Free occurrences in FOL

- Kunen p.99 points out to the difference between formula and sentence

  > Definition II.5.6: If φ is a **formula**, a *universal closure* is any **sentence** of the form $\forall x_1 \forall x_2 \ldots \forall x_n \varphi$, where $n > 0$.

  - **Q:** So then
    1. can an axiom in FOL contain free variables?
    2. How are free variables used in FOL formal proofs?
- Copi p.84 says

  > is a free occurrence. Thus we see that propositional functions may contain both free and bound occurrences of variables. On the other hand, all occur- rences of variables in propositions must be bound, since every proposition must be either true or false. A **propositional function** must contain at least one free occurrence of a variable, but no **proposition** can contain any free occurrences of any variable.

  - We must hence **clearly** differentiate propositional functions from propositions
  - **Note:** It is funny though how $\forall$ turns a (propositional) function into a proposition, by acting over all of the domain, and collapsing (with $\wedge$) the truth of the proposition, and even funnier with $\exists$ which is an $\vee$ without specifying the object in the domain for which truth holds! But careful! what we just wrote might be wrong, see **N1**.
- Wikipedia explains propositional functions

  > In propositional calculus, a propositional function is a sentence expressed in a way that would assume the value of true or false, except that within the sentence there is a **variable (x) that is not defined or specified, which leaves the statement undetermined**. The sentence may contain several such variables (e.g. n variables, in which case the function takes n arguments).

- mst:679744 *"When do free variables occur? Why allow them? What is the intuition behind them?"* explains:
  - Non-necessity of free variables in FOL:

    > Let's be clear. **You don't need to countenance free variables to set up standard first order logic**.

    > As far as the syntax goes, we needn't allow wffs with free variables.

        - **Note:** But we can, and maybe sometimes it is not clear if the syntax allows them without a closer look!
    - Syntax, constants and variables:

      > We can have the rule that if $\varphi(a)$ is a wff containing one or more occurrences of the **constant** a, then so is $\forall\xi\varphi(\xi)$ and $\exists\xi\varphi(\xi)$, where ξ is any **variable** and $\varphi(\xi)$ is of course the result of replacing each occurrence of a in $\varphi(a)$ with ξ.

        - But now we also have **constants** in the mix! And the **distinction between constant and variable is key** in this explanation! A wff with constants (so no free 'variables' but 'constants') only turns into a wff if the quantification is over a 'variable' that replaces the 'constant'! But this does not mean they need to appear in axioms (although they can). Instead, this is simply a way to define wff's in a way that prevents formula (with free variables) from being considered as sentences (or wff's).
    - Syntax vs. semantics:

      > "As far as the semantics goes, …"

        - **Note:** We also then need to be careful about the **semantic vs. syntactic** usage! not doing so is also a source of confusion! This also leads us to mst:481577
        - **(N1) Note:** But we seem NOT to understand the semantics at all!

          > "it is NOT the account that goes $\forall\xi\varphi(\xi)$ is true on I just if $\varphi(a)$ is true on I for every constant a. Our account of the semantics talks about extensions of I by assigning references to new constants, …"

            - And this leads us directly to mst:481577 *"Two styles of semantics for a first-order language: what's to choose?"*. Reading it tells us we do not know the distinction between and interpretation and a domain, what a 'fixed' interpretation is. Wikipedia is a good stop-gap: https://en.wikipedia.org/wiki/Interpretation_(logic).
            - Here we notice that we have not internalized the distinction between the domain of discourse and the signature, which is a set of non-logical symbols!
            - **(N2) Note:** We have to internalize that the domain, the semantic range of objects, is NOT part of the syntax. The **mapping IS the so-called interpretation**. What is also important to note is that the **interpretation is directional**! The **signature IS part of the syntax, the domain is NOT**. The interpretation is directional in the sense that it interprets the signature (be it constants, functions or relations). This means that __every element in the signature is mapped to some element in the domain_, but **this does not mean that every element in the domain is interpreting (reverse mapped to) a symbol in the signature**.

- Now given the semantics of the quantifiers being tied to truth for some domain symbols, **we might face the ‘problem’ that these symbols are not mapped to anything in the signature**, this is crucial and possibly explains an implicit misunderstanding that prevented us in the past from understanding certain readings about FOL.
- To solve that ‘problem’ there are two ways, these are the two ways mst:481577 means. Wikipedia explains it better:

  > Strictly speaking, a substitution instance such as the formula ϕ(d) mentioned above is not a formula in the original formal language of ϕ, because d is an element of the domain. There are two ways of handling this technical issue. **The first is to pass to a larger language in which each element of the domain is named by a constant symbol. The second is to add to the interpretation a function that assigns each variable to an element of the domain. Then the T-schema can quantify over variations of the original interpretation in which this variable assignment function is changed, instead of quantifying over substitution instances.**

  note that the T-schema is nothing but (the once very baffling statement)

  > The truth value of an arbitrary sentence is then defined inductively using the T-schema, which is a definition of first-order semantics developed by Alfred Tarski. The T-schema interprets the logical connectives using truth tables, as discussed above. Thus, for example, **ϕ & ψ is satisfied if and only if both ϕ and ψ are satisfied**.

- Another equivalent and short answer is

  > If you want something like ∀xϕ to be true just in case everything satisfies ϕ and at the same time allow that not every object has a name, then a Tarski style semantics should be your first choice.

- With this, **N1** is answered.

- The quote is long, but our conclusion from it is that we need not worry about free occurrences, and should consider them part of yet-to-be-completed propositions, always informal, being a short for some formal, fully quantified proposition. The shortcuts are listed in detail 2

  > So we can (and some authors do) treat both syntax and semantics without allowing wffs with free variables. Instead we can use “parameters”, i.e. additional constants (whose interpretation is not initially fixed). I think there are principled reasons for preferring this approach, but don’t need to argue that

> here: the present point is that it is certainly possible to do things this way.
>
> And seeing that this is so should help understand what is going on when – rather unfortunately perhaps – logic texts (always tempted by Bauhaus austerity!) decide not to use new constants or parameters which are typographically distinguished from variables, but prefer (on the grounds of economy) to re-use the same letters both for 'real' variables tying quantifiers to slots in predicates and for 'free' variables serving the role of parameters. This doubling up of roles for your xs and ys isn't necessary as we have seen, even if it is probably the majority practice (and also reflects some informal mathematical practice). And it can help – if this doubling seems confusing – to first tell the syntactical and semantic stories the other way, distinguishing variables and parameters, and then see the doubled-up role for your xs and ys as just a typographically economical trick.

- mst:2376437 explains:
  - First we read the comment:
    > Assume you want to prove ∀x,P(x). One convenient way is to say "Let x such that blabla… Therefore, P(x). Since there were no assumptions on x, we can conclude ∀x,P(x)". To do that in a formal proof, you need to drop the ∀, and reason on x, so that you'll have at some point in your proof something like Γ⊢P(x). To get to this point, you obviously need to be able to reason on formulas with free variables

    - So we see that this comment claims that we do get proof lines with free variables during a proof (but not axioms). We remember the complex rules of existential instantiation, but now, given what we learned here, we must wonder if what is instantiated is a constant, or a free variable. And this is indeed a point of confusion since at least one source (wikipedia) explains the we instantiate a constant symbol and not a free variable:
      > In one formal notation, the rule may be denoted by(∃x)Fx::Fa, where a is a new constant symbol that has not appeared in the proof.

      but it is not clear what is meant by *constant symbol*, the explanation is loose.
    - But we do fine an explicit warning in http://www.phil.cmu.edu/projects/logicandproofs/alpha/htmltest/m12_pred_derivations/translated_chapter12.html:
      > You might be wondering, though, why we'd want to be able to use any substitution instance, **including those where we instantiate with a free variable, rather than just those where the instantiating term is a constant**. **Well, this has to do with the way we deal with introducing universal**

> **quantifiers, so we'll actually hold off on answering this one for just a bit**.

This is in fact what we expected, but were not sure enough.

- The author say we instantiate a **term**, but what is a term? Is it a constant or a free variable (and does it matter). About this, we find

> The **term t could be either a constant or a variable**, but in either case we will end up interpreting the term by assigning it some value from the universe of discourse.

- The author explains that we choose to instantiate a *free variable* because we do not know what the object that satisfies the existential quantified true statement is (note how, unknowingly! that we noticed this in **N1**!!)! While explaining existential instantiation we read:

  > The pattern of inference we wish to capture for the elimination of existential quantifiers is closely related to that captured by the rule for disjunction elimination, which only makes sense, seeing as an existentially quantified sentence is much like an abbreviation for an arbitrarily long disjunction. As this analogy may foreshadow, existential elimination is going to involve an assumption that is similar to the assumption of one disjunct in the rule vE. In this case, however, **we assume a substitution instance of the quantified formula, a substitution instance of a very particular kind—we want our substitution instance to be arbitrary, since we don't know which individual satisfies it, only that one of them will. We thus assume a substitution instance where the instantiating term is a free variable**.

- The author also explains the need in UG

  > The pattern of inference we want to capture for introducing universal quantifiers is the following: If something is true for any individual you choose, no matter which individual that might be, then it is true for all individuals. **The tricky part is going to be isolating the sort of term that we consider to correspond to ''any individual you choose, no matter which individual that might be" and restricting our application of the rule of inference to that sort of term, and that sort alone.**

  > **Fortunately, finding the right sort of term is actually quite easy—we'll just use free variables once again, since variables are designed to be able to take on any value from a universe of discourse.**

- This was the last mention of free variables, and we really do get it now, especially with the last explanations.

- One has to be careful, in each book, what the choices are.
- The interplay between syntax and semantics, and hence between signature and domain, is crucial to appreciating the need for free variables. Our plan was to work with syntax alone, and that still works, but the justification for the choice of rules in this case, if one wants to understand it, relies on understanding the semantics as well, and that rigorously, as we said, the interplay between signature and domain. We should also never forget **N2**.
- For a completion of this, we must solve the examples in http://www.phil.cmu.edu/projects/logicandproofs/alpha/htmltest/m12_pred_derivations/translated_chapter12.html

Detail

1. Wikipedia explains propositional functions

   > Propositional functions are useful in set theory for the formation of sets. For example, in 1903 Bertrand Russell wrote in The Principles of Mathematics (page 106): "…it has become necessary to take propositional function as a primitive notion."

   > Later Russell examined the problem of whether propositional functions were predicative or not, and he proposed two theories to try to get at this question: the zig-zag theory and the ramified theory of types.[1]

   > According to Clarence Lewis, "A proposition is any expression which is either true or false; a propositional function is an expression, containing one or more variables, which becomes a proposition when each of the variables is replaced by some one of its values."[2] Lewis used the notion of propositional functions to introduce relations, for example, a propositional function of n variables is a relation of arity n. The case of n = 2 corresponds to binary relations, of which there are homogeneous relations (both variables from the same set) and heterogeneous relations.

2. > Let λ be the constant from the question formulation. Then, $\exists x.\ f(x,\lambda)=0$. In the last formula λ might appear free, but in reality comes from the context (which is quite short, so the example is kind of stupid, but nevermind).

   > The commutativity axiom $x+y=y+x$. In the usual formulation the x and y might appear free, but the convention in algebra is that free variables are quantified universally, i.e. the real formal statement is $\forall x.\forall y.\ x+y=y+x$. Of course this is long and tedious, so no one bothers to write those quantifiers.

> [During some proof…] Suppose that we got the solution x by magic, so we know that ∀y. F(y,x)=0, now observe that… Here, x might appear free, but it is not, only the proof is presented in a unconventional order (to make it easier, e.g. give more intuition what is happening) and proper quantification of x is introduced later. If we were to make it formal (and perhaps less understandable), then x would be just another bound variable.

> The derivative of a square: (x2)′=2x. Here x appears free, but it is a notation issue; here x2 denotes a function, namely (x↦x2), only that convention makes it implicit that x is in fact a parameter. In other words this could be (x↦x2)′=x↦2x, only it makes things much less clear.

> In logic Skolem constants may look like variables, but are not, i.e. those are constants or functions or some other weird objects, but not free variables.

Other Notes

1. We are continuing from https://gitlab.com/jadnohra/study/issues/17
2. An interesting distinction between PL and QL that we had not internalized:
   > "Unlike propositional logic, where every language is the same apart from a choice of a different set of propositional variables, there are many different first-order languages. Each first-order language is defined by a signature."
3. What is 'sentence based logic'? It seems to be a logic where formally, free variables are necessary, unlike FOL. mst:2376437
   > I'm curious what Causey's sentence only axioms and inferences are. I've only seen sentence based logic in inclusive logics. I'm not about to shell out 200 bucks for a book to find out though.
4. A universe of discourse should not be empty:
   > 1. (∀x)P(x) [Prem]
   > 2. P(w) [∀E: 1]
   > 3. (∃x)P(x) [∃I: 2]

   > It is very important to keep in mind here our restriction on acceptable universes of discourse—we agreed that we would only consider interpretations for predicate logic that had non-empty universes of discourse. If we did not make this restriction, then the above rule for introducing existential quantifiers would not be valid—in particular, any interpretation with an empty universe of discourse would provide a counterexample to any application of the rule, since absolutely no existentially quantified sentence is true

> on such an interpretation.

5. We could call many of the rules related to ‘fresh variables’, aliasing rules! Here is the rule for EI (http://www.phil.cmu.edu/projects/logicandproofs/alpha/htmltest/m12_pred_derivations/translated_chapter12.html)
   > v is a variable, v does not occur in P v does not occur in Q v does not occur in any (previous) undischarged assumptions.

6. > You might have noticed that there’s a fairly strong relationship between universal elimination and the notion of substitution instances. In fact, this relationship is one of the reasons that the rule for eliminating universal quantifiers is often called universal instantiation, rather than universal elimination.