Jad Nohra, Private Note. Revised: January 15, 2016

PL and QL, in Short

I PL

I.1 Within Semantics

1. A set of two boolean values is the beginning of semantics:

$$\{0, 1\}$$

The set is also shortly written as simply

$$2.$$

2. All the four functions $f_i : \{0,1\} \to \{0,1\}$ come next:

	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

$f_i : \{0,1\} \to \{0,1\}$

3. Finally, the sixteen functions $g_i : \{0,1\}^2 \to \{0,1\}$ are considered:

		g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
0	0	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
0	1	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
1	0	0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
1	1	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1

$g_i : \{0,1\}^2 \to \{0,1\}$

I.2 About Semantics

4. Each *formula$_\star$ is identified with a *function$_\star$

$$f : 2^n \to 2.$$

In this sense, we cannot differentiate (except by assigning distinct function names) between two *formulas$_\star$ that have a different *form$_\star$ but the same variables and same value for each variable assignment. Per example, $p \xrightarrow{f} p$ and $p \xrightarrow{g} p \vee p$ are the same function. But this is no different than one is accustomed to for *functions$_\star$, per example, $x \xrightarrow{f} x$ and $x \xrightarrow{g} x + 0$ are *the same function$_\star$.

5. In this sense, the connectives are the binary operators, the members of the set $op = \{f : 2^2 \to 2\}$.

6. The set of all ***tautologies*** is then $\tau = \{f : 2^n \to \{1\}\}$.

7. The set of all ***contradictions*** is $\bot = \{f : 2^n \to \{0\}\}$.

8. The set of all ***tautological consequences*** is then "$\vDash$" $= \{(\Gamma = \{f_i\}, g)$ such that $\{\underbrace{op_{\rightarrow}(f_i, g) \in \tau}_{\text{equiv. } op_{\rightarrow}(f_i,g)=1}, \forall i\}\}$.

9. Two formulas are ***tautologically equivalent*** simply if their functions are equal.

I.3 Within Syntax

10. The full list of syntactic symbols used to define well-formed-formulas, a sublist of which is used in each deductive system is:

$$\underbrace{\overbrace{),(}^{\text{parenthesis}}\ ,\ \overbrace{\bot,\top}^{\text{boolean constants}}\ ,\ \overbrace{\overbrace{\neg}^{\text{unary operator}}\ ,\overbrace{\downarrow,\not\leftarrow,\not\rightarrow,\wedge,\leftrightarrow,\not\leftrightarrow,\uparrow,\rightarrow,\leftarrow,\vee}^{\text{binary operators}}}^{\text{logical connectives}}}_{\text{logical constants}},\underbrace{p,q,r,s,p_1,q_1,r_1,s_1,\ldots}_{\text{variables}}.$$

We also freely use the following aliases for readability:

- [for (
-] for)
- | for $\uparrow$
- $\equiv$ for $\leftrightarrow$
- $=$ for $\leftrightarrow$

11. The structure of the grammatically accepted strings, called well-formed-forumals (WFFs), varies per deductive system. In BNF notation, with the expression

$$\mathrm{V} ::= p \mid q \mid r \mid s \mid p_1 \mid q_1 \mid r_1 \mid s_1 \mid \ \ldots$$

being shared among all, we have the following flavors of grammars:

Name	Constants	WFF (in BNF)
Mendelson (classical)	$\neg, \rightarrow,), ($	WFF :== V | $(\neg$E) | (E $\rightarrow$ E)
Dijkstra-Scholten (equational)	$\bot, \top, \neg, \rightarrow, \vee, \wedge, \leftrightarrow,), ($	$\circ$:== $\rightarrow$ | $\vee$ | $\wedge$ | $\leftrightarrow$ V' :== $\bot$ | $\top$ | V WFF :== V' | $(\neg$E) | (E $\circ$ E)
Gentzen	$\neg, \rightarrow, \vee, \wedge,), ($	$\circ$:== $\rightarrow$ | $\vee$ | $\wedge$ | $\leftrightarrow$ WFF :== V | $(\neg$E) | (E $\circ$ E)
Nicod (minimal)	$\uparrow,), ($	WFF :== V | (E $\uparrow$ E)
Nicod$^{\text{left-polish}}$	$\uparrow$	WFF :== V | $\uparrow$EE

I.4 Within the (Syntactic) Deductive System

12. In the following, we adopt the deductive system expounded in Rogers (who uses Mendelson's system) for illustration. We denote the set of all WFFs in question by $\mathcal{W}$.

13. A subset of all WFFs is chosen as axioms. This subset is infinite and described through axiom schemata using metavariables in $\mathcal{W}$

14. Per example the set of axioms $\mathcal{A}$ can be :

$$\mathcal{A} = \{\ A \to (B \to C)\ \} \cup \{\ [A \to (B \to C)] \to [(A \to B) \to (A \to C)]\ \} \cup \{\ [(\neg B \to \neg A)] \to [(\neg B \to A) \to B]\ \},$$
$$\text{with } A, B, C \in \mathcal{W}.$$

15. Another subset of all WFFs is chosen as the subset of (sound) inference rules $\mathcal{I}$. This subset is again infinite and describe through schemata using metavariables in $\mathcal{W}$. This set of *string transformations$_\star$ is used to create a closure of the axioms. The closure is the set of all 'theorems'.

16. As we try to tersely or formally characterize the *action$_\star$ of the $\mathcal{I}$ we find that we seem to be heading towards some kind of an algebra of strings, especially since the *substitutions$_\star$ happen recursively, and the axiom set is expanded by this, and this expansion has to be taken into account since it provides new axioms. Vaguely, the repeated application of a rule seems like an exponentiation. This leads us once more, but from a better point of view, to the Lindenbaum–Tarski algebra, to algebraic logic in general. We were completely ill-disposed to really understand what pointers like the following were telling us: "Starting with the propositional calculus with κ sentence symbols, form the Lindenbaum algebra (that is, the set of sentences in the propositional calculus modulo tautology). This construction yields a Boolean algebra. It is in fact the free Boolean algebra on κ generators. A truth assignment in propositional calculus is then a Boolean algebra homomorphism from this algebra to the two-element Boolean algebra." or "Given any linearly ordered set L with a least element, the interval algebra is the smallest algebra of subsets of L containing all of the half-open intervals [a, b) such that a is in L and b is either in L or equal to ∞. Interval algebras are useful in the study of Lindenbaum-Tarski algebras; every countable Boolean algebra is isomorphic to an interval algebra.". The talk here is about 'sentences' and not 'values' from the set {0,1}. We finally thought our way into this topic, it is time to investigate it. 'Algebraic Methods in Philosophical Logic' has sections called 'The algebra of strings' and 'the algebra of sentences', which is a good start. Or maybe 'Algebraic Methods of Mathematical Logic', 'Completeness Theory for Propositional Logics', 'Analysis and Synthesis of Logics: How to Cut and Paste Reasoning Systems'.

II QL

III Miscellanea

17. TODO: https://en.wikipedia.org/wiki/Two-element_Boolean_algebra says: "A powerful and nontrivial metatheorem states that any theorem of 2 holds for all Boolean algebras.[1] Conversely, an identity that holds for an arbitrary nontrivial Boolean algebra also holds in 2. Hence all the mathematical content of Boolean algebra is captured by 2. This theorem is useful because any equation in 2 can be verified by a decision procedure. Logicians refer to this fact as "2 is decidable". All known decision procedures require a number of steps that is an exponential function of the number of variables N appearing in the equation to be verified. Whether there exists a decision procedure whose steps are a polynomial function of N falls under the P = NP conjecture."