Inequalities

I Introduction

1. Months ago, we started with an exercise-study of [@c1], we did not reach very far. We were also discouraged to take a 'Mathematics Olympiad' approach by friends. However, we already stumble at the first proof requiring inequality in real analysis in [@c2] (theorem 1.7). We note that:

1. There is no literature treating how to organize proofs which includes multiple branches and points of failure and trial-error. By imitation, if one does not think about, one thinks that the proof attempt must be such that it resembles the final proof. Of course, this is not so at all, but then, one is left to their own devices created an ad-hock way to manage trial-and-error proof creation. One is supposed to extract, by practice the machinery that goes into this, by looking at many already packaged products (already finished, with no trial-and-error in sight) proofs. This is unreasonable and a hidden obstacle.
2. Inequality proofs carry a lot of resemblance to general proofs in their inverse nature and the combinatorial explosion of possible moves. As such, they form an excellent (and already very difficult) training ground.

II Heuristics Learned

2. In [@c7] we read the warning:

> Indeed, an examination of Olympiad inequalities suggests that probably three quarters of them can be handled with an astute application of the arithmetic-geometric means inequalities.
>
> The student who resorts to calculus to solve inequality problems runs three risks. The first is that, in missing the salient features of an inequality or optimization problem, she complicates the situation. The second is that the solution may not be complete; having found the condition for the vanishing of a derivative, the student may neglect giving an argument to justify the nature of the optimum. The latter danger is particularly pronounced if the student is equipped with the howitzer of Lagrange Multipliers; this is a neat technique, but often the classification of the optimum can be tricky. The third is that she might not develop the valuable ability to "read" algebraic expressions and develop an instinct for performing the most appropriate and effective manipulations.
>
> Another pitfall that occurs in this area is that students forget that the essence of solving a problem is to reduce it to something more elementary and straightforward. There is an unlimited supply of inequalities of ever increasing sophistication and power, and many students lack the maturity to adjust the strengths and generality of the tool to the situation at hand.

III Meta Technique

3. The content of mathematics is what is usually called mathematics. The meta-techniques of mathematics, and they do exist, despite their general insufficiency in the face of general proof. They are almost never taught or discussed. Each meta-technique applies to more than one specific proof and can therefore be useful to know.

4. In inequality proofs in real analysis, it is not mentioned that one is striding the line between the logical (quantifiers) and the arithmetical (algebra). Now the striding appears strange at first, but this is simply due to one forgetting the bridge between the two (which is heavily encoded as soon as one reaches the reals) and that one be working in a formal 'shortcut-language', e.g that of RA. In essence, when one is looking for some expression that proves an inequality, one is searching for a function, but one that is can be explicitly given by its form, as a witness.

5. Unlike with equalities, summarizing expressions by single variables can have detrimental effects as it makes decomposing them to act on their components (e.g by lower or upper bounding with simpler expressions) more opaque.

6. Usually, with inequalities, one is searching for a sequence of expressions, all between two edge interval expressions, with the goal expression having a certain form from which a witness can be extracted. (see note big-green-A). In the extensional sense, one could see this as an impossible-machine, verifying the witness variable by finding a number satisfying it for all numbers of all variables involved. Then, one is cutting through a certain pattern of these numbers with a closed-form expression (there is an infinity of choices) that is then the one used in the proof.

7. The rationals, by their density, make this a much larger exercise for the impossible-machine. But for a proof, one can maybe work with the rationals as fractions, and find a proof using the naturals, which one can use a hint for the case of reals?

8. During the first exercise-attempt at [@c1] we learned at least that one has an implicit wrong assumptions about a certain 'meta-linearity' in the sense of doing substitutions in any order gives the same result, which is a fatal assumption to certain proofs.

9. We still do not know how to manage trial-error proofs.

10. One idea is that going around in circles of reaching dead-ends in simultaneous inequality proofs is that one 'forgets' about some constraints while working out others, only to to be bit in the end. Efficiency is crucial here, and maybe one should 'carry' all constraints all the time at all lines.

11. One idea is to use a flag-like system (like for proofs) to help manage trial-and-error?

12. We start to realise that our main weakness is the following (needs a good name) paradox:

1. This is a a problem I wish to solve
2. I studied, I wish to use what I studied, on some level, a direct level or a meta-level. The proof that I have learned something is that I do not have to think too much about the problem
3. But this is a new problem, so what I retained from studying is probably not enough
4. But the, how to proceed in the face of the infinite-combinatorial-explosion?

5. Trying randomly, which heuristics am I using? meta-heuristics based on what I studied, so I am using what I studied
6. Proving to myself that I 'learned something' is an obstacle, and an invitation not to think 'about' 'this' problem
7. Forcing the implicit 'imitation of mathematical thinking' is an obstacle
8. As we now know only the content of mathematics is mathematics, the rest is reasoning, per example, reasoning about 'the problem at hand'
9. It is not useful to categorise the whole sum of the brain's capability into 'what was learned from studying' and 'what not'
10. It is better to approach the problem 'fully unarmed'. Any meta-level hint will happen by itelf, and freedom from assumptions (e.g transform of equality and inequality methods) is minimised.

From this, it seems that we have to revisit totally basic problem solving books (which we hated, that being an important indicator of 'mental laziness') and solve them fully. Candidate books for this are [@b1], [@b2], [@b3]. Inequality focused ones are [@c2] [Intuition in Mathematics and Physics: A Whiteheadian Approach]. Other notable readings for this 'switch' are [@d1], [@c7]. [@c5] might be a great memory aid since it is a unifier. [@c6] despite being of the 'wrong kind' might still be a surprisingly insight read on 'proving' and 'subtlety'.

First let us see how we fare with (Manfrino, Ortega, and Delgado 2009).

IV Exercises

Exercises with (Manfrino, Ortega, and Delgado 2009)

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