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# Quotes2, 104-204

1. Euler's death in 1783 was followed by a period of stagnation in mathematics. He had indeed solved everything: an unsurpassed treatment of infinite and differential calculus (Euler 1748, 1755), solvable integrals solved, solvable differential equa- tions solved (Euler 1768, 1769), the secrets of liquids (Euler 1755b), of mechan- ics (Euler 1736b, Lagrange 1788), of variational calculus (Euler 1744), of algebra (Euler 1770), unveiled. It seemed that no other task remained than to study about 30,000 pages of Euler's work." (Analysis by it's history)
2. "Bolzano points out that Gauss's first proof is lacking in rigor; he then gives in 1817 a "purely analytic proof of the theorem, that between two values which produce opposite signs, there exists at least one root of the equation" (Theorem III.3.5 below). In 1821, Cauchy establishes new requirements of rigor in his fa- mous "Cours d'Analyse". The questions are the following:
   – What is a derivative really? Answer: a limit.
   – What is an integral really? Answer: a limit.
   – What is an infinite series a1 + a2 + a3 + . . . really? Answer: a limit.
   This leads to
   – What is a limit? Answer: a number.
   And, finally, the last question: – What is a number?
   Weierstrass and his collaborators (Heine, Cantor), as well as Me ́ray, answer that question around 1870–1872. They also fill many gaps in Cauchy's proofs by clarifying the notions of uniform convergence (see picture below), uniform continuity, the term by term integration of infinite series, and the term by term differentiation of infinite series." (Analysis by it's history, Hairer)
3. "I find it really surprising that Mr. Weierstrass and Mr. Kronecker can attract so many students — between 15 and 20 — to lectures that are so difficult and at such a high level. (letter of Mittag-Leffler 1875, see Dugac 1978, p. 68)" (Analysis by it's history, Hairer)
4. "We give the last word to Abraham Robinson who was a pupil of A.A. Fraenkel, one of the most distinguished set theorists. Nevertheless Robinson recognized already 40 years ago: "(i) Infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless. (ii) Nevertheless, we should continue the business of mathematics 'as usual', i.e., we should act as if infinite totalities really existed." (Physical Constraints Of Numbers, W. Mueckenheim)
5. "It is impossible, however, to satisfy this condition for all natural numbers. It would require an unlimited amount of resources. But the universe is finite - at least that part available to us. Here is a simple means to realise the implications: First find out how many different natural numbers can be stored on a 10 GB hard disk. Then, step by step, expand the horizon to the 1011 neurons of your brain, to the 1028 atoms of your body, to the 1050 atoms of our earth, to the 1068 atoms of our galaxy, and finally to the 1078 protons within the universe. In principle, the whole universe could be turned into a big computer, but with far fewer resources than is usually expected without a thought be given to it." (Physical Constraints Of Numbers, W. Mueckenheim)
6. "Does a number exist in spite of the fact that we do not know and cannot know much more about it but that it should be a natural number? Does a poem exist, if nobody knows anything about it, except that it consists of 80 characters? Here is a poem which does exist:
   Flower, sweet primrose, show us your face; Now willow, t'is time to bud, make haste!
   H. C. Andersen wrote it in his fairy tale "The Snowman". We can read, enjoy, love, learn, forget, discuss or criticize it. But that is not possible for the large majority of 80-letters strings, namely those which have not been written as yet.
   The attempt to label each 80-digits number like the following
   12345678901234567890123456789012345678901234567890123456789012345678901234567890
   by a single proton only, would already consume more protons than the universe can supply. What you see is a number with no doubt. Each of those 80-digits strings can be noted on a small piece of paper. But all of them cannot even exist as individual ideas simultaneously, let alone as numbers with definite values. Given that photons and leptons can be recruited to store bits and given that the mysterious dark matter consists of particles which can be used for that purpose too, it is nevertheless quite impossible to encode the values of 10100 natural numbers in order to have them simultaneously available. (An advanced estimation, based on the Planck-length, leads to an upper limit of 10205 but the concrete number is quite irrelevant. So let us stay with 10100, the Googol.)" (Physical Constraints Of Numbers, W. Mueckenheim)
7. "Even some single numbers smaller than 210100 can never be stored, known, or thought of. In short they do not exist. (It must not be forgotten: Also one's head, brain, mind, and all thoughts belong to the interior of the universe.) To avoid misunderstanding: Of course, there is no largest natural number. By short cuts like 101010... we are able to express numbers with precisely known values surpassing any desired magnitude. But it will never be possible, by any means of future technology and of mathematical techniques, to know

that natural number P which consists of the first 10100 decimal digits of π (given π is a normal irrational without any pattern appearing in its n-adic expansion). This P will never be fully available. But what is a natural number the digits of which will never be known? P is an idea but it is not a number and, contrary to any 80-digits string, it will never be a number. It is even impossible to distinguish it from that number P', which is created by replacing the last digit of P by, say, 5. It will probably never be possible to decide, which of the following relations holds
P < P' or P = P' or P > P'.
If none of these relations can ever be verified, we can conclude, adopting a realistic philosophical position, that none of them is true." (Physical Constraints Of Numbers, W. Mueckenheim)

8. "2500 years ago Hippasos of Metapont discovered that there are properties which cannot be expressed by ratios of natural numbers. But does this automatically imply that irrational numbers exist? Shouldn't we be able to acknowledge that some things cannot be expressed by numbers? Who would be surprised to learn that some emotions do not fit into the set of numbers? Who would maintain that "justice" is the number 4 as the Pythagoreans insisted?" (Physical Constraints Of Numbers, W. Mueckenheim)
9. "The task to find the ratio between circumference and diameter of the ideal circle does exist as an idea, and it has a name. Euler called it p or c, until π became generally accepted - not least supported by Euler's adaption of Jones' π in his Introductio in Analaysin Infinitorum. But is this idea, named π, a number? Does the ratio of circumference and diameter exist in reality or in the brains of mathematicians? Surely not. In reality an ideal circle cannot exist because any real circle requires more mass than can be tolerated by an Euclidean space. On the other hand nobody will be able to imagine it. Not even the sharpest mind will be able to determine π better than up to two digits by pure imagination, few will be able to compute more than 5 digits of π in their mind using any desired algorithm, and even fewer will be able to memorize more than the first 50 digits (although it has been shown possible by Hideaki Tomoyori to know by heart 40000 digits) of π.
   Cauchy, Weierstrass, Cantor, and Dedekind attempted to give meaning to irrational numbers, well aware that there was more to do than to find suitable names. According to Cantor, ◊3 is not a number but only a symbol: "◊3 ist also nur ein Zeichen für eine Zahl, welche erst noch gefunden werden soll, nicht aber deren Definition. Letztere wird jedoch in meiner Weise etwa durch (1.7, 1.73, 1.732, ...) befriedigend gegeben" [2]. It is argued that ◊3 does exist, because it can be approximated to any desired precision by some sequence (an) such that for any positive ε we can find a natural number n0 such that Ian - ◊3I < ε for n ¥ n0. It is clear that Cantor and his contemporaries could not perceive the principle limits of their approach. But every present-day scientist should know that it is in principle impossible to approximate any irrational like ◊3 by ε < 1/210100. (An exception are irrationals like ◊3/210100. But that does not establish their existence.) Therefore, the request to achieve any desired precision fails in decimal and binary and any other fixed n-adic representation." (Physical Constraints Of Numbers, W. Mueckenheim)
10. " In reality an ideal circle cannot exist because any real circle requires more mass than can be tolerated by an Euclidean space." (Physical Constraints Of Numbers, W. Mueckenheim)
11. "For the last hundred years or so, mathematicians have finally ‘understood the infinite’–or so they think. Despite several thousand years of previous un- certainty, and very stiff opposition from notable mathematicians (Kronecker, Poincaré, Weyl, Brouwer and numerous others), the theory of infinite point sets and hierarchies of cardinal and ordinal numbers introduced by Cantor is now the established orthodoxy, and dominates modern logic, topology and analysis. Most mathematicians believe that there are infinite sets, but can’t demonstrate this claim, and have difficulty resolving the paradoxes that beset the subject a hundred years ago, except to assert that there are dubious ‘axioms’ that sup- posedly extract us from the quagmire. If you have read my recent diatribe Set Theory: Should You Believe? you will know that I no longer share this religion, and I tried to convince you that you shouldn’t either." (Numbers, Infinities and Infinitesimals, Wildberger)
12. "Modern mathematics as religion
   Modern mathematics doesn’t make complete sense. The unfortunate conse- quences include difficulty in deciding what to teach and how to teach it, many papers that are logically flawed, the challenge of recruiting young people to the subject, and an unfortunate teetering on the brink of irrelevance.
   If mathematics made complete sense it would be a lot easier to teach, and a lot easier to learn. Using flawed and ambiguous concepts, hiding confusions and circular reasoning, pulling theorems out of thin air to be justified ‘later’ (i.e. never) and relying on appeals to authority don’t help young people, they make things more difficult for them.
   If mathematics made complete sense there would be higher standards of rigour, with fewer but better books and papers published. That might make it easier for ordinary researchers to be confident of a small but meaningful contribution. If mathematics made complete sense then the physicists wouldn’t have to thrash around quite so wildly for the right mathematical theories for quantum field theory and string theory. Mathematics that makes complete sense tends to parallel the real world and be highly relevant to it, while mathematics that doesn’t make complete sense rarely ever hits the nail right on the head, although it can still be very useful.
   So where exactly are the logical problems? The troubles stem from the consistent refusal by the Academy to get serious about the foundational aspects of the subject, and are augmented by the twentieth centuries’ whole hearted and largely uncritical embrace of Set Theory.

Most of the problems with the foundational aspects arise from mathemati- cians' erroneous belief that they properly understand the content of public school and high school mathematics, and that further clarification and codification is largely unnecessary. Most (but not all) of the difficulties of Set Theory arise from the insistence that there exist 'infinite sets', and that it is the job of math- ematics to study them and use them.

In perpetuating these notions, modern mathematics takes on many of the aspects of a religion. It has its essential creed–namely Set Theory, and its unquestioned assumptions, namely that mathematics is based on 'Axioms', in particular the Zermelo-Fraenkel 'Axioms of Set Theory'. It has its anointed priesthood, the logicians, who specialize in studying the foundations of mathe- matics, a supposedly deep and difficult subject that requires years of devotion to master. Other mathematicians learn to invoke the official mantras when questioned by outsiders, but have only a hazy view about how the elementary aspects of the subject hang together logically.

Training of the young is like that in secret societies–immersion in the cult involves intensive undergraduate memorization of the standard thoughts before they are properly understood, so that comprehension often follows belief instead of the other (more healthy) way around. A long and often painful graduate school apprenticeship keeps the cadet busy jumping through the many required hoops, discourages critical thought about the foundations of the subject, but then gradually yields to the gentle acceptance and support of the brotherhood. The ever-present demons of inadequacy, failure and banishment are however never far from view, ensuring that most stay on the well-trodden path.

The large international conferences let the fellowship gather together and congratulate themselves on the uniformity and sanity of their world view, though to the rare outsider that sneaks into such events the proceedings no doubt seem characterized by jargon, mutual incomprehensibility and irrelevance to the outside world. The official doctrine is that all views and opinions are valued if they contain truth, and that ultimately only elegance and utility decide what gets studied. The reality is less ennobling–the usual hierarchical structures reward allegiance, conformity and technical mastery of the doctrines, elevate the interests of the powerful, and discourage dissent.

There is no evil intent or ugly conspiracy here–the practice is held in place by a mixture of well-meaning effort, inertia and self-interest. We humans have a fondness for believing what those around us do, and a willingness to mold our intellectual constructs to support those hypotheses which justify our habits and make us feel good." (Set Theory, Should you Believe?, Wildberger)

13. "The problem with foundations

The reason that mathematics doesn't make complete sense is quite easy to explain when we look at it from the educational side. Mathematicians, like everyone else, begin learning mathematics before kindergarten, with counting and basic shapes. Throughout the public and high school years (K-12) they are exposed to a mishmash of subjects and approaches, which in the better schools or with the better teachers involves learning about numbers, fractions, arith- metic, points, lines, triangles, circles, decimals, percentages, congruences, sets, functions, algebra, polynomials, parabolas, ellipses, hyperbolas, trigonometry, rates of change, probabilities, logarithms, exponentials, quadrilaterals, areas, volumes, vectors and perhaps some calculus. The treatment is non-rigorous, inconsistent and even sloppy. The aim is to get the average student through the material with a few procedures under their belts, not to provide a proper logical framework for those who might have an interest in a scientific or mathematical career.

In the first year of university the student encounters calculus more seri- ously and some linear algebra, perhaps with some discrete mathematics thrown in. Sometime in their second or third year, a dramatic change happens in the training of aspiring pure mathematicians. They start being introduced to the idea of rigorous thinking and proofs, and gradually become aware that they are not at the peak of intellectual achievement, but just at the foothills of a very onerous climb. Group theory, differential equations, fields, rings, topo- logical spaces, measure theory, operators, complex analysis, special functions, manifolds, Hilbert spaces, posets and lattices–it all piles up quickly. They learn to think about mathematics less as a jumble of facts to be memorized and algorithms to be mastered, but as a coherent logical structure. Assignment problems increasingly require serious thinking, and soon all but the very best are brain-tired and confused.

Do you suppose the curriculum at this point has time or inclination to return to the material they learnt in public school and high school, and finally organize it properly? When we start to get really picky about logical correctness, doesn't it make sense to go back and ensure that all those subjects that up to now have only been taught in a loose and cavalier fashion get a proper rigorous treatment? Isn't this the appropriate time to finally learn what a number in fact is, why exactly the laws of arithmetic hold, what the correct definitions of a line and a circle are, what we mean by a vector, a function, an area and all the rest? You might think so, but there are two very good reasons why this is nowhere done.

The first reason is that even the professors mostly don't know! They too have gone through a similar indoctrination, and never had to prove that multi- plication is associative, for example, or learnt what is the right order of topics in trigonometry. Of course they know how to solve all the problems in elemen- tary school texts, but this is quite different from being able to correct all the logical defects contained there, and give a complete and proper exposition of the material.

The modern mathematician walks around with her head full of the tight 3
logical relationships of the specialized theories she researches, with only a rudi- mentary understanding of the logical foundations underpinning the entire sub- ject. But the worst part is, she is largely unaware of this

inadequacy in her training. She and her colleagues really do believe they profoundly understand elementary mathematics. But a few well-chosen questions reveal that this is not so. Ask them just what a fraction is, or how to properly define an angle, or whether a polynomial is really a function or not, and see what kind of non- uniform rambling emerges! The more elementary the question, the more likely the answer involves a lot of philosophizing and bluster. The issue of the correct approach to the definition of a fraction is a particularly crucial one to public school education.
Mathematicians like to reassure themselves that foundational questions are resolved by some mumbo-jumbo about 'Axioms' (more on that later) but in reality successful mathematics requires familiarity with a large collection of 'el- ementary' concepts and underlying linguistic and notational conventions. These are often unwritten, but are part of the training of young people in the sub- ject. For example, an entire essay could be written on the use, implicit and explicit, of ordering and brackets in mathematical statements and equations. Codifying this kind of implicit syntax is a job professional mathematicians are not particularly interested in.
The second reason is that any attempt to lay out elementary mathematics properly would be resisted by both students and educators as not going forward, but backwards. Who wants to spend time worrying about the correct approach to polynomials when Measure theory and the Residue calculus beckon instead? The consequence is that a large amount of elementary mathematics is never properly taught anywhere.
But there are two foundational topics that are introduced in the early under- graduate years: infinite set theory and real numbers. Historically these are very controversial topics, fraught with logical difficulties which embroiled mathemati- cians for decades. The presentation these days is matter of fact–'an infinite set is a collection of mathematical objects which isn't finite' and 'a real number is an equivalence class of Cauchy sequences of rational numbers'.
Or some such nonsense. Set theory as presented to young people simply doesn't make sense, and the resultant approach to real numbers is in fact a joke! You heard it correctly–and I will try to explain shortly. The point here is that these logically dubious topics are slipped into the curriculum in an off-hand way when students are already overworked and awed by all the other material before them. There is not the time to ruminate and discuss the uncertainties of generations gone by. With a slick enough presentation, the whole thing goes down just like any other of the subjects they are struggling to learn. From then on till their retirement years, mathematicians have a busy schedule ahead of them, ensuring that few get around to critically examining the subject matter of their student days." (Set Theory, Should you Believe?, Wildberger)

14. "Let me remind you that mathematical theories are not in the habit of col- lapsing. We do not routinely say, “Did you hear that Pseudo-convex cohomology theory collapsed last week? What a shame! Such nice people too.” (Set Theory, Should you Believe?, Wildberger)
15. "The bulwark against such criticisms, we are told, is having the appropriate collection of 'Axioms'! It turns out, completely against the insights and deepest intuitions of the greatest mathematicians over thousands of years, that it all comes down to what you believe. Fortunately what we as good modern mathe- maticians believe has now been encoded and deeply entrenched in the 'Axioms of Zermelo—Fraenkel'." (Set Theory, Should you Believe?, Wildberger)
16. "Euclid may have called certain of his initial statements Axioms, but he had something else in mind. Euclid had a lot of geometrical facts which he wanted to organize as best as he could into a logical framework. Many decisions had to be made as to a convenient order of presentation. He rightfully decided that simpler and more basic facts should appear before complicated and difficult ones. So he contrived to organize things in a linear way, with most Propositions following from previous ones by logical reasoning alone, with the exception of certain initial statements that were taken to be self-evident. To Euclid, an Axiom was a fact that was sufficiently obvious to not require a proof. This is a quite different meaning to the use of the term today. Those formalists who claim that they are following in Euclid's illustrious footsteps by casting mathematics as a game played with symbols which are not given meaning are misrepresenting the situation." (Set Theory, Should you Believe?, Wildberger)
17. "Some will argue that a mathematician can do whatever she likes, as long as a logical contradiction doesn't result. But things are not so simple. Are we allowed to introduce all-seeing Leprechauns into mathematics as long as they seem to behave themselves and not cause contradictions? A far better approach to create beautiful and useful mathematics is to ensure that all basic concepts are entirely clear and straightforward right from the start. The onus is on us to demonstrate that our notions make sense, instead of challenging someone else to find a contradiction." (Set Theory, Should you Believe?, Wildberger)
18. "At some point, you are going to write down a number so vast that it requires all the particles of the universe (except for some minimal amount of what's left of you). May I humbly suggest you call this number w, in honour of the last person you vaporized to create it?
Now here is a dilemma. Once you have written down and marvelled at w in all its glory, where are you going to find w + 1? From this end of things–the working end–the endless sequence of natural numbers does not appear either natural nor endless. And where is the infinite set N? The answer is–nowhere. It doesn't exist. It is a convenient metaphysical fiction that allows mathematicians to be sloppy in formulating various questions and arguments. It allows us to avoid issues of specification and replace concrete understandings with woolly abstractions. What seems to be a happy and well behaved sequence when viewed from the beginning is more like an enormous fractal when viewed from the other end." (Set Theory, Should you Believe?, Wildberger)

19. "The second benefit would have been that our ties to computer science would be much stronger than they currently are. If we are ever going to get seri- ous about understanding the continuum–and I strongly feel we should–then we must address the critical issue of how to specify and handle the computa- tional procedures that determine points (i.e. decimal expansions). There is no avoiding the development of an appropriate theory of algorithms. How sad that mathematics lost the interesting and important subdiscipline of computer science largely because we preferred convenience to precision!" (Set Theory, Should you Believe?, Wildberger)
20. "Mathematical induction affords, more than anything else, the essential characteristic by which the finite is distinguished from the infinite. The principle of mathematical induction might be stated popularly in some such form as "what can be inferred from next to next can be inferred from first to last." This is true when the number of intermediate steps between first and last is finite, not otherwise. Anyone who has ever watched a goods train beginning to move will have noticed how the impulse is communicated with a jerk from each truck to the next, until as last even the hindmost truck is in motion. When the train is very long, it is a very long time before the last truck moves. If the train were infinitely long, there would be an infinite succession of jerks, and the time would never come when the whole train would be in motion. Nevertheless, if there were a series of trucks no longer than the series of inductive numbers..., every truck would begin to move sooner or later if the engine persevered, though there would always be other trucks further back which had not yet begun to move."

    This is a quote from B. Ru;ssell's Introduction to mathematical philosophy, pages 27-28, that I think describes well this limitation of induction:
    There are contexts in which a statement P(n) can be proved for all $n \in \mathbb{N}$ by induction, and has a counterpart $P(\infty)$ that is false. In other contexts, $P(\infty)$ may be true. But even then, induction on $\mathbb{N}$ does not prove the $P(\infty)$ case.
    Getting back to limits of functions, note for example that:
    A finite sum of continuous functions is continuous.
    A pointwise convergent series of continuous functions need not be continuous.
    But, a uniformly convergent series of continuous functions is continuous.
    So in this case, going from finite sums to infinite series requires new tools, different types of convergence, to obtain the desired properties. As for real analytic functions, I don't know what can be said along these lines. For complex analytic functions there are nicer results, such as the fact that a locally uniformly convergent sequence of complex analytic functions is complex analytic. In the real case, to give a stark contrast, every continuous function on a bounded interval is a uniform limit of polynomials (as analytic as you can get), but there are continuous functions that are differentiable nowhere. Similarly, a continuously differentiable function of period $2\pi$ is the uniform limit of its Fourier series, but continuously differentiable functions need not even be twice differentiable, let alone analytic." (Russel)
21. "If, on the other hand, there be mathematicians to whom these definitions and discussions seem to be an elabora- tion and complication of the simple, it may be well to remind them from the side of philosophy that here, as elsewhere, apparent simplic- ity may conceal a complexity which it is the business of somebody, whether philosopher or mathematician, or, like the author of this volume, both in one, to unravel." (J.H.Muirhead, in the 2010 editor's note of Introduction to Mathematical Philosophy by Bertrand Russell)
22. "Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increas- ing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analysing, to greater and greater abstract- ness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterises mathemat- ical philosophy as opposed to ordinary mathematics. But it should be understood that the distinction is one, not in the subject matter, but in the state of mind of the investigator. Early Greek geometers, passing from the empirical rules of Egyptian land-surveying to the general propositions by which those rules were found to be justifiable, and thence to Euclid's axioms and postulates, were engaged in math- ematical philosophy, according to the above definition; but when once the axioms and postulates had been reached, their deductive employment, as we find it in Euclid, belonged to mathematics in the ordinary sense. The distinction between mathematics and mathematical philosophy is one which depends upon the interest inspiring the research, and upon the stage which the research has reached; not upon the propositions with which the research is concerned." (Introduction to Mathematical Philosophy by Bertrand Russell)
23. "We may state the same distinction in another way. The most obvi- ous and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small nor very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using "simple" in a logical sense). And as we need two sorts of instruments, the telescope and the microscope, for the enlargement of our visual powers, so we need two sorts of instruments for the enlargement of our logical powers, one to take us forward to the higher

mathematics, the other to take us backward to the logical foundations of the things that we are inclined to take for granted in mathematics. We shall find that by analysing our ordinary mathematical notions we acquire fresh insight, new powers, and the means of reaching whole new mathematical sub- jects by adopting fresh lines of advance after our backward journey." (Introduction to Mathematical Philosophy by Bertrand Russell)

24. "The notion of order is one which has enormous importance in mathematics. Not only the integers, but also rational fractions and all real numbers have an order of magnitude, and this is essential to most of their mathematical properties. The order of points on a line is essential to geometry; so is the slightly more complicated order of lines through a point in a plane, or of planes through a line. Dimensions, in geometry, are a development of order. The conception of a limit, which underlies all higher mathematics, is a serial conception. There are parts of mathematics which do not depend upon the notion of order, but they are very few in comparison with the parts in which this notion is involved.
    In seeking a definition of order, the first thing to realise is that no set of terms has just one order to the exclusion of others. A set of terms has all the orders of which it is capable. Sometimes one order is so much more familiar and natural to our | thoughts that we are inclined to regard it as the order of that set of terms; but this is a mistake. The natural numbers—or the "inductive" numbers, as we shall also call them—occur to us most readily in order of magnitude; but they are capable of an infinite number of other arrangements. We might, for example, consider first all the odd numbers and then all the even numbers; or first 1, then all the even numbers, then all the odd multiples of 3, then all the multiples of 5 but not of 2 or 3, then all the multiples of 7 but not of 2 or 3 or 5, and so on through the whole series of primes. When we say that we "arrange" the numbers in these various orders, that is an inaccurate expression: what we really do is to turn our attention to certain relations between the natural numbers, which themselves generate such-and-such an arrangement. We can no more "arrange" the natural numbers than we can the starry heavens; but just as we may notice among the fixed stars either their order of brightness or their distribution in the sky, so there are various relations among numbers which may be observed, and which give rise to various different orders among numbers, all equally legitimate. And what is true of numbers is equally true of points on a line or of the moments of time: one order is more familiar, but others are equally valid. We might, for example, take first, on a line, all the points that have integral co-ordinates, then all those that have non-integral rational co-ordinates, then all those that have algebraic non-rational co-ordinates, and so on, through any set of complications we please. The resulting order will be one which the points of the line certainly have, whether we choose to notice it or not; the only thing that is arbitrary about the various orders of a set of terms is our attention, for the terms themselves have always all the orders of which they are capable." (Introduction to Mathematical Philosophy by Bertrand Russell)
25. "Because the post-Crusade European understanding of mathematics (and its concocted history) was so deeply influenced by theology, the absorption of the Indian calculus in Europe, and its adaptation to Western theology proved a difficult task. My contention is that mathematics is, today, a difficult subject to understand just because the complexities of theology have got intertwined with the straightforward secular understanding of mathematics that prevailed elsewhere." (Zeroism and calculus without limits, C.K Raju)
26. "The key claim is this: formal mathematics crumbles if it is interrogated from a Buddhist perspective." (Zeroism and calculus without limits, C.K Raju)
27. " The basic point is that any representation of anything real always discards or ignores a certain "non-representable" part. (One might say colloquially that every representation is "approximate", except that one does not know what is "exact".) " (Zeroism and calculus without limits, C.K Raju)
28. "An immediate consequence of paticca samuppada is anatmavada (no-soul-ism). The present is not caused by the past; it is only conditioned by the past. Therefore, the present is not implicit in the past (as it would be, for example, on the Samkhya-Yoga view21). Knowledge of the past is not implicit in the present (as it would be on Newtonian physics), and knowledge of the entire past is inadequate to determine the future. On the other hand, it is manifest (pratyaksa) that nothing stays constant for two instants. And the inference is not clear that beneath all this "surface" change, a human being has an eternally constant part—the soul. Our immediate concern with the soul relates to representation: if the soul—this supposedly constant part of a human being—exists, it is legitimate to represent events, as in everyday language, as this or that happening to a single individual. If the "soul" (in the above sense, as the constant part of a human being) does not exist, this everyday representation is no longer valid. It is a mere figure of speech with no underlying reality. So, this understanding of time directly leads to the problem of representation: how to represent something across time, when nothing about it stays constant?" (Zeroism and calculus without limits, C.K Raju)
29. "However, the idealist is not happy with this situation. He wants to make statements that are *eternally valid*, and he assumes that *it is possible to do so*. Therefore, he regards the process of specifying only a billion or a trillion digits as erroneous. He has no specific practical task in mind for which he regards this as erroneous. But, in his "mind's eye", he sees that this process still leaves out an infinity of unspecified decimal digits. Thus, there are an infinity of numbers which could possibly be confounded with the number we have in mind, and which we have specified only to a billion digits, for what is a billion compared to infinity? The idealist wants to assert that that there really is a single number
    pi or a single number sqrt2 , which can be uniquely specified. The problem is, as we have stated, that such specification requires an infinite process or a **supertask**. Though this supertask cannot be avoided, it can

be hidden. This is the strategy adopted in present-day formal mathematics: such supertasks are hidden underneath set theory. This set theory is the typical starting point of a math text, and the confusion begins right here: “a set is a collection of objects” is a common piece of nonsense found also in the current NCERT school texts." (Zeroism and calculus without limits, C.K Raju)

30. "The point of referring to a computer should now be clear: a computer makes manifest the impossibility of performing supertasks. We are not here referring to the axiom of choice or any such fancy transfinite induction principle. We are here referring to a simple process of specifying a real number." (Zeroism and calculus without limits, C.K Raju)
31. "A formal real number cannot be specified without appeal to set theory (or supertasks of some sort). It is therefore impossible for a formal real number to be represented on a computer. Formalists, therefore, declare the computer representation to be forever erroneous! Nevertheless, the fact is that computer representations of real numbers are adequate for absolutely all practical tasks today, without any exception. Therefore, declaring the computer to be forever erroneous reflects only the religious attitude that mathematics is perfect truth, and that this perfect truth can only be grasped metaphysically, for something is wrong with anything real or actually realizable. However, note that that while the Neoplatonic attitude had its merits, that is now made completely barren and devoid of all meaning in formalism. which just manipulates meaningless symbols according to an opaque grammar of supertasks." (Zeroism and calculus without limits, C.K Raju)
32. "Thus, it is possible to find an e different from zero such that -1 + (1 + e) = 0 while (-1+1) + e = e. For the stock IEEE floating point standard, we can take e = 0.0000001 or any smaller number. For double or higher precision arithmetic, this number has to be even smaller.
However, the failure of the associative “law” for floating points numbers does not mean that these numbers are criminals who violate laws! The key point is that it is the grandiose notion of associative “law” which is now to be regarded as erroneous—a useful simplification, but one that is not exactly valid, and admits an infinity of exceptions to the rule. The rule has to be applied with intelligence, as in everyday language, and not mechanically as in formalism. Therefore, no algebraic structure such as a field can be associated with real numbers: the formal mathematics of real numbers is forever erroneous!" (Zeroism and calculus without limits, C.K Raju)
33. "Eventually, through Tycho Brahe, then Kepler, and Galileo, and his student Cavalieri, this Indian work on calculus started circulating in Europe. Some mathematicians, such as Pascal and Fermat greeted it with enthusiasm. However, others like Descartes complained that this was not mathematics. Descartes wrote in his Geometry that
[T]he ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact.
From the Indian perspective this is a very strange statement, very hard for the human mind to grasp. Thus, Indians used a string or rope since the days of the sulba sutra. Measurement using the rope (rajju) was part of the mathematics syllabus traditionally taught to Indian children.
A string can obviously be used to measure a curved line. It can be straightened, and thus compared with a straight line. Therefore, it is very easy for any child to grasp the ratio of a curved and a straight line. Why did this major Western thinker find this simple thing so hard to understand?
First, Descartes took it for granted that the straight line was the natural figure, and that curved lines must necessarily be understood in terms of straight lines, and not the other way around. Second, he ruled out empirical procedures as not mathematics. According to his system of religious beliefs only the metaphysical could be perfect, and mathematics being perfect had to be metaphysical. It should be observed that this belief that mathematics ought to be metaphysical was unique to post-Crusade Christianity: Proclus, for example, did not subscribe to it, for he admitted the empirical at the beginning of mathematics, as in the proof of the Side-Angle-Side theorem (Elements 1.4, as it is called)." (Zeroism and calculus without limits, C.K Raju)
34. "As we have seen this difficulty of representation arises even with integers, or with anything else, but Descartes thought this was a problem specific to this new-fangled calculus, which lacked perfection, and hence was not quite mathematics. Perfection required, in his opinion, that each such straight-line segment should be infinitesimal, but then there ought to be an infinity of them. So, perfection required a supertask—that of summing the infinity of these infinitesimal lengths—and this, he thought, was beyond the human mind.25 After vacillating for a few years, Galileo concurred, and hence he left it to his student Cavalieri to take the credit or discredit for this disreputable sort of mathematics which was not perfect." (Zeroism and calculus without limits, C.K Raju)
35. "This “right sort of metaphysics” had a peculiar consequence for Newtonian physics. Compared to his predecessor, Barrow, who adopted a physical definition of time and time-measurement, and summarily rejected Augustine as a metaphysical “quack”, Newton reverted to a metaphysical and mathematical notion of time. His definition about “absolute, true, and mathematical time...” is not usually understood correctly, although all three adjectives make it clear that he is referring to a metaphysical and not a physical notion, and he confirms this by saying that it “flows on without regard to anything external”. Something physical cannot obviously have such a generalized disregard for anything external! This metaphysical notion of time meant that there was no proper way to measure time in Newtonian physics.26 The absence of a proper

definition of time was the reason for the eventual failure of Newtonian mechanics when it clashed with electromagnetic theory, and relativity had to be brought in." (Zeroism and calculus without limits, C.K Raju)

36. "Why did Newton need such a definition of time? We have to understand that he wanted to use calculus, and specifically time derivatives, to explain circular and elliptical orbits in terms of straight line motion. (Circular orbits alone could be "explained", without the need of the calculus, by postulating an inverse-square force law, and a "natural" straight line motion; Newton's success lay in extending this procedure to elliptical planetary orbits on the one hand, and to parabolic ballistic trajectories on the other.)" (Zeroism and calculus without limits, C.K Raju)
37. "But more than the practical applications, Newton was interested in rigour. It was critical to his understanding of calculus in terms of his theory of fluxions that time should flow, or be a "fluent" entity. This, he thought, made time infinitely divisible—at any rate it made "absolute, true and mathematical time" infinitely divisible—and such infinite divisibility was needed to justify that the supertasks needed for making the time- derivative meaningful could be performed. He thought the process would fail if time were discrete (for then the process of subdividing time, needed for taking time derivatives, would stop when subdivisions reached some finite, atomic proportions)." (Zeroism and calculus without limits, C.K Raju)
38. "Historically speaking, in his defence against Leibniz's charges of plagiarism, it is rigour for which Newton, writing anonymously about himself, claims credit. (The other thing he claims credit for is the sine series, which was obviously known from earlier, though not in Europe.) And, historians, today, once again credit Newton with the calculus on the grounds that he had rigorously proved the "fundamental theorem of calculus".
Of course, Newton was mistaken in thinking that this "fluency" of time provided a solution to the problem of supertasks. Many discerning people were aware of this, and Berkeley took it upon himself to tear Newton's theory of fluxions to bits, when there seemed a danger that Newton's views on the church would become public. In the event, Newton's History of the Church in 8 volumes, a result of 50 years of scholarship, was successfully suppressed, and Berkeley's criticism was subsequently played down.
Regardless of its probable motivation, and regardless of the subsequent attempts to play it down, Berkeley's criticism was valid. His argument was very simple and robust, and interesting. He assumes, along with Newton, Descartes, and Galileo, that mathematics is perfect, and cannot neglect even the smallest quantity. However, he goes along with Newton and allows that a quantity may be neglected if it is infinitesimal (whatever that might mean). But, he asks, if a quantity is to be set to zero at the end of the calculation, why not set it to zero in the beginning itself?
We know the modern answer to Berkeley's objections: that the ratio of two infinitesimals may be finite. A hundred years ago, when neither Non-Standard analysis nor non- Archimedean fields had come in, and infinitesimals were still formally disreputable, the idea was to try and define limits by playing on the space provided by the non-definition of 0/0. Dedekind's formal real numbers, or the continuum, by allowing infinite divisibility, seemed to be just the right framework for such limits. That is how advanced calculus (or elementary analysis) is still taught—by appealing to the completeness of the (formal) real numbers. But, of course, it was evident, even in Dedekind's time, that the construction of formal real numbers required set theory which was suspect for the infinitary processes it involved.
The axiomatisation of set theory has tamed those doubts in an interesting way. From curves, to numbers, the doubts have now been pushed into the domain of set theory which the average mathematician does not care about. So, like the professional theologians who were concerned with establishing the number of angels that could fit on the head of a pin, without bothering about fundamental questions as to the nature of God, the professional mathematician can merrily go on proving theorems without bothering about the supertasks used in set theory.
Secondly, on the philosophy of formalism the only question that mathematicians will accept about the process is that of consistency. The consistency of set theory has not been formally proved, of course, but mathematicians believe it is consistent. It is interesting to see the double standards involved here.
Thus, if supertasks were really regarded as really admissible, one should be able to apply them also in metamathematics. In that case, it would be a trivial matter to use some transfinite induction principle, such as Zorn's lemma, or Hausdorff maximality principle, to make set theory decidable. In such a case, the theory obviously cannot be consistent, by Godel's theorem, for it easily accommodates a statement which asserts its own negation. Thus, the consistency of set theory is maintained by means of a double standard: allow supertasks within mathematics, but not while talking about mathematics. It is interesting to note the theological parallel: it was exactly such a double standard that was needed to claim the consistency of the notion of an omnipotent, and omniscient God with the existence of evil in the world!" (Zeroism and calculus without limits, C.K Raju)
39. "The proposal on calculus without limits comes against this background. A rigorous formulation of the calculus does not need any supertasks. The need for supertasks was just a myth that arose in the West because of the idealistic belief in the "perfection" of mathematics, and the belief that this perfection could only be attained through metaphysics." (Zeroism and calculus without limits, C.K Raju)
40. "The proposal on calculus without limits comes against this background. A rigorous formulation of the calculus does not need any supertasks. The need for supertasks was just a myth that arose in the West because of the idealistic belief in the "perfection" of mathematics, and the belief that this perfection could only be attained through metaphysics.

Let us start, for example, with a key polemic used by Western historians, which relates to the "Fundamental theorem of Calculus". It is clear from the above considerations, that Newton could hardly have proved it, for his theory of fluxions was mistaken and had to be rejected. Also, the idea of the fundamental theorem of calculus supposes that we have an independent definition of the integral and the derivative, and the two are related by the theorem. However, where was Newton's definition of the integral? Such a definition became available only after limits (and this led to various complexities, for the fundamental theorem of calculus obviously does not hold with either the classical derivative and the Lebesgue integral, or with the Schwartz derivative and the Riemann integral). All that Newton had was the naïve idea of the integral as the anti-derivative, which naïve idea is the basis of the calculus today taught to students." (Zeroism and calculus without limits, C.K Raju)

41. "More to the point, the central question is: just what mathematics is about? If it is not metaphysics, or a branch of theology, its function is to help carry out calculations with a practical purpose in mind. From this perspective, what is the exact purpose that the fundamental theorem of calculus serves? At best, it enables one to solve the differential equations Newton used to formulate physics. However, the more appropriate thing for the calculus (and for Newtonian physics), then, is to have is a numerical technique for calculating the solution of ordinary differential equations. This was what Aryabhata developed, and which later came to be known as Euler's method of solving ordinary differential equations. Thus, instead of a theorem, we have a process. (Of course, the process can and has been improved since Aryabhata and Euler.)" (Zeroism and calculus without limits, C.K Raju)
42. "From the point of view of practical calculation, this process is far superior to the plethora of theorems that are needed to be able to solve the simplest ordinary differential equation. For example, the theory of the simple pendulum cannot be taught in schools, and is missed out even by most physics teachers, since it involves the Jacobian elliptic functions, which are difficult to teach and explain. Consequently, most people confound the simple pendulum with simple harmonic motion, the linear differential equations for which can easily be solved symbolically. However, with the process of numerical solution, the equations for the simple pendulum can be solved just as readily." (Zeroism and calculus without limits, C.K Raju)
43. "Similarly, there is the belief that a "closed form" solution is superior to a numerical solution. Now, to the extent, and only to the extent, that one is comparing a definite and well-established process of calculation (such as the calculation of sine values) with a new, and as-yet unclear numerical method, I am willing to go along with this. Often, such symbolic representation is just a fallback to the old days (namely 30 or 40 years ago) when a desktop-computation of the values of special functions (such as the Jacobian elliptic functions) was a complex task that one did not want to perform." (Zeroism and calculus without limits, C.K Raju)
44. "Now what exactly does this formal ability to do supertasks mean? Consider a series such as 1-1+1-1+.... What is the sum of this series? The partial sums of the series clearly oscillate between the values of 0 and 1 and hence the series is judged to be divergent on conventional analysis. However, in many practical situations it is convenient to define the
sum of this series as 1 . Such a definition may seem puzzling at first sight since the 2
series never attains that value, and the sum to any finite number of terms is always either 0 or 1. This is an example of an "asymptotically convergent" series. The point is that just like 0/0, the sum of an infinite number of terms has no arithmetical meaning of its own. Within the philosophy of formal mathematics, such a meaning has to be assigned by definition: and definitions, as everyone knows are bound to have some arbitrariness in them. The above definition is practically convenient and is often used in physics in the more sophisticated form .=1, where  is the Heaviside function and  is 2
the Dirac delta function. So we are back to the situation where there is no need to actually perform any supertasks. When a seeming supertask arises, what one needs, instead, is a process (a finite process) which needs to be carried out. The nature of this process (or, equivalently, the definition of such products of Schwartz distributions) is to be decided by practical value and empirical considerations. (The alternative is to rely on mathematical authority.)" (Zeroism and calculus without limits, C.K Raju)
45. "By: Doron Zeilberger

    For many years, and even today, topologists considered themselves the cream of mathematics. They monopolized editorial boards of `prestigious'(i.e. snooty, e.g. Acta, Annals, Invent.) journals and would-be-prestigious journals (which are even more pathetic, e.g. Duke J. and Israel J.), as well as the Fields Medal committee. Most people have heard Whitehead's epithet that `Graph Theory is the slum of topology', and many have agreed.

    I am pleased to be alive to witness the beginning of the trivialization of Topology. First came the Jones polynomials, that historically arose in another `fancy field': C* algebras, but this turned out to be a red herring, and thanks to the work of Lou Kauffman and others, they could have, and should have, arisen directly in elementary graph theory, being a very special case of the Tutte (chromatic) polynomials.

    Now, even more dramatically, we have the Seiberg-Witten invariants. It was Nati Seiberg who made the initial breakthrough, in the context of `exactly-solvable-models' in quantum field theory. Ed Witten, who speaks fluently both the language of topology and that of physics, realized the revolutionary significance.

Myself, I don't know either languages. Nevertheless, I am almost sure that, just like the Jones polynomials, the Seiberg-Witten invariant would be reducible to combinatorics and/or high-school algebra. Similar developments will occur all over mathematics, and will make many heavy tomes written by arrogant `high-brow' mathematicians, all obsolete, and unnecessary.

In this process, computer algebra, that great implementer of high-school algebra, will play a central role.

The king (abstract math) is dead. Long live the King (Concrete Mathematics). " (http://www.math.rutgers.edu/~zeilberg/Opinion1.html)

46. "(lrwiman@mailcity.com) Hi,

Based upon your website, I thought that you might find this anecdote interesting.

I'm a mathematics undergraduate who is working on various projects to various degrees in various fields. One of them is a problem in categorical algebra proposed to me by a professor. Yesterday, we talked about it for some time (about 2 hours), and we both got fairly tired of it (especially the lack of progress). Hence he asked me what else I've been working on. I mentioned that I'm giving a talk this week at our discrete mathematics colloquium, about Sturmian words and Beatty sequences (my small contribution to a problem of O'Bryant). I proceeded to describe the jist of what I had done, and he looked dismayed.

He then said "There should be a nice diagramatic way to do this," and showed that he didn't really understand what was going on at all. When I tried to explain where he went wrong, he got visably upset, and then said "well, I don't know how, but perhaps you should find it [the nice method]." I told him that I didn't think it fruitful to try and look for a way to visualize the problem in terms of commutative diagrams.

He told me that diagrams force you to look at concepts, and not special cases (derisively implying that that was what I had been looking at). The conversation went on in this vein, and moved to number theory. He said "I think number theorists don't look at numbers correctly; they are equivalence classes of sets. When you look at them on too small a level, you get lost in special cases and coincidences. They don't really know what they're looking at." This shocked and surprised me (as I once intended to be a number theorist, and still hold an intense interest in it). I changed the subject. I still want to work with my professor despite his bigotry.

I really don't understand this. He treated combinatorics and number theory as nothing more than a collection of meaningless special cases and coincidences. I've looked at your website in the past, and always assumed that the discrimination of the mathematical upper class (topologists, etc.) was something which you had exaggerated out of all control. Now I see that it is real, but I can't begin to fathom it. This sort of attitude seems to me to defeat the purpose of mathematics. A problem should be worked on iff it is interesting. I do find algebraic geometry considerably more interesting than graph theory. Yet this is only a matter of personal taste, so it seems unrelated to the importance of the mathematical results. " (http://www.math.rutgers.edu/~zeilberg/Opinion1.html)

47. "M. So, finally, we arrive at the following justification for real numbers. 1. We must go further than just the rationals. 2. When we do so we introduce certain procedures that give us new numbers. 3. Formalizing these, we end up with the monotone-sequences axiom, or something equivalent to it. 4. This axiom is not as precise as it seems, since the notion of an arbitrary monotone sequence, even of rationals, is not precise. 5. There is no need to make it precise, because we know how to reason in terms of arbitrary sequences. 6. That allows us to define the real numbers we have a use for, even if it gives us a lot of junk as well. 7. In fact, we don't really know what junk it does give us, and it's not even clear that it makes sense to ask." (A dialogue concerning the need for the real number system, Timothy Gower)

48. "Even while considering the universe to be finite, one can do mathematics symbolically as a game with a system of rules. If the game doesn't have enough pieces we just add new pieces, with new properties or allowed moves as required. All that matters is that the enlarged system is compatible with the old system; that the smaller game is a subgame of the large one; that the smaller system can be embedded in the larger one.

When does something exist ? Well if there isn't something from amongst the objects under consideration that has the properties we want then we just create new symbols and define how they relate to the old ones.

If we were a pythagorean and the only numbers that exist are rational numbers, then we wouldn't call 2√ a number, but if we were also finitist symbolicists then we could embed any collection of numbers into a collection that contains not only numbers but also "splodges" which is what we're going to call 2√. It's important to always keep in mind that 2√ isn't a number - it's a splodge. In this new system of arithmetic we've invented we can add numbers to splodges to get new splodges like 1+2√. What a fun game. Let's add some more splodges. We're bored with algebraic splodges so let's add some non-algebraic splodges like the one in your question. Of course that expression is a bit cumbersome so we'll give it the shorthand symbol π instead.

Given a splodge x, it would make calculus easier if there were a splodge x+o that was nearer to x than any other splodge, however that isn't possible so we embed the splodges in a larger system called the hypersplodges that contains not only splodges but also vapors, and contains not only the concept nearer but also the concept "nearer". Vapors like x+o are "nearer" to x than any splodge could ever be, and when you're finished using them they evaporate leaving just a splodge.

We want a splodge that satisfies x2+1=0 however there isn't one, so we embed the splodges in a larger system called weirdums in which we've added a piece called i with the rule that i2=−1, and under the new system we can "add" splodges to weirdums to get new weirdums like 1+i.

In solving differential equations we'd like a function which is zero everywhere except at a single point but which has a non-zero area under the curve. There is no function that behaves like this so we'll go to a larger system that contains not only functions but also spikes which do have the desired property because the larger system contains a rule about spikes which says they do. Conveniently certain calculations involving spikes cancel out leaving just functions.

A finitist or ultrafinitist shouldn't recognise the concept of infinite sets therefore the only sets are finite-sets and since all sets are finite the adjective finite is superfluous therefore from this point onwards we just use the term "set". Some people want to consider sets that contain things they haven't put in there themselves - which of course can't be done because a set only contains the items we've put there. So we embed the system of sets in a larger system that includes not only sets but also dafties. In this daft system the rules are that a daftie can have an "affinity" for things whether those things have been previously mentioned or not. Dafties have an affinity for things in the same way that sets contain things. A compatible embedding of a system of sets in the daft system means that a set has an affinity for the items it contains when the set is considered as part of the daft system, therefore by daft reasoning one can say things not only about dafties but also about sets. To each set one can attach a number. You can't do this with dafties so we embed the numbers in a larger system containing sinners and attach a sinner to each daftie. Sometimes there is a need for something that looks like a daftie but has no sinner - such things are called messes. A mess can have an affinity for collections of dafties that no daftie could have an affinity for. This could go on, but you need gobbledegook theory. The set system has zero gobbledegook. The daft system is level-1 gobbledegook. The messy system is level-2 gobbledegook. Finitists like to maintain a zero level of gobbledegook. Analysts are usually happy with one-level of gobbledegook and category-theorists are comfortable with any amount of gobbledegook.

There are two ways to compatibally extend a system:
1) a conservative extension adds new items but doesn't say anything new about the old items that couldn't be said before;
2) a progressive extension does say new things about the old items but only about things that were previously undecidable
P.S. We can combine the vapors, splodges and linedups in a system called the messysplodges but they haven't been studied much because they're a bit messy.
" (http://mathoverflow.net/questions/102237/what-is-the-status-of-irrational-numbers-within-finitism-ultrafinitism by Steve L. Cowan)

49. "Finitists like to maintain a zero level of gobbledegook. Analysts are usually happy with one-level of gobbledegook and category-theorists are comfortable with any amount of gobbledegook." (http://mathoverflow.net/questions/102237/what-is-the-status-of-irrational-numbers-within-finitism-ultrafinitism by Steve L. Cowan)
50. "An integer is a number without a fractional part, a number you could use to count things (although integers may also be negative). Mathematicians may distinguish between natural numbers and cardinal numbers, and linguists may distinguish between cardinal numbers and ordinal numbers, but these distinctions do not concern us here.
A real number is, for our purposes, simply a number with a fractional part. Since computers do not typically implement real numbers exactly, it is not necessary or even meaningful to distinguish between rational, irrational, and transcendental numbers. " (C Programming Notes, Steve Summit)
51. "``Cardinal Arithmetics is much older than Number Theory. People used to exchange things (in one-to-one-correspondence) way before there were numbers. Expressing numbers like 762 is already a sign of a very advanced civilization.''
``Given a conjecture, the best thing is to prove it. The second best thing is to disprove it. The third best thing is to prove that it is not possible to disprove it, since it will tell you not to waste your time trying to disprove it. That's what Gödel did for the Continuum Hypothesis.''
``There is an old maxim that says that two empires that are too large will collapse. The analog in set theory is that two different theories that are too powerful must necessarily contradict each other.''
----Saharon Shelah, Rutgers Univ. Colloqium, Oct. 26, 2001. "

52. "`The progress of mathematics can be viewed as progress from the infinite to the finite.'
--Gian-Carlo Rota (1983) [quoted by Xavier G. Viennot. Opening plenary talk, LACIM 2000, Sept. 7, 2000] "
53. "`The many mathematical theorems, that have finitary proofs, are certain, hence there is absolute certainty in mathematics. What is still problematic, like in Pythagoras's time, is the use of infinite objects.'
--Arnon Avron [Goedel's Theorems and the Foundations of Mathematics Problem (in Hebrew), Ministry of Defence, Israel, 1998, p. 167] "
54. "It cannot be said to be certain that there are in fact any infinite collections in the world. The assumption that there are is what we call the “axiom of infinity.” Although various ways suggest themselves by which we might hope to prove this axiom, there is reason to fear that they are all fallacious, and that there is no conclusive logical reason for believing it to be true. At the same time, there is certainly no logical reason against infinite collections, and we are therefore justified, in logic, in investigating the hypothesis that there are such collections." (Introduction to Mathematical Philosophy by Bertrand Russell)
55. "If we arrange the inductive numbers in a series in order
of magnitude, this series has no last term; but if n is an inductive number, every series whose field has n terms has a last term, as it is easy to prove. Such differences might be multiplied ad lib. Thus the number of inductive numbers is a new number, different from all of them, not possessing all inductive properties. It may happen that  has a certain | property, and that if n has it so has n + 1, and yet that this new number does not have it. The difficulties that so long delayed the theory of infinite numbers were largely due to the fact that some, at least, of the inductive properties were wrongly judged to be such as must belong to all numbers; indeed it was thought that they could not be denied without contradiction. The first step in understanding infinite numbers consists in realising the mistakenness of this view." (Introduction to Mathematical Philosophy by Bertrand Russell)
56. "One of the most striking instances of a “reflexion” is Royce’s illustration of the map: he imagines it decided to make a map of England upon a part of the surface of England. A map, if it is accurate, has a perfect one-one correspondence with its original; thus our map, which is part, is in one-one relation with the whole, and must contain the same number of points as the whole, which must therefore be a reflexive number. Royce is interested in the fact that the map, if it is correct, must contain a map of the map, which must in turn contain a map of the map of the map, and so on ad infinitum. This point is interesting, but need not occupy us at this moment. In fact, we shall do well to pass from picturesque illustrations to such as are more completely definite, and for this purpose we cannot do better than consider the number-series itself." (Introduction to Mathematical Philosophy by Bertrand Russell)
57. "Again, the relation of n to 2n, confined to inductive numbers, is one-one, has the whole of the inductive numbers for its domain, and the even inductive numbers alone for its converse domain. Hence the total number of inductive numbers is the same as the number of even inductive numbers. This property was used by Leibniz (and many others) as a proof that infinite numbers are impossible; it was thought self-contradictorythat|“thepartshouldbeequaltothewhole.”But  this is one of those phrases that depend for their plausibility upon an unperceived vagueness: the word “equal” has many meanings, but if it is taken to mean what we have called “similar,” there is no contradiction, since an infinite collection can perfectly well have parts similar to itself. Those who regard this as impossible have, unconsciously as a rule, attributed to numbers in general properties which can only be proved by mathematical induction, and which only their familiarity makes us regard, mistakenly, as true beyond the region of the finite." (Introduction to Mathematical Philosophy by Bertrand Russell)
58. "Whenever we can “reflect” a class into a part of itself, the same relation will necessarily reflect that part into a smaller part, and so on ad infinitum. For example, we can reflect, as we have just seen, all the inductive numbers into the even numbers; we can, by the same relation (that of n to 2n) reflect the even numbers into the multiples of 4, these into the multiples of 8, and so on. This is an abstract analogue to Royce’s problem of the map. The even numbers are a
“map” of all the inductive numbers; the multiples of 4 are a map of the map; the multiples of 8 are a map of the map of the map; and so on. If we had applied the same process to the relation of n to n + 1, our “map” would have consisted of all the inductive numbers except 0; the map of the map would have consisted of all from 2 onward, the map of the map of the map of all from 3 onward; and so on. The chief use of such illustrations is in order to become familiar with the idea of reflexive classes, so that apparently paradoxical arithmetical propositions can be readily translated into the language of reflexions and classes, in which the air of paradox is much less." (Introduction to Mathematical Philosophy by Bertrand Russell)
59. "Infact,as we shall see later, 2^aleph0 isaveryimportantnumber,namely, the number of terms in a series which has “continuity” in the sense in which this word is used by Cantor. Assuming space and time to be continuous in this sense (as we commonly do in analytical geometry and kinematics), this will be the number of points in space or of instants in time; it will also be the number of points in any finite portionofspace,whether|line,area,orvolume.After aleph0, 2.pow.aleph0 is the 2nd most important and interesting of infinite cardinal numbers." (Introduction to Mathematical Philosophy by Bertrand Russell)
60. "From the ambiguity of subtraction and division it results that negative numbers and ratios cannot be extended to infinite numbers. Addition, multiplication, and exponentiation proceed quite satisfacto- rily, but the inverse operations—subtraction, division, and extraction of roots—are ambiguous, and the notions that depend upon them fail when infinite numbers are concerned." (Introduction to Mathematical Philosophy by Bertrand Russell)

61. "This is so because, for Wittgenstein, “mathematical truth is created, not discovered48.” As such, there is no ‘real’ quasi-physical mathematical landscape that pre-exists Man, revealing itself through Set Theory. After all, mathematics is algorithmic, not descriptive metaphysics ; mathematics is the calculus49." (Wittgenstein And Labyrinth Of ‘Actual Infinity’: The Critique Of Transfinite Set Theory)
62. "Furthermore, the Diagonal Argument is nothing more than a constructive rule – that is, a rule for constructing certain kinds of numbers out of certain other kinds of numbers. It cannot generate an entirely different infinite set of numbers out of a first infinite set66. Thus, Cantor has only actually proved that he can construct a finite expansion through application of a rule-governed intension ; the constructive rule itself (even if infinitely applied) will never yield an infinite extension67 ! The Diagonal Method can thus prove neither that R exists as a gapless continuum68 (and, by extension, it cannot support the Continuum Hypothesis), nor that there is such a thing as an infinite cardinal (Cantor’s Theorem)69." (Wittgenstein And Labyrinth Of ‘Actual Infinity’: The Critique Of Transfinite Set Theory)
63. "A ‘set’ is nothing more than an abstract symbol for a list (‘extension’) generated by a rule (‘intension’). In the case of a transfinite set, then, the ‘infinite’ intension is simply a recursive rule for calculating certain kinds of results – one that does not have an ‘and then stop’ at the end. However, while the rule may not have a proper end, the extension cannot be considered infinite simply because the extension is precisely only what we have written down on the list, what we have calculated ; the law yields only the endless process, not the endless extension43." (Wittgenstein And Labyrinth Of ‘Actual Infinity’: The Critique Of Transfinite Set Theory)
64. "This is because, for Wittgenstein, as mathematics is an algorithm, the only mathematical reality is in the constructive proof-process, not the result71 – “[our] suspicion ought always to be aroused when a proof proves more than its means allow it ” ; the Diagonal Argument is then a “puffed-up proof72” because, while it may be infinitely applied, the completed collection of all its results can never be called infinite in the sense that Cantor intended. Thus, both by highlighting the nature of the intension/extension double-helix and by questioning the putative link between the bijective function and equinumerosity at the heart of Cantor’s set-theoretical approach73, Wittgenstein’s grammatical analysis has the effect of an axe dropping on the concretism of the Continuum Problem and Cantor’s Theorem : Cantor’s Theorem is logically false and the Continuum Hypothesis is not an ‘unsolved problem’ but, rather, merely a nonsensical pseudo- problem – which solves Hilbert’s first problem74." (Wittgenstein And Labyrinth Of ‘Actual Infinity’: The Critique Of Transfinite Set Theory)
65. "It’s almost unbelievable, the way in which a problem gets completely barricaded in by the misleading expressions which generation upon generation throw up for miles around it, so that it’s become virtually impossible to get at it. Wittgenstein75" (Wittgenstein And Labyrinth Of ‘Actual Infinity’: The Critique Of Transfinite Set Theory)
66. "I heartily agree. Too often the fine points of mathematical thinking are lost in calculus and numerics classes. I remember real and complex calculus as a tangled hair ball of rules of thumb, abuses of notation and, of course, rote memorization. Somehow, I never managed to learn how the reals relate to the rationals...only years later am I patching in all the holes. (Actually, I just forgot everything -- I let that old building fall down, and now must build a new one!)

    It's a real tragedy that calculus is required in disciplines where it's little used in comparison to the more fundamental skill of reasoning about relations. An emphasis on mathematical thinking, as opposed to mathematical techniques, would benefit not only our profession, but also the social sciences and business and economics, whose students are usually given a brief dip in stats and calc, with no effect on their general reasoning ability." (http://lambda-the-ultimate.org/node/2604)
67. "According to Cantor (1891), P(X) is ``bigger than'' X, since there is an injective function $x \to \{x\}$ in one direction but none the other way. He was ignoring Galileo's 1638 warning about ``how gravely one errs in trying to reason about infinities by using the same attributes that we apply to finites,'' in response to the observation that the squares form a proper but equinumerous subset of N, from which he concluded that ``equal, greater and less have no place in the infinite." Cantor's interpretation has prevailed (so far), even though it is well known that his motivations were religious at least as much as they were mathematical [Dau79]. Much has also been made of the self- referential nature of this and similar results such as Gödel's Incompleteness Theorem [Hof79]. We shall return to these matters at the end of the book." (Practical foundations of mathematics, Paul Taylor)
68. "It seems to me that the quantifiers and equalities on which Cantor's diagonalisation argument and the Burali-Forti paradox rely are stronger than are justified by logical intuition. We now have both the syntax and the semantics to weaken them.
    Such a revolution in mathematical presentation may be another century away: excluded middle and Choice were already on the agenda in 1908, but the consensus of that debate has yet to swing around to the position on which such developments depend. On the other hand, the tide of technology will drive mathematicians into publishing their arguments in computer-encoded form. Theorems which provide nirvana will lose out to those that are programs and do calculations for their users on demand. A new philosophical and semantic basis is needed to save mathematics from being reduced yet again to programming." (Practical foundations of mathematics, Paul Taylor)

69. "A constructive version of "the famous theorem of Cantor, that the real numbers are uncountable" is: "Let {an} be a sequence of real numbers. Let x0 and y0 be real numbers, x0 < y0. Then there exists a real number x with x0 ≤ x ≤ y0 and x ≠ an (n ∈ Z+) . . . The proof is essentially Cantor's 'diagonal' proof." (Theorem 1 in Errett Bishop, Foundations of Constructive Analysis, 1967, page 25.)" (http://en.wikipedia.org/wiki/Constructive_analysis)
70. "However, there is also a remarkable way that the project failed–apart, I mean, from the thwarting of the foundationalist aims that were so much a part of it. Consider this remark from Frege [15], and compare it to Arbib's [1], above:
A single [proof] step is often really a whole compendium, equivalent to several simple inferences, and into it there can still creep along with these some elements from intuition. In proofs as we know them, progress is by jumps, ... [and] the bigger the jump, the more diverse are the combinations it can represent of simple inferences with axioms derived from intuition ... [My] demand is not to be denied: every jump must be barred from our deductions.17 And consider this remark from Pasch in 1915:
A statement R1 is a consequent of B only because its derivation from the latter is completely independent of the meanings of the geometrical concepts occurring in it, so that the proof can be carried through without support from an actual or imagined diagram or from any sort of 'intuition'. A proof which does not meet this condition is no mathematical proof.18
After quoting Pasch's remark, Greaves ([17], p. 75) adds: 'This definition of acceptable proof in geometry is still current today'. Is it really? Perhaps this turns on what we're willing to describe as 'geometry'. Consider Spivak [38], p. 30-31. Among the exercises for chapter 1, we read: 'Consider the three surfaces shown below'. Below are three diagrams described as '(A) The infinite-holed torus', '(B) The doubly infinite-holed torus', and '(C) The infinite jail cell window'. We then read '(b) Surfaces (A) and (C) are homeomorphic! Hint: The region cut out by the lines in surface (C) is a cylinder, which occurs at the left of (A). Now draw in two more lines enclosing more holes, and consider the region between the two pairs'.19
It's clear that Frege and Pasch (and Hilbert, for that matter) were pressing for a significant practical change in the proof practices of ordinary mathematics. Traditional proofs in ordinary mathematics that contained large jumps based on intuitive considerations–such jumps indicated symptomatically, although hardly exclusively, by the presence of diagrams for guidance–were to be forever excised from respectable mathematical proof, and replaced by gapless derivations. This, as even an elementary inspection of current mathematics will make manifestly obvious, has not happened.20 Consequently this stronger demand of Frege, Pasch, and company, has been replaced by the (Steiner-Fallis) weaker constraint of 'in principle' translations. But this leaves unanswered the question of why the orig- inal program of changing the practice of mathematics failed. I want to suggest that it isn't merely a matter of the fact that such derivations–in practice–are too long.21 Rather, my suggestion is that the 'objects' that mathematical reasoning is prima facie about are essential to the capacity of the mathematician to create and understand mathematical proofs." (Is there still a Sense in which Mathematics can have Foundations? By Jody Azzouni)
71. "(iii) It explains–as required–the ubiquitous presence of tacit–but successful– reasoning in mathematics: the fact that so often tacit assumptions are only with great difficulty extracted from a piece of reasoning. The 'black box', 'off-the-shelf', availability of such inference packages prevents (easy) introspective recognition of what assumptions are presupposed in that package. Indeed, I suspect we recog- nize what assumptions–or other–are at work in an inference package by a kind of 'reverse engineering'. It isn't that we actually take apart the inference package (e.g., that we analyze exactly what tacit assumptions are being used when we visualize triangles on a sphere); rather, we design an axiom system in which as- sumptions are explicit–and show that the axiom system yields the same theorems that the inference package does–e.g., that triangles, according to the axiomatiza- tion, have the same properties that our inference package (with, perhaps, suitable plug-ins) attributes to them. Of course, as I mentioned earlier, there are many axiomatizations that yield the same theorems." (Is there still a Sense in which Mathematics can have Foundations? By Jody Azzouni)
72. "“Logic is not the ground on which I stand. How could it be? It would in turn need a foundation, which would involve principles much more intricate and less direct than those of mathematics itself. A mathematical construction ought to
be so immediate to the mind and its result so clear that it needs no foundation whatsoever.” (Heyting (1956), 6)"
73. "Heyting claims that constructivism avoids paradoxes—but he admits that construc-tivism is more restrictive than would really be necessary to avoid them (ibid., 11). And at any rate, paradox-avoidance is not the main motivation for the constructivists. Consistency in math is a plausible goal, for a foundational system so often described as being “secure.” Bishop claims that consistency is ensured in constructive math, up to human error, making it a non-issue for the constructivist mathematician (Bishop (1967), 353). But he never justifies any claim about the impossibility of deriving a contradiction through constructivist rules of inference. Without such a justification, constructivism is in no better position than classical math, paradoxes or no paradoxes. Brouwer is more verbose than Bishop on this topic, but similarly casts the issue off as irrelevant. Hesseling says, “If we have a set of logical axioms, Brouwer argues, and we can point out a
mathematical system of which the logical axioms can be considered to express properties, we know that no contradiction can occur, because a constructed math-ematical system cannot contain a contradiction.” (Hesseling (2003), 42)

Brouwer denies the necessity and validity of consistency proofs. He holds that such proofs only prove something about a formal linguistic system for communicating math, and that they don't prove anything about the mathematical content the system sets out to represent. There is not a trade-off, per se, between constructivism and classical math, since the choice to adopt constructivism is mainly a matter of ontological belief and preference. The primary methodological advantage of constructivism is that it offers us a rubric for deter-mining mathematical validity: namely, whether or not a construction has been performed. None of the constructivists, though, communicate their position in such a way that convinced mainstream mathematicians such a rubric is useful or necessary." (Constructivism: A Realistic Approach to Math? Mame Maloney)

74. "Occasionally in mathematical history, results surface that raise eyebrows and cause mathematicians to take a step back and say, “Wait, that shouldn't be so. We should work to avoid results like that!” Favorite examples of such results include Cantor's transfinite ordinals, the Banach-Tarski paradox, and the Axiom of Choice; all three of these results caused groups of mathematicians to balk. These mathematicians are signaling that the discipline has strayed too far from the realm of ideas accessible to the human mind, as major results are so counter-intuitive or inconceivable as to be deemed paradoxical. In some ways, this reaction makes sense; what is math (at least, in practice) if not a product of human minds? The constructivists, also known as intuitionists, are the quintessential conservative group; they took extreme pains to pare math down to exactly what is accessible to the human mind. In the end, though, their tactics failed to sway any more than a handful of other mathematicians, and classical math proceeded forth undeterred, counter-intuitive results and all. This result indicates to me that constructivism was, in fact, not a plausible approach to math—if for no other reason than its complete failure to convince working mathematicians." (Constructivism: A Realistic Approach to Math? Mame Maloney)
75. "The mathematical model of the concepts of physics doesn't stop there. Even supposing we have a satisfactory definition of the number line, this doesn't uniquely determine a definition of space. The first step in the modern definition of space is to consider the Cartesian product of the number line with itself three times (to get three dimensions). That is, a point in space is identified with a triple of numbers on the number line (x, y, z). Again, this is a model of space. (At this point I've not said anything about the geometry of this Cartesian product.) The next step is to give this Cartesian product a geometric structure, by defining the notion of distance and straight lines. Initially, Euclidean geometry was the model, but later other (curved) geometries were introduced. The last step in the model is the notion of a manifold, which is turn based on notions of continuity, topology, differentiability, and so forth. At each step, models inspired by intuition have been made. [I'd quite like to go through this in more detail highlighting the way in which these steps are modelling steps.] Although the structures formed in this way may not be false in the sense that one can derive contradictions in this system, they may fail to provide useful basic concepts for physics. It is in this sense that mathematics is a model of the basic concepts of physics." (http://thesamovar.net/philosophy/essays/quasiempiricalformalism)
76. "More mysteriously, mathematics can even be a model of itself. The modern concepts of real numbers, Cartesian products and so forth didn't exist for the ancient Greek mathematicians, and yet Euclidean geometry can be modelled in this language of modern mathematics. This reflects the fact that mathematical structures acquire a meaning independent of what they're modelling, and become interesting in themselves. [Does this paragraph make any sense?]" (http://thesamovar.net/philosophy/essays/quasiempiricalformalism)
77. "Quasi-empiricism can also throw some light on how mathematical conjectures are made. One of the Clay Mathematics Institute's Millennium Prize Problems offers one million dollars for a proof that the Navier-Stokes equations always have a smooth solution. Looking at the Navier-Stokes equations [put them in?] it might seem unclear as to why mathematicians believe there is a solution and why they think it so important to prove that there is one. The reason is that mathematicians believe that the Navier-Stokes equations govern fluid flow. Since we believe that the relevant mathematical concepts and equations do indeed accurately model fluid flow, and since we believe that it can never happen that the universe would cease to exist because there is no solution to an equation, we must believe that these equations always have a solution. A counterexample to this conjecture would be a heuristic falsifier, whereas a proof of the conjecture would justify studying the Navier-Stokes equation." (http://thesamovar.net/philosophy/essays/quasiempiricalformalism)
78. "If we depended upon consecutiveness for defining order, we should not be able to define the order of magnitude among fractions. But in fact the relations of greater and less among fractions do not demand generation from relations of consecutiveness, and the relations of greater and less among fractions have the three charac- teristics which we need for defining serial relations. In all such cases the order must be defined by means of a transitive relation, since only such a relation is able to leap over an infinite number of intermedi- ate terms. The method of consecutiveness, like that of counting for discovering the number of a collection, is appropriate to the finite; it may even be extended to certain infinite series, namely, those in which, though the total number of terms is infinite, the number of terms between any two is always finite; but it must not be regarded as general. Not only so, but care must be taken to eradicate from the imagination all habits of thought resulting from supposing it general. If this is not done, series in which there are no consecutive terms will remain difficult and puzzling. And such series are of vital importance for the understanding of continuity, space, time, and motion." (Introduction to Mathematical Philosophy by Bertrand Russell)

79. "Given any three points on a straight line in ordinary space, there must be one of them which is between the other two. This will not be the case with the points on a circle or any other closed curve, because, given any three points on a circle, we can travel from any one to any other without passing through the third. In fact, the notion "between" is characteristic of open series—or series in the strict sense as opposed to what may be called | "cyclic" series, where, as with people at the dinner-table, a sufficient journey brings us back to our starting-point. This notion of "between" may be chosen as the fundamental notion of ordinary geometry; " (Introduction to Mathematical Philosophy by Bertrand Russell)
80. "We shall employ still the notion of correlation: we shall assume that the domain of the one relation can be correlated with the domain
    of the other, and the converse domain with the converse domain; but that is not enough for the sort of resemblance which we desire to have between our two relations. What we desire is that, whenever either relation holds between two terms, the other relation shall hold between the correlates of these two terms. The easiest example of the sort of thing we desire is a map. When one place is north of another, the place on the map corresponding to the one is above the place
    on the map corresponding to the other; when one place is west of another, the place on the map corresponding to the one is to the left
    of the place on the map corresponding to the other; and so on. The structure of the map corresponds with that of |the country of which it is a map. The space-relations in the map have "likeness" to the space-relations in the country mapped. It is this kind of connection between relations that we wish to define." (Introduction to Mathematical Philosophy by Bertrand Russell)
81. "It follows from this that the mathematician need not concern him- self with the particular being or intrinsic nature of his points, lines, and planes, even when he is speculating as an applied mathematician. We may say that there is empirical evidence of the approximate truth of such parts of geometry as are not matters of definition. But there is no empirical evidence as to what a "point" is to be. It has to be some- thing that as nearly as possible satisfies our axioms, but it does not have to be "very small" or "without parts." Whether or not it is those things is a matter of indifference, so long as it satisfies the axioms. If we can, out of empirical material, construct a logical structure, no matter how complicated, which will satisfy our geometrical axioms, that structure may legitimately be called a "point." We must not say that there is nothing else that could legitimately be called a "point"; we must only say: "This object we have constructed is sufficient for the geometer; it may be one of many objects, any of which would be sufficient, but that is no concern of ours, since this object is enough to vindicate the empirical truth of geometry, in so far as geometry is not a matter of definition." This is only an illustration of the general principle that what matters in mathematics, and to a very great extent in physical science, is not the intrinsic nature of our terms, but the logical nature of their interrelations." (Introduction to Mathematical Philosophy by Bertrand Russell)
82. "Van Dalen (van Dalen 2005) remarks that, compared to the classical point of view, Brouwer's universe does not get beyond ω1 (there is no transfinite and actual infinity). But, what it lacks in 'height' is compensated in 'width' by the extra fine structure that is inherent to the intuitionist approach and its logic. This fine structure allows mathematically thinking the continuum in its very indeterminacy and errancy vis-à-vis discrete enumeration, and to do this without letting the continuum
    dissolve into an unintelligible mystery (Fraser 2006)." (From pure constructivism to imaginative constructivism – An analysis of Intuitionist Mathematics from the viewpoint of design theory, Kazacki)
83. "Poincaré expressed his doubts about the logical and axiomatic approaches of Russell and Hilbert: "The syllogism cannot reveal anything fundamentally new. If all mathematical propositions can be derived ones from others, how would mathematics not be reduced to an immense tautology?"" (From pure constructivism to imaginative constructivism – An analysis of Intuitionist Mathematics from the viewpoint of design theory, Kazacki)
84. "My purpose here is to provide still another philosophical perspective that I call conceptual structuralism for which the above conceptions of the continuum from geometry, analysis and set theory are essentially different. Comparisons will also be made with some other conceptions of the continuum, including phenomenological, non-standard, predicativist, intuitionist, and physical interpretations, all of which fail to satisfy as basic structural conceptions." (Conceptions of the Continuum, Feferman)
85. " Incidentally, Cantor (1883) referred to Dedekind's construction as "eigenartige", or "idiosyncratic" as Ewald (1996) p. 897 translates it. In Cantor's further discussion in comparison with other constructions of the reals, including his own, he says that Dedekind's has the "undeniable advantage ... that every number b corresponds to only a single cut. But this definition has the great disadvantage that the numbers of analysis never occur as 'cuts', but must be brought into this form with a great deal of artificiality and effort." (Ibid., p. 899)" (Conceptions of the Continuum, Feferman)
86. "1. The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time.
    2. Theoretical mathematics has its source in the recognition that these processes are independent of the materials or objects to which they are applied and that they are potentially endlessly repeatable.
    3. The basic conceptions of mathematics are of certain kinds of relatively simple ideal- world pictures which are not of objects in isolation but of structures, i.e. coherently conceived groups of objects interconnected by a few simple relations and operations. They are communicated and understood prior to any axiomatics,

indeed prior to any systematic logical development.
4. Some significant features of these structures are elicited directly from the world- pictures which describe them, while other features may be less certain. Mathematics needs little to get started and, once started, a little bit goes a long way.
5. Basic conceptions differ in their degree of clarity. One may speak of what is true in a given conception, but that notion of truth may be partial. Truth in full is applicable only to completely clear conceptions.
6. What is clear in a given conception is time dependent, both for the individual and historically.
7. Pure (theoretical) mathematics is a body of thought developed systematically by successive refinement and reflective expansion of basic structural conceptions.
8. The general ideas of order, succession, collection, relation, rule and operation are pre- mathematical; some implicit understanding of them is necessary to the understanding of mathematics.
9. The general idea of property is pre-logical; some implicit understanding of that and of the logical particles is also a prerequisite to the understanding of mathematics. The reasoning of mathematics is in principle logical, but in practice relies to a considerable extent on various forms of intuition in order to arrive at understanding and conviction.
10. The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other. Incoherent concepts, or ones which fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics. The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality." (Conceptions of the Continuum, Feferman)

87. " One standard set-theoretical conception of the continuum is as the set of all paths in T, or equivalently as the set 2N of all functions from N into the set {0, 1}. However, intuitively, it is natural to think of sequences (xn) of natural numbers as being generated by some sort of rule which tells us how, at each n, xn is to be evaluated given the values of xi for all i < n, or more generally simply as a rule which provides the value of xn for each n. On the other hand, the idea of a function from N into {0, 1} is based on a quite different intuition, namely as an arbitrary many-one correspondence considered extensionally, i.e. independently of how the correspondence may be effected, while rules are thought of intensionally, i.e. as specific procedures to carry out the requisite evaluations. Any such rule determines a function, but not conversely." (Conceptions of the Continuum, Feferman)
88. "from which the conception of number is to be obtained as ratios of time spans. In the end, though, Weyl is deeply dissatisfied with these efforts and concludes by writing: 'To the criticism that the intuition of the continuum in no way contains those logical principles on which we must rely for the exact definition of the concept “real number,” we respond that the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd. Nevertheless, those abstract schemata which supply us with mathematics must also underlie the exact science of domains of objects in which continua play a role. (Ibid., p. 108)'" (Weyl, according to Feferman, Conceptions of the Continuum)
89. "Whether one believed with Kant that axioms arose out of pure contemplation, or with Helmholtz that they were idealizations of experience, or with Riemann that they were hypothetical judgements about reality, in any event nobody doubted that axioms expressed truths about the properties of actual space and were to be used for the investigation of properties of actual space.
The developments of non-Euclidean and Riemannian geometry, and their subsequent application to the general theory of relativity by Einstein, dealt a death blow to this idea.9 This took place in the early twentieth century, and showed that on very large scales Euclidean geometry breaks down. Later it was also shown that Euclidean geometry must break down on small scales; how far physics has progressed towards the utter destruction of Kantian ideas about space deserves to be more widely appreciated by mathematicians and logicians. What follows is an explanation of the “Planck length” and its implications for the nature of space." (Constructivity, Computability, and the Continuum, Michael Beeson)
90. "Apparently Planck was the first to note that G ħ/c has the dimensions of length, but he offered no explanation. What follows is a simple calculation showing that distances smaller than this length cannot exist in the usual sense; i.e., spacetime cannot be considered to be smooth at that scale. The calcula- tion uses two fundamental equations: The uncertainty principle from quantum mechanics, and the Schwarzschild radius for the formation of a black hole, from general relativity. It is often stated that “general relativity and quantum mechan- ics are not consistent”, but seems not to be so well known to non-physicists that the inconsistency can be derived in one paragraph. (No claim of originality is made here; the argument is well-known to physicists and was shown to me by my friend Bob Piccioni.) These two equations will be combined to show that there is a minimum radius given by the Planck formula just mentioned, below which spacetime cannot be regarded as smooth. The smoothness of spacetime (possi- bly except at isolated singularities) is a fundamental starting point for general relativity, so this calculation shows the inconsistency of general relativity and quantum mechanics." (Constructivity, Computability, and the Continuum, Michael Beeson)
91. "The intuition of iterating a process is reducible to the concept of natural number, once the process to be iterated is understood.
This argument in support of Helmholtz’s view (and against Gödel’s) is not definitive, since the zooming processes do not account entirely for our intuition of the continuum. Here are some aspects not accounted

for: linearity, composi- tion, continuity, and fullness. By linearity, we mean the quality Euclid had in mind when he wrote that a line is that which has length but not breadth. We can zoom in on a plane or even on a self-similar fractal set. By composition we mean the question whether the continuum is composed of (infinitely many) points, each of which has zero length, but whose aggregation can make intervals of nonzero length. The alternative conception is that somehow these points need to be actively created “at run time”, as a computer scientist might say; per- haps by Brouwer’s “free choices” or by some sort of quantum-mechanical device. The zooming processes also do not address the continuity of the continuum, the property that Dedekind addressed with his definition of completeness (every cut determines a real) and Cauchy with his definition of completeness (Cauchy se- quences converge). By specifying that there are no visible gaps, we rule out the possibility that we are zooming in on some kind of fractal set rather than the true continuum. But, there are also invisible gaps to worry about: while zooming, how can we tell whether we are seeing the whole continuum or only, say, the rational
numbers? No matter how many times we zoom in on never becomes visible. We can call attention to it by placing a right isosceles triangle with its hypotenus on the number line. We can similarly call attention to any recursive real number; but how about the gap in the recursive reals at the limit of a Specker sequence? Can you visualize that gap? As the predicates used to define a Dedekind cut increase more and more in logical complexity, the existence of a point filling that cut seems less and less closely related to a fun- damental geometric intuition. Finally, the property of fullness of the continuum, as axiomatized in this paper using the fullness principle FP, seems similar to continuity, but distinct, since the recursive reals satisfy continuity (in the sense that recursively Cauchy sequences of recursive reals converge to recursive reals), but they do not satisfy FP. Fullness does not require reference to specific “gaps”, since it is defined by coverings. Perhaps the formulation of the fullness property and the recognition that it is not the same as continuity may help in future efforts to elicidate our intuitions about the continuum." (Constructivity, Computability, and the Continuum, Michael Beeson)

92. "Geometry was in attendance at the birth of logic in Euclid’s Elements, and the nature of the continuum was already giving philosophers difficulty before that (Zeno’s paradox). In the middle of the nineteenth century, Staudt (Geometrie der Lage 1847) took steps towards a modern deductive geometry; but Felix Klein ob- served in 1873 the difficulties about continuity in Staudt’s treatment, complaining of the necessity “to conceive points, also if these are defined by means of an in- finite process, as already existing.” The second half of the nineteenth century saw the development of an increasingly rigorous axiomatic approach to geom- etry, for example Pasch’s Vorlesung u ̇ber neuere Geometrie appeared in 1882; but as Freudenthal observes in his fascinating history ([8], pp. 106–107), Pasch had a number of Italian contemporaries: Veronese, Enrique, Pieri, Padoa. These developments set the stage for the appearance in 1899 of Hilbert’s Grundlagen der Geometrie [9]. 1 As Freudenthal says, the opinion is widespread that it was Hilbert who first gave a completely deductive logical system for Euclidean geom- etry, in which nothing was left to intuition. But in view of the achievements of his predecessors just mentioned, what was there left for Hilbert to do?" (Constructivity, Computability, and the Continuum, Michael Beeson)
93. "Hilbert’s book, on the other hand, begins with “Wir denken uns drie ver- schiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir Punkte; die dingen des zweiten Systems nennen wir Gerade. . .”. Freudenthal says “With this the umbilical cord between reality and geometry is severed.”" (Constructivity, Computability, and the Continuum, Michael Beeson)
94. "But a few years later, Brouwer was again attempting to characterize and elucidate the properties of the actual continuum. He wrestled for the first time with the problem that will concern us in this paper: How can we reconcile the computability of individual real numbers with our notion that the continuum itself is “full of points”, so full as to make a geometric line? Apparently he did not think that one can fill the gaps with computable numbers: he invented his theory of choice sequences. He said, for example, there is a real number 0.334434333444344..., where we are free to choose at any stage a 3 or a 4 ar- bitrarily, according to our free will. He did not require that we specify at any finite stage an algorithm for making all the rest of the choices.3 The necessity for functions defined on [0,1] to be defined on all choice sequences led to his continuity principle, that all such functions are continuous. This principle flatly contradicted classical mathematics, and was responsible in no small part for the well-known public relations problems of Brouwer’s intuitionistic mathematics.
Hermann Weyl was also concerned with the problem of “filling the gaps”." (Constructivity, Computability, and the Continuum, Michael Beeson)
95. "These three failures of important classical results of analysis might seem to be the death knell of Church’s thesis, since they appear to flatly contradict the principle of geometric completeness. Brouwer died in 1966, so he lived to see these results, but in his usual style, he never commented on them in print. They may have made him glad that he had developed the theory of choice sequences. Nevertheless, the Russian constructivists under the leader of Markov pursued the development of constructive mathematics assuming Church’s thesis for some decades. At the time of Brouwer’s death it appeared that your choices were:
(1) accept Brouwer’s theories, give up most of mathematics and give up talking to most mathematicians; or
(2) accept Church’s thesis, give up analysis and give up talking to most mathematicians; or
(3) reject constructive mathematics entirely.
This was not a difficult choice for most mathematicians; but Errett Bishop refused the prongs of this dilemma and published a book [2] in 1967 (the year after Brouwer’s death) in which he developed constructive

mathematics without using either Church’s thesis or choice sequences. Since he didn’t assume every real is recursive, the recursive counterexamples do not apply directly. Since he didn’t assume there are some non-recursive reals (e.g. choice sequences), the classical theorems are not directly contradicted. His idea was to show that by suitable choices of definitions, the constructive content of classical mathematics could be brought to the fore, and was substantial.
Logicians labored in the subsequent decade to analyze what Bishop had done, by constructing suitable formal theories and studying their formal inter- pretations. This work is summarized in [1]. These studies verified (for various formal theories) that Bishop’s work is indeed consistent with Church’s thesis as well as with classical mathematics, and is constructive in the sense that “when a person proves an integer to exist, he or she can produce that integer”. This is reflected in the “numerical existence property” of a formal theory T: if T proves $\exists x A(x)$ then for some numeral $\bar{n}$, T proves $A(\bar{n})$." (Constructivity, Computability, and the Continuum, Michael Beeson)

96. "Theorem of Mean Value Proof (Cauchy 1821) It is enough to see that the curve which has equation y=f(x) will meet one or more times the line y = c, inside the interval between a and b; now, it is evident that this will be what will happen when the hypotheses are met. QED
This proof is not a proof. It is not that the reasoning is faulty, it is the definitions that are missing: Cauchy does not have (yet) a rigorous notion of continuity, not of a curve (Weierstrass). He appeals to the evidence of threads and traces of pencil. Fortunately the theorem in Analysis is true, we can demonstrate it rigorously. Poinsot, in a course in the Ecole Polytechnique in 1815, believed he had demonstrated, in a similar fashion, that every continuous curve is differentiable everywhere, on the left or on the right. The counterexample is well known. Actually, at the beginning of the XIX century, the 'intuition' about the continuum in Mathematics needed to be made precise. The Ether of Physics was also in the scientific spirit of everyone, with the homogeneity of a perfect continuum. " (The Mathematical Continuum, From Intuition to Logic, Longo)
97. "For this reason 'the mathematician must have the courage of his inner convictions; he will affirm that the mathematical structures have an existence independent of the mind that has conceived them; … the platonist hypothesis … is … the most natural and philosophically the most economical' [Thom,1990;p.560]. Dana Scott more prudent said this to this author: 'it does no harm'.
The advantages of the platonic hypothesis in the 'linguistic synthesis' for the every-day communication amongst mathematicians are enormous, due to the efficacy of the objective signification that it can give to the language and to the crucial 'scribbles in the blackboard'. But the foundational and philosophical drawbacks that it entails are also very important, for all transcendent ontology disguises the historical and cognitive process, the project of intellectual construction, of which Mathematics is rich, and in particular the 'proof principles' and the 'construction principles' which are at the basis of its nature." (The Mathematical Continuum, From Intuition to Logic, Longo)
98. "Brouwer's Theorem. Every function on a closed interval [a,b] is uniformly continuous.
This, on the face of it, is in direct contradiction to classical mathematics, but once it is understood that Broudwer's theorem must be explained differently via the intuitionistic interpretation of the notions involved, an actual contradiction is avoided. `perhaps if different terminology had been used, classical mathematicians would not have found the intuitionistic redevelopment of analysis off-putting, if not downright puzzling." (Relationships between Constructive, Predicative and Classical Analysis, Feferman)
99. "Bishop criticized both non-constructive classical mathematics and intuitionism. He called non-constructive mathematics a 'scandal', particularly because of its 'deficiency in numerical meaning'. What he simply meant was that if you say something exists you ought to be able to produce it, and if you say there is a function which does something on the natural numbers then you ought to be able to produce a machine which calculates it out at each number. His criticism of intuitionism was its failure, simply, to convince mathematicians that there is a workable alternative to classical mathematics which provides this kind of numerical information (though intuitionistic reasoning also provides that in principle)" (Relationships between Constructive, Predicative and Classical Analysis, Feferman)
100. "Instead, Bishop begins by dealing with a very special class of functions of real numbers, namely those which are uniformly continuous on very compact interval. In this way, he finesses the whole issue of how one arrives at Brouwer's theorem by saying that those are the only functions, at least initially, that one is going to talk about. (So, the question is: if you just talk about those kinds of functions, are you going to be able to do a lot of interesting mathematics? That is, in fact, the case!)" (Relationships between Constructive, Predicative and Classical Analysis, Feferman)
101. "So if a classical mathematician reads Bishop's book he can say: well, I do not see what is very special about this. But what 'is' special is the way in which concepts are chosen and the way in which arguments are carried out. concepts are chosen so that there is a lot of witnessing information introduced in a way that is not customary in classical mathematics, where it is hidden, for instance, by implicit us of the Axiom of Choice. What Bishop does it to take that kind of information and make it a part of his explicit package of what his concepts are up to." (Relationships between Constructive, Predicative and Classical Analysis, Feferman)
102.