http://mathoverflow.net/questions/58696/why-study-lie-algebras

Here is a brief answer: Lie groups provide a way to express the concept of a continuous family of symmetries for geometric objects. Most, if not all, of differential geometry centers around this. By differentiating the Lie group action, you get a Lie algebra action, which is a linearization of the group action. As a linear object, a Lie algebra is often a lot easier to work with than working directly with the corresponding Lie group.

Whenever you do different kinds of differential geometry (Riemannian, Kahler, symplectic, etc.), there is always a Lie group and algebra lurking around either explicitly or implicitly.

It is possible to learn each particular specific geometry and work with the specific Lie group and algebra without learning anything about the general theory. However, it can be extremely useful to know the general theory and find common techniques that apply to different types of geometric structures.

Moreover, the general theory of Lie groups and algebras leads to a rich assortment of important explicit examples of geometric objects.

I consider Lie groups and algebras to be near or at the center of the mathematical universe and among th most important and useful mathematical objects I know. As far as I can tell, they play central roles in most other fields of mathematics and not just differential geometry.

ADDED: I have to say that I understand why this question needed to be asked. I don't think we introduce Lie groups and algebras properly to our students. They are missing from most if not all of the basic courses. Except for the orthogonal and possibly the unitary group, they are not mentioned much in differential geometry courses. They are too often introduced to students in a separate Lie group and algebra course, where everything is discussed too abstractly and too isolated from other subjects for my taste.

http://geocalc.clas.asu.edu/html/GeoCalc.html

http://geocalc.clas.asu.edu/html/UGC.html

Lie's motivation for studying Lie groups and Lie algebras was the solution of differential equations. Lie algebras arise as the infinitesimal symmetries of differential equations, and in analogy with Galois' work on polynomial equations, understanding such symmetries can help understand the solutions of the equations.

I found a nice discussion of some of these ideas in *Applications of Lie groups to differential equations* by Peter J. Olver, in Springer-Verlag's GTM series.

Chapter 19

The Shape of Differential Geometry in Geometric Calculus

David Hestenes

In mathematics and especially differential geometry, a **Kähler manifold** is a manifold with three mutually compatible structures; a complex structure, a Riemannian structure, and a symplectic structure. On a Kähler manifold *X* there exists Kähler potential and the Levi-Civita connection corresponding to the metric of *X* gives rise to a connection on the canonical line bundle.

Smooth projective algebraic varieties are examples of Kähler manifolds. By Kodaira embedding theorem, Kähler manifolds that have a positive line bundle can always be embedded into projective spaces.

In mathematics, a **symplectic manifold** is a smooth manifold, *M*, equipped with a closed nondegenerate differential 2-form, ω, called the symplectic form. The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds. For example, in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field, the set of all possible configurations of a system is modeled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

Any real-valued differentiable function, *H*, on a symplectic manifold can serve as an **energy function** or **Hamiltonian**. Associated to any Hamiltonian is a Hamiltonian vector field; the integral curves of the Hamiltonian vector field are solutions to Hamilton's equations. The Hamiltonian vector field defines a flow on the symplectic manifold, called a **Hamiltonian flow** or symplectomorphism. By Liouville's theorem, Hamiltonian flows preserve the volume form on the phase space.