Quotes4, 314-...

1. "With apologies to Samuel Johnson: It is the fate of those who toil at the lower employments of life, to be rather driven by the fear of evil, than attracted by the prospect of good ; to be exposed to censure, without hope of praise; to be disgraced by miscarriage, or punished by neglect, where success would have been without applause, and diligence without reward. Among these unhappy mortals is the translator of Latin mathematical works of days gone by; whom mankind have considered, not as the pupil, but the slave of science, the pioneer of literature, doomed only to remove rubbish and clear obstructions from the paths through which Learning and Genius press forward to conquest and glory, without a smile on the humble drudge that facilitates their progress. Every other author may aspire to praise; the translator can only hope to escape reproach, and even this negative recompense has been granted to a very few....

   General Introduction : The State of this Site Sept. 2012. See below for later additions. However, not withstanding the similarities of the present task with Johnson's remarks about compiling his dictionary, it is pleasing to note that for this website, around 3500 visits and 30,000 hits are made on a monthly basis, and that around 25,000 files are downloaded monthly to mathematicians and students of mathematics in around 130 countries, of which the U.S. accounts for approximately a quarter or more, on a regular basis. There is, of course, some seasonal variation depending on semester demand. The most popular files downloaded recently not in order have been Euler's Integration Ch. I , and some chapters of the Mechanica I & II , parts of Huygens' Horologium are very popular, some early Euler papers, esp. E025, and various parts of Gregorius and James Gregory's Optica Promota, and there is of course a constant demand for material on logarithms, and Harriot's book salvaged from his posthumous notes is of interest to browsers. Lately Jim Hanson's work on Napier's Bones and Promptuary have been very popular. This site is unique in that it provides the only translation available into English of a number of important works. Texts are presented with the understanding that they cannot be error-free, and reveal the translator's idiosyncrasies to some extent. Usually there are some notes to help you along, especially at the start. " (http://www.17centurymaths.com/)
2. "Eneström says that this work contains orthogonality conditions for rectangular corrdinates in space. He also says that Euler will express the coefficients of linear homogeneous transformations that let a sum of *n* square numbers remain constant using trigonometric functions for *n* = 3, 4, and 5, and also using rational parametric functions for *n* = 3 and 4. " (About E407 -- Problema algebraicum ob affectiones prorsus singulares memorabile, at http://www.math.dartmouth.edu/~euler/pages/E407.html)
   "The full title is ``{\it Problema algebraicum ob affectiones prorsus singulares memorabile},'' which means ``An algebraic problem which is memorable because of its rather noteworthy impressions." The problem is to find all linear transformations of N independent variables which preserve the sum of their squares.
   Euler solves the problem for $N = 2, 3, 4, 5$ and touches upon a solution for arbitrary N.
   Euler's treatment is purely algebraic; angles solely serve to parametrize the transformations.
   The three angles needed to parametrize the transformations in the case $N = 3$ became later well-known as ``Euler angles''. Euler angles are widely applied in solid geometry, physics, aeronautics and astronautics, and computer graphics.
   Presumably Euler did not have geometrical applications in mind when writing this paper. This conjecture is supported by three places in E407 where the transformation is a rotoreflection rather than a rotation; rotoreflections cannot be parametrized by angles in the way Euler uses them.
   The last four chapters (Ch. 33 - 36) treat rational parametrizations for the cases $N = 3$ and $N = 4$ and a certain class of magic squares of order four." (Mebius, Johan. "On Euler's 1770 paper ``{it Problema Algebraicum}'' (E407)")
   <Tag3>
3. "The classical quaternion representation theorem for rotations in 4D Euclidean space states that an arbitrary 4D rotation matrix is the product of a matrix representing left-multiplication by a unit quaternion and a matrix representing right-multiplication by a unit quaternion. This decomposition is unique up to the sign of the pair of component matrices.
   ...
   The history of this theorem is rather obscure. In 1770 Leonhard Euler ([EULE 1770]) publishes bilinear expressions for the 4th-order orthogonal substitution in terms of eight reals. As Euler himself writes, he obtained them by guesswork motivated by Diophantine Analysis. "
   (http://jemebius.home.xs4all.nl/So4hist.htm)
4. "The word degree originated with the Greeks. According to the historian of mathematics David Eugene Smith, they used the word μοιρα (moira), which the Arabs translated into daraja (akin to the Hebrew dar'ggah, a step on a ladder or scale); this in turn became the Latin de gradus, from which came the word degree. " (Maor, Trigonometric Delights) <Tag3>
5. "Geometric entities are of two kinds: those of a strictly qualitative nature, such as a point, a line, and a plane, and those that can be assigned a numerical value, a measure. To this last group belong a line

segment, whose measure is its length; a planar region, associated with its area; and a rotation, measured by its angle.
There is a certain ambiguity in the concept of angle, for it de- scribes both the qualitative idea of “separation” between two in- tersecting lines, and the numerical value of this separation—the measure of the angle. (Note that this is not so with the anal- ogous “separation” between two points, where the phrases line segment and length make the distinction clear.) Fortunately we need not worry about this ambiguity, for trigonometry is con- cerned only with the quantitative aspects of line segments and angles.1
The common unit of angular measure, the degree, is believed to have originated with the Babylonians. It is generally assumed that their division of a circle into 360 parts was based on the closeness of this number to the length of the year, 365 days. Another reason may have been the fact that a circle divides nat- urally into six equal parts, each subtending a chord equal to the radius (fig. 4). There is, however, no conclusive evidence to sup- port these hypotheses, and the exact origin of the 360-degree system may remain forever unknown.2 In any case, the system fitted well with the Babylonian sexagesimal (base 60) numera- tion system, which was later adopted by the Greeks and used by Ptolemy in his table of chords (see chapter 2)." (Maor, Trigonometric Delights) <Tag3>

6. "Even more important, the fact that a small angle and its sine are nearly equal numerically—the smaller the angle, the bet- ter the approximation—holds true only if the angle is measured in radians. For example, using a calculator we find that the sine of one degree 􏱒sin1◦􏱓 is 0.0174524; but if the 1◦ is con- verted to radians, we have 1◦ = 2π/360◦ ≈ 0􏱩0174533, so the angle and its sine agree to within one hundred thousandth. For an angle of 0􏱩5◦ (again expressed in radians) the agreement is within one millionth, and so on. It is this fact, expressed as limθ→0􏱒sinθ􏱓/θ = 1, that makes the radian measure so impor- tant in calculus." (Maor, Trigonometric Delights) <Tag3>
7. "The word radian is of modern vintage; it was coined in 1871 by James Thomson, brother of the famous physicist Lord Kelvin (William Thomson); it first appeared in print in examination questions set by him at Queen’s College in Belfast in 1873.7 Earlier suggestions were “rad” and “radial.”" (Maor, Trigonometric Delights) <Tag3>
8. "No one knows where the convention of measuring angles in a counterclockwise sense came from. It may have originated with our familiar coordinate system: a 90◦ counterclockwise turn takes us from the positive x-axis to the positive y-axis, but the same turn clockwise will take us from the positive x-axis to the negative y-axis. This choice, of course, is entirely arbitrary: had the x-axis been pointing to the left, or the y-axis down, the nat- ural choice would have been reversed. Even the word “clock- wise” is ambiguous: some years ago I saw an advertisement for a “counterclockwise clock” that runs backward but tells the time perfectly correctly (fig. 6). Intrigued, I ordered one and hung it in our kitchen, where it never fails to baffle our guests, who are convinced that some kind of trick is being played on them!" (Maor, Trigonometric Delights) <Tag3>
9. "When considered separately, line segments and angles behave in a simple manner: the combined length of two line segments placed end-to-end along the same line is the sum of the indi- vidual lengths, and the combined angular measure of two ro- tations about the same point in the plane is the sum of the individual rotations. It is only when we try to relate the two concepts that complications arise: the equally spaced rungs of a ladder, when viewed from a fixed point, do not form equal an- gles at the observer’s eye (fig. 7), and conversely, equal angles, when projected onto a straight line, do not intercept equal seg- ments (fig. 8).
Elementary plane trigonometry—roughly speak- ing, the trigonometry known by the sixteenth century—concerns itself with the quantitative relations between angles and line seg- ments, particularly in a triangle; indeed, the very word “trigo- nometry” comes from the Greek words trigonon = triangle, and metron = measure.1" (Maor, Trigonometric Delights)
10. "The exact dating of this 'table of Sines' is uncertain. These texts were regularly being revised and added to by different scholars. The similarity in the calculation to the Greek table of chords from Hipparchus (190-120 BCE) (we know his data came from the Babylonians) suggests that this Indian work appeared some time in the first century CE. Did the data in the Indian table come directly from the Babylonians, or via the Greeks? Nobody knows for sure. The important point is that the Indians made the technical and conceptual change from 'Chord' to 'Sine'." (http://nrich.maths.org/6843)
11. "Thales of Miletus (ca. 640–546 b.c.), the first of the long line of Greek philosophers and mathematicians, is said to have mea- sured the height of a pyramid by comparing the shadow it casts with that of a gnomon. As told by Plutarch in his Banquet of the Seven Wise Men, one of the guests said to Thales: Whereas he [the king of Egypt] honors you, he particularly admires you for the invention whereby, with little effort and by the aid of no mathematical instrument, you found so accurately the height of the pyramids. For, having fixed your staff erect at the point of the shadow cast by the pyramid, two triangles were formed by the tangent rays of the sun, and from this you showed that the ratio of one shadow to the other was equal to the ratio of the [height of the] pyramid to the staff.3
Again trigonometry was not directly involved, only the similarity of two right triangles. Still, this sort of “shadow reckoning” was fairly well known to the ancients and may be said to be the pre- cursor of trigonometry proper. Later, such simple methods were successfully applied to measure the dimensions of the earth, and later still, the distance to the stars (see chapter 5)." (Maor, Trigonometric Delights) <Tag3>
12. " As with many of the Greek scholars, Hipparchus’s work is known to us mainly through references by later writers, in this case the commentary on Ptolemy’s Almagest by Theon of Alexandria (ca. 390 a.d.). He was

born in the town of Nicaea (now Iznik in northwest Turkey) but spent most of his life on the island of Rhodes in the Aegean Sea, where he set up an observatory. Using instruments of his own invention, he determined the positions of some 1,000 stars in terms of their celestial longitude and latitude and recorded them on a map— the first accurate star atlas (he may have been led to this project by his observation, in the year 134 b.c. of a nova—an explod- ing star that became visible where none had been seen before). To classify stars according to their brightness, Hipparchus in- troduced a scale in which the brightest stars were given magni- tude 1 and the faintest magnitude 6; this scale, though revised and greatly extended in range, is still being used today. Hip- parchus is also credited with discovering the precession of the equinoxes—a slow circular motion of the celestial poles once every 26,700 years; this apparent motion is now known to be caused by a wobble of the earth's own axis (it was Newton who correctly explained this phenomenon on the basis of his theory of gravitation). " (Maor, Trigonometric Delights)

13. "As a mathematician, Hipparchus is credited with dividing a circle into 360° and being the first to use trigonometry. Using observations during the solar eclipse, probably in129 BC, he was able to work out the distance from the Earth to the Moon. The eclipse at Alexandria had been partial, four fifths of the sun had been covered, but at Syene it had been total. By measuring angles and applying distances between the two cities he was able to complete his calculations. Hipparchus also used mathematics to allow him to predict equinoxes, to accurately measure the length of the seasons and to calculate the size of the Moon and the Sun." (http://saburchill.com/HOS/astronomy/006.html)
14. "The theory of proportions is credited to Eudoxus (around 400–350 bce) and is expounded in Book V of Euclid's Elements. The purpose of the theory is to enable lengths (and other geometric quantities) to be treated as precisely as numbers, while admitting only the use of rational numbers. We saw the motivation for this in Section 1.5: the Greeks could not accept irrational numbers, but they accepted irrational geometric quantities such as the diagonal of the unit square. To simplify the exposition of the theory, let us call lengths rational if they are rational multiples of a fixed length.
The idea of Eudoxus was to say that a length λ is determined by those rational lengths less than it and those greater than it. To be precise, he says λ1 = λ2 if any rational length <λ1 is also <λ2, and vice versa. Likewise λ1 < λ2 if there is a rational length >λ1 but <λ2. This definition uses the rationals to give an infinitely sharp notion of length while avoiding any overt use of infinity. Of course the infinite set of rational lengths <λ is present in spirit, but Eudoxus avoids mentioning it by speaking of an arbitrary rational length <λ.
The theory of proportions was so successful that it delayed the development of a theory of real numbers for 2000 years. This was ironic,
because the theory of proportions can be used to define irrational numbers just as well as lengths. It was understandable though, because the common irrational lengths, such as the diagonal of the unit square, arise from
constructions that are intuitively clear and finite from the geometric point of view. Any arithmetic approach to sqrt(2), whether by sequences, decimals, or continued fractions, is infinite and therefore less intuitive. Until the 19th century this seemed a good reason for considering geometry to be a better foundation for mathematics than arithmetic. Then the problems of geometry came to a head, and mathematicians began to fear geometric in- tuition as much as they had previously feared infinity. There was a purge of geometric reasoning from the textbooks and industrious reconstruction of mathematics on the basis of numbers and sets of numbers. Set theory is discussed further in Chapter 24. Suffice to say, for the moment, that set theory depends on the acceptance of completed infinities."
The beauty of the theory of proportion was its adaptability to this new climate. Instead of rational lengths, take rational numbers. Instead of comparing existing irrational lengths by means of rational lengths, construct
irrational numbers from scratch using sets of rationals! The length sqrt(2) is determined by the two sets of positive rationals L√2 ={r:r2 <2}, U√2 ={r:r2 >2}. Dedekind (1872) decided to let sqrt(2) be this pair of sets! In general, let any
partition of the positive rationals into sets L, U such that any member of L is less than any member of U be a positive real number. This idea, now known as a Dedekind cut, is more than just a twist of Eudoxus; it gives a complete and uniform construction of all real numbers, or points on the line, using just the rationals. In short, it is an explanation of the continuous in terms of the discrete, finally resolving the fundamental conflict in Greek mathematics. Dedekind was understandably pleased with his achievement. He wrote
'The statement is so frequently made that the differential calcu- lus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given. . . . It then only remained to discover its true origin in the elements of arithmetic and thus at the same time secure a real definition of the essence of continuity. I succeeded Nov. 24 1858.' Dedekind (1872), p. 2"
(Stillwell, Mathematics and Its History). <Tag3>
Note by me: I found this while reflecting of the beautiful subtleties of cuts while studying them in the Zakon book. Per example, first defining the comparison of numbers in the completed field using set inclusion, which allows taking about supremums and infimums easily and comparing 'gap' numbers directly to rationals, and by the way showing that between any two different gaps there are rationals, that is subtly significant because it proves that a gap really defines exactly one 'new number' that 'fills it'. (Check: Another clever subtlety is using supremums only to define sums, etc.. while the direct naive approach would be to

define them as cuts, the result is surely the same, but the definition chosen saves some work and improves proof elegance.)

15. "Notice that “exhaustion” does not mean using an infinite sequence of steps to show that area is proportional to the square of the radius. Rather, one shows that any disproportionality can be refuted in a finite number of steps (by going to a suitable Pi). This is typical of the way in which exhaustion arguments avoid mention of limits and infinity." (Stillwell, Mathematics and Its History). <Tag4>
16. "During the 1830s, Hamilton and his colleagues Peacock, De Morgan, and John Graves pursued the idea of extending the concept of number. The ex- isting concept of number was already the result of a series of extensions— from natural and rational numbers to real and complex numbers—and Pea- cock observed that some principle of permanence was involved. It was tacitly agreed that certain properties of addition and multiplication should continue to hold with each extension of the number concept.
The “permanent” properties were not completely clear at the time, but most of them crystallized in the definition of a field given by Dedekind (1871). This concept had an independent origin, also around 1830, in the work of Galois on the theory of equations. So for convenience we start with the definition of a field and then explain its role in Hamilton’s search for an arithmetic of n-tuples." (Stillwell, Mathematics and Its History). <Origins>
17. "The concept of field was implicit in the work of Abel and Galois in the theory of equations, but it became explicit when Dedekind introduced number fields of finite degree as the setting for algebraic number theory. He saw that the ring of all algebraic integers is not a convenient ring, because it has no “primes.” This is because √α is an algebraic integer if α is, so there is always a nontrivial factorization α = √α √α in the ring of all algebraic integers. On the other hand, the algebraic integers in a field generated from a single algebraic number α of degree n,
.....
By drawing attention to the field Q(α) of degree n, Dedekind also brought to light some vector space structure: the basis 1,α,α2,...,αn−1 of Q[α], the linear independence of these basis elements over Q, and the dimension (equal to the degree) of Q[α] over Q. Despite the long history of linear algebra, dating back 2000 years in China at least, again it was the greater generality afforded by algebraic number theory that finally brought its fundamental concepts to light.
The next level of abstraction was reached in the 20th century and it was (in a new twist to Kronecker’s words) the work of a woman, Emmy Noether. In the 1920s she developed concepts for discussing common properties of different algebraic structures, such as groups and rings. One of the things groups and rings have in common is homomorphisms, or structure-preserving maps. A map φ : G → G′ is a homomorphism of groups if φ(gh) = φ(g)φ(h) for any g,h ∈ G. Similarly, a map φ : R → R′ is a homomorphism of rings if φ(r + s) = φ(r) + φ(s) and φ(rs) = φ(r)φ(s) for any r, s ∈ R. From this higher vantage point, normal subgroups (Sec- tion 19.2) and ideals can be seen as instances of the same concept. Each is the kernel of a homomorphism φ: the set of elements mapped by φ to the identity element (1 for a group, 0 for a ring)." (Stillwell, Mathematics and Its History). <Origins>
18. "David Hilbert (Figure 21.4) was born in 1862 in Ko ̈nigsberg and died in Go ̈ttingen in 1943. His father, Otto, was a judge, and David may have inherited his mathematical ability from his mother, about whom we know little except that her maiden name was Erdtmann. Ko ̈nigsberg was in the remote eastern part of Prussia (it is now Kaliningrad, a small, disconnected piece of Russia), but with a strong mathematical tradition dating back to Jacobi. When Hilbert attended university there in the 1880s he became friends with Hermann Minkowski, a former child mathematical prodigy two years his junior, and Adolf Hurwitz, who was three years older and a professor in Ko ̈nigsberg from 1884. The three used to discuss mathematics on long walks, and Hilbert seems to have picked up his basic mathemati- cal education in this way. In later life he made “mathematical walks” an important part of the education of his own students." (Stillwell, Mathematics and Its History).
19. "The invention of conic sections is attributed to Menaechmus (fourth century bce), a contemporary of Alexander the Great. Alexander is said to have asked Menaechmus for a crash course in geometry, but Menaechmus refused, saying, “There is no royal road to geometry.” Menaechmus used conic sections to give a very simple solution to the problem of duplicating the cube. In analytic notation, this can be described as finding the
intersection of the parabola y = 1 x2 with the hyperbola xy = 1. This yields 2 x1x2 =1 or x3 =2. 2
Although the Greeks accepted this as a “construction” for duplicating the cube, they apparently never discussed instruments for actually drawing conic sections. This is very puzzling since a natural generalization of the compass immediately suggests itself (Figure 2.8). The arm A is set at a fixed position relative to a plane P, while the other arm rotates about it at a fixed angle θ, generating a cone with A as its axis of symmetry. The pencil, which is free to slide in a sleeve on this second arm, traces the section of the cone lying in the plane P. According to Coolidge (1945), p. 149, this instrument for drawing conic sections was first described as late as 1000 ce by the Arab mathematician al-Kuji. Yet nearly all the theoretical facts one could wish to know about conic sections had already been worked out by Apollonius (around 250–200 bce)!"" (Stillwell, Mathematics and Its History). <Origins>
20. "Reasoning about infinity is one of the characteristic features of mathemat- ics as well as its main source of conflict. We saw, in Chapter 1, the conflict that arose from the discovery of irrationals, and in this chapter we shall see that the Greeks’ rejection of irrational numbers was just part of a gen- eral rejection of infinite

processes. In fact, until the late 19th century most mathematicians were reluctant to accept infinity as more than “potential.” The infinitude of a process, collection, or magnitude was understood as the possibility of its indefinite continuation, and no more—certainly not the possibility of eventual completion. For example, the natural numbers 1, 2, 3, . . ., can be accepted as a potential infinity—generated from 1 by the process of adding 1—without accepting that there is a completed totality {1, 2, 3, . . .}. The same applies to any sequence x1, x2, x3, . . . (of rational numbers, say), where xn+1 is obtained from xn by a definite rule.
And yet a beguiling possibility arises when xn tends to a limit x. If x is something we already accept—for geometric reasons, say—then it is very tempting to view x as somehow the “completion” of the sequence x1, x2, x3, ... It seems that the Greeks were afraid to draw such conclu- sions. According to tradition, they were frightened off by the paradoxes of Zeno, around 450 bce.
We know of Zeno’s arguments only through Aristotle, who quotes them in his Physics in order to refute them, and it is not clear what Zeno himself wished to achieve. Was there, for example, a tendency toward speculation about infinity that he disapproved of? His arguments are so extreme they could almost be parodies of loose arguments about infinity he heard among his contemporaries. Consider his first paradox, the dichotomy:
There is no motion because that which is moved must arrive at the middle (of its course) before it arrives at the end.
Aristotle, Physics, Book VI, '''Ch. 9
The full argument presumably is that before getting anywhere one must first get half way, and before that a quarter of the way, and before that one eighth of the way, ad infinitum. The completion of this infinite sequence of steps no longer seems impossible to most mathematicians, since it rep- resents nothing more than an infinite set of points within a finite interval. It must have frightened the Greeks though, because in all their proofs they were very careful to avoid completed infinities and limits.
The first mathematical processes we would recognize as infinite were probably devised by the Pythagoreans, for example, the recurrence rela- tions
xn+1 = xn + 2yn, yn+1 = xn + yn
for generating integer solutions of the equations x2 − 2y2 = ±1. We saw in Section 3.4 why it is likely that these relations arose from an attempt to understand sqrt(2) ,and it iseasy forus to see that xn/yn → sqrt(2) asn → ∞.
However, it is unlikely that the Pythagoreans would have viewed 2 as a “limit” or seen the sequence as a meaningful object at all. The most we can say is that, by stating a recurrence, the Pythagoreans implied a sequence with limit sqrt(2), but only a much later generation of mathematicians could accept the infinite sequence as such and appreciate its importance in
defining the limit.
In a problem where we would find it natural to reach a solution α by a limiting process, the Greeks would instead eliminate any solution but α. They would show that any number <α was too small and any number >α was too large to be the solution. In the following sections we shall study some examples of this style of proof and see how it ultimately bore fruit in the foundations of mathematics. As a method of finding solutions to problems, however, it was sterile: how does one guess the number α in the first place? When mathematicians returned to problems of finding limits in the 17th century, they found no use for the rigorous methods of the Greeks. The dubious 17th-century methods of infinitesimals were criticized by the Zeno of the time, Bishop Berkeley, but little was done to meet his objec- tions until much later, since infinitesimals did not seem to lead to incorrect results. It was Dedekind, Weierstrass, and others in the 19th century who eventually restored Greek standards of rigor.
The story of rigor lost and rigor regained took an amazing turn when a previously unknown manuscript of Archimedes, The Method, was dis- covered in 1906. In it he reveals that his deepest results were found using dubious infinitary arguments, and only later proved rigorously. Because, as he says, “It is of course easier to supply the proof when we have previously acquired some knowledge of the questions by the method, than it is to find it without any previous knowledge.”
The importance of this statement goes beyond its revelation that infinity can be used to discover results that are not initially accessible to logic.
Archimedes was probably the first mathematician candid enough to explain that there is a difference between the way theorems are discovered and the way they are proved." (Stillwell, Mathematics and Its History).

21. Note: I have noticed this a long time ago, but here it is from the mouth of Archimedes . Note that in this case method refers to infinitary method, and oh how this relates to Euler's work!
"“It is of course easier to supply the proof when we have previously acquired some knowledge of the questions by the method, than it is to find it without any previous knowledge.”" (Stillwell, Mathematics and Its History).
22. "This leads to an interesting observation: in the sex- agesimal (base 60) system, multiplying and dividing by 120 is analogous to multiplying and dividing by 20 in the decimal sys- tem: we simply multiply or divide by 2 and shift the point one place to the right or left, respectively. Thus equation (2) requires us to double the angle, look up the corresponding chord, and divide it by 2. To do this again and again becomes a chore, so it was only a matter of time before someone shortened this la- bor by tabulating half the chord as a function

of twice the angle, in other words our modern sine function.9 This task befell the Hindus." (Maor, Trigonometric Delights) <Origins>

23. "Moreover, if we compute the square root of each entry in column 4—that is, the ratio c/a = csc α—and then find the corresponding angle α, we discover that α increases steadily from just above 45◦ to 58◦. It thus seems that the author of our text was not only interested in finding Pythagorean triples, but also in determining the ratio c/a of the corresponding right triangles." (Maor, Trigonometric Delights) <AA>
24. "Plimpton 322 thus shows that the Babylonians were not only fa- miliar with the Pythagorean Theorem a thousand years before Pythagoras, but that they knew the rudiments of number theory and had the computational skills to put the theory into practice." (Maor, Trigonometric Delights) <AA>
25. "It is quite difficult to describe with certainty the beginning of trigonometry. . . . In general, one may say that the emphasis was placed first on astronomy, then shifted to spherical trigonometry, and finally moved on to plane trigonometry.
—Barnabas Hughes, Introduction to Regiomontanus'" (Maor, Trigonometric Delights)
26. "Mathematical documents from Ancient Egypt date back to 1900 B.C. The practical need to redraw field boundaries after the annual flooding of the Nile, and the fact that there was a small leisure class with time to think, helped to create a problem oriented, practical mathematics. A base-ten numeration system was able to handle positive whole numbers and some fractions. Algebra was developed only far enough to solve linear equations and, of course, calculate the volume of a pyramid. It is thought that only special cases of The Pythagorean Theorem were known; ropes knotted in the ratio 3:4:5 may have been used to construct right angles. " (http://fclass.vaniercollege.qc.ca/web/mathematics/about/history.htm) <Origins>
Note: Maybe it is this special case, and further ones, that then motivated the idea of a theorem? if that is the case it all starts with a 3:4:5 right triangle.
27. "We utilize furthermore the fact that the dimensions of our planetary system are so minute in comparison with the distances to the fixed stars which constitute the background of the celestial sphere that we commit no observable error at all if we keep either the sun or the earth in a fixed position with respect to the surrounding universe. Hence we will proceed in the following way.We shall start with the circular motion of the planets around the sun and then keep the earth fixed and ask for the resulting motion with respect to the earth. This will answer our question concerning the planetary phenomena." (Neugebauer. The Exact Sciences in Antiquity)
28. "All these effects act independently of each other and cause quite irregular patterns in the variation of the length of lunar months. It is one of the most brilliant achievements in the exact sciences of antiquity to have recognized the independence of these influences and to develop a theory which permits the pre- diction of their combined effects. Epping, Kugler, and Schaum- berger have indeed demonstrated that the lunar ephemerides of the Seleucid period follow in all essential steps the above outlined analysis." (Neugebauer. The Exact Sciences in Antiquity) <AA>
29. "47. The fundamental problem of the Babylonian lunar theory is determined by the calendar. So far as we know, the Babylonian calendar was at all periods truly lunar, that is to say, the "month" began with the evening when the new crescent was for the first time again visible shortly after sunset. Consequently the Ba- bylonian "day" also begins in the evening and the "first" of a month is the day of the first visibility. In this way the beginn- ing of a month is made dependent upon a natural phenome- non which is amenable to direct observation. This is certainly a very simple and natural definition, as simple as the concur- rent definition of the "day" as the time from one sunset to the nex.t. But as is often the case, a "natural" definition leads to exceedingly complicated problems as soon as one wishes to predict its consequences. This fact is drastically demonstrated in the case of the lunar months. A very short analysis will illus- trate the intrinsic difficulties." (Neugebauer. The Exact Sciences in Antiquity)
30. "Before describing the Babylonian planetary theory, we shall discuss the main features of the apparent movement of the planets from a modern point of view. We know that the planets move on ellipses around the sun, the earth being one of them. We shall derive from these facts the apparent motions as seen from the earth. In order to simplify our discussion, we shall replace all orbits by circles whose common center is the sun. The eccenbicities of the elliptic orbits are so small that a seale drawing that would fit this page would not show the difference between the elliptic and the circular orbits.
We utilize furthermore the fact that the dimensions of our planetary system are so minute in comparison with the distances to the fixed stars which constitute the background of the celestial sphere that we commit no observable error at all if we keep either the sun or the earth in a fixed position with respect to the surrounding universe. Hence we will proceed in the following way.We shall start with the circular motion of the planets around the sun and then keep the earth fixed and ask for the resulting motion with respect to the earth. This will answer our question concerning the planetary phenomena.
The first step is absolutely trivial. We know that the earth is a satellite of the sun, moving around it once in a year. In order to obtain the appearances seen from the earth we subtract from all motions the motion of the earth. Thus we see that by arresting the motion of the earth we obtain the appearance that the sun moves around the earth once per year. Its apparent path is called the ecliptic (cf. Fig. 12a and b).
Secondly we consider an "inner" planet. Mercury or Venus. which moves closer to the sun than the earth (Fig. 13a). If we stop the earth we need only repeat Fig. 12 in order to obtain again the motion of the sun. The orbit of the planet remains a circle with the sun in its center. Hence the geocentric description of the motion of an inner planet is given by a planet which moves on a little circle whose center is carried on a

larger circle whose center is the earth. The little circle is called an "epicycle". the large circle is the "deferent".

Finally we have an "outer" planet. Mars. Jupiter. or Saturn. whose orbit encloses the orbit of the earth (Fig. 14a). From the earth E the planet P appears to be moving on a circle whose center S moves around E. Thus we have again an epicyclic motion (Fig. 14b). In order to establish a closer similarity with the ease of the inner planets we introduce a point C such that the four points S. E. p. and C always form a parallelogram. SP is the radius of the planetary orbit; because EC = SP we see that C lies on a circle with center E. Similarly ES is the radius of the solar orbit, and, because ES = CP, we see that P lies on a circle around C. Thus the planet P moves on an epicycle whose center C travels on a deferent whose center is E (Fig. 14c). Thus we have established an exact analogue to the case of the inner planets. In both cases the planet has an epicyclic move- ment. In the case of the inner planets the center of the epicycle coincides with the sun. For the outer planets the center C of the epicycle moves around E with the same angular velocity as the planet moves around the sun, while the planet P moves on the epicycle around C with the same angular velocity as the sun moves around the earth." (Neugebauer. The Exact Sciences in Antiquity)

31. "For a finer theory of the planetary phenomena the above assumptions are too crude. It is easy, however, to see in what directions one should move in order to reach higher accuracy. The eccentricity of the orbits can be taken into consideration by assuming slightly eccentric positions of the earth with respect to the centers of the deferents. The latitude can be accounted for by giving the epicycles the proper inclination. Both devices were followed by the Greek astronomers." (Neugebauer. The Exact Sciences in Antiquity) <AA>
32. "The Babylonian method follows the exactly opposite arrange- ment. The f11"st goal consists in determining the "Greek-letter phenomena", and thereafter the longitude of the planet for an arbitrary moment t is found by interpolation." (Neugebauer. The Exact Sciences in Antiquity) <Origins>
33. "This difference in approach is, of course, the result of the historical development. The Babylonians were primarily in- terested in the appearance and disappearance of the planets in analogy to the first and last visibility of fixed stars-e. g. Sirius- and of the moon. It was the periodic recurrence of these phenom- ena and their fluctuations which they primarily attempted to determine. When Ptolemy developed his planetary theory, he had already at his disposal the geometrical methods by means of which the solar and lunar anomalies were explained very satis- factorily, and similar models had been used also for an at least qualitative explanation of the apparent planetary orbits. Thus it had become an obvious goal of theoretical astronomy to otter a strictly geometrical theory of the planetary motions as a whole and the characteristic phenomena lost much of their specific interest, especially after the Greek astronomers had developed enough observational experience to realize that horizon phenom- ena were the worst possible choice to provide the necessary empirical data." (Neugebauer. The Exact Sciences in Antiquity)
34. "57. Whatever phenomenon the Babylonian astronomers wanted to predict. it had to be determined within the existing lunar calendar. Suppose one had found that a planet would reappear 100 days from a given date. What date should be assigned to this moment? Obviously one should know whether the three intermediate lunar months were. perhaps. all only 29 days long. or all three were 30 days long, etc. This question could be an- swered perfectly well by lunar ephemerides whose goal it was to determine whether a given month was 29 or 30 days long. But planetary phenomena proceed very slowly. One single table for Jupiter or Saturn could easily cover 60 years and more. To determine calendar dates so far in advance would have meant the computation of complete lunar ephemerides for several decades. Furthermore. the actual computation of the planetary motion had to be based. in any ease. on a uniform time seale. All these difficulties were at once overcome by a very clever device. One used as unit of time the mean synodic month and
divided it in 30 equal parts. The Babylonians seem not to have had a special name for these units. referring to them simply as udays". Modern scholars have used the term "lunar days"; I shall use the corresponding term of Hindu astronomy. namely. utithi" .
The fact that the Babylonian calendar was a strictly lunar calendar has the effect that the total duration of a number of calendar months will not deviate more and more from the cor- responding total of mean SYnodic months. Dates expressed in tithis will never be far off from real calendar days. usually not more than ±1 day. Thus the Babylonian astronomers in their computations simply identified the results given in tithis with the dates in the real calendar. This is the standard procedure for all planetary texts.
The use of tithis implies that one did not try to reach. for the planetary phenomena. the same accuracy which was obtained in the lunar theory. While one went to great lengths to determine all possible influences upon the first and last visibility of the moon. we find no similar devices used for the planetary phe- nomena. The latitude of the planets. for instance. is nowhere taken into consideration for the planetary ephemerides." (Neugebauer. The Exact Sciences in Antiquity) <AA>
35. "The structure of our planetary system is indeed such that Rheticus could say "the planets show again and again all the phenomena which God desired to be seen from the earth". The investigations of Hill and Poincare have demonstrated that only slightly different initial conditions would have caused the moon to travel around the earth in a curve of the general shape given in Fig. 23(Jad: A figure of 8), and with a speed exceedingly low in the outer- most quadratures 01 and O. as compared with the motion at new and full moon. Nobody would have had the idea that the moon could rotate on a circle around the earth and all philosophers would have declared it as a logical necessity that a moon shows six half moons between two full moons. And what could have happened with our concepts of time if we were members of a double-star

system (perhaps with some uneven distribution of mass in our little satellite) is something that may be left to the imagination.

Actually, however, the initial conditions of our planetary system were chosen in such a way that all the satellites of the sun-and our own satellite as well-behave with great modesty. Their orbits can be closely approximated by circles such that the simplest possible model of a circular motion with constant speed leads immediately to very reasonable results for the descrip- tion of the solar and lunar phenomena. On the other hand, the deviations from the trivial circular orbits are just great enough to be observed and to challenge an explanation, but small enough such that again comparatively simple modifications of the trivial solution give satisfactory results. The successive approximations

of the Babylonian lunar and planetary theory reflect this situation perfectly. At the basis lies the counting of the periodically recurrent phenomena; the properly chosen periodic functions-zigzag functions or step functions-suffice to describe the deviation from a trivial mean motion." (Neugebauer. The Exact Sciences in Antiquity) <Origins>

36. "Perhaps a little before these methods were developed in Meso- potamia, perhaps almost simultaneously, a most decisive step in another direction was made by Eudoxus. The then recent discovery of the sphericity of the earth must have suggested a corresponding sphericity of the sky and a circular motion of the celestial bodies. Eudoxus's theory may have well started from the following consideration. The motion of sun and moon can be described as the combination of the uniform motions of two concentric spheres: one is the fast daily rotation about the poles of the equator; the other is slow and proceeds in opposite direction about an inclined axis which is perpendicular to the ecliptic. Eudoxus saw that a similar combination is capable of explaining, at least qualitatively, also the most striking phenomenon of planetary motion, the retrogradations. The motion of two homo- centric spheres allows of two trivial limiting types. I f the two axes are made to coincide, the body which is fixed to the equator of one of the two spheres moves simply in a circle with the difference velocity. If, secondly, the two opposite velocities are made of equal amount the body remains stationary as seen from the center.

    But one case remains to be investigated: what motion results from equal opposite velocities but inclined axes? Eudoxus found that the orbit is an 8-shaped curve (Fig. 24). Now one can superimpose a third rotation about an axis which is perpendicular to the plane of sYmmetry which represents the plane of the ecliptic. Conse- quently, the point P no longer follows a closed curve but proceeds with a certain mean velocity in the ecliptic. Simultaneously, however, there appears a periodic deviation from the ecliptic, or a motion in latitude. Finally, one will obtain retrogradations if the longitudinal component is less than the backward motion in the original figure eight. Thus it is demonstrated that, at least qualitatively, even the apparent irregularities of planetary motion can be described by a combination of circular motions of uniform angular velocity.

    In spite of the great importance, in principle, of the discovery of Eudoxus, it is quite obvious that a model of this type has grave shortcomings." (Neugebauer. The Exact Sciences in Antiquity) <Origins>
37. "For example, the observed retrogradations of the planets do not recur in curves of identical shape as would be the case in the Eudoxian model. Another difficulty lies in the large variation of the brightness of planets. that seemed to indicate corresponding variations in their distance from the earth. We do not know who first succeeded in explaining these and similar anomalies by means of a much more flexible modification of the theory of uniform circular motion. We know. however. that Apollonius (about 200 B.C.) used the simple device of viewing uniform circular motion. not from the center of the orbit. but from a slightly eccentric point. This obviously has the effect that the motion appears fastest where the circle is nearest to the observer and slowest at the opposite point. But Apollonius proved more. He demonstrated that an eccentric movement of this type can always be replaced by an epicyclic motion where the center of the epicycle moves on a circle with the observer at its center and with a radius of the epicycle equal to the eccentricity (cf. Fig. 25)." (Neugebauer. The Exact Sciences in Antiquity)
38. "Writings of the type oef Heron's "Geometry" were undoubtedly widespread in antiquity and formed the backbone of instruction in elementary mathematics. This explains the relatively large number of papyrus fragments containing such texts. As an example can be shown an unpublished papyrus of the CorneD CoDection (d. Pl. 12). The figure in the lower part of the right column is a typical example of the building up of a more complicated example from the simplest cases (cf. Fig. 28).

    It is precisely from the construction of such examples that we can demonstrate direct relations between geometrical treatises. For example, the concept of isosceles triangle is illustrated by a triangle of side 10 in Heron, Metriea I, XVII (opera III p. 48, 49); in Heron, Geometrica 10 (opera IV, p. 224, 225); al-Khwlrizmi, Algebra (Rosen p. 80); al-Blroni, Astrology 22 (Wright). The general triangle of sides 13, 14, and 15 is used by Heron, Metrica I, V and VIII; Geometriea 12; Mishnat ha-Middot 9 (Gandz p. 46); Mahlvlra VII, 53 (ed. Rangacharya p. 199); al-Khwlrizmi p. 82; Bhaseara, Ldlvati VI, 165, 168 (Colebrooke p. 71, 73). In fact this triangle is composed of two Pythagorean triangles of sides 13, 5, 12, and 15,9, 12 respectively. They occur again in two problems of the Pap. Ayer (Am. J. of Philology 19, 1898, p. 25 ft.)." (Neugebauer. The Exact Sciences in Antiquity)
39. "The historical sequence of these discoeries seems to be as follows. Since Old-Babylonian times the knowledge of solving the main types of quadratic equations existed. The discovery of irrational quantities led to the geometrization of these methods in the form of application of areas (4th century B.C.). Shortly afterwards the conic sections were discovered, as I think, from the investigation of sun dials (cf. p. 226). At

any rate the conic sections were at first considered as curves in space, unrelated to algebraic problems. Finally the relation to the application of areas was established. as found in Apollonius (3rd century B.C.). Figures which illustrate the configurations in space from which the relations between the plane areas were derived are given in Quellen und Studien zur Gesehiehte der Mathematik, Aht. B vol. 2 (1932) p. 220 f." (Neugebauer. The Exact Sciences in Antiquity) <Origins>

40. "ad 62. The theory of "applieation of areas" attained great importanee in ancient mathematies beeause of the diseovery that the eonie seetions could be incorporated in this theory. Indeed, our modem names ellipse, hyperbola, and parabola are direetly taken from this theory. The ease of the ellipse might be quoted as an example. Assume as given two "coordinated" directions (from which our use of the word "coordinates"), here denoted as the tE- and II-direetion (Fig. 29). Let E and 11 be two given parameters," (Neugebauer. The Exact Sciences in Antiquity) <Origins>
41. "It is also generally accepted that the essential turn in the development came about through the discussion of the conse- quences of the arithmetical fact that no ratio of two integers could be found such that its square had the value 2. The geometrical corollary that the diagonal of a square could not be "measured" by its side obviously caused serious discussion about the relation between geometrical and arithmetical proof. The "paradoxa" concerning continuity, both of space and time, made the relation to the whole problem of determination of area and volume evident. One way out might have been the assumption of a somewhat atomistic structure of geometrical objects by means of which the problem of area or volume would have been reduced, though in not too clear a fashion, to a counting of discrete elements, "atoms".
The reaction of the mathematicians against this type of specula- tion seems to have led to two major steps. First of all, one had to agree exactly on a system of basic assumptions from which alone the rest had to be deduced; this gave rise to the strictly axiomatic procedure. Secondly, it had become clear that one should consider the geometrical objects as the given entities such that the case of integer ratios appeared as a special case of only secondary interest; this led to the problem of how to formulate classical arithmetical and algebraic knowledge in geometrical language. The result is the familiar ·'geometrical algebra" of Greek mathematics. It is these two essential steps which are fully to the credit of the Greek mathematicians." (Neugebauer. The Exact Sciences in Antiquity) <Origins>
42. "The theory described so far was known to Hipparchus though refined in several respects by Ptolemy. The determination of the essential parameters was based on carefully selected observations of lunar eclipses and the results obtained were very satisfactory for the description of eclipses in general. Ptolemy, however, realized from a masterful analysis of observations of his own and of his predecessors that marked deviations from the predicted longitudes of the moon, reaching a maximal amount in the neighborhood of elongations of ± 90° from the sun, could occur. In other words he realized that the traditional lunar theory agreed with observations in the syzygies (conjunctions and oppositions) but could not explain longitudes near the quadratures, particularly for values of the anomaly" near ±90°. In these cases the diameter of the epicycle seemed to be enlarged over the value found at the syzygies.
The procedure which Ptolemy followed to cope with this situation is of interest in many respects. It provides us with a good insight into the mathematical and astronomical methodology of the time; the attitude toward a glaring defect of the theory is very revealing and has repercussions in Islamic astronomy and in the work of Copernicus; finally the method by which this inequality of the lunar motion was accounted for influenced also the planetary theory, both of Ptolemy and Copernicus.
As we have said above the observations suggested a dependence of the apparent diameter of the epicycle of the moon on the elongation from the sun. Such an effect could be produced by bringing the epicycle closer to the observer by the following mechanism (Fig. 34). Let Co be the position of the center of the lunar epicycle at conjunction with the mean sun such that ..." (Neugebauer. The Exact Sciences in Antiquity) <AA>
43. "Ptolemy concludes from these two inequalities, "And so, since it has been proved that the chord of an arc of 1 degree is both greater than and less than the same number of parts, clearly we shall have crd 1 degree = 1, 2' 50''; and by means of earlier proofs we get crd 1/2 degree = 0, 31' 25''." Toeplitz remarks [6] that Ptolemy's bland assertion of the absurdity "both greater than and less than" expresses a very modern viewpoint." (Brendan, How Ptolemy Constructed Trigonometry Tables) <AA>
44. "The most ambitious and complex of Ptolemy's instruments was the armillary sphere.
This comprised a set of seven concentric rings rotating around different axes, with the poles of each one anchored in its enclosing ring. The outermost ring was set parallel to the meridian (i.e. in a geographical north-south plane). The next could then be set to the celestial pole - that is, the pole around which the stars seem to move daily as viewed from the Earth. It would then correspond to the latitude of the place where the instrument was being used. A further ring was fixed to this such that its pole reproduced the obliquity of the ecliptic. Four inner rings could then be used to track the distances between particular planets and stars along and perpendicular to the ecliptic. Such angular separations could be read directly from the instrument.
Ptolemy also knew how to use an astrolabe, although he did not describe one in the Almagest. This immensely useful instrument was made and employed for centuries. It was essentially a plane projection of the heavens, adjustable for different latitudes. Once its "rete" (the skeletal disc pointing to stellar positions) had been set for one star's height, the time and other results could be read from the instrument. " (http://microcosmos.uchicago.edu/microcosmos_new/ptolemy/instruments.html) <AA>

45. "Among the simplest of these, and perhaps the most influential, was the device known as "Ptolemy's rulers." It was originally stipulated for measuring the angle between the Moon and a point directly overhead, i.e. the Moon's zenith distance.
A plumbline was used to set up a vertical column with a linear scale. One rigid arm was attached by a pivot to the bottom of the column, another similarly to the top. Two sights (e.g., fine slits) are fixed to the ends of the latter. The length of the upper arm equal to the distance along the column between the two pivots is marked off. The upper arm is then aligned to the Moon, and the lower set to cross it at the marked point. The distance along the lower arm can then be read from a scale and readily converted into the zenith angle." (http://microcosmos.uchicago.edu/microcosmos_new/ptolemy/instruments.html)
46. "A vertical line at any point can generally be determined by suspending a plumb line, which is a weight attached to the end of a string, and allowing it to come to rest. The pull of the earth, or gravity, as we call it, will stop the swinging of the plumb line. If we could see the line of the pull of the earth we should see that it passes through the motionless bob, the string and the support from which the string hangs. As the string is in the same line as the earth's pull, we say it hangs vertically.

    At some places the plumb line does not hang quite vertically. Where this is so we know that it is caused by the action of some other force, such as the attraction of a great mass like a mountain, or by the gravitational pull of the moon, or by the centrifugal force of the earth's rotation. Dr. Xcvii Maskelvne, a British astronomer, found that when he suspended two plumb lines near a mountain, one on the north and the other on the south, the angle between their directions was greater than the angle between two vertical plumb lines should be. On measuring he found that each plumb line was pulled slightly toward the mountain.

    We can understand this; but in India a very strange thing happens. When a plumb line is suspended in the southern regions, it hangs quite vertically; but when taken north, it is pulled, not toward the Himalaya Mountains, but away from the massive mountains, toward the southern plain.

    This behavior of the plumb line, so different from what we should expect, is due to the fact that the weight of the great table- land of southern India. and the material lying beneath it, is greater than the weight of the Himalaya Mountains and the material below them. The heavy plain attracts the bob of the plumb line away from the lighter mountains, massive though they appear to the eye. The unseen attraction is greater than the visible attraction." (http://www.chickjunk.com/does-a-plumb-line-always-hang-straight/) <IKIT>
47. "The deflections reflect the undulation of the geoid and gravity anomalies, for they depend on the gravity field and its inhomogeneities.

    VDs are usually determined astronomically. The true zenith is observed astronomically with respect to the stars, and the ellipsoidal zenith (theoretical vertical) by geodetic network computation, which always takes place on a reference ellipsoid. Additionally, the very local variations of the VD can be computed from gravimetric survey data and by means of digital terrain models (DTM), using a theory originally developed by Vening-Meinesz.

    VDs are used in astro-geodetic levelling, a geoid determination technique. As a vertical deflection describes the difference between the geoidal and ellipsoidal normals, it represents the horizontal gradient of the undulations of the geoid (i.e., the separation between geoid and reference ellipsoid). Given a starting value for the geoid undulation at one point, determining geoid undulations for an area becomes a matter for simple integration.

    In practice, the deflections are observed at special points with spacings of 20 or 50 kilometers. The densification is done by a combination of DTM models and areal gravimetry. Precise VD observations have accuracies of ±0.2″ (on high mountains ±0.5″), calculated values of about 1–2″.

    The maximal VD of Central Europe seems to be a point near the Großglockner (3,798 m), the highest peak of the Austrian Alps. The approx. values are ξ = +50″ and η = −30″. In the Himalaya region, very asymmetric peaks may have VDs up to 100″ (0.03°). In the rather flat area between Vienna and Hungary the values are less than 15", but scatter by ±10″ for irregular rock densities in the subsurface." (http://en.wikipedia.org/wiki/Vertical_direction) <IKIT>
48. "My attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first pre- sentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feel- ing of dissatisfaction was so overpowering that I made the

fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis." (Dedekind, Continuity and Irrational Numbers) <Origins>

49. "The development of the arithmetic of rational numbers is here presupposed, but still I think it worth while to call attention to certain important matters without discussion, so as to show at the outset the standpoint assumed in what follows. I regard the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetic act, that of counting, and counting it- self as nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already-formed individual to the con- secutive new one to be formed. The chain of these numbers forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. Addition is the combination of any arbitrary repeti- tions of the above-mentioned simplest act into a single act; from it in a similar way arises multiplication. While the performance of these two operations is always possible, that of the inverse operations, subtraction and division, proves to be limited. Whatever the immediate occasion may have been, whatever com- parisons or analogies with experience, or intuition, may have led thereto; it is certainly true that just this limitation in performing the indirect operations has in each case been the real motive for a new creative act; thus negative and fractional numbers have been created by the human mind; and in the system of all rational numbers there has been gained an instrument of infinitely greater perfection. This system, which I shall denote by R, possesses first of all a com- pleteness and self-containedness which I have designated in another place1 as characteristic of a body of numbers [Zahlk ̈orper] and which consists in this that the four fundamental operations are always performable with any two individu- als in R, i. e., the result is always an individual of R, the single case of division by the number zero being excepted." (Dedekind, Contibuity and Irrational Numbers)
50. "This analogy between rational numbers and the points of a straight line, as is well known, becomes a real correspondence when we select upon the straight line a definite origin or zero-point o and a definite unit of length for the mea- surement of segments. With the aid of the latter to every rational number a a corresponding length can be constructed and if we lay this off upon the straight line to the right or left of o according as a is positive or negative, we obtain a definite end-point p, which may be regarded as the point corresponding to the number a; to the rational number zero corresponds the point o. In this way to every rational number a, i. e., to every individual in R, corresponds one and only one point p, i. e., an individual in L. To the two numbers a, b respectively correspond the two points, p, q, and if a > b, then p lies to the right of q. To the laws i, ii, iii of the previous Section correspond completely the laws i, ii, iii of the present." (Dedekind, Continuity and Irrational Numbers)
51. "Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number. If the point p corresponds to the rational number a, then, as is well known, the length op is commensurable with the invariable unit of measure used in the construction, i. e., there exists a third length, a so-called common measure, of which these two lengths are integral multiples. But the ancient Greeks already knew and had demonstrated that there are lengths incommensurable with a given unit of length, e. g., the diagonal of the square whose side is the unit of length. If we lay off such a length from the point o upon the line we obtain an end-point which corresponds to no rational number. Since further it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length, we may affirm: The straight line L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals.
If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument R constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line.
The previous considerations are so familiar and well known to all that many will regard their repetition quite superfluous. Still I regarded this recapitulation as necessary to prepare properly for the main question. For, the way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitudes—which itself is nowhere carefully defined—and explains number as the result of measuring such a magnitude by another of the same" (Dedekind, Continuity and Irrational Numbers)
52. "The above comparison of the domain R of rational numbers with a straight
line has led to the recognition of the existence of gaps, of a certain incom- pleteness or discontinuity of the former, while we ascribe to the straight line completeness, absence of gaps, or continuity. In what then does this continu- ity consist? Everything must depend on the answer to this question, and only through it shall we obtain a scientific basis for the investigation of all continu- ous domains. By vague remarks upon the unbroken connection in the smallest parts obviously nothing is gained; the problem is to indicate a precise charac- teristic of continuity that can serve as the basis for valid deductions. For a long time I pondered over this in vain, but finally I found what I was seeking. This discovery will, perhaps, be differently estimated by different people; the majority may find its substance very commonplace. It consists of the following.
In the preceding section attention was called to the fact that every point p of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of

the other. I find the essence of continuity in the converse, i. e., in the following principle:
"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."
As already said I think I shall not err in assuming that every one will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that by this commonplace remark the secret of continuity is to be revealed. To this I may say that I am glad if every one finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power." (Dedekind, Continuity and Irrational Numbers) <Origins>

53. "The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there" (Dedekind, Continuity and Irrational Numbers) <Origins>
54. "In Section I it was pointed out that every rational number a effects a separation of the system R into two classes such that every number a1 of the first class A1 is less than every number a2 of the second class A2; the number a is either the greatest number of the class A1 or the least number of the class A2. If now any separation of the system R into two classes A1, A2 is given which possesses only this characteristic property that every number a1 in A1 is less than every number a2 in A2, then for brevity we shall call such a separation a cut [Schnitt] and designate it by (A1,A2). We can then say that every rational number a produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different; this cut possesses, besides, the property that either among the numbers of the first class there exists a greatest or among the numbers of the second class a least number. And conversely, if a cut possesses this property, then it is produced by this greatest or least rational number." (Dedekind, Continuity and Irrational Numbers) <Origins>
55. "Just as addition is defined, so can the other operations of the so-called elementary arithmetic be defined, viz., the formation of differences, products, quotients, powers, roots, logarithms, and in this way we arrive at real proofs of theorems (as, e. g., $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before. The excessive length that is to be feared in the definitions of the more complicated operations is partly inherent in the nature of the subject but can for the most part be avoided." (Dedekind, Continuity and Irrational Numbers)
56. "A system of continuous-time differential equations may have various discrete-time difference approximations with different time-step τ. Each of them has different dynamic properties. It has been found [2, 3] that computed results given by some
discrete-time difference schemes are parasitic, which have no physical meanings at all.
For example, when the exact time-dependent solution of a set of nonlinear differen-tial equations is known to be periodic, there is sometimes a range of the time-step τ where the computed solution of the finite difference equations is chaotic [3,9]. This kind of nonphysical parasitic solutions is called computational chaos (CC) [9]. By contraries, when the exact solution is known to be chaotic, computed solutions are however periodic within a range of time-step τ, and this numerical phenomenon is called computational periodicity (CP) [10]. So, computed dynamic behaviors observed for a finite time-step in some nonlinear discrete-time difference equations sometimes
have nothing to do with the exact solution of the original continuous-time differential equations at all, as pointed out by many researchers [2,3,9,10,18].
" (Liao, On the reliability of computed chaotic solutions of nonlinear differential equations)
57. "Still lengthier considerations seem to loom up when we attempt to adapt the numerous theorems of the arithmetic of rational numbers (as, e. g., the theorem $(a + b)c = ac + bc$) to any real numbers. This, however, is not the case. It is easy to see that it all reduces to showing that the arithmetic operations possess a certain continuity. What I mean by this statement may be expressed in the form of a general theorem:
"If the number λ is the result of an operation performed on the numbers α, β, γ, ... and λ lies within the interval L, then intervals A, B, C, ... can be taken within which lie the numbers α, β, γ, . . . such that the result of the same operation in which the numbers α, β, γ, . . . are replaced by arbitrary numbers of the intervals A, B, C, ... is always a number lying within the interval L." The forbidding clumsiness, however, which marks the statement of such a theorem convinces us that something must be brought in as an aid to expression; this is, in fact, attained in the most satisfactory way by introducing the ideas of variable magnitudes, functions, limiting values, and it would be best to base the definitions of even the simplest arithmetic operations upon these ideas, a matter which, however, cannot be carried further here."
(Dedekind, Continuity and Irrational Numbers) <Origins>
58. "In 1874, while on holiday in Interlaken, Dedekind met Cantor. Thus began an enduring relationship of mutual respect, and Dedekind became one of the very first mathematicians to admire Cantor's work on infinite sets, proving a valued ally in Cantor's battles with Kronecker, who was philosophically opposed to Cantor's transfinite numbers[citation needed].

If there existed a one-to-one correspondence between two sets, Dedekind said that the two sets were "similar." He invoked similarity to give the first precise definition of an infinite set: a set is infinite when it is "similar to a proper part of itself," in modern terminology, is equinumerous to one of its proper subsets.

(This[clarification needed] is known as Dedekind's theorem.[citation needed]) Thus the set N of natural numbers can be shown to be similar to the subset of N whose members are the squares of every member of N, (N → N2): ... " (http://en.wikipedia.org/wiki/Richard_Dedekind) <Origins>

59. "1879 and 1894 editions of the Vorlesungen included supplements introducing the notion of an ideal, fundamental to ring theory. (The word "Ring", introduced later by Hilbert, does not appear in Dedekind's work.) Dedekind defined an ideal as a subset of a set of numbers, composed of algebraic integers that satisfy polynomial equations with integer coefficients. The concept underwent further development in the hands of Hilbert and, especially, of Emmy Noether. Ideals generalize Ernst Eduard Kummer's ideal numbers, devised as part of Kummer's 1843 attempt to prove Fermat's Last Theorem. (Thus Dedekind can be said to have been Kummer's most important disciple.) In an 1882 article, Dedekind and Heinrich Martin Weber applied ideals to Riemann surfaces, giving an algebraic proof of the Riemann–Roch theorem." (http://en.wikipedia.org/wiki/Richard_Dedekind) <Origins>
60. "In 1888, he published a short monograph titled *Was sind und was sollen die Zahlen?* ("What are numbers and what should they be?" Ewald 1996: 790),[2] which included his definition of an infinite set. He also proposed an axiomatic foundation for the natural numbers, whose primitive notions were one and the successor function. The following year, Peano, citing Dedekind, formulated an equivalent but simpler set of axioms, now the standard ones." (http://en.wikipedia.org/wiki/Richard_Dedekind) <Origins>
61. "Representational Fluency. A trait I was not able to pinpoint exactly myself although I was circling around it: “One reason for this strong emphasis is that, according to Lesh (2000, p. 74), the idea of representational fluency is “at the heart of what it means to ‘understand’ many of the more important underlying mathematical constructs”. (Stewart - Thomas, Eigenvalues and Eigenvectors: Formal, Symbolic and Embodied Thinking)
62. "Authority -- Positive and negative: The first aspect of identity. One important aspect of the identities of authors as projected by their texts is the extent to which and the manner in which they claim to be authoritative within their community. If an author appears too tentative in his or her claims, less value might be placed on the results, whereas if one appears inappropriately self-assured, a reader might question the author's right to be so certain and even dismiss the work. Terms such as "clearly" and "obvious" are relative to the individuals using them (implying that this information is obvious to me but may not be so obvious to you). Our interpretation that the use of such terms serves as a claim to authority on the part of the writer (implying that this derivation is clear to me and I do not need to explain it further because if it is not clear to you, that is your fault not mine) is reinforced by the interview data, as demonstrated in some of the earlier quotes. We believe that, whatever the author's intent, the extent or absence of such words is one of the interpersonal aspects of the writing that will influence the ways in which the readers of the text will construct an image of the author and will consequently judge the worth of the text itself.

    This section exhibits a curious switch that makes it hard to argue with. It begins by attributing an arrogant attitude to the user of "clearly", and describes that attitude twice, in parenthetical imagined monologue, but ends only by saying that "whatever the author's intent" might be, the use of such words will generate an image of some sort in the reader, only an image of the author, not necessarily the real thing. Is the author who uses "clearly" parading his authority or is he merely subjecting his prose to misinterpretation of his motives? Burton and Morgan get away with claiming both, I suppose.

    I cannot deny the latter reading, any more than I can deny that some circle-squarers consider my denial of their discovery a defensive act of authority. But I will nonetheless deny that "clearly" is used to put down the readers. I ask my own readers here to find, if they can, samples of articles in the Proceedings of the American Mathematical Society (say) exhibiting such behavior. It is hard for me to believe that the writers of this analysis of the prose of mathematical research have had much experience reading any of it for its mathematical content – or not successfully, at any rate. I must explain to them that "clearly" is an abbreviation for some statements that would take up unnecessary space, and perhaps, even probably, interrupt the exposition in such a way as to make it harder (not easier) to read and understand. Further, that the author who uses "clearly" generally knows the statement is not "clear" to everyone, not to one person in a million actually, for it only needs to be clear to those who have a good chance at understanding his piece at all." (http://www.math.rochester.edu/people/faculty/rarm/jrme0.html)
63. Note of interest is the usage of 'reverses' in 3.
    "While it may be argued that these developments are simply different modes of thinking that grow in sophistication, I have come to describe them as ‘three worlds of mathematics’ that develop in sophistication in quite different ways.
    1. the conceptual-embodied world, based on perception of and reflection on properties of objects, initially seen and sensed in the real world but then imagined in the mind;
    2. the proceptual-symbolic world that grows out of the embodied world through action (such as counting) and is symbolised as thinkable concepts (such as number)
    that function both as processes to do and concepts to think about (procepts);
    3. the axiomatic-formal world (based on formal definitions and proof), which reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set-theoretic definitions."" (Tall, The Transition to Formal Thinking in Mathematics)

64. "Formal theories based on axioms often lead to structure theorems, which reveal that an axiomatic system (such as a vector space) has a more sophisticated embodiment and related symbolism—for instance a finite dimensional vector space is an n-dimensional coordinate system. In this way the theoretical framework turns full circle, building from embodiment and symbolism to formalism, returning once more to a more sophisticated form of embodiment and symbolism that, in turn, gives new ways of conceiving even more sophisticated mathematics." (Tall, The Transition to Formal Thinking in Mathematics)
65. "The study of the development of mathematical thinking is aided by several theoretical concepts to support our analysis. The human brain is highly sophisticated, but it is also surprisingly limited, being able to deal with only a small number of pieces of information at a time. In his famous paper, Miller (1956) suggested the number is around 7±2, based on a review of many articles published at the time. Personally I feel that it is much smaller than this; perhaps I could cope with more when I was younger – but I can't remember.
The human brain copes with this by connecting ideas together into 'thinkable concepts'. (Although all concepts are clearly thinkable, I use the two words together to focus on how the concept is held in the mind as a single entity at a single time.)
Compression into thinkable concepts occurs in several different ways. One, discussed by Lakoff (1987) in his book Women Fire and Dangerous Things, is categorisation, where concepts are connected in various ways in a category that itself becomes a thinkable concept. Sometimes the category may be represented by a specific case operating in a generic capacity such the equality 3 + 4 = 4 + 3 representing commutativity of addition.
Another mode of compression, described by Dubinsky and his colleagues (Cottrill et al, 1996), occurs in APOS theory where an ACTION is internalised as a PROCESS and is encapsulated into an OBJECT, connected to other knowledge within a SCHEMA; they also note that a SCHEMA may also be encapsulated as an OBJECT.
Following Davis (1983, p. 257), who used the term 'procedure' to mean a specific sequence of steps and a process as the overall input-output relationship that may be implemented by different procedures, Gray, Pitta, Pinto & Tall (1999) represented the successive compression from procedure through multi-procedure, process and procept, expanded in figure 3 to correspond to the SOLO taxonomy sequence: unistructural, multi- structural, relational, extended abstract, (Pegg & Tall, 2005).
This models the way in which a procedure—as a sequence of steps performed in time—is steadily enriched by developing alternative procedures to allow an efficient choice. The focus switches from the individual steps to the overall process, and may then be compressed as a procept to think about and to manipulate mentally in a flexible way.
Some students who have difficulty may become entrenched in a procedural approach, perhaps reaching a multi-procedural stage that can lead to procedural efficiency. Other students develop greater flexibility by seeing processes as a whole and compressing operations into thinkable concepts. This can lead to a spectrum of outcomes within a single group of learners between those who perform procedurally and those who develop greater flexibility. In arithmetic, Gray & Tall (1994) called this the proceptual divide.
The earlier work of Dubinsky and his colleagues (e.g. Cottrill et al., 1996) focused initially on a symbolic approach by programming a procedure as a function and then using the function as the input to another function. The data shows that, while the process level was often attained, encapsulation from process to object was more problematic.
A curriculum that focuses on symbolism and not on related embodiments may limit the vision of the learner who may learn to perform a procedure, even conceive of it as an overall process, but fail to be able to imagine or 'encapsulate' the process as an 'object'.
Widening the perspective to link symbolism to embodiment reveals that symbolic compression from procedure to process to object has an embodied counterpart. This happens when the actions involved operate on visible objects. The actions have an effect on the objects, for instance, when sharing them into equal shares, permuting them into a new arrangement, or translating an object on a plane. The 'effect' is the change from the initial state to the final state. The compression from procedure to process can be seen by shifting the focus of attention from the steps of a procedure to the effect of the procedure." (Tall, The Transition to Formal Thinking in Mathematics)
66. "The concept of real number is a blend of embodiment as a number line, symbolism as (infinite) decimals and formalism as a complete ordered field. Each has its own properties, some of which are in conflict. For instance, the number line develops in the embodied world from a physical line drawn with pencil and ruler to a 'perfect' platonic construction that has length but no thickness. This is a natural process of compression in which the focus of attention concentrates on the straightness of the line and the position of the lines and points. In Greek geometry, points and lines are different kinds of entity in which a point has position but no size and a point may by 'on' a line or not. The line is an entity in itself; it is not 'made up of points'.
Physically the number line can be traced with a finger and, as the finger passes from 1 to 2, it feels as if it goes through all the points in between. But when this is represented as decimals, each decimal expansion is a different point (except for the difficult case of recurring nines) and so it does not seem possible to imagine running through all the points between 1 and 2 in a finite time. There is also the counterfactual dilemma that, if the points have no size, how can even an infinite number of them make up the unit interval?
In the embodied world we may imagine a point as a very tiny mark made with a fine pencil, so practical points have an indeterminate small size even if theoretical points do not. Furthermore, if a point had no size

and a line no thickness, then we would not be able to see them. Prior to the introduction of the formal definition of real numbers, we live, perhaps somewhat uneasily, with the blend of a practical number line that we draw and imagine and a symbolic number system that can be represented by infinite decimals. Formally, the real numbers □ is an ordered field satisfying the completeness axiom. This involves entering a completely different world where addition is no longer defined by the algorithms of counting or decimal addition, instead it is simply asserted that for each pair of real numbers a, b, there is a third real number call the sum of a and b and denoted by a+b. Formally, it is possible to prove that there is, up to isomorphism, precisely one complete ordered field and that this can be represented by infinite decimals which are unique (except for the case where one decimal ends in an infinite sequence of nines and the other increases the previous place by one and ends in an infinite sequence of zeros). Thus it is possible for the human brain to recycle its former experiences and use the arithmetic of experience to blend the symbolic world with the formal world.

Personally I continue to be concerned that I ‘know’ things symbolically that I have never proved axiomatically. In the symbolic world, I ‘know’ that 210 is bigger than 103, because the first is 1024 and the second is 1000. But I have never proved this from the axioms for a complete ordered field or from the Peano postulates for the whole numbers. I am happy to accept that the familiar arithmetic of decimals is the unique arithmetic of the axiomatic complete ordered field because it fits together so coherently. But ‘acceptance’ is not mathematical proof.

In the transition from school arithmetic to formal mathematics we need to confront many issues such as this. Is it any wonder that Halmos in his book I want to be a mathematician remarked, ‘I never understood epsilon-delta analysis, I just got used to it.’ As mathematicians we begin to appreciate the purity and logic of the formal approach, but as human beings we should recognise the cognitive journey through embodiment and symbolism that enabled us to reach this viewpoint and helps us sustain it." (Tall, The Transition to Formal Thinking in Mathematics)

67. "I looked up 'measure' and 'integral' and 'probability' and 'statistics' in the card catalogue in the mathematics library, made a list of attractive titles, and then went to the parts of the stacks where those books lived. I didn't find all the books on my list in the stacks (does anyone ever?), but I found several. Scattered among them I found several others that looked interesting and that I hadn't noticed in the catalogue. I took down and looked at perhaps 30 or 40 books. Does this one look as if I could read it? Does the preface of that one exclude me (for not knowing enough, for not sharing its specialized purpose, or for any other reason)?, Is the third one in a language that I have difficulty with? In less than an hour I found a dozen promising books, I staggered to the checkout counter with them, and took them home.
I didn't read all those books. I didn't read any of them. Except for the books I have written, I can think of only two mathematics books I have ever read, in the sense of having read every word, the way I read Evelyn Waugh and Lewis Carroll. (They are Knopp's Theory of Functions, which I translated, and Carmicheal's Theory of Groups, which I proofread.) I did read the first 10 to 20 pages of all those books, and I dipped into other parts, skipping back and forth among them. (I wish I had read the first 10 pages of many more books - a splendid methematical educationcan be acquired that way.) Then I returned most of them to the library and came home with another huge armload." (Halmos, I Want to Be a Mathematician)
68. "While working at the Academy Euler met Katharina Gsell, who came to Russia when her father, a Swiss painter, came to work for Peter the Great. (Turnbull, p. 109) In 1734 they were married, and soon started a family. They had thirteen children, but only five survived childhood. Having a big family did not interfere with Euler's work. (Bell, p. 145)

    [Euler's] mind worked like lightning and was capable of intense concentration . Euler needed neither quiet nor solitude. Most of his work was done at home in the bedlam created by several small children noisily playing around his desk. Euler remained undisturbed and often rocked a baby with one hand while working out the most difficult problems with the other. He could be interrupted constantly and then easily proceed from where he had left off without losing either his train of thought or his temper. (Muir, p.141)

    This description shows the magnitude of Euler's talent. His thought processes did not require long periods of intense concentration.

    Since Euler had little or no problems concentrating with several interruptions, he must have been able to stop and then restart problems without losing large pieces of his progress. I believe this skill also characterizes an abstract thinker because Euler must have been able to remember the progress he made before he was interrupted without performing a time consuming review. This theory may be supported by the circumstances of Euler's life around 1766, when he became totally blind. This obstacle did not become a serious handicap for Euler or for his work. He was able to carry out long and tedious computations of fifteen digits in his head. (Turnbull, p. 111) This is a perfect example of an abstract thinker, according to my definition. At the beginning of this paper I defined an abstract thinker as a person that could visualize or in this case perform complicated mathematical concepts in their mind. "
    (http://www.math.rutgers.edu/~cherlin/History/Papers1999/graziosi.html)
69. "The world first became aware of Euler's abilities when he published a paper about the "masting of ships". Euler submitted this paper as an entry in the French Academy of Science's annual contest. In competition

not only with other graduate students but with many accomplished mathematicians and scientists, he was still able to win second prize. (Muir, p. 139) Presumably this paper discussed the physics and mathematics involved in the support of the mast, "a tall vertical spar that rises from the keel of a sailing vessel to support the sail and rigging." (American Heritage Dictionary, p. 512)

I consider this paper to be an example of Euler's abstract manner of thinking because he had very limited concrete knowledge, if any, of ships while writing this paper. Due to the fact that Euler lived in Switzerland, a land-locked country, he was not given the opportunity to see ocean-going ships. (Bell, p. 144) Therefore he wrote this paper without a concrete basis, namely a ship, to illustrate his findings or to verify his results. (Muir, p. 139) Since Euler was able to write an award winning paper on the masting of ships without apparently ever seeing one, I would definitely classify him as an abstract thinker. " (http://www.math.rutgers.edu/~cherlin/History/Papers1999/graziosi.html)

70. "An early Hindu work on astronomy, the Surya Siddhanta (ca. 400 a.d.), gives a table of half-chords based on Ptolemy's table (fig. 15). But the first work to refer explicitly to the sine as a function of an angle is the Aryabhatiya of Aryabhata (ca. 510), considered the earliest Hindu treatise on pure mathematics.1 In this work Aryabhata (also known as Aryabhata the elder; born 475 or 476, died ca. 550)2 uses the word ardha-jya for the half- chord, which he sometimes turns around to jya-ardha ("chord- half"); in due time he shortens it to jya or jiva.
Now begins an interesting etymological evolution that would finally lead to our modern word "sine." When the Arabs trans- lated the Aryabhatiya into their own language, they retained the word jiva without translating its meaning. In Arabic—as also in Hebrew—words consist mostly of consonants, the pronunciation of the missing vowels being understood through common usage. Thus jiva could also be pronounced as jiba or jaib, and jaib in Arabic means bosom, fold, or bay. When the Arabic version was translated into Latin, jaib was translated into sinus, which means bosom, bay, or curve (on lunar maps regions resembling bays are still described as sinus). We find the word sinus in the writings of Gherardo of Cremona (ca. 1114–1187), who translated many of the old Greek works, including the Almagest, from Arabic into Latin. Other writers followed, and soon the word sinus—or sine in its English version—became common in mathematical texts throughout Europe. The abbreviated notation sin was first used by Edmund Gunter (1581–1626), an English minister who later became professor of astronomy at Gresham College in London. In 1624 he invented a mechanical device, the "Gunter scale," for computing with logarithms—a forerunner of the familiar slide rule—and the notation sin (as well as tan) first appeared in a drawing describing his invention.3" (Maor, Trigonometric Delights) <Origins>
71. "The remaining five trigonometric functions have a more re- cent history. The cosine function, which we regard today as equal in importance to the sine, first arose from the need to compute the sine of the complementary angle. Aryabhata called it kotijya and used it in much the same way as trigonometric ta- bles of modern vintage did (until the hand-held calculator made them obsolete), by tabulating in the same column the sines of angles from 0◦ to 45◦ and the cosines of the complementary angles. The name cosinus originated with Edmund Gunter: he wrote co.sinus, which was modified to cosinus by John Newton (1622–1678), a teacher and author of mathematics textbooks (he is unrelated to Isaac Newton) in 1658. The abbreviated notation Cos was first used in 1674 by Sir Jonas Moore (1617–1679), an English mathematician and surveyor." (Maor, Trigonometric Delights) <Origins>
72. "The functions secant and cosecant came into being even later. They were first mentioned, without special names, in the works of the Arab scholar Abul-Wefa (940–998), who was also one of the first to construct a table of tangents; but they were of lit- tle use until navigational tables were computed in the fifteenth century. The first printed table of secants appeared in the work Canon doctrinae triangulorum (Leipzig, 1551) by Georg Joachim Rhæticus (1514–1576), who studied with Copernicus and be- came his first disciple; in this work all six trigonometric functions appear for the first time. The notation sec was suggested in 1626 by the French-born mathematician Albert Girard (1595–1632), who spent most of his life in Holland. (Girard was the first to understand the meaning of negative roots in geometric prob- lems; he also guessed that a polynomial has as many roots as its degree, and was an early advocate of the use of parentheses sec in algebra.) For sec A he wrote ???, with a similar notation for tan A, but for sin A and cos A he wrote A and a, respectively." (Maor, Trigonometric Delights) <Origins>
73. "The tangent and cotangent ratios, as we have seen, originated with the gnomon and shadow reckoning. But the treatment of these ratios as functions of an angle began with the Arabs. The first table of tangents and cotangents was constructed around 860 by Ahmed ibn Abdallah al-Mervazi, commonly known as Habash al-Hasib ("the computer"), who wrote on astronomy and astronomical instruments.5 The astronomer al-Battani (known in Europe as Albategnius; born in Battan,
Mesopotamia, ca. 858, died 929) gave a rule for finding the ele- vation of the sun above the horizon in terms of the length s of the shadow cast by a vertical gnomon of height h; his rule (ca. 920),
$s = h \sin (90^\circ - \varphi) / \sin \varphi$ is equivalent to the formula $s = h \cot \varphi$. Note that in expressing it he used only the sine function, the other functions having not yet been known by name. (It was through al-Battani's work that the Hindu half-chord function—our modern sine—became known in Europe.) Based on this rule, he constructed a "table of shadows"—essentially a table of cotangents—for each degree from 1◦ to 90◦." (Maor, Trigonometric Delights) <Origins>

74. "The modern name "tangent" did not make its debut until 1583, when Thomas Fincke (1561–1646), a Danish mathemati- cian, used it in his Geometria Rotundi; up until then, most Eu- ropean writers still used terms taken from shadow reckoning: umbra recta ("straight shadow") for the horizontal shadow cast by a vertical gnomon, and umbra versa ("turned shadow") for a vertical shadow cast by a gnomon attached to a wall. The word "cotangens" [sic] was first used by Edmund Gunter in 1620. Var- ious abbreviations were suggested for these functions, among them t and t co by William Oughtred (1657) and T and t by John Wallis (1693). But the first to use such abbreviations con- sistently was Richard Norwood (1590–1665), an English mathe- matician and surveyor; in a work on trigonometry published in London in 1631 he wrote: "In these examples s stands for sine: t for tangent: sc for sine complement [i.e., cosine]: tc for tangent complement: sec for secant." We note that even today there is no universal conformity of notation, and European texts often use "tg" and "ctg" for tangent and cotangent.
   The word "tangent" comes from the Latin tangere, to touch; its association with the tangent function may have come from the following observation: in a circle with center at O and radius r (fig. 16), let AB be the chord subtended by the central angle 2α, and OQ the bisector of this angle. Draw a line parallel to AB and tangent to the circle at Q, and extend OA and OB until they meet this line at C and D, respectively. We have AB = 2r sin α, CD = 2r tan α. showing that the tangent function is related to the tangent line in the same way as the sine function is to the chord. Indeed, this construction forms the basis of the modern definition of the six trigonometric functions on the unit circle." (Maor, Trigonometric Delights) <Origins>
75. "Through the Arabic translations of the classic Greek and Hindu texts, knowledge of algebra and trigonometry gradually spread to Europe. In the eighth century, Europe was introduced to the Hindu numerals—our modern decimal numeration system— through the writings of Mohammed ibn Musa al-Khowarizmi (ca. 780–ca. 840). The title of his great work, 'ilm al-jabr wa'l muqabalah ("the science of reduction and cancelation") was transliterated into our modern word "algebra," and his own name evolved into the word "algorithm." The Hindu-Arabic sys- tem was not immediately accepted by the public, who preferred to cling to the old Roman numerals. Scholars, however, saw the advantages of the new system and advocated it enthusiastically, and contests between "abacists," who computed with the good old abacus, and "algorists," who did the same symbolically with paper and pen, became a common feature of medieval Europe." (Maor, Trigonometric Delights) <Origins>
76. "It was mainly through Leonardo Fibonacci's exposition of the Hindu-Arabic numerals in his Liber Abaci (1202) that the decimal system finally took general hold in Europe. The first trigonometric tables using the new system were computed around 1460 by Georg von Peurbach (1423–1461). But it was his pupil Johann Mu ̈ller (1436–1476), known as Regiomon- tanus (because he was born in Ko ̈nigsberg, which in German means "the royal mountain") who wrote the first comprehen- sive treatise on trigonometry up to date. In his De triangulis omnimodis libri quinque ("of triangles of every kind in five books," ca. 1464)6 he developed the subject starting from ba- sic geometric concepts and leading to the definition of the sine function; he then showed how to solve any triangle—plane or spherical—using either the sine of an angle or the sine of its complement (the cosine). The Law of Sines is stated here in verbal form, and so is the rule for finding the area of a trian- gle, A = ab sin γ/2. Curiously the tangent function is absent, possibly because the main thrust of the work was spherical trigonometry where the sine function is dominant.
   De triangulis was the most influential work on trigonometry of its time; a copy of it reached Copernicus, who studied it thoroughly (see p. 45). However, another century would pass before the word "trigonometry" appeared in a title of a book. This honor goes to Bartholom ̈aus Pitiscus (1561–1613), a Ger- man clergyman whose main interest was mathematics. His book, Trigonometriae sive de dimensione triangulorum libri quinque (On trigonometry, or, concerning the properties of triangles, in five books), appeared in Frankfort in 1595. This brings us to the be- ginning of the seventeenth century, when trigonometry began to take the analytic character that it would retain ever since." (Maor, Trigonometric Delights) <Origins>
77. "His many travels brought him to Greece and Italy, where he visited Padua, Venice, and Rome. It was in Venice, in 1464, that he finished the work for which he is best remembered, On Trian- gles of Every Kind (see below). In addition to all these activities, Regiomontanus was also a practicing astrologer, seeing no con- tradiction between this activity and his scientific work (the great astronomer Johannes Kepler would do the same two centuries later). Around 1467 he was invited by King Mathias Huniades Corvinus of Hungary to serve as librarian of the newly founded royal library in Budapest; the king, who had just returned vic- torious from his war with the Turks and brought back with him many rare books as booty, found in Regiomontanus the ideal man to be in charge of these treasures. Shortly after Regiomon- tanus's arrival the king became ill, and his advisers predicted his imminent death. Regiomontanus, however, used his astro- logical skills to "diagnose" the illness as a mere heart weakness caused by a recent eclipse! Lo and behold, the king recovered and bestowed on Regiomontanus many rewards." (Maor, Trigonometric Delights)
78. "The world first became aware of Euler's abilities when he published a paper about the "masting of ships". Euler submitted this paper as an entry in the French Academy of Science's annual contest. In competition not only with other graduate students but with many accomplished mathematicians and scientists, he was still able to win second prize. (Muir, p. 139) Presumably this paper discussed the physics and mathematics involved in the support of the mast, "a tall vertical spar that rises from the keel of a sailing vessel to support the sail and rigging." (American Heritage Dictionary, p. 512)

I consider this paper to be an example of Euler's abstract manner of thinking because he had very limited concrete knowledge, if any, of ships while writing this paper. Due to the fact that Euler lived in Switzerland, a land-locked country, he was not given the opportunity to see ocean-going ships. (Bell, p. 144) Therefore he wrote this paper without a concrete basis, namely a ship, to illustrate his findings or to verify his results. (Muir, p. 139) Since Euler was able to write an award winning paper on the masting of ships without apparently ever seeing one, I would definitely classify him as an abstract thinker. " (http://www.math.rutgers.edu/~cherlin/History/Papers1999/graziosi.html)

79. "[Euler's] mind worked like lightning and was capable of intense concentration . Euler needed neither quiet nor solitude. Most of his work was done at home in the bedlam created by several small children noisily playing around his desk. Euler remained undisturbed and often rocked a baby with one hand while working out the most difficult problems with the other. He could be interrupted constantly and then easily proceed from where he had left off without losing either his train of thought or his temper. (Muir, p.141)

    This description shows the magnitude of Euler's talent. His thought processes did not require long periods of intense concentration.

    Since Euler had little or no problems concentrating with several interruptions, he must have been able to stop and then restart problems without losing large pieces of his progress. I believe this skill also characterizes an abstract thinker because Euler must have been able to remember the progress he made before he was interrupted without performing a time consuming review. This theory may be supported by the circumstances of Euler's life around 1766, when he became totally blind. This obstacle did not become a serious handicap for Euler or for his work. He was able to carry out long and tedious computations of fifteen digits in his head. (Turnbull, p. 111) This is a perfect example of an abstract thinker, according to my definition. At the beginning of this paper I defined an abstract thinker as a person that could visualize or in this case perform complicated mathematical concepts in their mind. " (http://www.math.rutgers.edu/~cherlin/History/Papers1999/graziosi.html)
80. Note Ptolemy's instruments, including the armillary.
    "Regiomontanus returned to his homeland in 1471 and settled in Nu ̈rnberg, close to his birthplace. This city, known for its long tradition of learning and trade, had just opened a printing press, and Regiomontanus saw the opportunities it offered to spread the written word of science (it was just a few years earlier, in 1454, that Johann Gutenberg had invented printing from mov- able type). He founded his own press and was about to embark on an ambitious printing program of scientific manuscripts, but these plans were cut short by his early death. He also founded an astronomical observatory and equipped it with the finest instruments the famed Nu ̈rnberg artisans could produce; these included armillary spheres and devices for measuring angular distances between celestial objects." (Maor, Trigonometric Delights)
81. "Regiomontanus was the first publisher of mathematical and astronomical books for commercial use. In 1474 he printed his Ephemerides, tables listing the position of the sun, moon, and planets for each day from 1475 to 1506. This work brought him great acclaim; Christopher Columbus had a copy of it on his fourth voyage to the New World and used it to predict the fa- mous lunar eclipse of February 29, 1504. The hostile natives had for some time refused Columbus’s men food and water, and he warned them that God would punish them and take away the moon’s light. His admonition was at first ridiculed, but when at the appointed hour the eclipse began, the terrified natives im- mediately repented and fell into submission." (Maor, Trigonometric Delights)
82. "In 1475 Pope Sixtus IV called upon Regiomontanus to come to Rome and help in revising the old Julian calendar, which was badly out of tune with the seasons. Reluctantly, he left his many duties and traveled to the Eternal City." (Maor, Trigonometric Delights)
83. "Regiomontanus’s most influential work was his De triangulis omnimodis (On triangles of every kind), a work in five parts (“books”) modeled after Euclid’s Elements (fig. 18). In this work he organized into a systematic body of knowledge the trigono- metric heritage of Ptolemy and the Hindu and Arab scholars. Book I begins with the definitions of basic concepts: quantity, ratio, equality, circle, arc, and chord. The sine function is intro- duced according to the Hindu definition: “When the arc and its chord are bisected, we call that half-chord the right sine of the half-arc.” Next comes a list of axioms, followed by fifty-six theo- rems dealing with geometric solutions of plane triangles. Much of this material deals with geometry rather than trigonome- try, but Theorem 20 introduces the use of the sine function in solving a right triangle." (Maor, Trigonometric Delights) <Origins>
84. "Trigonometry proper begins in Book II with the enunciation of the Law of Sines; as with all other rules, this is stated liter- ally rather than in symbols, but the formulation is as clear as in any present-day textbook. The sine law is then used to solve the cases SAA and SSA of an oblique triangle. Here also ap- pears for the first time, though implicitly, the formula for the area of a triangle in terms of two sides and the included angle: “If the area of a triangle is given together with the rectangu- lar product of the two sides, then either the angle opposite the base becomes known, or [that angle] together with [its] known [exterior] equals two right angles.”6 In modern formulation this says that from the formula A = 􏱒bc sin α􏱓/2 one can find either α or 􏱒180◦ −α􏱓 if the area A and the product of two sides b and c are given." (Maor, Trigonometric Delights) <Origins>

85. "In words that might have been taken from a modern textbook, he admonishes his readers to study the book carefully, as its subject matter is a necessary prerequisite to an understanding of the heavens:
You, who wish to study great and wonderous things, who wonder about the movement of the stars, must read these theorems about triangles. . . . For no one can bypass the science of triangles and reach a satisfying knowledge of the stars. . . . A new student should neither be frightened nor despair. . . . And where a theorem may present some problem, he may always look down to the numerical examples for help.8
Regiomontanus completed writing On Triangles in 1464, but it was not published until 1533, more than half a century af- ter his death." (Maor, Trigonometric Delights)
86. "He presented Copernicus with an inscribed copy of On Triangles, which the great master thoroughly stud- ied; this copy survived and shows numerous marginal notes in Copernicus's handwriting.9 Later, Tycho Brahe (1546–1601), the great Danish observational astronomer, used the work as the ba- sis for calculating the position of the famous nova (new star) in Cassiopeia, whose appearance in 1572 he was fortunate to wit- ness. Regiomontanus's work thus laid much of the mathematical groundwork that helped astronomers shape our new view of the universe." (Maor, Trigonometric Delights)
87. "In 1471 Regiomontanus posed the following problem in a let- ter to Christian Roder, a professor at the university of Erfurt: "At what point on the ground does a perpendicularly suspended rod appear largest [i.e., subtends the greatest visual angle]?" It has been claimed that this was the first extreme problem in the history of mathematics since antiquity.10" (Maor, Trigonometric Delights)
88. "To understand the course of mathematics, we need to adopt a differentangle of view. What was profound was that results were contextualized sothat they ceased
to be inconsistency threats. The Löwenheim-Skolemparadox; Skolem's w-inconsistent model of Peano arithmetic (also theconjunction of Gödel's completeness and
incompleteness theorems); w-inconsistency of Quine's "New Foundations" set theory; independence of theAxiom of Choice; etc. But mathematics had always
proceeded like this: e.g.Dedekind had taken Galileo's paradox as the definition of infinity.Here, in fact, is one reason why the paraconsistent school is
lacking. Itcompletely misses what actually happened to contradictions when theycropped up: they were reclassified from contradictions to Axioms.Let us reflect
on this. It is known that any contradiction can be neutralizedby making a distinction which previously nobody had encountered orwanted. What is not understood
is that any outcome in mathematics whichviolates or masks original intentions regarding a structure can be construedas a contradiction if a way is found to
express those intentions as anaxiom. (E.g. uniqueness of the nonnegative whole numbers.) But: instead ofavailing themselves of the opportunities to convict
themselves ofinconsistency, mathematicians abandoned aspects of the original intentions.A promised vindication which fails can be considered as a sign
ofcharlatanism, of course." (Flynt, Is Mathematics a Scientific Discipline)
89. "The geometric continuum is "filled" with non-recursive members, even though we cannot prove their individual existence. Perhaps we should say, the continuum is not-not filled with non-recursive members. These unspecifiable points corre-
spond, perhaps, to "generic" reals; or perhaps, to Brouwer's choice sequences; or perhaps, some of them can be generated by quantum-mechanical processes; or perhaps, they are figments of our mathematical imagination. This conclusion, however, does not necessarily destroy the basic premises of constructive mathe-matics, nor does it even necessitate accepting classically false axioms as Brouwer
did. The principle F P , for example, is an example of a system expressing some of our intuition about the non-recursive "gap-fillers" in the continuum, and still pos- sessing the usual properties of constructive systems. There may be other, stronger axiom systems that capture yet more of our intuition about the continuum.
In searching for such additional principles, it may be fruitful to examine the source of our intuitions about the continuum. At any rate, our intuition about the continuum is not related to the physical space we inhabit, but only to our mental
conceptions about a possible idealization of that space, since modern physics tells us that physical space cannot be coordinatizable and indefinitely divisible." (Beeson, Constructivity, Computability, and the Continuum)
90. "Some of this first paragraph concerns Dedekind's philosophical foundations for his theory. I would just as soon ignore it, but he uses it in paragraph 66 to prove that infinite sets exist. Here's Dedekind's first few sentences of this paragraph, as translated by Beman.
In what follows I understand by thing every object of our thought. In order to be able easily to speak of things, we designate them by symbols, e.g., by letters, and we venture to speak briefly of the thing a of a simply, when we mean the thing denoted by a and not at all the letter a itself. A thing is completely determined by all that can be affirmed or thought concerning it. A thing a is the same as b (identical with b), and b the same as a, when all that can be thought concerning a can also be thought concernting b, and when all that is true of b can also be thought of a.
The explanation of how symbols denote things is just fine; what I object to is his concept of things being

objects of our thought. That's an innocent concept, but in paragraph 66 it's used to justify the astounding theorem that infinite sets exist. See Dedekind's Preface to the first edition, appended at the end of these notes, for further explanation of Dedekind's philosophy." (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?) <Justif>

91. "Interestingly, Dedekind says does not want to consider a set with no elements: "we intend here for certain reasons wholly to exclude the empty system which contains no element at all, although for other investigations it may be appropriate to imagine such a system."" (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?) <Curious>
92. "Dedekind says that he will consider sets [Systeme] (translated as system by Beman, but the usual word in English is now set), denoted with uppercase letters such as S and T, that have elements, the things mentioned in paragraph 1. There's a fair amount of philosophy that's of interest for historical reasons, so I'll quote the first half of the paragraph and a footnote as translated by Beman.
It very frequently happens that different things, a,b,c,... for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a system S; we call the things a,b,c... elements of the system S, they are contained in S; conversely, S consists of these elements. Such a system S (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing [1]; it is completely determined when with respect to every thing it is determined whether it is an element of S or not.
This point is footnoted.
In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances. I mention this expressly because Kronecker not long ago (Crelle's Journal, Vol. 99, pp. 334–336) has endeavored to impose certain limitations upon the free formation of concepts in mathematics which I do not believe to be justified; but there seems to be no call to enter upon this matter with more detail until the distinguished mathematician shall have published his reasons for the necessity or merely the expediency of these limitations.
Note that Dedekind says a set is a thing, which allows for a set of sets. Also, with the understanding that a set is determined by its elements, he declares that two sets are equal, S = T , when they have exactly the same elements.
He mentions that the set S that contains only one element a should not be considered to be the same as the thing a itself. We would say the singleton set {a} is not the same as a. Even though Dedekind uses the same notation for a as for {a}—indeed he never uses curly braces for sets—I'll make the distinction in these notes." (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?) <Origins> <Justif>
93. "¶5. Theorem. If A ⊆ B and B ⊆ A, then A = B." (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?) <Origins>
94. "Dedekind's set theory is naive in the sense that there are no axioms for it. Zermelo developed axioms for set theory some time after this work of Dedekind's. Some sort of axiom is needed to justify the existence of a union of sets. Other axioms are needed, of course, but I don't think it's necessary to mention them, except in paragraph [66]." (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?)
95. "Since he doesn't allow the empty set, he has to say something about intersections when there is no common element. When there is no common element, he says that the symbol 􏰋{A, B, C, . . .} is meaningless. "We shall however almost always in theorems concerning intersections leave it to the reader to add in thought the condition of their existence and to discover the proper interpretation of these theorems for the case of non-existence." It seems to me it would have been easier to accept the empty set as a valid set." (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?) <Justif>
96. "II. Functions on a set
¶¶21–25. In this section Dedekind defines a function, or transformation, on a set, describes compsition of functions, and develops basic properties of them.
¶21. Definition. A function (or transformation) φ with domain a set S is a rule that assigns to each element s of S a value φ(s), called the image (or transform) of s. We also say φ maps s to φ(s)." (Joyce, Notes on Richard Dedekind's Was sind und was sollen die Zahlen?) <Origins>
97. Note that since we now reached Dedekind Cuts in Zakon, we can finally better notice the subtlety that Russell is talking about (not that we find it any better in Schizophrenic mode). The difference in construction also reminds us of Bishop. Also note that in Zakon's book the original 'postulation' variant is used, it dates from 1888, while Russell's paper dates from 1919.
"From the habit of being influenced by spatial imagination, people have supposed that series must have limits in cases where it seems odd if they do not. Thus, perceiving that there was no rational limit to the ratios whose square is less than 2, they allowed themselves to "postulate" an irrational limit, which was to fill the Dedekind gap. Dedekind, in the above-mentioned work, set up the axiom that the gap must always be filled, i.e. that every section must have a boundary. It is for this reason that series where his axiom is verified are called
"Dedekindian." But there are an infinite number of series for which it is not verified.
The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil.
It is clear that an irrational Dedekind cut in some way "represents"
an irrational. In order to make use of this, which to begin with is no more than a vague feeling, we must find

some way of eliciting from
it a precise definition; and in order to do this, we must disabuse our minds of the notion that an irrational must be the limit of a set of ratios. Just as ratios whose denominator is 1 are not identical with integers, so those rational | numbers which can be greater or less 72 than irrationals, or can have irrationals as their limits, must not be identified with ratios. We have to define a new kind of numbers called
"real numbers," of which some will be rational and some irrational. Those that are rational "correspond" to ratios, in the same kind of way in which the ratio n/1 corresponds to the integer n; but they are not the same as ratios. In order to decide what they are to be, let us observe that an irrational is represented by an irrational cut, and a cut is represented by its lower section. Let us confine ourselves to cuts in which the lower section has no maximum; in this case we will call the lower section a "segment." Then those segments that correspond to ratios are those that consist of all ratios less than the ratio they correspond to, which is their boundary; while those that represent irrationals are those that have no boundary. Segments, both those that have boundaries and those that do not, are such that, of any two pertaining to one series, one must be part of the other; hence they can all be arranged in a series by the relation of whole and part. A series in which there are Dedekind gaps, i.e. in which there are segments that have no boundary, will give rise to more segments than it has terms, since each term will define a segment having that term for boundary, and then the segments without boundaries will be extra.
We are now in a position to define a real number and an irrational number.
A "real number" is a segment of the series of ratios in order of magnitude.
An "irrational number" is a segment of the series of ratios which has no boundary.
A "rational real number" is a segment of the series of ratios which has a boundary.
...
The above definition of real numbers is an example of "construc- tion" as against "postulation," of which we had another example in the definition of cardinal numbers. The great advantage of this method is that it requires no new assumptions, but enables us to proceed deductively from the original apparatus of logic." (Russell, Introduction to Mathematical Philosophy) <Origins>

98. "With regard to limits, we may distinguish various grades of what may be called "continuity" in a series. The word "continuity" had been used for a long time, but had remained without any precise definition until the time of Dedekind and Cantor. Each of these two men gave a precise significance to the term, but Cantor's definition is narrower than Dedekind's: a series which has Cantorian continuity must have Dedekindian continuity, but the converse does not hold." (Russell, Introduction to Mathematical Philosophy)
99. Note: this is often the case, but explicitly stated this time.
"We are thus led to a closer investigation of series with respect to limits. This investigation was made by Cantor and formed the basis of his definition of continuity, although, in its simplest form, this definition somewhat conceals the considerations which have given rise to it. We shall, therefore, first travel through some of Cantor's conceptions in this subject before giving his definition of continuity." (Russell, Introduction to Mathematical Philosophy)
100. "With regard to limits, we may distinguish various grades of what may be called "continuity" in a series.
The word "continuity" had been used for a long time, but had remained without any precise definition until the time of Dedekind and Cantor.
Each of these two men gave a precise significance to the term, but Cantor'sdefinition is narrower than Dedekind's: a series which has Cantorian continuity must have Dedekindian continuity, but the converse does not hold.The first definition that would naturally occur to a man seekinga precise meaning for the continuity of series would be to define itas consisting in what we have called "compactness," i.e.
in the factthat between any two terms of the series there are others.
But this would be an inadequate definition, because of the existence of "gaps"in series such as the series of ratios.
We saw in Chapter VII.
that thereare innumerable ways in which the series of ratios can be dividedinto two parts, of which one wholly precedes the other, and of which thefirsthasnolastterm,|whilethesecondhasnofirstterm.Such a state of a↵airs seems contrary to the vague feelingwe have as to what should characterise "continuity," and, what is more, it shows that the series of ratios is not the sort of series that is needed for many mathematical purposes.
Take geometry, for example: we wish to be able to say that when two straight lines cross each other they have a point in common, but if the series of points on a line were similar to the series of ratios, the two lines might cross in a "gap" and have no point in common.
This is a crude example, but many others might be given to show that compactness is inadequate as a mathematical definition of continuity.It was the needs of geometry, as much as anything, that led to the definition of "Dedekindian" continuity.
It will be remembered that we defined a series as Dedekindian when every sub-class of the field has a boundary.
(It is sucient to assume that there is always an upper boundary, or that there is always a lower boundary.
If one of these is assumed, the other can be deduced.) That is to say, a series is Dedekindian when there are no gaps.
The absence of gaps may arise either through terms having successors, or through the existence of limits in

the absence of maxima.
Thus a finite series or a well-ordered series is Dedekindian, and so is the series of real numbers.
The former sort of Dedekindian series is excluded by assuming that our series is compact; in that case our series must have a property which may, for many purposes, be fittingly called continuity.
Thus we are led to the definition:A series has “Dedekindian continuity” when it is Dedekindian and compact.But this definition is stilltoo wide for many purposes.
Suppose,for example, that we desire to be able to assign such properties to geometrical space as shall make it certain that every point can be specified by means of co-ordinates which are real numbers: this is not insured by Dedekindian continuity alone.
We want to be sure that every point which cannot be specified by rational co-ordinates can be specifiedasthelimitofaprogressionofpoints|whoseco-ordinates are rational, and this is a further property which our definition doesnot enable us to deduce.We are thus led to a closer investigation of series with respect to limits.
This investigation was made by Cantor and formed the basis of his definition of continuity, although, in its simplest form, this definition somewhat conceals the considerations which have given rise to it.
We shall, therefore, first travel through some of Cantor’s conceptions in this subject before giving his definition of continuity.Cantor defines aseries as “perfect” when all its points are limiting- points and all its limiting-points belong to it.
But this definition does not express quite accurately what he means.
There is no correction required so far as concerns the property that all its points are to be limiting-points; this is a property belonging to compact series, and to no others if all points are to be upper limiting- or all lower limiting- points.
But if it is only assumed that they are limiting-points one way, without specifying which, there will be other series that will have the property in question—for example, the series of decimals in which a decimal ending in a recurring 9 is distinguished from the corresponding terminating decimal and placed immediately before it.
Such a series is very nearly compact, but has exceptional terms which are consecutive, and of which the first has no immediate predecessor, while the second has no immediate successor.
Apart from such series, the series in which every point is a limiting-point are compact series; and this holds without qualification if it is specified that every point is to be an upper limiting-point (or that every point is to be a lower limiting-point).Although Cantor does not explicitly consider the matter, we must distinguish di↵erent kinds of limiting-points according to the nature of the smallest sub-series by which they can be defined.
Cantor assumes that they are to be defined by progressions, or by regressions (which are the converses of progressions).
When every member of our series is the limit of a progression or regression, Cantor calls our series “condensed in itself” (insichdicht).
|Wecomenowtothesecondpropertybywhichperfectionwasto 103 be defined, namely, the property which Cantor calls that of being “closed” (abgeschlossen).
This, as we saw, was first defined as consistingin the fact that all the limiting-points of a series belong to it.
But this only has any e↵ective significance if our series is given as containedin some other larger series (as is the case, e.g., with a selection ofreal numbers), and limiting-points are taken in relation to the largerseries.
Otherwise, if a series is considered simply on its own account, it cannot fail to contain its limiting-points.
What Cantor means is not exactly what he says; indeed, on other occasions he says something rather di↵erent, which is what he means.
What he really means isthat every subordinate series which is of the sort that might be expected to have a limit does have a limit within the given series; i.e.
every subordinate series which has no maximum has a limit, i.e.
every subordinate series has a boundary.
But Cantor does not state this for every subordinate series, but only for progressions and regressions.
(It is not clear how far he recognises that this is a limitation.) Thus, finally, we find that the definition we want is the following:—A series is said to be “closed” (abgeschlossen) when every progres- sion or regression contained in the series has a limit in the series.We then have the further definition:—A series is “perfect” when it is condensed in itself and closed, i.e.
when every term is the limit of a progression or regression, and every progression or regression contained in the series has a limit in the series.In seeking a definition of continuity, what Cantor has in mindis the search for a definition which shall apply to the series of real numbers and to any series similar to that, but to no others.
Forthis purpose we have to add a further property.
Among the real numbers some are rational, some are irrational; although the numberof irrationals is greater than the number of rationals, yet there are rationals between any two real numbers, however | little the two may 104 di↵er.
The number of rationals, as we saw, is @0.
This gives a further property which suces tocharacterise continuity completely, namely,the property of

containing a class of @0 members in such a way that some of this class occur between any two terms of our series, however near together.
This property, added to perfection, suces to definea class of series which are all similar and are in fact a serial number.
This class Cantor defines as that of continuous series.We may slightly simplify his definition.
To begin with, we say:A “median class” of a series is a sub-class of the field such that members of it are to be found between any two terms of the series.Thus the rationals are a median class in the series of real numbers.
It is obvious that there cannot be median classes except in compact series.We then find that Cantor’s definition is equivalent to the follow- ing:—A series is“continuous” when (1) it is Dedekindian, (2) it contains a median class having @0 terms.To avoid confusion, we shall speak of this kind as “Cantorian continuity.” It will be seen that it implies Dedekindian continuity, but the converse is not the case.
All series having Cantorian continuity are similar, but not all series having Dedekindian continuity.The notions of limit and continuity which we have been defining must not be confounded with the notions of the limit of a function for approaches to a given argument, or the continuity of a function in the neighbourhood of a given argument.
These are di↵erent notions, very important, but derivative from the above and more complicated.
The continuity of motion (if motion is continuous) is an instance of the continuity of a function; on the other hand, the continuity of spaceand time (if they are continuous) is an instance of the continuity of series, or (to speak more cautiously) of a kind of continuity whichcan, by sucient mathematical | manipulation, be reduced to the 105 continuity of series.
In view of the fundamental importance of motionin applied mathematics, as well as for other reasons, it will be wellto deal briefly with the notions of limits and continuity as appliedto functions; but this subject will be best reserved for a separate chapter.The definitions of continuity which we have been considering, namely, those of Dedekind and Cantor, do not correspond very closely to the vague idea which is associated with the word in the mind of the man in the street or the philosopher.
They conceive continuity rather as absence of separateness, the sort of general obliteration of distinctions which characterises a thick fog.
A fog gives an impression of vastness without definite multiplicity or division.
It is this sort of thing that a metaphysician means by “continuity,” declaring it, very truly, to be characteristic of his mental life and of that of children and animals.The general idea vaguely indicated by the word “continuity” whenso employed, or by the word “flux,” is one which is certainly quite di↵erent from that which we have been defining.
Take, for example,the series of real numbers.
Each is what it is, quite definitely and uncompromisingly; it does not pass over by imperceptible degreesinto another; it is a hard, separate unit, and its distance from every other unit is finite, though it can be made less than any given finite amount assigned in advance.
The question of the relation betweenthe kind of continuity existing among the real numbers and the kind exhibited, e.g.
by what we see at a given time, is a dicultand intricate one.
It is not to be maintained that the two kinds are simply identical, but it may, I think, be very well maintained thatthe mathematical conception which we have been considering in this chapter gives the abstract logical scheme to which it mustbe possibleto bring empirical material by suitable manipulation, if that materialis to be called “continuous” in any precisely definable sense.
It wouldbe quite impossible | to justify this thesis within the limits of the 106 present volume.
The reader who is interested may read an attemptto justify it as regards timein particular by the present author inthe Monist for 19145, as well as in parts of Our Knowledge of the External World.
With these indications, we must leave this problem, interesting as it is, in order to return to topics more closely connected with mathematics." (Russell, Introduction to Mathematical Philosophy)

101. " We certainly cannot obtain this result empirically, or apply it, as Dedekind does, to “meine Gedankenwelt”—the world of my thoughts.
If we were concerned to examine fully the relation of idea and object, we should have to enter upon a number of psychological and logical inquiries, which are not relevant to our main purpose. But a few further points should be noted. If “idea” is to be understood logically, it may be identical with the object, or it may stand for a description (in the sense to be explained in a subsequent chapter). In the former case the argument fails, because it was essential to the proof of reflexiveness that object and idea should be distinct. In the second case the argument also fails, because the relation of object and description is not | one-one: there are innumerable correct de- 140 scriptions of any given object. Socrates (e.g.) may be described as
“the master of Plato,” or as “the philosopher who drank the hem- lock,” or as “the husband of Xantippe.” If—to take up the remaining hypothesis—“idea” is to be interpreted psychologically, it must be maintained that there is not any one definite psychological entity which could be called the idea of the object: there are innumerable beliefs and attitudes, each of which could be called an idea of the object in the sense in which we might say “my idea of Socrates is quite di↵erent from yours,” but there is not any central entity (ex- cept Socrates himself) to bind together various “ideas of Socrates,” and thus there is not any such one-one

relation of idea and object as the argument supposes. Nor, of course, as we have already noted, is it true psychologically that there are ideas (in however extended a sense) of more than a tiny proportion of the things in the world. For all these reasons, the above argument in favour of the logical existence of reflexive classes must be rejected." (Russell, Introduction to Mathematical Philosophy)

102.