Notes on Euclid's Elements

"We have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine."
Proclus

"and further they say that Ptolemy once asked him (Euclid) if there was in geometry a way shorter than that of the elements ; he replied that there was no royal road to geometry."
Proclus

"The more I hasten to cover the ground, I said, the more slowly I travel"
Plato

"What methods have men invented throughout the centuries to deal with geometrical questions?"
J.L. Coolidge

"beyond the grasp of sense perception"
"an explicit rejection of sense perception as a source of knowledge"
Pierre Beaudry
(http://www.wlym.com/archive/pedagogicals/greeks.html)

"Many propositions were doubtless first discovered by drawing all sorts of figures and lines in them,and observing relations of equality, between parts."
T.L. Heath

"With Apollonius the main body of Greek geometry is complete, and we may therefore fairly say that four centuries sufficed to complete it.
But some one will say, how did all this come about? What special aptitude had the Greeks for mathematics ? The answer to this question is that their genius for mathematics was simply one aspect of their genius for philosophy. Their mathematics indeed constituted a large part of their philosophy down to Plato. Both had the same origin."
T.L. Heath

"A still more essential fact is that the Greeks were a race of thinkers. It was not enough for them to know the fact. they wanted to know the why and wherefore (the 8ia tl), and they never rested until they were able to give a rational explanation, or what appeared to them to be such, of every fact or phenomenon."
T.L. Heath

*"but the theory had to be modified from time to time to suit observed new facts ; they had continually in mind the necessity of * saving the phenomena' (to use the stereotyped phrase of Greek astronomy)."*
T.L. Heath

"for travelling over the country there are royal roads and roads for common citizens. but in geometry there is one road for all"

The idea behind the project of going through the all the propositions (but not the proofs) of the Elements came after deciding to finally demystify the concept of angle and trigonometry and put the subject on the right philosophical and intuitive track. The decision came during the tracker number 1 (https://sites.google.com/site/77neuronsprojectperelman/jad/theplan/tracker).
Take the idea of reconstructing Ptolemy's table of chords per example. It makes no sense to undertake this without an understanding of Geometry in Euclid's 'axiomatic' way, and not as the ruins left over from my education. Before starting these notes, the readings indicated in the tracker already demystified much of what I wanted. Triangle similarity per example regained a prominent position in my head after having been a hidden source of worry in my life so far.
Another motivation is the fact that this book was a cornerstone of mathematical education up until the 19th century, at which point abstractions sadly started to become the standard line of teaching. This means that most of the excellent mathematicians I know knew this book thoroughly.

Note(1). I finally (while reading documents in the tracker, along with the constant reading of Whittaker's Theories of the Aether) understood that mathematical definitions are not explanations, nor philosophical justifications. When they try, they always fail (compare Euclid's to Hilbert's definitions of angle). It was only very late that we realized this and decided for axiomatic approaches (Hilbert era) and leave the relation to actually human observation and experience outside of the scope. I now finally agree with this approach, simply because I have seen that the other way has proven impossible and useless (in Physics as well). We define the mathematical objects as exactly what they need to obey the manipulations we intend to do with them, to differentiate them and relate them to the other objects. Not more. The definitions relate then to these mental objects and nothing else to begin with. Does this nudge me towards Platonism when I was always against it? (See Penrose's view on the three worlds in 'Road to Reality'?) I think not, but I need to think about this later.

Note. This quote ties in nicely with other quotes I read in the Tracker about differentiating angle (magnitude) and measure. It is also clearer and more explicit.

> *Treating angles as magnitudes should not be confused with measuring angles. The angles themselves are the magnitudes. The only measurement of angles in the Elements is in terms of right angles (defined in the next definition). Degree measurement and radian measurement were not used until later. In terms of degrees a right angle is 90°, while in terms of radians a right angle is pi/2 radians.*
> *Throughout ancient Greek mathematics, only positive magnitudes were considered. Zero and negative magnitudes were not conceived. For the most part, a lack of zero and negative magnitudes complicates mathematics, but occasionally simplifies it. In any case, the power of a mathematics without zero and negative magnitudes is no less in the sense that any statement made using the language of zero or negative magnitudes can be translated into a statement that doesn't use them, although the translated statement may be longer and less understandable.*
> *Although in modern mathematics, angles can be positive, negative, or zero, and can be greater than a full circle (360° or 2 pi radians), in the Elements angles are always greater than zero and*

> *less than two right angles (180° or pi radians), except perhaps in one interpretation of proposition III.20 where the central angle of a circle could be greater than two right angles.*

What is a magnitude? It is a thing that we can perform arithmetic on, and compare (think ordered field). A measure is a more physical concept, the relation between a standard magnitude (of measure one per example), and another. A magnitude is more abstract, while a measure is more concrete; we measure centimeters, grams, etc... and our standard measure is totally arbitrary, but consistent with the imposed by abstract magnitude. When it does not, the quantity involved probably stops being seen as measurable. I find it difficult to find things that could not be called measurable. Is beauty measurable? If not, is it because firstly, there is no standard magnitude like centimeter? In the sense that beauty is subjective? and secondly because even if it was, and we decided on an algorithm to measure beauty, beauty would still not add up? Per example, overlaying two paintings of beauty measure five can produce a painting of measure one, but also of measure six depending on the paintings? Is there a more subtle example than beauty?

Note(2). Symmetry is of course everywhere and is an unspoken basis for many mathematical concepts. I know of books dedicated to this idea. Unfortunately I am always overloaded. Hopefully I will read one of them in the future. Right angles are extremely important; they form the basis for understanding and deriving all other angles, and function as a unit (or a half or quarter, it does not really matter) for rotations, just as one is a 'unit' for numbers.

> *When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.*

Again, avoiding explanations of what it means for angles to be equal:

> *Nowhere does Euclid explicitly state what it means for angles to be equal, or for that matter, for lines, plane figures, or solids to be equal, although much can be determined by the way he uses equality.*

And I guess also explanations of greater and lesser angles, even though the author does not explicitly confirm my guess like he did in the above quote:

> *Def. 11. An obtuse angle is an angle greater than a right angle.*
> *Def. 12. An acute angle is an angle less than a right angle.*

Note. Two of the more nebulous definitions are *eventually* made rigorous with topology.

> *Def. 13. A boundary is that which is an extremity of anything.*
> *Def. 14. A figure is that which is contained by any boundary or boundaries.*

These are rather nebulous definitions since they are based on the undefined terms "extremity" and "contained by." Euclid deals with two kinds of figures in the Elements: plane figures and solid figures. Plane figures are defined in the upcoming definitions: circles and semicircles in [I.Def.15](I.Def.15) and I.Def.18, rectilinear figures in [I.Def.19](I.Def.19) and particular kinds of rectilinear figures such as triangles and quadrilaterals following that. Specific solid figures such as spheres, cones, pyramids, and various polyhedra are defined in Book XI. Plane figures are not solid figures since they are not contained by any boundaries in space. Thus, implicit to the concept of figure is the ambient plane or space of the figure.

Extremities, boundaries, and topology

Euclid deals with three kinds of extremities, or boundaries. There are the ends of lines (I.Def.3), the edges of surfaces (I.Def.3), and the surfaces of solids (XI.Def.2). A finite line has two points as its boundaries. A circle is defined in I.Def.15 as is a plane figure and has its circumference as its boundary. A sphere is defined in XI.Def.14 as a solid figure and has a spherical surface as its boundary.

The modern subject of topology studies space in a different way than geometry does. The geometric concepts of straightness, distance, and angle are excluded from topology, but the concept of boundary is central to topology. In topology, a sphere remains a sphere even when it's squeezed or stretched.

Not everything has a boundary. For instance, the circumference of a circle has no boundary. Also a spherical surface has no boundary. In topology, a finite region with no boundary is called a cycle. Circles and spherical surfaces are cycles. In general, if something is a boundary, it has no boundary itself. So boundaries are cycles. But not all cycles are boundaries.

Topology uses observation to distinguish various spaces. For instance, on a spherical surface, every circle is the boundary of a region on that surface. But on a toroidal surface (rotate a circle around a line in the plane of the circle that doesn't meet the circle), there are circles (for instance, that circle mentioned parenthetically) that don't bound any region on the surface. Thus, spherical surfaces are topologically different from toroidal surfaces.

Figures and their boundaries

The definition of figure needs to be fleshed out. In order to be a figure, a region must be bounded, that is, held in by a boundary. For instance, an infinite plane is unbounded, so it is not intended to be a figure. Neither is the region between two parallel lines even though that region has the two parallel lines as its extremities.

Other figures may be considered if other ambient spaces are allowed, although Euclid only uses plane and solid figures. For a one-dimensional example, a line segment could be considered to be a figure in an infinite line with its endpoints as its boundary. Also, a hemisphere could be considered to be a figure on the surface of a sphere with the equator as its boundary.

Note. Definition and Existence; I have the feeling that deep inside, I keep forgetting that they are not equivalent, and I hope to fix it by noting it yet another time.

A definition such as this describes what circles are. Definitions do not guarantee the existence of the things they define. The existence of circles follows from a postulate, namely, Post.3.

Usually, existence, when not proved is given in a postulate, such as the brief postulate 1:

To draw a straight line from any point to any point.

Note. Again, symmetry to the rescue, helping avoid an explanation of *equal* for the sides of an equilateral or isosceles triangle.

Def. 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
...
Definition 20 classifies triangles by their symmetries, …

The scalene triangle C has no symmetries, but the isosceles triangle B has a bilateral symmetry. The equilateral triangle A not only has three bilateral symmetries, but also has 120°-rotational symmetries.

By the way, this is how Hilbert introduces equality of segments.

This axiom makes possible the introduction into geometry of the idea of continuity. In order to state this axiom, we must first establish a convention concerning the equality of two segments. For this purpose, we can either base our idea of equality upon the axioms relating to the congruence of segments and define as "equal" the correspondingly congruent segments, or upon the basis of groups I and II, we may determine how, by suitable constructions (see Chap. V, § 24), a segment is to be laid off from a point of a given straight line so that a new, definite segment is obtained "equal" to it. [2]

With congruence defined in the following axioms.

The axioms of this group define the idea of congruence or displacement.
Segments stand in a certain relation to one another which is described by the word "congruent."

IV,I. If A,B are two points on a straight line a, and if A' is a point up on the same or another straight line a0, then, upon a given side of A' on the straight line a0, we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment A'B'. We indicate this relation by writing

$$AB \equiv A'B'.$$

Every segment is congruent to itself; that is, we always have

$$AB \equiv AB.$$

We can state the above axiom briefly by saying that every segment can be laid off upon a given side of a given point of a given straight line in one and and only one way.

IV, 2. If a segment AB is congruent to the segment A'B' and also to the segment A''B'', then the segment A'B' is congruent to the segment A''B''; that is,

$$\text{if } AB \equiv A'B' \text{ and } AB \equiv A''B'', \text{ then } A'B' \equiv A''B''.$$

IV, 3. Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let A'B' and B'C' be two segments of the same or of another

straight line a' having, likewise, no point other than B' in common.
Then, if AB ≡ A'B' and BC ≡ B'C', we have AC ≡ A'C'. [2]

Later he goes on to relate congruence to equality by using Pascal's theorem.

Pascal's theorem, which was demonstrated in the last section, puts us in a position to introduce into geometry a method of calculating with segments, in which all of the rules for calculating with real numbers remain valid without any modification.

Instead of the word "congruent" and the sign ≡, we make use, in the algebra of segments, of the word "equal" and the sign =.
If A,B,C are three points of a straight line and if B lies between A and C, then we say that c = AC is the sum of the two segments a = AB and b = BC. We indicate this by writing
$$c = a + b.$$
The segments a and b are said to be smaller than c, which fact we indicate by writing a<c, b<c. On the other hand, c is said to be larger than a and b, and we indicate this by writing c>a, c>b. [2]

Pascal's theorem itself is the following, and Hilbert takes four pages to prove it.

Theorem 21. (Pascal's theorem.) Given the two sets of points A, B, C and A', B', C' so situated respectively upon two intersecting straight lines that none of them fall at the intersection of these lines. If CB' is parallel to BC' and CA' is also parallel to AC', then BA' is parallel to AB'. [2]

Hilbert's congruence axioms are a bit unsettling at first. They do not seem to provide enough constraints. But we have to look at them as describing a relations, in terms of how we will use it, what we require of it, compared to other relations in the spirit of Note(1). Let us try to understand it. Given a segment and a point, there is one and only one other point that forms a congruent segment. That certainly captures a part of the concept, especially with the use of 'only one'. Notice how subtle the construction is, hinging entirely on 'only one'. There is no doubt 'only one' makes the whole differences in an infinity of mathematical proofs and statements, but this one specifically feels very subtle. As if capturing the essence of an almost invisible ghost. Given this axiom, we still can imagine the first segment to have 'double the length' (although such a concept is not at the point defined at all) of the first segment, and the axiom would not be broken. However, any of the other two axioms would. On the one hand a segment is congruent to itself, the idea of 'double' is thereby expelled from congruence. On the other hand, if a segment a is congruent to b, and b to c, so that b is double a and c is double b, then c is quadruple a, but then it is not congruent because it is not double. All of this is of course quite informal, but it brings out the ingenuity and elegance of the axioms. They demarcate exactly the bounds of 'equality', a concept we consider intuitive and fully comprehend. An illusion. This is not surprising though, for I have already concluded some time ago that intuition really means a prolonged hands-on experience, an understanding of behavior, usually not more.

Note. Should the uniqueness of a straight line between two points a theorem? an axiom?

Although it doesn't explicitly say so, there is a unique line between the two points. Since

Euclid uses this postulate as if it includes the uniqueness as part of it, he really ought to have stated the uniqueness explicitly.

Hilbert mentions the axiom:

Two distinct points A and B always completely determine a straight line a. We write AB = a or BA = a.

then, after mentioning more axioms, he says:

Theorem 1. Two straight lines of a plane have either one point or no point in common; two planes have no point in common or a straight line in common; a plane and a straight line not lying in it have no point or one point in common.

I am not sure how this theorem ‘easily follows’ from the axioms above it.

Note. A nice graph of Euclid’s Elements I and II is included in [4], it can be seen as a measure of the complexity of book I.

Note. Less emphasis will be put on Hilbert’s foundations from now on, because of reasons explained[1].

Note(3). This postulate would have appeared totally strange for me in the past. But reading about Hilbert’s system[1] , I have a better understanding, and so did Euclid already at the time. After explaining what a right angle is (Note(2)), we postulate that we allow ourselves to treat two angles in ‘two different places’ as sharing something in common, a certain magnitude, angle, independently of their position. The angle magnitude is a local feature of any two lines, extracted. The postulate in one shot, excludes all that has nothing to do with the concept of angular magnitude.

Postulate 4: That all right angles equal one another.

Next, Joyce explains how measuring angles follows from the right angle, starting the possibility of ‘measure’.

This postulate forms the basis of angle measurement. The only angle measurement that occurs in the Elements is in terms of right angles.

Notice that the second sentence means, that we could have multiple systems of measure, like radians, and that is not used in the book. This is in fact nice, since I thought to myself at the beginning of these notes that one way to think of angle is as a measure of turns, one turn being treated as unity. Apart from problems with angles larger than half a turn, which, it seems Euclid does not consider, only going up to half a turn (which makes sense), my ‘one turn’ idea is quite close, the subtle difference still being very important.

Note. The famous parallel postulate, which I now ‘get’ thanks to undertaking this project**.** I also

[1] See the section ‘The Philosophy of Hilbert’s axiomatic system’

realize something I did not before, when the lines are parallel, the line falling on the other two can take any direction, the postulate still holds. For when it does not fall with a right angle, on both sides, each two interior angles will 'clearly' have one of them larger than right.

> *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Note. For Proposition I.1, it struck me that I did not notice any of the 'holes' Joyce mentions. But this should not be surprising, the holes are so natural to assume (because of experience), and this is also the reason why they were not seen as holes. Of course, understanding them is very important. What is nice though is the small number of postulates (that Joyce suggests) with which they could be fixed.

> *It is surprising that such a short, clear, and understandable proof can be so full of holes. These are logical gaps where statements are made with insufficient justification. Having the first proof in the Elements this proposition has probably received more criticism over the centuries than any other.*
>
> ***Why does the point C exist****? Near the beginning of the proof, the point C is mentioned where the circles are supposed to intersect, but there is no justification for its existence. The only one of Euclid's postulate that says a point exists the parallel postulate, and that postulate is not relevant here. Indeed, some postulate is needed for that conclusion, such as "****If the sum of the radii of two circles is greater than the line joining their centers, then the two circles intersect****." Such a postulate is also needed in Proposition I.22. There are models of geometry in which the circles do not intersect. Thus, other postulates not mentioned by Euclid are required. In Book III, Euclid takes some care in analyzing the possible ways that circles can meet, but even with more care, there are missing postulates.*
>
> ***Why is ABC a plane figure****? After concluding the three straight lines AC, AB, and BC are equal, what is the justification that they contain a plane figure ABC? Recall that a triangle is a plane figure bounded by contained by three lines. These lines have not been shown to lie in a plane and that the entire figure lies in a plane. It is proposition XI.1 that claims that all parts of a line lie in a plane, and XI.2 that claims that the entire triangle lie in a plane. Logically, they should precede I.1. The reason they don't, of course, is that those propositions belong to solid geometry, and plane geometry is developed first in the Elements, also, no doubt, plane geometry developed first historically.*
>
> ***Why does ABC contain an equilateral triangle****? Proclus relates that early on there were critiques of the proof and describes that of Zeno of Sidon, an Epicurean philosopher of the early first century B.C.E. (not to be confused with Zeno of Elea famous of the paradoxes who lived long before Euclid), and whose criticisms, Proclus says, were refuted in a book by Posidonius. The critique is sound, however, and the refutation faulty.*
> *Zeno of Sidon criticized the proof because it was not shown that the sides do not meet before they reach the vertices. Suppose AC and BC meet at E before they reach C, that is, the straight lines AEC and BEC have a common segment EC. Then they would contain a triangle ABE which is not equilateral, but isosceles.*

> *Zeno recognized that in order to destroy his counterexample it was necessary to assume that straight lines cannot have a common segment. Proclus relates a supposed proof of that statement, the same one found in proposition XI.1, but it is faulty. Proclus and Posidonius quoted properties of lines and circles that were never proven and never explicitly assumed as postulates.*
>
> *The possibilities that haven't been excluded are much more numerous than Zeno's example. The sides could meet numerous times and the region they contain could look like a necklace of bubbles. What needs to be shown (or assumed as a postulate) is that* ***two infinitely extended straight lines can meet in at most one point****.*

Note. Proposition I.2 is very clever indeed[2]. It also motivates this whole project. Indeed, imagine asking a layman to prove it, he would simply say it is obvious, or maybe using a non-collapsing compass. The point is to prove it within some axiomatic system: Euclid's. Only then is it meaningful to attack Ptolemy's chord table (and someday sleep in peace).

Note. In Note(3), we said that 'we allow ourselves to treat two angles in 'two different places' as sharing something in common, a certain magnitude'; Prop I.3 implicitly says the same about length. I can see how this can be used for segment addition, but I am not sure about segment equality, I will stay alert.

Note. Quote Prop 4. part 'The method of superposition', and then say: Very liberal, almost random choice of postulates, does it matter? why not take all of geom. as postulates? because 'all of' is infinite, but is that the only reason? I guess so. 'all of' being all of our simplest mind would see obvious. all of what experience the most basic experience slaps us with in the face. all what we cannot go on without, while trying to prove what the skeptics in us do not accept as a 'nature's slap'. As simple as that.

Note. The 'method of superposition' would have immediately refused by me in the past, and rightfully so. However, I can now see it in two contrasting ways. To explain the first and accepting way, I must first notice that there seems to be in general a quite liberal, almost random choice of what is a postulate. But although this would have seemed shocking to me in the past, it does not anymore. It is in fact, from a viewpoint that is outside of the human experience quite normal that this choice be random. It only becomes a matter of discussion once the choice is guided by human experience. How is the choice made? For the lack of a better word, what is chosen as a postulate is what *nature slaps us with in the face,* taking all that is necessary to prove all what we want to prove, and then some more. From this point of view, the method of superposition by which the proof is done is quite natural and one agree with Euclid. A crazy idea does present itself: why not take every reasonable thing we are thinking of proving as a postulate, does that really make it worse? In applied practice not, but it terms of mathematical beauty and formalism definitely.

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[2] See the section "A collection of brilliant constructions"

The second an rejecting view is obvious. However, we can still try to explain superposition, inspired from Note(3), by the idea that what is asked for is to consider the triangles *in themselves* independently of their situation in the world, independently of all 'global coordinates' differences, only using relatives. A close reading of the *proof* with this point of view in mind is indeed more appealing and *acceptable.*

> *If the triangle ABC is superposed on the triangle DEF, and if the point A is placed on the point D and the straight line AB on DE, then the point B also coincides with E, because AB equals DE. Again, AB coinciding with DE, the straight line AC also coincides with DF, because the angle BAC equals the angle EDF. Hence the point C also coincides with the point F, because AC again equals DF.*
> *But B also coincides with E, hence the base BC coincides with the base EF and equals it. C.N.4*
> *Thus the whole triangle ABC coincides with the whole triangle DEF and equals it. C.N.4*
> *And the remaining angles also coincide with the remaining angles and equal them, the angle ABC equals the angle DEF, and the angle ACB equals the angle DFE.*
> *n comparison to my ideash re is the author's take: on 'superposition':, noting the interesting reason 'using group theory is not appropriate to an elementary exposition of Euclidean geometry' for Dodgson to refuse using group theory in this context:The method of proof used in this proposition is sometimes called "superposition." It apparently is not a method that Euclid prefers since he so rarely uses it, only here in I.4 and in I.8 and III.24, but not in many other propositions in which he could have used it.*

Here is Joyce's comment for comparison, noting the interesting reason '*using group theory is not appropriate to an elementary exposition of Euclidean geometry*' of Dodgson's reservation to using Group theory.

> *It is not entirely clear what is meant by "superposing a triangle on a triangle" means. It has been variously interpreted as actually moving one triangle to cover the other or as simply associating parts of one triangle with parts of the other. For the two triangles illustrated in the figure, you can actually slide one over the other in a continuous motion within the plane. Note, however, that if one triangle is the mirror image of the other, then any continuous motion would require moving one triangle outside of the plane. But the triangles don't have to be same plane to begin with, and they often are not in the same plane when this proposition is invoked in the books on solid geometry.*
>
> *Whatever the intended meaning of superposition may be, there are no postulates to allow any conclusions based on superposition. One possibility is to add postulates based on a group of transformations of space, or if restricted to plane geometry, on a group of transformations of the plane. Charles Dodgson (a.k.a. Lewis Carroll) would have said that using group theory is not appropriate to an elementary exposition of Euclidean geometry. Heath has described a more elementary conservative basis in his commentary on this proposition.*
>
> *Yet another alternative is to simply take this proposition as a postulate, or part of it as a postulate. For instance, Hilbert in his Foundations of Geometry takes as given that under the hypotheses of this proposition that the remaining angles equal the remaining angles. Then,*

Hilbert proves that the base equals the base.

Note. For the proof of Proposition I.5, after noting the usual need for creative addition of auxiliary elements, in this case extending lines and creating points (not needed in Pappus's version), we pause to observe the following major stumbling block to cross The Pons asinorum ('asse's bridge'). Where would the confidence come from to make claims concerning subtraction of angles? Nowhere before was this concept established. Less evidently, the concept of producing a segment larger than a given segment was only hinted at in Joyce's comment in I.3. One could add the triangle's segments to themselves, doubling them. But also note that with an 'arbitrary' segment, would could use shorter segment, but then angle subtraction would have to be turned to addition. It is such problems that make it impossible to find the proofs instead of taking Euclid's and 'agreeing with them'. One would per example stop at angle subtraction, decide that since this was not introduced, one must find another proof, but with the limited arsenal at this stage, this might turn out to be impossible. The geometry taking as a logical progression is misleading (*irreführend),* and is better taken 'as a whole', a system proven? consistent. It becomes clearer why this had to be refounded.
Finally we note that when taking the segments to be equal to the original sides, the proof in effect 'degenerates' into Pappus's.
Because of all what was mentioned, the label 'Pons asinorum' is unjustified.

I thought of a proof that uses a circle and yet introduced arcs, since a circle is defined as containing all equal lines, so one could always pass a circle around ABC with A as center. One could then claim, that the arcs subtended by the two angles in question must be equal since each is the full circle, from which, for each angle, two identical arcs (the arc BC and its opposing arc) are subtracted.

Note. For proposition I.5 discussed above, notice that Pappus's proof is actually a good example of 'formalizing' a proof that uses a feeling the symmetry. The subtlety is nicely pointed out by Joyce.

The difficulty lies in treating one triangle as two, or in making a correspondence between a triangle and itself, but not the correspondence of identity.

Note. Here is an interesting warning about Book X.

The book contains 115 propositions none of which is recognizable at first sight. There is general agreement that the difficulty and the limitations of geometric algebra contributed to the decay of Greek mathematics (Van der Waerden, Science Awakening, p.265.) Author like Archimedes and Apollonius were too difficult to read. However, Van der Waerden disputes that it was a lack of understanding of irrationality which drove the Greek mathematicians into the dead-end street of geometric algebra. Rather it was the discovery of irrationality, e.g. the diagonal of a square is incommensurable with the side of the square, and a strict, logical concept of number which was the root cause. [6]

Additionally, Mader provides[7] a simplified version of an axiomatic system, inspired by Hilbert's. This might be interesting to go though.

Note. While reading SSS congruence in Prop I.8, it occurred to me to find any non-euclidean geometries where congruences are different, I found[3] a beautifully easy example of a non-euclidean geometry where SAS fails. The nice thing about this geometry is that it is very 'accessible' and shows how changing the metric alone can have a dramatic effect.

> *The metric (distance formula) underlying what has become known as taxicab geometry was first proposed as a means of creating a non-Euclidean geometry by Herman Minkowski (1864-1909) early in the 20th century. (Minkowski was an early teacher of Albert Einstein.) The metric was one of a whole family of metrics Minkowski proposed to easily create non-Euclidean geometries.*
>
> *Taxicab geometry only fails one of the axioms or postulates of Euclidean geometry, but it does so in grand style. In Euclidean geometry, if two sides and the included angle of two triangles are congruent, then the triangles are congruent. This is called the SAS (side-angle-side) property for triangles. (Other congruence properties are ASA and AAS.) This assumption does not hold in taxicab geometry, and one example can prove it fails dramatically. In Figure 1, two sides and all three angles of the two triangles are congruent (see the Angles page for more about taxicab angles). But, the third sides are not congruent. So, taxicab geometry not only fails the SAS property, but even an ASASA property does not guarantee congruence of triangles. (Note that this is the case for both traditional and pure taxicab geometry.)*

Also interesting are the geometry's (optional) own angles.

> *As described in the Definitions page, traditional taxicab geometry was defined to use Euclidean angles. But, in its most general form, the measure of an angle is defined as arc length along the unit circle in the geometry. If we use this definition for angular measure, then we quickly see that taxicab geometry will have its own measure for angles that is very different from Euclidean geometry since circles in taxicab geometry are very different than Euclidean circles.*

Also note the alternative names: *The taxicab metric is also known as rectilinear distance, L1 distance or l1 norm (see Lp space), city block distance, Manhattan distance, or Manhattan length, with corresponding variations in the name of the geometry.*

Note. For bisecting the angle in proposition I.9, I imagined an alternative construction: After drawing a circle intersecting the two lines in A and B, using a non-collapsible compass (or Euclid's alternative device), we again draw circles from A and B intersecting the opposite sides at C and D, now the intersection of BD and AC joined with O, bisects the angle at O.

Note. It is somehow fascinating how a definition of right angle by symmetry can be readily used to prove that an angle is right. In prop. I.11, we simply prove that the angles on both sides of a

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[3] http://taxicabgeometry.net

line with another it intersects are equal, hence they are right.

Note. In the proof of Prop. I.14, which I successfully proved mentally before checking, Joyce says that we are for the first time using the postulate that all right angles are equal. I did not notice this, but this is to be expected, since I already (because of my history) think of angles as magnitudes on the most fundamental level, and not as a statement of symmetry by which Euclid defines them. This and many other basic differences in thinking are something I have to cope with and continuously make explicit during this undertaking.

Note. Prop I.15,

> *If two straight lines cut one another, then they make the vertical angles equal to one another.*

and the previous related ones, could have been made more trivial if we have more precisely defined angle arithmetic, in the way Joyce suggests in a comment in Prop I.13

> *So, one way a sum of angles occurs is when the two angles have a common vertex (B in this case) and a common side (BA in this case), and the angles lie on opposite sides of their common side. Thus, addition of angles can be performed by joining adjacent angles.*

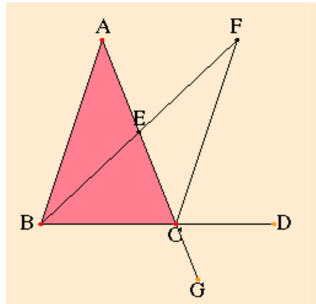
In fact, we could also made things simpler by starting from a straight line having an 'angle' of 'half a turn', and explaining that intersecting with another line, a form of angle arithmetic, cuts that angle into two how sum is 'half the turn'. For Euclid however, '180 degrees' was not an angle as Joyce explains, and so was any sum of angles larger than it, in which case it was treated as a sum of angles.

> *That sum being mentioned is a straight angle, which is not to be considered as an angle according to Euclid. It is a formal sum equal to two right angles. In other propositions formal sums of four right angles occur. These and larger formal sums are not angles themselves, merely sums of angles. Only if an angle sum is less than two right angles can it be identified with a single angle.*

This is again, a basic difference to be coping with.
It might be interesting to try to think of reasons why Euclid thought this way. One could be the perceived additional clarity of his way, another could be subtle difficulties where two lines becomes one if they intersect and have an inclination of a 'straight angle'.
Finally, I notice that there is something that feels like an advantage (I might be wrong) about using the 'straight angle' as a basis: it is the only non-zero angle (ignoring turn multiples) whose 'absolute value' is the same from both directions CW and CCW. Unlike turning a right angle CW and CCW giving different results, turning a straight angle CW and CCW gives the same result.

Note. I am now reading the third chapter in Zakon's basics book, and it talks about the geometry of vector spaces. Of course, the discussion is very different than this one, but all of this, the vectorial approach (zakon) and analytic euclidean geometry, are 'models' of the 'same thing' and now I see that acknowledging this fact explicitly and making it precise has a lot of value. A fun thought is thinking of angle bisection in all three contexts and in trigonometry.

Note(4). The proof of Prop I.16 (*In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.*) is worth stopping at for more than a reason.



1. I mentally tried to prove it, but ended up using the parallel postulate the following way: Since AB and AC intersect at A, they are not parallel. Hence, the sum of angles ABC and ACE is less than two right angles. In addition the sum of ACE and ACD is two right angles, so ABC is less than ACD. The same idea can be used for the angle BAC.

2. Euclid's proof highlights a philosophy of geometric proofs that is worth to keep in mind, because it is is actually quite similar to one of my favorite patterns: invariants and form (change of form as used in variable substitution being one example). In the context of the elements, change of form of the problem can be interpreted as a change of location of certain elements of the problem, preserving some wanted properties. In this case, the whole proof consists of changing the location of the angle BAC to ACF. This allows comparing BAC to ACD by 'proxy', by 'change of location', by 'substitution' if we allow the slight stretch of interpretation. This provides the equivalent of the 'philosophy' or proof by algebraic manipulation. Also, it provides hints to the creative phase needed, during which the creation of structures to help with the proof is done. Making the phase a bit more creative and more methodical with the appeal to 'proof by wishful thinking' (I still have to document my list of 'proof techniques'). In fact, I am sometimes inclined to think that all of book I is 'wishful thinking' approach to prove that the sum of the angles in a triangle in two right angles, all the propositions having been chosen for that ultimate purpose inside an orchestrated axiomatic (in Euclidean standards) system.
Addendum. Upon seeing prop. I.20 (*In any triangle the sum of any two sides is greater than the remaining one.*), I would include it (the triangle inequality) into the list of Euclid's actual goals, alongside the previous one.

3. I wrote to professor Joyce asking if the fact the F turns out on the upper side of BD is not something that needs to be justified (by a postulate).

4. As to Joyce's note about planar elliptic geometry, I remind myself of this quote, and make Coolidge's book a priority that comes after the Whittaker books.

> *What methods have men invented throughout the centuries to deal with geometrical questions? [8]*

When reflected upon, this quote turns out to be quite important for me. It puts down any kind of 'geometry' as 'a' 'method', 'invented'. The goal always being 'understanding' geometrical questions. This allows me to relate Euclid, Hilbert, Riemann, and all others in a much nicer way than I had done before, even though I cannot quite explain how.
In this sense, Euclid's method provides a better (or a base) understanding of some geometrical

questions, leaving many other geometrical questions (see Coolidge's book for an extensive list) unanswered. This makes it 'a' method, not 'the' method. And this connects nicely to our previous remarks about the multiple 'definitions of angle' and the 'The philosophy of Hilbert's axiomatic system' in the appendix.

5. A very important removal of a latent block happens at this stage. I do not think I ever believed that 'synthetic geometry' can provide statements about magnitudes, this might have been the reason for a certain 'fear of never understanding it'. But now I can see that it can, how it can and why it can. with Book I culminating in a statement about the sum of angles in a triangle. Note that synthetic geometry is defined by Coolidge as follows:

> *Synthetic Geometry. This is the geometry of practically all the predecessors of Descartes, the geometry with practically no algebraic substructure, where we consider figures directly and not through equations. It is also the geometry of the groups of motions and collineations. [8]*

In fact, a couple of propositions later, we found a proposition and a related note that, taken together, beautifully show an example of what can be done partially: ordering compared to a more exact relationship. Prop I.19 proves that

> *In any triangle the side opposite the greater angle is greater.*

And the note explains

> *Although some of the geometric underpinnings of trigonometry appear in the Elements, trigonometry itself does not. Trigonometry makes its appearance among later Greek mathematics where the the basic trigonometric function is the chord, which is related to the sine.*
> *Without going into details, the law of sines contains more precise information about the relation between angles and sides of a triangle than this and the last proposition did. The law of sines states that*
>
> *(sin A)/BC = (sin B)/AC = (sin C)/AB.*
>
> *Alternately, the first equation may be read a proportion*
>
> *sin A is to sin B as BC is to AC.*
>
> *In other words, the sine of an angle in a triangle is proportional to the opposite side. (Proportions aren't defined in the Elements until Book V.)*

Also note that the triangle inequality is another beautiful example of what *can* be proved with synthetic geometry.

Note. After taking the first step in the paragraph 'A list of all proof figures in the Elements', and especially after an obvious initial grouping holding the two goal theorems in mind (which indeed

dwarfs Book I, making it feel extremely accessible, short of simple). paralleles immediately started to emerge between this study and the state of the mind with respect to the completed Linear Algebra book. This is not surprising; on the contrary, such a level of organization of a network of local relations was the goal. However, the material in this study appears more immediately accessible, and that is purely because of experience (even if illusory one hard wired by the educational system). This reaffirms our ideas about the idea that one can have the same level of 'intuition' about any abstract topic as with any topic that is closer to the human experience, or to visualization. This brings up the following question: what kinds of adventures should one craft, and either planning or spontaneously engage in during study. Taking as a rather weak example eigenvalues.
What I also noticed is that in fact, most mathematics I know, even the more 'abstract' ones are all amenable to visualization so far. In fact, visualization is a major implicit component in inventing proofs.

Note. It became clear to me that the triangular inequalities, even though more basic, are not as entrenched as 'the sum of angles in 180'. But this again, is merely a consequence of the educational system. The inequalities now take their rightful place in my mind, preceding the more special result of sum of angles. In fact, the parallel postulate now appears in an ever intensifying light. All the inequalities holding without it, and the equality with it. This brings all other geometries closer together since most inequality properties probably hold in all of them. From this angle, it is surprising it took so long to acknowledge them. Surely, thinking of them was accessible to even the greeks (geometry on a sphere). But I speculate that it is the entrenched obviousness of Euclidean geometry which blocked the way, making it seem the 'reference' for all other geometries. And of course, derivations between one and the other are always possible. But the mental block to deciding that geometry is 'a method for tackling geometrical questions' as Coolidge puts it, is both very large, and very fruitful.

Note. Even at this early stage, I feel that a read through the sections of [8] related to Euclid's elements might prove enlightening.

Note. I was able to solve the problem on minimum distance (by Heron of Alexandria), using propositions up to I.20, where the problem is suggested by Joyce. I had to struggle for two days until I found a construction and proof. In the end it was exactly as the book proposes.
I had tackled this problem before (without success) while reading Whittaker, where it appears during the history of the theories of light, in old theories about reflections and refractions. During my attempt, I was planning to use the equality of the two incident angles to either build an isosceles triangle or congruent triangles, but once I found the proof I realized the actual usage that I did not expect. To show that a line made of three points (AEB') is actually straight. For a figure see the list of proofs in the appendix.

Note. Here is a good elucidation of how the Greeks differentiated between the 'definitions' and the 'common notions'.

> *He adopts the distinction already made by Aristotle, namely, that the common notions are*

> *truths applicable to all sciences whereas the postulates apply only to geometry.*
> *As we have noted, Aristotle said that the postulates need not be known to be true but that their truth would be tested by whether the results deduced from them agreed with reality. Proclus even speaks of all of mathematics as hypothetical; that is, it merely deduces what must follow from the assumptions, whether or not the latter are true.*
> *Presumably Euclid accepted Aristotle's views concerning the truth of the postulates. However, in the subsequent history of mathematics, both the postulates and the common notions were accepted as unquestionable truths, at least until the advent of non-Euclidean geometry.[9]*

Note. The parallel postulate was ‘Euclid’s own’.

> *Postulate 5 is Euclid's own; it is a mark of his genius that he recognized its necessity. Many Greeks objected to this postulate because it was not clearly self-evident.*
> *The attempts to prove it form the other axioms and postulates--which, according to Proclus, commenced even in Euclid's own time--all failed.[9]*

Note. The ‘unavoidable’ superposition. (I was right, it still partially appears in Hilbert’s)

> *Common notion 4, which is the basis for proof by superposition, is geometrical in character and should be a postulate. Euclid uses superposition in Book I, Propositions 4 and 8, though apparently he was unhappy about the method; he could have used it to prove Proposition 26 (a.s.a=a.s.a and s.a.a=s.a.a), but instead uses a longer proof. He probably found the method in the works of older geometers and did not know how to avoid it. More axioms were added to Euclid's by Pappus and others who found Euclid's set inadequate.[9]*

Note. The relationship to irrationals comes in Book II, with a purpose.

> *The outstanding material in Book II is the contribution to geometrical algebra. We have already pointed out that the Greeks did not recognize the existence of irrational numbers and so could not handle all lengths, areas, angles, and volumes numerically. In Book II all quantities are represented geometrically, and thereby the problem of assigning numerical values is avoided.*

Note. Coolidge[8] mentions the remarkable recent discovery of the method by which Archimedes arrived at the formulas for volumes, before then proving them by his method of exhaustion. The formulas came from his work on mechanics. He explains them in a long document called ‘THE METHOD OF ARCHIMEDES TREATING OF MECHANICAL PROBLEMS — TO ERATOSTHENES’. It is usually labeled as *the method of equilibrium.*
It turns out, the method is basically integration, of course without the modern concept of a limit. Coolidge also points out that the *method of exhaustion*, explaining how it foreshadows

> *Let us see exactly what we have, mathematically speaking, in this method of exhaustion. I think the title is misleading; we do not actually 'exhaust' the difference between one figure and another; we show that it can be reduced so far as to imply the existence of an inequality which we subsequently discover to be inadmissible. I personally see in it a foreshadowing of the technique of delta and epsilon, which is so fundamental in modern mathematics.[8]*

rigorous limits. We notice how close it is to proofs that a certain number is the least upper bound of a sequence. It actually is one such proof, only in a different language. Yet again we see how the Greeks and in general human thinking, is the same across the ages. And that the subtle refinement along the ages makes-- if we are honest-- an unexpectedly large difference.

> *Archimedes tells us that this is the method by which he discovered the facts but that it seemed to him more satisfactory to give proofs by the more rigorous method of exhaustion. He did not really believe that a sphere was made up of a number of very think cylindrical slices. He was too honest to try to slur over the difficulty by saying there were an infinite number of them. With the modern method of limits the method of equilibrium can be made perfectly rigorous by a process essentially the same as integration. But what conscience to reject the method as he had it as not rigorous, when lesser men, such as Cavalieri and Roberval whom we shall discuss later, believed much less accurate thinking was rigorous. And what a mind to develop both this and the Eudoxus method of exhaustion. In those days there were giants in the land.[8]*

A digestible explanation of the method, with illustrations is provided by Assis[11].

Note. Assis[12] is a whole book dedicated to a possible path to experimentally retrace the path Archimedes might have taken to study the center of gravity and the laws of the lever. The basic law of the lever has always fascinated me. So I found it worthwhile to demystify it, especially after reading that Archimedes proved it. Once more, reading that such a law can be proved triggers a very strange and uncomfortable feeling. The first time I felt it was after reaching Prop. I.20 in Euclid. Here, there was a proof of the triangle inequality. It was before me, I even proved it myself, still I could not believe that one could actually prove it. The hidden problem is the idea that all these observations are too basic to be proven. That assuming some of them and proving the rest is a trick to be able to say that something was proved, since assuming what it is to be proven and proving what is postulated might (sometimes) work as well. There is an ever recurring implicit need for a cause-effect chain. Despite the illusion that we had gotten ridden of it during the linear algebra book, where a straight chain of reasoning had to become more of a network of inter-dependent truths. From my readings about the history of physics, I find that there should be no shame to think like that. In fact, it is necessary to bring to light the fact that this constriction might after all be absurd. We have to simply remember our goal here, elucidated by Heath when he explains why Euclid's work is called 'Elements'. Greek mathematics, as opposed to other antique ones, for whatever philosophical reasons, brought in the necessity to prove, or better, the *need to prove.* This can never be achieved without, by some philosophical method, first extracting some *elements* and making them the base of the system. The justification for the choice of elements cannot be mathematical. And it is indeed frightening to see that as soon as this is done, much of what seems to be unrelated fact or truth, becomes a necessary consequence of some elements. The idea is both obvious and scaringly deep. And the sooner one explicitly grasps it the better.

For the particular case of Archimedes' law of the lever, the proof makes creative use of what he sets as a very convincing postulate that

If magnitudes at certain distances be in equilibrium, (other) mag-nitudes equal to them will also be in equilibrium at the same dis-tances.

By which he means

By "magnitudes equal to other magnitudes," Archimedes wished to say "magnitudes of the same weight." And by "magni-tudes at the same distances," Archimedes understood "magni tudes the centers of gravity of which lie at the same distances from the fulcrum."
Suppose a system of bodies keeps a balance in equilibrium. According to this postulate, Archimedes can replace a certain body A suspended by the beam through its center of gravity located at a horizontal distance d from the vertical plane passing through the fulcrum, with another body B which has the same weight as A, without disturbing equilibrium, provided it is also suspended by the beam at its CG which is at the same horizontal distance d from the vertical plane passing through the fulcrum.

In addition the following proved proposition is used.

If two equal magnitudes have not the same centre of gravity, the centre of gravity of the magnitude composed of the two magnitudes will be the middle point of the straight line joining the centres of gravity of the magnitudes

There is so much intellect at work, separating the elements; one must bow in respect and learn.

Let us emphasize that separating the elements is not a choice of elements that are the cause of the others. There is no causation at work. Logical implication does not amount to causation

In logic, the technical use of the word "implies" means "to be a sufficient circumstance." This is the meaning intended by statisticians when they say causation is not certain. Indeed, p implies q has the technical meaning of logical implication: if p then q symbolized as p → q. That is "if circumstance p is true, then q necessarily follows." In this sense, it is always correct to say "Correlation does not imply causation." (wikipedia)

Let us, for the occasion, add some useful quotes about correlation and causation to crystallize our conclusion.

Edward Tufte, in a criticism of the brevity of "correlation does not imply causation," deprecates the use of "is" to relate correlation and causation (as in "Correlation is not causation"), citing its inaccuracy as incomplete.[1] While it is not the case that correlation is causation, simply stating their nonequivalence omits information about their relationship. Tufte suggests that the shortest true statement that can be made about causality and correlation is one of the following:[4]

"Empirically observed covariation is a necessary but not sufficient condition for causality."
"Correlation is not causation but it sure is a hint."

For any two correlated events A and B, the following relationships are possible:

** A causes B;*
** B causes A;*
** A and B are consequences of a common cause, but do not cause each other;*
** There is no connection between A and B; the correlation is coincidental.*

Less clear-cut correlations are also possible. (wikipedia)

The following has nothing with Euclid, but I record it here as a reminder of an inquiry into 'causality and deductive systems'

The causal sets programme is an approach to quantum gravity. Its founding principle is that spacetime is fundamentally discrete and that the spacetime events are related by a partial order. This partial order has the physical meaning of the causality relations between spacetime events.

The programme is based on a theorem[1] by David Malament that states that if there is a bijective map between two past and future distinguishing spacetimes that preserves their causal structure then the map is a conformal isomorphism. The conformal factor that is left undetermined is related to the volume of regions in the spacetime. This volume factor can be recovered by specifying a volume element for each spacetime point. The volume of a spacetime region could then be found by counting the number of points in that region.

Causal sets was initiated by Rafael Sorkin who continues to be the main proponent of the programme. He has coined the slogan "Order + Number = Geometry" to characterise the above argument. The programme provides a theory in which spacetime is fundamentally discrete while retaining local Lorentz invariance. (wikipedia)

Note. After more reading about the history of Greek Mathematics in Heath[14], I realized that it is not true that my mind seems to have little tolerance for lack of formalism or lack of exactitude. The reality of the matter is that the intolerance is towards lack of causation. In effect, refusal of intellectual magic. If something is stated as true, or given as a solution, I want to see how it was arrived at by human means. Surely, the reason for this is intellectual pride--the only good kind of pride. Again, it boils down to me always asking why. But I am now in a position to refine the question. Why so far has implicitly meant a search for cause. On the human level, this question makes sense: why did the mathematician think this way; what were the circumstances of his knowledge and state of mind. This however, as we have been noting, is the wrong question to ask on the purely mathematical level. There is no real causation in a formal deductive system; correlation does not capture what happens either. 'Why' must hereafter be replaced by 'How does this fit into a (known or unknown) deductive system', for lack of a better formulation.

With all this in mind, the question of the existence of undecidable problems in Euclidean plane geometry is inevitable. They do exist of course. Here is an example which reduces such a problem to a known undecidable problem. This is quite ahead of me and a good motivation.

UNDECIDABILITY IN EUCLIDEAN GEOMETRY

Harvey M. Friedman
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There is a shortage of elementary decision problems known to be recursively unsolvable. Here we give an example from Euclidean geometry that is "almost linear" and potentially meaningful in high school.

1. Recursive Unsolvability.
2. Counting.

1. RECURSIVE UNSOLVABILITY.
We work entirely in the Euclidean plane, R^2. A line is a line in R^2 which extends infinitely in both directions. A rational line is a line with two distinct points whose coordinates are rational.
An integral line system is a finite set of rational lines whose intersection points have integer coordinates.

We say that f is an equivalence between integral line systems S,T if and only if f is a bijection from S onto T such that for any L_1,...,L_k in S, L_1,...,L_k have a common point if and only if f(L_1),...,f(L_k) have a common point.

Two integral line systems are said to be equivalent if and only if there is an equivalence between them.

PROBLEM A. Is a given integral line system equivalent to one containing the perimeter of the unit square?
THEOREM 1. Problem A is not algorithmically solvable.

The proof uses the negative solution to Hilbert's Tenth Problem.
If we instead consider rational line systems only, which are simply finite sets of rational lines, then the solvability of the problem is equivalent to Hilbert's Tenth Problem for the rationals.

2. COUNTING.
PROBLEM B. Count the number of integral line systems with n lines, containing the perimeter of the unit square, up to equivalence.

THEOREM 2. There exists n such that the count cannot be carried out in ZFC.

It is not clear how small n can be taken in Theorem 2.
(http://www.math.osu.edu/~friedman.8/)

Note. With this project, I have virtually reached bottom of how much in the past one can go as far as Mathematics. It seems it is also time to demystify conic sections. As they also originate in that time. Reading Coolidge[8], it seems to me that in order to do that, I must in the first place stop thinking that only point, line and plane are accessible to my intellect. Which seems again yet another old, solidified assumption. A useful state of mind is the idea that in order to stop asking myself the question why mathematicians investigated this or that idea, is simply to

understand the fact that mathematicians live for intellectual challenge. There is no shortage of actual answers in terms of practical reasons for investigations (wish to measure, wish to draw for art or for architecture such that ..., wish to expand or generalize what is known...), but even if there were none, this answer would always be enough.
As for curves that are not point, line and plane, I realized that a way to marry the developments of Geometry throughout the ages, one must first think of what a *curve* is in general (in a very rough and non-formal way). Is it an equation (cartesian)? Is it a graph? Is it a construction? It is all of these, and even a little bit of thought into the various ways of describing it brings the following to light.
In Greek Geometry, one relates features to each other (be it strictly with straightedge and compass or by other devices). As an example, a line can be drawn in relation to two points, a circle in relation to one point. The figure of a theorem or a proof is several such features, all related to each other (We are getting close to the idea of quadratic closure from Stillwell's book). Now we have seen that bringing in extra (hidden but crucial) features is often an important proof device. In fact, the *hidden feature* usually is what constrains the whole construction and provides a cause-effect path to the theorem. Now if we take Greek constructions, and add Algebra (Viete) we get a step closer to Analytic geometry. We know that the orthonormal coordinate system was never really used by Descartes the way it is taught in schools. Nevertheless, if we consider those two perpendicular lines and the origin in which they meet, are they not again, a *hidden feature*? They definitely are, and everything is then related to them. If we think even more about descriptions of curves, we thus consider describing them relative to the coordinate system we just established, using equations. Another way would be to describe them relative to other features, I think this is the method of locus in Greek geometry. I never had to draw a halfway accurate parabola in my life. Imagine a Greek trying to do that, with no cartesian coordinate system (not to forget that the Greeks and the Egyptians did use coordinate systems for astronomy and scaling figures), he probably would have to create a way to construct it relative to other constructible curves; as a locus.
Yet another way to describe a curve would be in relation to nothing but itself (it turns out this is called a mechanical curve). Starting from a point, we describe the relation to the next. Notice that we probably want that relation to be *constant* from point to point, not depending on any coordinates, that is, the position of the point in relation to anything that is not the curve itself. The value of this observation is that it justifies calculus. The relation being *constant*, it must be extractable (calculating the derivative) from the other kinds of descriptions. This also gives us a philosophical hint into problems with singularities, where the *self referential* description probably breaks down. There is a catch though; if all we start from is a point, we have no way of orienting ourselves in space. The *self referential* relation would have to determine an *infinitesimal vector* with the dimensionality of the space in question along which to reach the next point. But this requires a coordinate system except if we start not from a point, but from a sufficiently large number of points as to create a reference, but this defeats the purpose. It therefore makes sense that such descriptions could not have been used before the advent of such a generic but fully determined reference.

Note. This quote from Heath shows that he indeed agrees to the most obvious idea one would have about the origin of conic sections.

The question arises, how did Menaechmus come to think of obtain- ing curves by cutting a cone 1 On this we have no information whatever. Democritus had indeed spoken of a section of a cone parallel and very near to the base, which of course would be a circle, since the cone would certainly be the right circular cone. But it is probable enough that the attention of the Greeks, whose observation nothing escaped, would be attracted to the shape of a section of a cone or a cylinder by a plane obliquely inclined to the axis when it occurred, as it often would, in real life ; the case where the solid was cut right through, which would show an ellipse, would presumably be noticed first, and some attempt would be made to investigate the nature and geometrical measure of the elongation of the figure in relation to the circular sections of the same solid ; these would in the first instance be most easily ascertained when the solid was a right cylinder ; it would then be a natural question to investigate whether the curve arrived at by cutting the cone had the same property as that obtained by cutting the cylinder. As we have seen, the observation that an ellipse can be obtained from a cylinder as well as a cone is actually made by Euclid in his Phaeno- mena: 'if, says Euclid, ' a cone or a cylinder be cut by a plane not parallel to the base, the resulting section is a section of an acute-angled cone which is similar to a Ovpeos (shield).' After this would doubtless follow the question what sort of curves they are which are produced if we cut a cone by a plane which does not cut through the cone completely, but is either parallel or not parallel to a generator of the cone, whether these curves have the same property with the ellipse and with one another, and, if not, what exactly are their fundamental properties respec- tively.[13]

As to the origin of the study of conic sections, Doubling the cube (A very obvious geometric question to seek an answer for) was the initial drive. Doubling the cube can be reduced to the problem of find two mean proportionals and this is what Menaechmus was aiming to construct with his sections.

We have seen that Menaechmus solved the problem of the two mean proportionals (and therefore the duplication of the cube) by means of conic sections, and that he is credited with the discovery of the three curves ; for the epigram of Eratosthenes speaks of ' the triads of Menaechmus ', whereas of course only two conies, the parabola and the rectangular hyperbola, actually appear in Menaechmus's solutions. The question arises, how did Menaechmus come to think of obtain- ing curves by cutting a cone. [13]

The link between the duplication of the cube and the two mean proportionals was provided by Hippocrates of Chios[13].

A quick summary of the duplication problem section [14] can be found at http://www.cs.mcgill.ca/~cs507/projects/1998/zafiroff/ .

Geometers took up the question and made no progress for a long time, until Hippocrates of Chios showed that the problem was reducible to that of finding two mean proportionals in continued proportion between two given straight lines. Again, after a time, the Delians were told by the oracle that, if they would get rid of a certain plague, they should construct an altar of double the size of the existing one. They consulted therefore Plato who replied that the oracle meant, not that god wanted an altar of double the size, but that he intended, in setting them the task, to shame the Greecs for their neglect of mathematics and their *contempt for geometry. According to Plutarch, Plato would have referred the Delians to Eudoxus and Helicon of Cyzicus for a solution. The problem was thereafter studied in the Academy, and constructions of the problem of the two means were proposed in the 4th century B.C. by Archytas, Eudoxus, Menaechmus and even (though erroneously) to Plato himself. After Hippocrates' reduction of the problem, it was always attacked in the form of finding two mean proportional between two given straight line segments.*

In seeking the solutions of problems, geometers developed a special technique, which they called "analysis." They assumed the problem to have been solved and then, by investigating the properties of this solution, worked back to find an equivalent problem that could be solved on the basis of the givens. To obtain the formally correct solution of the original problem, then, geometers reversed the procedure: first the data were used to solve the equivalent problem derived in the analysis, and from the solution obtained the original problem was then solved. In contrast to analysis, this reversed procedure is called "synthesis." Menaechmus' cube duplication is an example of analysis: he assumed the mean proportionals x and y and then discovered them to be equivalent to the result of intersecting the three curves whose construction he could take as known. (The synthesis consists of introducing the curves, finding their intersection, and showing that this solves the problem.) It is clear that geometers of the 4th century BC were well acquainted with this method, but Euclid provides only syntheses, never analyses, of the problems solved in the Elements. Certainly in the cases of the more complicated constructions, however, there can be little doubt that some form of analysis preceded the theses presented in the Elements.

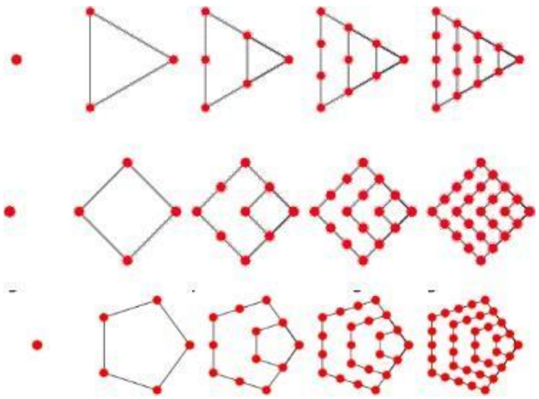
Hippocrate showed that the problem could be reduced to that of finding two mean proportionals: if for a given line segment of length a it is necessary to find x such that x3 = 2a3, line segments of lengths x and y respectively may be sought such that a:x = x:y = y:2a; for then a3/x3 = (a/x)3 = (a/x)(x/y)(y/2a) = a/2a = 1/2.

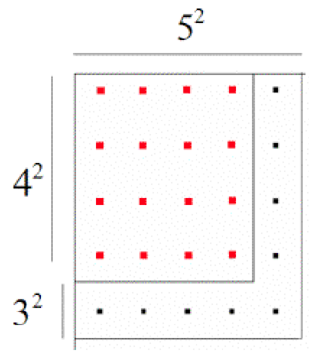
The extremely creative solution of Archytas, involving the intersection of three surfaces, I can only explain to myself in one way. Since the algebraic equivalent is actually not complicated, it seems to me that the Greek geometers thought geometrically, the way we think algebraically. We can never really apprehend their thinking, or imagine to solve the problems the way they did and keep our sanity, except if we first really spend time cultivating this kind of *geometro-algebraic* thinking (where the algebra is implicit). This might have benefits, since it makes our image-based thinking closer to usefulness by enriching it with more exact relations to proportion and algebra. We are again noticing that synthetic geometry is far from being impotent, or confined to seemingly qualitative results. Studying the solutions of this problem would be an excellent way of achieving this.

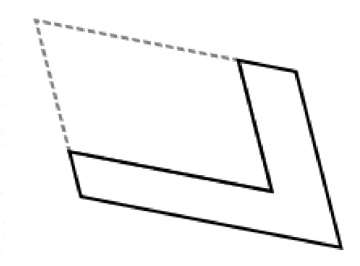
The above solution is a remarkable achievement when it is remembered that Archytas flourished in the first half of the fourth century b.c, at which time Greek geometry vas still in its infancy. It is quite easy, however, for us to represent the solution analytically.[14]

Note. Polygonal numbers (Greek figurative numbers), the gnomon.
Notice that the gnomon is actually a method of, of multiples uses.Notice the similarity between the 3,4,5 gnomon and square root extraction (of course geometrical, but equivalent to the

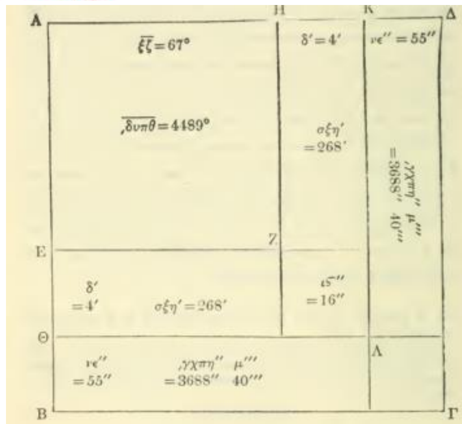
obvious algebraic equivalent).







A gnomon as in Euclid book II



Note. I am reading through Heath[14] in quite some detail, I feel the foundational benefits in terms of philosophy, sources, basic mathematical thinking, and comparative improvements are immense. By ‘in quite some detail’ I mean that I am not bypassing the details of what is being said and only going for the main ideas. So far this has been very useful, pleasant, and indeed, making me go faster and not slower.

> *The more I hasten to cover the ground, I said, the more slowly I travel. (Plato) [14]*

Note. Skimming through Coolidge[8]. Geometry through all epochs is infinitely interesting, and I have no clue about it.

Appendix

Definitions of geometry

L.J. Coolidge

It seems to me that the whole subject of geometry, if we make the very doubtful assumption that any one can give an exact definition of what geometry is, falls into four main divisions:

1) Synthetic Geometry. This is the geometry of practically all the predecessors of Descartes, the geometry with practically no algebraic substructure, where we consider figures directly and not through equations. It is also the geometry of the groups of motions and collineations.
2) Algebraic geometry. Here we study the properties of geometrical figures through algebraic relations connecting their coordinates or their equations. The group is at first the linear one, afterwards that of one-to-one algebraic transformations, the birational group.
3) Differential geometry. Here we study the properties of figures discovered by the processes of differential calculus. The group is that of one-to-one analytic transformations.
4) Topology. The group here is usually the homeomorphic one of one-to-one continuous transformations.

It is evident that these distinctions are not, in practice, as sharp as they here appear. [8]

Definitions of angle

Euclid

A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

He adds in the next definition:

And when the lines containing the angle are straight, the angle is called rectilinear.

Hilbert

Let alpha be any arbitrary plane and h, k any two distinct half-rays lying in alpha and emanating from the point O so as to form a part of two different straight lines. We call the system formed by these two half-rays h, k an angle and represent it by the symbol <(h, k) or <(k, h). From axioms II, 1-5, it follows readily that the half-rays h and k, taken together with the point O, divide the remaining points of the plane a into two regions having the following property: If A is a point of one region and B a point of the other, then every broken line joining A and B either passes through O or has a point in common with one of the half-rays h, k. If, however, A, A0 both lie within the same region, then it is always possible to join these two points by a broken line which neither passes through O nor has a point in common with either of the half-rays h, k. One of these two regions is distinguished from the other in that the segment joining any two points of this region lies entirely within the region. The region so *characterised is called the interior of the angle <(h,k). To distinguish the other region from this, we call it the exterior of the angle <(h,k). The half rays h and k are called the sides of the angle, and the point O is called the vertex of the angle. [2]*

This is similar to one of the ideas I had a long time ago trying to think what angles really are. Of course, such an extremely elementary definition requires a lot of development to reach a practical level, something I would have never been able to do.

Angle as a shape (not invariant under location), the inclination in the faces of a pyramid.

*[Thales] is said to have been the first to have known and to have enunciated [the theorem] that the angles at the base of any isosceles triangle are equal, though in the more archaic manner he described the equal angles as similar.^*
^ This theorem is Eucl. i. 5, the famous pons asinorum. Heath notes {H.G.M. i. 131) : "It has been suggested that the use of the word ' similar ' to describe the equal angles of an isosceles triangle indicates that Thales did not yet conceive of an angle as a magnitude, but as a figure having a certain shape, a view which would agree closely vith the idea of the Egyptian se-qet, ' that which makes the nature,' in the sense of determining a similar or the same inclination in the faces of pyramids."[14]

Invalid proofs in Euclid's Elements

Proposition XI.2, proving that two lines form a plane.

Books XI through XIII, there is sometimes mentioned a "plane of reference," and proposition XI.2 claims that two intersecting lines determine a plane as does any triangle (but its proof fails completely).

The philosophy of Hilbert's axiomatic system

Hilbert took a view of meaning which was diametrically opposed to that held by Frege, an opposition expressed most clearly in their differing attitudes toward axioms. According to Frege, axioms should be self-evident truths. This is should be clear what they are about and that they are true of this. This attitude is illustrated quite clearly in Frege's letter to Hilbert on geometry, where he objects to the latter's claim that the axioms of order 'define' the concept 'between'. According to Frege, this is to get the matter the wrong way round:
Every definition contains a sign (an expression, a word), that beforehand has no meaning [Bedeutung], and to which meaning is first given by the definition. After this has been done, one can make a self-evident proposition out of the definition, which can be used like an axiom.

But the definition itself cannot be an axiom, since these are supposed to express truths about their subjects, and for Frege this presupposes that the term standing for the subjects already *have a meaning. He goes on:*

Thus, axioms and theorems can never attempt to fix the meaning [Bedeutung] of a sign or word which appears in them; rather, this must already stand fixed.

Frege's attitude is, in effect, already visible in his earlier work on the natural number concept. We arrive at significant basic propositions governing number by analyzing the number concept finely enough; but then in addition we have to be able to say what the numbers are, what numerical terms refer to in some absolute sense. Moreover, the tendency to make reference the linchpin of meaning is clear in subsequent general work on the philosophy of language. These sense of sentences is explained on the lines of the sense of singular terms, and this is explained ultimately in terms of the reference of the terms, for the sense of a term is 'the mode of representation' of the thing the term stands for. Of course, as Frege well knew, this theory does not sit well with the recognition that we can understand sentences which have no truth-value, say sentences containing terms with no reference. But Frege does say that a properly regimented and precise language will contain no such terms, and it was clear that his treatment of mathematics, given the invention of the Begriffsschrift, was intended to furnish just such a language. More importantly here, such an account is precisely what he assumes in the case of geometry. Frege say:

The other sentences (axioms, basic laws, theorems) ought to contain no word of sign whose sense and reference [Sinn und Bedeutung], or whose contribution to the expression for the thought, does not already stand fully fixed, so that there is no doubt about the sense of the proposition, of the thought expressed therein.

Thus knowing what object is denoted, and knowing this somehow independently of the axioms, seems crucial for understanding any expression in which the denoting term appears. As I have pointed out above, Frege had an account of this in the case of numerical terms by the way of a theory of 'logical objects' (extensions of concepts), but it is not at all clear what the answer would be in other cases or that any kind of general answer is possible.

For geometry, it looks as if the Frege position amounts to the view that it must be possible to explain what terms like 'point', 'line', 'plane' stand for in some direct way, say by appeal to intuition, indepdendently of the special axioms governing them and their consequences. This would make the ontological theory for the terms something which is separate from the mathematical of physical theory. It was this extra component which Hilbert took to be problematic, unrealizable and unnecessary. It was problematic even in the case of natural number where Frege claimed to have an answer. At first sight, it looks as if the answer Frege gives is connected in a significant way to arithmetical theory, for it appears to a large extent to depend on the basic analysis of the way natural number terms apply to concepts. It thus looks as if Frege was not asking for an extra component, but really just filling in something basic about the notion of number, that is, selecting objects to stand for the equivalence classes determined by the equinumerosity relation over the extensions of concepts. However, there are two remarks one should make about this. First, it seems clear from Dedekind's work and, say, from Boolos's recent elementary reconstruction of the theory using second-order logic, that this extra, ontological, component is not necessary, even if partially 'theory driven'. Second, even though it produces some more theory to answer the further ontological question, the Frege account does not succeed in answering the question in a suasive way. For,

even after being told that 'numbers are logical objects of such-and-such kind', one might still ask 'How do you know that you have really got hold of 'purely logical' objects?'. In other words, as Dummett suggests, Frege puts forward no epistemological theory which explains how we can know logical truths: he held that that the …..
Compare what Hilbert says in his reply to the Frege letter already cited:

Here lies the cardinal point of the misunderstanding. I which to assume nothing as known. I see in my explanation in section 1 the definition of the concepts 'point', 'line', 'plane', if one takes all axioms of group I-V as characteristic. If one undertakes to search for other definitions of 'point', perhaps through paraphrases like extensionless etc., then I would have to oppose such an enterprise in the most decisive way. One would then be searching for something which one could never find because it is simply not there. Everything would get lost, become confused and vague,and degenerate into a game of hide-and-seek.
Faced with some notions which require explanation, one gives the best and most complete theory one can. What else can or should be done over and above this? If we have something to say about intractable notions like 'infinity' in mathematics or 'force' in physics, we surely want to include any important insights which derive from this in our theories. Why would there be any such insights which are not incorporated directly in the theory, and is this not just what Frege's view, when pushed, is demanding?
Thus Hilbert's position, implicit in his [1894] and [1899], and much more explicit in the correspondence with Frege, was quite different. As we have seen, Hilbert claimed, in effect, that we do not need any kind of direct or extra insight into the referents of the basic terms of mathematics in order to be able to understand mathematical theories: the axioms governing the concepts involved contain all the insight we get or need. According to Hilbert, the axioms of geometry or real numbers are hence more like collective 'implicit definitions' of the terms that figure in them than 'basic truths' in Frege's sense. In his correspondence with Frege, Hilbert clearly intends this notion of definition in a strong sense, namely that the axiom system completely fixes the meaning of the concepts involved. For example, he writes to Frege:
On the contrary, to wish to give a definition of a point in three lines is an impossibility, since rather the whole body of axioms gives the complete definition. Every axiom contributes something, and every new axiom alters the concept. ''Point' in Euclidean, non-Euclidean, Archimedean, non-Archimedean geometry is in each case something different.
This is interesting, not least because it gives us the beginnings of an explanation as to why for Hilbert the completeness of an axiom system, in the sense of negation completeness, is such an important and desirable property. For if the axioms determine the meaning of a concept completely, then if they cannot settle everything we would like to know about the concept, i.e. answer every question posable in the appropriate language, then nothing can. For changing the axioms, say by adding new principles, will change the concept, and this will thus strictly speaking render the new system incapable of answering any old questions. But more importantly in this context, this shows that the notion of fixed reference to objects, such a crucial part of Frege's theory of meaning, is left out of Hilbert's account. According to Hilbert, there does not have to be any absoluteness or fixity of reference in order for mathematical theories to be properly understood. Rather, the axiom system alone determines the meaning; and this must mean too that any structure which happens to satisfy the axioms specified will be a system of the right sort, say a geometry or a number system, whether this structure is something realised by material nature or some other part of mathematics. Hilbert puts it this way:

> *You say that my concepts, e.g. 'point', 'between', are not uniquely fixed; e.g., on p. 20 'between' is differently interpreted, and a point is there a number pair. --Agreed; but it is nevertheless self-evident that every theory is only a framework or schema of concepts, together with their necessary relations to one another, while the basic elements can be thought of in an arbitrary way. If among my points I imagine some system of things, like the system: love, law, chimney-sweep, …, and then assume only my collection of axioms as relations between these things, then all my theorems hold also for these things, e.g., Pythagoras's theorem. In other words; every theory can always be applied to infinitely many systems of basic elements. On needs only to apply a reversible one-one [eindeutige] transformation, and to establish that the axioms are correspondingly the same for the transformed things. … All the statements of electricity theory hold naturally of any other system of things which one puts in place of the concepts 'magnetism', 'electricity', … if the axioms in question are satisfied. The feature I have mentioned can never be a defect of a theory (rather only a powerful advantage), and is in any case unavoidable.[3]*

TODO: continue transcribing here, there are many important ideas ahead (http://books.google.de/books?id=KhUIY9sZVjkC&pg=PA196&lpg=PA196&dq=hilbert%27s+definition+of+congruence&source=bl&ots=TBTj2tY-T1&sig=t8hQVQ2evRk873iWr1Ce_z_JHEg&hl=en&sa=X&ei=QJB5Ud-vBInJtAaH4YHAAg&redir_esc=y#v=onepage&q=hilbert%27s%20definition%20of%20congruence&f=false)

With this quote, I hope to settle the constant thinking about the essence of definitions and axioms. It shows Hilbert's philosophy, and goes a long way to demystify how and why he chose his axioms and definitions to be the way they are, and not any different, out of many possibilities. In a way, Euclid *could* have written his much closer to Hilbert's. There may be a technical mastery of a different level, but I think the main difference is a philosophical one. This quotes also encourages to have an even more discerning eye when reading any axiom or definition, or any supposed intuitive explanation. On the other hand, one must also conclude from it the *futility* of this kind of higher mathematics, in terms of effectiveness and usefulness over *ordinary, Euclidean* mathematics. The judge must be the results and the fruitfulness of the approach, and, for me, the best way to cope is still considering mathematics to be the integral of its own development (by humans). An angle is not exactly and only what Euclid defined it to be, nor what Hilbert defined. Angle is a personal human experience, including all definitions and history that one might have read about it; simply everything that relates to it, and equally, *not* anything that has been experienced to be different from it (again, including all what was read). Nihilism aside, angle is what one will do with it next.

Additionally to the above quote, we must always stay aware of the following:

> *Hilbert's Foundations of Geometry (1899) is often – and rightly – seen as a landmark in the development of the so called axiomatic method. One of the main innovations in Hilbert's Foundations is that meta theoretic issues such as the questions of consistency and independence of axioms are for the firsttime systematically treated in a way that has since then become standard. Ina famous letter to Frege he writes:I was forced to construct my system of axioms by the following necessity:I wanted to provide an opportunity for understanding those geometric propositions which I consider to be the most important*

products of geo-metric investigations – that the axiom of parallels is not a consequence of the remaining axioms; similarily the Archimedean axiom; etc. I wantedto answer the question whether it is possible to prove the propositionthat two equal rectangles having the same base line also have equal sides.In fact, I wanted to create the possibility of understanding and answer-ing such questions as why the sum of the angles of a triangle is two rightangles and how this fact is related to the axioms of parallels. ([28] p. 10)Hilbert's perspective in this passage is of decidedly metatheoretical character in the sense that questions about what can and what can not be proved fromsome given set of axioms, are posed from a point of view external to geometrical investigations properly so called. The purpose of Hilbert's axiomatization of geometry in his Festschrift therefore was not just to provide a basis for geometryfrom which every geometrical truth could be proved, but rather it was from the very beginning aiming at metatheoretical properties of Euclidean geometry and subtheories of Euclidean geometry. In a similar spirit he writes in a small paper titled
¨ Uber den Satz von der Gleichheit der Basiswinkel im gleichschenklingen Dreieck (dating from the same period):Unter der axiomatischen Erforschung einer mathematischen Wahrheitverstehe ich eine Untersuchung, welche nicht dahin ziehlt, im Zusammen-hange mit jener Wahrheit neue oder allgemeinere S¨atze zu entdecken,sondern die vielmehr die Stellung jenes Satzes innerhalb des Systemsder bekannten Wahrheiten und ihren logischen Zusammmenhang in derWeise klarzulegen sucht, dass sich sicher angeben l¨asst, welche Voraus-setzungen zur Begr¨undung jener Wahrheit notwendig und hinreichendsind. ([22] p. 119)
At the heart of Hilbert's methodology lies his consequent — what has since be-come to be known as model theoretic approach to axiom systems. Geometric axioms are no longer seen as true propositions which are immediate from our spatial intuition, but are now considered to be like conditions in being satisfied by some interpretations and not by others. [5]

This is of great help. We must yet again understand something by its history, this time, Hilbert's foundations. As the quote explains, the goal really was to be able to analyze the resulting system meta-theoretically. As we already have seen, Hilbert's axioms do not shed any extra intuitive light on obscure definitions or postulates of Euclid, and that is not its goal. Of course, its goal is of a much higher sophistication, one that we are not yet concerned about since they are way above us, even though extremely interesting in the long run. Despite all this, the same paper provides a very intuitive explanation of the general idea of how this was done. I will divide the quotes in two, the beginning of the second part containing a useful short explanation.

Hilbert's model theoretic approach to axiomatic theories is the key for his in-dependence proofs, for it is clear that, in order to prove the unprovability of some proposition φ from other propositions S , one cannot "go through" all possible proofs and check that none of them is actually a proof of φ using only propositions from the set S.
Hence, in the absence of proof theoretical meth-ods properly so called, the only way to prove an axiom φ to be independent from a group of axioms S is to produce a countermodel, i.e. an interpretation,in which every axiom in the group S is true, but φ is false. The conceptual presupposition for such a strategy seems to be that the domain of the theory S (i.e. the set of objects the theory is supposed to talk about) and it's basic concepts are free to be reinterpreted. Roughly this means that, although one may have an intended interpretation in mind when setting up the axioms, this intended interpretation is no longer privileged among other interpretations that might satisfy the axioms. Intuitions about an intended

interpretation of some given discourse have only heuristic value in setting up the axioms and drawing attention to possibly fruitful applications, but they are irrelevant as far as the logical content
of the thereby established axiomatic theory is concerned.To get a feeling for Hilbert's method, let us look at the following example from Hilbert's Festschrift: Here, Hilbert wants to show that the axiom of complete-ness is independent from the rest of the axioms for Euclidean geometry. The axiom of completeness is a maximal axiom and (roughly) states that the systemof things the theory talks about (i.e. points, lines) cannot be extended while still satisfying the remaining axioms. (If added to the remaining axioms, the axiom of completeness
therefore guarantees that the resulting system captures the "usual" structure of Euclidean space up to isomorphism.) To show that the axiom of completeness is independent along the lines indicated above, Hilbert reinterprets the primitive concepts of Euclidean geometry as follows:

1. a is a point " is reinterpreted by "a is a pair (x,y) of algebraic num-bers, i.e. numbers x and y that can be constructed by repeated applica-tions of the four basic arithmetic operations together with the operation $|\surd(1+\xi^2)|$ *from the number 1"*
2. "bis a line" is reinterpreted by "bis the ratio (u:v:w) of three such algebraic numbers"
3. "the point a is incident with the line b" is reinterpreted by "ais a pair(x,y), such that...,b is the ratio (u:v:w)... and ux+vy+z= 0"

It can be shown now that under this reinterpretation every axiom of Euclidean geometry, except the axiom of completeness, is satisfied. (Just add some non-algebraic number and make sure to "close off" under the four basic arithmetic operations and the operation $|\surd(1+\xi^2)|$
Before we can see more clearly what Frege's troubles with this kind of independence proofs was, let us first clearly state what is involved conceptually in independence proofs `a la Hilbert. [5]

And

As far as Hilbert is concerned, an independence proof of an axiom φ from a set of axioms S is supposed to show that neither φ nor ¬ φ are provable from S, i.e. that no sequence of logical inferences might bring us from S to φ or its negation. This is done by doing two things: First by providing an interpretation I with respect to which all axioms in S as well as φ come out true, and second by providing another interpretation J with respect to which all axioms in S are true while φ is false. Here an interpretation is specified by a set of objects D forming the domain of objects of the theory, together with a specification of the denotations of the primitive concepts over the given domain D . Now, appealing to the informal concept of semantic consequence, φ is a semantical consequence of S if and only if φ is true in every interpretation in which all the sentences in S are true. Therefore, I witnesses that ¬ φ is not a semantic consequence from S and J that φ is no semantic consequence of S either. Summing up, on this straightforward model-theoretic reading of Hilbert's independence proofs, he is relying on

1. the informal notion of provability
2. the informal notion of an interpretation , and the notion of truth with respect to an interpretation 3. the relation of semantic consequence , defined in terms of all possible interpretations and
4. the soundness of the intuitive notion of proof with respect to the informal semantic

consequence relation

Hilbert, however, never adresses any of these presuppositions explicitly (at leastduring his dispute with Frege and some time after) but instead takes them to bepart of ordinary mathematics. [5]

Frege's opinion on Hilbert's proof:

It is necessary to dwell on the distinction genuine/pseudo axioms, because even careful writers on this issue sometimes mix things up.
Keeping in mindthis distinction is important not just for a faithful interpretation of Frege, but necessary in evaluating (and appreciating) his positive account on the problem of how to prove the independence of genuine axioms.
*Before looking closer at Frege's own proposal how independence proofs should be handled, it might be useful to look at some points surrounding this issue.We have seen that one of the main targets of Frege's criticism of Hilbert is the fact that according to Frege Hilbert's independence arguments lack the kind of stringency which he expects from an argument to count as a proof:**As it stands, we remain completely in the dark as to what he [Hilbert]really believes he has proved and which logical and extralogical laws and expedients he needs for this**. ([28] pp. 111-112)In particular Hilbert's loose talk about interpretations, the appeal to the infor-mal semantical consequence relation and the assumption of informal soundness(with respect to informal provability) seem to be what Frege has in mind here.It is obvious, at least as far as the 1906-paper is concerned, that **Frege does not find anything particularly wrong with the mathematical content of Hilbert's arguments but rather with their presentation and the language***
***in which they are stated (specifically the "interpretation-talk")**. This is not only indicated by his remark that "the question may still be raised whether, taking Hilbert's result as a starting point, we might not arrive at a proof of independence of the real axioms" ([28] p. 103), but it is obvious from the idea lying behind his own proposal. That is, Frege is not so much worried about the truth of the independence results established by Hilbert, but with the means*
of establish-ing them. In particular, Frege, at least in the 1906-paper, does not show anyqualifications that proofs of independence are somehow impossible.
What is really at stake here is the form that independence arguments concerning real axioms should take if they should count as genuine proofs. [5]

A collection of selected quotes (From the Elements)

Other uses of a straightedge can be imagined. For instance, it might be marked at two points on it, then fit into a diagram so that the two points fall on two lines, perhaps curved. This operation is an example of "neusis" or "verging" where lines are adjusted to fit the diagram. For instance, Archimedes, who lived in the century after Euclid, used neusis in several constructions in his work On Spirals. In the Book of Lemmas, attributed by Thabit ibn-Qurra to Archimedes, neusis is used to trisect an angle.

A collection of brilliant constructions

All of Euclid's constructions are clever, many are brilliant; still we only choose a few for the sake of sanity.

This construction happens despite the 'collapsing compass'. One could very vaguely say that the 'idea' is that the length is 'transferred bit by bit'.

I.2 To place a straight line equal to a given straight line with one end at a given point.

I.3 directly builds on this to create the construction that can transfer distances.

To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Pre-history, works and people that influenced the Elements

According to Proclus (410-485 C.E.) in his Commentary on Book I, Hippocrates of Chios (fl. ca. 430 B.C.E.) was the first to write an Elements. Leon and Theudius also wrote versions before Euclid (fl. ca. 295 B.C.E.). These other Elements have all been lost since Euclid's replaced them. It is conceivable that in some of these earlier versions the construction in proposition I.2 was not known, so this proposition would instead have been a postulate (a stronger version of I.Post.3). Once the construction in I.2 was discovered, the current weaker I.Post.3 would do. Then again, I.2 might go back to the time of Hippocrates.

This, and almost all historical information about the elements comes from T.L. Heath's various books, that often rely on Heiberg. In this case, the above is a summary of the following.

Proclus, On Evclid i., ed. Friedlein 64. 16-70. 18
Since it behoves us to examine the beginnings both of the arts and of the sciences with reference to the present cycle [of the universe], we say that according to most accounts geometry was first discovered among the Egyptians,^ taking its origin from the measure- ment of areas. For they found it necessary by reason of the rising of the Nile, which wiped out everybody's proper boundaries. Nor is there any- thing surprising in that the discovery both of this and of the other sciences should have its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the per- fect. Thus the transition from perception to reason- ing and from reasoning to understanding is natural. Just as exact knowledge of numbers received its origin among the Phoenicians by reason of trade and contracts, even so geometry was discovered among the Egyptians for the aforesaid reason. Thales ^ was the first to go to Egypt and bring back to Greece this study ; he himself discovered many propositions, and disclosed the underlying principles of many others to his successors, in some cases his method being more general, in others more empirical. After him Ameristus,^ the brother of the poet Stesichorus, is mentioned as having touched the study of geometry, and Hippias of Elis " spoke of him as having acquired a reputation for geometry. After these Pythagoras ^ transformed this study into the form of a hberal education, examining its principles from the beginning and tracking down the theorems immaterially and intellectually ; he it 38 who dis- covered the theory of proportionals ^ and the construc- tion

of the cosmic figures. After him Anaxagoras of Clazomenae ^ touched many questions affecting geometry, and so did Oenopides of Chios,^ being a little younger than Anaxagoras, both of Avhom Plato mentioned in the Rivals ^ as having acquired a reputa- tion for mathematics, AfterthemHippocratesofChios,^ whodiscovered the quadrature of the lune, and Theodorus of Cyrene^ becamedistinguishedingeometry. ForHippocrates is the first of those mentioned as having compiled elemeyits.^ Plato,^ ^vho came after them, made the other branches of mathematics as \vell as geometry- take a very great step forward by his zeal for them and it is obvious how he filled his writings >vith mathematical arguments and everywhere stirred up admiration for mathematics in those Avho took up philosophy. At this time also Hved Leodamas of Thasos ^ and Archytas of Taras ^ and Theaetetus of Athens,^ by whom the theorems were increased and an advance was made towards a more scientific grouping.*

Younger than Leodamas were Neoclides and his pupil Leon, vho added many things to those known before them, so that Leon was able to make a collec- tion of the elements in which he was more careful in respect both of the number and of the utility of the things proved ; he also discovered diorismi, showing when the problem investigated can be solved and whennot.^ EudoxusofCnidos,aUttleyoungerthan Leon and an associate of Plato's school, was the first to increase the number of the so-called general theorems ; to the three proportions he added another three, and increased the number of theorems about the section, which had their origin with Plato, apply- ing the method of analysis to them.^ Amyclas of Heraclea,^ one of the friends of Plato, and Men- aechmus," a pupil of Eudoxus Avho had associated with Plato, and his brother Dinostratus ^ made the whole of geometry still more perfect. Theudius ^ of Magnesia seemed to excel both in mathematics and in the rest of philosophy ; for he made an admirable arrangement of elements and made many particular propositions more general. Again, Athenaeus of Cyzicus, who lived about those times, became famous in other branches of mathematics but mostly in geometry. They spent their time together in the Academy, conducting their investigations in common. Hermotimus^ ofColophonadvancedfarthertheinvestigations begun by Eudoxus and Theaetetus ; he discovered many propositions in the elevients and compiledsomeportionofthetheoryofloci. Philippus of \iedma,'^ a disciple of Plato and by him diverted to mathematics, not only made his investigations accord- ing to Plato's directions but set himself to do such things as he thought would fit in with the philosophy of Plato.*

Those who have compiled histories carry the development of this science up to this point. Not much younger than these is Euclid, who put together the elements, arranging in order many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefutable demonstration the things which had been only loosely proved by his predeces- sors. This man lived in the time of the first Ptolemy ; for Archimedes, who came immediately after the first Ptolemy, makes mention of Euclid ; and further they say that Ptolemy once asked him if there was in geometry a way shorter than that of the elements ; he replied that there was no royal road to geometry.^ He is therefore younger than the pupils of Plato, but olderthanEratosthenesandArchimedes. Forthese men were contemporaries, as Eratosthenes' some- where says. In his aim he was a Platonist, being in sympathy with this philosophy, whence it comes that he made the end of the whole Elements the construc- tion of the so-called Platonic figures.^ There are many other mathematical writings by this man, Monderful in their accuracy and replete with scientific investigations. Such are the Optics and Catoptrics, and the Eleme?its of Music, and again the book Ofi Divisions.'^ He deserves admiration pre-eminently in the compilation of his Elements of Geometry on account of the order and of the selection both of the theorems and of the problems made with a view to the elements. For he* *included not everything which he could have said, but only such things as he could set dovn as elements. And he used all the various forms of syllogisms, some getting their plausibility from the first principles," some setting out from demonstrative proofs, all being irrefutable and accurate and in harmony with science. In addition to these he used all the dialectical methods, the divisional in the dis- covery of figures, the definitive in the existential arguments, the demonstrative in the passages from first principles to the things sought, and the analytic in the converse process from the things sought to the first principles. And the various species of conver- sions,^ both of the simpler (propositions) and of the more complex, are in this treatise accurately set forth and skilfully investigated, what wholes can be con- verted with wholes, what wholes M'ith parts and conversely, and what as parts with parts. Again, mention must be made of the continuity of the proofs, the disposition and arrangement of the things which precede and those which follow, and the power with which he treats each detail. Have you, adding or subtracting accidentally, fallen avay unawares from science, carried into the opposite error and into ignorance ? Since many things seem to conform with the truth and to follow from scientific principles, but lead away from the principles into error and deceive the more superficial, he has handed down methods for the clear-sighted understanding of these matters also, and with these methods in our posses- sion we can train beginners in the discovery of para- logisms and avoid being misled. The treatise in which he gave this machinery to us he entitled [the book] of Pseiiclaria,^ enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of the error vith practical illustration. This book is therefore purgative and disciplinary, while the Elements contains an irrefutable and complete guide to the actual scientific investigation of geometrical matters. [14]*

Let us, confining ourselves to the main subject of pure geometry by way of example, anticipate so far as to mark certain definite stages in its development, with the intervals separating them. In Thales's time (about 600 B. c.) we find
the first glimmerings of a theory of geometry, in the theorems that a circle is bisected by any diameter, that an isosceles triangle has the angles opposite to the equal sides equal, and (if Thales really discovered this) that the angle in a semicircle is a right angle. Rather more than half a century later
Pythagoras was taking the first steps towards the theory of numbers and continuing the work of making geometry a theoretical science ; he it was who first made geometry one of the subjects of a liberal education. The Pythagoreans, before the next century was out (i. e. before, say, 450 B. c), had practi- cally completed the subject-matter of Books I -II, IV, VI (and perhaps III) of Euclid's Elements, including all the essentials
*of the * geometrical algebra ' which remained fundamental in Greek geometry ; the only drawback was that their theory of proportion was not applicable to incommensurable but only to commensurable magnitudes, so that it proved inadequate as soon as the incommensurable came to be discovered. In the same fifth century the difficult problems of doubling the cube and trisecting any angle, which are beyond the geometry of the straight line and circle, were not only mooted but solved theoretically, the former problem having been first reduced to that of finding two mean proportionals in continued*
proportion (Hippocrates of Chios) and then solved by a remarkable construction in three dimensions (Archytas), while the latter was solved by means of the curve of Hippias of Elis known as the quadratrix ; the problem of squaring the circle was also attempted, and Hippocrates, as a contribution to it, discovered and squared three out of the five lunes which

can be squared by means of the straight line and circle. In the fourth century Eudoxus discovered the great theory of proportion expounded in Euclid, Book V, and laid down the principles of the method of exhaustion for measuring areas and volumes ; the conic sections and their fundamental properties were discovered by Menaechmus; the theory of irrationals (probably discovered, so far as V2 is concerned, by the early Pythagoreans) was generalized by Theaetetus ; and the geometry of the sphere was worked out in systematic trea- tises. About the end of the century Euclid wrote his Elements in thirteen Books. The next century, the third, is that of Archimedes, who may be said to have anticipated the integral calculus, since, by performing what are practi- cally integrations, he found the area of a parabolic segment and of a spiral, the surface and volume of a sphere and a segment of a sphere, the volume of any segment of the solids of revolution of the second degree, the centres of gravity of a semicircle, a parabolic segment, any segment of a paraboloid of revolution," and any segment of a sphere or spheroid. Apollonius of Perga, the ' great geometer ', about 200 B. c, completed the theory of geometrical conies, with specialized investigations of normals as maxima and minima leading quite easily to the determination of the circle of curvature at any point of a conic and of the equation of the evolute of the conic, which with us is part of analytical conies. With Apollonius the main body of Greek geometry is complete, and we may therefore fairly say that four centuries sufficed to complete it.

But some one will say, how did all this come about? What special aptitude had the Greeks for mathematics ? The answer to this question is that their genius for mathematics was simply one aspect of their genius for philosophy. Their mathematics indeed constituted a large part of their philo- sophy down to Plato. Both had the same origin. [14]

The amazing travels of Pythagoras, and the important fact that even if the stories are not true, they ‘reflect throughout the Greek spirit and outlook’.

But the same avidity for learning is best of all illustrated by the similar tradition with regard to Pythagoras's travels. Iamblichus,in his account of the life of Pythagoras,3 says that Thales, admiring his remarkable ability, communi-

cated to him all that he knew, but. pleading his own age and failing strength, advised him for his better instruction to go and study with the Egyptian priests. Pythagoras, visiting Sidon on the way, both because it was his birthplace and because he properly thought that the passage to Egypt would be easier by that route, consorted there with the descendants of Mochus, the natural philosopher and prophet, and with the other Phoenician hierophants, and was initiated into all the rites practised in Biblus, Tyre, and in many parts of Syria, a regimen to which he submitted, not out of religious enthusiasm, 'as you might think' (a>? av ns ocrrXco? i>7roAa/?oi), but much more through love and desire for philosophic inquiry, and in order to secure that he should not overlook any fragment of knowledge worth acquiring that might lie hidden in the mysteries or ceremonies of divine worship ; then, understanding that what he found in Phoenicia was in some sort an offshoot or descendant of the wisdom of the priests of Egypt, he concluded that he should acquire learning more pure and more sublime by going to the fountain-head in Egypt itself.

1 There ', continues the story, ' he studied with the priests and prophets and instructed himself on every possible topic, neglecting no item of the instruction favoured by the best judges, no individual man among those who were famous for their knowledge, no rite practised in the country wherever it was, and leaving no place unexplored where he thought he could discover something more. . . . And so he spent 22 years in the shrines throughout Egypt, pursuing astronomy and geometry and, of set purpose and not by fits and starts or

casually, entering into all the rites of divine worship, until he was taken captive by Cambyses's force and carried off to Babylon, where again he consorted with the Magi, a willing pupil of willing masters. By them he was fully instructed in their solemn rites and religious worship, and in their midst he attained to the highest eminence in arithmetic, music, and the other branches of learning. After twelve years more thus spent he returned to Samos, being then about 56 years old.'
Whether these stories are true in their details or not is a matter of no consequence. They represent the traditional and universal view of the Greeks themselves regarding the beginnings of their philosophy, and they reflect throughout the Greek spirit and outlook. [14]

All the following are by Thales (before Euclid).

A circle is bisected by the diameter.

They say that Thales was the first to demonstrate " that the circle is bisected by the diameter, the cause of the bisection being the unimpeded passage of the straight hne through the centre.

The word "demonstrate" (^)must not be taken too literally. Even Euclid did not demonstrate this property of the circle, but stated it as the 17th definition of his first book. Thalcs probably was the first to point out this property. Cantor {Gesch. d. Math, i^., pp. 109, 140) and Heath (II.O.M. i. 131) sug- gest that his attention may have been drawn to it by figures of circles divided into equal sectors by a number of diameters. Such figures are found on Egyptian monuments and vessels brought by Asiatic tributary kings in the time of the eighteenth dynasty. [16]

Angles as a magnitude (invariant under location), angles as a shape.

*[Thales] is said to have been the first to have known and to have enunciated [the theorem] that the angles at the base of any isosceles triangle are equal, though in the more archaic manner he described the equal angles as similar.^*

^ This theorem is Eucl. i. 5, the famous pons asinorum. Heath notes {H.G.M. i. 131) : "It has been suggested that the use of the word ' similar ' to describe the equal angles of an isosceles triangle indicates that Thales did not yet conceive of an angle as a magnitude, but as a figure having a certain shape, a view which would agree closely vith the idea of the Egyptian se-qet, ' that which makes the nature,' in the sense of determining a similar or the same inclination in the faces of pyramids." [16]

The vertical and opposite angles are equal.
This theorem, that when two straight lines cut one another the vertical and opposite angles are equal, was first discovered, as Eudemus says, by Thales, though the scientific demonstration was improved by the ^^iter of the Elements/'a [16]

Equality of triangles.

Eudemus in his History of Geometry attributes this theorem to Thales. For he says that the *method by which Thales showed how to find the distance of ships at sea necessarily involves this method.^*

** The method by which Thales used the theorem referred to, Eucl. i. 26, to find the distance of a ship from the shore, has given rise to many conjectures. The most attractive is that of Heath {The Thirteen Elements of Euclid's Elements^ i., p. 305, H.G.M. i. 133). He supposes that the observer had a rough instrument made of a straight stick and a cross- piece fastened to it so as to be capable of turning about the fastening in such a manner so that it could form any angle with the stick and would remain where it vras put. The observer, standing on the top of a tower or some other eminence on the shore, would fix the stick in the upright position and direct the cross-piece towards the ship. Leaving the cross-piece at this angle, he would turn the stick round, keeping it vertical, until the cross-piece pointed to some object on the land, which would be noted. The distance between the foot of the tower and this object would, by Eucl. i. 26, be equal to the distance of the ship. Apparently this method is found in many practical geometries during the first century of printing. [16]*

A more colorful version (not by Heath), with some excellent philosophical quotes:

"The heavens are filled with gods." -Thales of Miletus

Let me start with a provocative question: what do the height of the Egyptian pyramids and the distance to the Moon have in common? How did Thales of Miletus (600 B.C.) discover a principle by means of which one is able to determine the height of the Egyptian pyramids (without the use of the Pythagorean theorem), as well as the distance to the Moon, given a knowledge of its size? If you can answer those questions, then you have made what Lyndon LaRouche calls a "discovery of principle." That is, what you have discovered is not some THING, some physical reality, which can be acquired through sense perception, and be measured with a ruler, in some way. No! You have discovered an idea, a Platonic idea, in the form of a principle of measure, a principle of congruence, or concordance, between ideas and actions on the universe, which can only be developed by the human mind. Recall in this connection, what Plato said about the teaching of astronomy, and the ordering principle of the heavens in the {Republic}, VII, 529, d,e, and {Laws}, X, 899 b.

What is required for the discovery of such causality is a higher cardinality; and that cardinality is no less than addressing the subjective power of the human mind to become the causal agency, which commands the ordering of the Universal Blazonry of the Heavens. This is the specific task that Thales, Kepler, and Gauss, especially, all three challenged themselves with, each in his own way: to develop a truly scientific notion of universal congruence beyond the grasp of sense perception. This also means, that each of us must muster the courage to do exactly the opposite that the evil Francis Bacon proposed when he said that man should not "give out a dream of his own imagination for a pattern of the world."

But this principle, in one form or another, implies the resolution of Plato's ontological paradox of the One and the Many. This means that whatever the nature of the discovery, it must imply three conditions of axiomatic change: 1) it must involve a multiplicity, a Many of some sort, 2) it must imply an axiomatic break with a previous set of assumptions, 3) it must be determined by a One that bounds the process of discovery from the outside. To put it in a nutshell, you need a Many, a Discontinuity of perception, and a One. From the advanced standpoint of

Lyndon LaRouche, those are the three necessary conditions that must make up the Platonic Idea of any discovery of principle.

Thales of Miletus lived 600 years before Christ and was recognized as one of the Seven Sages of Greece during Solon's archonship of Athens. It was Thales who forecast the solar eclipse of May 28, 585 B.C. which put an end to the protracted war between the Lydians and the Medes, and ultimately settled a lasting peace between them. Not only did Thales know when eclipses would occur, but he also knew that the cycles of the Sun, the Earth, and the Moon had to concur in the plane of the ecliptic in order to cause such eclipses. This knowledge was outstanding, since no one at that time understood what eclipses were all about. Furthermore, Thales is reputed to have created the first almanac, giving the solstices, the equinoxes, the phases of the moon, and a long-range calendar with eclipse and weather prediction. He invented a means of steering the course of ships on the sea, and and a way to determine their distances from shore by sighting them from a tower.

Thales developed a very elegant and simple theorem which is so elementary that its simple beauty and generality leave blind -- mentally blind that is -- those who don't investigate its purpose. His idea resembles a changing geometrical figure, which is never the same: sometimes it is a triangle, sometimes a line, sometimes overlapping shadows and, more often than not, it also takes the form of an array of spheres circumscribed by cones. Like the principle of water, which he took as the basis of his philosophy, it is forever changing; however, in all cases and everywhere, it remains the same. As his follower Heraclitus said: "You never bathe in the same river twice," and yet, it is always the same river. Underlying all of the changes, there remains a mental congruence which is not apprehensible by sense perception, a principle of similarity and proportion, which enabled him to establish certain astronomical measurements involving the determinations of lunar and solar eclipses. The Thales Theorem can be stated as follows:

ANY LINE PARALLEL TO ONE OF THE SIDES OF A TRIANGLE WILL DIVIDE THE TWO OTHER SIDES IN PROPORTIONAL SEGMENTS, AND WILL DETERMINE ANOTHER TRIANGLE SIMILAR TO THE FIRST.

…
In the three previous discoveries, the reader will have noticed that the singularities are of the same type, because they all contain an explicit discovery of a One with reference to a Many, in the form of self-similarity and proportionality, and they explicitly contain a discontinuity, an explicit rejection of sense perception as a source of knowledge. These are the conditions that Lyndon LaRouche has established to "test," so to speak, the validity of a hypothesis; the resolution of the ontological paradox of the One and the Many is the litmus test that bears upon all of the crucial discoveries in history.

(http://www.wlym.com/archive/pedagogicals/greeks.html) (Pierre Beaudry)

A collection of notes on Greek mathematics, trivialized by algebra.

The Pythagoreans looked upon a square number n^2 as a race course formed of successive numbers from 1 up to n (the turning point) and back again through (n-1), (n-2) and so on to 1 (as the goal), in this way: 1+2+3+…+(n-1)+n+(n-1)+…+1. As an example we have
1+2+3+…+10+9+…+2+1=10^2 ,
10+20+30+…+100+…+10=10^3, , 100+200+300+…+1000+…+100=10^4
and so on.
It was in virtue of these relations that the Pyhagoreans spoke of 10 as the unit of the second course, 100 as the unit of the third course and so on. [14]

Greekomania

All the following information about why the greeks were able to do what they did, is important at least because very clear similarities between the reasons can be made at an individual level, why one man and not the other. This is a journey into the pure essence of what it is to be a philosopher, a mathematician, a scientist; an elucidation of the requirements and necessary or favorable circumstances.

Among the different Greek stocks the Ionians who settled
on the coast of Asia Minor were the most favourably situated in respect both of natural gifts and of environment for initiat- ing philosophy and theoretical science. When the colonizing spirit first arises in a nation and fresh fields for activity and development are sought, it is naturally the younger, more enterprising and more courageous spirits who volunteer to leave their homes and try their fortune in new countries similarly, on the intellectual side, the colonists will be at least the equals of those who stay at home, and, being the least wedded to traditional and antiquated ideas, they will be the most capable of striking out new lines. So it was with the Greeks who founded settlements in Asia Minor. The geographical position of these settlements, connected with the mother country by intervening islands, forming stepping-
stones as it were from the one to the other, kept them in continual touch with the mother country ; and at the same time their geographical horizon was enormously extended by the development of commerce over the whole of the Mediter- ranean. The most adventurous seafarers among the Greeks of Asia Minor, the Phocaeans, plied their trade successfully as far as the Pillars of Hercules, after they had explored the Adriatic sea, the west coast of Italy, and the coasts of the Ligurians and Iberians. They are said to have founded Massalia, the most important Greek colony in the western countries, as early as 600 B.C. Cyrene, on the Libyan coast, was founded in the last third of the seventh century. The Milesians had, soon after 800 b. a, made settlements on the east coast of the Black Sea (Sinope was founded in 785) ; the first Greek settlements in Sicily were made from Euboea and Corinth soon after the middle of the eighth century (Syracuse 734). The ancient acquaintance of the Greeks with the south coast of Asia Minor and with Cyprus, and the establishment of close relations with Egypt, in which the Milesians had a large share, belongs to the time of the reign of Psammetichus I (664-610 B.C.), and many Greeks had settled in that country. The free communications thus

existing with the whole of the known world enabled complete information to be collected with regard to the different conditions, customs and beliefs prevailing in the various countries and races ; and, in parti- cular, the Ionian Greeks had the inestimable advantage of being in contact, directly and indirectly, with two ancient civilizations, the Babylonian and the Egyptian. [14]

It is in fact true, as Gomperz says that the first steps on the road of scientific inquiry were, so far as we know from history, never accomplished except where the existence of an organized caste of priests and scholars secured the necessary industry, with the equally indispensable continuity of tradi- tion. But in those very places the first steps were generally the last also, because the scientific doctrines so attained tend, through their identification with religious prescriptions, to become only too easily, like the latter, mere lifeless dogmas. It was a fortunate chance for the unhindered spiritual de- velopment of the Greek people that, while their predecessors in civilization had an organized priesthood, the Greeks never had. To begin with, they could exercise with perfect freedom their power of unerring eclecticism in the assimilation of every kind of lore. ' It remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East, and threw all the fantastic rubbish on one side.' 1 For the same reason, while using the earlier work of Egyptians and Babylonians as a basis, the Greek genius could take an independent upward course free from every kind of restraint and venture on a flight which was destined to carry it to the highest achievements.
The Greeks then, with their unclouded clearness of mind' and their freedom of thought, untrammelled by any ' Bible ' or its equivalent, were alone capable of creating the sciences as they did create them, i.e. as living things based on sound first principles and capable of indefinite development. [14]

It was a great boast, but a true one, which the author of the Epinomis made when he said, ' Let us take it as an axiom that, whatever the Greeks take from the barbarians, they bring it to fuller perfection '. 2 He has been speaking of the extent to which the Greeks had been able to explain the relative motions and speeds of the sun, moon and planets, while admitting that there was still much progress to be made before absolute certainty could be achieved.

It seems unreasonable that the Greeks, for some reason, were more exact observers than other civilizations. Therefore, the right explanation is that the Greeks, by virtue of their philosophy (or by other means) learned the extremely high value of exact observation. The advantage of this point of view is that such an attitude can be controlled (and learned) on the individual level.

Religion and Science, the Greek view.

'Let no Greek ever be afraid that we ought not at any time to studv things divine because we are mortal. We ought to maintain the very contrary view, namely, that God cannot possibly be without intelligence or be ignorant of human nature : rather he knows that, when he teaches them, men will follow him and learn what they are taught. And he is of course perfectly aware that he does teach us, and that we learn, the very subject we are now discussing, number and counting; if he failed to know this, he would show the greatest want of intelligence ; the God we speak of would in fact not know himself, if he took it amiss that a man capable of learning should learn, and if he did not rejoice unreservedly with one who

became good by divine influence.'

they pointed out that, whereas such things as rhetoric ad poetry and the whole popular … can be understood even by one who has not learnt them, the subjects by the special name of … (mathematics) cannot be known by any one who has not first gone through a course of instruction in them; they concluded that it was for this reason that these studies were called … (mathematics).

Applied Mathematics

In applied mathematics Aristotle recognizes optics and mechanics in addition to astronomy and harmonics. He calls optics, harmonics, and astronomy the more physical (branches) of mathematics,1 and observes that these subjects and mechanics depend for the proofs of their propositions upon the pure mathematical subjects, optics on geometry, mechanics on geometry or stereometry, and harmonics on arithmetic ; simi- larly, he says, Phaenomena (that is, observational astronomy) depend on (theoretical) astronomy.

We are told that it was one of the early Pythagoreans, unnamed, who first taught geometry for money : ' One of the Pythagoreans lost his property, and when this misfortune befell him he was allowed to make money by teaching geometry.' 1 We may fairly conclude that Hippocrates of Chios, the first writer of Elements, who also made himself famous by his quadrature of lunes, his reduction of the duplication of the cube to the problem of finding two mean proportionals, and his proof that the areas of circles are in the ratio of the squares on their diameters, also taught for money and for a like reason. One version of the story is that he was a merchant, but lost all his property through being captured by a pirate vessel. He then came to Athens to prosecute the offenders and, during a long stay, attended lectures, finally attaining such proficiency in geometry that he tried to square the circle.2 Aristotle has the different version that he allowed himself to be defrauded of a large sum by custom-house officers at Byzantium, thereby proving, in Aristotle's opinion, that, though a good geometer, he was stupid and incompetent in the business of ordinary life.

Spartans

As a detail, we are told that he got no fees for his lectures in Sparta, and that the Spartans could not endure lectures on astronomy or geometry or logistic; it was only a small minority of them who could even count ; what they liked was history and archaeology.

Pythagorean triples, right angles, gnomons, observation, dots (discrete) Plato, Euclid and Egypt (pyramids?).

To return to Pythagoras. Whether he learnt the fact from Egypt or not, Pythagoras was certainly aware that, while 3^2 +4^2 =5^2, any triangle with its sides in the ratio of the numbers 3, 4, 5 is right angled. This fact could not but add strength to his conviction that all *things were numbers, for it established a connexion between numbers and the angles of geometrical figures. It would also inevitably lead to an attempt to find other square numbers besides 52 which are the sum of two squares, or, in other words, to find other sets of three integral numbers which can be made the sides of right-angled triangles ; and herein we have the beginning of the indeterminate analysis which reached so high a stage of development in Diophantus. In view of the fact that the sum of any number of successive terms of the series of odd numbers 1, 3, 5, 7 . . . beginning from 1 is a square, it was only necessary to pick out of this series the odd numbers which are themselves squares; for if we take one of these, say 9, the addition of this square to the square which is the sum of all the preceding odd numbers makes the square number which is the sum of the odd numbers up to the number (9) that we have taken. But it would be natural to seek a formula which should enable all the three numbers of a set to be immediately written down, and such a formula is actually attributed to Pythagoras.*

This formula amounts to the statement that,

if m be any odd number, m^2 + (m-1)^2/2 = (m+1)^2/2

Pythagoras would presumably arrive at this method of formation in the following way.

Observing that the gnomon put round n2 is 2n+l, he would only have to make 2n+1 a square.

If we suppose that 2n+1 =m2 we obtain n=1\2(m2—1), and therefore It follows that

m^2 + (m-1)^2/2 = (m+1)^2/2

Another formula, devised for the same purpose, is attributed to Plato,1 namely

(2m)2+(m2-l)2 ==(m2+l)2.

We could obtain this formula from that of Pythagoras by doubling the sides of each square in the latter ; but it would be incomplete if so obtained, for in Pythagoras's formula mis necessarily odd, whereas in Plato's it need not be. As Pytha- goras's formula was most probably obtained from the gnomons of dots, it is tempting to suppose that Plato's was similarly evolved.

Consider the square with n dots in its side relation to the next smaller square (n-1)^2 and the next larger (n+1)^2. Then n^2 exceeds (n-1)^2 by the gnomon 2n-1, but falls short of (n+1)^2 by the gnomon 2n+1. Therefore the square (n+1)^2 exceeds the square (n-1)^2 by the sum of the two gnomons 2n-1 and 2n+1, which is 4n.

That is, 4n+(n-1)^2=(n+1)^2.

The formulae of Pythagoras and Plato supplement each other. Euclid's solution (X, Lemma following Prop. 28) is more general, amounting to the following.

If AB be a straight line bisected at C and produced to D, then (Eucl. II. 6)

AD.DB+CB2 =CD2 which we may write thus

uv=c2—b2 ,

where u—c+b, v=c—b, and consequently

c =1/2(u+v)} 6=1/2(u—v).

In order that uv may be a square, says Euclid, u and v must, if they are not actually squares, be ' similar plane num- bers', and further they must be either both odd or both even in order that b (and c also) may be a whole number.

Similar plane ' numbers are of course numbers which are the product of two factors proportional in pairs, as Trip, np and mq. nq, or mnp2 and mnq2. Provided,then,that these numbers are both even or both odd, m2n2p2q2+((mnp2-mnq2)/2)2=((mnp2+mnq2)/2)2 is the solution, which includes both the Pythagorean and the Platonic formulae.

A theorem of Archytas (a hint about previous books)

Another interesting theorem relative to geometric means evidently goes back' to the Pythagoreans. If we have two numbers in the ratio known as twiuopLos, or superparticularis, i.e. the ratio of n+\ to n, there can be no number which is ameanproportionalbetweenthem. ThetheoremisProp.3of Euclid's Sectio Canonist and Boetius has preserved a proof of it by Archytas, which is substantially identical with that of Euclid.3 The proof will be given later (pp. 215-16). So far as this chapter is concerned, the importance of the proposition lies in the fact that it implies the existence, at least as early as the date of Archytas (about 430-365 B.C.), of an Elements of Arithmetic in the form which we call Euclidean ; and no doubt text-books of the sort existed even before Archytas, which probably Archytas himself and others after him im- proved and developed in their turn.

I have left out a large section about the theory of numbers (unit, dyads, odd-even classifications) and theory of proportionals (ten types of means, etc..) without quoting. All of it is trivialized by modern algebraic notation. However, that could only have been seen in retrospect. Things have to be studied to the extreme and these two theories were. Now, they seem very mystic (polygonal numbers, friendly numbers, …) but who could have predicted their lack of application (so far)? This will always be a ‘problem’; that going to the extremes studying a certain topic will be judged very useless by most when there is a lack of direct application. It is fun to imagine how much more advanced greek mathematics would have been had a ‘philosophy of notation’ been added to their philosophy of mathematics, based on the idea that there is a certain limit to the complexity even the best human brain can juggle at one time, and that it is very advantageous to trivialize as much as possible all of the current knowledge (even when it hurts), in order to elevate the theory higher and higher by making the building blocks trivial (to stack).

Another idea about the two theories mentioned above is that maybe, this highlights an advantage of the axiomatic method, even if it might mean loss of meaning and turn mathematics into a symbol game. It is possible that the theories were rendered over-complicated because of the philosophical stance of trying to give meaning to everything, link it to the universe, when the link is not justified, or not readily captured with the current level of technology. For sure, we might agree that ‘all things are number’, but the link between unity, and the universe passes through an infinitely number of layers. A dry axiomatic system would have probably been quite advantageous, if not controversial with the lack of a philosophical defense. But this would have been difficult since, unlike in more modern times, there was no evidence of how crippling it is to try to require direct meaning from axiomatic systems.

The proof of √2 being irrational, or better said, the *discovery* of irrationals by the Pythagoreans, takes on a whole new dimension. All the basic blocks of the proof, down to the classification of numbers into even and odd, had to be discovered and studied.
Also note how the irrationals were classified as geometric, and from the historical perspective it makes a lot of sense. The hidden difficulties and problems caused by this (having to use geometry for theories and topics that are basically ‘non-geometrical’) philosophical stance that makes lots of sense could only be understood later, and highlight the achievement and value of classical real analysis.

We mentioned above the dictum of Proclus (if the reading dXoyoav is right) that Pythagoras

discovered the theory, or study, of irrationals. This subject was regarded by the Greeks as belonging to geometry rather than arithmetic. The irrationals in Euclid, Book X, are straight lines or areas, and Proclus mentions as special topics in geometry matters relating (1) to positions (for numbers have no position), (2) to contacts (for tangency is between continuous things), and (3) to irrational straight lines (for where there is division ad infinitum, there also is the irrational).4 I shall therefore postpone to Chapter V on the Pythagorean geometry the question of the date of the discovery of the theory of irra- tionals. But it is certain that the incommensurability of the diagonal of a square with its side, that is, the ' irrationality of V2, was discovered in the school of Pythagoras, and it is more appropriate to deal with this particular case here, both because the traditional proof of the fact depends on the elementary theory of numbers, and because the Pythagoreans
invented a method of obtaining an infinite series of arith- metical ratios approaching more and more closely to the value of y/2.
The actual method by which the Pythagoreans proved the fact that V2 is incommensurable with 1 was doubtless that indicated by Aristotle, a reductio ad absurdum showing that, if the diagonal of a square is commensurable with its side, it will follow that the same number is both odd and even.

A good guess at how the Pythagoreans arrived at their √(2) approximation is given by Lazlo [16]. His paper shows that it is not trivial to guess how they arrived at their formula, which is related to Pythagorean triples and the Pell Equation.

In the famous passage of the Republic (546 c) dealing with the geometrical number Plato distinguishes between the 'irrational diameter of 5 ', i.e. the diagonal of a square having 5 for its side, or \/(50), and what he calls the 'rational diameter ' of 5. The square of the ' rational diameter ' is less by 1 than the square of the ' irrational diameter ', and is there- fore 49, so that the 'rational diameter 'is 7; that is, Plato referstothefactthat2. 52—
72=1,andhehasinmindthe particular pair of side- and diameter- numbers, 5 and 7, which mustthereforehavebeenknownbeforehistime. Astheproof of the property of these numbers in general is found, as Proclus says, in the geometrical theorem of Eucl. II. 10, it is a fair inference that that theorem is Pythagorean, and was prob- ably invented for the special purpose. [14]

Rope-stretchers and 3,4,5

Many Greek writers besides Proclus give a similar account of the origin of geometry. Herodotus says that Sesostris (Ramses II, circa 1300 B.C.) distributed the land among all the Egyptians in equal rectangular plots, on which he levied an annual tax ; when therefore the river swept away a portion of a plot and the owner applied for a corresponding reduction in the tax, surveyors had to be sent down to certify what the reduction in the area had been. ' This, in my opinion (SoKeei fjioi) ', he continues, ' was the origin of geometry, which then passed into Greece.' 1 The same story, a little amplified, is repeated by other writers, Heron of Alexandria,2 Diodorus Siculus,3 and Strabo.4 True, all these statements (even if that in Proclus was taken directly from Eudemus's History of
Geometry) may all be founded on the passage of Herodotus, and Herodotus may have stated as his own inference what he was told in Egypt ; for Diodorus gives it as an Egyptian tradition that geometry and astronomy were the discoveries of Egypt, and says that the Egyptian priests claimed Solon, Pythagoras, Plato, Democritus, Oenopides of Chios, and Eudoxus as their pupils. But the Egyptian claim to the discoveries was never disputed by the Greeks. In Plato's

Pkaedrus Socrates is made to say that he had heard that the Egyptian god Theuth was the first to invent arithmetic, the science of calculation, geometry, and astronomy.5 Similarly Aristotle says that the mathematical arts first took shape in
Egypt, though he. gives as the reason, not the practical need which arose for a scientific method of measuring land, but the fact that in Egypt there was a leisured class, the priests, who could spare time for such things. 6 Democritus boasted that no one of his time had excelled him ' in making lines into figures and proving their properties, not even the so-called Harpe- donaptae in Egypt '. 7 This word, compounded of two Greek words, dprreSovrj and oltttuv, means ' rope-stretchers ' or ' rope-fasteners'; and, while it is clear from the passage that the persons referred to were clever geometers, the word reveals a characteristic modus operandi. The Egyptians were ex- tremely careful about the orientation of their temples, and the use of ropes and pegs for marking out the limits, e.g. corners, of the sacred precincts is portrayed in all pictures of the laying of foundation stones of temples.1 The operation of ' rope-stretching ' is mentioned in an inscription on leather in the Berlin Museum as having been in use as early as Amenemhat I (say 2300 B.C.).2 Now it was the practice of ancient Indian and probably also of Chinese geometers to make, for instance, a right angle by stretching a rope divided into three lengths in the ratio of the sides of a right- angled triangle in rational numbers, e.g. 3, 4, 5, in such a way that the three portions formed a triangle, when of course a right angle would be formed at the point where the two smaller sides meet.
There seems to be no doubt that the Egyptians knew that the triangle (3, 4, 5), the sides of which are so related that the square on the greatest side is equal to the sum of the squares on the other two, is right-angled ; if this is so, they were acquainted with at least one case of the famous proposition of Pythagoras.

The empirical mensurations of the Egyptians

The most important available source of information about Egyptian mathematics is the Papyrus Rhind, written probably about 1700 B.C. but copied from an original of the time of King Amenemhat III (Twelfth Dynasty), say 2200 B.C. The geo- metry in this ' guide for calculation, a means of ascertaining everything, of elucidating all obscurities, all mysteries, all
difficulties', as it calls itself, is rough mensuration.

…

This temple was planned out in 237 B.C.; the in- scriptions which refer to the assignment of plots of ground to the priests belong to the reign of Ptolemy XI, Alexander I (107-88 B.C.). From so much of these inscriptions as were published by Lepsius1 we gather that ... was a formula for the area of a quadrilateral the sides of which in order are a, b, c, d.

…

We come now (4) to the mensuration of circles as found
in the Papyrus Rhind. If d is the diameter, the area is
given as ... it follows that the value of ir is taken as … 3.16, very nearly. A somewhat different value for w has been inferred from measurements of certain heaps of grain or of spaces which they fill. Unfortunately the shape of these spaces or heaps cannot be determined with certainty.

…

orchardt suggests that the formula for the measurement of a hemisphere was got by repeated practical measurements of heaps of corn built up as nearly as possible in that form, in which case the inaccuracy in the figure for it is not surprising. With this problem from the Kahun

papyri must be compared No. 43 from the Papyrus Rhind. A curious feature in the measurements of stores or heaps of corn in the Papyrus Rhind is the fact, not as yet satisfactorily ex- plained, that the area of the base (square or circular) is first found and is then regularly multiplied, not into the ' height itself, but into •§ times the height.
…
As to this Eisenlohr can only suggest that the circle of diameter k which was accessible for measurement was not the real or mean circular section, and that allowance had to be made for this, or that the base was not a circle of diameter k but an ellipse with -^ k and k as major and minor axes. But such explanations can hardly be applied to the factor (¾)^2 in the Kahun case if the latter is really the case of a hemispherical space as suggested. Whatever the true explanation may be, it is clear that these rules of measure- ment must have been empirical and that there was little or no geometry about them.

se-qet, to the seconds.

Much more important geometrically are certain calculations with reference to the proportions of pyramids (Nos. 56-9 of the Papyrus Rhind) and a monu- ment (No. 60). In the case of the pyramid two lines in the .j\c figure are distinguished, (1) ukha-thebt, which is evidently V some line in the base, and piv-em-us or per-em-us ('height'), a word from which the name Trvpa\xis may have been derived.1 The object of the problems is to find a certain relation called se-qet, literally ' that which makes the nature ', i. e. that which determines the proportions of the pyramid.
…..
But, lastly, the se-qet in No. 56 is ff and, if te-qet is taken in the sense of cot HFEi this gives for the angle HFE the value of 54° 14/ 16", which is precisely, to the seconds, the slope of the lower half of the southern stone pyramid of Dakshur; in Nos. 57-9 the se-qet, J, is the co-tangent of an angle of 53° 7/ 48", which again is exactly the slope of the second pyramid of Gizeh as measured by Flinders Petrie ; and the se-qet in No. 60, which is £, is the cotangent of an angle of 75° 57' 50", corresponding exactly to the slope of the Mastaba-tombs of the Ancient Empire and of the sides of the Medum pyramid.1
These measurements of se-qet indicate at all events a rule- of-thumb use of geometrical proportion, and connect themselves naturally enough with the story of Thales's method of measuring the heights of pyramids.

Thales

At the beginning of the summary of Proclus we are told that THALES (624-547 B.C.)
'first went to Egypt and thence introduced this study (geometry) into Greece. He discovered many propositions himself, and instructed his successors in the principles under- lying many others, his method of attack being in some cases more general (i. e. more theoretical or scientific), in others more empirical (alo-OrjTiKdorepoi', more in the nature of simple inspection or observation).' 2
With Thales, therefore, geometry first becomes a deductive science depending on general propositions ; this agrees with what Plutarch says of him as one of the Seven Wise Men
' he was apparently the only one of these whose wisdom stepped, in speculation, beyond the limits of practical utility the rest acquired the reputation of wisdom in politics.' 3
(Not that Thales was inferior to the others in political wisdom. Two stories illustrate the contrary. He tried to save Ionia by urging the separate states to form a federation with a

capital at Teos, that being the most central place in Ionia. And when Croesus sent envoys to Miletus to propose an alliance, Thales dissuaded his fellow-citizens from accepting the proposal, with the result that, when Cyrus conquered, the city was saved.)
(a) Measurement of height of pyramid.
The accounts of Thales's method of measuring the heights of pyramids vary. The earliest and simplest version is that of Hieronymus, a pupil of Aristotle, quoted by Diogenes Laertius
' Hieronymus says that he even succeeded in measuring the pyramids by observation of the length of their shadow at the moment when our shadows are equal to our own height.'
Pliny says that
' Thales discovered how to obtain the height of pyramids and all other similar objects, namely, by measuring the shadow of the object at the time when a body and its shadow are equal in length.' 2

Invariance of the se-qet

the solution is itself a se-qet calculation, justlikethatinNo.57ofAhmes'shandbook. Inthelatter problem the base and the se-qet fire given, and we have to find the height. So in Thales's problem we get a certain se-qet by dividing the measured length of the shadow of the stick by the length of the stick itself ; we then only require to know the distance between the point of the shadow corre- sponding to the apex of the pyramid and the centre of the base of the pyramid in order to determine the height; the only difficulty would be to measure or estimate the distance from the apex of the shadow to the centre of the base.
…
(2) Tradition credited him with the first statement of the theorem (Eucl. I. 5) that the angles at the base of any isosceles triangle are equal, although he used the more archaic term ' similar ' instead of ' equal '. 2

We never learn...

Next, leaving the cross-piece at the angle so found, he would turn the stick round, while keeping it vertical, until the cross-piece pointed to some visible object on the shore, which would be mentally noted ; after this it would only
be necessary to measure the distance of the object from the foot of the tower, which distance would, by Eucl. I. 26, be equal to the distance of the ship. It appears that this precise method is found in so many practical geometries of the first century of printing that it must be assumed to have long been a common expedient. There is a story that one of Napoleon's engineers won the Imperial favour by quickly measuring, in precisely this way, the width of a stream that blocked the progress of the army.

On overview of the contents of the 13 books

Kline, in ‘Mathematical thoughts, From ancient to Modern Times’ (MT), provides a most comprehensive summary of what would be a full reading of Heath’s books, and other similar details references. Additionally, the points of focus he chooses, and his commentaries, are very relevant to this research. In fact, his conclusions are the ones I would be after finding (If I had the skill to do so). From the title of his book it is already obvious that he is set out to answer my main question: How a number of ideas evolved through time to become modern Mathematics. How, from the most basic Mathematics accessible to almost every human, and directly

springing into existence out of practical means, we step by step, contribution by contribution, each (almost all of the time) ‘logical’ but also very brilliant, created the current body of Mathematics. The journey pinpoints all the kinds of thinking --philosophical, sophistical, pragmatic, technical, observational, bold-- and knowledge -- mathematical, practical, extensive -- that I do not possess and I hope to acquire.
All quotes in this section are from MT except when otherwise noted.

> *Books VII, VIII, IX treat the theory of numbers, that is, the properties of whole numbers and the ratios of whole numbers. These three books are the only ones in the Elements that treat arithmetic as such. In Them Euclid represents numbers as line segments and the product of two numbers as a rectangle, but the arguments do not depend on the geometry. The statements and proofs are verbal as opposed to the modern symbolic form.*
>
> *Many of the definitions and theorems, particularly those on proportion, duplicate what was done in Book V. Hence historians have considered the question of why Euclid proves all over again propositions for numbers instead of referring to propositions already proven in Book V.*

I have thought about the Books II to IV, and V (TODO) and the geometric algebra (symbolic algebra presented and proved in a geometric way) therein. I wondered if not, in fact, Euclid and the Greeks did have our modern algebra in mind, or at least, that they first knew the results from arithmetic (like in book VII-IX) and desired to prove them for proportions (geometric real numbers) (Book II to IV), and also for magnitudes (a not completely successful attempt at real numbers) (Book V). I am also sure about this. The answer to why Euclid proves similar propositions over again is because, even though arithmetic books come later,
the order here is misleading. The idea would be that the theorems are known from arithmetic, and Euclid then proves that these theorems also hold for proportions and irrationals, thereby ‘extending the number system’, like when we prove that dedekind cuts are an ordered field. For me, this demystifies the origin of Euclid’s Geometric Algebra, I am sure it did not come directly out of thinking about Geometry.

The relation between number and magnitude in Euclid is elucidated as much as possible by Kline

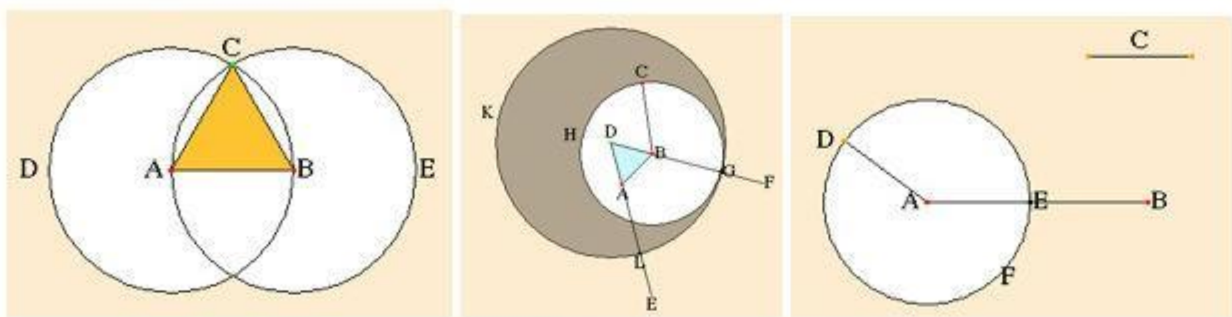
Direct (and long) proofs of the major theorems

Once I am done with book I, I will here attempt to reconstruct the two major theorems mentioned in Note(4).2 without help. (TODO)

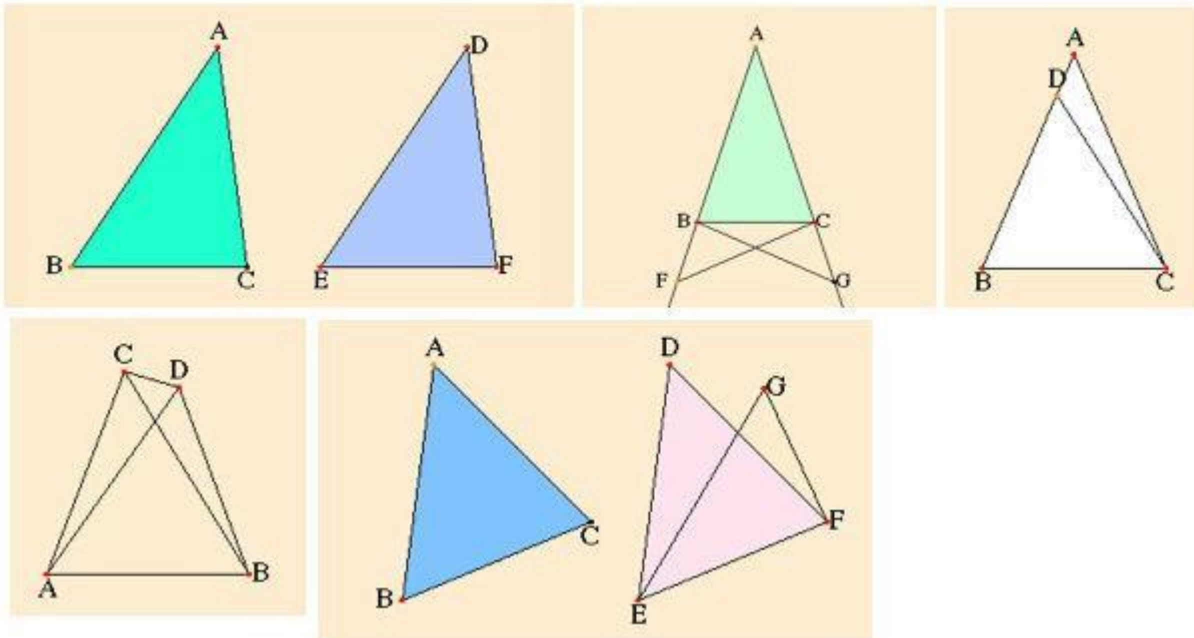
A list of all proof figures in the Elements

One of the good things about this undertaking is that I feel it should be possible to 'memorize' all the proofs for the long term because of the visual character of the Elements. Contrasting this to symbolic proofs that are harder to visualize and are therefore not 'memorized' at all, or, when 'memorized', then by some mix of linguistic, pictorial (symbols) and relational memory that I still cannot clearly describe.
For this purpose, I will list here all the figures of the proposition proofs, in the order of which they occur, taken as snapshots from the website.
The idea is to go over them visually (and rather quickly), be able to 'spot the proposition', 'spot the construction', 'spot the proof'. It will be interesting to see if this has any equivalent for the less visual disciplines.

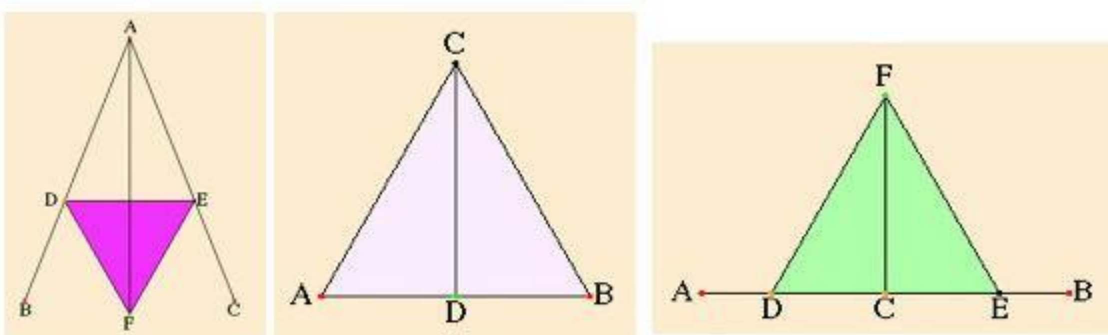
Moving segments (1-3)



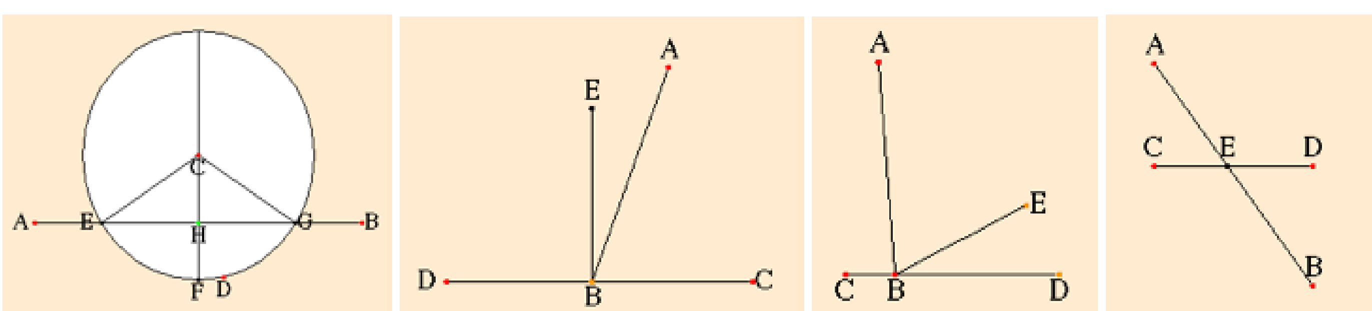
Congruences: SAS, SSS (4-8)



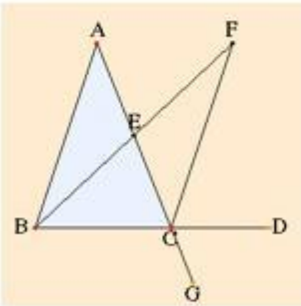
Bisection: segment and angle (9-11)



Angular (12-15)



Triangular inequalities (16-20)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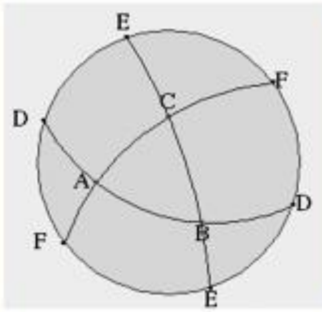
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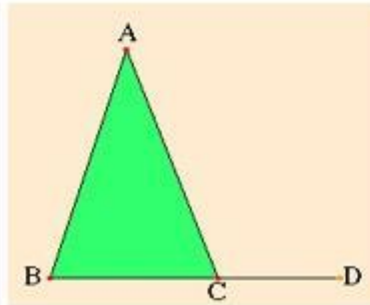
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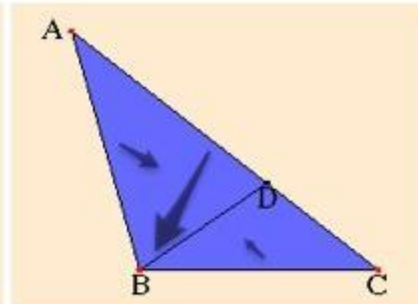
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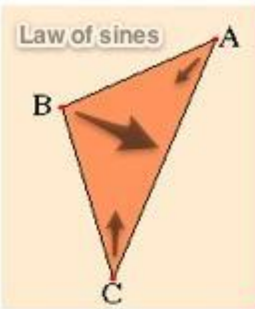
Law of sines
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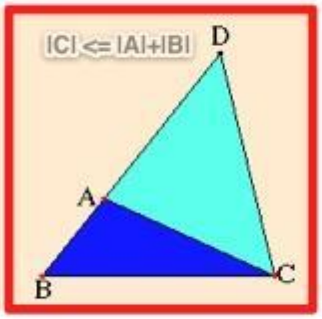
C


|C| <= |A|+|B|
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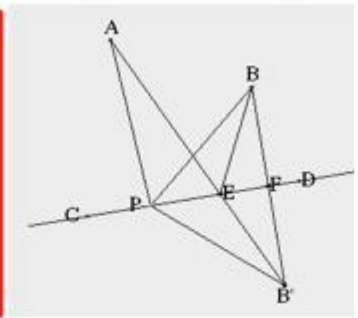
A
B
C
P
E
F
D
B'

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