MA728 Vector Spaces

Final Course Memo

Jad Nohra

1 The Algebra

1. Vector spaces belong to linear *algebra* because indeed, we try to see most of its sets and relations (mostly operators) from the point of view of algebraic operators and their properties.

2. In this course we mainly study finite dimensional vector spaces. What makes them special as sets, is the existence of a function

$$\dim$$

$$\dim : \mathcal{V} \to \mathbb{N}$$

that assigns to each space a natural number, and that number, by looking at the properties of this function and its interplay with other objects and functions, is truly an indication of *size*.

To actually construct this function, we hinge on counting elements in a *basis*.

3. A vector space is *finite-dimensional* if it is spanned by a finite list of vectors. Otherwise, it is called *infinite-dimensional*. It turns out that this is equivalent to having a basis.

$$\texttt{fin}\ V$$

$$\triangleq\ \exists\, v_1 \cdots v_n : V = \operatorname{span}(v)$$

4. Per example, when we *restrict* ourselves to linear functions, we can use dimension to characterize injectivity and surjectivity, and vice versa, from the naturally sounding viewpoint that there can be no injective mapping from two space with difference dimensions, and there can be no surjective mapping from a space to one of higher dimension. This is made very precise in (18.).

5. We have further properties when we consider interactions between dim and the subset relation. Indeed, subsets have smaller or equal dimension.

$$U \subseteq V\ \vdash\ \dim(U) \leq \dim(V)$$

6. The word 'fundemantal' in 'fundamental' theorems is confusing. It is better to think of it as indicating a 'crowning' theorem which culminates the efforts of a theory, and has ample use within the theory (bringing together some but not necessarily all of the important concepts and properties) and outside of the theory.

7. In fact, the dim operator has 'norm-like' properties, where it reminds of the triangle inequality since

$$\dim\,(U+V) \leq \dim\,(U) + \dim\,(V)$$

actually we can say more: the fundamental theorem of finite dimensional vector spaces:

$$\dim\,(U+V) = \dim\,(U) + \dim\,(V) - \dim\,(U \cap V)$$

This is easy to remember by analogy with the number of elements in the union of two finite sets, where

$$\begin{aligned}\#(U \cup V) &= \#(U_{proper}) + \#(V_{proper}) + \#(UV_{common}) \\ &= \#(U_{prop} + UV_{cmn}) + \#(V_{prop} + UV_{cmn}) - \#(UV_{cmn})\end{aligned}$$

We also note here that indeed, one can proove that the intersection is a space:

$$U \cap V \in \mathcal{V} \text{ where } U, V \subseteq P$$

and U, V inherit the operations from P.

8. In the dimension over sum inequality above, knowing the fundamental theorem above, we can say that equality is achieved iff.

$$\dim(U \cap V) = 0.$$

Here we need to remember that the only vector space of dimension zero, by definition is the one that contains a single element, and with basis that has no element, that is

$$U \cap V = \{0\}$$
$$\text{Bases}(V) = \emptyset$$

9. But note that we also have

$$\dim\,(U \times V) = \dim\,(U) + \dim\,(V)$$

This reminds us of both the norm and the complex conjugate operators

op	$+$	$\times$
dim	$\leq (= if direct)$	$=$
norm	$\leq$	
conjugate	$=$	$=$

10. The definition of sum for vector spaces. We need to 'join' vector spaces somehow, this is a specially defined sum, again an 'algebraic' operator, this time not between elements of a vector space, but between spaces

$$U + V$$

that is, the operator 'plus' over pairs of vector spaces:

$$op+ : \mathcal{V}^2 \to \mathcal{V}$$

The sum is the *smallest* vector space containing both spaces. Smallest means that:

- any set fulfilling the needed condition must be larger or equal to the sum

- the condition being: containing both spaces and being a space

11. The definition of the product of two spaces, follows the usual cartesian product. But we notice that the resulting set is actually also a space.

$$U \times V \triangleq \{(u,v) : u \in U, v \in V\}$$

This is exactly the definition of a Cartesian product, but when we define the product of vector spaces, we also define it along with the two operators. So here, the product, seen as a function, is not 'only' a function from set pairs to sets

$$\text{op}\times : (\mathcal{U},\mathcal{U}) \to \mathcal{U}$$

where

$$\mathcal{U} \text{ is the universe of sets}$$

But actually

$$\text{op}\times : (\mathcal{V},\mathcal{V}) \to \mathcal{V}$$

With

$$\begin{array}{rrrcl} \text{op}+: & (U \times V, U \times V) \to & U \times V & \triangleq & (u_1,v_1) + (u_2,v_2) = (u_1 + v_1, u_2 + v_2) \\ \text{op}.: & (F, U \times V) \to & U \times V & \triangleq & \lambda(u,v) = (\lambda u, \lambda v) \end{array}$$

12. The definition of the sum of two spaces, imitates definition of the product space, but using a more 'algebraic' operation. space.

$$U + V \triangleq \{u + v : u \in U, v \in V\},$$

and notice that this is very related to the product space, since we can also write it as:

$$\begin{array}{rcl} U + V & \triangleq & \{\Gamma(w) : w \in U \times V\} \\ \Gamma : U \times V \to P & \triangleq & (u,v) \to u + v \\ \text{where } U, V \subseteq P & & \end{array}$$

13.

$$\texttt{direct}\ sum$$

14. We can also define a quotient space. The best way to think about this is the ability to 'divide the space' by some subspace, such that in the new space the subspace we divided by 'collapses' to 'identity'. This is like 'normalizing' the set of rationals by dividing all its elements by some rational q, such that in the new space, q is mapped to 1. It is actually possible to do this, and still have the resulting set be a space. We can see how this is useful for 'removing' something from a space (actually collapsing it to the special vector of additive identity).

To construct a quotient space, we construct a 'quotient map' π

$$\pi_U : V \to \{U' : U' \leq V, U' \parallel U\}$$

$$U' \parallel U \triangleq \exists v \in V : U' = \text{apply}_U(v)$$

with

$$\text{apply}_U(v) : V \to \text{Subsets}(V)$$
$$\text{apply}_U(v) \triangleq \{v + u : u \in U\}$$

and

$$V/U \triangleq \{U' : U' \leq V, U' \parallel U\}$$
$$v + U \triangleq \text{apply}_U(v)$$

With $U \leq V$ is a (not very common) notation for U being a subspace of V.

The π map turns the wanted subspace (in the original space) to the identity vector (in the quotient space). Any vector in the original subspace can be added to the wanted subspace, this creates a new 'parallel' subset (not subspace) and the function turns this subset into a vector. There are many vectors that produce the same 'parallel' subset when applied to the desired subspace, and these vectors form an equivalence class. We find out that two such vectors (in the original space) are equivalent iff. their difference is in the desired subspace. Given this, we can 'represent' vectors in the quotient space (that is images of 'parallel' subsets in the original space) by

$$U' = U + [v]$$

where $[v]$ is the equivalence class of vectors mentioned, or simply take v, a representative of the class.

15. As promised, we can 'slash away' the desired subspace from the original space, and this is indeed captured by the dimension:

$$\dim(V/U) = \dim(V) - \dim(U)$$

Indeed the basis remaining after removal of the basis of the subspace, is mapped by π to a basis of the quotient space.

This can be especially elegant when we per example 'remove' that is, 'divide by' the null space of a linear function, resulting in an injective space, (TODO: if we also limit ourselves to the range of the function, we end up with a space on which the function is an invertible?)

16. We note that when thinking about a set being a vector space, we should first focus on the *existence* of two operators on the elements of the set. These are

$$\text{op+} : \text{V}^2 \to \text{V}$$
$$\text{op.} : (\text{F}, \text{V}) \to \text{V}$$

$$\texttt{v-space}\ v \Leftrightarrow (\exists \text{op+}, \text{op.} \ldots)$$

We invent our shorthand for remembering the main properties that these operators must have:

+	closed	ident (0)	inv	comm	assoc	Also + and . must be both-ways distributive
.	closed	ident (1)			assoc	

Note that it is better to see . more as a single-sided function composition than multiplication.

17. Closure and the existence of an additive zero are particularly important because in the case of a subset inheriting the operations, to prove that it is a subspace, we need to only prove these three properties (this is a theorem).

18. Coming back to functions and dimension, we have a fundamental theorem, telling us that linear functions *project* size, and the lost dimension between the domain and the

$$\text{rng } space$$

is exactly the dimension of the

$$\text{null } space$$

of the function.

$$\dim(\text{dom}(T)) = \dim(\text{null}(T)) + \dim(\text{rng}(T))$$

This is a certain kind of 'conservation of size', since the 'size' of the domain, either goes to the range, or the the null space

19. How does the fundemantal theorem directly link to injectivity? This is captured by

$$\dim(\text{null}(T))$$

- For linear maps, the question of injectivity can be narrowed down to knowing if any non-zero vector maps to zero.
- We know by a theorem that $T(0)$ must be 0.
- So if any non-zero vector maps to 0, more than one vector map to zero, and the map is injective.
- In fact, the set of all vectors mapping to zero is a space: $\text{null}(T)$.

20. How does the fundemantal theorem directly link to surjecitvity? This is captured by

$$\dim(\text{rng}(T))$$

- Careful in noting that this is the range we are talking about, and it is a space.
- Therefore, we can compare its size to the codomain, which tells us what we need about surjectivity.

21. Additionally, the beauty of duality appears, and it turns out that the set of linear functions over a certain domain-codomain pair is a vector space itself

$$\mathcal{L}(V, W) \in \mathcal{V}$$

22. Following the algebraic traits of vector spaces, we can go quite further by taking powers of linear operators. For that, we need to be able to apply an operator multiple times, and there, a requirement is that the domain should be *invariant* under the linear map, making the map an operator. And there emerges the fact that we can have an algebra for 'linear operator multiplication', which is at heart, composition of functions, but of course over vector spaces, the functions themselves being linear.

$$\mathcal{L}(V) \triangleq \mathcal{L}(V, V)$$

23. This book does not go into depth or rigor with respect to the relation between operator powers and polynomials. This is better treated in books such as Shilov or Broida-Williamson. But basically, the composition of an operator with itself has multiplication-like properties. Specifically, non-commutative multiplication. This makes the vector space become a non-commutative algebra. Where certain factorizations theorems apply. When the field is that of complex numbers, this is especially easy since polynomials can be fully factorized.

24. We can therefore talk about powers of

$$T \in \mathcal{L}(V),$$

namely

$$T^n \in \mathcal{L}(V)$$

and about factoring *polynomials in* T, the factorizations look like

$$(T - \lambda_1 I) \cdots (T - \lambda_n I)$$

and these λ scalars are called *eigenvalues*, they do have many useful properties.

25. Note that the factorizations of polynomials in T as shown above, still carry the lack of commutativity, but it turns out that this does not disable us from using many properties of polynomials over scalars, including the ability to factorize.

26. A specific example (use lambda-sqrd=9) exercise...

2 Basis

27. By the operations on vectors, one can take *linear combinations* of a finite list vectors. The set of all possible linear combinations (by exhausting over the elements of the field) is called the span of that list. It is in fact a space.

$$\begin{aligned} \text{span}(v) &: V^{n=\text{tuple-size}(v)} \to V \\ \text{span}(v) &\stackrel{\Delta}{=} \{a_i v_i : a \in F^n\} \\ x_i &\stackrel{\Delta}{=} \text{the i-th element in the list } x \\ \text{span}(\emptyset) &\stackrel{\Delta}{=} \{0\} \end{aligned}$$

$$\texttt{lin-indep}\ (v) \Leftrightarrow (\text{lin-combs}_a(v) = 0 \Leftrightarrow a = 0)$$

28. Some vector spaces can be exhausted by linear combination of some n-tuples (lists).

- We call such spaces finite-dimensional.
- There is a minimum size n of such a tuple for a finite vector space.

- Any such n-tuple is called a *Basis*

$$\text{Bases}(V) : V \to \text{n-tuples of } V$$
$$\text{Bases}(V) \stackrel{\Delta}{=} \text{The n-tuple spans } V$$
Note that this is a relation, not a function!

29. A basis is characterized by that fact that representations of vectors in it are unique.

$$\text{rep}_b(v) : V \to F^n$$
$$\text{rep}_b(v) \stackrel{\Delta}{=} a : v = a_i b_i$$
Note that a is well defined because the representations are unique.

The fact that representations with respect to a basis are unique is a useful proof tool where we can equate all scalars from two different representations of the same vector. The ability to manipulate representations (adding, etc.) is exactly where the properties of the operators on vectors play an important role. When the representation is not with respect to a basis, it is a set valued function.

30. As is not unusual with vector space, we can reduce the proof of uniqueuess of represetnations of all vectors in a space to a proof of uniqueuess of representation of zero. We call a tuple for which the representation of 0 is unique *linearly independent*. This is a characteristic of a basis, in addition to span. It is therefore a theorem that

$$\begin{aligned}
&w_i \text{ are linearly independent} \\
&\qquad \stackrel{\Delta}{=} \text{rep}_w \text{ is a function (not a relation)} \\
&\qquad \stackrel{\Delta}{=} \text{rep}_w(v) \text{ is unique for any } v \in \text{span}(w) \\
&\qquad \Leftrightarrow \text{rep}_w(0) = 0 \in F^n
\end{aligned}$$

31. Note that it is the properties of the operators on the vector space provide all the manipulation potential for vector representations. This is behind many of the results of finite-dimensional linear algebra.

32. In fact

- A linearly independent list that has the 'right size' is a basis.
- A linearly independent list can be brought up to the 'right size'.
- A spanning list of the 'right size' is a basis.
- A spanning list can be brought down to the 'right size'.
- All bases have the same 'right size', which is the dim of the space.
- $\dim(V) \stackrel{\Delta}{=} n : \exists$ an n-tuple basis for V

Note that rep is a multi-valued function. This is often used to solve exercises related to uniqueness, as it gives a set of n conditions that hold simultaneously.

3 Isomorphism

33. In this book, we define *Isomorphism* by the existence of an invertible linear map. But we then find it equivalent to bijectivity.

$$
\begin{aligned}
&\texttt{inv}\ T \in \mathcal{L}(V,W) \\
&\qquad \triangleq \ \wedge \begin{cases} \exists L : LT = I, & I \in \mathcal{L}(V) \quad \text{(left inverse)} \\ \exists R : TR = I, & I \in \mathcal{L}(W) \quad \text{(right inverse)} \end{cases} \\
&\qquad \Leftrightarrow \ \wedge \begin{cases} \texttt{inj}\ T \\ \texttt{surj}\ T \end{cases}
\end{aligned}
$$

34. When the map is an operator, we have a very useful theorem, that gives us three choices of tackling certain proofs. Additionally, being biconditionals, these are equivalent definitions.

$$
\begin{aligned}
&T \in \mathcal{L}(V) \\
&\qquad \vdash \quad \texttt{inv}\ T \Leftrightarrow \texttt{inj}\ T \Leftrightarrow \texttt{surj}\ T
\end{aligned}
$$

35. Isomorphism also interacts characteristically with dimension

$$\texttt{iso}\ U, V \ \Leftrightarrow\ \dim(U) = \dim(V)$$

Note here that the needed isomorphism for this proof is easily found: The one that maps the equally sized bases to each other one to one.

4 Matrices

36. A matrix is a product of two tuples

$$F^{m,n} \triangleq F^m \times F^n$$

37. Having chosen a basis for a space, each vector can be characterized by the n-tuple scalar from its representation. This tuple can be put in a matrix in a way that allows the operators on vectors to be mirrored by operations on the matrices. With

$$
\begin{aligned}
b &\in \text{Bases}(V), & n &= \dim(V) \\
v &\in V, & v &= a_i b_i
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathcal{M}_B \\
&\qquad : V \to F^n \\
&\qquad \triangleq v = a_i b_i \to \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}
\end{aligned}
$$

38. Linear maps are themselves vector spaces. Linear maps on finite-dimensional spaces are spaces that are themselves finite-dimensional. There is also a 'matrix map' that maps them into matrices. In this case, we need to choose two bases, one for the domain and one for the codomain of the map.

With

$$\begin{aligned} T &\in \mathcal{L}(V,W) \\ T &: b_i \to a_{i,j} b'_j \\ b &\in \text{Bases}(V), \quad \dim(V) = m \\ b' &\in \text{Bases}(W), \quad \dim(W) = n \end{aligned}$$

We have

$$\begin{aligned} &\mathcal{M}_{b,b'} \\ &\quad : \mathcal{L}(V,W) \to F^{m,n} \\ &\quad \triangleq T \to \begin{pmatrix} \text{rep}_{b'}(Tb_1) & \cdots & \text{rep}_{b'}(Tb_m) \end{pmatrix} = \begin{pmatrix} a1,1 & & a_{m,1} \\ & \ddots & \\ a_{1,n} & & a_{m,n} \end{pmatrix} \end{aligned}$$

39. The matrices themselves, that is, the sets $F^{m,n}$ are vector spaces.

40. The maps $\mathcal{M}_{.}$ and $\mathcal{M}_{.,.}$ are linear. In fact, they are isomorphisms.

$$\begin{aligned} \mathcal{M}_{b,b'}(T+S) &= \mathcal{M}_{b,b'}(T) + \mathcal{M}_{b,b'}(S) \\ \mathcal{M}_{b,b'}(\lambda T) &= \lambda \mathcal{M}_{b,b'}(T) \end{aligned}$$

41. The composition of linear maps is also captured by the multiplication of matrices. In fact, the structure of the matrices, along with the rule of multiplication, are so desgined such that every fits. Namely, such that matrix maps preserves the algebraic operator of linear map 'multiplication' into the corresponding the operator of matrix 'multiplication'.

$$\mathcal{M}_{b,b'}(TS) = \mathcal{M}_{c,b'}(T)\mathcal{M}_{b,c}(S)$$

5 Polynomials

42. Using linear algebra and a single foreign theorem from complex numbers, we use this section in the book to show that in $\mathbb{C}$, every polynomial can be factored into as many zeros (roots) as its degree.

$$a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_n)$$

43. In $\mathbb{R}$, the situation is similar up to the fact that some degrees have to be of degree two, because they are irreducible and cannot be factored further.

$$a_0 + a_1 x + \cdots + a_n x^n = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1)(x^2 + b_M x + c_M)$$

44. When we say polynomial over C, we mean that both coefficients and variables are in $\mathbb{C}$

45. We should not forget that the number of coefficients is one more than the degree of the polynomial

46. Several theorems must be remembered that lead up to the two facts above:

1. Every nonzero polynomial over $\mathbb{C}$ has at a zero. This is not a linear algbera proof, it comes from complex analysis.
2. λ is a zero of $p(z)$ iff. $p(z) = q(z)(z - \lambda)$ for some $q(z)$.
3. A polynomial is the zero function iff. all coefficients are zero.
4. There is a division algorithm for polymials such that if $s \neq 0$, then there exist q, r such that
$$p = qs + r$$
with $\deg(r) < \deg(s)$.
5. A polynomial of degree n has at most n distinct factors.
6. Polynomials with real coefficients (but not necessarily real variables) have zeros in pairs.
7. In $\mathbb{R}$, a quadratic polynomial $x^2 + bx + c$ can be factored into two real roots iff.
$$b^2 - 4c \geq 0$$

6 Eigenvalues

47. For the definition of an eigenvalue, with

$$T \in \mathcal{L}(V)$$
$$\lambda \in F$$

We have

$$\begin{aligned}
&\texttt{eig}\ \lambda \text{ of } T \\
&\qquad \triangleq \exists v \neq 0 : Tv = \lambda v \\
&\qquad \Leftrightarrow \exists v \neq 0 : Tv - \lambda v = 0 \\
&\qquad \Leftrightarrow (T - I) \text{ not } \texttt{inj} \\
&\qquad\quad (\text{equivalently not } \texttt{surj}\ , \text{equivalently not } \texttt{inv}\ )
\end{aligned}$$

Such a v is then called the eigenvector `eigv` corresponding to λ.

48. Eigenvectors corresponding to different eigenvalues are linearly independent. With

$$T \in \mathcal{L}(V)$$
$$\texttt{eig}\ \lambda_i,\ i = 1 \to n, \text{ distinct}$$
$$\texttt{eigv}\ v_i, \text{ , corresponding to } \lambda_i$$

We have

$$\texttt{lin-indep}\ v_1, \cdots, v_n$$

Because of this, clearly, V cannot have more than $\dim(V)$ eigenvalues, because otherwise, v would be a basis with more than $\dim(V)$ elements.

49. In fact, the set of all the eigenvectors of a certain eigenvalue form a vector space. Hence, the space V is a direct sum of such eigenspaces.

7 Inner Product Spaces

8 Miscellanea

50. 5.14

51. A linear map is injective is equivalent to $\text{null}(T) = \{0\}$. Hence, it is a 'forall' property on all nonzero vectors. Hence, when a map is not injective, we have an existence property: some nonzero vector has a zero image.

9 Definitions, Theorems and Lemmata

9.1 1.A. $\mathbb{R}^n$ and $\mathbb{C}^n$

52. Definition of complex numbers.

53. Properties of complex numbers.

54. Definition of subtraction and division of complex Numbers. By virtue of existence of inverses.

55. Convention that F is $\mathbb{R}$ or $\mathbb{C}$.

56. Definition of list, length, also called n-tuple.

Even though this is a notational shortcut for a function from a finite number, we can also nicely think of this function in the extensive sense, which makes the notational shortcut more easily amenable to rigorous thinking on the fly.

57. Definition of F^n, including addition, 0, additive inverse, scalar multiplication, proof of commutativity.

No mention of cartesian product, using the list term as defined above. No mention that this is a vector space, since those have not been defined yet.

9.2 1.B. Definition of Vector Space

58. Definition of addition and scalar multiplication.

Including implicit properties of existence and closedness.

59. Definition of vector.

60. Definition of real vector space and complex vector space. With note that by default we mean a vector space on F.

61. Example of F^∞.

62. Definition of F^S as the set of functions from S to F.

As a memory help, this the generalization of F^n from n being the finite set $1, \cdots, n$ to any set S.

63. Theorems. Unique additive identity, additive inverse, zero times vector is zero, number time the zero vector is zero, -1 times vector is additive inverse.

9.3 1.C. Subspaces

64. Definition of a subspace.

All that is needed is 0 membership and closures under the two operators.

65. Definition of sums of subsets.

66. Theorem that the sum of subsets is the smallest space containing both sets.

67. Definition of direct sum, by unique representability.

Conceptually, it may be good to think about direct sums as any subset of a vector space that is also a space, and especially as ones that only intersect at zero. It is clear that this is the least intersection any two subspaces can have since they both must contain zero. And indeed in this case they are direct. It must be remembered then that interstection being zero implies unique representation but as an iff! Which can be done with a small mental proof by contradiction.

We use 'sum' to be algebraic, but it is important to both see the sum as an operation on spaces, but also as a 'set/space' which is only expressed as the 'sum' of two other 'spaces/sets'. So here, the algebra acts on the subsets of a space that are spaces themselves, these are the elements of this algebra. This possibly is a key in being able to take our thinking to be 'direct' on a higher level of abstraction, by always seeing the higher level structure mentally/graphically as composed of elements and not sets, and mentally seing them as points in a bag. This time, the points being subspaces. This reminds us of the topology book.

Sometimes the relation between the 'higher level' points and the lower level ones is a bit more evolved than being a subspace. This is the case of quotient spaces, but there the picture is surprisingly simple as well: All the parallel subsets! become points. The space we are dividing by is the zero vector! So we remember to see the higher level as point, and we remember the 'function' mapping between lower and higher level points. In general, many times this will formally reduce to working in power sets (of powers sets ...) with constaints making only some of the power sets the ones to be considered, or even having hybrid structures containing elements from different 'powers'.

68. Theorem that direct sum by unique representation is equivalent to unique representation of zero.

69. Theorem that direct sum by unique representation is equivalent to intersection only containing zero.

70.

Note that subspace sums are introduced even before talking about span, linear independence and bases. Why? To teach us that these concpets make sense not only for finite dimensional spaces! Our mental proofs of the theorems above used bases, and thought of the subspaces as grouping of elements from some base, but this thinking is only valid for finite-dimensional spaces!

9.4 2.A. Span and Linear Independence

71.

Section 2 has as title 'Finite Dimensional Spaces' so it is important to keep in mind that most concpets that follow are only meaningful in this context, even the concept of span. Why? Probably because in infinite dimensional spaces, although one may take the span of a finite set of vectors, the concept is potentially too weak to be able to describe all vectors in the space.

72. Definition of a list of vectors, or linear combination and of span. This includes the special definition of the span of the empty set being zero.

73. Theorem that the smallest set containing all vectors in a list is the span.

By analogy, we could see a sum of spaces as the span of a list of spaces. This is indeed meaningful and not wrong, even if not very elegant, since multiplying a space by a scalar, if defined, would result in the same space by closure.

74. Definition of the word 'spans'.

75. Definition of finite-dimensional vector space as being spanned by a list of vectors.

76. Definition of a polynomal in $\mathcal{P}(F)$.

In an exercise we were confused about $\mathcal{P}(F)$ where we thought elements of it can be infinite in degree but that is not true. They must be finite, they are not 'series', but they can have arbitrary finite degree, unlike $\mathcal{P}_m(F)$.

77. Definition of the degree of a polynomial.

78. Definition of $\mathcal{P}_m(F)$.

79. Proof that $\mathcal{P}_m(F)$ is a vector space.

80. Definition of infinite-dimensional vector space. One that is not finite-dimensional.

81. Definition of linear indepdendence by linear combination equating to zero iff all scalars are zero. With the special definition that the empty list is also considered linearly independent.

This is equivlent to no vector being a linear combianation of other vectors, but the equivalent definition given is more usable, or at least more canonical, in proofs (it seems). Also, when formally expressed, the book's definition is much more elegant. Maybe it is a bit less intuitive, but why? Because it is easier for us to think of relations between the vectors, than by constraints? Or because the definition itself containts a bidirectional? Or because the definition as stated feels slightly too powerful and untrue on first sight? And hence requires 'memoerization' being less natural? Or maybe because we feel the term 'linear independence' is quite tied to the natural definition and that the book definition should be a theorem?.

Thinking about it, it is very special that being spanned by any n-tuple and the properties of a vector space alone give birth to the basis! It is very important to level up in 'algebraic' thinking and see things from this 'structure' perspective, than to use our old view where we run to hug the concept of basis and base all our internal thinking around it. This works and is very practicaly, but conceptually not the highest point to view things from.

82. Definition of linear dependence.

83. Linear dependence lemma.

remember that the spans staying the same after removal is a key part of the lemma.

84. Theorem than there is no linearly independent list that has more elements than the spanning list of the space.

This reminds is of the notes about the linear independence definition, and is more key conceptually/structurally/algebraically than seems as first, because it 'closes the door' on the size of linearly independent list, opening the door towards the basis concept.

85. Theorem that finite-dimensional spaces can only have finite-dimensional subspaces.

9.5 2.B. Bases

86. Definition of a Basis. By uniqueness of representation.

87. Theorem that a spanning list contains (and can be reduced to) a basis.

88. Theorem that every finite-dimensional space has a basis.

89. Theorem that every linearly-indendent list extends to a basis .

90. Theorem that every space and a subspace of it can be completed by a third subspace with a direct sum.

91.

Note how this section is conceptually unproblemmatic due to the historic hugging of this concept, at the expense of having the highest/strongest/structural/algebraic vantage point!

9.6 2.C. Dimension

92. Theorem that any two bases have the same length. Or in future terminology, that we can define dimension using basis.

93. Definition of dimension.

94. Theorem that subspaces have less-or-equal dimension.

95. Theorem that a linearly-independent list with the right length is a basis.

96. Theorem that a spanning list with the right length is a basis.

97. Theorem that the dimension of a sum is less-or-equal to the sum of dimensions. In fact, the sum of the dimensions minues the dimension of the intersection, which is a space.

9.7 3A. The Vector Space of Linear Maps

98. Definition of linear map.

There is a missing proof in the book, that linear maps as defined, are morphisms.

99. Notation for the set of all linear maps $V \to W$. $\mathcal{L}(V, W)$.

100. Linear map is defined by its action on a basis.

101. Definition of addition and scalar multiplication on $\mathcal{L}(V, W)$.

102. Theorem that $\mathcal{L}(V, W)$ is a vector space.

103. Definition of the product of linear maps.

Once invariance is introduced and $V = W$ this opens the door to operators and an algebra.

104. Theorems about the algebra of linear maps: The multiplication is associative, has identity, distributive. (But not commutative).

105. Theorem that $T(0) = 0$.

9.8 3B. Null Spaces and Ranges

106. Definition of null space.

107. Theorem that the null space is a space.

108. Definition of injective function.

109. Theorem that for a linear function, injectivity is equivalent to null space only containing zero.

110. Definition of range.

111. Theorem that the range is a subspace (of the codomain).

112. Definition of surjective function.

113. Theorem that $\dim(\text{dom}(T)) = \dim(\text{null}(T)) + \dim(\text{rng}(T))$.

114. Theorem that a map to a small space is not injective.

115. Theorem that a mat to a larger space is not surjective.

116. Theorem that a homogeneous system of linear equations always has a non-trivial solution if it has more variables than equations.

To recall this quickly, think of a linear combination of a linearly dependent list of vectors.

117. Theorem that a inhomogeneous system of linear equations may have no solution (i.e has no solution for some choice of constant terms), if it has more equations than variables.

To recall this quickly, think of two 'very long' vectors having to represent an arbitrary vector.

9.9 3.C. Matrices

118. Definition of a matrix $A_{j,k}$.

119. Definition of a matrix of a linear map $\mathcal{M}(T)$.

120. Definition of matrix addition, scalar multiplication.

121. Theorem that the matrix of a sum of maps is the sum of their matrices. Similarly for scalar multiplicatio.

122. Theorem that $\dim(F^{m,n}) = mn$.

123. Definition of matrix multiplication.

124. Definition of the matrix of linear maps.

125. Theorems about columns and rows of matrices that are trivial in summation notation.

9.10 3.D. Invertibility and Isomorphic Vector Spaces

126. Definition of invertible map, by the existence of another *linear* map that is both left right inverse.

We had the misconception that this is a general definition using left and rigth inverses because V, W may be of different dimensions, but this is not possible! They can only be different spaces with equal dimensions, although this is nowhere stated, but can be easily proved.

127. Theorem that an invertible map has a unique inverse.

128. Definition of notation T^{-1}.

129. Theorem that invertibility is equivalent to injectivity and surjectivity.

130. Definition of isomophism, by the existence of invertible maps.

131. Theorem that dimension equality is equivalent to isomophism.

132. Theorem that $\mathcal{L}(V, W)$ is isomorphic to $F^{\dim(V),\dim(W)}$.

133. Theorem that $\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W)$.

134. Definition of the matrix of a vector $\mathcal{M}(v)$.

135. Theorem about columns of matrices related to multiplication of matrix by vector.

136. Theorem that linear maps act like matrix multiplication, by how matrix multiplication was defined.

137. Definition of operators, elements in $\mathcal{L}(V)$.

138. Theorem that for operators, invertibility, injectivity and surjectivity can only occur together (iff.).

9.11 3.E. Products and Quotients of Vector Spaces

139. Definition of a product of vector spaces.

140. Theorem that the product of vector spaces is a product.

141. Theorem that the dimension of a product is the sum of the dimensions.

142. Theorem about direct sum equivalence of injectivity of the Γ function (for n spaces).

143. Theorem that a sum being direct is equivalent to dimensions adding up (for n spaces).

144. Definition of $v + U$.

145. Definition of affine (parallel) set.

146. Definition of quotient space.

147. Theorem that two affine subsets parallel to U are either equal or disjoint. Includes the fruitful $v + U = w + U$ iff. $v - w \in U$.

148. Definition of addition and scalar multiplication of quotient spaces, by hinging on $v + U$ notation.

149. Theorem that quotient spaces are vector spaces.

150. Definition of quotient map π, which maps v to $v + U$.

151. Theorem that $\dim(V/U) = \dim(V) - \dim(U)$.

152. Definition of $\tilde{T} : V/\text{null}(T) \to W$, which slices away the null space of T by passing to quotient space. $\tilde{T}(v + \text{null}(T)) = Tv$.

153. $\tilde{T}$ is linear, injective, has same range as T, and domain iso to range of T.

9.12 4.A. Polynomials

9.12.1 Complex Conjugate and Absolute Value

154. Definition of real and imaginary part functions.

155. Definition of complex conjugate and absolute value.

156. Properties of complex numbers, with some proofs.

9.12.2 Uniqueness of Coefficients for Polynomials

157. A polynomial is the zero function iff. all coefficients are zero.

158. There is a division algorithm for polynomials.

To recall the condition on the degrees, remember that the rest is smaller than the divisor in rational division .

9.12.3 Zeros of Polynomials

159. Definition of the zero of a polynomial.

160. Definition of a factor of a polynomial.

161. Theorem that each zero corresponds to a (1-degree) factor.

162. Theorem that a polynomial has at most as many zeros as its degree.

9.12.4 Factorization of Polynomials over $\mathbb{C}$.

163. Out-of-material theorem that every nonconstant polynomial with complex coefficients has a zero. This is the ′fundamental theorem of Algebra.

Note that it is enough that the coefficients are complex.

164. Theorem that polynomials over $\mathbb{C}$ can be fully factorized.

9.12.5 Factorization of Polynomials over $\mathbb{R}$.

165. Theorem that polynomials with real coefficients have zeros in pairs.

166. Theorem that a quadratic equation in $\mathbb{R}$ is solvable depending on the sign of the discriminant.

167. Theorem that polynomials over $\mathbb{C}$ can be fully factorized with degree 1 and 2 factors.

9.13 5.A. Invariant Subspaces

168. Definition of Invariant subspace.

169. Definition of eigenvalue.

170. Proof of characteristics equivalent to eigenvalue, By the operator $T - \lambda I$ having a non-trivial null space.

171. Definition of eigenvector.

172. Theorem that eigenvectors of different eigenvalues are linearly indepdendent.

173. Theorem that the number of eigenvalues is less-or-equal than the dimension.

174. From an operator T and a space U that is invariant under it. Definition of the opertor restricted to a subspace $T_{|U}$ .Definition of the operator on the quotient space T/U.

The importance here is that invariance in crucial to these being well-defined.

TODO: clarify recall.

9.14 5.B. Eigenvectors and Upper-Triangular Matrices

175. Definition of the power of an operator T^m.

176. Definition of a polynomial over an operator $p(T)$.

177. Definition of a product of polynomials over F, pq.

178. Theorem on multiplicative properties of polynomials over T.

clarify commutativity, note that focusing on operators gets us into a place where non-commutativity of function composition stops being a hurdle, since we reapply the same function.

179. Theorem that on a complex space, every operator has an eigenvalue.

180. Definition of the matrix of an operator $\mathcal{M}(T)$, the only difference from the general definition is that we define only when the same basis is chosen for domain and codomain.

181. Definition of a diagonal matrix.

182. Definition of an upper-triangular matrix.

Note the standard notation with a zero at one corner and a star at the other.

183. Theorem about two conditions equivalent to the matrix of an operator T being upper-triangular, by span and invariance.

For recall, simply recall the rep Tb_i columns. For a less graphical view, think of it as the representation of Tb_i having zero for all $j > i$ which is quite elegant.

This is not internalized enough.

184. Theorem that over $\mathbb{C}$, every operator has a upper-triangular matrix.

Note that this is not true over $\mathbb{R}$

Prove this.

185. Theorem that we can check the invertibility of an operator by looking at any representation of it that is upper-triangular. Then the operator is invertible iff. all diagonal entries are non-zero.

186. Theorem that the eigenvalues of an operator are precisely the diagonal entries of its upper-triangular matrix if it has one.

We personally note that this means that all upper-triangular matrix of an operator have the same diagonal.

9.15 5.C. Eigenspaces and Diagonal Matrices

187. Definition of an eigenspace.

188. Theorem that the sum of eigenspaces is direct.

189. Definition of a diagonalizable operator.

190. Multiple conditions equivalent to diagonalizability.

TODO: memorize, internalize

191. Theorem that the existence of $\dim(V)$ distinct eigenvalues im[lies diagonalizability.

9.16 6.A. Inner Products and Norms

192. Definition of the dot product in $\mathbb{R}^n$.

193. Definition of the inner product.

194. Definition of an inner product space.

195. Properties of an inner product, with some proofs.

196. Definition of norm.

197. Properties of norm, with proofs.

198. Definition of orthgonality.

199. Theorem that 0 is orthogonal to any vector and is the only vector orthogonal to itself.

internalize

200. Theorem: The Pythagorean theorem.

201. Theorem that a certain specific decomposition is always orthogonal.

202. Theorem: The Cauchy-Schwarz Inequality.

203. Theorem: The triangle inequality.

204. Theorem: The parallelogram equality.

9.17 6.B. Orthonormal Bases

205. Proof about the norm of an orthonormal linear combination.

206. Proof that an orthonormal list is linearly independent.

207. Definition of an orthonormal basis.

208. Lemma that an orthonormal list of the right length is a basis.

209. Theorem about how to write a vector's representation wrp to an orthonormal basis, by taking inner products.

210. Thereom about the Gram-Schmidt procedure making an linearly independent list orthonomal without changing its span.

211. Lemma that an orthonormal basis always exists, by applying Gram-Schmidt.

212. Lemma that an orthonormal list extends to an orthonormal basis.

213. Theorem that if an operator has a UT matrix, then it has a UT matrix wrp an orthonormal basis.

214. Theorem: Schur's theorem, that any operator over a finite dimensional complex vector space has a UT matrix over some orthonormal basis.

215. Definition of a linear functional, by being element of $\mathcal{L}(V, F)$. **216.** We used to think the codomain needs to be one dimensional, but not necessarily F.

217. Theorem: The Reisz representation theorem. A linear functional ϕ can be characterized by a unique vector u such that $\phi(v) =< u, v >$.

10 Textbook

Linear Algebra Done Right. third edition, 2015. Sheldon Axler. Video lectures are also available: "https://www.youtube.com/playlist?list=PLGAnmvB9m7zOBVCZBUUmSinFV0wEir2Vw

Contents

1 **The Algebra** 1

2 **Basis** 6

3 **Isomorphism** 8

4 **Matrices** 8

5 **Polynomials** 9

6 **Eigenvalues** 10

7 **Inner Product Spaces** 11

8 **Miscellanea** 11

9 **Definitions, Theorems and Lemmata** 11
- 9.1 1.A. $\mathbb{R}^n$ and $\mathbb{C}^n$ . . . 11
- 9.2 1.B. Definition of Vector Space . . . 11
- 9.3 1.C. Subspaces . . . 12
- 9.4 2.A. Span and Linear Independence . . . 13
- 9.5 2.B. Bases . . . 14
- 9.6 2.C. Dimension . . . 14
- 9.7 3A. The Vector Space of Linear Maps . . . 15
- 9.8 3B. Null Spaces and Ranges . . . 15
- 9.9 3.C. Matrices . . . 16
- 9.10 3.D. Invertibility and Isomorphic Vector Spaces . . . 16
- 9.11 3.E. Products and Quotients of Vector Spaces . . . 17
- 9.12 4.A. Polynomials . . . 17
  - 9.12.1 Complex Conjugate and Absolute Value . . . 17
  - 9.12.2 Uniqueness of Coefficients for Polynomials . . . 17
  - 9.12.3 Zeros of Polynomials . . . 18
  - 9.12.4 Factorization of Polynomials over $\mathbb{C}$. . . . 18
  - 9.12.5 Factorization of Polynomials over $\mathbb{R}$. . . . 18
- 9.13 5.A. Invariant Subspaces . . . 18
- 9.14 5.B. Eigenvectors and Upper-Triangular Matrices . . . 19
- 9.15 5.C. Eigenspaces and Diagonal Matrices . . . 19
- 9.16 6.A. Inner Products and Norms . . . 20
- 9.17 6.B. Orthonormal Bases . . . 20

10 **Textbook** 21