The kinds of assertion statements employed in various kinds of deduction systems may be very roughly characterised as follows.

(1) **Hilbert style**. Assertions of the form "$\vdash \beta$" for some logical formula β.

(2) **Gentzen style**. Conditional assertions are the "elementary statements" of the system.

(2.1) **Natural deduction**. Assertions of the form "$\alpha_1, \alpha_2, \ldots \alpha_m \vdash \beta$" for logical formulas α_i and β, for $i \in \mathbb{N}_m$, where $m \in \mathbb{Z}_0^+$.

(2.2) **Sequent calculus**. Assertions of the form "$\alpha_1, \alpha_2, \ldots \alpha_m \vdash \beta_1, \beta_2, \ldots \beta_n$" for logical formulas α_i and β_j, for $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, where $m, n \in \mathbb{Z}_0^+$.

The Logic of the Diagonal Argument

Many mathematicians aggressively maintain that there can be no doubt of the validity of this proof, whereas others do not admit it. I personally cannot see an iota of appeal in this proof ... my mind will not do the things that it is obviously expected to do if this is indeed a proof.

—P .W. Bridgman (1955), p. 101

GRADUS
AD
PARNASSUM,
Sive
MANUDUCTIO
AD
COMPOSITIONEM MUSICÆ
REGULAREM,
Methodo novâ, ac certâ, nondum antè tam exacto ordine in lucem edita :
Elaborata à
JOANNE JOSEPHO FUX,

Theorem 11.7.2 effectively says that if the set of points where a proposition is true is a closed set (i.e. $\forall x \in X, ((\{s \in X;\ s < x\} \subseteq S) \Rightarrow x \in S))$, and the set of all points where the proposition is false is a closed set (i.e. $\forall Y \in \mathbb{P}(X) \setminus \{\emptyset\}, \exists z \in Y, \forall y \in Y, z \leq y$), then either the proposition is true everywhere or false everywhere. This is because the two sets are disjoint. (If the set X is non-empty, it is easily shown that the proposition cannot be false everywhere. The set X must have a minimum z, which gives $\{s \in X;\ s < z\} = \emptyset \subseteq S$. Therefore $z \in S$. I.e. the property is always true for the minimum element z of X.)

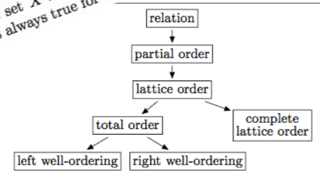

Figure 11.6.1 Family tree for classes of order relations, including various well-orderings

11.6.6 Remark: *Motivation for focusing on left well-orderings.*
The definition of a well-ordering is not symmetric. A relation R can be a well-ordering while its inverse R^{-1} is not. Thus in Definition 11.6.1, for example, the phrase "unique minimum element" would need to be replaced by "unique maximum element" if "$\leq$" is replaced by the corresponding representation "$\geq$". The mathematics of well-ordering would be the same with either "minimum" or "maximum" in Definition 11.6.1. But then it would be necessary to make conversions for each application.

Probably one good reason why one-sided well-orderings are customarily defined rather than two-sided ones is the fact that transfinite induction (in Section 11.7) requires only a one-sided well-ordering. Transfinite induction is similar to the standard induction on the non-negative integers. Every non-empty subset of the non-negative integers, with the usual order, is guaranteed to have a minimum element. This is why standard induction works.

Transfinite induction is not the only application of well-orderings. The existence of the infimum of any non-empty set is guaranteed if the order is a left well-ordering. Similarly, the existence of the supremum is guaranteed if the order is a right well-ordering.

Transfinite induction as a means of proof and transfinite recursion as a means of definition are now a commonplace in the toolkit of any pure mathematician: textbooks of classical analysis such as Hobson 1921 contain frequent illustrations.