Potter.

- (p7). Formalist and Realist do not mean the same for mathmicians and philers. We are formalists once we establish a formal system, but realists during the establishment phase 'what we mean', while not refusing any formal system whatsoever.
- (p7). If Lakatos really did 'attach' the axiomatic 'method', then we must wonder what the 'method' exactly is.
- (p6). Formally eliminable definitions is a good idea to remember. Also, the names are simply all wrong, axioms are simply 'bootstraps', 'structural initial conditions'
- (p7). For a realist, truth then is not a 'toy', and 'existence' actually means that. We do not belong to this camp. To a philer though, a constructivist is not a realist.
- (p7). Note that both Platonists and Constructivists are realists. They differ on what 'existence' 'means'.
- (p8). Nice: the difference is if the mathical obcts owe their existence to the signs used to express them. If not, then truth about thms about them might be independent of signs, and hence true inaccessible by any means thms do then 'exist' (Platonist).
- (p8). Formalism emerged during the 19th Century (with multiple geoms, etc.). It is exemplified by the starting if in: "if any structure satisfs the axms, then it satisfs the thms" (of course, we must add that the logic itself must be taken in the 'if').
- (p8). Formalism has had many names: implicationism, deductivism, if-thenism, eliminative structuralism.
- (p8-9). Holds answers or at least order and validation into our early foundational questions. This is gold. At no point, then, will mathematics assert anything unconditionally.
- (p9). We disagree that thoroughgoing implicationism is a harsh discipline. There is simply a lot of missing, ignore but very necessary work to be done to link models to their applications.
- (p10). We do agree with Mayberry, there is no foundation. Foundation presupposes a certain kind of root based structure, one which is surely to simple to encompass mathematics.
- (p10). I don't know how Frege 'demolished' formalism, the extreme case of implicationism in which mathematics is no more than a game played with symbols. Maybe he attacked the 'no more', but I don't think any one really means 'no more'. They mean that only ultimately, the hope is that some times, it is not only a game played with symbols but a bit more: a game that models something real and hence useful.
- (p10). The difference between formalism and postulationism is subtle at first sight, but is actually very important for our logic studies, let us try to understand it properly. At the very least, the connection between the want for consistency and the meaning inference is applaudable and should be remembered. As for the conviction of the truth of T, it is a conviction!

- (p10). We don't understand this "For the great advantage of postulationism over implicationism is that if we are indeed entitled to postulate objects with the requisite properties, anything we deduce concerning these objects will be true unconditionally.". Maybe we are misunderstanding formalism and it is so that formalism really disallows anything from being a model of anything else.
- (p11). Balaguer's quote is the kind of statement that we by now regard as meaningless. Existence is 'existence', a word in our meta-game.
- (p11). If there can be such a large difference in opinion between what is realist, formalist and full-blooded platonist, then the whole classification topic falls. Just another game.
- (p12). A statement we can use confidently and remember distinguishing FOL from SOL. "Calling this calculus ‘first-order’ marks that the variables we use as placeholders in quantified sentences have objects as their intended range....First-order predicate calculus is thus contrasted with the second-order version which permits in addition the use of quantified variables ranging over properties of objects."
- (p9). Validating quote of a bootstrapping example (including Poincare as a bonus): "One way of thinking of a structure is as a certain sort of set. So when we discuss the properties of structures satisfying the axioms of set theory, we seem already to be presupposing the notion of set. This is a version of an objection that is sometimes called Poincare´'s petitio because Poicare´ (1906) advanced it against an attempt that had been made to use mathematical induction in the course of a justification of the axioms of arithmetic. In its crudest form this objection is easily evaded if we are sufficiently clear about what we are doing. There is no direct circularity if we presuppose sets in our study of sets (or induction in our study of induction) since the first occurrence of the word is in the metalanguage, the second in the object language."
- (p13) (MAP) By this lucid explanation, we put Goedel's first incompleteness theorem onto our logic map. "I have already mentioned the popularity of the formalist standpoint among mathematicians, and logicians are not exempt from the temptation. If one formalizes the rules of inference, it is important nonetheless not to lose sight of the fact that they remain rules of inference — rules for reasoning from meaningful premises to meaningful conclusions. It is undoubtedly significant, however, that a formalization of first-order logic is available at all. This marks a striking contrast between the levels of logic, since in the second-order case only the formation rules are completely formalizable, not the inference rules: it is a consequence of Go¨del’s first in- completeness theorem that for each system of formal rules we might propose there is a second-order logical inference we can recognize as valid which is not justified by that system of rules."
    - Is it maybe that the formalization of the inference rules amounts to semantic completeness? The question then becomes: What is the difference between the formalization of formation rules and the formalization of inference rules. And the thing to remember is that the latter is possible in FOL but not in SOL.
- (p13) (MAP) Yet another linked to the above item to add is the link to the countability, model and Loewenheim/Skolem. The item is that there are larger systems that FOL in which the inference rule can be formalized. "Notice, though, that even if there is some reason to regard formalizability as a requirement our logic should satisfy (a question to which we shall re- turn shortly), this does not suffice to pick out first-order logic uniquely, since there are other, larger systems with the same property. An elegant

theorem due to Lindströ m (1969) shows that we must indeed restrict ourselves to reas- oning in first-order logic if we require our logic to satisfy in addition the Loewenheim/Skolem property that any set of sentences which has a model has a countable model. But, as Tharp (1975) has argued, it is hard to see why we should wish to impose this condition straight off. Tharp attempts instead to derive it from conditions on the quantifiers of our logic, but fails in turn (it seems to me) to motivate these further conditions satisfactorily."

- (p13). For once, someone explains why we need schemes to begin with and it has much to do with fininfinity. A scheme is a finite way to describe an 'infinite' number of axioms by 'template'. This happens across the interface of meta-langugage to formal language. We need a name for this because it is not the usual 'translation' because it seems to 'translate' something in the formal language while 'creating it' at the same time. But this is maybe necessary every time we 'create a new language' or better isolate a game within our largest language for the specific purpose of using it as a language (formal or not). Maybe the seeming infinity of the process is best understood by noticing that the scheme (template) becomes an axiom when it is 'instanced' and there is an infinity of instance because there is an infinity of objects for variables to range on. So although we already had an infinity of those without complaint, we complain about an infinity of axioms. But then there is no need to complain about anything at all. If anything, we should think about why we did not complain about the infinity of the variable range, and here we are back to the infinity of the natural numbers, in connection with the infinite process paradox of words given a finite list of symbols but unrestricted length, or, the model of either lack of death or timeless word construction process. Maybe the best is to say there is no infinity involved here at all in this game. just a template that can be instantiated, along with no restricted length on words. We seem to be going in large scale circles, but this is the reason why we are making this study and charting the logic 'map'.
- (p14). Even more gold here.
  - First validating our previous note saying the following: "The presence of an axiom scheme of this form in a system does nothing to interfere with the system’s formal character: it remains the case that the theorems of a first-order theory of this kind will be recursively enumerable, in contrast to the second-order case, because it will still be a mechanical matter to check whether any given finite string of symbols is an instance of the scheme or not"
  - Second showing that indeed it does all stem from the uncountability of the power set of naturals, which has been the main topic for us in this quest, and the motivation for looking into Intuitionism. There is a nice trivializer thrown in as well. There are 'more' properties than objects, directly tied to the power set: "Kreisel (1967a, p. 145) has suggested — on what evidence it is unclear — that this weakness ‘came as a surprise’ when it was discovered by logicians in the 1910s and 1920s, but there is a clear sense in which it should not have been in the least surprising, for we shall prove later in this book that if there are infinitely many objects, then (at least on the standard understanding of the second-order quantifier) there are uncountably many properties those objects may have. A first-order scheme, on the other hand, can only have countably many instances (assuming, as we normally do, that the language of the theory is countable). So it is to be expected that the first-order theory will assert much less than the second-order one does." (relate this to our written notes on P(E) and intuitionism).
- (p14). The poor poor realist: "But for a realist there will be a further question as to how

we finite beings can ever succeed in forming a com- mitment to the truth of all the infinitely many instances of the scheme."

- (p15). I do not really understand the difference between the first and second order versions, but one important new fact came to mind. Why do we need an axiom for mathematical induction? Why is it not a theorem? The answer is that at this level, we are constructing formally the natural numbers, and this axiom is a statement about their structure, describing their well ordering. In other words, their well ordering is solely expressed with this axiom, without out, we have no axiom 'meaning' the well ordering. If one on the other hand takes well ordering as an axiom, one can prove mathematical induction (within the proper framework).
- Note that if 'what we mean' comes first, then we agree to a 'semantic' precursor to any formal one. However, we do not see this as a necessity, only a common process. One could create a formal game for the sake of it.
- (p16). (hist). An important remark about the history of the distinction between first and second order logic. Frege at his time per example, passed between the two without a twitch. "The distinction between first- and second-order logic that we have been dis- cussing was originally made by Peirce, and was certainly familiar to Frege, but neither of them treated the distinction as especially significant: as van Heijenoort (1977, p. 185) has remarked, 'When Frege passes from first-order logic to a higher-order logic, there is hardly a ripple.' The distinction is given greater primacy by Hilbert and Ackermann (1928), who treat first- and higher-order logic in separate chapters, but the idea that first-order logic is in any way privileged as having a radically different status seems not to have emerged until it became clear in the 1930s that first-order logic has a complete formalization but second-order logic does not.". We should remember the last sentence, and it does not surprise us because of finitistic considerations. Here, it is not about the 'compression' (computability, finite codification) of the reals, but of a certain logic.
- (p16). Quite surprising, especially the last part: "The result of this was that by the 1960s it had become standard to state mathematical theories in first-order form using axiom schemes. Since then second-order logic has been very little studied by mathematicians (although recently there seems to have been renewed interest in it, at least among logicians)."
- (p16). We must disagree and say that this is a bit misguided, the answer being: imitation. "Yet even when the question is simplified in this manner, it is surprisingly hard to say for sure what motivated mathematicians to choose first-order over second-order logic, as the texts one might expect to give reas- ons for the choice say almost nothing about it."