Solve

I General

1. What if you really have run out of things to try? [@c2]

> A question that students frequently ask is, “When I’m stuck and I have no idea at all what to do next, how can I continue to work on a problem?” I know of only one really good answer. It is advice due to Polya. If you can’t solve a problem, then there is an easier problem you can’t solve: find it.

II Lecturing

2. Can you give a simple example about the importance of quantifier order?

$$(\exists x)(\forall y),\ \ y < x.\text{true in } \mathbb{Q}$$
$$(\forall x)(\exists y),\ \ y < x.\text{false in } \mathbb{Q}$$

3. Give the $\epsilon - \delta$ definition: For a function f, at a point a

$$f \text{ continuous at } a := \ (\forall \epsilon > 0)(\exists \delta > 0) : (\forall x)\rho(x,a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

Note that this can be proved by rewriting δ as a function of ϵ:

$$f \text{ continuous at } a := \ (\exists \delta : \mathbb{R}^+ \to \mathbb{R}^+) : (\forall \epsilon \in \mathbb{R}^+)(\forall x)\rho(x,a) < \delta(\epsilon) \implies \rho(f(x), f(a)) < \epsilon$$

Would this work?

$$f \text{ continuous at } a := \ (\forall x)\left(\exists \epsilon : \begin{cases} \mathbb{R}^+ \to \mathbb{R}^+ \\ x \to 0 \implies \epsilon(x) \to 0 \end{cases}\right) : (\forall \delta_{\mathbb{R}+})\rho(x,a) < \delta \implies \rho(f(x), f(a)) < \epsilon(\delta)$$

$$f \text{ continuous at } a := \ (\forall x)\left(\exists \epsilon^{\mathbb{R}^+ \to \mathbb{R}^+} : \frac{x \to 0}{\epsilon(x) \to 0}\right) : (\forall \delta_{\mathbb{R}+})\frac{\rho(x,a) < \delta}{\rho(f(x), f(a)) < \epsilon(\delta)}$$

Clearly, the standard is the most beautiful. What about this?

$$f \text{ continuous at } a := \ (\forall \epsilon > 0)(\exists \delta > 0) : (\forall x)\rho(x,a) < \delta \implies \rho(f(x), f(a)) < \epsilon$$

III Factoring Polynomials with ?

IV Inequalities with ?

V Sequence Limits with (Ginzburg 1963)

4. $\frac{n^2-n+4}{3n^2+2n+2} \to ?$

- Brute: guess 1/3. Check, by forward calculation and transitive inequalities (conditioned on expressions ultimately being positive as n grows).
- Corner: Using limit identities directly leads to $\inf / \inf$ so we give up. What could it bring to simply factor out both numerator and denominator?!
- Misconception: A ratio can only divide numbers not infinities. This is not true for limits where if the expressions fit, we might be able to find a common 'component' in numerator and denominator and since this is a 'form' not a number (and not zero), the ratio can divide it and turn it to one, hence getting rid of the infinities.
- Method: For simple rational sequences.
- Solution: $\frac{n^2-n+4}{3n^2+2n+2} = \frac{n^2(1-1/n+4/n^2)}{n^2(3+2/n+2/n^2} = \frac{(1-1/n+4/n^2)}{(3+2/n+2/n^2}$, then use limit identities.

5. Discussing accumulation points

- Brute: prove that once some are found, there are no others.
- Corner: split the sequence into a number of subsequences (that exhaust the sequence), then exhibit limits (by limit arithmetic) of those, you are done.

6. Comparing limits of two (recurrence related) sequences

- Take

$$\begin{cases} a_1 > b_1 > 0 \\ a_{n+1} = \dfrac{a_n + b_n}{2} \\ b_{n+1} = \sqrt{a_n b_n} \end{cases}$$

- Assumed you showed that a_n is monotonously decreasing and bounded, b_n mon. increasing and bounded, hence both sequences do have limits.
- You are asked to prove that $(a_n - b_n) \to 0$.
- Now Take $a_n \to \alpha$, $b_n \to \beta$, then $(a_n - b_n) \to (\alpha - \beta)$, so if we show that $\alpha = \beta$ we are done.
- Brute: Try to find a form for $r_n = (a_n - b_n)$ where the recurrence is gotten ridden of, so $r_{n+1} = f(n)$, not $r_{n+1} = f(a_n, b_n)$. We did this in our notes showing that $\frac{r_{n+1}}{r_n} < \frac{1}{2}$ ...
- Corner: To get rid of the recurrence with limits (move a recurrence out of a limit), we use:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

Note that the above is something that I would never write or start with or think of using in a proof as a useful identity! Then

$$\begin{aligned}
\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{a_n + b_n}{2} \quad (= \alpha) \\
&= \frac{\alpha + \beta}{2} \\
\alpha &= \frac{\alpha + \beta}{2} \\
\alpha &= \beta
\end{aligned}$$

VI Function Limits with (Ginzburg 1963)

VII Series Limits with (Ginzburg 1963)

VIII Differentials (also applied) with (Ginzburg 1963)

IX Taylor Series with (Ginzburg 1963)

X Basic Numerical Integrator Convergence with ?

XI Advanced Calculus Prerequisites with (*A ProblemText in Advanced Calculus*)

XII Path to [Intermediate Dynamics for Engineers] [VLDU] -> [Div, Grad, Curl, and All That: An Informal Text on Vector Calculus] -> [A Brief on Tensor Analysis] -> [Intermediate Dynamics for Engineers]

XIII Other Books

7. What about [A ProblemText in Advanced Calculus]?

> To mention just one example there is the theorem concerning change of order of differentiation whose conclusion is $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ This well-known result to this day still receives clumsy, impenetrable, and even incorrect proofs. I make an attempt, here and elsewhere in the text, to lead students to proofs that are conceptually clear, even when they may not be the shortest or most elegant. And throughout I try very hard to show that mathematics is about ideas and not about manipulation of symbols.

XIV Test Books

8. All of the above are absolute pre-requisites for the test books below. If we can solve these, we shall not call ourselves incapable:

1. A Collection of Problems on Complex Analysis
    - Is this support material enough?
    - Complex Function Theory (Sarason)
    - Markushevich's book Short Course of the Theory of Analytic Functions
2. A Collection of Problems on the Equations of Mathematical Physics
3. A Collection of Problems in Mathematical Physics

Bibliography

A ProblemText in Advanced Calculus.

Ginzburg, Abraham. 1963. ***Calculus: Problems and Solutions***. Courier Corporation.