An Alternative Asymptotic Notation

1. It is important to internalize asymptotic notation such that it becomes second nature. In our opinion, there are obstacles to this happening naturally which can in fact be remedied. We try to do this here.

2. Consider the relation between x^2 and x^3. It is both true that x^3 'grows faster' than x^2 and that x^3 'vanishes faster' than x^2. Of course, we 'mean' the former in terms of x increasing to infinity, and the latter in terms of x decreasing to zero, 'vanishing'.

3. The difference between the cases above is captured by having to specify what x tends to when using the little-o notation but this does not change the fact that it disallows us to write what we would like to, a simple 'asymptotic' inequality, since then we could write $x^3 > x^2$ for the first case, contradicting $x^3 < x^2$ for the second. This is not a problem because we have to specify what x tends to, but then we loose the simple notation. Is there a better way? None of the known notations solve this.

4. The problem is that the definitions are too general. Per example,

$$f(x) \in o(g(x)) \text{ as } x \to x_0$$

is defined as

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

For the limit of this ratio (assuming $g(x)$ eventually either keeps 'growing' or 'shrinking' as x tends to x_0) to go to zero we have two possibilities

1. If the denominator is 'growing', the numerator is either 'shrinking' or 'growing slower' than it. In short, the denominator is 'growing faster' than the numerator.
2. If the denominator is 'shrinking', the numerator is 'shrinking faster' than the denominator.

It is the (powerful and arguably elegant) mixing of these two possibilities into the same relation that causes latent confusion and refusal to assimilate it as 'second nature' by the cerebra of simple-minded humans such as the author, despite the differentiation depending on whether x tends to a number or to infinity. Admittedly, the generality allows talking not only about x tending to something, but could also about $h(x)$ tending to $k(x)$. Haivng said that, we have never seen such uses. Similarly, what $f(x) \in o(g(x))$ means would be quite convoluted if we chose $g(x)$ to be $1/x$ or $-x$. Such uses we have never seen either. $g(x)$ is usually assumed to be 'eventually' monotonically increasing in x.

5. Given the above, we propose to only ever consider two cases $x \to 0$ and $x \to \infty$ and recommend eventual increasing monotonicity of $g(x)$ short of requiring it. We relate the case $x \to 0$ to the symbol where the exaggerated accent suggests that '$(x - x_0)$ is going down (to zero)'. In similar vain, we use for $x \to \infty$ the symbol

$$\underline{\diagup}$$

noting that we are able to reverse the symbols in the same way we do for ordinary inequalities. Furthermore, we use non-strict inequalities to differentiate between little-o and big-o.

6. Given appropriate choice of typography, we can superimpose both symbols as shown in the asymptotic inequalities (1) and (2). This can be considered as both the 'highlight' of this notation and its usefulness in clearly exposing the 'issues' we mentioned.

7. This notation translates in the following manner to Landau notation:

$(x - x_0) \to 0,\quad f(x) \in o(g(x))$		$f \overline{\diagdown} g$	$g \overline{\diagup} f$
$(x - x_0) \to 0,\quad f(x) \in O(g(x))$		$f \overline{\overline{\diagdown}} g$	$g \overline{\overline{\diagup}} f$
$x \to \infty,\quad f(x) \in o(g(x))$		$f \underline{\diagup} g$	$g \underline{\diagdown} f$
$x \to \infty,\quad f(x) \in O(g(x))$		$f \underline{\underline{\diagup}} g$	$g \underline{\underline{\diagdown}} f$

and can be faithfully read this way:

$f \overline{\diagdown} g$	f vanishes faster than g
$f \overline{\diagup} g$	f vanishes slower than g
$f \underline{\diagdown} g$	f grows faster than g
$f \underline{\diagup} g$	f grows slower than g

8. In this notation we have:

$$x^3 \underline{\diagdown} x^2$$
$$x^2 \underline{\diagup} x^3$$
$$x^3 \overline{\diagdown} x^2$$
$$x^2 \overline{\diagup} x^3$$

and we can go (mischievously) further writing:

$$x^3 \overline{\underline{\diagdown}} x^2 \tag{1}$$

$$x^2 \overline{\underline{\diagup}} x^3. \tag{2}$$

References

Hildebrand, A.J. "Asymptotic Analysis Lecture Notes, Math 595, Fall 2009."