Note. The innocent sentence by Kline “Though it is not intuitively satisfying to view the line in this manner, the idea is logically sound.” resonated in my mind. This problematic had been an invisible problem. I did solve it successfully while studying Zakon’s book, but never mentioned this problem in this clear manner. The sentence itself occurs in the chapter about the origins of projective geometry; a very enlightening chapter.

My idea of ‘controlled schizophrenia’ can play a very useful role here; the role of seeking and obtaining ‘intuitive’ understanding at times, and working fully from a logical point of view oblivious of intuition at others.
From Zakon’s book, the notes I wrote do indicate that there is a very clear and intuitive way to understand the whole foundations laid out, provided one realizes the idea of ‘meta-numbers’ I developed.

I must point out that intuition in mathematics, a problem I dealt with during my crysis ‘What is understanding’, is a mere illusion coming from a frequent interaction, just like usual understanding is an illusion as well. But with Kline’s sentence, I would say that a good part of studying mathematics is turning learned logical systems into embodied personal intuition.
I focus here again on ‘learning’ because I totally embraced the Greek idea that comes at the base of the name ‘mathematics’ as that which must first be learned, make it *the* distinction compared to other sciences.

Note. Again in relation to projective geometry and the above, I take up this paragraph from Kline:

> *What is far more significant than Descartes's insight into construction problems and their classification is the importance he assigned to algebra.*
> *This key makes it possible to recognizes the typical problems of geometry and to being together problems in that geometrical form would not appear to be related at all. Algebra brings to geometry the most natural principles of classification and the most natural hierarchy of method. Not only can questions of solvability and geometrical constructibility be decided elegantly, quickly, and fully from the parallel algebra, but without it they cannot be decided at all. Thus, system and structure were transferred from geometry to algebra.*

It is as if the greeks were not really doing geometry at all, but algebra ‘projected to’ geometry, ‘projected’ onto a plane, or three dimensional space. Just as with projective geometry where circles appear as special cases of projections of conics, involution and harmonic sets as projecting to bisection. Another example is a pentagon as a special case of a hexagon:

> *Pascal, by seeking the relationships of different figures, such as the hexagon and the pentagon, also sought some common approach to these figures. In fact, he was supposed to have deduced some 400 corollaries from his theorem on the hexagon, by examining the consequences of the theorem for related figures.*

Note. In my notes on Euclid’s elements, I maintained a section called ‘Greekomania’ or ‘the amazing greeks’. I am happy to find out I am not the only one with respect to the astonishment specifically about how they found the results to prove. I also had concluded that they must have known algebra as we know it, derived their theorems using it, and then only published and proved them geometrically. I am almost surely wrong in that, but here is the comforting quote:

> *In the edition of 1659-61, van Schooten actually gave the algebraic form of a transformation of coordinates from one base line (x-axis) to another. He was so impressed with the power of Descartes's method that he claimed the Greek geometers had used it to derive their results. Having the algebraic work, the Greeks, according to van Schooten, saw how to obtain the results synthetically--he showed how this could be done--and then published their synthetic methods, which are less perspicuous than the algebraic, to amaze the world. Van Schooten may have been misled by the word "analysis," which to the Greeks meant analyzing a problem, and the term "analytic geometry," which specifically described Descartes's use of algebra as a method.*

I must disagree on two points though, assuming this is true, the Greeks would have done this due to their requirements of rigor and foundation. Secondly, Van Schooten being misled by the word “analysis” seems to be a moot point, even not knowing a lot about him.

Note. In some respect, and very vaguely, coordinate geometry is to synthetic geometry what matrix algebra is to linear algebra. One needs a choice of coordinate system in the former and a basis in the latter.

Note. A very enlightening paragraph about the terminology confusion pertaining to the word “analysis”.

> *The fact that algebra was built up on an empirical basis has led to confusion in mathematical terminology. The subject created nu Fermat and Descartes is usually referred to as analytic geometry. The work "analytic" is inappropriate; coordinate geometry or algebraic geometry (which now has another meaning) would be more suitable. The word "analysis" had been used since Plato's time to mean the process of analyzing by working backward from what is to be proved until one arrives at something known. In this sense it was opposed to "synthesis," which describes the deductive presentation. About 1590 Vieta rejected the word "algebra" as having no meaning in the European language and proposed the term "analysis" (Chap. 13, sec. 8); the suggestion was not adopted. However, for him and for Descartes, the word "analysis" was still somewhat appropriate to describe the application of algebra to geometry because algebra served to analyze the geometric construction problem. One assumed the desired geometric length was known, found an equation that this length satisfied, manipulated the equation, and then saw how to construct the required length. Thus Jacques Ozanam (1640-1717) said in his Dictionary (1690) that moderns did their analysis by algebra. In the famous eighteenth-century Encyclopedie, D'Alembert used "algebra" and "analysis" as synonyms. Gradually, "analysis" came to mean the algebraic method, though the new coordinate geometry, up to about the end of the eighteenth century, was most often formally described as the application of algebra to geometry. By the end of the century the term "analytic geometry" became standard and was frequently used in titles of books.*
>
> *However, as algebra became the dominant subject, mathematicians came to regard it as having a much greater function that the analysis of a problem in the Greek sense. In the eighteenth century the view that algebra as applied to geometry was more that a tool--that algebra itself was a basic method of introducing and studying curves and surfaces (the supposed view of Fermat as opposed to Descartes)-- won out, as a result of the work of Euler, Lagrange and Monge. Hence the term "analytic geometry" implied proof as well as the use of the algebraic method.*

Consequently we now speak of analytic geometry as opposed to synthetic geometry, and we no longer mean that one is a method of invention and the other of proof. Both are deductive.
In the meantime the calculus and extensions such as infinite series entered mathematics. Both Newton and Leibniz regarded the calculus as an extension of algebra; it was the algebra of the infinite, or the algebra that dealt with an infinite number of terms, as in the case of infinite series. As late as 1797, Lagrange, in Theorie des fonctions analytiques, said that the calculus and its developments were only a generalization of elementary algebra. Since algebra and analysis had been synonyms, the calculus was referred to as analysis. In a famous calculus text of 1748 Euler used the term "infinitesimal analysis" to describe calculus. This term was used until the late nineteenth century, when the word "analysis" was adopted to describe calculus and those branches of mathematics built on it. Thus we are left with a confusing situation in which the term "analysis" embraces all the developments based on limits, but "analytic geometry" involves no limit processes.