Jad Nohra, Private Note. Revised: July 18, 2016

Continuity

in Jester Notation

1. y is an ***r-enlargement*** of x.

- y subsumes all the elements of x under the r relation.

$$\doteqdot \left\{\begin{matrix} {}_\circ\boldsymbol{x} \\ {}_\circ\boldsymbol{y} \\ {}^{\cdot}\boldsymbol{r}_\circ \end{matrix}\right\} \frac{{}^{[\,.\boldsymbol{\alpha}^*_x]}\boldsymbol{r}_y}{x \overset{r}{\leq} y}$$

2. A ***set-split*** of x.

$$\doteqdot \left\{{}_\circ\boldsymbol{x}\right\} \frac{x \;=\; \cup \begin{matrix} \diagup\ \triangle \\ \diagdown\ \triangledown \end{matrix}}{x \prec^{\triangle}_{\triangledown}}$$

3. r is a ***loosly-in*** relation. [1]

- r is such that when it holds between a point and a set
  1. and its set is r-enlarged, it holds between the point and the larger set.
  2. and its set is $\in$-split, it holds between the point and at least one part.
- r is
  1. r-enlargement resistant.
  2. $\in$-splitting resistant.

$$\doteqdot \left\{\begin{matrix} {}_{.}\boldsymbol{\alpha}^* \\ {}_\circ\boldsymbol{\beta}^+ \\ {}^{\cdot}\boldsymbol{r}_\circ \equiv \circledast \end{matrix}\right\} \frac{\left[\Rightarrow \dfrac{{}^{\alpha}\boldsymbol{r}_\beta + \left[\beta \overset{r}{\leq} \varphi\right]}{{}^{\alpha}\boldsymbol{r}_\varphi}\right] + \left[\Rightarrow \dfrac{{}^{\alpha}\boldsymbol{r}_{\left[\beta \prec^{\triangle}_{\triangledown}\right]}}{\vee \begin{matrix} \diagup\ {}^{\alpha}\boldsymbol{r}_\triangle \\ \diagdown\ {}^{\alpha}\boldsymbol{r}_\triangledown \end{matrix}}\right] \Bigg\| \left[\Rightarrow \dfrac{\alpha \circledast (\beta \leq \varphi)}{\alpha \circledast \varphi}\right] + \left[\Rightarrow \dfrac{\alpha \circledast (\triangle + \triangledown)}{(\alpha \circledast \triangle) \vee (\alpha \circledast \triangledown)}\right]}{{}_{\mathcal{L}}\!\bowtie^{\boldsymbol{r}}}$$

4. x ***joins*** y. [2]

1

- Add discrete and concrete choice examples.
- The challenge is an abstract axiomatic definition of asymptotically zero distance!

2

- dict: this is 'near' in @c4
- Add into the notation a way to show that $x \cap y \not\equiv x \bowtie y$
- Note that strictly-in is a (silly/trivial) kind of loosly in.

- x is said to join y not only when it's in it, but also if it's loosly† in it.
- The join relation is the closure of the loosly-in relation with respect to 'strictly-in' set membership.

$$\div \left\{\begin{matrix} {}_{.}\boldsymbol{x} \\ {}_{\circ}\boldsymbol{y} \end{matrix}\right\} \frac{\vee \begin{matrix} \diagup\ \boldsymbol{x}_{y} \\ \diagdown\ {}^{x}_{\mathcal{L}}\bowtie^{\dagger}_{y} \end{matrix}}{{}^{x}\bowtie_{y}}$$

5. $\overline{x}$ is the ***closure*** of x.

- The closure of x is x enlarged with its join† points.

$$\div \left\{\begin{matrix} {}_{\circ}\boldsymbol{x} \\ {}_{.}\boldsymbol{\alpha}^{*} \end{matrix}\right\} \frac{\cup \begin{matrix} \diagup\ \boldsymbol{x} \\ \diagdown\ \left\{ \boldsymbol{\alpha}^{*} \bowtie^{\dagger}_{x} \right\} \end{matrix}}{\overline{\boldsymbol{x}}}$$

6. $\widetilde{x}$ is the ***border*** of x.

- The border of x is what is ***common*** between its and its complement's closures.
- The border of x is what is ***shared*** between its closed enlargement and its complement's closed enlargement.
- To bootstrap the concept of border of a set (out of its closure), we close (if not already so) both the set and its complement, then take the intersection of those two closed sets, by which one can ***cross*** between the set and its complement. [3]

$$\div \left\{ {}_{\circ}\boldsymbol{x} \right\} \frac{\cap \begin{matrix} \diagup\ \overline{+\boldsymbol{x}} \\ \diagdown\ \overline{-\boldsymbol{x}} \end{matrix}}{\widetilde{\boldsymbol{x}}}$$

7. $\widehat{x}$ is the ***interior*** of x.

- The interior of x is x without its border.

$$\div \left\{ {}_{\circ}\boldsymbol{x} \right\} \frac{- \begin{matrix} \diagup\ \boldsymbol{x} \\ \diagdown\ \widetilde{\boldsymbol{x}} \end{matrix}}{\widehat{\boldsymbol{x}}}$$

8. x is ***closed***.

- x is a closure.

$$\div \left\{ {}_{\circ}\boldsymbol{x} \right\} \frac{= \begin{matrix} \diagup\ \overline{\boldsymbol{x}} \\ \diagdown\ \boldsymbol{x} \end{matrix}}{{}_{\mathcal{C}}\boldsymbol{x}}$$

9. x is ***open***.

[3] Stress that here is where the complement comes into play, motivated by the 'outward-looking' part of topology, the relation of a set to the 'outside' Maybe better, simlply because the definition 'loosly-in' but not 'strictly-in' fails for closed sets. The only way to extract their border is looking at their complement ('open') sets.

- x has a rest-closure, its rest is closed, it is ***rest-closed***.
- A set can be both open and closed, obviously this only happens when its border is empty, that is, what is ***common*** (what can be crossed†) between it and its complement is empty, that is, both it and its complement are closed (and also then open).

$$\doteqdot \left\{ {}_\circ\boldsymbol{x} \right\} \frac{= \begin{matrix} \diagup\ \widehat{\boldsymbol{x}} \\ \diagdown\ \boldsymbol{x} \end{matrix} \ \Bigg\| \ {}_{\mathcal{C}}[-\boldsymbol{x}]}{{}_{\mathcal{O}}\boldsymbol{x}}$$

10. x is a ***limit*** of y.

- x is join†-recoverable from y after its removal from the latter.
- x ***proper-joins*** y.

$$\doteqdot \left\{ \begin{matrix} {}_.\boldsymbol{x} \\ {}_\circ\boldsymbol{y} \end{matrix} \right\} \frac{{}^{x}_{\bowtie}\gamma^{\dagger}[y - x]}{y \xrightarrow{\gamma} x \ \Big\| \ {}^{x}(\bowtie)_{y}}$$

11. y is a ***filter base*** on x

- y is a bunch of x epsets with a set in common.

$$\doteqdot \left\{ \begin{matrix} {}_\circ\boldsymbol{x} \\ {}_\circ\boldsymbol{y} \end{matrix} \right\} \frac{\boldsymbol{y}_{\mathcal{P}^+_x} + \left[\left[\cap \begin{matrix} \diagup\ {}_\circ\triangle^*_y \\ \diagdown\ {}_\circ\nabla^*_y \end{matrix} \right]^\star \ \Bigg\| \ \overline{\boldsymbol{y}}^{\cap_n} \right]}{{}^{y}\underline{\boldsymbol{\mathcal{F}}}_x}$$

12. x, y are ***disjoint***

- x, y are set-separated.
- x, y have no intersection.

$$\doteqdot \frac{\not\cap \begin{matrix} \diagup\ {}_\circ\boldsymbol{x} \\ \diagdown\ {}_\circ\boldsymbol{y} \end{matrix}}{{}_{\mathcal{S}}\boldsymbol{\Upsilon}^x_y}$$

13. y is a ***neighborhood*** of x

- y contains an open set that contains x.

$$\doteqdot \frac{{}^{[\cdot\|\circ]}\boldsymbol{x}\ {}_{\mathcal{O}}\boldsymbol{\alpha}^\star_\circ\boldsymbol{y}}{{}^{y}\boldsymbol{\mathcal{N}}_x} \ \Bigg\| \ {}^{4}$$

14. Closure and Boundary, first near, filter-base and then filter! @c4,p110, Theorem 6.2. In a topological space X, a point x is near a subset A, if and only if, there is a filter base B in A converging to x. http://en.citizendium.org/wiki/Neighbourhood_(topology) https://en.wikipedia.org/wiki/Topological_

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[4]

- https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)#Topology_from_neighbourhoods
- Open: does not contain it's natural border at places. Closed: does contain.. . Clopen: easy and obvious. Open and not clopen: easy, etc. !!!

space#Definition https://en.wikipedia.org/wiki/Characterizations_of_the_category_of_topological_spaces#Definition_via_closeness_relation https://en.wikipedia.org/wiki/Closeness_(mathematics)#Closeness_relation_between_a_point_and_a_set [5]

15. x, y are ***separated***

- x, y are topo-separated
- Neither of x, y intersects the closure of the other.
- x, y are set-separated, and neither contains a limit point of the other.

$$\doteqdot \frac{\left[{}_{\mathcal{S}}\boldsymbol{\Upsilon} \begin{array}{l} \diagup \ \overline{\triangle} \\ \diagdown \ \triangledown \end{array} \Big| {}_{\circ}[\boldsymbol{x}, \boldsymbol{y}] \right] \Big\| \left[{}_{\mathcal{S}}\boldsymbol{\Upsilon}^{x}_{y} + \not\cap \begin{array}{l} \diagup \ .\boldsymbol{\alpha}^{*}_{\frac{}{\triangle}} \\ \diagdown \ \triangledown \end{array} \right]}{{}_{\mathcal{T}}\boldsymbol{\Upsilon}^{x}_{y}}$$

16. x, y are ***boundary-separated***

- x, y are closure-separated.
- x, y are separated, and so are their boundaries.

$$\doteqdot \frac{\left[{}_{\mathcal{S}}\boldsymbol{\Upsilon} \begin{array}{l} \diagup \ {}_{\circ}\overline{\boldsymbol{x}} \\ \diagdown \ {}_{\circ}\overline{\boldsymbol{y}} \end{array} \right] \Big\| \left[{}_{\mathcal{T}}\boldsymbol{\Upsilon}^{x}_{y} + \not\cap \begin{array}{l} \diagup \ \partial x \\ \diagdown \ \partial y \end{array} \right]}{{}_{\partial}\boldsymbol{\Upsilon}^{x}_{y}}$$

17. x, y are ***function-separated*** @c20,3.1

18. x is Connected

$$\doteqdot \frac{{}^{[\mathcal{S}+\mathcal{T}]}\boldsymbol{x} + \left[x \neq \cup \begin{array}{l} \diagup \ {}^{[\mathcal{O}\|\mathcal{C}]+}\boldsymbol{y}^{*} \\ \diagdown \ {}^{[\mathcal{O}\|\mathcal{C}]+}\boldsymbol{z}^{*+\not\cap y} \end{array} \right]}{{}^{x}\boldsymbol{\varpi}}\text{[6]} \Bigg\| \doteqdot \frac{\doteqdot \overline{disc}}{{}^{x}\boldsymbol{\varpi}}\text{[7]} \Bigg\|$$

[5]

- continue here, also see neighborhood system in note above, it's the same idea? the wiki page https://en.wikipedia.org/wiki/Neighbourhood_system is silent about closure and limit points.
- @c4,p110
- don't forget the 'better' categorical space [Characterizations of the category of topological spaces], and https://books.google.de/books?id=1ttmCRCerVUC&pg=PA198&dq=topology+limit+filter+closure

[6]

- WIP
- https://proofwiki.org/wiki/Definition:Connected_(Topology)/Subset
- If we use the boundary definition, we need to explicitely pass to a subspace
- Let us do the above but also use the induced-subspace-free version as in the reference (def. 1), which then needs the concept of separation, which requires closure.

[7] This is a bad definition (for now) because:

- It is a negative definition, we better define disconnected first
- It uses the for now confusing notion of 'open' and relies on 'induced (subspace)' topology, such that [0,1] is open in the [0,1]

19. x is ***near*** y. [8]

- x is either in y or it joins it, given a choice of join.

$$\doteqdot \left\{\begin{matrix} {}_{\cdot}\boldsymbol{x} \\ {}_{\circ}\boldsymbol{y}^{+} \\ {}_{\bowtie}\boldsymbol{\gamma}^{\dagger} \end{matrix}\right\} \frac{\vee \begin{matrix} \diagup \; \boldsymbol{x}_y \\ \diagdown \; {}^{x}\boldsymbol{\gamma}_y \end{matrix}}{{}^{x}\boldsymbol{\nu}_y^{\gamma}}$$

20. y is an Ultrafilter on x

$$\doteqdot \frac{\left[\doteqdot \dfrac{y_{\mathcal{P}_x^+} + \overline{\cap}_n + \overline{\supset}}{y_{\mathcal{F}_x}}\right] + \left[\forall \dfrac{\boldsymbol{z}_x^*}{\mathbf{1}_y(z) + \mathbf{1}_y(-z) = 1}\right]}{y_{\mathcal{U}_x}}$$

[9]

21. Conceptual continuity

$$\doteqdot \frac{\sim}{\underbrace{\sim_{\mathcal{M}}}_{\text{topological continuity}} \quad \Big| \quad \underbrace{\sim_{\mathcal{W}}}_{\text{pen continuity}}}$$

22. Mathematical continuity

$$\doteqdot \frac{\left[\doteqdot \dfrac{f_{X \to Y}^{\mathcal{T}} + \overrightarrow{\varnothing}}{\sim_D}\right] + \overrightarrow{\text{near}}}{\sim_C}$$

23. $\sim\ :\ [\sim_{\mathcal{M}} : \sim_n^{\text{top}}] + [\sim_{\mathcal{W}} :$ 'pen continuity'$]$

24. Continuity: {Generalized continuities} & $\mathcal{W}$(pen continuity)

25. There are many mathematical concepts that all (called ***generalized continuities*** in topology), when translated, project to model the 'real world' continuity, also called 'pen continuity' [@c19].

26. Choosing one mathematical concept of continuity and declaring it as the definition of 'continuity' (as opposed to 'uniform continuity' or 'Darboux property') is mostly bias.

subspace of R, since it is the union of (-1,1] and [0, 2), http://www-history.mcs.st-and.ac.uk/~john/MT4522/Lectures/L14.html, http://www-history.mcs.st-and.ac.uk/~john/MT4522/Lectures/L19.html

- we must first use nearness topology and/or set theoretic/discrete measure/distance until we develop the right reading in terms of boundary containment

[8] Add into the notation a way to show that $x \cap y \not\equiv x\nu y$ We don't need this for closure since as we see in @c4,p6 points in the set are near, but must not necessarily be limits (consider a ball and an isolated point, it will be in the closure but is not part of the added closure points). To show this in the alt. definition of closure using the DefinClosuExt We need this though to define the usual 'limit' @c4,p6

[9]

- Filter, WHAT IS A FILTER THAT IS NOT AN ULTRAFILTER? cofinite filter! (filters as additive measure), also 'neighborhood filter', add 'how to read' as 'have sthg in common' and in terms of dist?

- Also see @c20,p17

27. The ***Darboux property*** (viz. the ***intermediate value property, IVP or IVT***), seems to be the most minimal, loose and basic notion of continuity.

28. The ***Brouwer fixed-point*** theorem in one dimension is the intermediate value theorem.

29. The intermediate value theorem implies that on any great circle around the world, for any continuous quantity (e.g temperature), there exist two ***antipodal*** points with the same value.

Proof: Take f to be any continuous function on a circle. Draw a line through the center of the circle, intersecting it at two opposite points A and B. Let d be defined by the difference $f(A)-f(B)$. If the line is rotated 180 degrees, the value d will be obtained instead. Due to the intermediate value theorem there must be some intermediate rotation angle for which $d=0$, and as a consequence $f(A)=f(B)$ at this angle.

This is the special case of the ***Borsuk-Ulam*** theorem.

30. In the case of functions mapping $\mathbb{R}$ to $\mathbb{R}$, the following proper inclusion holds [@c8]:

$$\mathcal{C} \subset \text{Ext} \subset \text{ACS} \subset \text{Conn} \subset \mathcal{D}$$

where

- $\mathcal{C}$: continuous functions
- Ext: extendable functions (Darboux-like functions)
- ACS: almost-continuous functions in the sense of Stallings
- Conn: connectivity functions
- $\mathcal{D}$: Darboux functions

$\mathcal{W}$:	World
$\mathcal{M}$:	Mathematical
$\circ, \mathcal{S}$:	Set
$\cdot, \mathcal{P}$:	Point
$\mathcal{T}$:	Topological
$\overrightarrow{\text{x}}$:	Preserves x
$x^{\dagger}$:	x is a choice
x^{*}:	$\forall x$
$x^{\star}$:	$\exists x$, alt. $x \neq \emptyset$
x^{+}:	x excluding $\emptyset$
$x^{(+)}$:	x proper: excluding $\emptyset$, $-\emptyset$
$\triangle, \triangledown$:	Permutation of 'one', 'other'

$\mathcal{O}$:	Open set (topologically)
$\mathcal{C}$:	Closed set (topologically)
$\not{\mathcal{O}}$:	Connected set (topologically)
$\mathcal{N}$:	Neighborhood (topologically)
$\square$:	Compact set (topologically)
$\boldsymbol{b}(x)$:	Boundary of x (topologically)

$\mathcal{F}$:	Filter, WHAT IS A FILTER THAT IS
$\mathcal{U}$:	Ultrafilter
$\sim$:	Continuity (conceptually)
$\sim_D$:	Darboux continuity
$\sim_C$:	Standard continuity

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