Preliminary Notes on Everything

(*The second most futile but equally crucial note document)*

It contains candidate notes for 'Notes on Everything'. Unlike in that document, most notes here are either superseded by better ones, or found to be wrong after more analysis, or are too vague or badly formulated. It is nevertheless valuable to keep all these 'negative results' because they help converge to the correct ideas.

Understanding, Knowing, Thinking

1. {WEAK} Why do we need the visual real number line along with a static of dynamic point scene to think about limiting processes and convergence in analysis. It seems that what we need is a picture, but that is not the real need. To pass from idea A to idea B through a link of transformation L, we must have all of A,L,B in mind simultaneously (maybe B only partially since it is still forming). This seems impossible when thinking verbally (is it really?). It is very possible with a mental image because of the parallel treatment of it, or at least the working of very short term memory giving the illusion of parallelism. Is this the only real possibility of thinking from A to B we have? Can formal symbols be treated like these images? Maybe sometimes and with training. Are the symbols stripped of 'meaning' and reduced to the allowed transformations on them at the time of processing? Is this related to the speed of visual processing compared to the storage time-frame of short term memory? Pronouncing a word mentally might already too slow for it to fit in this memory while another linking work follows? How do we get around this limitation? At the same time, we know we DO use verbal thinking. This is unclear.

2. {WEAK} All modes of thinking are valuable: Visual, formal/symbolic/manipulatory, relational, causal, dynamic/process/sequential, exhaustive/random/exploratory.
    a. Since the possibilities of symbolic transformations are usually large, it is a good idea to guide this mode of thought with strong relational thinking, or wishful (proof) thinking (Euler mode).
    b. The exhaustive mode should not be excluded or forgotten (Euler mode).
3. {VAGUE} Why do we consider differentiation, integration, etc. ? We do because of applied mathematics, and not by exhaustion of mathematics. On a lower level, limit, number, etc. are also motivated by applied mathematics. They exist purely by and for themselves, but of all existing creatures, the choice is necessarily motivated by the applied, directly or indirectly.

   Possible VQ. Lakatos p13 note1. It was mineralogy!

4. {VAGUE} Are the 'details' of the proofs (e.g in analysis for 'convergence', 'completeness', etc.) details? Or are they the 'essence' of number, function, etc.? They would be if they were the 'only essence' but are they<open>? I think not, or let us simply say, some may, some may be not. Those that are not we could think of as 'accidentally true', but motivated by applied mathematics. Feeling accidental makes them appear as details and therefore difficult to remember without effort [1.] We have to 'be the number', 'be the limit', 'be the point of continuity', 'be the sequence' in order to internalize the essence of the details/proofs. Is this 'retainable' for the long term? Why does it feel desirable? To allow deep and flowing thinking. It is strange that they feel like details, whereas the theorems that they prove do not.

5. It is amazing how stubborn the primitive mind is. Even after having understood and accepted a concept that is very general, it still contents itself to apply only within the restricted domain through which it was initially discovered, which is fatal. I was led to this observation after reading about Euler's treatment of asymptotic development of series (computing a sum for a very large n) and using calculus to manipulate the series, which is, for the stubborn primitive mind quite a surprise, despite having *understood* calculus in its generality, which was obviously only an illusion of understanding. Either the calculus is accepted or not, and as soon as it is, it must be linked to the essence of *functions* as a very fruitful classification and powerful tool when it applies. [EHM p.251]

6. Surprise is an excellent indicator of lack of understanding. I was surprised while studying basic algebra of the amount of restriction and consequences that emanate from the polynomials being a group. A simple analogy helps retain the idea. Imagining an infinity of particles in a bowl, the shape of the individual particles plays a determining role in the structure of the material. We can say the same about the mathematical structure. Knowing this so far surprising fact is important for removing the bias of underestimating the scale of structural consequences resulting from even the tiniest local feature.

7. Add note on studying the 'how' being an excellent path to 'understanding' the 'why'. A long term past blocker, also for the Greeks, freed by Archimedes (or was it Galileo), reference?

8. Related to our englobing approach to modes of thinking, is the quote from [UQ p.213]

   "*… a sort of Rome/Athens contrast. Weierstrass and Riemann exemplify the two styles. Weierstrass, of the Berlin school, could not blow his nose without offering a meticulous eight-page proof of the event's necessity. Riemann, on the other hand, threw out astonishing visions of functions roaming wildly over the complex plane, of curved spaces, and of self-intersecting surfaces, pausing occasionally to drop in a hurried proof when protocol demanded it."*
   In our developing philosophy of thinking, schizophrenia is what should be used here, we should be both, and both to extremes. Additionally, such statements will actually mislead most readers. They might think that Weierstrass by nature did not have the wild-creative

capabilities of Riemann, while Riemann did not have the respect or skills of hard-proof like Weierstrass. But this is surely wrong, they both had it all, it is only their style, their choices of problems, their presentation that differed. In our case, our hallucinations seems to be coming from the 'Goettingen (Riemann)' school, and the lesson to learn for our hallucination engine is that it should be even bolder, extreme and confident.

9. Related to 8, note who treacherous our biases and brains are by default. The functions do roam there like Riemann' envisioned', it is clear to anybody dealing with them. However, the mere fact of never having observed it with the biological eye, is enough to stop most of us from having such visions. The next component is a good technique to make something out of them that has at least the germs of amenability to mathematical expression.

10. An easy way to improve memory (I hope), knowing it is a hierarchy of pair relations, is to also focus on qualifying the kinds of relations.

11. Related to 3. and confirming strongly, also related to [GAGC 1], is [UQ 4], a very important quote indeed. We already knew that there is a large difference between current and historical motivations, at least because of many things being already solved. We also knew that how many if not all seemingly abstract and out of the air breakthroughs were very practically oriented, revealed by computation or not need for it. Imagine here Newton doing numerical integration by hand, etc. But the quote [UQ p.258] really brings the point home. It is a great bias of the computer age, and especially for a programmer, that computation is exactly what needs to be automated, and not at all that '*computation can be fun*'. There is nothing wrong with that, but we need to be aware of it, and, at the very least, use the computer to do these hint providing computations or drawings, at the slightest appearance of an idea that would benefit from them. Let us here repeat and stress the validating part of the quote.
    "*It is important to realize that the common man, except in special professional cases (painter, architect, Greek altar designer, etc.) really has no direct need for geometry, even the simplest Euclidean geometry: Lines, circles, intersections, angle determinations. Let alone conics and conic sections. Historically, much of Geometry was initially motivated by needs, and of course then by curiosity, and the determination of the mathematical mind to clarify and explore."*
    This quote also pinpoints a very specific computer-age weakness, which is at the trigger of much of my needs to read about the history of geometry, but also a sign that the weakness is in the process of being healed, especially with the current reading of GAGC, UQ (unexpectedly), and HGM (Coolidge), which started to become accessible after GAGC and MCMM.
    "The interest in curves was a great mathematical growth point in the middle 19th century, nourished by algebra and calculus as well as by geometry. It is an easy interest to acquire, or rather it was in the days before math software came up,"

12.

Writing

1. {REFORMULATE} Before 'writing' an idea, 'meta-think' about it. What are the 'objects'? What relations do we want to express? What are the words that best capture the relations? What is the sentence structure that captures the relations?

2. {WEAK} Remember Russell. Shortest clearest meaning.

“Analysis by Its History” (2008), Ernst Hairer, Gerhard Wanner

1. {WEAK} Hairer says 'What is a limit? A number'. We must add: 'What is a number? A limit (in the Cauchy sense)'. However, some numbers are exclusively accessible as limits, others are also accessible by other means: induction for integers, field axioms for rationals.

I. Sequences, Real Numbers, Series

1. {SHALLOW} A sequence can be visualized either two dimensionally and statically (index, value) or one dimensionally and dynamically (moving values). Another possibility is a cute Sheep herd, either looking at their next neighbors or at some cluster point.

2. Keep in mind that all the proofs like B/W and the supremum of sets with Cauchy sequences are proofs of existence, despite the fact that they use recursive construction. These constructions are not computational but existential. This is important for the right perspective and understanding.

 * TODO nested intervals and Cauchy seq.
 * TODO "Hauptlehrsatz" analysis the interplay between domain and range.

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§ Notes on Metric Spaces

 * Quote from TODO.

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§ Notes on Z.A.I (Zakon, Analysis vol. I)

“Analysis I”, Zakon

1. The 'invariant' view on limits seems to be triggered by a wish to put some familiarity into a seemingly 'still' very wild concept. And so it is sad that this has to be repeated, but also necessary. Maybe the new algebraic point of view finally casts this into the right position. The concept is 'necessary' by the unconstrained filling of all Dedekind gaps, which is logically the least biased kind of 'filling', that is: completion. The same can be said about limits of conv sequences. Note that the 'limits' of non-conv seqs could be also studied, they would not be numbers, they would include the two infinities, then sets of two reals (eg alt 0,1), etc.
   In any case, with such a wide creature as the real field, studying existence is the most basic that can be done, existence of solutions to any kinds of eqs, or systems of eqs, eg diff eqs. It should not be surprising that this can be done, just as converg can be determined even though the proc is inf. We are back to the old phil. Disbelief of the possibility of study of inf, finitely. But the answer is simple (ser kurosh note): from finite props we can def wild inf set. Similarly, by finit props of a process, we can determine its convergence.

2. The bounded set in Zakon, must finally get a life of its own. Related to the note above, there are structures through which the finite study of infinite limits is made possible. The bounded set is one such fundamental structure. A wild set, with one single finitist quantity of information about it: a bound. We can see the abstraction here. It is not important how this bound was found, but this is simply a classification of (infinite) sets, those about which we have this very simple additional bit of information. Logically this seems to come next after arbitrary sets. In a way, it would be nice to have (write) a Russellian kind of treatment, not of number, as the one we read, but of the concepts in Zakon I and II.

3. There is no 'guarantee' that such a finit. study (reduction of an infin seq to a fin conv result) can always be done, but it is always necessary to try. Without the unbiased understanding from Note 2, this is not even attempted.

4. The book 'Numbers' (Ebbinghaus et al), presents with utmost clarity and completeness, without getting bogged down by details, the three routes to modern real numbers, including their very old origins (starting p.27). What is especially new is the nested internal route, which we have not uncovered ourselves, focusing only on Dedekind Cuts

and Cauchy sequences. The compactness of the presentation fits perfectly with our method of expanding for clarity and compacting for (binary) interconnections. The fog around nested intervals has surely been cleared. This chapter is the perfect complement for Zakon's book.

“Éléments d'histoire des mathématiques” (1960), Nicolas Bourbaki

1. There is no doubt that the formal definition of derivative is the best possible expression of the intuitive concepts of tangent, rate of change, linear approximation, etc. and pinpoints a very specific and unique relationship between two functions. Nevertheless, the weak human mind persists in challenging the idea. During one such resistant episode I noticed that the popular description that the derivative is the best linear approximation is not a very clear one. We need to define what *best* means, and with the standard definition, *best* is geared towards smaller and smaller intervals. One could imagine a practical situation where *best* is the derivative that when integrated by assuming uniform change for a specific interval *dx*, gives the most correct answer. The correct answer being the value of the standard integral. This might be weak argument because we seek generality and not dependence of the derivative on some specific value of *dx* and so we might invoke Occam’s razor in favor of the standard definition. Despite the weakness, this leads us to the idea of assuming for a very short moment that everything physical including *change* or *time* turned out to be discrete, the derivative immediately stops giving the right answer when applied. We should see the matter differently yet: The formal definition is the only definition that leads to an *everywhere true* {???} formula for the derivative when it exists, allowing the calculation of the derivative at any point. And when that is not what is desired, as in the crude and contrived example above, where change is uniform during the smallest discrete time interval, there is no problem to begin with since we can directly use the differential equation without passing to the limit.

2. My idea of using two horizontal number lines to think about continuity also works for derivatives.

3. Despite trying to take a historic route, AH still has to make sacrifices to keep itself short. I have noticed this when comparing what I know about the initial discoveries of series from AH and my reading of this book. I have tried to internalize a specific interplay between the theorems discovered at the time, especially noting the use of the binomial theorem. However, this book showed me the story is much more complicated and makes even AH appear to be artificial. I am more and more understanding the value of studying the modern treatment and understanding the often artificiality of trying to situate myself back in time. I can never *really* do that without making it a serious and effort taking goal by itself. The proofs from the period will almost never be reproducible or even seem like rigorous proofs, at least the ones from this period. Also, I would have to

assimilate all the faux pas that were taken, and all the obscure methods and proofs, which although illuminating for the understanding of how the brains of the greats worked, requires quite some time. This is even less desirable when a final methods ends up superseding and trivializing a considerable amount of previous work, like per example the discovery of the link between integrals and derivatives. These are real difficulties. An alternative is only reading for culture, or at least not taking the exercises too seriously when not necessary. When searching for the *essence* it is turning out that it is to be found in the future and not in the past. This is especially true when with regard to the essence of the mathematical object itself in contrast to the essence of its historic and human development. These are two different things.
On the other hand, not knowing the history at all makes the study too pointless.

4. The modern point of view would be the best one if only it be complemented with two studies (which is not the case). A purely historical study, and this is something that we have long established, but also a meta-mathematical one in the sense I will now explain. I am not talking about meta-mathematics in the traditional sense of logical study, but about a discipline that *proves* that each axiom stated in mathematics makes sense and how it relates to all things non-mathematical that it is related to. By proving I mean establishing logically and as rigorously as possible the meaning of the axiom, the reasons for a specific choice of words from all possible choices that are not mathematically equivalent and might non-mathematically (and therefore vaguely) mean the same thing. I suppose that during mathematical instruction this is left to the instructor's creativity and to a part of the exercises, while during non-mathematical instruction the link remains in the domain of magic despite the honest attempts to show that it is right because it makes a lot of sense in this and that special case which we know is no proof at all. As particular examples I have in mind the definition of a derivative for which examining the subtleties [1.] easily raises doubts to whether this is what we really intuitively mean by it, by tangents in graphs, by rates of change in physics, etc. Other examples are continuity, the definition of a metric, etc.
   Ultimately, what I have in mind is a discipline that *meta-proves every* axiom used. It will indeed turn out that the axioms are what we really mean but to get to that stage with full conviction and logical rigorousness quite some work will have to be done. Work of immense importance when it comes banishing magic and illusion. By magic I mean the feeling that mathematics works (either applied to external disciplines or to internal branches) because of some obscure relation or trick that only the magician (mathematician), or maybe even *no one* knows about but it simply there because it was found (either by *genius,* or by trial and error), and that the relation to the actual application in mind (even when mathematical), never having been logically analyzed to exhaustion is also burdened with a sense of magic. By illusion I mean a blind trust that the axioms correspond exactly to any application at hand as soon as they are found to be the closest axioms available from the arsenal or all axioms and theories, and that the theory is always perfectly adapted with no limitations nor possibilities of refinement or subtle modifications with great difference in results. Such *meta-proofs* are not readily available. The explanation for that are manifold. The excitement of using what is already

working, the justified trust in the work of the greats and the successful use by multiple generations, the difficulty of making such proofs indisputable, the justified fear of *falling* into meta-physics. It is therefore all natural that instead of facing the admittedly scary task of creating such proofs, they are left for the instructors to handle, keeping mathematics clean, and the minds of the students *dirty* and burdened, sometimes forever.
We have to admit that for some students, mentally predisposed to accept axioms as given without ever inquiring why, the whole problem does not present itself. But such a predisposition is in my opinion no good thing. Only a minority possesses such predisposition, and for all the rest our two complementary studies are immediately necessary. Such predisposition is mistakenly praised, seen an innate readiness for abstraction.In reality, it is a uncritical attitude towards authority and as such cannot be commended.

I found out weeks later that my idea of meta-proof is not new at all, so I am happy about these v-quotes:

a. Justifying Definitions in Mathematics—Going Beyond Lakatos, Charlotte Werndl*
b. A Critique of a Formalist-Mechanist Version of the Justification of Arguments in Mathematicians' Proof Practices, Yehuda Rav
c. What is Dialectical Philosophy of Mathematics?, BRENDAN LARVOR*
d. http://en.wikipedia.org/wiki/Imre_Lakatos. With the specific example of his work: ‘Proofs and Refutations’
e. http://archives.lse.ac.uk/TreeBrowse.aspx?src=CalmView.Catalog&field=RefNo&key=LAKATOS
f. We need to add our ideas about ‘technical result’ and their validating quote also due to Lakatos. The quote not only validates but explains and elaborates much better than I (unsurprisingly). ‘...consider the Carath´eodory definition of measur able sets, another proof-generated definition Lakatos discusses. The mat hematician Halmos [1950, p. 44] remarks on this definition: “The greatest just ification of this apparently complicated concept is, however, its possibly sur prising but absolute complete success as a tool of proving the extension t heorem”. Lakatos [1976, p. 153] comments: as we learn from the second part [Halmos’s remark above], this concept is a proof-generated concept in Carath´eodory’s theor em about the extension of measures [...]. So whether it is intuitive or not is not at all interesting: its rationale lies not in its intuitive- ness but in its proof-ancestor.’ (arXiv:1310.1625v1 [math.HO] 6 Oct 2013 Justifying Definitions in Mathematics—Going Beyond Lakatos)
g. Linking perfectly to our doubts about uniform convergence is this quote: ‘Lakatos [1976, pp. 144–146] argues that uniform convergence is proof- generated, also by referring to textbooks. This definition falls und er the subject ‘convergence and divergence of series and sequences of functions’’ (arXiv:1310.1625v1 [math.HO] 6 Oct 2013 Justifying Definitions in Mathematics—Going Beyond Lakatos)
h. VQ, Proofs and Refutations p9, notes. This relates to my criticism of exercises left to reader while the theorems presented in the section. This is not

mathematics.

i. google, dialectical mathematics.
j. The ultimate VQ. Proofs and Refutations p15, note2.

5. As soon as the limit based definition of derivative and integral is understood, all the known elementary derivatives and integrals appear in a new light: they are very special cases. There is no necessity that the limit forms be *equal* to nice finite equations. On the contrary, the fact that the derivatives and integrals of elementary functions are elementary appears as quite surprising instead of intuitive.
   Even at this early stage, the three-body problem, a driving topic for my search of truth, implodes to dust conceptually but luckily not technically. It is also more and more evident that computability is, in our age, the more relevant classification. Before computers, computability went hand in hand with the existence of simple to compute by hand, finite formulas. From this perspective the history of the related mathematical developments makes complete sense.

6. TODO!!! Is it not a good idea to study MMCM as a book?
   It seems that it is as the same time specific and example oriented and abstract and advanced.
   Instead of swallowing whole theories/books in their encompassing character why not this approach? seriously consider this!
7. TODO!!!
   a. Limits for processes, compared to the natural approximation of gt2 which is process free as least on a first view, even if it can be reduced to a process like limit.
   b. The historic path proceeds from confusion to clarity, from hints to essence and not the other way round. It does however show the essence of human thought and its processes.
   c. The meaningfulness of derivatives, that is rate of change related concepts, is forced upon the early investigators, while the essence remains elusive, until Noether's theorem. It is only with this theorem that the fact that the specific derivative limit definition means what it does in physics and can be applied is explained. If only by reducing it to the fact of symmetry (assumed or proved).
      Real analysis does exist mathematically in and for itself, and there is no problem with this. It simply has to be remembered that the relation to this specific limit formula relates to real world applications by a matter of fact and not logical necessity. It is not the historic beginnings of the calculus that explain the relationship; what they document is the beginning of the human acknowledgment of it and their attempts of quantization, for goals of prediction and understanding, both qualitative and quantitative.
      In other words, the specific choice of our limit formula amongst any other which we questioned in note 1 [1.] is explained (reduced to a more elementary assumption that does not involve rate of change but only *static* symmetry) by

Noether's theorem. The conservation laws that entail produce the physically meaningful phenomenon of velocity which was the link to the scientific observations and experiments, which in turn transformed into 17^{th} century calculus. Having understood this frees the mind from the burden of trying to attach *meaning* to the formal definitions and instead simply observing the historic evolution that led to them. After fully understanding Noether's theorem, we will have come full circle and gave them as much essential meaning as possible, while at the same time having observed the human discovery of it.
The principle of minimization of potential energy is of similar (to symmetry) status that inevitably leads to variational calculus.

d. The same applies for integration, whose essence is additivity and invariance under a (specific?) set of transformations. These two things completely characterize integration. Indefinite integrals that are the elementary ones and that makes complete sense in terms of classification. Definite ones a special narrow case, and the relation to derivatives in the narrow? scope (see EHM) where it applies quite special and almost accidental, regardless of its historic importance.

8. Our idea about analysis being about what can be said finitely to be true about infinite processes automatically includes limits of functions among others.
9. The counter-intuitive difficulties of real analysis based calculus are without exception non-mathematical and their resolutions are therefore non-mathematical. The choice of the formal definitions, the existence of a meaningful rate of change as the parameter approaches zero, the rate of change at a point being related to the form of the function but not to any specific neighborhood since any single change at neighbor point does not affect the derivative. The first difficulty is resolved by the note 7.c [7.c]. All the other difficulties are simply caused by wrong intuitions about limits, having their sources in the arithmetic of the finite. We must not deny that even the notation invites the beginner to confusion; an expression like $\lim_{h\to 0} h/h$ necessarily evokes the memory of a division by zero.
10. Rate of change is the cornerstone of the derivative's definition. Let us first consider change itself, and then its rate. The philosophy of change and motion, which is extremely rich, is not *settled*. It may never be resolved and even if it would, it will not turn out to be as simple as the layman's understanding of change. In any case, is change exist as a concept by logical necessity? I feel that it is so, as soon as any form of regularity and consistency exhibiting existence is assumed (e.g human consciousness). If change is a logical necessity, what makes its rate special? Is its specialty also logically necessary? I feel not. Purely abstract mathematical truth does entail the specialty of rate (of change) by Noether's theorem when we interpret it in reverse. What I mean is that even though the theorem is built upon the Lagrangian, which is built upon the definition of a derivative, I note the fact that the implication of the theorem "the derivative of the conserved quantity *as formally defined* must be zero" distinguishes the formal definition from any other random limit formulas and gives it meaning. But Noether's theorem is based on the assumption of symmetry, a property that itself is static, independent of change and more basic that it. Nevertheless the factuality of symmetry is purely empirical. Therefore, while the formal definition of derivative does have a mathematical

existence in and for itself, its *meta-proof* has both logical and empirical components.

“Mathematics. Its Content, Methods and Meaning” (1900-1963), A.D. Aleksandrov, A.N. Kolmogorov, M.A. Lavrent’ev

1. Given the absence of a known functional form of the solutions of a differential equation, either by a proof, or by a current state of affairs, what other ways do we have to study it? Numerical approximations are one option, and they are a common computer-age alternative. Another option is topology and an immensely powerful one. We can imagine how, stuck in terms of a functional representation, Poincare and others turned to still wanting to know more, and that some properties of the solutions are intuitively accessible from looking at the vector field of phase space of the differential equation sampled at a number of points. As soon as this intuition has to be turned into mathematics, we enter topology. Herein lies the applicational and historical birth of it and it makes complete sense.

§.II. Analysis

* TODO add notes from email? about the existence of a derivative in the formal sense is the right classification of function where avg velocity makes at all sense in the first place! This is a possible resolution of our problem.
* TODO paradox between conservation of energy and discrete/computing universe. (maybe this helps? http://mathpages.com/home/kmath637/kmath637.htm)

1. (p66) "Mathematical analysis is the branch of mathematics that provides methods for quantitative investigation of various processes of change, motion and dependence of one magnitude on another". Although purposefully crude, the by now understood extensive generality of analysis puts the expression 'dependence of one magnitude on another' under a new light.

2. (p67) Let us compare the example given about gravity (determining the 'instantaneous' velocity during vertical free fall) to our doubts about the 'meaning' of derivative and tangent. The example is down to earth and makes a lot of sense, and this hints leads us to think that our intuitive problem with the formal definition of derivative is essentially a intuitive problem about infinity, i.e., the infinite divisibility of intervals.

3. (p68) If we think about what we are doing here to determine the force acting on a wall of a water reservoir using calculus, it turns out to be a mental travesti at very first sight, and this is surely an intuitive obstacle that prevents many from taking this path of thinking to resolve this problem. The reason for the non-intuitiveness comes once more from our finitist biases, in the sense that the thinking process we are about to describe is never

used and indeed a sure dead end (and therefore to be avoided) when dealing with finite (discrete) problems. It feels wrong to start on purpose with a 'wrong' statement (that has an error term in it) and to expect that by manipulating it we might arrive at the right result (despite the fact that this has been used by humans since a long time ago e.g., the Greeks). This strange method evokes paradox-like descriptions like: The perfect approximation, the non-approximate approximation. Passing to friendlier ones like: The limit of an approximation, the removal of error in an initial approximation, the refinement of error in an approximation, the transformation of an initial approximation to strictly better ones, the mindful addition of a finite (but precisely expressible) error followed by its progressive elimination from an expression. Even friendlier ways to say it are: The decomposition of a complex problem into simpler ones, but this time an infinitely of finitely describable ones. An even better way to see it, and which brings in the limit very clearly is: Relating a clearly definite finite creature (the weight of the column of water at a 'point') to a necessarily infinite one (the force on the surface formed by all the points in a wall') by a process called limit which, when it converges, logically identifies a unique answer (in this case a number). This last sentence might as well be the essence of a large part of basic analysis.

4. Another latent discomfort that this reading elucidates is one of absolute and relative differences. Clearly lim $h \to 0$ f(x+h)-f(x) is zero. Nevertheless $\lim_{h\to 0}$ (f(x+h)-f(x)) / h must not necessarily be so. Again this is a finitist bias, and it is very obvious that the above is true in infinitesimal analysis by considering even the simplest function such as f(x)=x. It is amazing how long this latent discomfort has survived.
   a. The usage of the absolute difference limit is silent when comparing two functions. For all functions it is zero, at least for all functions where it makes sense, which brings us to its usage for defining continuity. It is there where it has meaning. In this case, it is not the value of the difference that we are interested in, but in the possibility of obtaining any desired (as small as desired) difference by perturbing x.
   b. As for the relative difference, we may say that the hint to form this ratio comes readily from intuition, in terms of direction when moving, that is the direction of the velocity vector, and also from tangents. We may imagine the point f(x) as not only a point, but one endowed with a direction, just like a human (not a fully symmetrical creature, i.e., not a sphere and not a point) moving 'forward'.
   c. TODO add note from email? about singularities! this is related to dividing by almost 'zero', to neighborhoods, to local behavior, to topology.

5. (p67) "uav = Δs/Δt = gt + (1/2).g.Δt . Letting Δt approach zero we obtain an average velocity which approaches as close as we like to the true velocity at the point A".

   I find 'true velocity' to be nowhere properly defined. Such sentences make it sound like we know what 'true/instantaneous velocity" is, either intuitively, or that we have defined it, or maybe even that we understand it philosophically. But we know that philosophically the topic of change is not settled, that intuitively it is vague even qualitatively due its inseparability with infinity. The definition given is one of a limit, which renders sentences like the above circular in their attempt to explain true velocity.

Average velocity seems less problematic, maybe precisely because it need not involve limits (at first sight, before needing to measure some square root of two distance) is purely arithmetical. The average velocity multiplied by time gives the position difference.

As we know from the definitions of a function limit, they ultimately involve a sequence. TODO fix this: the essence of the limit of a function is that of a decreasing h, this can be turned into a sequence and is then equivalent (iff.) but this is only a variation in description amenable to easier analysis. We also feel nested intervals at play here, or simply any strictly decreasing function of h (e.g 1/x sampled, giving the sequence definition). TODO <rel. to function limit discussion and resolve>. Note that for integration, the presence of a sequence occurs more naturally (TODO with resolution).

The average velocity is easily accepted as 'uav = gt + (1/2).g.Δt'. This is the velocity at a specific point (time t) and is a function of Δt. In other words, for each Δt there is generally a different velocity.

Conceptually we could say that the 'true velocity' is the velocity with which the 'particle' (TODO!!! slope/direction vs. speed!!! slope in x/t graph is 'speed' !! latent confusion!!) will continue with if abruptly any 'constraints' disappeared (TODO vague!! in which graph???). In that case, we can use the 'left' limit and get 'uav = gt - (1/2).g.Δt' which must give the same result (by the definition of a limit with is 'double sided' by default). Very informally, is this the velocity at which the particle would continue? Any specific Δti, with a velocity of gt - (1/2).g.Δti' is contradicted by any different Δtj! 'Geometrically' the tangent would intersect the trajectory for any specific Δti. The only uav that cannot be contradicted is one for which no 'next smaller' Δti exists which brings us to the problem of infinite divisibility and its resolution by limits. Visualizing the problem for a curve going downwards, we see a supremum-like situation concerning the 'ultimate' Δti, and we assume that at an ultimate stage, the slope function is 'nice' and non-oscillating(†). At that stage, for every two distinct Δt, if one is smaller than the other, the slope at it is higher. If we now exclude all the specific Δti velocities, and we exclude all slopes that are clearly larger than all of them, we arrive at the true slope in a manner exactly corresponding to the definition of a supremum, and that reminds of the methods of Archimedes. A natural classification of functions 'nice' with regards to slope ensues, and clearly, they all have to be 'continuous' to begin with (the absolute changes must always exist) (TODO check).

When this limit exists, it is clearly special amongst all others in many senses.

a. Closest to all with no 'tangent intersection'.
b. Upper bound (least).
c. Independence of Δti and in fact of any punctual change of the function outside the point itself.

d. (††) Any Δti-specific velocity 'misses' the behavior in the interval [0,Δti], in other words, an infinity of functions have the same Δti-velocity at that point but are arbitrarily different. Therefore Δti-specific velocities cannot in general give 'complete' information about a function, they cannot completely characterize it. This means that we cannot 'recover' the full function(†††) from these velocities, unlike the 'true' velocities as proven in analysis by integration(††) (up to a constant).
e. We cannot 'integrate' the Δti-specific 'derivatives', can we even express them?
f. Maybe most importantly (and this occurred to me one day after the initial note) that it is only for the functions that admit a 'true' velocity (differentiable) can we talk about the meaningfulness of the average velocities, because for all other functions, the average velocities are so 'wild' and 'incoherent' that they lose (intuitive, physical) meaning. This is a very strong and strict relationship and even very intuitive.

6. We conclude that the justification 'true velocity' is a mathematical one and not in principle physical <TODO rel. energy conserv paradox>. It is nevertheless true that the more precisely we can physically measure Δt, the closer the measured velocity gets to the 'true' one. At least for the time being, we have precisely separated the mathematics from the 'intuition' (or better said described the actual relationship), from 'justification', and from 'origins'. We have also established their relationship 'meta-proved'. The fundamental theorem of calculus appears in a new light, especially because the note ††(††) and we see the >>necessity<< of limits without which the theorem would not work at all. This is obvious, but in fact subtle.

    a. The physical measurement is only meaningful when the function is well behaved and quite monotonous at the Δt resolution.
    b. In the cases of functions that are so 'oscillatory' that they are beyond the physical measurement resolution, the correspondence to 'true' velocity breaks down, and this is surely not uncommon during physical investigation on the 'border' between classical mechanics and newer developments. As a barbarically crude example, imagine an oscillation period of 10^-100 sec.
    c. Talking about 'material particles', we ultimately get out of classical mechanics but our conclusions still apply.
    We have here done more than justify 'true velocity' for a specific example and freed it from the original meaning of 'velocity'.

7. We must at once and forever stop being irritated by the zero in lim h→0 and this is fully justified by this note. <rel. singularity TODO>

    a. (†) We can see how the existence of a derivative means 'eventually monotonous', 'eventually non-oscillatory' (warning: subtle with the example of sin(1.nx)/n). We can take this a bit further and say 'eventually limiting to a straight line'.

b. (p119) (†††) Our analysis of the unique ability of the derivative among the discrete forms to recover the full information about a function has several consequences.
c. The FTC establishes more precisely the 'recovery' and also the opposite direction: differentiation. When possible, the primitive is then pinpointed up to a constant. This relates very tightly the differentiation and integration. In other words, up to to constant, describing the derivative or describing the primitive amount to describing the 'same thing'. They are almost two expressions of the same function (again, when possible). The immediate practical use is that when the derivative of some function is easier to describe than the function itself, we are very close to describing the function. This is exactly the idea applied to the determination of volumes in p119-120 or force in p67. Note that in these examples, what we have easy access to is the differentiated form (surely a common situation) and not the derivative itself. We can however easily extract the derivative. The important difference between the two is analyzed in the next note.

8. (p119) In the previous note, we precisely described the difference between the average velocities (Δti-specific) and the derivative, the sampled and the limit. It gets much better: A precise relation beween the two forms is obtainable, very interesting and useful. It is in principle related to our idea of 'recovery'. Since we can recover the whole function, we can also recover information about the Δti-specific form. In other words, we can precisly relat the derivative form to any of the approximate forms of which there is an infinity. The discrete Δti-forms are from now on called differentials, usually denoted by $\Delta y = f(x+\Delta x)-f(x)$ for an increment of Δx with $y=f(x)$. The discrete rate of change is the $\Delta y/\Delta x$, let us denote it by F'.
   a. There 'error' between F' and f' is $\alpha = F' - f'$, and is a function of both Δx and x. It follows from our definitio of f' that $\lim_{\Delta x \to 0} \Delta y/\Delta x = f'$. Therefore, $\lim_{\Delta x \to 0} \alpha = \lim_{\Delta x \to 0} \Delta y/\Delta x - f' = 0$. Since α tends to 0 as Δx does, at good enough approximation, both of them are less than one, and so their product $\alpha.\Delta x$ is less than Δx, it tends to zero faster than Δx, and $\alpha.\Delta x$ tends to become negligible compared to Δx. So if Δx is called first order infinitesimal, $\alpha.\Delta x$ is of lower order, all in 17th century parlance which can be rigorously proved using limits. We have actually established the sought for precise relationship already. $\Delta y = f'.\Delta x + \alpha.\Delta x$ with $\alpha=g(x,\Delta x)$. What this means is that if we have an expression of Δy (a differential) as a function of Δx, and group all factors of power Δx^1 together, the sum of these factors must equal f'. We say that f' is the linear, or first order, component of Δy. By 'the linear' we means the 'only' since $\alpha.\Delta x$ must be of lower order. This 'detail' is crucial.
   b. 17th century parlance 'tends to' is very intuitive and powerful. It hints to a simple arithmetic with limits that has to be used whenever the rules for it (established by modern analysis) hold.
   c. Relating this to our previous note, it is because of this precise relationship that we can determine volumes, starting with differentials.

d. Note. Differentials are thoroughly (more thoroughly than one could imagine) treated in Euler's <TODO> which I studied for a short while. It makes sense to try to make them the finitist base of real analysis.

9. Relating the two preceding notes, and coming back to the advantage of the rate of change limit compared to the change with respect to 'recovery' and differentiation of functions, is it not so that, except from being relating to the intuitive concept of velocity, there is maybe no reason to give the exact form of derivative the exclusivity of essence in calculus. Is it not maybe true that any function g of the form g(f(x+Δx) - f(x)) that has a limit not equal to zero has an equal claim to a related FTC? We note that all higher order derivatives are a special case of this form. Is this in any way related to the concept of convolution? <TODO>

10. This note occurred after rightly taking a dip into Abstract Algebra and many realizations about structure (GAGC 1:21, UQ: 1:15, ATBS: 1:3, A:1:4). In analysis, by the 'algebraic/analytic' necessity of using complete ordered fields, and the necessity of formal definitions of differentiation and integration, we see that the variable with respect to which we diff/integ, are from a complete ordered field. Now the links of this abstract theory to the physical, ones, which we analyzed in the notes below, invites us to see the limit as the only and necessary device for the formal expression. It is an illusion to think that the limits are only a temporary link between a function and its closed form integral. This is a very confusing and naive view. The limit is always there, in the essence of these creatures. (Note that the plan of internalizing limits by studying finite abstract algebra seems to be working, in a way that is not obvious to describe).We have to differentiate between many different points of view that can be easily confused:
    a. The abstract theory of real analysis.
    b. Physics of the real world, where ultimately, it seems that time must logically be discrete, otherwise time jumps in infinite steps (Zeno), or time does not exist as such (find related quote in Stabi). Despite that, thermodynamics, Noether's theorem, quantum physics, statistical mechanics, are the link towards the justification of our notes about velocity above. Thus this is without justification not related to a.
    c. Computation, which is a very algebraic discipline, and it's necessity given current technology, in providing numerical solutions to problems in b. backed theoretically by a.
11. Continuing 10, we spotted the word 'KAM, as a Theory of Dynamical Systems'. Given our abstract algebra readings, we must ask, in the vein of 'what is an algebra', what is a theory (of dynamical systems). It is a very justified question. In any case, applying structural and algebraic thinking to what 'dynamical systems' might be formally, very fast brings us to the idea of semi-groups and free groups, and puts the introductions to the related books we have into an understandable point of view. With the simple systems where starting from some point x, we get f(x), which becomes x, etc.. we immediately see the algebraic structure, and the hopping along f, and the importance of fixed points. We also immediately notice the practical case of f only being known through its

differential, but as we know now, the differential can in the limit completely determine its 'integrate', To study this, we finally get to the idea of the common term tangent bundles, etc. In short, it was very important to start studying abstract algebra. It is no surprise that Arnold wrote his book about the quintic.

12.

§.VII. Curves and Surfaces

1. Thinking about geometry, we can apply our ideas of the most general kind of function: A relation between a domain and a co-domain. Even restricting ourselves to functions that are amenable to a representations (while not confusing the two), it is clear that Greek and analytic geometries can only ever cover an extremely limited sets of representable functions. Functions expressed in relations to themselves <rel. our previous notes about representations of function in notes on Euclid’s elements> differentially expand this limited domain, and bring us a step closer to the curves and surfaces encountered in real life. There are of course even more general curves and surfaces. There is also the modern computational option of attacking the problem using simulation, but that itself can only be built on mathematics.
   As usual the best way to think about this difference between the geometries is as classification of representable(?) functions.
2. We can be more specific about the birth of differential geometry by saying that it is the moment at which analysis became well enough studied that it started to have application in geometry, as this quote explains. “By the middle of the 18th century, the differential and integral calculus had been sufficiently developed … that the way was open for more profound applications to geometry” (p.59).
3. ‘Curves and surfaces’ fall into ‘differential geometry’ and this directly relates us to analysis. <rel. analysis>
4. Even the definition of surface already links us to topology which is the discipline that provides the rigorous way to define them (p.58). <rel. topology>
5. We start from ‘differential’ properties ‘at a point’. Without our deep analysis of the concept of neighborhood in our real analysis studies, the exact meaning and depth of this would have passed unperceived (p.59.).
6. “Thus the direction of a curve at a point is determined by its tangent at that point and the amount by which it twists is described by its curvature”. This quote triggers a thought that unfortunately I did not think of myself until now. A tangent, a first derivative, of a given curve can be the same for a whole infinity of other curves as well, all different. So it relates all of these curves that share this point. Without looking at global differences, we can step out of the first derivative, which already gave us the information it can, and search elsewhere. Thinking visually, what differentiates these curves is the ‘curvature’. If we want to bind ourselves to neighborhoods and differential properties, we have then to pass to the second derivative. Notice how this justifies that fact that curvature is related to second and not first derivatives. Visually, it is curvature that differentiates between the

curves with the same first derivative. Also notice how this relates to Expansions such as Taylor's or power series, and hints at the fact that if we take an infinite number of additional degrees of curvature, we might be able to fully describe the function globally (up to a constant). I suppose this does not apply unconditionally and is subject to further classification (analytic functions? <rel. §I.p128).
In direct relation to what we just noticed is the quote that follows: "only in its later developments does it proceed to the study of their properties "in the large" i.e., in their entire extent."

7. This is a validating quote about the application triggers of an 'abstract' discipline. "This upsurge was due to the demands of mechanics, physics, and astronomy, i.e., in the final analysis to the needs of technology and industry, for which the available results of elementary geometry were completely insufficient" (p.60).

Studying

1. It might be good to establish a rule of only starting the exercises and section theorem proofs after one week has passed since reading the material of a section. This makes sure we are not *cheating* in terms of very short term memory, while still not making the task too difficult. A longer period might be too hard to cope with and we are aware that we cannot fully internalize all the details although we would like to. One week seems like a fair compromise.
2. After the note <rel. TODO note on paper>, I can now clarify an 'issue' with the historic method. It is undeniable that each and every 'historical' topic is as interesting as any other (including modern) topic in and for itself. This was made clear to me by reading about Euler's correspondences ('Euler and Modern Science', chapter 'Leonhard Euler in Correspondence with Clairaut, d'Alembert, and Lagrange) and Euler's scientific work on bridges ('Euler and I.P. Kulibin). Although this is obvious, for some reason, the specific idea took this long to surface. It is the idea that, understanding apart, if obtaining new results and contributing is to be at all a goal, then it creates a large difference between the otherwise equally interesting old and new problems. Old problems are solved and new ones are not. New ones usually require a full understanding of old ones, but not necessarily their history. We know that the history is a great part of learning how to think, but if that is not an issue anymore (i.e. we now learned how to think), or never was (we are a genius), then the 'modern treatment' becomes the best known shortcut towards the new problems. This is probably a reason for the modern presentation, but also a big part of its failure. The shortcut is almost always 'too short', almost like teleporting, nothing is retained along the way. This, at least for me, is a problem. But I finally have a complete solution to this problem <rel. TOD note>.
   Note. From our historical studies I know that any 'published' item (e.g. proof) is always a very abridged version of longer work, including failures. Sometimes, very long, year-long, decany-long work, that results from thinking and (relational on multiple levels) knowledge. After having surveyed this, I see that what I had to learn is how to think, and

convince myself (gain confidence) that I can.

3. VQuotes from ‘’ in the paragraph ‘PHILOSOPHY OF MATHEMATICS IN THE TWENTY-FIRST CENTURY’. I found this after sensing a hidden, albeit harmless, background motivation of dialectical materialism in MCMM, triggered by the feeling that too much bias is used to sway mathematics into the realm of material human activity, which I agree with, but also find incomplete. This lead me to ‘Philosophy, Learning and the Mathematics Curriculum: Dialectical Materialism and Pragmatism Related to Chinese and American Mathematics Curriculums’ and to the contrast between dialectical materialism, and Dewey’s instrumental pragmatism, which seems to be related to the foundations of educational philosophy.

4. Related to the above, here is a clear quote (although not new in content) about some formalist origins of the current educational system.

   Formalism, as a philosophical movement, is steeped in the abstract, symbols- oriented conception of mathematics. One who subscribes to a Deweyan philosophy of mathematics might find it difficult to understand the formalist perspective and so a word on its origins is in order. As mathematics developed and pure mathematics became less immediately and obviously tied to the natural world (think of set theory or non-Euclidean geometry), it became harder to make sense of or justify the foundations of mathematics. What resulted was a reconception of mathematics in the early part of the twentieth century that sought to explain mathematics in terms of following rules or axioms. In Mathematics: The Loss of Certainty, Morris Kline explained the origins of the formalist movement. Mathematician David Hilbert developed the idea that mathematics could avoid the ambiguities and confusions that some branches of purely abstract mathematics introduced by conceiving of math- ematics in purely formal or symbolic (as opposed to intuitively meaningful) terms. Kline viewed this move as an effort to hold on to the old conception of mathematics as perfect and certain: Whether or not the symbols represent intuitively meaningful objects, all signs and symbols of concepts and operations are freed of meaning (according to formalism). For the purpose of foundations the elements of mathematical thought are the symbols and the propositions, which are combinations or strings of symbols. Thus, the formalists sought to buy certainty at a price, the price of dealing with meaningless symbols.8 While symbolic formalism has been incredibly useful in the areas of computer programming and artificial intelligence, its function in a few areas does not qualify it as the correct way of thinking about the nature and origin of mathematics. Dewey would, I am sure, applaud its use as a means of improving our lives, but probably caution dyed-in-the-wool formalists not to confuse function and ontology (I will develop this assertion in the following section).9

Hallucinations

1. According to this hallucination, my goal should be nothing less than the destruction of human mathematics as we know it.
    a. The idea that mathematical truth is a guarantee of immortality is wrong, as shown by Lakatos. It is only a matter of time before all the greats suffer the fate of obscure figures from the ancient ages of mathematics.
    b. Achieving this is related to the ‘technological singularity’. We know that a large part of genius is mental energy, something machines do not lack. The other part is turning what is currently ‘computation’ to ‘thinking’, that is, computation on the human (above than average, philosophical that is) level. This is related to the singularity more specifically in terms of self-conscious machines. Machines that have a (built-in by creation) reason (bias, goal) to act, to think. Why would not a machine nowadays ‘invent’ Euclidean geometry? Partially because of the sense experience, the bias to live, to compete for survival on a higher than animalistic level. That achieved, different machines, or even one machine with controlled schizophrenia, can employ multiple meta-methods (heuristics) to solve its problems. Be it the belief in the continuity of truth (Newton Machine), the desired for classification and finding of simple yet englobing truths (Euler Machine), etc.
2. This is related to the lake (notes still on paper, and in lakatos pdf). After reaching p.86,87, where concept vs. definition is treated. It occurred to me that the lake is of an even richer structure, where two tags that have the same content (tag the same creatures) (include the same set), but are expressed differently (see Existence of the World for a discussion of this topic) are not ‘the same’ but they are still identical in content. Per example, the tag ‘{0}, recursive add 1’ or {0,1}, recursive add 1’ have the same content but are not the same. These are trivial versions of the different theorems and the terms they introduce (Eulerian polyh, Lagragian polyh, etc.). In essence, these are ordinal vs. cardinal differences. The two tags, seen as set generating processes, can be seen ordinally or cardinally. When the set is infinite, we cannot enumerate it, but we might able to describe it. Nevertheless, Ordinally, the two processes might be very different. In lake terminology tagging creatures in a different order (e.g polyh constructions), but potentially tagging all the same creatures. The ordinal/cardinal view of tags is all what a large part of Lakatos (until p.86) is about. This is a great addition to the idea of the lake.
3. Note on the recurrent question of method of proof. The dichotomy between local and global plays an important role during proof formation. This reminds us of the equally important role it takes in Lakatos’s refutation theory. Imagine the simple analogy of starting in a green square on a board, with the goal to reach a red one. Looking at the immediate surroundings, seeing nothing but the green color, there is no clue for orientation. This is a good first step based on a local analysis of the situation. What usually happens is that we need to take a meta level view of the board. This necessarily erase details because of memory and cognitive limitations but given that trying all routes is impossible, it is a good idea. It might turn out that the color gets redder going North. In

terms of tags, the path north might be associated with a certain set of tags, the local search then looks for neighboring tiles that, although green are also closer to the above mentioned set of tags. Note that we can have situations where are first no path at all is clear, but the solution is the same. On memorizing the proof, the idea becomes finding amongst possible paths one that is characterized and strongly differentiated by a minimal set of tags that can be followed locally, and therefore, mechanically. Finding this set of tags sometimes gives the feeling of ‘true and deep understanding’.

4. Both an analog device and its digital simulator are computers. One computes with electrical circuits viz. differential equations, while the other computes arithmetically. Each computer has a set of computable functions. The sets intersect but are not equal. In audio equipment simulation (e.g tube amplifier simulators, etc.), the goal is to extend the algorithms and computational power of the digital computer so that its set becomes a superset of the analog computer. In this sense, do not mixed-computing devices make a lot of sense? By augmenting digital computers with analog, quantum, etc. computers, we might be able to cover a wide range. Other ideas come to mind: Is an analog circuit able to simulate the three body problem? with and without collision singularities?
    a. Why do we say ‘functions’? We can think of the result of computation as simply ‘itself’. Then we could study it ‘intrinsically’. But what we have in mind is for example an output signal. In that case, ‘time’ or the ‘zero signal’, and ideally the real number ‘line’ become the ‘substratum’ of the work to be done. In other words, since the output is not ‘nothing’, but a ‘continuum’ of ‘one dimensionally determined quantity’ (number), what we can talk about is the computational output being a ‘function’.
5. in UQ, the author explains how there is much more structure in Groups and Rings, compared to Fields. It is maybe nice to think that limits and lubs are the most basic, fundamental, interesting and important ‘structures’ in Fields.

Notes on ‘The Genesis of the Abstract Group Concept’ (Hans Wussing)

1. It is important to realize that the common man, except in special professional cases (painter, architect, Greek altar designer, etc.) really has no direct need for geometry, even the simplest Euclidean geometry: Lines, circles, intersections, angle determinations. Let alone conics and conic sections. Historically, much of Geometry was initially motivated by needs, and of course then by curiosity, and the determination of the mathematical mind to clarify and explore.
2. For projective geometry in particular, as soon as contact is made, one immediately needs to ask (but soon hides) the question: What is a projection. What this means based on our thinking methods is asking: when is an operation a projection and when is it not. Beginner textbook examples of projections are many, from projection in the plane, to space. But what allows us to say that tracing lines from a point in a plane is a projection? This is indeed a crucial question to ask, whose answer leads to a motivation for the

Erlangen Program. Each colinearity preserving transformation is a projection. This was not at all obvious for the longest time.

3. Linear Algebra was served to us on a golden plate without us knowing it. But as we know, ‘early-biasing’ can be turned into an advantage. It was natural to wonder why a linear map, so axiomatically defined, has anything to do with ‘lines’. There was a tendency to investigate this. Now we know that any attempt to declare this as obvious (starting from a pure and unlearned but logically thinking human mind) is a crime. Any path to investigate this would actually be tracing backwards the history of the historical development. A better way is to learn about the history. A re-reading of Linear Algebra after this might be quite beneficial.
4. Geometry as ’the study of properties invariant under certain transformations’ is a misleading sentence that requires explanation. What is meant is of course not transformation of certain figures. Taking a circle and transforming it by a ‘projection’ can be very well done in Euclidean geometry, and that has been confusing us until we finally analyzed the matter. We have already established ‘determination’ as a basic meta concept and pillar of mathematical investigations. Many Euclidean geometry ‘problems’ (usually artificial as explained in 1.) are of ‘determination’. What is determined by what, what freedoms are left, etc. In any case, let us take any problem in Euclidean geometry. There are so far hidden transformations that can be applied to the problem as a whole, that do keep the problem as it is. The transformations are ‘hidden’ because they are so natural due to our Euclidean sense experience (a bias). The figure of the problem drawn on a plane paper, would be solved the same way no matter how the paper is rotated, no matter how the paper is translated (by adjoining it to other papers), etc. The problem can certainly change, that is, the solution would change or become invalid (e.g the determinacy relations would change), if the paper is projected, skewed, etc, and the problem on the transformed paper is studied. Being aware of this finally explains what is meant by the first sentence of this paragraph. Additionally, we can now understand what is meant by a Euclidean plane, compared to some projective plane. Taking all these Euclidean transformations above, and applying it to a plane as a plane embedding all possible figures and problems, the plane itself will always be mapped to itself. Translating the ‘Euclidean plane’ gives the (not a) ‘Euclidean plane’. So the plane as a ‘double extension’, taken with the set of allowed transformations, gives the studied Geometry. Classifying all possible geometries is of course the big daunting task, very nicely explained (when one can follow) in Coolidge’s book.
5. Remarking on 1. we notice that interests in physics and computing lead us naturally to high dimensional manifolds/geometry. This allows us to tolerate the artificiality of ‘template geometrical problems’ of textbooks, since they are introductory versions of the geometries we are really interested in, learning to walk before learning to run.
6. Remarking on 4. Notice the massive strength of the experience bias. As soon as we understand what ‘Geometries’ are, and their inseparability from ‘transformations’, the debate of the geometry of physical space finally makes more sense. Notice the immense bias imposed by experience on the ‘obviousness’ of consistency of experience, irrespective of location, orientation, etc. Irrespective of linear velocity, of ‘choice of coordinate system’. The possibility of repeating experiments, of doing the same things in

multiple places and times. Are these true 'invariants'? It all makes sense now, and the group of transformations of physical space makes sense. The 'relativity', in its basic question of independence of speed of light of the speed of the emitter poses even less of a problem now. It stopped posing a problem after the last physical reading period, but it is clearer now. Is it right to say then, that the discovery of the mathematical nature of physical space can only be achieved 'intrinsically' by us since we live in it, and that this also gives support for the possibility of intrinsic study? (Logical support apart).

7. Remarking on 5, imagine a person measuring the distance between two parallel sides of a road, disappearing vertically into the horizon in relation to another person observing. To the first person, the distance is constant, to the second person, the distance is shrinking, which person is the 'reference'? This is an example of the bias towards a geometry or another based on experience.
8. I now understand why some of my goal books have been inaccessible after many attempts at soft reading. After the Erlangen Program era, the language of physics has become inseparable from modern geometry, whereas in the now distant past, and in high-school physics, apart from analytic coordinate geometry, it was very accessible algebraically. Of course there are algebraic counterparts and representations to all the geometric concepts used. However, understanding the required geometry, and especially the (finally developing) modern geometric point of view, is crucial.
9. Where do I find 'Oeuvres de Camille Jordan', and what are the 174 different groups! mentioned in p.197
10. "A rotation fixes one point in the plane and turns the rest of it some angle around that point. In general a rotation could be by any angle, but for patterns like we have, the angle has to divide 360°, and a little more analysis finds further restrictions. In fact, the angle of rotation can only be 180°, 120°, 90°, or 60°." says David E. Joyce. Indeed, a 7-fold symmetry (30°) is impossible: "5-fold, 7-fold and other symmetries are not possible in crystals, because you can't fill space with 5-sided objects". The world falls on my head.
11. In view of the (rightful, identifying, classifying and philosophically necessary) explosion of 'groups' in abstract group theory, it is not wrong to guide oneself by 'application'. Although knowing the 'artificial' simple templates is still useful. A perfect 'application' can be motivated by the following paper, which constitutes a logical continuation to our three-body-problem quest, and a link to the 'computing age'. Luckily (but not surprisingly), it is also related to variational calculus. The paper is: "An introduction to Lie group integrators { basics, new developments and applications" (Elena Celledoni, Håkon Marthinsen, Brynjulf Owren) (http://arxiv.org/abs/1207.0069).
12. Remarking on 11, it is better to know that some 'thing' is a group, a certain type of group, than not knowing anything about it. {very weak, but the best I can do for now}.
13. Clean email notes:
    a. being a group is a infint precise delimitation of all the elements, therefore it is an excellent description/tag. Even when know what the thing 'is' so elegantly has no 'application'!
    b. the now understood 'surprise' about the small 'restrictions' on elemwnts of a set/their relations leading to very specific and global features of structures (cogs

image, static timeless equilibrium,..). This is intimatwly relayed to group theory, exposing the substructures and inner structures, etc.

c. groups are one of the most basic structural elements that separate describable (finitely) sets from 'random' 'table based' 'structureless' 'lawless' sets. Drop one requirement and we go haywire (it seems), as relations can go 'wild' ??? yes indeed (but not exactly): http://en.wikipedia.org/wiki/Magma_%28algebra%29. we are getting close to: http://en.wikipedia.org/wiki/Category_theory
d. A 'line' in eucl plane as one element in the 'set' of eucl geom?
e. The 'set of groups/sets' with 'applications in our universe'

14. A nice ‘summary’ of the problem of distance and at the same time motivation for Geometric Algebra is (Beyond EuclideanGeometry) (http://www.mrao.cam.ac.uk/~clifford/pages/Geometry.pdf):
    a. Affine geometry is a subset of projective geometry. Euclidean geometry is a subset of affine geometry. How do we recover Euclidean geometry from projective? Need to find a way to impose a distance measure.
    b. Only distance measure in projective geometry is the cross ratio. Start with 2 points and form line through them. Intersect this line with the fundamental conic to get 2 further points X and Y. Form cross ratio. Define distance by …
    c. Cayley & Klein found that different fundamental conics would give Euclidean, spherical and hyperbolic geometries. United the main classical geometries. But there is a major price to pay for this unification: All points have complex coordinates! Would like to do better, and using GA we can!
15. Remarking on ‘structural limitations’, remember the old surprise of the limited number of regular polyhedra. Contrast this to regular polygons. This is fundamental and mentioned in p.198.
16. In general, it seems we are on the way of finding ‘all’ ‘structured’ sets and their ‘subsets’ that can be found. What a clearly necessary thing to do, and how much of an impossibility and absurdity it can seem when one first learns of sets!
17. An application of our ‘embracing’ and non-preferential multiple views on any ‘thing’, understanding that the theory of forms is related tightly to geometry but that they are two different and both fruitful points of view is important, and indeed crucial to eliminate any illusions of understanding, when learning about one view and deciding to ignore the other, considering it as already known.
18. p.199 is a glance into future details, and the relation between the quintic (degree five) and the regular polyhedra (degree 5-2=3). Also note the quote about the ‘value of intuitive sense’ (we might disagree, based on our theory of understanding).
    “The great value of these examples lies in the fact that they furnish an intuitive sense of the intrinsically abstract conceptions of substitution theory.”
    “By means of this representation one establishes a remarkable connection between the theory of equations of nth degree and the theory of covariants of n elements of a space of n-2 dimensions, so that each theory can be viewed as an image of the other.”
19. Notice that after this reading, topology finally takes its right conceptual place, being even less restrictive than colinearity (find related page, Monge?). Topological classification (genus, etc.) and gluing also take their place. Algebraic topology, the use of algebra (a

specific use of symbols) as a topological proof/classification tool is clearer as well.

20. p. 201 highlights the relation to Poincare, through Fuchs's investigations on algebraically integrable linear differential equations, with results that inspired Klein for a certain time, before going back to his main line of relating the quintic to the icosahedron. (With this in mind, Arnold's course for high school students appears slightly pompous). Note that Poincare came after Fuchs, Klein and Lie and that on p.223, topology appears: *'The evolution of the concept of a (finite) continuous group to that of a topological group is associated with an extremely fruitful development of its content.'*
21.

Notes on 'Unknown Quantity' (John Derbyshire)

1. I seem to have a kind of love-hate relationship with the author's style.

2. This book, especially Part 2 and beyond, contains the mental image one has after a mathematics bachelor from rough detail perspective. Based on our philosophy of learning, this is perfect for accelerating our organic learning by planting all the right seeds and rough connections.

3. The 'Algebraic Geometry' chapter is an excellent root structure builder, it even ends with a very good concrete (even if maybe simplistic) example of duality (line-point) in projective space. So are the three preceding chapters. I have to build a mental image of these chapters. The chapters on the quintic is also of this kind.
4. Related to our sentence "*The transformations are 'hidden' because they are so natural due to our Euclidean sense experience (a bias).*" [GAGC note.4], the author provides an equally good additional way to see the idea in [UQ p.250].

   "I spoke of moving that ellipse around the plane. Well, that is one way of looking at what I did. Another would be to think of the ellipse sitting serene and immobile in its plane while the coordinate system moves…"

   Note that this teaches us another (again sadly hidden so far) lesson about coordinates. They are a relation, and therefore 'relative'. We can see the thing that has its coordinates given as moving, while a coordinate basis is fixed, or vice versa. It it mathematically and (logically for many purposes) 'the same' transformation.

5. [UQ p.257]
   "*The interest in curves was a great mathematical growth point in the middle 19th century, nourished by algebra and calculus as well as by geometry. It is an easy interest to acquire, or rather it was in the days before math software came up, that days when you had to work hard, using a lot of computation and lot of insight, to turn an algebraic equation into a curve on a sheet of paper. Who knew, for example, that this rather*

pedestrian algebraic equation of the fourth degree, $4(x^2 + y^2 -2y)^2 + (x^2 - y^2) (x-1)(2x-3)=0$, represents, when you plot y against x, the lovely ampersand shape shown in Figure 13-2? Well, I knew, having plotted that curve with pencil, graph paper, and slide rule during my youthful obsession with Cundy and Rollett's Mathematical Models, which gives full coverage to plane curves as well as three-dimensional figures.

… the now-lost practices of careful numerical calculation and graphical plotting offer--offered--peculiar and intense satisfactions. This is not just my opinion, either; it was shared by no less a figure than Carl Friedrich Gauss. Professor Harold Edwards makes this point very well, and quotes Gauss on it, in Section 4.2 of his book Fermat's Last Theorem. Professor Edwards:

Kummer, like all other great mathematicians, was an avid computer, and he was led to his discoveries not by abstract reflection but by the accumulated experience of dealing with many specific computational examples. The practice of computation is in rather low repute today, and the idea that computation can be fun is rarely spoken aloud. Yet Gauss once said that he thought it was superfluous to publish a complete table of the classification of binary quadratic forms "because (1) anyone, after a little practice, can easily, without much expenditure of time, compute for himself a table of any particular determinant, if he should happen to want it… (2) because the work has a certain charm of its own, so that it is a real pleasure to spend a quarter of an hour in doing it for one's self, and the more so, because (3) it is very seldom that there is any occasion to do it." Once could also point to instances of Newton and Riemann doing long computations just for the fun of it…. [A]nyone who takes the time to do the computations … should find that they and the theory which Kummer drew from them are well within his grasp and he may even, though he need not admit it loud, find the process enjoyable. "

6. Gems for including invariants into our root structure are on pages 260,261,262. These gems all start from a small discussion about a couple of invariants of conics in a previous chapter, an metric example being of course their area, related to determinants, but also some projective invariants. As soon as one is hooked, one naturally wants to find a method to finding all invariants, which is probably where all this leads in the end. My first contact (and immediate realization of their fundamental importance with all fascination) was with eigenvalues and determinants during the study of Linear Algebra (Hefferon).

   "Salmon was another 'avid computer'. In the second edition of his Higher Algebra he included and invariant he had worked out for a general curve of the sixth degree. If you look at the invariants for the general conic I gave in my primer, you will believe that this was not mean feat. In fact, it fills 13 pages of Salmon's book.

   I mention this as a useful reminder that this mid-century fascination with curves and surfaces was being fed in part from pure algebra and was feeding some results back in turn. The invariants I described in my primer were first conceived in entirely algebraic terms; the geometrical interpretations were secondary.

You can add two polynomials, or subtract them, or multiply them, … It's a ring. The study of invariants in polynomials is therefore really just a study of the structure of rings. Nobody thought like that in the 19th century. The first glimpses of the larger river--theory of rings--into which the smaller one--theory of invariants--was going to flow, came in the late 1880s, with the work of Paul Gordan and David Hilbert.

7. It surely gets very interesting after p.261 ! Something big is coming.
8. The book has to be re-read, mapped and memorized, with proper notes for the missing parts
9. [UQ p.268] Locally flat makes more sense now after our understanding of derivative. Is locally flat not simply meaning: where in the limit, some calculus (at least linear approximation) is possible? This brings us back to our 'thing studied relatively to itself' notion'. Note that we have a vague idea of why 'Euclidean flat' can be used as a preferred 'reference' locally, maybe it is simply because of its linearity, which seems to be related to the simplest algebraic kind of operation, of determination between two quantities y=a.x, where a is a constant. Or is it simply because of the fact that Rn has the 'simplest' 'structure'? also computationally? the fact of the independence of all the extensions, one for each dimension? Probably, it is but one choice, the least encumbered one because of the facts above.
10. Out of all 'must-read' classics, [UQ] suggests a subset, which we should surely read, in an order dictated by having the 'language' prerequisites. (1) Jordan's Treatise on Algebraic Substitutions and Equations "was being by mathematicians everywhere" (p.271), noting that this was on our radar since Linear Algebra, but our knowledge was hopelessly lacking, maybe [UQ] and [MCMM] fixed that a bit. (2) Klein's "Comparative Considerations of Newer Geometric Researches" (p.275). (3) A glance at the books from 5. (4) Maybe Coolidge's Treatise on Algebraic Plane Curves (p.278), and also A Treatise on the Circle and the Sphere. (5) todo…
11. Note that pages 266,... finally being the germ of a root structures for the analysis-topology bridge, a source of many 'wow' notes during the [EHM] (Bourbaki) readings.
12. Camille Jordan seems to be the one who came up with the idea, confronted with Riemann's complicated surfaces, to study looping paths on them. A concept introduced quite early in all topology books I have browsed. As with all good things, this links to Poincare, who 'algebrized' the idea. [p.281]
13. Poincarre's algebrization, using families of Jordan loops, turns out to be amenable to groups (of these families) [p.282]. This is an excellent root structure a direct connection between topology and group theory, the fundamental groups of the sphere and the torus (C_∞ x C_∞) are great mental images. especially the torus fundamental group, the the multiplicity of winding (imagine winding multiple times around one axis before getting to the starting point, each loop with a different number of windings is 'different', unlike for a sphere).
14. Loop-back! I understand the basic content of the Poincare conjecture that Perelman proved! [p.283]
15. Algebra (Saunders Mac Lane, Garret Birkhoff), as recommended by the author, seems to be the right book for our studies. It is difficult not to fall in love (at least from a

distance) into such a dissection of mathematical creatures, small to big, making a 'short story long', turning 'complication' into simple but long stories, because that fits exactly with our philosophy of understanding.

16.

Notes on 'Abel's Theorem in Problems and Solutions' (Arnold)

1. I have reading the book, including mentally trying to solve the exercises, and I stumbled upon this question, which shows excellently the importance of structure:

   '*85. Suppose that a group G contains exactly 31 elements. How many subgroups does it contain?*' (p.25)

2. Here is yet another structural gem, which feels 'familiar', but thanks to mathematics, can be expressed so precisely.

   '*Hence in the group of symmetries of a triangle ABC the subgroup of rotations is normal, whereas the group generated by the reflection with respect to the altitude drawn from A to the base BC (containing two elements) is not normal.'* (p.28)

3. It seems that taking up the geometric examples and using them as a memorization helper for the concepts introduced might be a good idea.

Notes on 'Algebra' (Mac Lane, Birkhoff) (3rd edition)

1. *'A permutation on a set X is a bijection X->X'* (p.44)
   This was one of the largest slaps in the face since a while. I have been fixing my views on morphisms in terms of their structural characteristics (since I know highly appreciate structure). But this is such an obvious but elusive, very strong relationship between 'permutation' and bijection, that is unaffected by passage to the infinite. What this short sentence means is that for a set, the set of all possible bijections to itself is in fact nothing else than the set of all possible permutations. This is not obvious at first because one does not think of a set as having order, while a premutation requires it. However, one does not need to artificially construct such order (e.g by create another set, pairing the elements to natural numbers). Instead, each distinct bijection is taken as an element, and the set of this is the set of permutations. In a finite set this is so clear, and the natural obstacle-less application to infinite sets is very awakening at the time of reading.

Bijections and permutation will never be the same again. Also, the permutahedron for *n*=4, its exhilaratingly beautiful structure, and the relation of this structure drawn on a plane to the projection of the three-dimensional Truncated octahedron (archimedean solid) is a genuine slap in the face. This is once more a showcase of ‘structure’ (see Notes Abel’s Theorem in Problems and Solutions 1,2,3, etc.)

2. This is so obvious but I never noticed it. Morphisms are generalizations of basic relation properties (injection, surjection). It is simply that a set has no ‘structure’, and therefore it does not make sense to speak of morphisms, so the structural requirements disappear, all is left is the set-centric relation properties. This became clear to me while reading: *‘There are analogous groups of symmetries for algebraic systems’* (p.46)

3. This is yet another great example of structure: *‘$Aut(Z_4) \cong Z_2$’* (p.46)

4. It is scary how latent this mental error has been. Function composition is provably associative (p.6), and at the same time, there are non-associative algebras, per example, vector cross products, that are clearly not associative. To be exact, such an algebra is an *exterior algebra* or *Grassmann algebra*, where in R^3 the cross product is taken as the multiplication operation. Amazingly, up until now, functions and binary operations have been confused. Function composition can act as a product, and serve as the product operation for a group. In that case, it is acting as a necessarily associative binary operation. A function on a set S acts on elements of S. A binary operation on a set S acts on elements of SxS. So for integers, a.b.c contains two binary operations: mul(b,c) and mul(a, mul(b,c)). When function composition is used, for three function we get a.b.c, with apply(a, apply(b, apply(c, identity))), which is not necessarily the same situation as for the binary operation. In other words, function composition always *produces* a binary operation on pairs of functions, but not the other way round. The difference in the necessity of associativity is an immense structural difference, due to the higher restrictiveness (not obvious) of composition. A small finite table of a binary operation probably makes this clear enough.

5. In a previous hallucination (find?), I expressed the view that a large part of real analysis seems to be specialized abstract algebra. This was confirmed by the section 'Convergence in Ordered Fields' (p.269). In fact this books answers most of the hanging questions I had so far, concerning linear algebra, invariants, determinants, etc. Including solvability of polynomials an Galois theory. I had to smile when I saw the related section, followed by 'Compass and Ruler' section, both towards the end of the book, in stark contrast to my history oriented books, and rightfully so. My study method seems to be working as I wanted it to.
   Finally, note that Differential equations are too specialized to be treated in this book, but of course also benefit largely from abstract algebra (e.g differential forms http://en.wikipedia.org/wiki/Differential_forms, dynamical systems (MCMM.11), inspired by the success with Galois theory, which is something I guessed and validated (find quote?).