Note. It is not new that for setting foot into number theory with genuine interest, the fact that it is a mind game seemingly disconnected from reality or application. This has been expressed by Jacobi:

> *It is true that Fourrier had the opinion that the principal object of mathematics was public use and the explanation of natural phenomena; but a philosopher like him out to know that the sole object of science is the honor of the human spirit and that under this view a problem of [the theory of] numbers is worth as much as a problem on the system of the world.*[3]

This attitude is the most common attitude for a 'normal human being', including many mathematicians. Fermat was clearly not 'common' and was surprised:

> *He complained on one occasion that hardly anyone propounded or understood arithmetical questions* [1]

Even if Jacobi is right, with my limited time, fully indulging in this topic feels like luxury, and waste of resources. Nevertheless, I must partially tackle it, and this is possible with full motivation due to the following reasons.

1. It is an essential part of understand the solution of the roots of the quintic in Stillwell's book.
2. It's basic understanding rounds off my historical perspective and leaves one less gap in it.
3. It provides an excellent way to relativize the other topics I study. An early specific example is Fermat's 'method of infinite descent'[1.p.275] which, when viewed relatively to analysis and calculus, highlights the fact that dealing with a field is very different than dealing with a ring or a group (congruences), and hence, it is not strange that the techniques have to differ in essence. This explains why, when one is first confronted with the subject (proving things restricted to integers), there is a strong feeling of powerlessness. The feeling comes from the inapplicability of the so far studied techniques which are all 'field' and 'continuous' related to put it roughly.
4. It is a natural departure point for abstract algebra, providing real examples of groups (congruence mod primes), rings, etc.
5. The powerlessness mentioned is only felt by weak (common) minds. Euler and others, who obtained results, show once more the importance of a bias free, custom-to-problem application of mental powers is essential. So number theory is yet another excellent study of this 'meta-method'.

The fact that It is a great mental exercise does not provide the needed motivation since most of my studies are.

Note. On observations and 'empiricism'. Euler points out that

> *'Summary. It is not a little paradox that in the part of mathematics which is usually called pure so much depends on observations, which people think to have to do only with external objects a*
> *ecting our senses. Since numbers in themselves must refer uniquely to pure intellect, it i does not seem worth investigating the value of observations and quasi experiments in their study. However it can be shown with strong reasons that the greatest*

part of the properties of numbers we have come to know have been noted at rst only through observations, long before their truth has been concermed by strict proofs. And there are even many of them which we know but we are not yet able to prove: we have come to their knowledge only through observations. So it is clear that in the science of numbers, which is still greatly incomplete, a lot has to be expected from observations, to nd new properties of numbers for which later a proof has to be worked out . . . Such knowledge depending only on observations, in case a proof is lacking, must be carefully distinguished from truth, and based only on induction. There is no lack of examples in which induction alone has led to errors. Whenever we come to know a property of numbers through observations, based only on induction, beware not to take it as true, but take the opportunity to investigate it carefully and show its truth or falsity, in any case a useful deed'

Let us now turn to a concrete example, showing the kind of observations and the number of examples taken for them.

In this research Euler wants to characterize the numbers which can be written as ***aa+ 2bb****, a and b relatively prime. Similar questions arise in connection with the study of Pythagorean triples. So Euler lists* ***all such numbers up to 500****, and begins to make observations on the table he is considering:*

- that if a prime number is there, that is it if it can be written as sum of a square and twice a square, this presentation is unique;

- that if a prime number n can be so written, the same holds for 2n;

- that if an odd number can be so written, the same is true for its double, and conversely, for an even number and its half;

- that if two numbers can be so written, the same is true for their product;

- that the prime divisors of such a number are of the same form;

and so on, to arrive through this series of remarks to the fact that

- the prime divisors of such a number, if it has any, are of the form ***8n+1 or 8n+3****, and so on, finally arriving at a proposition armed by Fermat, without proof, that*

- prime numbers of the form 8n+1 or 8n+3 can be written as aa+ 2bb and they only.

Having arrived at such a conjecture, Euler first ***checks it up to 1000*** *but then proceeds to give a prove of it, through a series of algebraic arguments, only in one step using the method of infinite descent.*

That's typical of Euler, who did not underrate proofs.'

References

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3. M.T vol. 3
4. Gabriele Lolli, Experimental methods in proofs. (http://www2.dm.unito.it/paginepersonali/lolli/articoli/Esperimenti2.pdf)
5.