Notes on Almost Everything, Courtasy of Harold.

http://mathoverflow.net/questions/4648/when-to-pick-a-basis/4900#4900

http://faculty.luther.edu/~macdonal/GA&GC.pdf

Advanced Calculus: A Differential Forms Approach

This is almost perfect, but it should be better explicitely delineated like this:
- Basic calculable (linear) low dim model
- Handling higher dimensions
- Handling curvature

http://math.nyu.edu/faculty/edwardsd/books.htm

http://math.nyu.edu/faculty/edwardsd/talks.htm

I wrote the book because I believed that differential forms provided the most natural and enlightening approach

The fundamental postulate of Einstein's theory of special relativity is simply this: *All laws of physics should, like Maxwell's laws of electrodynamics, be unchanged by Lorentz transformations of the coordinates.* The motivation of this postulate is, briefly, as follows:

$$dx\,dy = (a\,du+b\,dv)(a'\,du+b'dv) = aa'\,du\,du + ab'\,du\,dv + ba'\,dv\,du + bb'\,dv\,dv = (ab'-a'b)\,du\,dv.$$

The 2-form $(ab'-a'b)\,du\,dv$ is called 'the pullback of the 2-form $dx\,dy$ under the affine map (1)'. The name 'pullback' derives from the fact that the map goes from the uv-plane to the xy-plane while the 2-form $dx\,dy$ on the xy-p

The affine mapping

$$x = u$$
$$y = 0$$

For example, in seeking a 'relativistic' version of the fundamental law

force = mass × acceleration

one must make a fundamental change in one's conception of 'mass'. Einstein asserts, in fact, that "mass and energy are essentially alike", even though the original idea of mass was *inertia*, which is virtually the *opposite* of energy. The argument by which Einstein arrived at this amazing conclusion was roughly as follows:

ne entire uv-plane to the x-axis. Thus the ny polygon is a 'polygon' of zero area. This the formula $dx\,dy = 0 \cdot du\,dv$ given by (2).

ullback goes in the direction (xy to u e direction of the map itself (uv to xy). ck should be considered to be define raic rules $du\,du = 0$, $du\,dv = -dv\,d$ $= a\,du + b\,dv$, etc., and not by tl Then the same rules serve to define tl a 2-form $A\,dy\,dz + B\,dz\,dx + C\,dx$ under an affine map

$$x = au + bv + c$$
$$y = a'u + b'v + c'$$
$$z = a''u + b''v + c''$$

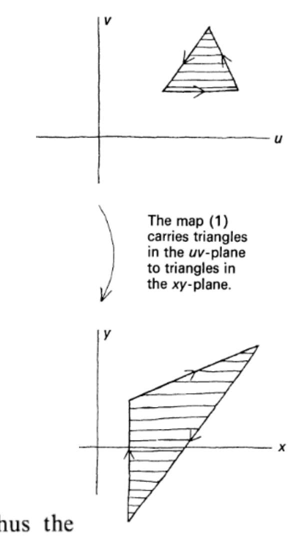
of the uv-plane to xyz-space. (A 2-form in xyz pul back under the map to give a 2-form in uv.) One mere performs the substitution and applies the algebraic rul to obtain

pullbacks have analogous interpretations. The connection between this geometrical interpretation of pullback and its actual algebraic definition has been indicated by plausibility arguments. A rigorous statement and proof are given in Chapter 6.

The Evaluation of Two-Forms Pullbacks

The algebraic rules which govern computations with forms all stem from the following fact: Let

$$(1)\qquad x = au + bv + c, \quad y = a'u + b'v + c'$$



The map (1) carries triangles in the uv-plane to triangles in the xy-plane.

nction $y = f(x)$, in which range domain are 1-dimensional, can

be a function assigning to each point of the uv-plane a point of the xy-plane (where a, b, c, a', b', c' are fixed numbers). A function from the uv-plane to the xy-plane is also called a *mapping** or a *map* instead of a function, and a mapping of the simple form (1) (in which the expressions for x, y in terms of u, v are polynomials of the first degree in u, v) is called an *affine mapping*. Given an oriented polygon in the uv-plane, its image under the affine mapping (1) is an oriented polygon in the xy-plane. For example, the oriented triangle with vertices (u_0, v_0), (u_1, v_1), (u_2, v_2) is carried by the mapping (1) to the oriented triangle (x_0, y_0), (x_1, y_1), (x_2, y_2) where

$$x_0 = au_0 + bv_0 + c$$
$$y_0 = a'u_0 + b'v_0 + c'$$

and similarly for x_1, y_1, x_2, y_2. It will be shown that the oriented area of the image of any oriented polygon under the map (1) is $ab' - a'b$ times the oriented area of the polygon itself. That is, the map (1) 'multiplies oriented areas by $ab' - a'b$', a fact which is conveniently summarized by the formula

$$(2)\qquad dx\,dy = (ab'-a'b)\,du\,dv.$$

However, in most cases it is as easy to carry out the substitution directly as it is to use this formula.

The 3-form $dx\,dy\,dz$ can be interpreted as the function 'oriented volume' assigning numbers to oriented three-dimensional figures in xyz-space in the same way that $dx\,dy$ is the oriented area of two-dimensional figures in the xy-plane and in the same way that dx is the oriented length of intervals of the x-axis. The principal fact about 3-forms which is needed to establish the plausibility of

Roughly speaking, homology theory is devoted to the question, "When is a k-form exact?" That is, "Given a k-form ω, under what conditions is there a $(k-1)$-form σ such that $\omega = d\sigma$?"

Let two affine maps

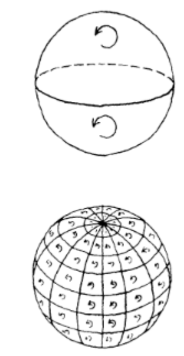
$$u = \alpha r + \beta s + \gamma t + \zeta \qquad x = au + bv + cw + e$$
$$v = \alpha' r + \beta' s + \gamma' t + \zeta' \quad \text{and} \quad y = a'u + b'v + c'w + e'$$
$$w = \alpha'' r + \beta'' s + \gamma'' t + \zeta'' \qquad z = a''u + b''v + c''w + e''$$

be given. Then the pullback of $dx\,dy\,dz$ under the composed map (of rst-space to xyz-space) is equal to the pullback under the first map (of rst-space to uvw-space) of the pullback under the second map (of uvw-space to xyz-space) of $dx\,dy\,dz$. In short, the pullback under a composed map is equal to the pullback of the pullback.

In the opinion of many mathematicians, the theorems of §9.4 are logically unacceptable because they are nonconstructive existence theorems, that is, theorems which assert that something or other 'exists' without telling how to find it explicitly. These mathematicians hold that it is pointless to say that something 'exists' if there is no way of finding it. For example, they hold that it is pointless to assert that an infinite sequence in a compact set has a point of accumulation (the Bolzano-Weierstrass Theorem) because there may be no way whatsoever of finding a point of accumulation. Either a point of accumulation can actually be found (in which case the theorem can be improved on) or there is no way to find a point of accumulation (in which case the theorem is futile).

If one adopts this constructive view of mathematical existence then several of the theorems of this book must be modified (and the theorems of §9.4 must be rejected altogether). However, the modifications are not as extensive as one might at first imagine, and the useful theorems of calculus survive it intact. In fact, a careful, constructive restatement of the theorems of calculus clarifies them and heightens their usefulness.

The proof that "the integral $\int_R A(x, y)\,dx\,dy$ of a continuous 2-form $A\,dx\,dy$ over a rectangle R of the xy-plane converges" (§2.3 and §6.3) used the theorem that a continuous function on R is necessarily uniformly



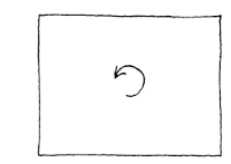
The intuitive meaning of the 2-form $dx\,dy$ on xyz-space is 'oriented area of the projection on the xy-plane'—a function assigning numbers to surfaces in xyz-space. It was on the basis of this intuitive idea that the algebraic rules governing 2-forms and their pullbacks under affine maps were derived in Chapter 1. Similarly, the algebra of 3-forms was based on 'oriented volume'. In Chapter 4, the algebra of k-forms was defined as a natural extension of the algebra of 2-forms and 3-forms, and this algebra was found to be very useful in stating and proving such basic theorems as the Chain Rule, the Implicit Function Theorem, and the method of Lagrange multipliers in Chapter 5. However, it has not yet been proved that 2-forms actually do describe areas or that 3-forms describe volumes, when areas and volumes are defined—as they must be—by integrals. This section is devoted to proving that the pullback operation on k-forms, as defined algebraically in Chapter 4, does indeed have a meaning in terms of 'k-dimensional volume'*, as defined by an integral in the obvious way:

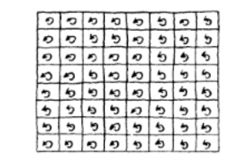
Definition

Let D be a bounded subset of $x_1x_2 \ldots x_k$-space. The *k-dimensional volume* of D, denoted $\int_D dx_1\,dx_2 \ldots dx_k$, is defined as follows: Let B be a number such that all coordinates of all points of D are less than B in absolute value. In other words, let B be a number such that D is contained in the k-dimensional cube $\{(x_1, x_2, \ldots, x_k): |x_i| \le B, i = 1, 2, \ldots, k\}$. An *approximating sum* $\sum(\alpha)$ to $\int_D dx_1\,dx_2 \ldots dx_k$ is formed by choosing

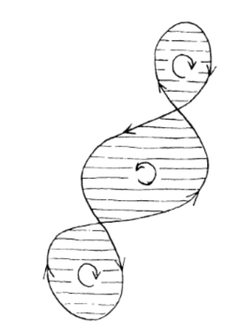
(α) (i) a subdivision of each of the k intervals $\{-B \le x_i \le B\}$ into small subintervals, thereby subdividing the cube $\{|x_i| \le B\}$ into k-dimensional 'rectangles' which will be denoted generically by R_α, and
(ii) a point P_α in each of the 'rectangles' R_α,

Definition of Certain Simple Integrals. Convergence and the Cauchy Criterion





**The notation $\int_a^b f(x)\,dx$ denotes, of course, the integral of the 1-form $f(x)\,dx$ over the interval $\{a \le x \le b\}$ oriented from a to b. Unfortunately there is no such convenient notation for indicating orientations of 2-dimensional integrals.*



A reasonable closed curve in the plane is the boundary of an oriented domain.

er to show that $dx\,dy\,dz$ can be interpreted as volume it is necessary to have an intuitive idea -dimensional figures can be oriented. To see is done it is useful to reformulate the idea of the n of 2-dimensional figures as follows: An n of a plane can be specified by giving three near points $P_0P_1P_2$. Two orientations $P_0P_1P_2$, re said to agree if the points $P_0P_1P_2$ can be $P_0'P_1'P_2'$ in such a way that throughout the ne three points remain non-collinear. Otherwise itations are said to be opposite. Then it is cally plausible that the orientations $P_0P_1P_2$ and are opposite (do not agree) and that every on agrees either with $P_0P_1P_2$ or with $P_1P_0P_2$. Thus all orientations $P_0'P_1'P_2'$ are divided into two classes by $P_0P_1P_2$—those which agree with $P_0P_1P_2$ and those which agree with $P_1P_0P_2$. In the xy-plane these classes are called clockwise and counterclockwise—the counterclockwise orientations being those which agree with the orientation (0, 0), (1, 0), (0, 1).

In the same way, an orientation of space can be described by giving four non-coplanar points $P_0P_1P_2P_3$. Two orientations $P_0P_1P_2P_3$ and $P_0'P_1'P_2'P_3'$ agree if the points of one can be moved to the points of the other keeping them non-coplanar all the while. All orientations fall into two classes such that two orientations in the same class agree. In xyz-space these classes are called left-handed and right-handed, an orientation being called right-handed if it agrees with the orientation (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1).*

A 'flow' is an imaginary physical phenomenon in which space is filled with a moving fluid which consists of infinitely many particles. Such a phenomenon can be described mathematically in two quite different ways—by following the particles, and by standing still and counting the particles as they go by.

Banach space has formed the basis of most of the theorems and proofs of this book.

For any real number $p \ge 1$ the 'p-norm'

$$|x|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

also defines a Banach space structure on R^n. The triangle inequality $|x + y|_p \le |x|_p + |y|_p$ is *Minkowski's Inequality* proved in Chapter 5 by the method of Lagrange multipliers (§5.4, Exercise 9). For $p = 2$ the p-norm is the

Similarly, the statement of Stokes' Theorem can be amplified so that it becomes constructively true. The proof of the Implicit Function Theorem given in §7.1 is constructive,† so that this theorem is perfectly acceptable from the constructivist point of view. However, not every theorem can be interpreted constructively. A very surprising exception is the trichotomy law of §9.1, that is, the 'law' that *every real number is either positive, negative, or zero*. The following example shows that this 'law' is not entirely self-evident:

exception has ever been found. Consider number $r = .a_1a_2a_3a_4 \cdots = \dfrac{a_1}{10} + \dfrac{a_2}{100} + \dfrac{a_3}{1000} + \cdots$ defined by

$$a_n = \begin{cases} 0 & \text{if } 2n + 4 \text{ can be written as the sum of two primes,} \\ 1 & \text{otherwise.} \end{cases}$$

The Goldbach conjecture is that $r = 0$.

Millions of decimal places of r are known, and they are all zero. However, in order to prove $r = 0$ it is necessary to prove the Goldbach conjecture, and in order to prove $r > 0$ it is necessary to disprove the Goldbach conjecture. But it is quite conceivable that human (or inhuman) intelligence will never succeed either in proving or in disproving the Goldbach conjecture. Thus it may be that neither the statement $r = 0$ nor the statement $r > 0$ will ever be proved. The constructivist position is that it is pointless to assert, as the trichotomy law does, that either $r = 0$ or $r > 0$. What one means is simply that the statements $r = 0$ and $r > 0$ are contradictory, that is, that both cannot be true. To put this statement in the form of the trichotomy law gives the mistaken impression that the Goldbach conjecture necessarily can be resolved one way or the other.

What is involved is the so-called law of the excluded middle. If one proves that the denial of a statement is false, is one justified in concluding that the statement is true? Surprisingly enough, the answer is "no" if all statements are interpreted constructively. One might conceivably prove, for example, that the assumption