1. The semantics of PL, unlike the syntactic deductive systems of it, are less prone to having flavors and we therefore present them first.

2. A set of two boolean values is the beginning of semantics:

$$\{0,1\}$$

3. All the four functions $f_i : \{0,1\} \to \{0,1\}$ come next:

	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

$f_i : \{0,1\} \to \{0,1\}$

4. Finally, the sixteen functions $g_i : \{0,1\}^2 \to \{0,1\}$ are considered:

		g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
0	0	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
0	1	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
1	0	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
1	1	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1

$g_i : \{0,1\}^2 \to \{0,1\}$

5. A functional point of view of some semantic concepts gives the following terse definitions:

- Each 'formula' is identified with a function $f : 2^n \to 2$, where '2' denotes the set {0,1}. In this sense, we cannot differentiate (except by assigned distinct function names) between two 'formulas' that have a different 'form' but the same variables and same value for each variable assignment. Per example, $p \xrightarrow{f} p$ and $p \xrightarrow{g} p \vee p$ are the same function.
- In this sense, the connectives are the binary operators, the members of the set $\{\circ : 2^2 \to 2\}$.
- The set of all *tautologies* is then $\{\tau : 2^n \to \{1\}\}$.
- The set of all *contradictions* is $\{\bot : 2^n \to \{0\}\}$.
- The set of all *tautological consequences* is then $\{(\Gamma = \{f_i\}, g)$ such that $\{\to (f_i, g) = 1, \forall i\}\}$.
- Two formulas are tautologically equivalent simply if their functions are equal.

6. The full list of syntactic symbols used to define well-formed-formulas, a sublist of which is used in each deductive system is:

$$\underbrace{\overbrace{),(}^{\text{parenthesis}}\ ,\ \overbrace{\bot,\top}^{\text{boolean constants}}\ ,\ \overbrace{\overbrace{\neg}^{\text{unary operator}}\ ,\overbrace{\downarrow,\nleftarrow,\nrightarrow,\wedge,\leftrightarrow,\nleftrightarrow,\uparrow,\rightarrow,\leftarrow,\vee}^{\text{binary operators}}}^{\text{logical connectives}}}_{\text{logical constants}},\underbrace{p,q,r,s,p_1,q_1,r_1,s_1,\ldots}_{\text{variables}}.$$

We also freely use the following aliases for readability:

- [for (
-] for)
- | for $\uparrow$
- $\equiv$ for $\leftrightarrow$
- $=$ for $\leftrightarrow$

7. Gluing the "semantic", the 'syntactic', and ˶informal human˝, we have the following (universally agreed upon) translation -what we mean- table:

- ˶false˝, "0", "f_1", "g_1", '$\bot$'
- ˶identity˝, "f_2"
- ˶negation˝, "f_3", '$\neg$'
- ˶joint denial˝, "g_2", '$\downarrow$'
- ˶converse nonimplication˝, ˶not but˝, "g_3", '$\nleftarrow$'
- ˶material nonimplication˝, ˶abjunction˝, ˶but not˝, "g_4", '$\nrightarrow$'
- ˶conjunction˝, ˶and˝, "g_5", '$\wedge$'
- ˶not p˝, "g_6"
- ˶not q˝, "g_7"
- ˶biconditional˝, ˶logical equality˝, ˶iff.˝, "g_8", '$\leftrightarrow$'
- ˶exclusive disjunction˝, ˶exclusive or˝, "g_9", '$\nleftrightarrow$'
- ˶q˝, "g_{10}"
- ˶p˝, "g_{11}"
- ˶alternative denial˝, "g_{12}", '$\uparrow$'
- ˶material conditional˝, ˶if then†˝, ˶implication†˝, "g_{13}", '$\rightarrow$'
- ˶converse implication˝, ˶if†˝, "g_{14}", '$\leftarrow$'
- ˶disjunction˝, ˶or˝, "g_15", '$\vee$'
- ˶true˝, "1", "f_4", "g_{16}", '$\top$'

8. The structure of the grammatically accepted strings, called well-formed-forumals (WFFs), varies per deductive system. In BNF notation, with the expression

$$\mathrm{V} ::= p \mid q \mid r \mid s \mid p_1 \mid q_1 \mid r_1 \mid s_1 \mid \ \ldots$$

being shared among all, we have the following flavors:

Name	Constants	WFF (in BNF)
Mendelson (classical)	$\neg, \rightarrow,), ($	WFF :== V | ($\neg$E) | (E $\rightarrow$ E)
Dijkstra-Scholten (equational)	$\bot, \top, \neg, \rightarrow, \vee, \wedge, \leftrightarrow,), ($	$\circ$:== $\rightarrow$ | $\vee$ | $\wedge$ | $\leftrightarrow$ V' :== $\bot$ | $\top$ | V WFF :== V' | ($\neg$E) | (E $\circ$ E)
Gentzen	$\neg, \rightarrow, \vee, \wedge,), ($	$\circ$:== $\rightarrow$ | $\vee$ | $\wedge$ | $\leftrightarrow$ WFF :== V | ($\neg$E) | (E $\circ$ E)
Nicod (minimal)	$\uparrow,), ($	WFF :== V | (E $\uparrow$ E)
Nicod$^{\text{left-polish}}$	$\uparrow$	WFF :== V | $\uparrow$EE

9. The axiom schemata and inference rules of each deductive flavor complete the basic descriptions thereof. The capital letters used stand for WFFs and in that sense, each axiom schema stands for infinitely many axioms (similarly for inference rules), which are obtained by replacing the capital letters with all possible WFFs.

Name	Axiom Schemata & Inference Rules
Classical	— Axiom Schemata $A \to (B \to A)$ $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$ $[\neg B \to \neg A] \to [(\neg B \to A) \to B]$ — Inference Rules $A,\ A \to B \ \vdash\ B$ ("modus ponens")
Equational	— Axiom Schemata $[A \equiv (B \equiv C)] \quad \equiv \quad [(A \equiv B) \equiv C]$ ("associativity of $\equiv$") $(A \equiv B) \quad \equiv \quad (B \equiv A)$ ("symmetry of $\equiv$") $\top \quad \equiv \quad (\bot \equiv \bot)$ ("property of $\top, \bot$") $\neg A \quad \equiv \quad (A \equiv \bot)$ ("property of $\neg$") $A \vee (B \vee C) \quad \equiv \quad (A \vee B) \vee C$ ("associativity of $\vee$") $(A \vee B) \quad \equiv \quad (B \vee A)$ ("symmetry of $\vee$") $(A \vee A) \quad \equiv \quad A$ ("idempotency of $\vee$") $[A \vee (B \equiv C)] \quad \equiv \quad [(A \vee B) \equiv (A \vee C)]$ ("distributivity of $\vee$ over $\equiv$") $A \vee \neg A$ ("excluded middle") $A \wedge B \quad \equiv \quad A \equiv B \equiv A \vee B$ ("property of $\wedge$, viz. golden rule (10)") $A \to B \quad \equiv \quad A \vee B \equiv B$ ("property of $\to$") — Gries Inference Rules $A = B \ \vdash\ C[p := A] \equiv C[p := B]$ ("Leibniz") $A = B \ \&\ B = C \ \vdash\ A = C$ ("transitivity") $A \ \vdash\ A[p := B]$ ("substitution") — Tourlakis Inference Rules $A \equiv B \ \vdash\ C[p := A] \equiv C[p := B]$ ("Leibniz") $A \ \&\ A \equiv B \ \vdash\ B$ ("equinanimity")
Minimal	— Axiom Schemata $(A\vert(B\vert C))\vert((D\vert(D\vert D))\vert((E\vert B)\vert((A\vert E)\vert(A\vert E))))$ — Inference Rules $A, A\vert(B\vert C) \ \vdash\ C$ ("Nicod's modus ponens")
Minimal$^{\mathrm{P}}$	— Axiom Schemata $\vert\vert A\vert BC\vert\vert D\vert DD\vert\vert EB\vert\vert AE\vert AE$ — Inference Rules $A, \vert A\vert BC \ \vdash\ C$

	— Inference Rules			
	$A,\ A \to B$	$\vdash$	B	(“modus ponens”)
	$\neg B,\ A \to B$	$\vdash$	A	(“modus tollens”)
	$A \to B,\ B \to C$	$\vdash$	$A \to C$	(“pure hypothetical syllogism”)
	$\neg A,\ A \vee B$	$\vdash$	B	(“disjunctive syllogism”)
	$A \vee C,\ (A \to B) \wedge (C \to D)$	$\vdash$	$B \vee D$	(“constructive dilemma”)
	$A \wedge B$	$\vdash$	A	(“simplification”)
	$A,\ B$	$\vdash$	$A \wedge B$	(“conjunction”)
	A	$\vdash$	$A \vee B$	(“addition”)
	— Replacement Rules			
	$\neg(A \wedge B)$	$::$	$\neg A \vee \neg B$	(“De Morgan’s rule”)
	$\neg(A \vee B)$	$::$	$\neg A \wedge \neg B$	(“De Morgan’s rule”)
Gentzen’s ND (natural deduction)	$A \vee B$	$::$	$B \vee A$	(“commutativity”)
	$A \wedge B$	$::$	$B \wedge A$	(“commutativity”)
	$A \vee (B \vee C)$	$::$	$(A \vee B) \vee C$	(“associativity”)
	$A \wedge (B \wedge C)$	$::$	$(A \wedge B) \wedge C$	(“associativity”)
	$A \wedge (B \vee C)$	$::$	$(A \wedge B) \vee (A \wedge C)$	(“distribution”)
	$A \vee (B \wedge C)$	$::$	$(A \vee B) \wedge (A \vee C)$	(“distribution”)
	A	$::$	$\neg\neg A$	(“double negation”)
	$A \to B$	$::$	$\neg B \to \neg A$	(“transposition”)
	$A \to B$	$::$	$\neg A \vee B$	(“material implication”)
	$A \leftrightarrow B$	$::$	$(A \to B) \wedge (B \to A)$	(“material equivalence”)
	$A \leftrightarrow B$	$::$	$(A \wedge B) \vee (\neg A \wedge \neg B)$	(“material equivalence”)
	$(A \wedge B) \to C$	$::$	$A \to (B \to C)$	(“exportation”)
	A	$::$	$A \vee A$	(“tautology”)
	A	$::$	$A \wedge A$	(“tautology”)
	— Inference Rules			
	$\Gamma, A, B, \Delta \Rightarrow \Lambda$	$\vdash$	$\Gamma, A \wedge B, \Delta \Rightarrow \Lambda$	(“$\wedge$ left”)
	$\Gamma \Rightarrow \Delta, A, \Lambda\ \ \&\ \ \Gamma \Rightarrow \Delta, B, \Lambda$	$\vdash$	$\Gamma \Rightarrow \Delta, A \wedge B, \Lambda$	(“$\wedge$ right”)
	$\Gamma, A, \Delta \to \Lambda\ \ \&\ \ \Gamma, B, \Delta \Rightarrow \Lambda$	$\vdash$	$\Gamma, A \vee B, \Delta \Rightarrow \Lambda$	(“$\vee$ left”)
Gentzen’s SC (sequent calculus)	$\Gamma \Rightarrow \Delta, A, B, \Lambda$	$\vdash$	$\Gamma \Rightarrow \Delta, A \vee B, \Lambda$	(“$\vee$ right”)
	$\Gamma, \Delta \Rightarrow A, \Lambda\ \ \&\ \ B, \Gamma, \Delta \Rightarrow \Lambda$	$\vdash$	$\Gamma, A \to B, \Delta \Rightarrow \Lambda$	(“$\to$ left”)
	$A, \Gamma \Rightarrow B, \Delta, \Lambda$	$\vdash$	$\Gamma \Rightarrow \Delta, A \to B, \Lambda$	(“$\to$ right”)
	$\Gamma, \Delta \Rightarrow A, \Lambda$	$\vdash$	$\Gamma, \neg A, \Delta \Rightarrow \Lambda$	(“$\neg$ left”)
	$A, \Gamma \Rightarrow \Delta, \Lambda$	$\vdash$	$\Gamma \Rightarrow \Delta, \neg A, \Lambda$	(“$\neg$ right”)

10. The golden rule of equational calculus is motivated by Gries as follows: “We chose the Golden rule to define $\wedge$ because it is amazingly versatile, given the associativity and symmetry of $\equiv$. For example, one view is that it defines $p \wedge q$ as $p \equiv q \equiv p \vee q$, but it can also be rewritten as $(p \equiv q) \equiv (p \wedge q \equiv p \vee q)$, which indicates that p and q are equal iff their conjunction and disjunction are equal. With the Golden rule, we can prove a host of theorems that relate A to the already-defined operators.”

11. It seems that in equational logic WFFs, ‘$\equiv$’ (should) be read as either “iff.” or “equals”, depending on the emphasis.

12. It will be the goal of the syntactic deductive system to syntactically correspond to all semantical tautologies. In other words, to each distinct boolean function of any arity that has range {1} and not {0,1} or {0}, there must correspond one distinct syntactic WFF with the exact same variables while conforming to the translations table (7). The reason this is the goal is that by "1" we mean ˝true˝ so in other words, the goal will be to syntactically identify all WFFs that are unconditionally ˝true˝: to identify all ˝facts˝ of PL.

13. TODO: understand the fuss in the introduction of http://www.cs.tau.ac.il/~aa/articles/gentzen_jar.pdf

14. TODO: Hook 'computers and languages, theory and practice' the BNF chapter is a good hook into context-free-grammars, reg-exp, bnf, etc.

15. LINKS: gentzen SC: ftp://ftp.cis.upenn.edu/pub/cis510/public_html/tcl/book-63-64.pdf and http://www.cs.tau.ac.il/~aa/articles/gentzen_jar.pdf, gentzen ND: 'A Concise Introduction to Logic' and http://www.mathpath.org/proof/proof.inference.htm, nicod, mendelson: http://www.trinity.edu/cbrown/logic/alter.pdf

16. TODO: gentzen ND vs. SC https://rd.host.cs.st-andrews.ac.uk/seminars/vonPlato-GCS.pdf, http://plato.stanford.edu/entries/proof-theory-development/#NatDedSeqCal, https://homepage.univie.ac.at/christian.damboeck/ps06/clemente_nat_ded.pdf

17. TODO: golden rule: http://www.philosophy.ed.ac.uk/undergraduate/documents/Natural_deduction_rules_propositional.pdf and 'Logic (Paul Tomassi).

18. TODO!!!: http://www-personal.usyd.edu.au/~njjsmith/papers/smith-freges-js-logic.pdf, the origins of $\vdash$

19. ADD: http://emilkirkegaard.dk/en/wp-content/uploads/Paul-Tomassi-Logic.pdf (in-english)

20. TODO: these may be good hooks into a model theory application, and again into links with algebra and cat: 'First-order equational logic consists of quantifier-free terms of ordinary first-order logic, with equality as the only predicate symbol. The model theory of this logic was developed into Universal algebra by Birkhoff, Grätzer and Cohn. It was later made into a branch of category theory by Lawvere ("algebraic theories").[1]' and http://www.cl.cam.ac.uk/~mpf23/papers/Types/soat.pdf

21. TODO: 'reading turnstiles'!!!! https://books.google.de/books?id=70W4Q-kzdicC&pg=PA44&lpg=PA44&dq=inference+rule+using+turnstile&source=bl&ots=e0J_rynDCv&sig=hI5oOurmL8f2bJDIqX7KHENWGi4&hl=en&sa=X&ved=0ahUKEwj6w___S-KHKAhVFQhQKHd2dB9gQ6AEIOjAF#v=onepage&q=inference%20rule%20using%20turnstile&f=false, " "So far, only a few textbooks have been written with the pluralist approach; this book is one of them. Though pluralistic in content, this book is old-fashioned style: It is intented for beginners, so we use philosophy for motivation and we attempt to keep algebra to a minimum."", also by same author and a MUST judging from the preface!!!!! totally in line with our attitudes : 'Handbook of Analysis and Its Foundations: A Handbook'

22. TODO: We need a whole section on 'how to read this!!!'. this comes from an important question we had, which led us to Schechter. From him we read the 'how to read' trivializer of no LEM: 'we know how to prove Goldbach's conjecture or we know how to disprove Goldbach's conjecture.'

23. TODO: very interesting for later: http://www.cs.cornell.edu/home/kreitz/pdf/96ki-matrixprover.pdf

24. TODO: check where we stand with this: http://www.math.vanderbilt.edu/%7Eschectex/papers/difficult.html