Valid 'syntax' for a 'number'

- It is a **hard truth** that: the existence of finitely describable infinite summation process whose elements are all positive but whose sum is clearly bounded

- This sheds light on the the syntax of numbers: 'series' cannot be excluded in general from being 'numbers'

- The quest for delimiting those 'forms' (convergence series) is open

- Logically (using the semantic no-leprechaun assumption) we can prove that we can keep adding parts of 1 forever without ever reaching 1, by always taking the half of the rest.

- In fact, $\sum 1/2^n = 1$. One proof uses the fact that the series has the property for all N, it equals $1-(1/2^N)$. Looking at this series from the increasing point of view, we are continuously increasing. Looking at it from the point of view of 1, we can provabley get arbitrarily near to it, and we can provably never exceed it.

- In other words, the operators on numbers, are **not exclusive** to the forms of number that seem to include include finite processes: naturals, rationals, algebraic forms of the two.

- This gives us an **unexpected richness** of our numbers. Series can be **split into a finite part** and a series, if the original series converges, we can analyze the rest series, since the original series supports the number operators. Furthermore, we can model **infinitely precise** approximations (e.g integrals) that are a very useful language for science.

- Whether infinite precision is philosophically viable is of **no interest to formal mathematics**! Here, as we pointed out, we only care about the non-exclusivity of 'numeric' forms in satisfying arithmetical operators. In this sense, reading limits as 'precision infinite' rather then 'something becomes zero' is the better alternative. This approach should be taken when talking about rectangles whose heights 'tend to zero', replacing it by 'almost-rectangles (on the top) approximated to infinite precision'. This makes sense since the approximation will not suddenly and 'discontinuously' drop to zero because the bases become zero. All this may be a good explicit answer to Berkeley.

- When bootstrapping the numerical concept of integration from geometry, we must **abandon the concept of area**, and 'go native'. This is a game switch.

- In a confused manner, the author tries to explain what we outline here. We also note that based on our '**split**' observation above, it is not true that '+' does not **mean the usual**, it is, when a split part is taken outside the 'infinitely precise form', which better be represented by the summation symbol $\sum$ and not '+', or alternatively, ending with a '...'. (a '...' in the middle is just a compressed normal

The calculating power of calculus comes from this dual nature of the integral. It can be viewed as a limit of sums of products or as the inverse process of differentiation. [A.p.6]

George Berkeley aptly described infinitesimals as "ghosts of departed quantities." He would object, "Now to conceive a quantity infinitely small, that is, infinitely less than any sensible or imaginable quantity or than any the least finite magnitude is , I confess, above my capacity." [A.p.6]

Fourier tried defining the definite integral of a nonnegative function as the area between the graph of the function and the x-axis, but that begs thequestion of what we mean by area. [A.p.7]

*Weierstrass had shown that if the **series [of functions]** converges **uniformly**, then **term-by-term integration** is valid. The problem with this result is that the most interesting series, especially **Fourier series**, often do not converge uniformly and yet term-by-term integration is valid. Uniform convergence is sufficient, but it is **very far from necessary**. As we shall see, finding useful conditions under which term-by-term integration [of a series of functions] is valid is **very difficult so long as we cling to the Riemann integral**. As Lebesgue would show in the opening years of the twentieth century, his definition of the integral yields a **simple, elegant solution, the Lebesgue dominated convergence theorem**. [A.p.10]*

is valid for any x, as long as the denominator on the right is not 0. Equation (2.11) is something very different. It is a statement about successive approximations. The equality does not mean what it usually does. The symbol + no longer means quite the same. [B.p.20]

Convergence Basics

- TODO: work out, using the following pointers.
- Cauchy convergence and the 'idea' of convergence are covered in [M-ST:728749]

- This can be easily related to our 'non-exclusivity' of numbers, by being more subtle about the operations and the 'allowed' numbers, and ties us to completeness. With this the question 'can every cauchy-incomplete space be completed' becomes non-trivial in the sense that the operations, once extended to the new completion elements must not loose their properties.

- Provide a good picture of the above!

Can a series 'switch' convergence?

- Yes it can. This is treated in [B.42.43].
- TODO: complete from the reference, do not forget to mention the ratio test, relate to convergence test not knowing the limit
- B.42 exhibits a series that first seems to converge, then diverges.

A: A Radical Approach to Lebesgue Integration
B. A Radical Approach to Real Analysis
C. Nonstandard Analysis (Alain Robert)
D. Symbolic Integration Tutorial (Bronstein)
E. Visual Complex Analysis (Needham)
F Proofs and Refutations (Lakatos).

Integral as an Antiderivative

- Popular as a definition in the past, intuitive and pedagogical

- Mostly adequate for calculus, does not cause much loss of power there (used sometimes even until 1960 for calculus)

- At the end of the day, inadequate as a mathematical definition

Add our convergence figure

Cauchy (1820), Fourier: Such a definition is too restrictive.

Various Definitions of 'Integral'

"Integral calculus is the inverse of differential calculus. Its goal is to restore the functions from their differential coefficients." (1802), [A.p.7]

Fourier wanted to apply his methods to arbitrary functions. Not all functions have antiderivatives that can be repressed in terms of standard functions. [A.p.7]

Even in a situation in which a function has no explicit algebraic formulation, it is possible to make sense of its integral, provided the function is continuous. ...

Dirichlet stretched the concept of function to that of a rule that can be individually defined for each value of the domain. Once this conception of function is accepted, the gates are opened to verys trange functions. At the very least, integrability can no longer be assumed. [A.p.3]

"Not all functions have anti-derivatives that can be expressed in terms of standard functions." [A.p.7] : Draw this!

Poisson defined the definite integral as the difference of the values of the antiderivative. It would seemt here is nothing to prove.

The characteristic function for the rationals (Dirichlet, 1820)

- Dirichlet produced this functions to show that 'not all functions are integrable' , as a counter-example to non-proved assumptions related to the possibility of finding the terms for Fourier series, these terms being given by integrals

- Is not Cauchy integrable

- Also not Riemann integrable despite Riemann's attempts

- Violates the linearity of the integral operator for those definitions

- Departs from the 'function as (algebraic) form' tradition, this is a switch of game, from form to existence, especially since the Cauchy definition is not 'form-based', and probably cannot be. It does so by being based on a pointillistic predicate, which is not 'interval'-based

- **Continuity** came to replace the 'form-based' property as the saving regularity, (piecewise) **monotonicity** along some interval (lack of **oscillation** eventually) was also needed by Dirichlet to prove some convergence theorem.

The Cauchy Definition

- Is the first 'arithmetic' translation of the geometric concept of area under a curved graph. It uses the concept of infinitely precise approximation, carried by a finitely describable form, whose distance to some number can be computed, relative to a arbitrary precision increase process

- A decisive insight is that convergence can be proven before finding what it is converged to. In short, there are properties that guarantee convergence, that rely on **quantifiable facts about the converging object itself**, instead of relying on distance to a limit. It must be noted though that the arbitrariness must first be fixed, a **formal witness of convergence found**, usually, by choosing the way infinite precision is reached, by choosing some **homogeneous finite formula for the intervals, parametrized by the parameter that carries the increase in precision**. If one such witness is found that is amenable to a convergence proof, the job is done. Must arbitrarily chosen fom might be useless for this purpose. The majority of ways in which the approximation object is 'morphed' are beyond finite description though.

Poisson's (Failed) Definition

- Poisson defined the definite integral of f as F(b)-F(a), F being a (any) antiderivative.

- Poisson shows that if F is such that F' = f, then using Taylor expansion of F, that F(b)-F(a) is a kind of infinite sum. But to prove that this sum is bounded, he seems to need Leibniz's conception of the integral as an infinite sum of products. *[A.p.10]*

- In short, the infinite summation is an 'essence' of integration. The infinity being necessary for infinite precision.

The Riemann Definition

- The Rieman definition merely finds a general way in which Cauchy's definition can be relaxed in the sense of form, while preserving(?) all its intent and meaning.

- 'Merely', the impact is large since a larger number of results can be found due to this relaxation.

- The exact correspondence of the two forms is not trivial to prove though (lid: cauchy_equiv_riemann)

The 'Fundamental Theoreom of Calculus'

- A history of the name:
  - *"the fundamental proposition of the theory of definite integrals." (Poisson, 1820)*
  - Cauchy mentions FTC in 1823, but simply by passing, not calling it theorem or proposition.
  - *"Fundamental principles of the integral Calculus" (Lardner, 1825)*
  - *"Fundamental theorem for integrals" (Freycinet, 1860)*
  - *"Fundamental theorem of integral calculus" (du Bois-Reymond, 1876)*
  - *Ditto but in enlish (Hobson, 1907)*
  - *"Fundamental theorem of calculus" (Hardy, 1908)*

- The two ways to see the FTC are *[A.p.9]*:
  - As evaluation: The definite integral of f on [a,b] is F(b)-F[a], where f is the derivative of F (any of them) at every point in the interval.
  - As antiderivative: TODO, we don't understand this, find an example

- **The FTC, under various definitions of integral**, holds under various conditions.

- For **Riemann integration, the conditions for FTC holding are complicated.**

Continuity at a single point

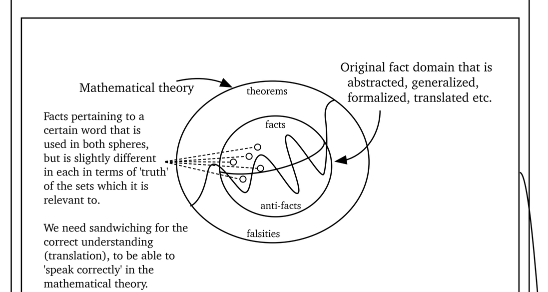
- Yes this is possible and very easy to construction by using the characteristic function of rationals, multiplied by x. This is continuous only at zero
- This teaches us not to have misconceptions about non-infinitesimal sizes of neighborhoods where the 'continuity holds'. In the case above continuity holds only at a point, even though its intuition is neighborhood based, and its definition hinges on neighborhoods.
- **The definition hinging on neighborhoods does not logically mean that any neighborhood of non-infinitesimal size must exist where the predicate being defined and hinging in its definitions on neighborhoods holds'.**
- **Even stronger, this leads us to emphasize an 'obvious' fact, that only known theorems are theorems, nothign else! In other words, anything that is not a known theorem in all probability does not hold! Except in very special cases, cases too special where combinations of properties make something hold, but where such special cases are of no general, textbook or utility relevance**
- **The above gives a pointer on the (now even more interesting) way in which infinitesimals restore intuition. One must study this in order to learn the power of manipulating formal systems to 'rename' concepts. The neighborhood in question for the predicate may be infinitesimal! This is a renaming of 'for all epsilon', a figure of speech**
- **Simply put, disbelief of continuity at a single point is nothing but a blunt misunderstanding of 'for all there exists', leading to thinking about concrete neighborhoods. Let us call this the *concretisation fallacy***
- Would a concept of '**radius of continuity**' solve this, making it possible for this radius to be zero at a continuous point, just like convergence of series?
- The characteristic function is not a necessary component of the function continuous at a single point. There is no theorem about this so it is not true until proven. Conway's Base-13 function can also be used

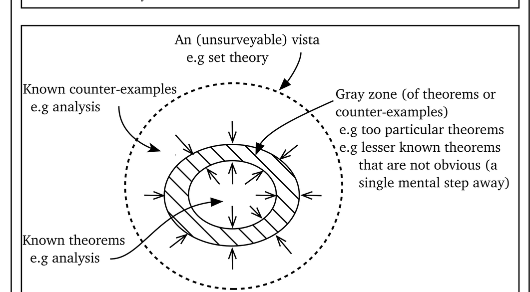
f is continuous at x if the graph of f "infinitesimally near" x varies only infinitesimally from a straight (horizontal, in fact) line. [M-ST:999320]

The notion of S-continuity is easy to manipulate from a logical point of view (there is no inversion of quantifiers). [C.51]

[Counterexamples in Analysis] (Book)

- This invaluable book teaches us an 'obvious' lesson: Everything that is NOT a theorem, does not hold. Per example:
  - There is no theorem about a 'finite size of neighborhood
  - There is no theorem about contibuity not being possible at a single point





- This means that the '**gray zone**' is real, that is is imperative to known as many 'counter-examples' as possible to keep the gray zone to a minimum, by **sandwiching** it.

- The **gray zone is expensive**, any question such as the single-point continuity one has to be worked out by us, or researched, which is time consuming.

- TODO: copy more examples from the book

$$\not\Rightarrow \frac{\mathbb{F}}{\mathcal{O} \mid \mathcal{C} \mid \mathcal{A}}, \quad \not\Rightarrow \frac{\mathbb{F}+\mathcal{O}}{!\mathcal{O} \mid \mathcal{C} \mid \mathcal{Q}\text{-dense}}, \quad \not\Rightarrow \frac{\mathbb{F}+\mathcal{O}+\text{C-}\mathcal{C}}{\mathcal{C}}$$

$$\not\Rightarrow \frac{f_{\mathbb{F}\mathcal{O}} + \vdash\!\dashv + \wedge}{\square \mid \mathbb{W} \mid \text{IVT}}, \quad \not\Rightarrow \frac{f_{\mathbb{F}\mathcal{O}} + \square + \wedge}{\mathbb{W}}, \quad \not\Rightarrow \frac{f_{\mathbb{F}\mathcal{O}} + \mathbb{W}}{\overline{\uparrow}}, \quad \not\Rightarrow \frac{f_{\mathbb{F}\mathcal{O}} + \sim}{\text{Rolle}}$$

How 'undifferentiable' can a continuous function be?

- Belief until around 1850: continuous is differentiable at most points, that is at most over a sparse infinity of points is the function undifferentiable.

- 1875 (Darboux, du-Bois Reymond): continuous nowhere differentiable.

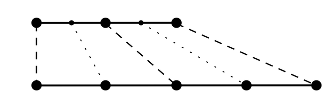
- Weierstrass: continuous monotonic undifferentiable at any algebraic number. In fact one can prove that such a function is differentiable at 'most' points.

- We still need to verify this, but basically, **continuity** and **lack of oscillation (viz. monotonicity)** gives us what we 'expect', that is, what we can draw (see an old note of ours). Or maybe differently, oscillation has been shown to be so wild that it breaks **differentiability** for continuous functions, and that can be accomodated such that it happens at all points.

- TODO: relate this to integration

Point and Neighborhood

- A misunderstanding of the relation between point and neighborhood is easily a '**deceitful misconception**'.
- The misconception start with two human experience based misinterpretations



- Consider the function y=2x in the figure.
  - This function is 'broken' in the discrete setting: the points 1,3 of y have no pre-image. In a certain sense, we do not have a one-to-one mapping of points. Clearly, since this is a 'one to two' map. But the graph of y=2x in the non-discrete setting is clearly 'one-to-one'.
  - We can fix this by 're-discretizing' adding the points 0.5,1.5 of x and 'rescaling' the whole figure to 'whole numbers'
  - But now the 'intervals' are broken: we have a one-to-one mapping of points, but the intervals are stretched.

- In the discrete setting, an interval IS the set of its points

- With the rationals, we cannot say this anymore in the following sense:
  - consider y=2x on the interval [0,1]
  - The 'measure' of the interval is doubled, the interval is 'stretched', each sub-interval is 'stretched'
  - But points cannot be 'stretched', in any case a point 'stretched to two' is still not an interval. Did we just arrive at something like measure theory? **Any finite number of points is not an interval, and hence does not even BEGIN to have 'measure'!**
  - If an interval IS the set of its points, then if the interval is 'stretched', so are the points, however, this is not so, since y=2x is 'one-to-one', indeed, to each point exactly one is mapped. A paradox is '**where do these points come from?**'
  - In any case, an interval contains infinitely-many points, this is our usual 'change of type by limit'
  - So infinitely many points stretch to infinitely many, and although the interval 'measure' was stretched, the relation between the two sets of points is still 'one-to-one', nothign was stretched here. It is as if infinity of points has an infinite reserve of hidden points
  - Note that there is also no 'one-to-one' function between [0,1] in the reals and [0,1] in the rationals

- What do we do with all of the above?

- The **relation in (set-theoretic) mathematics between interval and point is set theoretic**. An interval is a special kind of infinite set of points

- It is great that we can 'model' y=2x at will in the naturals, rationals and reals

- We achieve this by 'packaging' potential infinity by formalism and syntax

- This is possible due to the 'translation', nay 'reinterpretation', nay 'pointing a finite string (syntax) into any semantical (e.g 'infinite') concept' in a consistent manner

- Note that this is the way we can use 'exponentiation' (power-set) to 'reinterpret it' for infinite entities to give us the uncountable reals
- The relation is set-theoretic and made possible by the ordering of the number system, allowing two numbers to define a whole infinite set of numbers

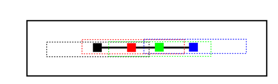
- By this, mathematics allows to handle this paradox with no 'inconsistency'

- It does this, when looking at this humanly, in the reverse manner our discrete figure works. All points are 'already there', at least in the sense that we can pinpoint any two corresponding points when we need to. What is double though, 'stretched' is still the interval. But all of this is 'human interpretation' of the very powerful set theoretic vista of the reals. Nevertheless, one must learn to 'read' this without the discomfort of this 'conceptual for the human'-'inversion'.

- But this 'reading' can be fixed. **We leave the 'number of points', and we enter a 'cardinality' of points. 'one-to-one' functions preserve cardinality of the sets, 'one-to-one' is a description on the level of cardinality of points, and is NOT a description on the level of 'measure' of intervals**. This is the useful reading.

- The above means that **we cannot recover the 'measure' of the interval from the (not number!) infinite (or power-infinite) cardinality when the interval is seen as a set. This must be done differently. Some extra structure must probably be added on top of the pure 'set' structure. Per example, subtracting the 'numbers' that stand for the interval's endpoints**.

- TODO: is there more to be said?



- This disconnect between point and neighborhood can be easily found to model something. Imagine a chain of four one-dimensional particles, that is kept together by some kind of electric force. The 'wire' can be stretched, and hence the 'interval measure' is stretched, nevertheless, it still contains only four particles, and is 'gapless' in the sense that the force will prevent anything to go through it. Is not a point the wrong model for these real 'particles'? again, what is the right model for a 'real particle', who knows! **it depends on the scale (operators) we are interested in**!

- A '**crippling misconception**' is also that of thinking of the 'things' of somethings modelled by mathematics to be modelled by some kind of a set, and only the things. By '**logically packaging infinity**' it would really seem as if the effectiveness of mathematics is unreasonable. But! not only the 'things' but also the 'operators on things' have to be 'reinterpreted' in the modelling, and they usually are when the model is useful. So the effects of the operators on the things in what is being modelled mirrors the effects of the operators on the things in the mathematical model. The 'packaging' **stops being merely a 'useful figure of speech' because the operator comes along**. It is not that the mathematical thing is a figure of speech of the modelled things, but that **the thing-operator system mathematically mirrors the thing-operator being modelled**. The mathematical model becomes a concrete and effective one, and not a figure of speech.
This is related in some manner to the 'non-exclusivity of number form'.

- This is possible because the set-theoretic vista is simply so vast that it allows us to 'effectively' model so much, in a way that supports proofs.

- Note that in rigid-body simulation, it is not true that even with exact solutions we are modelling reality, but we are modelling a theoretical 'rigid-body world' that is closer to the real 'rigid-body world' than any approximate solution simulation, because the reinterpretation of operators in terms of the billions of particles and scale of forces turns out to be best matching for the 'exact solutions in the real number system', at that scale, not at any other.

Method and Integration and Taylor Series

- It is important not to forget that 'practical integration' (when the goal is closed-form) is only related to analysis in two ways:
  - The analytic manipulations allowed
  - The proofs of convergence when something needs to be 'checked'

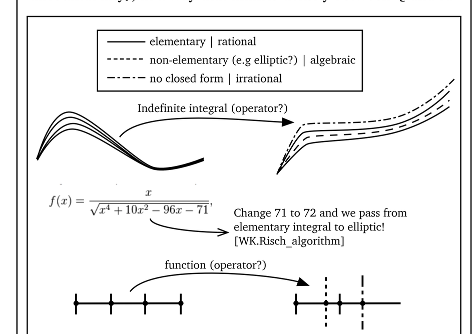
But the discovery of the integral, be it indefinite or not, is usually not the domain of analysis, so the topic of this section is somewhat 'out-of-topic'. This is clearly why it is not treated in analysis books, and if, then only as 'IQ test' exercises because the topic is a great and deep mine for that. Or is it? How methodical can we get?

- Since 1970, there is a method for deciding if an elementary function has an indefinite elementary integral, and finding it.

- The whole topic is a generalization of the 'implicit-explicit' problem. Here explicit means the removal of the integral sign. Apart from that, it is quite the same, with the operations being different.

- In a certain way, for those elementary function with no elementery indefinite integral, the 'implicit' form cannot be made 'explicit'.

- It is a 'crippling-misconception' (but at the same time a leading discovery method!) to extrapolate the way integration behave over polynomials in one way: Changing factors does not change the kind of integral, and this is simply not true in general! In a certain way, there is 'discontinuity' between the 'closed-formness' and the 'geometric-curveness'. Maybe the right faithful picture is geometric curves as finely closer together as we like, with 'closed-form' being available for only some of them (by uncountability), in a way that mimics density between Q and R.



- At the same time, extrapolating from integration over finite to infinite polynomials, and bypassing rigor temporarily, has been a leading (discovery) 'method' in the old days.

- In the old (Newton) times, methods for discovery were:
  - geometric hints as to relations between functions and their indefinite integrals
  - explosion into power series by means such as the (unproved) generalized binomial series
  - manipulation 'term-by-term' of power series
  - find explicit relation between infinite series
  - ...
  - TODO: be more specific using Hairer's book

- Our theory of 'mathematics as **artefact** (lost technology)' applies beautifully here. Since modern methods necessitate search for patterns, while some older (non-rigorous) methods provided the original discoveries by geometric or 'unrigorously extrapolative' means. The effects are still here, as we can see in Gowers's quote and even the title of its article.

- Nowadays, even though a method exists (**Risch**), it is of course, like many methods, only suitable for computers. For humans:
  - change of variables, which is just another manipulation to explicitize
  - rules of calculus, directly or indirectly
  - finding patters in taylor series expansions and turning them into closed form [Irresitible Integrals (Boros)], all with the help of computers [D]

- Note that much of the same goes for finding 'closed-form' taylor series. One can always find ones by calculus, but this only gives it term by term. Again, ancient techniques and computers help.

- It seems that the **full partical faction method** is the conceptual basis for all computer algorithms, we should at least study it as presented in [D.6]. TODO.

- Also see [Table of Integrals, Series, and Products] and [Table of Integrals, Series, and Products (Related Papers)] that we scraped.

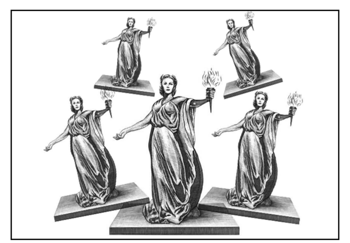
While this problem was studied extensively by Abel and Liouville during the last century, the difficulties posed by algebraic functions caused Hardy (1916) to state that "there is reason to suppose that no such method can be given". This conjecture was eventually disproved by Risch (1970), who de- scribed an algorithm for this problem in a series of reports [12, 13, 14, 15]. In the past 30 years, this procedure has been repeatedly improved, extended and refined, yielding practical algorithms that are now becoming standard and are implemented in most of the major computer algebra systems. [D.5]

The Full Partial Fraction Algorithm. This method, which dates back to Newton, Leibniz and Bernoulli, should not be used in practice, yet it remains the method found in most calculus texts and is often taught. Its major drawback is the factorization of the denominator of the integrand over the real or complex numbers. We outline it because it provides the theoretical foundations for all the subsequent algorithms.
... Note that this alternative is applicable to coefficients in any field K,
... Thus, this approach can be seen as expanding the integrand into series around all its poles (including ∞), then integrating the series termwise, and then interpolating for the answer, by summing all the polar terms, obtaining the integral of (3).
[D.6]

It is a standard trick for evaluating difficult integrals along the real line to consider a closed-contour and "blow-up" the complex part till it vanishes, leaving us with the residues picked up along the way. [M-ST.441881]

One final remark is that this length calculation explains why the usual substitution of cos(tau) for t in an integral of the form I[a,b][1/sqrt(1-t^2)] is not a piece of unmotivated magic. [Gowers: How do the power-series definitions of sin and cos relate to their geometrical interpretations?*]*

Carry the (Artefact) Torch



- The question of why ln(x) is defined as the integral of I(1,x,1/t, dt) traditionally produces theses answers:
  - Because this is the definition of ln(x). (e(x) is then defined as its inverse)
  - Because ln(x) is the inverse of e(x) and e'(x)=e(x), the justification of which is by definition
  - Because the properties of ln(x) can be shown geometrically to correspond to the properties of how the area under 1/x behaves. (this is closest to the original old discovery)

- This is an excellent example of mathematical artefact. Here, we finesse the theory into 'carry the torch'.

- Mathematics can be seen as the figure above, a million torches waiting to either be used by you, multiplied by you, or passed on to you.

- There is no way a human can carry all torches, there are too many of them (someday, a computer will)

- In this sense, the mathematical education is merely an organized effort to teach you how to receive a small number of prototypical torches. No more.

- It is crucial not to get blinded by the million torches, and not to try to carry all the torches. Most must be used and not carried.

- One goal of unifying mathematics is to give anyone new the ability to recreate as many torches from almost scratch, instead of receiving them as artefacts.

Learn: ArtefactElementary

- This learn is about tracing some elementary artefacts and analyzing their methods
- TODO: Answer the question about who found the form, refine the Leibnezian transumations using [3] (it does not stop here), continue with Newton's 'Resolution of affected equations', do all this using the bookmarks: **[31-March-2016], and the written notes.**

[1] (Bos) Johann Bernoulli on Exponential Curves
[2] (Klein) Elementary Mathematics from an Advanced Viewpoint
[3] (Dennis et.) Pascal and Leibniz: Sines, Circles, and Transmutations

$$e = \sum_{k=1}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} (1+\tfrac{1}{n})^n = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \max(\sqrt[x]{x})$$
$$: \lim_{h\to 0} \frac{e^h-1}{h} = 1 : \int_1^e \frac{1}{t} dt = 1$$

Leibniz gives differential rules for $\mathrm{d}(u+v)$, $\mathrm{d}(uv)$, $\mathrm{d}(\frac{u}{v})$

[1] Nieuwentijt criticizes the abscence of $\mathrm{d}(u^v)$

Leibniz and Bernoulli (indep.) show: $(u^v)' = u^v \log(u) v' + v u^{v-1} v'$

For u constant and $v = x$, $(a^x)' = a^x \log(a)$
When $\log(a) = 1$ we solved $(a^x)' = a^x$

$$(a^x)' = a^x \underbrace{\lim_{h\to 0} \frac{a^h-1}{h}}$$

$\lim_{h\to 0} \frac{a^x(a^h-1)}{h} = a^x \lim_{h\to 0} \frac{(a^h-1)}{h}$

so $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

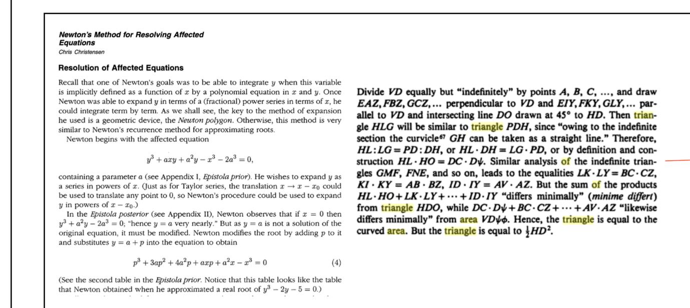
Mercator, Newton, Gregory (indep.)

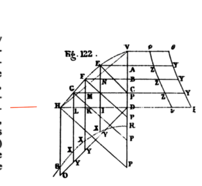
[2.151] Newton finds this by reverting $\log(x)$

Who found that $\exp(x) = \log^{-1}(x)$ has the *form* a^x ???

$$\log(1+x) = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$
$$\underbrace{\log^{-1}(x)}_{(e^x)} = 1 + \frac{x^1}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$e = \sum_{k=1}^{\infty} \frac{1}{k!}$$

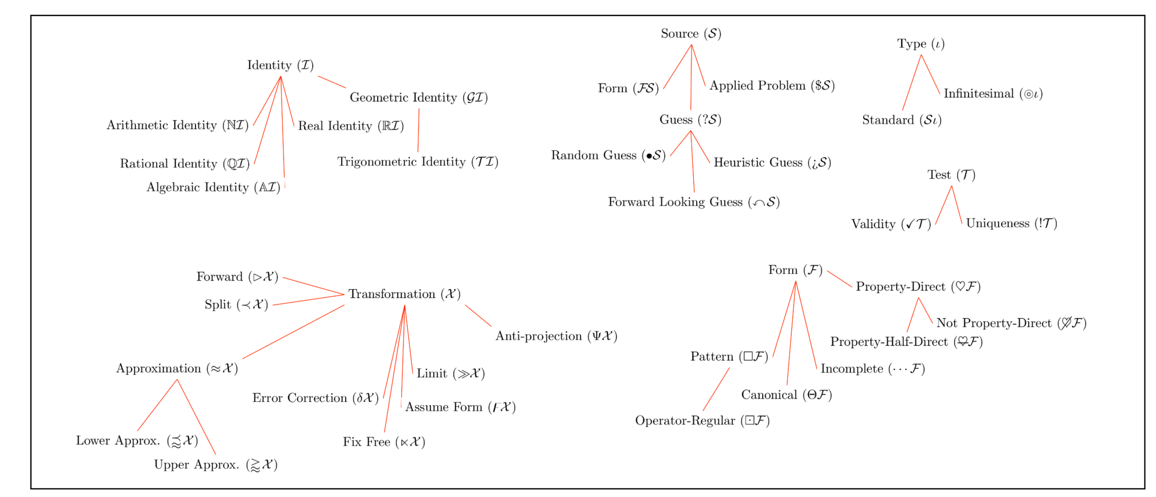
$$e = \lim_{n\to\infty} (1+\tfrac{1}{n})^n$$
$$e^x = \lim_{n\to\infty} (1+\tfrac{x}{n})^n$$

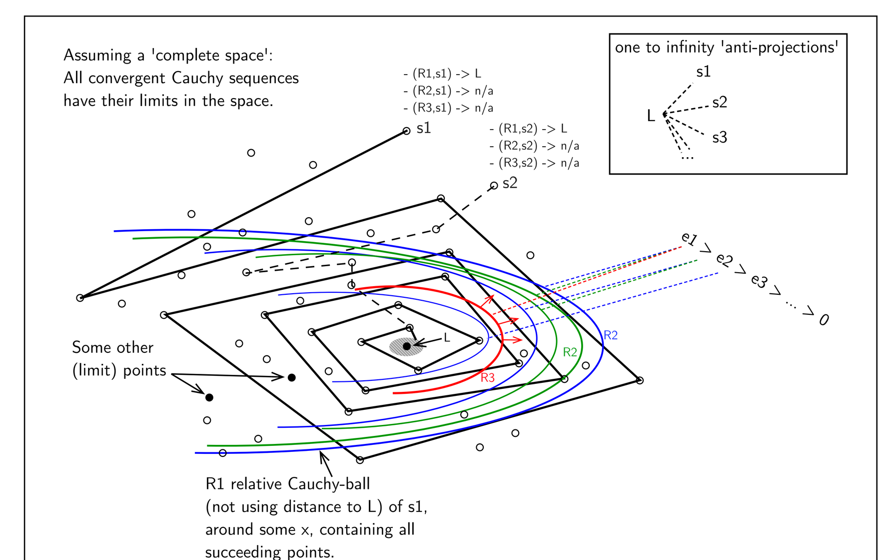




Linear Leibnezian Transmutation [3.5,6]

Curved Leibnezian Transmutation [3.9,10]





Paths to Unifrom Convergence

- By finding the flaw for 'max'
- Lakatosian
- Natural dist
- ???

TODO

It turns out that this questions <mystery of convergence of real power series> has a beatifully simple answer, but only if we investigate it in the complex plane. If we instead restrict ourselves to the real linve ... then the relationship ... is utterly mysterious. Historically, it was precisely this mystery that led Cauchy to several of his breakthroughs in complex analysis. [E.64]

To see that there is a mystery, consider the power series representations of the functions G(x) = 1/(1-x^2) (fig 12.a) and H(x) = 1/(1+x^2) (fig 12.b). ... both series have the same interval of convergence, -1 < x < 1.

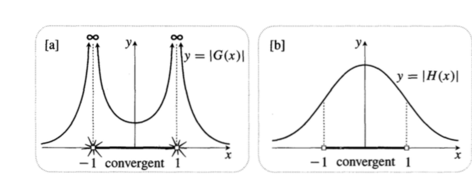


Figure [12]

Fourier's words about the perpendiculars in the graph are telling. He considered these limit functions to be (in some sense) continuous. In fact, Fourier certainly regarded anything as a continuous function if its graph could be drawn with a pencil which is not lifted from the paper. Thus Fourier would not have regarded himself as having constructed counterexamples to Cauchy's continuity axiom.2 It was only in the light of Cauchy's subsequent characterisation of continuity that the limit functions in some of Fourier's series came to be regarded as discontinuous, and thus that the series themselves came to be seen as counterexamples to Cauchy's conjecture. Given this new, and counterintuitive definition of continuity, Fourier's innocent continuous drawings seemed to become wicked counterexamples to the old, long established continuity principle.

*Cauchy's definition certainly translated the homely concept of continuity into arithmetical language in such a way that 'ordinary commonsense' could only be shocked.1 **What sort of continuity is it that implies that if we rotate the graph of a continuous function a little, it turns into a discontinuous one?** [F.130]*

So if we replace the intuitive concept of continuity by the Cauchy concept then (and only then!) does the axiom of continuity seem to be contradicted by Fourier's results. This looks like a strong, perhaps decisive, argument against Cauchy's new definitions (not only of continuity, but also other concepts like that of limit). No wonder then that Cauchy wanted to show that he could indeed prove the continuity axiom in his new interpretation of it, thereby providing the evidence that his definition satisfies this most stringent adequacy requirement. [F. 130]

Editors note: What is violated here is, perhaps, not our intuitive notion of continuity, but rather our belief that any graph representing a function would still represent some function when slightly rotated. Fourier's curve is continuous from an intuitive point of view, and this intuition can still be accounted for by the e, S definition of continuity (with which Cauchy is usually credited); for Fourier's curve, complete with perpendiculars, is parametrically representable by two continuous functions. [F.130]*

Power series, uniform convergence, Asymptotics, NSA, complex analysis

- TODO: An excellent problem based hook into complex analysis is found in [E], bluntly showing why convergence of power series is mysterious as long as one stays on the real line.

- An excellent illustration of 'fact-line distortion' is continuity's arithmetic definition, which results in that a continuous curve if rotated can become discontinuous, which has nothing to do with 'lifting the pencil'. What we are actually seeing here, is a set-theory-map-based, a relation (coordinate-relationship) based view, as opposed to a geometric-based, coordinate-free-based, object(curve)-based view. There is no 'coordinate-free univariate analysis', probably because it 'could exist' had things gone differently, but we have all we need with real analysis to cover all bases, perhaps by recourse to '**coordinate-free multivariate analysis**'!!!! Which is then what 'our intuition is' about the convergence to a 'discontinuous' curve, of 'geometric', 'curve-native' convergence, and not 'coordinate-based' convergence!

- The above is a step short of coordinate-free differential geometry, since we are not dealing necessarily with differentiable things, but only continuous ones? Actually, what one has intuitively in mind is the **topological definition of a continuous curve**. In other words, one has: **univariate analysis, topology, coordinate-free differential geometry**. Univariate analysis being the most 'basic' in the sense that it is the closest to set-theoretic maps, and has the least 'geometric' character. Change of coordinates plays a weak (no?) role here. We then assemble the later disciplines from the former. All this might constitute a 'crippling misconception' about what one is studying in real analysis. Cauchy and friends '**separated' this discipline from the other two, and that is a very important fearure of it that must be explicitly stated.**

- Relate 'modes of convergence', 'true-at-the-limit fallacy', converge in multiple senses? yes pointwise, uniform, etc. we should not forget again, the 'operator', convergence with respect to which distance, convergent with respect to which predicate, which distance operator is the right one for a certain predicate?

- For power series, pointwise and uniform convergence coincide [F.135]. This validates our ideas about 'projection': the concept ways of convergence must be unprojected from what it looks like on power series.

Convergence

- **Attention** has to be paid, and no element 'assumed', 'disregared', or 'underappreciated' in the **actors in the concepts of convergenece**, and more precisely, in any of its formal translations. This means that the metric, the space, the types the metric works on, are all equally important.

- In terms of metric, it is important to notice that as soon as it is possible to **find a valid metric (from previous analysis, we know that the concept of a metric is that it does not allow 'backdoors' for the < operator)** for a type, or two types (e.g distance from a point to a function)! One can already think about convergence. This is 'subtle'. Per example: **how on earth can one study the convergence of a sequence of rationals to sqrt(2)?! One cannot calculate the distance of any rational to sqrt(2)! Here again, the implicit/explicit play is the answer. We know implicitly a property of sqrt(2) and how sqrt behaves under the metric: if 1^2 < sqrt(2)^2, then 1 < sqrt(2). This allows us to complete a type (rationals) by another (irrationals), without full explicit access to the metric induced distances**.

- One should never forget the 'anti-projection' nature of things, the **infinitely many sequences that converge to some L**, in other words, 'formal' **sequence representations of a 'number' L**, in other words, many problem solutions in this domain will amount to **finding! the one that has a nice compressible form that is amenable to formal convergence** proof given the formal manipulations provided by the metric at hand.

- The figure shown only make sense '**complete spaces**' (in the Cauchy sense). But we know how wild topological spaces are so this is no probem at all.

- We found it '**wrong' that convergece is defined by distance to a certain limit**! Should it not be that a sequence can be seen as convergent in and by itslef? By the metric 'radius' of the subsets as we go down going down to zero, and hence pointing to a single 'limit' (if it is part of the space)? Yes, this is exactly Cauchy-completeness! We wil call such balls, not constructed from the limit point but from the property of the sequence itself, and hence every ball centered at the first point in the sequence that is the start of the subsequence considered, '**Cauchy-ball**'.

Why are there different **kinds of convergence**? As soon as we talk about functions, we have 'pointwise' and 'uniform'. This already tells us there are probably infinitely-many kinds of convergence. We note that the terminology is confusing though. When one think about the 'natural' distance between two things, one, all things being equal is considering these things in their enitreness. The distance between two functions should naturally not have much to do with the one between their 'points', since the relation between the two is a very set-theoretically, logically oriented one which is a bias. In this sense the 'natural' distance between two 'curves' is 'd-infty' (sup-metric), not a pointwise 'd-2'. In any case, this is bias as well. But it is very important to note that the **concept itself has NOTHING to do with a logical game, the commutation between 'for all' and 'there exists'**, that is a 'mere' logical translation, albeit a necessary one of course. In short, **the sequence of curves does 'curve-converge' to a curve, the sequence of (continuous) functions does 'function-converge' to a (discontinuous) function. It took a lot of turmoil to realize this, by Lakatos we know that this is a process of finding the definition of function, etc.**