Covariance

I Natural, Canonical, Standard

1. We mean ’natural’ in the category theoretical sense: a natural transformation of functors.

2. We mean ’canonical’ in the sense that something does not involve making any choice beyond the given data.

3. A construction can be natural without being canonical and vice versa. In short, the two concepts are unrelated in general.

4. There is vagueness in the way the word ’natural’ is used in the mathematical literature: ’Otherwise, I agree, it is hard to tell what people mean by ”natural”’. Per example, on wikipedia, it is stated that: ’the standard basis (also called natural basis or canonical basis) for a Euclidean space ...’, without justifying any of the adjectives, but this kind of vagueness is also found in textbooks, depending on the focus and level.

5. Each of the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is a natural construction. The first of them can be described as the initial (0,1)-Algebra.

6. We use $\mathbb{F}$ to mean both $\mathbb{R}$ and $\mathbb{C}$.

7. $\mathbb{F}^n$ is a vector spaces and it affords a standard basis. This standard basis is indeed natural.

II Inner Product Space

8. A vector space can be equipped with an inner product, making it an inner product space.

9. Not every vector space (VS) can be turned into an inner product space (IPS).

- An IP makes no sense if the VS’s ring is not $\mathbb{R}$ or $\mathbb{C}$.
- More precisely, a field with an ordered subfield should suffice. This is then related to conjugation and the Hermitian inner product, but we withhold.

- What is needed for IP is a symmetric bilinear form but there exists some freedom of definition here: ’An inner product on a real vector space is an example of a symmetric bilinear form. (For some authors, an inner product on a real vector space is precisely a positive definite symmetric bilinear form. Other authors relax the positive definiteness to nondegeneracy. Perhaps some authors even drop the nondegeneracy condition (citation?).)’.
- For an infinite dimensional space, the axiom of choice is additionally needed so that one obtains a Hamel basis.
- For finite dimensional VSs over $\mathbb{R}$ or $\mathbb{C}$, none of this is a problem, and they can be turned into IPSs. In the infinite dimensional case, this brings us to Hilbert spaces.
- IPSs are one step away from being metric (?).

10. We denote the inner product of two vectors as

$$\mathbf{u} \cdot \mathbf{v}$$

11. In $\mathbb{C}^n$, and using the summation convention, we define the standard (where the bases are dictated to be the natural ones) inner product as

$$\mathbf{u} \cdot \mathbf{v} = u^i \overline{v}_i.$$

This makes it an inner product. In $\mathbb{R}^n$ this simplifies to

$$\mathbf{u} \cdot \mathbf{v} = u^i v_i.$$

12. Even with the standard bases on $\mathbb{F}^n$, we have multiple inner products such as the the weighted inner products which are of the form

$$\mathbf{u} \cdot \mathbf{v} = \sum w_i u_i v_i$$

for some fixed weight vector $\mathbf{w}$.

13. ’Length and orthogonality are, to a great extent, what we define them to be. Only if we use standard inner products in $\mathbb{R}^3$ do length and orthogonality correspond to physical length and 90° angles.’.

14. A real inner product space is called a Euclidean space and a unitary space is an inner product space for which the scalars are the complex numbers.

15. The property of the inner product as being the length of the projection along the orthogonal multiplied by the length of the base holds for all inner products.

this is good!! http://math.stackexchange.com/questions/1110637/is-the-standard-scalar-product-in-a-coordinate-space-basis-independent We might see it a tiny bit differently, but this comment is very good. Also, we can finally say that a ‘reciprocal basis’ Also this seems useful, also as a hook into functional analysis:

http://www.ams.org/bookstore/pspdf/gsm-156-prev.pdf Same for the hilbert spaces reference.

is this the ‘answer’??http://math.stackexchange.com/questions/643617/inner-product-formulation-of-a-dual-basis

http://math.stackexchange.com/questions/86876/notation-of-dual-pairings-and-inner-products?!?!?! umbral!

maybe these help (choice of bilinear form): http://www.math.uconn.edu/~kconrad/blurbs/linmultialg/bilinearform.pdf

Every bilinear form B on V defines a pair of linear maps from V to its dual space V∗. Define B1, B2: V → V∗ by (there is only one dual space! but there are many bilinear forms) https://en.wikipedia.org/wiki/Bilinear_form https://en.wikipedia.org/wiki/Dual_space https://unapologetic.wordpress.com/2009/04/15/real-inner-products/

or maybe this: http://math.stackexchange.com/questions/383534/dual-basis-existence-and-uniqueness

http://www.math.umn.edu/~garrett/m/algebra/notes/25.pdf so abstractly unique, but concretely not? (that is coord-free vs coord based?). is this it?

III Reciprocal Basis

16. To define a reciprocal basis for a vector space the latter must be an inner product space.

This is it: http://mathoverflow.net/questions/56938/what-does-the-adjective-natural-actually-mean

IV Natural (Canonical) Isomorphism

- http://math.stackexchange.com/questions/622589/in-categorical-terms-why-is-there-no-canonical-isomorphism-from-a-finite-dimens
- http://math.stackexchange.com/questions/105490/isomorphisms-between-a-finite-dimensional-vector-space-and-its-dual
- Vectk category: https://en.wikipedia.org/wiki/Category_(mathematics)
- http://math.stackexchange.com/questions/234127/natural-isomorphism-in-linear-algebra
- http://mathoverflow.net/questions/139388/example-of-an-unnatural-isomorphism
- http://math.stackexchange.com/questions/670159/why-do-we-need-dual-space/670181#670181
- http://math.stackexchange.com/questions/700224/how-can-a-functor-which-preserves-products-not-be-natural/700324#700324

- http://math.stackexchange.com/questions/1230736/prove-that-the-isomorphism-between-vector-spaces-and-their-duals-is-not-natural
- http://math.stackexchange.com/questions/622589/in-categorical-terms-why-is-there-no-canonical-isomorphism-from-a-finite-dimens
- http://mathoverflow.net/questions/19644/what-is-the-definition-of-canonical/121989#121989
- http://math.stackexchange.com/questions/551820/a-finite-dimensional-vector-space-that-is-not-naturally-isomorphic-to-its-dual?rq=1
- http://math.stackexchange.com/questions/236035/yoneda-lemma-applied-to-finite-dimensional-vector-spaces?rq=1
- http://math.stackexchange.com/questions/622589/in-categorical-terms-why-is-there-no-canonical-isomorphism-from-a-finite-dimens?rq=1
- http://math.stackexchange.com/questions/1230736/prove-that-the-isomorphism-between-vector-spaces-and-their-duals-is-not-natural?lq=1
- http://ncatlab.org/nlab/show/dinatural+transformation

V Dual space

- https://en.wikipedia.org/wiki/Dual_pair

VI Covectors and Physics

17. inner, dot, scalar, confusion in physics: http://math.stackexchange.com/questions/754449/is-there-any-distinction-between-these-products-scalar-dot-inner?rq=1 , https://www.physicsforums.com/threads/inne products-and-the-dual-space.735158/

18. http://physics.stackexchange.com/questions/131348/is-force-a-contravariant-vector-or-a-covariant-vector-or-either?rq=1

VII Miscellaneous

19. The reals seen as a vector space over the rationals provide for a good (?) example of an infinite dimensional vector space and links us to the Hamel basis.

20. 'According to Mac Lane (as I remember it from Categories for the Working Mathematician), Eilenberg and Mac Lane invented categories so they could talk about functors, and they wanted to talk about functors so they could define "natural."'.

\begin{note} 'he subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual" or "conjugate" space T(L). Let L be a finite-dimensional real vector space, while its conjugate TiL) is, as is customary, the vector space of all real valued linear functions t on L. Since this conjugate T{L) is in its turn a real vector space with the same dimension as L, it is clear that L and T(L) are isomor- phic. But such an isomorphism cannot be exhibited until one chooses a defi- nite set of basis vectors for L, and furthermore the isomorphism which results will differ for different choices of this basis.' (http://www.ams.org/journals/tran/1945-058-00/S0002-9947-1945-0013131-6/S0002-9947-1945-0013131-6.pdf) \end{note}

21. Inner, scalar and dot product make for a confused terminology.

- http://math.stackexchange.com/questions/476738/difference-between-dot-product-and-inner-product
- http://math.stackexchange.com/questions/754449/is-there-any-distinction-between-these-products-scalar-dot-inner?rq=1
- https://www.physicsforums.com/threads/inner-products-and-the-dual-space.735158/
- `"https://books.google.de/books?id=Nh9apVawii4C&pg=PA177&lpg=PA177&dq=reciprocal+basis+indef`

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"Hilbert Spaces." http://www.seas.upenn.edu/~dorny/VectorSp/HilbertSpaces237-331.pdf.

"Hilbert Spaces and Dagger Categories." https://qchu.wordpress.com/2012/06/25/hilbert-spaces-and-dagger-categories/.

"Inner Product for Vector Space over Arbitrary Field." http://math.stackexchange.com/questions/216630/inner-product-for-vector-space-over-arbitrary-field.

"Is There a Vector Space That Cannot Be an Inner Product Space?" http://math.stackexchange.com/questions/247425/is-there-a-vector-space-that-cannot-be-an-inner-product-space.

"Real Inner Product Spaces." https://rutherglen.science.mq.edu.au/wchen/lnlafolder/la09.pdf.

"Rigidity of the Category of Fields." http://math.stackexchange.com/questions/634010/rigidity-of-the-category-of-fields.

"The Category of Inner Product Spaces." https://unapologetic.wordpress.com/2009/05/06/the-category-of-inner-product-spaces/.