# Differential Forms

## I Eternal Words

**1.** "Is there a relationship between macroscopic and microscopic circulation? In the above counterexamples, we've shown cases where the macroscopic circulation seemed to be unrelated to the microscopic circulation (i.e., the curl). This observations may lead one to wonder if there is any relationship between the macroscopic and microscopic circulation. First of all, roughly speaking, the microscopic circulation measured by the curl is the average macroscopic circulation around a loop in the limit that the size of the loop shrinks to zero. Therefore, assuming the vector field is continuous, the macroscopic circulation should begin to approach the microscopic circulation as one looks at smaller loops. Moreover, it turns out that there is a fundamental relationship between macroscopic circulation and curl, but we designed these counterexamples to sidestep that relationship. Note that for each of these counterexamples, the vector field was not defined along the z-axis. ***If, for example, the vector field was continuously differentiable everywhere, then there is a fundamental relationship between curl and macroscopic circulation called Stokes' theorem. If Stokes' theorem applied, then we'd know that the macroscopic circulation around any loop would be determined by the curl inside the loop***. Understanding Stokes' theorem, however, requires quite a bit more background than this presentation of the curl, including line integrals and surface integrals of vector fields." [Math-Insight, Subtleties about curl].

In other words continuous differentiability ($C^1$) is regularity enough to make pointillistic (microscopic) properties link to macroscopic (average, integral) ones.

## II Vector Fields as a Stepping Stone

**2.** We have paused this study for a while, after tarting 'pointillistic.idraw', then on to 'Gather continuity', to 'cont_short.lzt'. We then turned back to classical mechanics, discovering Landau's beautiful axiomatic treatment, logged in a the tag session 'least_action', then finally back to differential forms.

**3.** This section is about using vector fields (from the more standard vector calculus) as a stepping stone since: "The term "vector field" is usually understood to mean a contravariant vector field, whereas a covariant vector field is often known as a differential form of degree 1, or a "differential 1-form". Then roughly speaking, one may say that velocities are vectors and gradients are forms." [@c6,p.1222]. This stepping stone has less rabbit holes than the discrete approach, the algebaic approach, or the rigorous infinitesimal approach. All being approaches we have slightly started exploring.

**4.** A solid introduction to vector fields might start with Schey's [@c7].

**5.** Let us not forget that differentials of functions [Wikipedia] (e.g $dy = f'(x)dx$) can be approached as:

1. A kind of differential form, specifically, the exterior derivative of a function.
2. Nilpotent elements of commutative rings.
3. Smooth models of set theory.
4. Infinitesimals.

**6.** Let us not forget that in differential geometry, a 'vector' is not (merely) the algebraic 'vector'.

**7.** So if the whole point of differential forms is to be integrated and satisfy the generalized Stokes', and if forms are just covariant versions of fiels, then fields could be studied first in [@c7].

**8.** Why can't we just turn then the covariant forms into contravariant fields? We 'can' possibly by index-lowering (turning from contra to cov, etc.) ***[mst-192198]***, but we suppose that things like gradients of scalar fields are 'naturally' covar, and when the relation between the gradient and the coordinates is abstracted away and made coordinate free, the covar property must still figure somewhere.

**9.** For fixing focus, we can make a specific topic that of ***fully understanding the difference between the gradient as a vector and the gradient as a 1-form***, inspired by "There is no simpler 1-form than the gradient, "df" of a function. Gradient a 1 -form? How so? Hasn't one always known the gradient as a vector? Yes, indeed, but only because one was not familiar with the more appropriate 1 -form concept."[@c7].

**10.** From our initial analysis of explanations of variance using velocity and gradient as examples, we can make the following (tentative) statement: In the coordinate-based formulation of a certain 'thing', the 'thing' depends on other 'things', recursively. If we decide to take one of these other 'things' calling it 'thingx' and change it to another 'thingy', in such a manner that one can go back and fourth between the two (viz. one can see 'thingx' and 'thingy' as two different coordinations of some coordinate-free thing, or one can see 'thingx' as a reference 'thing' and 'thingy' as a transformation of it), there are two way in which the 'change of coordinates' can happen.

In fact, the use of ***the word 'change' (of coordinates) is 'irreführend'***, when one changes the oil of an engine, new oil replaces old oil, and this is what one might initially happens mathematically when we mean 'change of coordinates'. However, what we do is more of a 'chaining', an ***'insertion'*** of a coordinate transformation ***function, which we 'name' as a 'coordinate'***. This is obvious from all explanations since if per example '$x$' is replaced with '$y$', there would be no '$dx$' figuring in expressions in the new coordinate system, but there is, and it is actually used to define variance. The relation between 'thingx' and 'thingy' can be seen as 'x' being a function of 'y', but also as 'y' being a function of 'x', all things being equal, no one version takes preference.

In that sense, the insertion can take place as either '$x \to yx$' or '$x \to xy$', in the following sense, considering a chain of dependencies (with juxtaposition as composition), of functions:

$$A_x = fg \cdots x \cdots qr$$

$$A_{yx} = fg \cdots yx \cdots qr$$
$$A_{xy} = fg \cdots xy \cdots qr$$

When one works in coordinate-free (axiomatic) characterizations, one has no access to the expressions above in a 'numerical' manner, but one still wants to express what one 'means' when one for the specific kind of thing $A$, one talks about 'change of coordinates'. ***Both $A_{yx}$ and $A_{xy}$ are equally legitimate mathematically, but one fixes one of them based on what one has in mind when one speaks of $A$. One of them is called covariant, the other contravariant***. When differentiability is possible, taking the derivative of $A$, one will obtain a product of $\frac{d\cdots}{d\cdots}$, if $\frac{dy}{dx}$ figures the 'thing' is called contravariant and if $\frac{dx}{dy}$ figures instead, then covariant.

Having understood this, it is obvious that in the multi-dimensional case, a mix of variance can happen.

The usual simple explanations of variance in the realm of differential geometry focus on 'Velocity' being contra and 'Gradient' being co. All explanations [1] we saw do no hit the point on the head. We try to do this here succinctly.

***Velocity***: Let $x$ be a function of $t$, and let 'change of coordinates' mean working with $y$ which is obtained from (dependent on) $x$. Let $V_x$ be the 'Velocity' (w.r.t time $t$) in the 'old coordinate system' and '$V_y$' be the 'Velocity' in the 'new one':

$$\longrightarrow y$$

$$\longrightarrow x$$

$$\longrightarrow t$$

Then

$$V_x = \frac{dx}{dt}$$

[1] http://www.idius.net/tutorials/tensor-calculus/contra-vs-covariant/, http://www.mathpages.com/home/kmath398/kmath398.htm

$$V_y = \frac{dy}{dx}\frac{dx}{dt} = V_x \frac{dy}{dx}$$

'$V$ is contrav'.

***Gradient***: Let $\alpha$ be a function of $x$, and let 'coc' mean working with $\alpha$ in terms of $y$, such that $y$ is rooted into $x$ ($x$ depends on $y$) on which $\alpha$ depends. Let $\alpha_x$ be the gradient of $\alpha$ in the 'old coordinate system' with respect to the 'input variable' ($x$), and let $\alpha_y$ be the gradient of $\alpha$ in the 'new coordinate system' with respect to the 'new' 'input variable' ($y$):

$$\longrightarrow \alpha$$

$$\longrightarrow x$$

$$\longrightarrow y$$

Then

$$\alpha_x = \frac{d\alpha}{dx}$$

$$\alpha_y = \frac{d\alpha}{dx}\frac{dx}{dy} = \alpha_x \frac{dx}{dy}$$

'$\alpha$ is cov'.

In a certain sense, we never 'insert' the transformation function 'between the two related things', but we either add it at the 'top' (contrav) or at the 'bottom' (covar). This makes sense, since the we cannot 'split' what is 'originally' given, but we do 'reroute it'. In any case, it might be worth checking our interpretation's consistency with some articles [2] .

How does this relate to purely linear algebraic explanations [3] that project to coordinates using 'perpendiculars' (for cov) versus 'parallels to other axis' (for contrav)? We argue that these explanations give the illusion that is no relation between the two explanations and that basically, this is abuse of the same words for similar concepts in different contexts.

This is a promising hook, which we continue at covar_preshort_1.

[2] http://physics.stackexchange.com/questions/131348/

[3] http://mathpages.com/rr/s5-02/5-02.htm

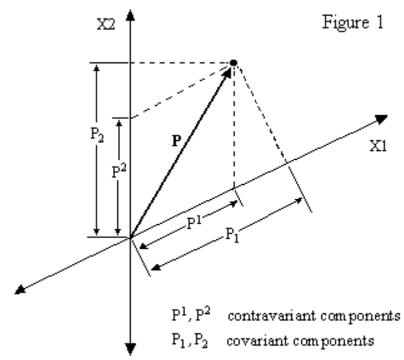


Figure 1

# III Rudiments

## III.1 Real Numbers (Musings on)

**11.** This section was written before our insights into the whole nature of mathematics coming from our study of formal logic. Especially the insights related to set theory, limits, transfinite induction, and the real meaning of 'abstract', the currency of axiomatic theories and the formal (syntactical) nature of axiomatized mathematics.

**12.** Dedekind's construction you are logically beautiful it's true, but Cauchy's converging sequences, even if scarred by equivalence, are still more beautiful than you. At least until a really nice short and human proof is found for all the constructions needed. We call these sequences of recurring kind: type switch by homogeneous translation requiring an infinity. We can merge the types after the bootstrapping, mixing sequences of rationals with sequences of reals and calling all of them real.

**13.** The reals require (non-methodical) creativity by definition.

- The Dedekind cuts are far from constructive, they prove existence, supported by the logic used with its infinity axioms.
- Concretely, when on wants to actually find real numbers that are not rationals, one resorts to limits.

- Limits are defined by an equality, that is, the answer, the limit, can only be ***checked*** to be a limit using infinite sequences.

- Worse, there is an infinity of choices even in crafting the sequence.

- We conclude that in general, a calculation through a limit is a tough question that always needs a creative (problem-specific, ad-hok) answer.

- This is exactly where most tough questions in this field will fall. The existence of special limits such as solutions to special differential equations.

- By this, we see that what we have been trying to work around using notation is in general impossible.

- In fact, the book on constructible reals tries to partially refine the notation of reals, those that we might find with our failed notation.

**14.** Since there is an 'abstract' property based description of the derivative operator, why is there not one for the 'limit operator' which is, just like the differential operator, linear. Maybe the reason is that such an operator evades description through properties and is hence lost in the non-existent journals of negative results. The closest we find to this are bounded linear operators on Banach spaces and band dominated operators on the Hilbert space of all square summable functions.

**15.** For our notation, we could see limits as projecting a function of a variable to a constant where the variable has been 'infinitesimalized' to zero. We see now that in the limit of $f(x, dx)$ where $dx$ tends to zero, $x$ is really a constant. This might give us access to a possible notation since all limits can be re-expressed as tending to zero. Maybe even more, we could also fix the kind of sequence (e.g $1/n$ as n tends to infinity) for purposes of canonical notation.

**16.** From all of the above we conclude that:

1. Numbers obtained from limits are identification of one open end of an interval (the number which a limit is tending to) with one closed end of an interval with the help of a relation between the two intervals.

2. What is required is that, although, for some reason, the domain interval cannot be closed, the governing function between the intervals is 'stable' at the 'added' range limit point within any neighborhood smaller than some given when where stability (might be unsigned) starts to hold.

3. ***Better said, remember the fact that their exist functions of the naturals that do not have to tend to infinity as the domain grows forever. There is no problem imagining this (e.g. the constant function). In fact this is all there is to it. The approximation of $\pi$ by upper and lower circumferences is nothing but this. The circumference cannot cease to be an approximation until the segments have become of zero length, an infinity of points that nevertheless is not dust, as long as the segments have lengths, we can have a better approximation and that one is limited by two finite numbers from above and below, and since the circumference is not made of segments, the segments must be infinitesimated, type switched to infinity from a finite number of straight segments to one curved segment by the way of a length dust infinity so dense that it has length, there is logically nothing against an infinity of zero length segments to have a non-zero length and that is the logic that we need and use, the zero length segments do not exist (except if we identify them with the process since it is they and they are it, and that gives us the nilpotentns), they are simply abstractions of a (homogeneous bounded) process of translation since no real number can ever by finitely written down and maybe this is more essential than we thought until now since the numbers that we can compress are the ones that our logic affords us to write as fininfinte limits, a logic whose axioms are such that we mean that what holds for our finifinites holds for the uncompressible real numbers which is not unreasonable as we can approximate the uncompressible one with***

***finifinites. Limits are nothing but this. The problem it seems, simply is again, that we have been unable to include infinity under 'numbers'. They 'defy algebra' in other words, algebra does not model type switching through infinity. In a sense, that is what we have managed using limits.***

4. Did we go even further in our abstraction that identifying those processes with numbers? By also including 'real' real numbers, numbers that we cannot identify with any processes? Maybe, by saying 'the complete ordered field', but this is a compression of the property of all (ideal) elements with those properties, so again finite, if on a second level, so we did describe at least very inexactly, those 'real' reals, they fill in any gaps in the real ordered field left by the compressible reals.

5. This is the n'th time where we try to reformulate the above. We should stick to the accepted formulation, that is the one rooted in the axioms of set theory.

6. An old fact is our note that real analysis is the use of (a certain) logical framework to tell us that arithmetic can be done on limits, that is, real numbers that are themselves more limits than numbers as the untrained would see them. In other words, it teaches us that this addition of a point to the interval.

7. TODO: Finally we are able to deal with clarity with '@c8 p. 109 Theorem 8'. It is a perfect example of the usefulness of the 'harmless' completion by appeal to 'the' 'complete ordered field', and we now understand the subtlety of this theorem and its relation to LEM without it blinding us.

## III.2 Tangents

**17.** One can arrive at the definition of the derivative by a human limiting process of secants.

**18.** One can then, resisting further interpretation, state that integration of limits of secants re-accumulates this or that.

**19.** The explanation 'best approximation' is devious. Best approximation given which interval? In the limit? In the limit the interval is a point. No. Best approximation in an escaping manner (through limits) in the sense that, thinking from a one sided point of view and in one dimension, whatever secant is given that might be better along a chosen interval in terms of distance between it and the curve compared to distance between the tangent, one can always find a smaller interval where the tangent wins.

**20.** Can we justify the intuition of symmetry for tangents? Actually we can.

- We make the point that eventually, we require that the function is monotone, not fractal and not discontinuous. The exception being that our point of interest is a supermum, in which case we require the function to be eventually monotone on both sides of it.
- Looking from one side, we take the limit of secants, this is intuitive given the monotonicity, it is a supremum, the kind of 'largest that is lower than all' from the days the greeks.
- Looking from the other side, we do the same.
- How come they match, how come the 'best approximation' is the same line on both sides? They don't have to, this is exactly the prerequisite for a function to be differentiable and not merely continuous.
- But what we just described is exactly the 'hidden' symmetry we are after.

**21.** What follows might be wrong (do mean the dot production of the directional derivative? and the relation to total derivative?) Actually, we are not wrong, refine this by looking at VLDU p109 (with proof), 116, 118. And our 'Notes on Dynamics' p35 note*, 36, 37', 42-43.

**22.** Can we justify the 'tangent by properties' approach requiring linearity? Yes, even in one dimension. Again thinking one sided, we require that the slope given any 'positive' direction (yes, this is not coordinate-free thinking) is the negative of the slope in the opposite direction. In this way, the symmetry is equivalent to the linearity property. The scaling without changing sign just indicates the fact that the tangent is a line.

**23.** Passing to higher dimensions and the property of linearity there, we can see that for two dimensions (in terms of domain), all we need is two independent slopes to describe our 'best linear' plane. Now the plane is obtained by linear combinations of the two slopes, and hence we require that the linear combinations continue to yield best approximations along one dimensional lines in the domain and this justifies the property and generalizes it.

### III.3 Functional Analysis†

**24.** Let us think of functions as objects. We already mentioned that by properties, the unary differential operator is linear: $D(af+g) = aD(f) + D(g)$. Now this is indeed quite special if one thinks of all possible operators. On the other hand, is every linear operator a differential? We can immediately see that the answer is no since the integral is also linear. Furthermore, every integral transform (of which there are quite a few) is also linear. So to pinpoint the linear operators that are differential, we need another property, this would be the chain rule$^?$.

**25.** Here is a list of some of the integral transform types: Fourrier, Hartley, Mellin, Laplace, Weierstrass, Hankel, Abel, Hilbert, Poisson and Identity.

**26.** Non-linear differential operators? Yes they do exist. An example is the 'Schwarzian derivative'†. It is related to conformal mapping of circular arc polygons and second order ordinary differential equations.

**27.** All this leads us directly to Hilbert spaces, square integrable functions and bounded linear operators.

**28.** Can one study tensors over 'functions', over Hilbert spaces? Yes.

**29.** In general functional analysis becomes conceptually unproblematic because of our better understanding of mathematical logic, semantics and the axiom of choice which has become not only harmless, but necessary (but not religiously or exclusively by the 'No Choice Rule').

**30.** The 'foundational theorems' of the field are:

- The Banach–Steinhaus theorem or uniform boundedness theorem.
- The Hahn-Banach theorem.
- The Open mapping theorem.
- The spectral theorem. Related to operator theory, the study and classification of linear operators on function spaces. This includes a generalization of matrix diagonalization and eigendecomposition. This study depends heavily on the topology of function spaces.
- The closed graph theorem.

## IV Discrete Differential Forms with (Grady L.J.)

**31.** We already know this, at least since our new understanding of 'abstract', but here is a good quote on infinitesimals as (conceptual and formal) representations. “However, the expression of the Fundamental Theorem of Calculus does not require any concept of limit or infinitesimal but essentially states a topological property of the integral that can be phrased for either continuous or discrete spaces”

**32.** “From an early stage in the development of vector calculus, there was interest in discretizing the equations of vector calculus so that they could be solved in pieces. A major motivation for this approach is that many of the differential operators are linear. In this case, linearity implies that the act of applying an operator to a function may be subdivided into small, local operators and then reassembled to produce the result.”

**33.** “Both the definitions of the differential form and the corresponding derivative operator are invariant (in the sense of tensor analysis) to changes of coordinates and do not require the specification of a metric. In the next section, we shall demonstrate that the derivative of differential forms is a generalization of the common differential dx defined along one coordinate (e.g., x) to ***a differential measuring the change of a function along all coordinates (e.g., x, y, and z) simultaneously***. Once this for- malism is established, the instantiation of the Fundamental Theorem in the discrete setting is direct and transparent."

**34.** "The generalization of the derivative provided by the theory of differential forms in arbitrary dimensions is motivated by the requirement that it must measure how a function changes in all directions simultaneously, just as df/dx measures how a function f changes in the x coordinate direction. This requirement ***leads directly to the antisymmetry property of differential forms*** and ***the exterior algebra that is based on the measurement of volume enclosed by a set of vectors***, which we demonstrate below."

**35.** Here is a true grand summary: "We begin by recalling the basic definitions of p-vectors and p-forms in n dimensions as antisym- metric tensors of contravariant and covariant type, respectively, and the metric tensor that provides a mapping between p-vectors and p-forms. The exterior algebra of antisymmetric tensors motivates the exterior derivative for differential forms, which leads to Generalized Stokes' Theorem and the integrability conditions encapsulated by the Poincaré lemma. Finally, the Hodge star operator provides a mechanism for computing the inner product of two p-forms." We note that the metric tensor sounds like a bridge between the syntatic forms and their 'interpretation' as measure! Here again, inspired by our formal logic studies.

**36.** Another grand summary, relating differential forms to tensors: "In this section we survey the theory of higher-order vectors, their associated dual quantities, known as forms, and the corresponding algebra. These so-called p-vectors and p-forms are special cases of antisymmetric tensors, so this theory could be viewed simply as a subset of tensor analysis, yet their elegant geometric interpretation and their suitability for faithfully representing the physical character of many of the quantities studied in several branches of physics makes them a powerful addition to the tools available for modeling physical systems."

**37.** Interpreting [@c1, p.17], we discover a trivial reading for the exterior product (51).

**38.** A fast path for the subtle, into algebrizing things under integrals by mere observations of what one usually finds there is: "In the expression a above, we notice the absence of terms in dzdy, dxdz, dydx, which suggests symmetry or skew-symmetry. The further absence of terms dxdx, ... strongly suggests the latter." [@c2, p.16]

**39.** ***Scatter*** Having a bit of difficulty with [@c2, p.7], we scatter (X.1) through similar material.

**40.** Perhpaps, an even simpler way to contemplate Differential Forms from a discrete point of view is to look at Oliver Knill's work, which also links us to quantum calculus.

# V Discrete Differential Forms with Knill

**41** (If Archimedes would have known)**.** is our initial source with Knill.

**42.** Now that we know more about many things, including integration, we can have an unbiased and non-destabilizing look at quantum calculus.

**43.** "This result is is also in the continuum the easiest version of Stokes theorem. Technically, one should talk about 1-forms instead of vector fields. The one forms are anti-commutative functions on edges. [40] is an exhibit of three theorems (Green-Stokes, Gauss-Bonnet [37], Poincaŕ e-Hopf [38]), where everything is defined and proven on two pages. See also the overview [42]."

**44** (37)**.** O. Knill. A graph theoretical Gauss-Bonnet-Chern theorem. [38] O. Knill. A graph theoretical Poincar ́ e-Hopf theorem. [39] O. Knill. On index expectation and curvature for networks. [40] O. Knill. The theorems of Green-Stokes,Gauss-Bonnet and Poincare-Hopf in Graph Theory. [41] O. Knill. The Dirac operator of a graph. [42] O. Knill. Classical mathematical structures within topological graph theory.

**45.** It seems that all what is needed from 'orientation' is to be able to do integration 'consistently' and maybe that would be the better name for 'orientation'.

# VI Tensor Products with (Winitzki)

**46.** All is bound together with the Theoreom at the top of [@c3, p.56]. Bind this into notes. TBC.

**47.** Re-add Winitzki's papers, it seems we forgot about them.

# VII Humanly Spoken

**48.** Just like distance is speed multiplied by time, the point of the theory of the integral is to keep this idea alive. Therefore, humanly spoken we read:

$$\frac{G = \int_a^b f(x)\,dx}{G \text{ is } f \text{ (multiplied) by } x}.$$

Above, $x$ humanly spoken is meant to refer to the 'type' of $x$, the value being $b - a$. We note that the occurance of summation, and hence the appearance of (the infinitesimal concept) $dx$ is just one device of calculation, due to curvature. Related are (31) and (32)

**49.** It is true that in the above, we have a dissymetry that leads us to ask: why not the integral of something like $f(x)\,g(x)$? We think that this is indeed what we get when we try to calculate integrals in general, and might be one point, using a different notation, that generalizations try to tackle (maybe without knowing it).

**50.** Similarly to the indefinite integral above, we read the generalized Stokes theorem as simply as:

$$\frac{\int_\Omega dw = \int_{\partial\Omega} w}{dw \text{ by } \Omega \text{ equals } w \text{ by } \Omega\text{'s boundary}}.$$

**51.** The exterior or wedge product has a simple reading:

$$\frac{v \wedge w}{\text{The volume formed by } v \text{ and } w}$$

**52.** Not a traditional entry in this section, but finally somebody spells out something we felt would be mentioned very often but is not so. Bressoud says in [@c4]: "The lesson is that differential forms exist to be integrated. This is where we shall look for the definition of a differential."

# VIII Orientation

**53.** It seems our understanding of orientation has been totally under-developed, and that orientation is a major part of understanding differential forms, specifically, orientation in arbitrary dimensions. Our idea to try to understand this better by looking into discrete differential forms will probably not work; upon further reading, we realize it could work: (11)).

**54.** We find that the actual existence of tesselations in arbitrary dimensions is important.

**55.** We find that orientation has to do with 'induced' gluing, that is cancelling of 'borders', in the same way it happens when cutting an interval into multiple, but while have the FTC fact that intervals can be oriented, and that going in 'reverse' negates the integral.

**56.** If we want to 'cut' the arbitrary-dimensional domain of integration into pieces, how would we make sure we are not entering into the situation where some 'interval' must be negated? What about integrating over a Moebius strip? We are confirmed:

**57.** We think that going to linear algebra (and determinants) is a good tactic since linear algebra has no bias towards what we can intuitively imagine geometrically, and hence the determinant might give the right hints to pass to higher dimensions.

**58.** We find that our old find with Bachman is actually a good short path, although 'good' definitions of orientation, (keep the Moebius strip in mind) actually are references we already had, but since we were not concerned about orientation, we totally missed them:

- “The problem is that there is no way to figure out signs - It would be like trying to integrate a function from R to R without knowing whether you were moving forward or backward. What you CAN actually integrate are pseudo-differential forms. The whole point of choosing an orientation is to turn a differential form into a psuedo-differential form. For those, I recommend the wonderful short story by John Baez found here.”, “You can only integrate an n-form over a smooth n-dimensional manifold if it is equipped with an ORIENTATION. You may be so used to this that you’ve come to accept the orientation as an inevitable prerequisite for integration. But it’s not true! Integration of pseudo n-forms works perfectly fine on any smooth manifold, even an unoriented or unorientable one. It’s only if you make the mistake of trying to integrate an N-FORM” - he practically spat the term out in disgust - ”that you’ll need an orientation. And all the orientation does is let you convert your n-form to a pseudo n-form! Correcting one bad move with another....”, “Why didn’t any of the 10+ books I have on differential geometry and forms mention this?!”, “What do you expect if you read so few books?!”, “But surely even those miserable tomes said you could only integrate an n-form on an n-manifoldthat was *oriented*. No? And if so, surely it was your jobto wonder whether it was possible or not to do integrals on anunoriented manifold! You should have pondered a Moebius strip, and asked yourself: ‘What’s to prevent me from doing integralson this thing?’ You’d chop it up into coordinate charts, do the integral on each piece with the help of a partition of unity,and then add up the results.... And after a little thought, you would have discovered the answer: ***it all works perfectly fine, *if* you stick in an extra minus sign to describe how your n-form transforms under a change of coordinates whose Jacobian has negative determinant! In other words, you should use the absolute value of the Jacobian, just like the change of variables formula in multivariable calculus tells you to!*** But when you do this, you’ve reinvented pseudo n-forms. And then, after asking around a bit, you’d have discovered that everyone... everyone who counts, that is... has already realized this!””, “”But... but if it’s *that* simple, why don’t my textbooks talk about it?””, “They certainly drop the necessary clues. As for why they don’t *emphasize* this stuff, well, this is just one of those tricks we Wizards use to distinguish the people who think for themselves from those who fall for any plausible line of claptrap. In fact, every time Halloween falls on a full moon - like this year - we get together and agree on what facts like this we will keep secret, precisely to see who rediscovers it for themselves. This year we...”, “Anyway, I hope you see now that people who think they’re really integrating n-forms are just slightly less naive than the people who think they’re really integrating functions. You can only integrate these things if your n-manifold is equipped with extra structure... and this extra structure is only being used to convert these things to pseudo n-forms”, “”What do you think I am, a walking journal article? I give you all the clues you need to figure everythingvout for yourself, and you want *references*?? Here - HERE’s your much-beloved REFERENCES!”
- At least, the above show that we can think a little and that we are not naive. It confirms our Moebius strip ’faithful picture’. None of this hypocrisy is new to us, and we do not find it amusing. All the jokes about references only confirm one thing, our law of conservation of inflicted pain. Not everybody has all the time in the world to rediscover whatever is considered ’non-naive, clever and advanced’ at their day and age.
- A validating quote of our thought about can be found in [@c5,p.84] with figure to go with it.

The relation to permutations has to be arrived at. It seems that there is an ’alternative’ to Generalized Stokes for nonorientable manifolds, this might be related to Rahm and cohomology. As for nonoriented manifolds, ’densities’ must be used: “First of all, the things that you actually integrate are densities, which are the differential geometric counterparts of measures. No orientation is needed. A degree n form on an n-dimensional manifold is almost a density, but not quite. We need an orientation to associate to the top degree form a density. This is what you ultimately integrate when you integrate a form. For more details see page 105 of these notes.” (http://www3.nd.edu/~lnicolae/Lectures.pdf, http://mathoverflow.net/questions/90455) Orientation links:

1. What is the best way explain to undergraduates that all 1-dimensional manifolds are orientable? http://mathoverflow.net/questions/54645
2. Non-orientable 1-dimensional (non-hausdorff) manifold http://math.stackexchange.com/questions/635980
3. 'The Geometry of Physics' says: “We shall say that a manifold Mn is orientable if we can cover M by coordinate patches having positive Jacobians in each overlap. We can then declare the given coordinate bases to be positively oriented, and we then say that we have oriented the manifold.”

When not approached through parametrization and Jacobians, but topologically, by gluing, with or without simplices, we enter the realm of combinatorial topology. Our main question is the reason for the translation between permutations and the jacobian approach, we guess the translation happens through the properties of determinants specifically, why does this equivalence class interest us? “We say that two permutations are equivalent iff they have the same signature. Consequently, there are two equiv- alence classes of permutations: Those of even signature and those of odd signature.” (http://www.cis.upenn.edu/~cis610/complex1.pdf)

1. Intuitive Combinatorial Topology https://books.google.de/books?id=bk_UBwAAQBAJ&pg=PA53&lpg=PA53&dq=combinatorial+topology+gluing&source=bl&ots=WQKZPURUaj&sig=ANxdt_k_DUnrqeUkPeMef7UpdfM&hl=en&sa=X&ved=0ahUKEwiMqs7egfXKAhUCPQ8KHTvHAMMQ6AEILzAC#v=onepage&q=combinatorial%20topology%20gluing&f=false
2. ***How is orientation 'induced' and how is gluing possible at all, and related if at all and when, to orientation? When thinking about Moebius again, how do we describe the difference between Modebius-like (nonorientable) and a cylinder-like (orientable) in higher dimensions? If we think about it a little, we find immediately that this is related to parametrization since we need to describe a loop on Moebius and on the cylinder.***
3. An Introduction to Homology by Nadathur. http://math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Nadathur.pdf
4. A possibly good crash: http://www.dyinglovegrape.com/math/topology_data_3.php (Applying Topology to Data, Part 3: Orientation, Chains, Cycles, and Boundaries!)
5. For induced structures, these might be useful: http://www.iiserpune.ac.in/~vmallick/2013s1/mth622/top_2_01_07.pdf (Orientation, manifolds with boundary, induced structures), Inducing orientations on boundary manifolds (http://math.stackexchange.com/questions/377236)
6. A maybe too basic resource is 'The Orientation Manifesto' (http://www.math.cornell.edu/~goldberg/Notes/Orientation.pdf)
7. 'Basic Concepts of Algebraic Topology' (Croom)
8. Is it possible to define orientability using orientation preserving loops? http://math.stackexchange.com/questions/620165
9. ***Matroids*** might also be related!
   (a) https://en.wikipedia.org/wiki/Matroid
   (b) https://en.wikipedia.org/wiki/Convex_polytope
   (c) https://en.wikipedia.org/wiki/Oriented_matroid
   (d) https://ncatlab.org/nlab/show/orientation

(e) 'Oriented Matroids and Triangulations of Convex Polytopes' (has to do with representability)

10. Convex Polytopes:

(a) 'Enumerating Triangulations of Convex Polytopes' http://www.emis.de/journals/DMTCS/pdfpapers/dmAA0107.pdf (mentions circuits)

(b) 'Basic Properties of Convex Polytopes' http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.136.2061&rep=rep1&type=pdf

(c) 'Regular Triangulations of Convex Polytopes' (https://www.math.ucdavis.edu/~deloera/MISC/BIBLIOTECA/trunk/LeeCarl/leeregtriang.pdf)

(d) 'Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra' p.60,Thm.3.1, "Existence of Triangulations: ***Every convex polytope can be triangulated using no new vertices***".

(e) 'An elementary illustrated introduction to simplicial sets'

(f) All this spill stems from our realization that we ***simply do not even understand convex polytopes in higher dimensions***).

11. This is validating! "Let X ≈ |K| be a combinatorial d-manifold. We say that X is orientable if it is possible to assign an orientation to all of its cells (d-simplices) so that ***whenever two cells σ1 and σ2 have a common facet, σ, the two orientations induced by σ1 and σ2 on σ are opposite***. A combinatorial d-manifold together with a specific orientation of its cells is called an oriented manifold. If X is not orientable we say that it is non-orientable." 'Notes on Convex Sets, Polytopes, Polyhedra, Combinatorial Topology, Voronoi Diagrams and Delaunay Triangulations'. http://www.cis.upenn.edu/~jean/combtopol-n.pdf

12. Related to the above, 'Fundamentals of Mathematics: Geometry' (Behnke) might be providing a fully geometric view of the matter of orientation (including covers, starting p.8). He says "If the circle is traversed in the positive sense, the origin (together with the whole interior of the circle) lies to the left of the direction of traversal, ...", "6. The situation in space is analogous. A plane divides the space into twohalfspaces. Each of these two halfspaces is convex and becomes nonconvexwhen a single point (not on the plane) of the other halfspace is added to it.The space becomes oriented (left- and right-handed screws are ...", "The above discussion for the plane can be repeated here, and we can proceded analogously in higher dimensions. The n-dimensional space is oriented by an oredred set of n-oriented lines (an n-lateral), the even and odd permutations of which preserve of reverse the orientation of the space."

**59.** With all of this said, we have to do our homework as to basic multi-variable calculus, for this, we have to solve Bachman's and then return to differential forms proper. Note that the 'problem' with [VLDU] is that it treats multi-variable calculus in a 'unified' way and hence using differential forms, this makes it a blocker at this stage, since we wish to look at differential forms after treating vector-calculus and orientation separately.

**60.** Perhaps, before even considering differential forms, we should 'really understand' (for the n'th time) ***linear forms*** https://en.wikipedia.org/wiki/Linear_form. They are also expounded in 'Fundamentals of Mathematics, Volume I'.

# IX Integration

**61.** Reading in [DGR] comparing to [Integration - A Functional Approach], we conclude an important way to 'speak' the relation between rationals and irrationals in $R$, in a way that explains away the author's perplexity (p.1011): "It is perhaps somewhat perplexing that the rational numbers have measure zero. Since the rational numbers are a dense subset of the real numbers, there seems to be no space at all between the rational numbers. The "contiguous stretches" of irrational numbers all have zero length. So it might seem that the irrational numbers should have zero measure too!

Yet the irrationals in the interval [0,1] have measure 1 while the rationals have measure 0.”. The right way to think about this is to remember that not only there is a rational/irrationals between two irrationals/rationals, but more importantly that there is an infinity of both rationals and irrationals between every two rationals and irrationals. In the sense a statement about betweenness once we get out of the finite domain is too weak to give clues about the kinds of infinity involved. So there might very well be (and it is so) a countable infinity of rationals between any two irrationals, as well as an accompanying uncountable infinity of irrationals.

**62.** Based on the above, maybe ’measure’ is partially about capturing this set theoretic state of being of our ultimate model of physical ’bendable ruler’ for integration purposes.

**63.** The author of DGR quotes Bell on his frustration about the number of integrals with no closed form: “[. . . ] the problem of evaluating □ f (x) dx for comparatively innocent-looking functions f (x) may be beyond our powers. It does not follow that an “answer” exists at all in terms of known functions when an f(x) is chosen at random—the odds against such a chance are an infinity of the worst sort (“non-denumerable”) to one. When a physical problem leads to one of these nightmares approximate methods are applied which give the result within the desired accuracy.”. But let us think for a minute about simply functions in general. If we consider all possible functions on $R$, how many have a closed form? The situation is the same. One should see it in reverse. The fact that the definitions of integral sometimes yield closed forms is interesting.

**64.** A question related to the above is: how would a non-closed-form function that ’occurs’ in the real world manifest itself? We would not even be able to write it down in any way. Supposedly, some such functions do exist, but are not totally inaccessible: they are accessible through ’closed-form’ systems of implicit differential equations or minimization problems.

**65.** What is the relation between integrating differential forms and the theory of integration?

**66.** “ the class of Riemann-integrable functions is not closed under taking limits.” [A Crash Course on the Lebesgue Integral and Measure Theory]

**67.** One can talk about ’characterizations’ of the different integrals (per example as linear functionals [CHARACTERIZATION OF RADON INTEGRALS AS LINEAR FUNCTIONALS]), what are the different characterizations? Also, how are integrals defined in the setting of synthetic differential geometry, what is the relation of these definitions with integration theory?

**68.** What is the ’geometry’ of integrals once one moves from Riemann integrals? We also read that basically, once one moves away, the concept of ’geometric’ area starts to make little sense, so we define the integral at will. Still, we cannot find a characterization basic to all integrals, maybe the synthetic approach yields some pointers. Also, as we know, solutions to PDEs and the theory of the fourrier transform generated drive the Lebesgue, so then, the definition of an ’integral’ might be related to the application, to ’what is meant’ by integrating.

**69.** About the ’geometry’ of it we read: “The geometric interpretation of the Lebesgue integral is in a sense more natural, as its theoretical context generalizes the notions of length, area, and volume. Imagine the function which takes the value cc on the intevral [a,b]ab and 00 otherwise. By definition, the Lebesgue integral of this function ffis the measure (length) of the interval, times the value ff takes on it. Suppose now that ff were 00outside a set AA which is not an interval, but rather a stranger set made of intervals and points. The integral of this function would still be the measure of AA (its generalized length) times cc. Basically, the Lebesgue integral abstracts from literally taking the integral to be the area under the curve.In one of the comments, the terrific book A Garden of Integrals by Burke was mentioned. Another nice book exploring different integrals is The Integrals of Lebesgue, Denjoy, Perron, and Henstock, by Gordon.”

**70.** “In terms of a geometric interpretation, one way to look at it is as follows. Both integrals compute the volume under surfaces, or area under curves corresponding to ”nice” functions for which you can think of these regions as areas or volumes. But the distinction is that Riemann’s integral does it by adding up volumes of vertical rectangles of equal width arranged so that they approximate the region (the domain is partitioned), whereas Lebesgue’s integral does it by adding up volumes of horizontal rectangles of equal height arranged so that they approximate the region. In both cases the assumption of equal height/width can be dropped but you get the idea."

**71.** "Further, I would claim that, in fact, "the integral" people mostly use is not so much formally defined by any particular set-up, but is characterized, perhaps passively, in a naive-category-theory style (or, those might be my words) by what properties are expected. That is, for many purposes, we truly don't care about the "definition" of "integral", because we know what we expect of "integrals", and we know that people have proven that there are such things that work that way under mild hypotheses...For all its virtues (in my opinion/taste), this "characterization" approach seems harder for beginners to understand, so the "usual" mathematical education leaves people with definitions...My own preferred "integral" is what some people call a "weak" integral, or Gelfand-Pettis integral (to give credit where credit is due), characterized by λ(∫Xf(x)dx)=∫Xλ(f(x))dxλXfxdxXλfxdx for VV-valued ff, for all λλ in V□V, for topological vector space VV, for measure space XX. This may seem to beg the question, but wait a moment: when ff is continuous, compactly-supported, and VV is quasi-complete, locally convex, widely-documented arguments (e.g., my functional analysis notes at my web site) prove existence and uniqueness, granted exactly existence and uniqueness of integrals of continuous, compactly-supported scalar-valued functions on XX. Thus, whatever sort of integral we care to contrive for the latter will give a Gelfand-Pettis integral.Well, we can use Lebesgue's construction, or we can cite Riesz' theorem, that every continuous functional on Coc(X)CcoX (Edit: whose topology is upsetting to many: a colimit of Frechet spaces. But, srsly, it's not so difficult) is given by "an integral" (somewhat as Bourbaki takes as definition).Either way, we know what we want, after all.An example of a contrast is the "Bochner/strong" integral, which has the appeal that it emulates Riemann's construction, and, thus, directly engages with traditional ... worries? But, after the dust settles, there is still a bit of work to do to prove that the (as-yet-unspoken) desiderata are obtained. Further, surprisingly often, in practice, the "weak" integral's characterization proves to be all that one really wants/needs! Who knew? :)"

**72.** This is good motivation: "Given a set of real numbers, 1i(E) will represent its Lebesgue measure. Before defining this concept, let's consider the properties that it should have. ... Unfortunately,it is not possible to define a measure that satisfies all of these properties for all subsets of real numbers. See the remark following Theorem 1.10. The difficulty lies in property(4). Since this property is essential to guarantee the linearity of the Lebesgue integral, it is necessary to restrict the collection of sets and consider only those for which all of the properties are valid. In other words, some sets will not have a Lebesgue measure." [The Integrals of Lebesgue, Denjoy, Perron, and Henstock]

**73.** ***Recovering a functions from its derivative*** might be a good answer to 'something' that underlines all various kinds of integrals. [Garden, p.16].

**74.** Reading on in the the [Garden], we see the need for a figure showing how an 'integral form' is atomized and rebuilt, and how this works because the definition gives a freedom of choice and because there are infinitely many 'forms' and 'set theoretic things' that converge, that is 'project' to the same limit. Some of them are 'useful'.

**75.** How is uniform continuity expressed in NSA?

**76.** This is a pivotal (and untold story) that forms a great root structure for the garden of integrals:
" While studying convergence problems for Fourier series, Dirichlet began to consider functions that assumed one value on the rationals and a different value on the irrationals. For example, suppose [... $i_R$ is the indicator function for rationals and $i_I$ of irrationals ...] Certainly

$$1 = \int_0^1 1\,dx = \int_0^1 (i_R + i_I)(x)\,dx,$$

and linearity of the integral would require that

$$1 = \int_0^1 i_R(x)\,dx + \int_0^1 i_I(x)\,dx.$$

But these two Cauchy integrals are not defined. Dirichlet had discussions with Riemann to try to find an integration process that would overcome that difficulty. ***Riemann did not find*** such an integration process

(that would be discovered by Lebesgue), but did develop another integration process more powerful than Cauchy's, Riemann's process is the subject of the next chapter." [Graden, p.42].
We notice here multiple things:

- The indicator functions do not play nicely with the Cauchy's intervals for the domains.
- It is not clear what we want the separated integrals to be, while it is clear that the joined integral should be 1. In other words, the translation of the area concept from 'synthetic geometry' stops here. It is not possible to even define this separations 'geometrically' since it is purely 'number'-based. The problem cannot be expressed geometrically, nor can it be drawn. If anything, drawing it would tell us that the plots of both parts are 'lines' and both part integrals should be equal to 1, giving a sum of 2. Hence, in our translation, we leave the domain of 'geometry' and search for an extended 'coordinate' based extension to our concept of integrals. The linearity characetrization, an abstract property (as we know, the currency of abstract mathematics) offers itself as a good way to do this.
- Riemann failed to solve this, but what is never told is that he did try. This was one of his motivations.
- Maybe the essence of the hint to move on from the geometric domain of Cauchy's intervals is that within the number domain, a predicate was found that cannot be 'intervalized', that is too 'pointillistic', too 'numerical'. This is a ***game switch***. The next game is thus to follow the property of linearity among others. The integrals that were discovered using Cauchy's methods are a 'nice' class of the general pointillistic class.

**77.** It seems important to keep in mind the difference between a definition of integral and the tightly related (iff) criteria of integrability. The definitions so far define a process, an algorithm, while the criteria are predicates on the function being integrated, that bypass the need to find the actual integral but still can be used to prove the existence of it. At first sight, this seems interesting and needs more examination. Why is this possible?

**78.** Knowing more about the smallness of the difference between Caushy's and Riemann's integral, the obvious question comes to mind about why exactly does Rieman's have any advantages? We feel that the answer will be one of purely 'formal' reasons. In the freedom of choice of find convergence series that does 'project' to the point of convergence, one is bound to be able to capture those of formal finite form, and Riemann's seems to provide more (algebraic) freedom in finding them. However, we must note that (http://math.stackexchange.com/questions/326197/
the-equivalence-between-cauchy-integral-and-riemann-integral-for-bounded
"D.C.Gillespie proved the theorem in 1915 (Annals of Mathematics, Vol.17) and what a proof! To propose the proof as an exercise in a calculus book seems rather strange ... However see exercise 2.1.19 in Bressoud's A Radical Approach to Lebesgue's Theory of Integration. There is a hint on page 300. Can it help ? See also theorem 1 in Kristensen, Poulsen, Reich A characterization of Riemann-Integrability, The American Mathematical Monthly, vol.69, No.6, pp. 498-505. But the story is the same!"
"Gillespie's is just the one I've mentioned in my text by hyperlink article. After some efforts, now I believe that the elementary proof of squeezing Riemann sums through Cauchy sums would be meaningless, even though there were." It seems this questions is specifically treated in the potentiall excellent book: [A Radical Approach to Lebesgue's Theory of Integration].

**79.** TODO: create timeline using [@c4], including p.41-45. and p.6-8.

**80.** Bressoud says in [@c4], talking about a topic we thought was not relevant to integration so far: "The surprising answer is No. In fact,in a sense that later will be made precise, a continuous, monotonic function is differentiable at "most" values of x. There are very important subtleties lurking behind this fourth [Continuity and Differentiability] question.". Maybe looking back to our note (76) we can try to 'answer' why, why is this actually full of subtelties? More importantly, is there any hope of a faithful error-less 'understanding' of real analysis?

**81.** Once we reach complex analysis, Needham's 'Visual Complex Analysis', as recommended in http://math.stackexchange.com/questions/4054/intuitive-explanation-of-cauchys-integral-formula-in-complex-analysis might be very useful. It turns out that the generalized Stokes' theorem (and the curl of something being zero), can play a role in the intuitive understanding of Cauchy's integration formula.

# X Scatters

## X.1 Scatter 1

**82.** Scatter material is:

1. VLDU p.499-512
2. Morita p.61-70
3. Frankel p.xxxii, p.73-76
4. Lovelock, p.131-136
5. Applied exterior calculus, p.77-93
6. Linear Algebra via Exterior Products, p.11-151

## X.2 Scatter 2

**83.** While studying integration through 'pointillistic.idraw', we found that there is an equally rich path in terms of analytic functions that is also crucial for understanding. However, studying this should come after finishing 'pointillistic'.

**84.** Scatter material is:

1. http://math.stackexchange.com/questions/620290/is-it-possible-for-a-function-to-be-smooth-everywhere-analytic-nowhere-yet-tay

## X.3 Scatter 3

Why 'smooth manifolds', why 'smooth coordinate transformations', why not 'analytic', etc. Scatter material is:

1. https://en.wikipedia.org/wiki/Non-analytic_smooth_function
2. https://en.wikipedia.org/wiki/Partition_of_unity
3. Search:lack of analycity in differential geometry
4. http://mathoverflow.net/questions/14877
5. http://mathoverflow.net/questions/110410
6. http://mathoverflow.net/questions/72210
7. http://mathoverflow.net/questions/8789
8. http://mathoverflow.net/questions/90656

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Grady L.J., Polimeni J.R. "Discrete Calculus, Applied Analysis on Graphs for Computational Science."

H., Flanders. "Differential Forms with Applications to the Physical Sciences."

Winitzki, Sergei. "Linear Algebra via Exterior Products."