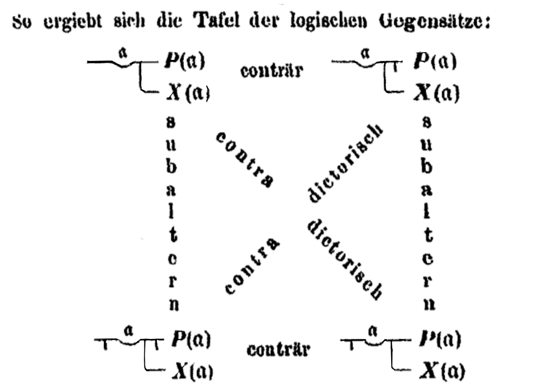

Figure 7: Frege's square of opposition The conträr below is an erratum: It should read subconträr [5]

94. We can say even more about induction than we said in note (91) by looking at the author's set theoretical definition of 'well-formed-formulas'. It is probably so that using induction along with a limitation clause is the 'one-and-only' way to ***formally define*** constructive infinite sets ***within set theory***, without recourse to any meta mixture. There is nothing wrong with this, just like there is nothing wrong with 'induction' having been chosen as the 'one-and-only' formal set-theoretic notion of 'algorithm', these single choices are necessary and sufficient.

95. Nice is the author's use of a more concrete intersection defintion instead of 'smallest set' for a 'limitation clause' in his construction where he says: "(SC3) Limitation clause. The set of formulas is the intersection of all sets constructed with clauses (SC1) and (SC2), i.e. sets of proto-formulas."

96. The author puts ***substitution*** (of 'letters of propositions' meaning 'variables letters') and ***unique readability*** in their proper place, highlighting the hidden essential character of the latter. He again ***criticizes the unique readability's standard explanation***. He says: "If parentheses were not used in the definition (DSC), then the two formulas $(F \wedge (G \wedge H))$ and $((F \wedge G) \wedge H))$ will be the same formula $F \wedge G \wedge H$, i.e. one and the same sequence. Lukasiewicz has introduced a more "economical" (...) method which permits to distinguish these two formulas (...). Whitehead

[4]"Johannesmagistris-square" by Peter Damian - Own work. Licensed under Public Domain via Commons - https://commons.wikimedia.org/wiki/File:Johannesmagistris-square.jpg#/media/File:Johannesmagistris-square.jpg

[5]Halle: L. Nebert. Diagram from p24 of book. Describes "square of opposition" for his system. Taken from purely mechanical scan performed via gallica.bnf.fr; some defects of the scan repaired by hand by myself.

and Russell were using also a . "notation" without parentheses but with dots (due to Peano). ***For all these "notations"*** it is possible to prove that "there is only one way to waite the the same formula". ***This confuse formulation of the confusely called "unique readability theorem" can be properly understood only in the light of algebraic tools***. (...) ***behind this question of "notation" is hidden a mathematical property which is the real essence of the architecture of the set of zero-order formulas***."

97. The author beautifully and intuitively characterizes the concept of an ***absolutely free algebra*** and links the ***zero-order formulas*** to it by explaining: "The set of zero-order formulas defined by (DSC) has a fundamental property which can be roughly described as follows: ***If we associate values to atomic formulas, and we have a process which tells us what is the value of a compound formula when the values of its direct components are known, then we can associate values to all formulas. This property corresponds to the concept of absolutely free algebra***."

98. The author goes on to emphasize that the 'URT' is nothing but of a proof of 'UFA': " ***The "unique readability theorem" consists just in proving that the set of zero order formulas constructed with (DSC) is an absolutely free algebra***. It turns out that it is nothing more than that : a.ccidental properties of (DSC) related for example to the concept of sequence can be forgotten without any trouble".

99. In effect the author is suggesting that all 'notations' are 'representations' and that there is an essence outside of those notations, and that essence is not an 'arbitrary choice of one of these notations'. This is possible exactly because the concept of a UFA abstracts the form away, and leaves the property (which is common to all notations, used unused or undiscovered).

100. The author baffles us with 'universal free algebra'. A study of the possible more basic concept of 'free group' leads us to groups, generators, relators, the 'word problem' and back to Tarski with the following possible references:

- 'Algebraic Methods of Mathematical Logic' (Rieger)
- 'The Geometry of the Word Problem for Finitely Generated Groups'
- 'The Word Problem And The Isomorphism Problem for Groups' (Stillwell)
- Also, [@c18]

101 (@c18)**.** seems to be our best first resource for 'universal algebra', 'relational algebra' and 'world algebra'. It even has a section of concrete free algebras and how to think about them.

102. TODO: follow [@c18]

III Mathematical Logic with (Tourlakis)

III.1 Miscellanea

103. PL is decidable but intractable, QL is undecidable and although a Pascal program can if run indefinitely, produce all 'theorems of logic' (p. 7). In any case, this is a good hook into computability, which for the first time, does not appear so conceptually unreachable. In fact, we might use Tourlakis' book for that.

104. Goedel's theorem happens on the meta-level, at least as introduced by Tourlakis.

105. The main, and perhaps only, problem to solve on the syntactic level, is tightly related to the main feature of 'discussions': their infinite potential extensibility, the passage from T1 to T2[21]. Given our usually strict rules for syntax, induction on complexity is of great help and solves all(?) problems. We have to see this process as unfolding the syntax tree and explaining why (in the absence of Leprechauns) some property or definition holds at every branch without having to enumerate all possible trees and branches (which is impossible).

106. It is best, when considering the mental movement to 'semantics' after treating 'pure syntax', is to imagine metatheoretical constructs, hooking onto 'any' branch in the syntactic tree, and relating it to some 'semantic' construct, and doing that usually using induction in order to handle 'meta-formally' the fact that the syntax is a 'complex' tree (difficulties which might be alleviated with an infix notation and the absence of a unary operator, but the induction of course is essential and will still be there). The author's discussion about "But why the fuss of assigning the values f and t to the (formal) variables and constants?" is certainly endearing and confidence building, but we find that by our whole previous analysis, the answers are even simpler than the ones the author gives.

107. Given the above, snippets like this one do not bother us at all anymore: "v(p) = whatever we originally assigned to p; v(T) = t v(F) = f; v(!A)= F!(v(A)); v(A*B) = F*(v(A),v(B)); ... ". Here v and F are 'funny' functions, with domain in the syntactic alphabet, and codomain in the semantic world however that is seen (metatheory, set theory, informal, ...).

108. Caution: Tourlakis' 'state' can be confusing. A state 'v' fixes the values of all variables. At the same time, a state is used as a 'function' acting on syntactic formulas in a manner that is inductive. In this sense, it would be better to use notation such as v_i for some i to distinguish between states. Of course, this could lead to yet more explanations by the author since he insists there are in general infinitely many variables, and hence, v is an infinite list/set of tuples, a countable function? This caution leads us to a subtlety. Tourlakis never really 'defines' the state 'v', only its usage. He is shying away from it due to his belief it cannot it properly 'formally' captured in this context, and refers to his set theory book for that. This is a bootstrapping problem, and he refuses to go for a bootstrapping 'naive set theory', or anything equivalent that would be necessary to capture the definition of 'v', at least informally and metatheoretically. To formally 'functions' some kind of set theory is needed, which would turn Tourkalis's enterprise into a 'graph in equilibrium', rather than a 'tree with foundations'. But with our current knowledge, we can live with this.

109. Continuing the above note and analysing the path of his proofs in [@c13b] reveals that we are right. His defintion 'I.3.4 Definition (Propositional Valuations)' requires (as he only explains in a footnote) his metatheorem 'I.2.13 Metatheorem (Definition by Recursion)' which mentions sets:"Let (T,R) be unambiguous and Cl(T, R) in A, where A is some set. Let also Y be a set, ..." and indeed at the beginning of the containing chapter he says: "It is inevitable that the language of sets intrudes in this chapter (as it indeed does in all mathematics) and, more importantly, some of the results of (informal) set theory are needed here (especially in our proofs of the completeness and compactness metatheorems). Conversely, formal set theory of volume 2 needs some of the results developed here. This "chicken or egg" phenomenon is often called "bootstrapping" (not to be confused with "circularity" – which it is not (Only informal, or naive, set theory notation and results are needed in Chapter I at the meta-level, i.e, outside the formal system that logic is.)), the term suggesting one pulling oneself up by one's bootstraps. (I am told that Baron Münchhausen was the first one to apply this technique, with success.)"

110. This is needlessly confusing, making unnecessary and biased claims: "Boolean logic is primarily interested in those formulae that are true (t) in all possible states." ([@c13 p.32])

111. The definition of satisfiability ("1.3.11 Definition. A formula A is satisfiable iff ... A set of formulae Γ is satisfiable iff ..."), linking syntax and semantics (since we use the semantic 't' and not the syntactic 'T') has been, it seems, a main blocker to our progress. The purpose of it is the following: Since our goal is deductive systems, we must define 'consequence', and this must within the syntactic-semantic glue since it involves (semantic) truth. Now tautology is 'absolute truth', while consequence, intuitively, is 'relative truth'. The whole point of the definition is defining exactly this within the glue, so that we can use the definition to procede to our deductive system, which is, after all, a system of discussion-settling 'consequence'.

112. Keeping in mind Tourlakis' comments on pages xiii, xiv is crucial towards working his book properly. Among other things, he says

1. "While the approach in this volume is truly formal, just like Bourbaki's, it is not as terse; we are guilty of the opposite tendency! We also believe that, unlike the seasoned practitioner, the undergraduate mathematics, computer science, and philosophy students need some reassur-

ance that the form-manipulation proof-writing tools presented here indeed prove (mathematical) ”truths”, ”all truths”, and ”nothing but truths”. This means that we cannot run away from the most basic and fundamental metatheoretical results. After all, every practitioner needs to know a few things about the properties of his tools; this will make him more effective in their use.”

2. “Why are soundness and completeness relevant to the needs of the user? Complete- ness of prepositional logic, along with its soundness, give us the much-needed—in the interest of user-friendliness—license to mix semantic and syntactic tools in formal proofs without sacrificing mathematical rigor. Indeed, this license (to use propositional semantic tools) is extended even in predicate logic, and is made possible by the trick of adding and removing quantifiers (”for all” and ”for some”). On the other hand, soundness of the two logics allows the user to disprove statements by constructing so-called countermodels.”

3. “For the above reason, equational-style proofs receive a thorough exposition in this volume. It is my intention to endow the reader with enough machinery that will make him proficient in both styles of proof, but more importantly, will enable him to choose the style that is best suited to writing a proof for any particular theorem.”

113. What exactly does p.79 break in our current understanding? did we not use any ’tautologies’ to derive our other rules of inference Inf1 and Inf2? A possible answer lies in [@c16 p.21], which ’defines’ a deductive system, and removes our bias that it ’has’ to be constructed from rules that are derived themselves from tautologies. In fact, we ourselves said this before, the goal is ’discussion settling’ while only work in the ’syntactic’ and in the ’true’. How we do this does not matter, and this explains that Inf1 and Inf2 do not need to be ’functional’. The quote we refer to is: “We now return to the syntax of P, and define first the class of axioms of P. What we want to do is to distinguish a class of formulas of P from which all of the tautologies of P (and no other formulas) can be derived, by means of applying certain rules of inference which we shall subsequently specify. Once we succeed in doing this we will have two ways of showing that a given formula A is a tautology; viz., by means of the truth-table test, and by means of deriving A from the axioms of P.”

114. The above debiases us even more, we must push forth our ’game’ and ’utility’ project. The history of all formal systems of all possible ’logics’ and ’mathematics’ might be too cumbersome to ’learn’, we must instead look at the even bigger picture if possible. Nevertheless, while Tourlakis will help us choosing one system, learning it, and understanding Goedel’s results with it, [@c16] promises a soft and contrasting survey. We should try to create a diagram out of it if possible at all.

115. TBC

III.2 Equational Logic

116. It seems to us that Tourlakis has read [@c15] quite carefully.

117. The DS system is quite enlightening in a very specific manner: the choice of its inference rules. TBC.

118. The two inference rules that Tourlakis adopts are in fact, related to the Equational Logic of Dijkstra and Scholten (without Tourlakis saying it). This is made clear in the paper ’The Formal System of Dijkstra and Scholten’ (Rocha). Gries in [@c15] and ’Equational logic: A great pedagogical tool for teaching a skill in logic’ is obviously also a proponent. Rocha says: “Rule Equanimity is the stronger version of the traditional Modus Ponens but it is based on equivalence. Rule Leibniz enables the above-mentioned substitution of ‘equals for equals’ in DS. ”. Rocha also gives some context: “The logic of E. W. Dijkstra and C. S. Scholten has been shown to be useful in program correctness proofs and has attracted a substantial following in research, teaching, and programming. However, there is confusion regarding this logic to the point in which, for some time, it was not considered a logic, as logicians use the word. The main objections arise from the fact that: (i) symbolic manipulations seem to be based on the meaning of the terms involved, and (ii) some notation and the proof style of the logic are different, to some extent, from those found in the traditional use of logic. This paper presents the Dijkstra-Scholten logic as a formal system, and explains its proof-theoretic foundations as a formal system, thus avoiding any confusion regarding term manipulation, notation, and proof style. The formal system is shown to be sound and complete, mainly, by using rewriting and narrowing based decision and semi-decision procedures for, respectively, propositional and first-order logic previously developed by C. Rocha and J. Meseguer." and then "I feel deeply grateful to Jose Meseguer for our friendship and his tutelage, en- couragement, and support during our collaborative research. In honoring him for his 65th birthday, I have chosen a topic close to our common interest in logic, algebraic specification, and mechanical theorem proving. Mechanizing the logic of E. W. Dijkstra and C. S. Scholten was the first topic of research during my Ph.D. studies under Jośe's guidance starting in the autumn of 2006; it was also a fruitful 'excuse' to learn and use Maude as a formal tool for the first time."

119. The reason for the name 'Leibniz Rule' is indeed directly attributable to Leibniz, according to Gries: "Def. 1. Two terms are the same (eadem) if one can be substituted for the other without altering the truth of any statement (salva veritate). If we have A and B, and A enters into some true proposition, and the substitution of B for A wherever it appears results in a new proposition that is likewise true, and if this can be done for every proposition, then A and B are said to be the same; and conversely, if A and B are the same, they can be substituted for one another as I have said. Terms that are the same are also called coincident (coinsidentia); A and A are, of course, said to be the same, but if A and B are the same, they are called coincident. Def. 2. Terms that are not the same, that is, terms that cannot always be substituted for one another, are different (diversa). Corollary. Whence also, whatever terms are not different are the same. Charact.1. A co B signifies that A and B are the same, or coincident. Charact. 1. A non B signifies that A and B are different. (From [29, page 291], which is an English translation of the Latin version of Leibniz's work found in [19]. Note that Leibniz used the sign co for equality.)"

120. Related to the above, we find in [@c15 p.58] "Modus ponens (see (3.77)) is Latin for Method of the bridge. In many propositional calculi, a form of Modus ponens is one of the major inference rules this is discussed in more detail on Sec. 6.2. Modus ponens takes a back seat in our calculus because of our emphasis on equational reasoning. Nevertheless, it is extremely useful at times.".

121. In 'Equational logic: A great pedagogical tool for teaching a skill in logic', Gries interestingly explains that "Most propositional logics are built around inference rule Modus Ponens: ... Implication assumes real significance, while equivalence is treated as a second-class citizen b equiv c is merely an abbreviation for b implies c A c implies b. And, people begin using implication in complicated ways where use of equivalence would be clearer and shorter. Further, the main method of a proof is to break a formula into its little subcomponents and then con- struct a new formula from them. This is often not the simplest method of proof. Equational logic E uses substitution of equals of equals (Leibniz) as a main inference rule instead of Modus Ponens. Here are the four inference rules of E. ... Using Leibniz, manipulations are done by substituting (perhaps large) components of a formula by something equal to them, and this often results in shorter and more un- derstandable proofs. Further, proofs generally follow a form that people have learned in high school: a proof is basically a sequence of substitution of equals for equals."

122. All this leads us to the intriguing 'Z' https://en.wikipedia.org/wiki/Z_notation.

123. All this is a very interesting path that leads us to quite recent papers relating to ***Word Problems***, ***Complexity Classes*** and ***Boolean Circuits***.

- https://www.cs.utexas.edu/users/vl/papers/calc.ps "This note is about the "calculational style" of presenting proofs introduced by Dijkstra and Scholten and adopted in some books on theoretical computer science. We define the concept of a calculation, which is a formal counterpart of the idea of a calculational proof. The definition is in terms of a new formalization DS of predicate logic. Any proof tree in the system DS can be represented as a sequence of calculations. This fact shows that any logically valid predicate formula has a calculational proof"

- http://philpapers.org/rec/BOHILA "Dijkstra and Scholten have proposed a formalization of classical predicate logic on a novel deductive system as an alternative to Hilbert's style of proof and Gentzen's deductive systems. In this context we call it CED (Calculus of Equational Deduction). This deductive method promotes logical equivalence over implication and shows that there are easy ways to prove predicate formulas without the introduction of hypotheses or metamathematical tools such as the deduction theorem. Moreover, syntactic considerations (in Dijkstra's words, "letting the symbols do the work") have led to the "calculational style," an impressive array of techniques for elegant proof constructions. In this paper, we formalize intuitionistic predicate logic according to CED with similar success. In this system (I-CED), we prove Leibniz's principle for intuitionistic logic and also prove that any (intuitionistic) valid formula of predicate logic can be proved in I-CED"

- http://philpapers.org/rec/LIFOCP (see 'similar')

- http://philpapers.org/rec/SHETRO-4

- http://philpapers.org/rec/GUITTD, http://arxiv.org/pdf/math/0612089v1.pdf, " In this document, we study a 3-polygraphic translation for the proofs of SKS, a formal system for classical propositional logic. We prove that the free 3-category generated by this 3-polygraph describes the proofs of classical propositional logic modulo structural bureaucracy. We give a 3-dimensional generalization of Penrose diagrams and use it to provide several pictures of a proof. We sketch how local transformations of proofs yield a non contrived example of 4-dimensional rewriting."

- https://en.wikipedia.org/wiki/List_of_complexity_classes, https://en.wikipedia.org/wiki/Boolean_circuit

- http://iml.univ-mrs.fr/~lafont/pub/circuits.pdf "Boolean circuits are used to represent programs on finite data. Reversible Boolean circuits and quantum Boolean circuits have been introduced to modelize some physical aspects of computation. Those notions are essential in complexity theory, but we claim that a deep mathematical theory is needed to make progress in this area. For that purpose, the recent developments of knot theory is a major source of inspiration. Following the ideas of Burroni, we consider logical gates as generators for some algebraic structure with two compositions, and we are interested in the relations satisfied by those generators. For that purpose, we introduce canonical forms and rewriting systems. Up to now, we have mainly studied the basic case and the linear case, but we hope that our methods can be used to get presentations by generators and relations for the (reversible) classical case and for the (unitary) quantum case."

- http://www.pps.univ-paris-diderot.fr/~burroni/mapage/highwordpb.pdf "Higher dimensional word problems with applications to equational logic"

- 'Classic Algebra: Word Problems Algebraic Solutions' https://en.wikipedia.org/wiki/Word_problem_for_groups "The word problem was one of the first examples of an unsolvable problem to be found not in mathematical logic or the theory of algorithms, but in one of the central branches of classical mathematics, algebra. As a result of its unsolvability, several other problems in combinatorial group theory have been shown to be unsolvable as well."

- Has formal mathematics 'kidnapped' mathematics?
  - Unrelated but very intriguing is Baofu and his books including: 'The Future of Post-Human Mathematical Logic'. "Why should mathematical logic be grounded on the basis of some formal requirements in the way that it has been developed since its classical emergence as a hybrid field of mathematics and logic in the 19th century or earlier? Contrary to conventional wisdom, the

foundation of mathematic logic has been grounded on some false (or dogmatic) assumptions which have much impoverished the pursuit of knowledge."

- TODO: From Baofu, extract a diagram of the five 'disciplines' of logic as he tabulated it and concisely described it, noting what is 'more syntax' and what 'more semantics'.
- Is Gentzen our new hero? 'The collected papers of Gerhard Gentzen'
- 'Plato's Ghost: The Modernist Transformation of Mathematics'
- Is this the ultimate victory of the 'computer'?
- 'Set Theory from Cantor to Cohen' by Kanamori is surprisingly encompassing, and links us to 'measure theory' through ordinals with good sentences such as:
  * "Zermelo's axiomatization also shifted the focus away from the transfinite numbers to an abstract view of sets structured solely by $\in$ and simple operations."
  * "The functions in these classes are now known as the Baire functions, and this was the first stratification into a transfinite hierarchy after Cantor"
  * "Lebesgue's thesis [1902] is fundamental for modern integration theory as the source of his concept of measurability. Inspired in part by Borel's ideas but notably containing non-constructive aspects, Lebesgue's concept of measurable set through its closure un- der countable unions subsumed the Borel sets, and his analytic definition of measurable function through its closure under pointwise limits subsumed the Baire functions."
  * "In the memoir [1905] Lebesgue investigated the Baire functions, stressing that they are exactly the functions definable via analytic expressions (in a sense made precise). He first established a correlation with the Borel sets by showing that they are exactly the pre- images of open intervals via Baire functions."
- Kanamori has other interesting works such as: bullets: 'The Higher Infinite. Large Cardinals in Set Theory from their Beginnings' TBC 'The emergence of descriptive set theory' TBC
- All this shows us that we were right, and it is pointless to proceded deeper into mathematics without understanding formal set theory and the transfinite.
- The following, from 'Review: Gerhard Gentzen, Die Gegenwärtige Lage in der mathematischen Grundlagenforschung. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie' (J. Barkley Rosser) is extremely telling: "The first book is a well written summary of the present status of foundations, and contains one of the most lucid accounts of the Brouwer viewpoint that the present reviewer has seen. The distinction between the Brouwer and Hubert schools is presented from the point of view of their treatment of the infinite. For Brouwer, who always insists on finite constructibility, the infinite exists only in the sense that he can at any time take a larger (finite) set than any which he has taken hitherto. Hubert would treat of infinite sets by the same methods used for finite sets, as if he could com- prehend them in their entirety. Gentzen refers to this point of view as the "as if" point of view. He presents various paradoxes which arise when the "as if" method is used without proper care. This of course opens the question of what is "proper care." In the nature of things, the Brouwer method must fail to produce a paradox, since it never leaves the domain of the constructive finite. However the Brouwer method does not produce sufficient mathematical theory for physical and engineering uses. So Brouwer's method must be described as "excessive care." A

proposed way out of the difficulty is to base the ”as if” method on an appropriate formal system, and use the Brouwer method to prove that the formal system is with- out a contradiction. For none of the various formal systems so far proposed has such a proof of freedom from contradiction been given. More serious still, a well known theorem of Gödel says that if a logic L is used to prove the freedom from contradiction of a logic L2, then L must in some respects be stronger than L2. So the above pro- gram will fall through unless one can point out some respect in which the Brouwer method is stronger than the ”as if” method. Gentzen thinks he has found it. His idea is to use the Brouwer method, involving the use of transfinite induction up to a certain ordinal a, to prove the freedom from contradiction of that part of the ”as if” method which involves transfinite induction only up to an appropriate smaller ordinal /3. If 0 is fairly large, the resulting ”as if” method, though restricted, should be adequate for physics and engineering. In the second book, Gentzen illustrates the above proposal by using the Brouwer method, with induction up to eo, to prove the freedom from contradiction of number theory with induction up to any ordinal less than eo. An important gap in the proof is the absence of a constructive proof that induction is valid up to eo. Gentzen himself comments on this gap, and expresses the belief that it will shortly be filled. The present proof of freedom from contradiction is made considerably simpler than the earlier proof (in Mathematische Annalen—see above) by using Gentzen’s LK-calculus, rather than his NK-calculus. The proof is too complicated to be sketched here. However it is worth saying that what Gentzen does is to describe a means of attaching an ordinal number (less than €o) to any proof of number theory. He then describes how, if one had a proof of a contradiction, one could find a second proof of a contradiction having a smaller or- dinal number than the first proof.”

- Rosser also has another interesting book (Logic for Mathematicians) about which a review says: “I like a book that is what it says it is, and Logic for Mathematicians fits that description. This is an immersion into pure logic for non-specialists, and it succeeds in its declared goal. Those who are fascinated by dense mathematical notation will be cheered by this book. Rosser has made a conscious decision to use symbolic logic, “because we do not know otherwise to attain the desired precision” (p. vii). This reasonable decision leads to such compact expressions as (A,γ)(α,β):α || γ on α.β || γ.A on β. □ .α = β for the parallel postulate (p. 177). While there is certainly a case to be made for the clear exposition of language, it is at times fascinating to see the economy of notation in the symbolic logic version of common mathematical statements. ”
- About Rosser: “In 1936, he proved Rosser’s trick, a stronger version of Gödel’s first incompleteness theorem which shows that the requirement for ω-consistency may be weakened to consistency. Rather than using the liar paradox sentence equivalent to ”I am not provable,” he used a sentence that stated ”For every proof of me, there is a shorter proof of my negation”.”
- More Rosser: http://www.jstor.org/stable/1968669?seq=1#page_scan_tab_contents (paste introduction!) and ’Simplified Independence Proofs: Boolean Valued Models of Set Theory,’
- What Gentzen did is also explained in ’Hilbert’s Programs and Beyond’
- Also interesting is ’Formalized Mathematics’ by John Harrison, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.18.2798&rep=rep1&type=pdf, as a historical summary, but also for this ’unicornian’

quote: “But even Goedel’s undecidable sentence for a simple number theory is really just a theoretical pathology. Paris and Harrington (1991) have come a bit closer to a realistic mathematical statement that cannot be proved in a simple first order arithmetic; even this relies on a rather artificial encoding of a combinatorial result in number theory. It seems we are nowhere near finding a mathematically interesting incompleteness in ZFC set theory, by which we mean a statement unprovable in ZFC but known for other reasons to be true, as distinct from statements like the Continuum Hypothesis which have no more been decided by non-formalistic methods.”

– TBC

IV Trivializers with (Rogers 1971)

124. Here is a perfect overview of the basic construction of ***formal inference systems***, validating our views (except for ’translation’): “In our approach to each of the various areas of logic, and thus to the sentential logic in particular, w e shall proceed by developing a certain formal system of logic (or a whole type of formal systems of logic). This will be done in each case in a certain order. First, we shall take up a certain part of the syntax of that system of logic. Here we characterize principally the symbols and formu- las of that system of logic in an exact way. In particular, in characterizing, or distinguishing, certain of the expressions of that system as formulas, no reference is made to any interpretation of those expressions. The second step in setting forth a logical system will consist in providing the semantics of that system. Here we first specify in an exact fashion just how the expressions of that system are to be interpreted; then define a number of important semantic concepts, and establish a number of basic results con- cerning those concepts. Most importantly, we here define the fundamental concept of a logically valid formula within that system. In the case of the sentential logic, the logically valid formulas are the tautologies. Finally, we return to the syntactical approach, and attempt to characterize syntactically this class of valid formulas, which we have just defined semantically. We attempt to do this by laying down certain formulas as axioms — that is, as formulas accepted without proof. We then specify certain rules of inference, and define as theorems those formulas within the system which can be derived from those axioms by means of those rules of inference. The attempt is to do all this in such a way that the theorems of the particular system will coincide with the valid formulas of that system. In the case of the sentential logic this turns out to be possible. Here the class of valid formulas can be successfully characterized by syntactical means. This also remains true for that branch of logic taken up in Chapters II and III; viz., the first-order predicate logic. It turns out, however, no longer to be possible with respect to the logic of Chapter IV; viz., the second-order predicate logic. Here the syntactical approach falls short of the semantical approach; that is, here the class of logically valid formulas can be characterized only semantically.”

125. Here is a good summary of the terms ***truth functional, extensional, principle of extension, non-extensional, modal, two-valued***. “Philosophers sometimes speak of the truth-value of a sentence as its extension. W e have just seen that all contexts within the sentential logic are truth-functional contexts. For this reason, these contexts are often called extensional contexts, and the Replacement Principle is referred to as a principle of extensionality for the sentential logic. Further, the sentential logic itself is said to be an extensional logic, in the sense that all o f its contexts are extensional. A logic which contained belief contexts, on the other hand, would be a non-extensional logic in this sense. And modal logic, in which the concepts of necessity and possibility are studied, is another example of a non-extensional logic. For though ’Two plus two equals four’ and ’Snow is white,’ for example, are both true, when we replace the former by the latter in the true sentence ’Necessarily, two plus two equals four,’ the result is a sentence which is false: namely, the sentence ’Necessarily, snow is white.’ Though non-existensional logics are often philosophically important, for the purposes of orthodox mathematics it is not necessary to take up the study of such logics, and all of the logical systems which we shall consider in this book are extensional in some appropriate sense. As one final introductory observation, we remark that in addition to the standard two-valued approach to the sentential logic, in which the only recognized sentential values are truth and falsity, logicians have also studied many-valued approaches to the sentential logic, in which three or more sentential values are recognized. And in addition there is the intuitionistic approach to the sentential logic, which departs from the orthodox sentential logic in not accepting without restriction the law of excluded middle, according to which every sentence is either true or false. But we shall not here be further concerned with these alternatives to the orthodox sentential logic."

126. A good explanation of the ***'paradoxes of material implication'*** some of which are

- $A \supset (B \supset A)$
- $\sim A \supset (A \supset B)$
- $(A \supset B) \vee (B \supset A)$

is: "The first two of these three schemata are sometimes referred to as 'paradoxes of material implication.' And, indeed, if we were to read the symbol ' 3 ' as 'implies' (or 'materially implies'), then all three of these schemata would seem to be paradoxical; not in the sense that they were contradictory, but in the sense that they were highly counter-intuitive. For then the first of these schemata would apparently say that any true sentence is implied by any sentence whatsoever; the second, that a false sentence implies any sentence whatsoever; and the third, that for any two sentences, at least one implies the other. The paradoxical appearance largely disappears, however, if we read the symbol 'D', not as 'implies', but only as 'if ... then'. The first of these schemata then simply expresses the fact that a conditional is true if its consequent is true; the second, that a conditional is true if its antecedent is false; and the third, that for any two sentences at least one of the conditionals between them is true."

127. About the above we note that the sentential logic's connective are 'translations' of the english words that have the same text. so sentential 'or' is not english 'or', etc.

128. The first in the list above is called the ***'principle of explosion'***, which is introduced in wikipedia by "The principle of explosion (Latin: ex falso (sequitur) quodlibet (EFQ), "from falsehood, anything (follows)", or ex contradictione (sequitur) quodlibet (ECQ), "from contradiction, anything (follows)"), or the principle of Pseudo-Scotus, is the law of classical logic, intuitionistic logic and similar logical systems, according to which any statement can be proven from a contradiction."

129. Trivializer for ***effective, algorithm***: "It is clear that the construction of truth-tables provides us with a perfectly general test for determining whether a formula of P is a tautology. For any formula A, if A receives the value truth in each of the rows in its truth-table, then A is a tautology; if A receives the value falsity in at least one of these rows, then A is not a tautology. We shall speak of the truth-table test as a mechanical test, or an effective test. The concept of an effective test or procedure is an intuitive concept which can be given an exact analysis within mathematical contexts, and we shall consider such an analysis (in terms of recursive functions) in Chapter VIII. Until we reach that chapter, we shall use only the intuitive concept of the mechanical, or the effective. By way of explanation of that concept, perhaps a number of illustrations and informal remarks will suffice. Familiar mathematics provides us with a large number of effective procedures. For example, the procedures for determin- ing the sum and product of any two numbers, the procedure for extracting square roots, and the procedure for solving quadratic equations, are effective procedures. Such effective procedures are often called algorithms. These procedures are effective or algo- rithmic in the sense that they provide us with instructions for ascertaining something or other in a systematic, step-by-step manner. Any concept which is defined in such a way that there is an effective procedure for determining whether that concept applies in any particular case is called an effectively defined concept. Thus, the concepts of a formula of P, or of a tautology of P, are effectively defined concepts. Where there exists no general procedure for determining whether a concept applies, on the other hand, ingenuity is required. We shall from this point forward repeatedly draw upon these informal concepts of an effective procedure and an effectively defined concept."

130. Looking at the above, what we have been calling 'artihmetical, or computational' is 'effective', and within 'effective' there are all the degrees of complexity.

131. Note the definition of logical validity in the context of sentential logic by tautology, hinting it might be something else in other systems: "Most importantly, we here define the fundamental concept of a logically valid formula within that system. In the case of the sentential logic, the logically valid formulas are the tautologies. ". In predicate logic, the author refers us back to Leibniz's concept saying that "A formula A is valid within F1 - or logically valid, or a logical truth, within F 1 — if and only if A is true under every interpretation. Here we have a precise analysis (restricted to formulas of F 1) of Leibniz' famous informal concept of 'being true in all possible worlds.' Intuitively, a valid formula is one that is true by virtue of logical considerations alone; or one that is true under all logically possible conditions.". Finally, he relates the two contexts, validating our thoughts about 'hidden' generalization with "The semantical concepts introduced in our discussion of the sentential logic can easily be shown to be special cases of the above concepts, and are readily transferred to the predicate logic. Thus, for example, the tautologies of F1 are a special case of the logically valid formulas of F1 , and tautological implication is a special case of logical implication."

132. A very important distinction is the one between ***logical implication*** and ***tautological implication***, the author says "Tautological implication is a special form of logical implication; viz., logical implication by virtue of the meanings of the sentential connectives. We shall define a general concept of logical implica- tion in Chapter II. Tautological implication will there be formally subsumed under logical implication. And similarly for tautological equivalence and logical equivalence.". The detail of 'assignment of truth values to variables' is probably what is special to sentential logic and does not go far enough for predicate logic.

133. A pragmatic explanation of ***'definitional equivalence'***: "It will be noticed that the only sentential connectives which occur within the above axiom schemata are the connectives for negation and for the conditional. As for the remaining connectives, we must either add further axiom schemata in which they appear, or in some way correlate them with the connectives for negation and the conditional. We shall here choose the latter course. We shall say that a formula A is definitionally equivalent to another formula B if and only if there are formulas A1 , B1, A2 , and B2 such that A and B are alike except that A contains an occurrence of A1 at some place where B contains an occurrence of B1 and either (a) $A1$ is $A2 \vee B2$ and $B1$ is $\sim A2 \supset B2$, or (b) $A1$ is $A2 \wedge B2$ and $B1$ is $\sim (A2 \supset \sim B2)$, or (c) $A1$ is $A2 \equiv B2$ and $B1$ is $(A2 \supset B2) \wedge (B2 \supset A2)$ "

134. It is very sad that we only know notice a hidden subtlety within statements such as: "Thus, for example, within the sentential logic the difference in logical structure between the sentences 'Seven is greater than six' and 'All men are mortal' cannot be exhibited in any way. Clearly, any system of logic which is satisfactory, however,must enable us to exhibit this difference. As we shall see, the first-order predicate logic permits us to do this.". We never noticed the utter importance of the existence of 'functions and predicates' in QL. With this, such predicates can be turned into '<' and we can logically reason about ***'7 > 6'***. This is so obvious that one should cry. Probably, set theory can also be introduced in this manner.

135. A question to myself, can an inconsistency inducing syntactic contradiction like $\vdash A \wedge \neg A$ even be 'expressed' semantically in PL? It seems that the 'rules' for creating truth tables do not even permit to start expressing it.

136. A better name for 'tautological consequence' is 'semantic consequence' that is a 'semantic' oriented definition of the commmon-informal concept of consequence. But there is no need to memorize this better name anymore because, after we have delved into 'how to read' (XIII) with Rogers, we know that 'theorem' means 'proof theory' means 'syntax' and 'tautology' means 'model theory' means 'semantics'.

137. This is not suggested by the book, but we note that there is a very simple way to define some of the semantic concepts if one takes a 'functional' point of view.

- Each 'formula' is identified with a function f : $2^n \rightarrow 2$, where '2' denotes the set {0,1}. In this sense, we cannot differentiate (except by assigned distinct function names) between two 'formulas' that have a different 'form' but the same variables and same value for each variable assignment. Per example, $p \xrightarrow{f} p$ and $p \xrightarrow{g} p \vee p$ are the same function.

- In this sense, the connectives are the binary operators, the members of the set $\{\circ : 2^2 \rightarrow 2\}$.

- The set of all ***tautologies*** is then $\{\tau : 2^n \rightarrow \{1\}\}$.

- The set of all ***contradictions*** is $\{\perp : 2^n \rightarrow \{0\}\}$.

- The set of all ***tautological consequences*** is then $\{(\Gamma = \{f_i\}, g)$ such that $\{\rightarrow (f_i, g) = 1, \forall i\}\}$.

- Two formulas are tautologically equivalent simply if their functions are equal.

138. In our first treatment, even up to axiomatic set theory, we will not be concerned with completeness proofs. This will come during the second treatment.

139. We can trivialize the choices of a deductive system in the following we: we choose an initial set of forms (objects) and an initial set of rules (transformations), such that the closure of these forms under the rules.

140. By the example used on p.23, we immediately see why we need a 'natural deduction'. A human suffers an immediate loss of all 'proof powers' when for the first time forced to use the book's deductive system. Even the translation from informal to formal seems elusive.

141. WIP

1. " As we shall see later, however, the semantical concepts for the predicate logic differ in an important respect from the semantical concepts of the sentential logic. These latter concepts are effectively defined concepts (for all formulas A and finite classes of formulas Γ) . The former concepts, however, are not effectively defined concepts. That is, there are no effective procedures for determining in every case whether the semantical concepts for the predicate logic apply to that case. Thus, for example, the concept of valid formula of F 1 is not an effectively defined concept, though the concept of a tautology o f F 1 is an effectively defined concept. Truth-tables provide us with an effective test for determining whether or not any given formula A is a tautology; but there is no corresponding effective test for determining whether or not A is valid. In a great many cases we can determine, of course, whether or not A is valid; but it is known that there is no effective procedure for determining this in every case."

2. Note that in the above is an example of the separation between logical and tautological consequence. Also, the note points to an obvious possible mental obstacle, since the untrained human mind things 'effectively'. We have thought in the past that the semantic definition of truth through all interpretations would make it impossible to actually find 'rules of inference', since this would need checking all the infinitely possible interpretations, but of course, this is naive, there are other ways to prove that something is a rule of inference: by the (metatheoretical) properties of the logical symbols of F^1.

3. The standard formal theories used to formalize metatheoretic proofs are PRA, and WKL_0 as explained in http://math.stackexchange.com/questions/1592569/formalizing-the-meta-language-of-first-order-logic-and-studying-it-as-a where the answer to the question "We've a formal system say First order Logic, we reason about it in our meta-language using our meta-logic. We study its properties as a mathematical object. We prove theorems like ... Now, I wonder, Why not to formalize the meta language itself which we use to argue about FOL. ..." is "t is completely standard to study formalized metatheories. Historically, this is particularly of interest from the point of view of mathematical finitism - most proof systems for first-order logic are completely finitistic, based only on symbol manipulation, and so their formalization is somewhat straightforward. Formalization of the

metatheory is also of interest to help us see the mathematical techniques required to prove theorems of logic such as the completeness theorem. Rather than making a formal metatheory that quantifies directly over strings, as suggested in the question, it has been more common to work with Primitive Recursive Arithmetic, whose basic objects are natural numbers. There is a tight link between formulas and natural numbers, as I explained in this answer. PRA is often studied as a quantifier-free theory, with a large collection of function-building symbols; it is typically considered to be "the" formal standard for a finitistic theory. We know for example, that a formalized version of the incompleteness theorem is provable in PRA. PRA cannot talk about infinite sets, and so cannot talk about infinite models. To look at logical theorems such as the completeness theorem, we need to move to slightly stronger theories. We could use ZFC set theory, but that is far more than we need to study the countable logical theories that arise in practice. It has been common to use theories of second-order arithmetic, which can talk about both natural numbers and sets of natural numbers, to study the completeness theorem. We know that, in this context, the completeness theorem for countable theories is equivalent to a theory of second-order arithmetic known as □□□0, relative to a weaker base theory. The way to get into this subject is to first learn quite a bit of proof theory, after you have a strong background in basic mathematical logic. There are not any resources I am aware of that are readable without quite a bit of background. Even introductory proof theory texts often assume quite a bit of mathematical maturity and exposure to mathematical logic, and are written at a graduate level. So there is no royal road to the area, unfortunately, although much beautiful work has been done. To get a small sense of what exists, you can read about the formalization of the incompleteness theorem in Smorynski's article in the Handbook of Mathematical Logic. That article only describes one part of a much larger body of work, however. The overall body of work on formalized metatheories is spread out in many places, and described from many points of view. You can find some of these in the Handbook of Proof Theory, and in the following texts: Petr Hájek and Pavel Pudlák, Metamathematics of First-Order Arithmetic, Springer, 1998, Stephen G. Simpson, Subsystems of Second-Order Arithmetic, Springer, 1999, Ulrich Kohlenbach, Applied Proof Theory: Proof Interpretations and their Use in Mathematics, Springer, 2008. All of these are intended for graduate or postgraduate mathematicians."

4. 'Petr Hájek, Pavel Pudlák, Metamathematics of First-Order Arithmetic' looks very interesting and to the point.

V Redeeming Cantor's Paradise

142. Historically, we have been of the view that Cantor's paradise is more like a hell, we could not at all agree with Hilbert, and we read a lot that aligns with this view. We can now do better.

143. In mathematical pre-history, it was ostensible concepts that provided what mathematical theories would model and idealize.

144. Formal systems are actually not at all 'kidnapping' mathematics, because they allow for departure points for theories that can be shown solid (if found consistent either absolutely or relatively) that are not necessarily coming directly from the real world.

145. Attachement to the real world is yet another bias. Before formal systems, there was no clear way to bootstrap theories apart from looking at the real world, but after them this became possible.

146. It is this that caused the mathematical explosion after Cantor, and now we finally see the 'paradise', along with the part of it related to following the consequences of a 'set theory' formalized at first exactly to capture what informal mathematics there was, and ending with astounding consequences such as measure theory[123], of course 'conditionally on set theory'.

147. These consequences are also all the reasons for our past troubles with certain proofs, certain non-constructions, and certain 'creative' limit contraptions. But all these are simply consequences of the formal system, no more, no less.

148. Seeing this is impossible for somebody who did not study formal systems, who still does not see the difference between 'syntax' (textually structured systems) and 'semantics' (systems structured in any other way), all within the larger informal human language.

VI The Game of PL With (Smith 2003)

VI.1 Syntax

149. The proposed game here is that of the syntax of propositional logic as a wwf game. The rules are well known.

150. We note that, with a bit of peek into the relation of this game with {T,F}, that we must forcefully be able to capture all possible ways of combining two 'things' in order to produce a 'thing'. In this sense, any complete set of combinators will do.

VI.2 Semantics

151. Here again is an unnecessary convolution and complication using terms such as 'interpretation' and 'valuation' and ending with statements such as: "Indeed, many logic books in effect ignore the contents of wffs and immediately focus just on what we are calling valuations (perhaps confusingly, they then call the valuations 'interpretations')."

152. The simple truth about the game of PL semantics is that that that assignment from atoms to {T,F} is a function and not a relation, and that the combinators form a boolean algebra which preserves this fact, such that the assignment from wffs to TF is also a function and not a relation. What remains is to arbitrarily fix the rule for each combinator. Usually, one puts a halo of informal glue around the formal combinators, one that is not as ambiguous as the full language of humans and uses that to justify 'truth tables'.

VI.3 Miscellanea

153. Yet another misfortune of type erasure in the literature is the length of explanation needed to explain the difference between **Socrates** and **"Socrates"**. In effect all what is needed is to unerase types and say that the former is an alias for the man Socrates whereas the latter is an alias for the name Socrates. The same goes for the symbol $\vee$ and the combinator $\vee$.

154. Quine quotes are the most proper (if too annoying) device for 'simple macro expansion'.

155. A lengthy discussion of truth functionality can again be simplified using our real world grounding. As we said, we create our PL game and then observe what and how much it expresses. Obviously, some informal sentences are not modeled at all by the PL game. Also, since the PL game is boolean combinator complete, all 'truth functional' informal unambiguous sentences find PL translations.

156. Tautologies, yes, thank you, by our algebraic game and choice of logic (exactly T (x) or F). Any lengthy discussion on this is again confused by lack of typing and grounding.

VI.4 Deductive Systems

157. How is Socrates doing? PL cannot formalize typing and types!!!! 'Socrates is of type man'! This is THE point we have not nailed before. Although PL did help with many formalization challenges. Move on to PL as 'obvious/useless' deductive? Revise 'valid inference, sound inference'. Pass from PL to PLC (we felt something was going on...)

158. Informal implication is yet another boolean operator, it is 'truth functional' and has no natural property that makes it more important than any other combinator or choice of combinators. However, since this is about discussion settling, where from experience causality like implication is very human, we use PL's therefore as 'the' deductive operator.

VI.5 Tautological Entailment

VII *Historisches* with (Ferreiros 2001)

159. "First-Order Logic is not a 'natural unity', i.e., a system the scope and limits of which could be justified solely by rational argument. There is reason to think that, like so many other conceptual systems, First-Order Logic—hereafter FOL—is the sound and satisfactory outcome of a fascinating combination of rational argument and historical contingencies." We might add: that lead to a graph-like structure in equilibrium (not a tree) and hence which must be understood as a whole.

160. "But as any trained historian knows, excessive passion for the technical details, and concentration upon the emergence of the main results of a discipline (as judged from our standpoint), may easily lead to whiggism. In this way, it seems, historians of logic have tended to forget some of the complexities with which the historical road to modern logic confronts us."

161. On FOL and set theory "Let me give a simple example. In the early decades of the twentieth century, the paradoxes of logic and set theory grew to almost legendary proportions. It was usually emphasized that their discovery had been a great and consequential event in the development of logic. Meanwhile, the secondary literature of the second half of that century on the history of FOL hardly mentions the paradoxes at all. In the midst of a plethora of technical details concerning language, theory and metatheory, one may not even find a single mention of set theory. One can explain the discrepancy by considering that the secondary literature frequently presupposes the present notion of logic, as if it were ahistorical. While concentrating upon technical details, historians have tended to forget changes in the overall conception of the subject. To counterbalance this tendency, I

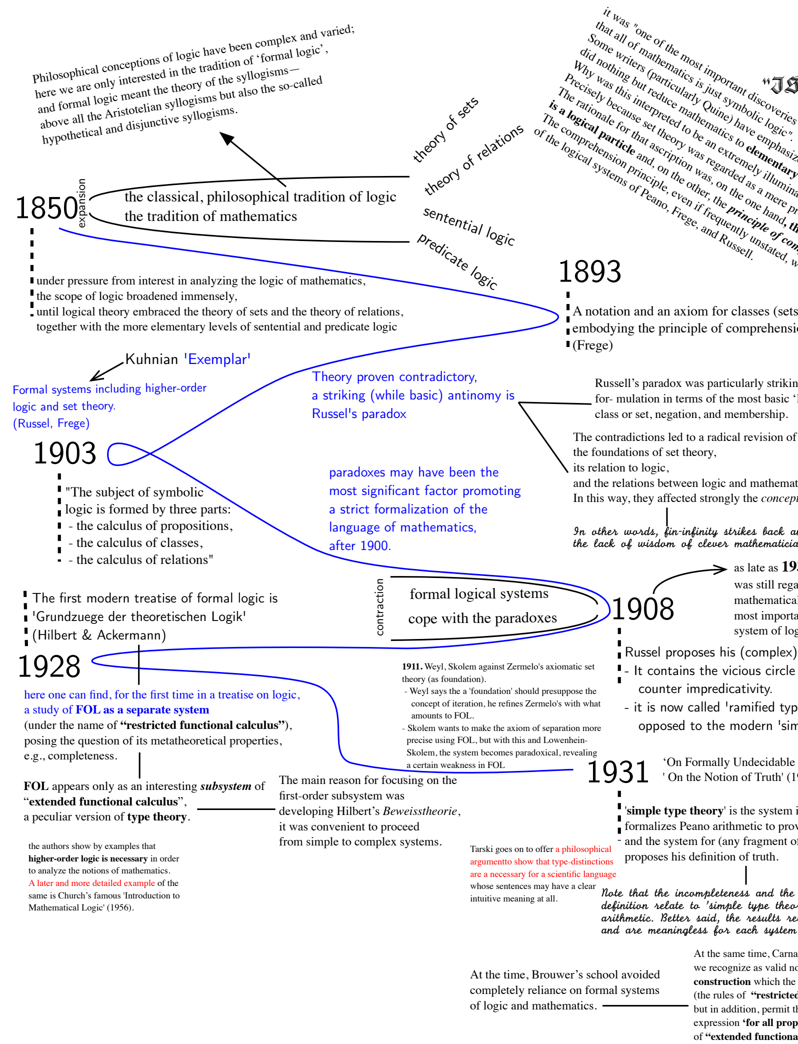

Figure 8: First-Order-Logic 'exemplar' Timeline

shall look at the development of modern logic, so to say, from above (focusing on its relations with set theory and the foundations of mathematics) and not from below (focusing only on the elementary levels of logical theory)."

162. We are actually starting to understand and not dislike set theory as it is. What it enables is a very large very little structured vista from which sets and relations can be pinpointed, or more constructively simply claimed to exist by a sort of homogeneity, as things in theories. Of course, here we face the usual difficult of 'picking without coordinates' but that is not the topic. In fact, the vista is made even less restrictive and actually have 'even less structure' exactly (and counter-intuitively) by adding the axioms related to infinity. In this way we do not impose the finity on the vista and from this point of view it is less structured. We remember hear the 'canonical embedding' chain: natural,integer,rational,real,complex.

VIII Logical Humor with (Fomenko)

163. "Sort of. Obsessing over logical certainty can be maddening as history would tell."

164. The modern logic solution out of the paradoxes in a nutshell [@c10]:
"Returning now to Russell's paradox, in his 1929 doctoral thesis, Gödel proved the completeness theorem which was first stated by Hilbert and Ackermann in Principles of Mathematical Logic thus: "Every valid logical expression is provable. Equivalently, every logical expression is either satisfiable or refutable." From then since, for a statement to be a logical mathematical statement, one has to determine that it cannot take both a true and a false value. A statement should be either true or it should be false. If not, it is not a logical mathematical statement. Thus my statement about the Barber of Seville shaving all men and only those men who do not shave themselves is not a mathematical statement. The combined works of Ernst Zermelo (1871-1953) and Abraham Fraenkel (1891-1965) next offered other means of getting around Russell's paradox by relaxing mathematical logic somewhat and shifting from an axiom schema of comprehension to an axiom schema of specification (implied by the axiom schema of replacement and the axiom of the empty set). This meant that solely a logically definable subclass of a set is a set. The relationship between hypothesis and facts must be logical. Thus my Barber of Seville is not a set. It is rather what Zermelo-Fraenkel refers to as a proper class—which in ZF set theory terms means: It is definitely not a set. This new focus on the axiom schema of specification marked the beginning of axiomatic set theory. Gödel next demonstrated that Zermelo and Fraenkel's way of going about mathematical logic was not a bad idea by publishing his Consistency Proof for the Generalized Continuum Hypothesis in 1939 and The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory in 1940. (In 1963, Paul Cohen—a formalist—proved that the Continuum Hypothesis is independed from the axioms of set theory.) One should stress that mathematicians need only to be convinced that a statement cannot be both true and false in order to consider it—i.e., it is not necessary to prove that the statement cannot be both true and false in order to consider it. In fact, Gödel had circa 1931 asserted that there are mathematical statements that cannot be proven within a logical system, thus there will always be some uncertainties. Should one consider the Goldbach conjecture (1742), the distinction is easily grasped: It is not known whether the Goldbach conjecture is provable, but it is very clear that there is no option aside from it being a yes or a no. The Goldbach conjecture is good and sound mathematics. In the words of my mathematician friend as we reflected on logic and its impact for workaday mathematicians on a cool and rainy October evening: "Everything was fine after that." "

165. "Kurt Gödel (1906-1978) wrote: Theorem VI. For every ω-consistent recursive class κ of FORMULAS there are recursive CLASS SIGNS r, such that neither v Gen r nor Neg(v Gen r) belongs to Flg(κ) (where v is the FREE VARIABLE of r). [1] Thus the world first read about the existence of undecidable mathematical propositions according to Gödel. Kurt Gödel went on to refine his statement of undecidability, suggesting two so-called incompleteness theorems."

166. ““Since the Zermelo-Fraenkel set-theory axiom walled off Russell’s paradox, no other problem has shown up,” I was puckishly reminded”

167. “Since his first brush with mathematics and in opposition to the pervasive sentimental idealism of his day, mathematician and logician Bertrand Russell (1872-1970) felt that mathematics lacked rigor and ought to be rebuilt on a firm logical foundation—that is, from the bottom up. His mathematical genius and good looks making up for what he lacked in social insights, Russell ideas were well received regardless of personal eccentricities. One of Russell’s master work was his 1910 Principia Mathematica, which he spent ten years co-writing with Alfred North Whitehead, all the while courting Whitehead’s wife.”

168. “Published in three volumes, the gargantuan book contains 362 pages with an ancilliary consequence that 1+1=2.”

169. “Another source added that Zermelo was already aware of the paradox, but had kept it a secret to go along with other faculty members at the University of Göttingen, among them David Hilbert (1862- 1943).”

170. “ In the 20th century, composer Arnold Schoenberg (1874-1951) wrote music strictly based on 12 tone row matrices and spearheaded a certain formalism in musical composition that reverberated throughout the century. Likewise, 20th century logician Gerhard Gentzen (1909 -1945) worked from the rules to the theorem. Getzen’s 1936 proof of Peano’s axioms using combinatorial methods marked the beginning of ordinal proof theory. The logician’s remains a somewhat unusual way to go about doing mathematics, just as a 12 tone row matrice remains an unusual begining for a piece. Still, medals and prizes are awarded to mathematicians who can take a look at a number of examples and find a (consistent) theorem that explains them all. At the end of the day, regardless of the method that has been employed, there is one thing we have all come to agree on. It is that mathematics is inherently filled with creative potential.”

171. “undecidable problems are not uncommon. In fact, undecidable problems are ubiquitous in mathematics, as Chaim Goodman-Strauss (University of Arkansas) states in [4], a paper he submitted to the Notices in 2009 and that prompted my current writing. This ubiquity confounds formal decidability. We might remember from Gödel’s thesis that every valid logical expression is provable. No proof exists from standard set theory arguments that amounts to “I can’t decide it.” Yet it is the case that unruly undecidable problems tend to surround us and this, I believe, is due to our creative tendencies. ”

172. “Research mathematicians delight in constructing mathematical objects with certain properties to fit conjectures in an attempt to prove them, and then to pluck or poke at these constructions to see how they fare. This may not lead to a Fields medal, but it is a lot of fun. What is less so, often impractical—and in some cases as it turns out, impossible—is for mathematicians to go back to the generating axioms of their discipline and check that the whole system, including their new theorem is consistent . There are computer programs that allegedly could now do this for humans, but one wonders if the practice of thus checking proofs should be institutionalized or even frequent.”

173. TODO: Can’t Decide? Undecide! (Chaim Goodman-Strauss). A good hook into tiling, undecidability, universal machines, NP-hard, and other nice things.

IX Topology and Logic with (Vickers)

174. Just like we finally understood that mathematics is unsurveyable, and how mind-bogglingly large the background vista of sets is, we finally understand that topology is not far off. It is also very wild. We must remember that it is a generalization of somtimes very unrelated things, even if they are all ’topologies’. This is validated in <TODO: mathoverflow quote by guy explaining you need to add adjective>.

175. In particular, ’clopen’ is simply quite wild. Nevertheless, we need to find a good way to ’talk’ about topology. <TODO: mo>. The measuring analogy is good, but so is Vickers.

176. TODO: great time to look at Nets and Filters.

177. It is good to know that the above bothered at least one mathematician <grot quote> and it is very informative to look at the desolate work about tame topology.

178. In a way that nicely mirrors going from the wilderness of sets to the tamer geometric logic, in a way that is related to topology, Vikers says:

179. Logically stronger than is a great way to talk about 'algebraic implication' in general.

180. TODO: Viker's great summary of Lindenbaum and other things, also of 'free'.

181. TODO: functions are just sets, what is so special? they move us from a set to the next! in poset, the edges are simply all the elements of the chosen set (function eq).

X Spills

X.1 'Limited' Set Theory

182. Let us think of the usual counterpart of syntax, that is, set theory. What would happen if we constrained ourselves to a 'finite' or 'countable' set theory? Maybe this idea is meaningless, but the questions leads us to several interesting (and not completely new) threads:

- Myhill's constructive set theory
- 'The strength of extensionality II — Weak weak set theories without infinity' (Sato), with very interesting diagrams. http://www.sciencedirect.com/science/article/pii/S0168007210001296
- 'Set Theory: Constructive and Intuitionistic ZF' http://plato.stanford.edu/entries/set-theory-constructive/#AxiSysCZFIZF
- 'Constructive Set Theory from a Weak Tarski Universe'

183. 'Constructive Set Theory from a Weak Tarski Universe' contains a very possible hook into homotopy type theory, with quotes such as:

- "Simplicial sets are an abstract combinatorial generalisation of Euclidean simplicial complexes in which for each di- mension a set of n-faces is provided, together with degeneracy and face maps that describe the structure of the complex: which faces are glued and which vertices are collapsed. This sequence of sets can be organised in a functor describing simplicial sets as a presheaf category."
- "FOUNDATIONS of mathematics are usually settled in one of these three context: set theory, type theory or category theory. On the categorical side there are two quite different approaches: one focuses on the category of all categories whereas the second relies on the notion of topos. Topoi are well-behaved categories modelled on the key examples of the category of sets and the categories of sheaves over a topological space, they can be conceived as categories of continuously variable sets. Type theory is certainly the less known of these three outside the circle of logicians, for this reason we briefly sketch some of its features."
- "The simplicial set given by two vertices linked by a 1-face is the simplicial interval, which allows to define homotopies between simplical maps, and to develop a theory of homotopy for simplical sets. A careful analysis of the common features shared by topological and simplicial homotopy theory yields to the definition of model struc- ture on a category that is an abstract playground for homotopy even in absence of an interval. A model structure consists in three distinguished classes of maps modelled on (topological) weak equivalences, Serre fibrations and cofibration. We will see that identity types of Martin-Löf type theory, can be interpreted categorically using model structures."

X.2 Axioms and Syntax

184. Upon reading the axioms of a ring in Stillwell's 'Elements of Algebra' again after a long while we notice multiple things.

185. We now see the axioms as purely syntactic, whereas before, we seem to have closely associated them to their prototypical representation. This is a misunderstanding of many dangers.

1. The benefit of the abstract is exactly to 'forget' the originally prototypical 'concrete' thing that can be described using the axioms.
2. The above mirrors exactly the relations between syntax and semantics in logic. In that sense, interpretations are *similar$_\star$ to representations.
3. We notice that the assumed domain of things is very important. The more axioms something obeys, the smaller the part of the domain will (probably) be that satisfies it. The part could be degenerate, hence the importance of proving the existence of objects that satisfy a set of axioms.
4. If the set of axioms is of the form '... = ...' (as is often the case), we are lead to some kind of a generalisation of relations algebras.
5. It is astounding how tightly 'arithmetic and numbers' are engrained in that they easily mentally 'kidnapped' axioms systems describing them, a very detrimental effect. Indeed, algebra seems to be very much concerned with 'form' and not 'number' except when specifically mixed.
6. Algebra is a very treacherously rich topic, Rich being an understatement. A witness is http://mathoverflow.net/questions/90700/where-is-number-theory-used-in-the-rest-of-mathematics/90746#90746

XI Quantifiers, Quantifiers, Quantifiers

186. "How about quantifier $\forall$? A non-empty set of objects is assigned to it. Let me call it a "Q-function." One way to address the main question of the paper (i.e. whether quantifiers are logical constants) is to raise the following question: Are there fundamental differences between p-functions and a Q-function? In the previous section, we identified the following four semantic factors which distinguish the logical constants (i.e. sentential connectives) and the other units (i.e. sentential sym- bols)."

187. "Enderton [1, pp. 69–70] puts quantifier symbol $\forall$ into the parameter group, along with predicate symbols, constant symbols, and function symbols. Etchemendy, who is against the logical constants debate itself, experiments both with fixed and various meanings for $\forall$. (Etchemendy [2, Chs. 5 & 8].)"

XII Consequence Relation

188. In this section, we finally fully elucidate the difference between logical consequence ($\vDash$) and proof theoretic, also called syntactic, ($\vdash$) consequence.

189. According to [@c12], Goedel's completeness theorem an be summarised by: $K \vDash S$ iff. $K \vdash S$.

190. Definitions of logical consequence, also called tautological consequence. Given a formula B and a set of formulas Γ.

1. We say that $\Gamma \vDash B$ if every truth assignment that satisfies Γ also satisfies B. To prove $\Gamma \vDash B$, proceed as follows: Let ϕ be a truth assignment such that $\phi(A) = T$ for every $A \in \Gamma$; show that $\phi(B) = T$. (Hodel p. 61)

191. Notice that in effect, Traski's truth in every model is nothing but the above, where a model is a truth assignment ϕ.

192. What the above note tells us is that we have missing detail in our figure 1: Taking our world as the semantic world, the semantic theory to be formalized where truth can be talked about, there too, we 'define' what we mean by consequence, and this amounts to 'semantic consequence'. What has been confusing us is the use of the word 'model', which implied something more than a 'truth assignment'. What we are saying here is that we usually work in worlds in which their is a fruitful concept of 'consequence' and not random unrelated truths, and this makes sense since when a world is not so, there is little point in formalizing it as proofs (out of compression, homogeneity), informal or formal, are in such a world impossible.

193. Definitions of proof theoretic (syntactic) consequence. Given a formal system F, and a set of formulas Γ.

1. we write $\Gamma \vdash_F A_n$ when we have a proof in F using Γ. That is, when we have a finite sequence $A_1, \ldots, A_n$ of F such that A_k is either an axiom of a inferred conclusion of Γ and/or preceding formulas.

194. So then, looking at our figure 4, we see that $\vdash$ is a 'proof' fully within F, while $\vDash$ is a 'proof' fully within H. To prove soundness and completeness, we will usually have to work outside F and H to relate the correspondence between all such 'proofs'.

195. We must also warn that logical and tautological implication are not the same thing[132].

XIII 'Reading' Logical Statements with (Schechter 2005)

196. With the help of [@c17, p.45] we note that $\Rightarrow$, $\Leftrightarrow$ are 'usually' used to denote informal implication when working in the meta-theory. Per example, one proves soundness using

$$\vdash A \;\Rightarrow\; \vDash A,$$

and (semantic) completeness using

$$\vDash A \;\Rightarrow\; \vdash A.$$

197. To 'informally read' all the variations of 'implication' properly, here is a nice list of examples from Echter.

1. Tautology. The formula $A \to B$ ***is always true***″.
$$\vDash A \to B$$
2. Inference. Whenever A is true then B is true″.
$$A \vDash B$$
3. Admissibility. If some substitution of formulas for metavariables makes A an always-true formula, then the same substitution makes B always-true″.
$$\vDash A \;\Rightarrow\; \vDash B$$
4. Theorem. The formula $A \to B$ can be derived from nothing (except for the logical axioms) ″.
$$\vdash A \to B$$
5. Inference. From formula A ***we can derive*** formula B″.
$$A \vdash B$$
6. Admissibility. If some substitution of formulas for metavariables makes A into a formula that can be de- rived from nothing, then the same substitution also makes B derivable from nothing.″
$$\vdash A \;\Rightarrow\; \vdash B$$
7. Detachment. Turning a theorem into an inference rule″, works in 'most logics of interest'.
$$\vdash A \to B \;\Rightarrow\; A \vdash B.$$

   However, doing it the other way round is the subject of three deduction principles presented in the book. In classical logic, all three principles are valid.

   - In constructive logic, this hinges on constructive implication.
   - In three-valued Lukasiewicz logic this is $A \vdash B \;\Rightarrow\; A \to (A \to B)$.
   - In 'relevant logic', we have $A \vdash B \;\Rightarrow\; \vdash \overline{A} \vee B$.

8. Note that the above is an interaction between the theory's connective and the meta-theory's one. There is no reason why this too would not be formalized.

9. The absence of LEM in constructive logic can also be 'informally read' (and hence informally understood) quite easily with this trivializer: "***we know how to prove*** Goldbach's conjecture or ***we know how to disprove*** Goldbach's conjecture". Clearly, we may very well 'not know' either even if their are negations of each other.

198. Add Schechter's 'relevance' note.

199. Add Schecther's page 10 logics.

200. Add more Schechter.

201. There are many ways to use the word 'completeness' (https://en.wikipedia.org/wiki/Completeness_(logic)), the relevant ones are:

1. Expressive completeness. A formal language is expressively complete if it can express the subject matter for which it is intended.

2. Functional completeness. A set of logical connectives associated with a formal system is functionally complete if it can express all propositional functions.

3. Semantic completeness. $\vDash A \;\Leftrightarrow\; \vdash A$.

4. Strong completeness. TODO.

5. Refutation completeness. TODO.

6. Syntactical completeness. TODO.

XIV The Demise of the WFF

202. Introducing Algebraic Logic, this time, knowing what we are reading. We reached ourselves the vague feeling of algebrization as we described the 'closure' in a deductive system, but failed to nail it down formally. The ensuing research produced a flood of results, some of which we have been encountering repeatedly before but were ill-situtated to even read properly.

203. Here are some accusative quotes from [@c18]:

- "Many authors talk of "expressions" instead of strings, but this neologism leads to the eventual need to distinguish those "expressions" which are well-formed (i.e.,grammatical) from those that are not, with the resultant barbarism "wel-formed-expression"."

- "In logic texts, the usual inductive definition of sentences for sentential logic says that sentences are generated from sentences, that is from other (welformed) expressions, and not,as in the examples above, from nonsensical strings."

- The introduction, p.3 and on is very important to read and internalize at this stage. It allows us not to fall into implicit mistakes while reading other related expositions.

204. What about Curry's [@c19]? it appears very close to our attitudes.

205. What is an ***Algebra***, what is a ***calculus*** and what is not? [@c19] says: "The term 'algebra' is used in this book as a name for a system with free variables but no bound variables. Thus the system of Example 5 (Sec. 2C3) is an algebra and is aptly called propositional algebra. In contradistinction the term 'calculus' will, as a rule, be used to describe a system with bound variables, so that it is suitable to speak of a calculus of l conversion, a predicate calculus, etc. These terms agree with ordinary mathematical usage, where the distinguishing characteristic of the infinitesimal calculus, as opposed to elementary algebra, is the presence of bound variables in the former."

206. What characterizes a ***logical algebra***? At least one thing is the ***idempotency*** of all its elements under meet and join: "The principal operations considered in this chapter-and the only ones admitted until we come to Sec. C-are two binary ones called meet and join and indicated, respectively, by the infixes '$\wedge$' and '$\vee$'. The obs are then constructions from the atoms by these two operations. The operations will turn out to be commutative and associative, and they may or may not have certain properties analogous to the distributive law of ordinary algebra. What is peculiar about these algebras, however, is that the operations will be idempotent; i.e., we shall have $a \wedge a = a, a \vee a = a$ for all obs a. These laws hold for most algebras which have logical interest, and therefore they may be regarded as characterizing logical algebras in the sense of Sec. 1A."

207. TODO: We have to check the referecences in [@c19, p.158,158], especially: "Couturat [ALg]".

208. The investigation leads us to Anschauung 4 (II).

XV ZFC

XV.1 Intended Interpretation

209. Finally we reach enough knowledge to ask ourselves: 'What is the intended interpretation of ZFC?'. The answer is **V**, the class of hereditary well-founded sets, also called the von Neumann universe.

210. Note that the attribution of **V** to the 'v' in von Neumann, is erroneous, the ideas goes back to as far as Whitehead and Russell.

211. In this pictorial representation of the first four levels of **V**, an empty box is the empty set. Level $\mathbf{V}_5$ contains $2^{16} = 65536$ elements, while $\mathbf{V}_6$ contains 2^{65536} elements, more than the number of atoms in the known universe. This reminds us of the 'vista'[162] we talked about.

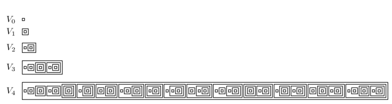

Figure 9: **V**

212.

213 (@c11)**.** seems to have the right attitude to tells us more about this. We read:

- TODO, relate this to our translation figures: "The intended interpretation of set theory will be further discussed in I11 $4. We turn now to ad hoc interpretations; this is the basis of all consistency proofs in this book. If S is any set of sentences, we may show S is consistent by producing any interpretation under which all sentences of S are true. Usually, E will be still interpreted as membership, but the domain of dis- course will be some sub-domain of the hereditary sets. Thus, we shall produce one interpretation for ZFC +CH and another for ZFC +i C H without ever deciding whether CH is true in the intended interpretation. The justification for this method of producing consistency proofs is the easy direction of the Godel Completeness Theorem; that if S holds in some interpretation, then S is consistent. The reason this theorem holds is that the rules of formal deduction are set up so that if S t 4, then 4 must be true under any interpretation which makes all sentences in S true. If we fix an interpretation in which S holds, then any sentence false in that interpreta- tion is not provable from S. Since 14and 4 cannot both hold in a given interpretation, S cannot prove both q5 and The nontrivial direction of the Godel Completeness Theorem is that if S is consistent, then S holds in some interpretation, whose domain of dis- course may be taken to be a countable set (but we may not be able to inter- pret E as real membership). We do not need this result in our work, but it is of interest, since it shows that the notion of consistency is not tied to a particular development of formal derivability. In fact, if we allow infinitistic methods in the metatheory, we may dispense entirely with formal proofs and define S to be consistent iff S holds in some interpretation, and define S 1q5 iffq5 is true in every interpretation which makes all sentences of S true. It is then much easier to see when S t 4.In this approach, the Compact- ness Theorem (2.1) becomes a deep result rather than a trivial remark."

- But here, we move to Kunen's newer edition.

- TODO. We have to pursue this later. It's prerequisites indeed include PL/QL, but this is a bit confusing at the moment, because we thought that the semantics of QL necessitate set theory.

XVI Durchbruch with (Leary and Kristiansen 2015)

214. We now fully understand what a formal language is, a formal structure, their relation, the informal set theory used on the meta-level to define them, the fact that proofs about the syntax of the formal language can be formalized using a weak formal theory (PRA).

215. We understand that for the formal language, 'set' does not have to be mentioned in principal, all that is necessary is that the meta-formal definitions, theorems and proofs be effective.

216. We note that the constants of the formal language are mapped to elements in the structure. Formal language variables do not directly map to anything in the structure, the definition of formal truth allows them to be related to the elements in the structure to which the formal constants map.

217. We understand the 'problem' with free variables, and how a 'programming language' oriented terminology makes many of these concepts much more readable to a programmer.

218. We understand the reasons for the definition being the way they are.

219. We understand what a sentence (function with no arguments) and a nonsentence (function with arguments) are.

220. We understand the different signs of implication with perfect clarity.

221. We understand what a 'model' is and how a whole structure is (or is not) a 'model' for a single formula.

222. We understand that nonlogical 'truth' comes exclusively from the structure.

223. We understand that there is not 'total freedom' in what can be consistently made 'true' in the structure and how this influences the model. There is an easily describable essence of what happens here, and it is to some extent explained in the golden [@c26].

224. We fully understand the wiki page on the von Neumann Hierarchy, much better than before. We conceptually understand $V_W, V_{W+W}, V_{W+1}(\mathbb{N}), V_{W+2}(\mathbb{R})$.

225. We can read about what forcing is without confusion.

226. From [@26] we note the important of the mention of w-consitency, which would be needed to formalize the proof, and is stronger than 'plain' consistency itself.

227. We fully understand this quote from (http://math.stackexchange.com/questions/173735):

> I think an important answer is still not present so I am going to type it. This is somewhat standard knowledge in the field of foundations but is not always adequately described in lower level texts.
>
> When we formalize the syntax of formal systems, we often talk about the set of formulas. But this is just a way of speaking; there is no ontological commitment to "sets" as in ZFC. What is really going on is an "inductive definition". To understand this you have to temporarily forget about ZFC and just think about strings that are written on paper.
>
> The inductive definition of a "propositional formula" might say that the set of formulas is the smallest class of strings such that:
>
> - Every variable letter is a formula (presumably we have already defined a set of variable letters).
>
> - If A is a formula, so is ¬(A). Note: this is a string with 3 more symbols than A.
>
> - If A and B are formulas, so is (A□B). Note this adds 3 more symbols to the ones in A and B.

This definition can certainly be read as a definition in ZFC. But it can also be read in a different way. The definition can be used to generate a completely effective procedure that a human can carry out to tell whether an arbitrary string is a formula (a proof along these lines, which constructs a parsing procedure and proves its validity, is in Enderton's logic textbook).

In this way, we can understand inductive definitions in a completely effective way without any recourse to set theory. When someone says ”Let A be a formula” they mean to consider the situation in which I have in front of me a string written on a piece of paper, which my parsing algorithm says is a correct formula. I can perform that algorithm without any knowledge of ”sets” or ZFC.

Another important example is ”formal proofs”. Again, I can treat these simply as strings to be manipulated, and I have a parsing algorithm that can tell whether a given string is a formal proof. The various syntactic metatheorems of first-order logic are also effective. For example the deduction theorem gives a direct algorithm to convert one sort of proof into another sort of proof. The algorithmic nature of these metatheorems is not always emphasized in lower-level texts - but for example it is very important in contexts like automated theorem proving.

So if you examine a logic textbook, you will see that all the syntactic aspects of basic first order logic are given by inductive definitions, and the algorithms given to manipulate them are completely effective. Authors usually do not dwell on this, both because it is completely standard and because they do not want to overwhelm the reader at first. So the convention is to write definitions ”as if” they are definitions in set theory, and allow the readers who know what's going on to read the definitions as formal inductive definitions instead. When read as inductive definitions, these definitions would make sense even to the fringe of mathematicians who don't think that any infinite sets exist but who are willing to study algorithms that manipulate individual finite strings.

Here are two more examples of the syntactic algorithms implicit in certain theorems:

- Gödel's incompleteness theorem actually gives an effective algorithm that can convert any PA-proof of Con(PA) into a PA-proof of $0=1$. So, under the assumption there is no proof of the latter kind, there is no proof of the former kind.

- The method of forcing in ZFC actually gives an effective algorithm that can turn any proof of $0=1$ from the assumptions of ZFC and the continuum hypothesis into a proof of $0=1$ from ZFC alone. Again, this gives a relative consistency result.

Results like the previous two bullets are often called ”finitary relative consistency proofs”. Here ”finitary” should be read to mean ”providing an effective algorithm to manipulate strings of symbols”.

This viewpoint helps explain where weak theories of arithmetic such as PRA enter into the study of foundations. Suppose we want to ask ”what axioms are required to prove that the algorithms we have constructed will do what they are supposed to do?”. It turns out that very weak theories of arithmetic are able to prove that these symbolic manipulations work correctly. PRA is a particular theory of arithmetic that is on one hand very weak (from the point of view of stronger theories like PA or ZFC) but at the same time is able to prove that (formalized versions of) the syntactic algorithms work correctly, and which is often used for this purpose.

228. We understand that one could formalise 'number theory' using a formal language and some model such as Peano's. We understand that one can very easily formalise set theory by the inclusion of a single '$\in$' symbol on top of the (QL) formal language in the book. We understand (not stated) that number theory can then, alternatively to the first approach, be also formalised by building it inside the formalised set theory.

XVII Study in a Nutshell

229. From 'Combinatorial Set Theory, With a Gentle Introduction to Forcing', we extract an 'in-short' chapter on FOL and ZFC: 'FOL and ZFC in a nutshell' [FZnut]. We study this before continuing with Set Theory exercises. These should be taken from one of

1. 'Naive Set Theory'
2. 'The Axioms of Set Theory' (An appendix)
3. 'An Outline of Set Theory'

230. What is the importance and use of the 'deduction theorem' in [ZFnut p.32]

231. Here is a reproduced note from [ZFnut p.35]

> When did the quantifiers acquire their metaformal meanings such that we can prove thing about their fixed assigned meanings, a form of simple/lvl-0/metaformal/informal set theory?
>
> We search for the answer in [FItML]. The Chaff at p.14 with 'worry' is a first hint where to look. Indeed this happens in Def.1.7.4 at p.34 where we way per example:
>
> If φ is (»V«x)(q) and, »for each element a of« A, S |= α[β(χ|α)].
>
> The accompanying Chaff confirms: "Chaff: Also notice that we have at last tied together the syntax and the semantics of our language! The definition above is the place where we formally put the meanings on the symbols that we will use, so that V means "or" and V means "for all""
>
> The 'Soundness Theorem' tells us that the way we defined language, structure, truth, proof, metaformal/informal glue (see above), a strong statement can be made about the whole: That there is no model (not to confuse with structure: a model of the proof's axioms is a structure in which the axioms hold) given our constraints of what model is —and what the glue/transfer metaformally is— such that T syntactically proves phi, while phi in the model (the structure in which T holds) is not semantically true .

232. We finally have a clear understanding of Goedel's three theorems. A very unexpected serendipity and a proof that our 'in-short using the right appendix' is fruitful, while bouncing from a more verbal (and/or friendly) book for support.

XVIII Rabbit Holes

XVIII.1 The Reals Are Not a First Order Theory

233. In [@c17, p.42], Schechter says that "Actually, the usual axioms for the real numbers do not make a "first-order theory". Most of the axioms, such as $x + y = y + x$, are about numbers; but the least upper bound property is an axiom about numbers and about set of numbers. It is not possible to describe the real numbers as a first-order theory. One might expect that a simpler system, such as the integers, can be fully explained by a first-order theory, but Goedel showed that even that is not possible."

234. Here we are confused and we ask: Is not set theory based on a first-order language?

235. Compare the following from http://math.stackexchange.com/questions/102961/is-there-a-first-order-logic-for-calculus to our first note, things are clearly not that simple. “Real analysis, including set theory, can be formalised within ZFC. As a first-order theory, ZFC has a countable model by the Löwenheim-Skolem theorem, so the answer to your question is yes. You ask whether there is a ”logic for calculus”, which is a little bit misleading: we usually think of a logic as a language together with a proof system and a semantics. First and second order logic are logics or formal systems under this description. What is needed for calculus in addition to the basic formal system is a signature consisting of those non-logical constants required to pick out those relations, functions and constant elements which we would need to refer to in order to axiomatise the structures involved (in this case, the complete ordered field), and a set of axioms sufficient to prove the theorems of calculus. As I said above, the background theory employed is usually that of set theory, but in fact that’s not necessary: one could drop the powerset axiom and in its place just assert the existence of R, and perhaps the set of functions 2^R, depending on how much one wanted to prove.” We also read: “As others have mentioned, you aren’t looking for a logic in which to do analysis, but a signature and a model. In this respect you must be somewhat careful: the structure (R,0,1,+,dot,leq) will not give you analysis. In fact, it admits quantifier elimination and is decidable, so it is quite boring. The classical structure of analysis is the structure (R,Z,0,1,+,dot,leq), with a ”name” for the integers. This allows tuple coding, and all the good stuff that comes with it. Of course, by Lowenhiem-Skolem, you can get countable models that admit analysis.”

236. Some answers to this questions is found in https://johncarlosbaez.wordpress.com/2014/09/08/the-logic-of-real-and-complex-numbers/. TODO!!!! very important.

XIX Crumbs

XIX.1 Fast Reading of (Schechter 2005)

237. p.238,239 is eye opening.

238. p.290 finally allows us to grasp ’finite refutation’ and an application of the topological fact of the ’finite representation principle (p. 142)’ in an excellent manner. Is this in some manner (vague or not) related to (235) ?

XX Miscellanea

239. It may be that Kolmogorov complexity is the right, although very much longer path, towards incompleteness[@c6]. At the same time this makes us appreciate the brutal and direct sharpness of the much shorter path.

240. “Mathematical logic is what logic, through twenty-five centuries and a few transformations, has become today.” (Jean van Heijenoort)

241. “In this approach, the Compactness Theorem (2.1) becomes a deep result rather than a trivial remark.” [@c11]

242. The ”Belief is not mathematics” rule. This automatically excludes all platonism-like arguments to justify arbitrary and ’orthodox’ choices.

243. ’ Die Architektur der Mathematik: Denken in Strukturen’ says that there are three fundamental types of mathematical structures:

1. Order structures
2. Algebraic structures
3. Topological structures

Per example, let M be a set. Now look at M as structured respectively by

1. A relation, may be seen as a directed graph

2. An operation, may be seen as a Cayley table
3. A distinction of subset, may be seen as a fragmented Venn diagram

These structures are the 'mother structures' of Bourbaki. This is taken from https://www.researchgate.net/post/What_is_the_essential_difference_between_Algebra_and_Topology, also here is a failed discussion at delineating algebra and analysis: https://www.researchgate.net/post/What_is_difference_between_Algebra_and_Analysis

XXI Status

244. We have decided, despite being crucial and interesting, to forego Gentzen style sequent calculus and natural deduction and again follow Tourkalis (read seriously, including his set theory book) and Rogers (read leisurely). But not before first reading Schechter's pluralist logic book. Continue here: 115.

245. After a very long time, we were lead back here. After re-reading this very useful document, we followed the advice of 'Teach Yourself Logic 2017' and landed on [@c25], which in combination with our re-reading, answered what seems to be all the main difficulties we have here, in a very simple and direct manner. We summarize them in XVI.

246. For the moment, we pose here. We could continue with [@c25] to attack the Goedel's proofs, or alternatively in [@c27] as recommended in 'Teach Yourself', although reading [@c26] before that would be useful. In general it seems it would be worthwhile to try to devise a special trivialising notation which could potentially make the proofs look very easy. From [@c27] we already see it is a irrational-numbers-like diagonlization, proving the escape of truth-in-the-structure elements from the countable number of proofs in the formal language, this hinging on diophantine-like-questions.
We pose here because all that we cared about for the moment is renewing our understanding of QL (FoL) in order to build into our notations elements of it and able to express axiomatic set theory on top to be able to render arguments and proofs in our notation. This is because we now understand existence proofs and definitions in a new light.
for the purpose of a short but articulated treatment of axiomatic ST, we found golden the appendix section of [@c28]. A strange but maybe excellent book is 'outline of set theory'.

Bibliography

Beklemishev, Lev D. 2007. ***Provability, Computability and Reflection, Volume 152 (Studies in Logic and the Foundations of Mathematics)***. New York, NY, USA: Elsevier Science Inc.

Béziau, Jean-Yves. "The Mathematical Structure of Logical Syntax."

Brooks, Kyle. "Propositional Logics and Their Algebraic Equivalents."

Chiswell, Ian, and Wilfrid Hodges. 2007. ***Mathematical Logic***. Vol. 3. OUP Oxford.

Curry, Haskell B. 1977. ***Foundations of Mathematical Logic***. Dover.

Dunn, J. Michael, and Gary M. Hardegree. 2001. ***Algebraic Methods in Philosophical Logic***. Oxford: Clarendon Press.

Etchemendy, John. "The Concept of Logical Consequence."

Ferreiros, Jose. 2001. "The Road to Modern Logic-An Interpretation." ***The Bulletin of Symbolic Logic*** 7. Association for Symbolic Logic.

Fomenko, A.T. "On Undecidability and Millipedes with Art." https://wumath.wustl.edu/news-newsletter/articles/999.

Gilbert, William J., and W. Keith Nicholson. 2004. ***Modern Algebra with Applications***. Singapure: John Wiley; Sons.

Gries, David, and Fred B. Schneider. 1993. ***A Logical Approach to Discrete Math (Monographs in Computer Science)***. Hardcover; Springer.

Hajnal Andreka, Istvan Nemeti. "Reducing First-Order Logic to Df3, Free Algebras."

Hodel, Richard E. ***An Introduction to Mathematical Logic***.

J. M. Font, D. Pigozzi, R. Jansana. "A Survey of Abstract Algebraic Logic."

Kharazishvili, Alexander B. 2014. ***Set Theoretical Aspects of Real Analysis***. CRC Press.

Leary, Christopher C, and Lars Kristiansen. 2015. ***A Friendly Introduction to Mathematical Logic***. Lulu.com.

Li, Ming, and Paul Vitanyi. 2008. ***An Introduction to Kolmogorov Complexity and Its Applications***.

Mendelson, Elliot. 2009. ***Introduction to Mathematical Logic***. 5th ed. Chapman; Hall.

Nohra, Jad. "Notebook Big-Blue."

Post, Emil L. 1941. ***On the Two-Valued Iterative Systems of Mathematical Logic***. Princeton University Press.

Reckhow, Robert A. 1974. "On the Lengths of Proofs in the Propositional Calculus."

Rogers, Robert. 1971. ***Mathematical Logic and Formalized Theories a Survey of Basic Concepts and Results***. Monograph Collection (Matt - Pseudo).

Schechter, Eric. 2005. ***Classical and Nonclassical Logics: An Introduction to the Mathematics of Propositions***. Princeton, NJ, USA: Princeton University Press.

Smith, Peter. 2003. ***An Introduction to Formal Logic***. Cambridge University Press.

Tourlakis. ***Mathematical Logic***.

———. ***Lectures in Logic and Set Theory, Volume I***.

Vickers, Steven. "Topology via Logic."

Wagner, Roy. 2008. "Post-Structural Readings of a Logico-Mathematical Text." ***Perspectives on Science*** 16 (2). MIT Press: 196–230.