First Exploitation

I Introduction

1.
1. Differential forms
   - (a) Tensors with [CItLA] [...]
     - i. Variance [✓]
   - (b) Vector calculus with [VLDU] and [olver]
     - i. Gradient [...]
     - ii. Div
     - iii. Curl
   - (c) Differential forms [...](notes:6,8,9,10)
   - (d) Integration on forms and relation to grad-div-curl [...](notes:2,3,4,5)
   - (e) Integration on manifolds and change of coordinates with [IDfE]
   - (f) The relation between multiple integrals and differential forms. [???]
   - (g) Differential forms without Tensors (eating the dessert first) with [NoDF]
     - i. A. Summation notation exploitation dessert with [SCVA]
2. Linearization
   - (a) o-notation with [tWoA] [...]
   - (b) Taylor with [tWoA] [...]
   - (c) Multivar Taylor
   - (d) Remainder [...]
   - (e) Numerical convergence with [EoNA]
   - (f) Convergence and eigenvalues with [aRoPMT]
   - (g) Application with 'modeling' and [EoNA]
3. Variational Calculus
4. Target
   - (a) Check any misconceptions against Tarantola's [EoP]
   - (b) Read [GaM]
   - (c) Read 'PBD' and [mihai]

II Detail

2. We explored the basics of measure theory in [meas_short], but stopped before reaching integration. Despite this, we now have a good idea on the $\mathbb{N}$-arithmetic and its 'pathologies' that need to be worked around. This gives us a good eye when we follow the FTC through vec-calc up to integration on mfolds, keeping in mind these claim from [pointillistic]:

1. The FTC, under various definitions of integral, holds under various conditions. For Riemann integration, the conditions for FTC holding are complicated.
2. The characteristic function of $\mathbb{Q}$
   - Dirichlet produced this function to show that 'not all functions are integrable', as a counter-example to non-proved assumptions related to the possibility of finding the terms for Fourier series, these terms being given by integrals.
   - Is not Cauchy integrable.
   - Also not Riemann integrable despite Riemann's attempts.
   - Violates the linearity of the integral operator for those definitions.

We suspect simply that conditions for barring non-measurable sets —those that the Riemann integral cannot handle— simply disappear with Lebesgue, no more, no less. In other words, Lebesgue integration is the mathematical achievement (as part of measure theory) of pinning down the pathology that had been plaguing definition of integrals and making it possible to sweep them under the rug. A generalization of even that is the Khinchin integral.

3. Having learning a good deal of measure theory to better understand measures such as the Lebesgue measure and the concept of push-forward and push-back (through the example of random variables in probability), we can have a clearer view of diff-forms in contrast. This view is expounded nicely in:

- https://en.wikipedia.org/wiki/Volume_form
- http://math.stackexchange.com/questions/49641
- http://math.stackexchange.com/questions/1948401

4. A good compact presentation tackling explicitly how differential forms generalize vector calculus can be found in Peter Petersen's 'Manifold Theory'(http://www.math.ucla.edu/~{}petersen/manifolds.pdf). A first reading shows that we are not yet at the level of reading it properly. A trivializer from [VLDU, p.544] reads:

> Now we come to the construction that gives the theory of forms its power, making possible a fundamental theorem of calculus in higher dimensions. We have already discussed integrals for forms. A derivative for forms also exists. This derivative, often called the exterior derivative, generalizes the derivative of ordinary functions. We will first discuss the exterior derivative in general; later we will see that the three differential operators of vector calculus (div, curl, and grad) are embodiments of the exterior derivative.

Pictorially:

$$f^{\uparrow integ}_{\downarrow deriv} \text{ generalizes to } dform^{\uparrow integ}_{\downarrow ext.deriv}$$

The trivializer is followed by an key equivalent formulation of a derivative, suitable to diff-forms. It indicates that for diff-forms, understanding the integration first is useful, since ext-derivative can be built on top of that.

5. The 'abuse of notation' of

$$\frac{\partial}{\partial x_i}$$

as basis vector for tangent spaces is fully treated in the hand-written note [ISM/DN/1]. This also includes full treatment of the

$$\frac{d}{dt}|_{t=0}$$

formulation used in the definition of vector calculus derivatives. Strangely, the key too this is our new understanding of set-theory and fol in the axiomatic and logical sense. A note from [SODE-I, p.101] attributed to Cartan shows that this understanding was not always there, as these were seen as 'purely symbolic', meaning that as 'ink on paper shortcuts of longer notation'.

6. A reason to celebrate. We understand differential forms. From the background of the current open books [ItSM], [MMCM], [SODE-I] and [MT], we finally can read wikipedia's 'Differential form' page without confusion or misunderstanding. We decide to test our understanding by doing exercises in Sadun's short [NoDF].

7. Unfortunately, we got stuck with the assumed proof for 'mixed partials', after which we decided to 'graduate' by restarting Zakon from where we left off with exercises. Unfortunately, we realized that there is no point in doing real analysis if we do not master inequalities. Unfortunately, we found that we do not find 'substitutions', which are at the heart of basic inequality proofs 'intuitive' and hence they would be impossible to master (tag: inequality, search: substituion). This lead us all the way back to formal logic. Right now the plan is to work through 'Mathematical Logic' from Chiswell to get more confidence in the formal sense when doing substitutions. Pathetic overkill? Yes. We took that dive and exited it after 4 months, documented in 'Compute'. We restart with a fresh motivation for 1g.

8. Let us not forget Weinreich's Geometrical Vectors. These are sometimes cryptic and seem to be explained in more detail in Tiee's works [Contravariance, Covariance, Densities and All That], who cites [Geometrical Vectors], [Applied Differential Geometry] and [Gravitation] when talking about this. Another work that talks about this and that we already had but forgotten is Bachman's [A Geometric Approach to Differential Forms] (the book version not the paper version, comes with nice exercises and might be good for study).

9. Note that in [Guide to Geometric Algebra in Practice], the last section is called 'Towards Coordinate-

Free Differential Geometry' and that is interesting, as it promises an elegant approach.

10. In only 20 pages, the introduction to differential forms in [Discrete Calculus, Applied Analysis on Graphs for Computational Science] is a well prepared summary!