Note. A coherent descent procedure.

> *The algorithm is in the same family as the algorithms described originally by Barr, Gilbert, and Wolfe [l], [2], [29] and may be viewed as a descent procedure which works on the distance between elementary polytopes contained in the convex sets. We have devised a special subalgorithm for evaluating the distance between the elementary polytopes. It contributes significantly to the overall efficiency of the algorithm. An important feature of the algorithm is its very general initialization features. When used to detect collision along a continuous path in the configuration space which describes the position and orienta- tion of the objects, they allow significant reductions in the total computation time. The algorithm has good numerical proper-ties and a thoroughly tested Fortran subroutine is available.[1]*

Note. In the paper, the term *convex radius* does not occur.The related concept used is that of a *spherical extension.*

> *If the distance between objects A and B is known, so is the distance between their spherical extensions.*
>
> *Spherical extensions are valuable for several reasons. They may be used to cover an object with a shell of safety: if x K‘, it is clear that the distance between x and K exceeds r. More importantly, they may lead to economical representations of complex objects. Object A in Fig. 1 is a simple example. It is the union of two spheres (extensions of points) and a circular cylinder with spherical end caps (an extension of a line segment). Another example is a solid rectangular plate of thickness 2r with round edges; it is modeled by an r- spherical extension of a planar polytope with four vertices. Similarly, more general wire-frame objects can be given rounded representations.*

Note. The first page of the preliminaries in [1] seems abstract at first, but it is actually intuitive and easy to imagine geometrically once deciphered.

Carathéodory's theorem is the basic theorem to understand for the GJK algorithm, and although it needs the construction of a convex hull, and GJK avoids precisely that, it is still important and very intuitive to understand.

Carathéodory's theorem applies to points in the convex hull, but it can of course be applied indirectly to points outside the hull in the following sense. For any point (the origin in GJK) in R^n, we assume it is in the convex hull, and find the n-simplex hull containing it. If the point was indeed in the hull, it will be inside the simplex. If it is not inside the simplex (easy to check), then it was outside H (it is easy to see that the algorithm can proceed with no problems in the latter case as well).

With this tool, we can already, given the minkowski sum, decide if the two polytopes intersect. We are however interested in the distance when they do not (and eventually the penetration when they do). For that purpose, an Carathéodory's theorem is not enough. We need to describe the closest point on the sum to the origin as a convex combination. We do not know what the point is beforehand, in contrast with just determining intersection where we know we

are concerned about the origin. This is where supporting hyperplanes come into play.
When the closest point C is on the boundary (which it is when there is no intersection), it must be on a feature of the hull. A feature has at most the dimension of n-1 simplex no matter how many points it is composed of, and therefore, C is the convex combination of n points of the hull (those from the feature) again by Carathéodory's theorem.

Note. The distance sub-algorithm, as described in the paper is indeed quite cryptic to the uninitiated. The origins formula described in (30) have to be found somewhere else. Indeed Olvang [5] provides the missing explanation, as well as an alternative voronoi based sub-algorithm. The first key point is the recurrent idea that for affine hulls, closest point problems (minimization), can be replaced by orthogonality conditions (in direct consequence of the pythagorean theorem). The second is that the constraint of a point orthogonal to an affine hull of a simplex and in the simplex (convex combination) turns out to be beautifully a linear algebraic problem.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ (\mathbf{x}_2-\mathbf{x}_1)\cdot\mathbf{x}_1 & (\mathbf{x}_2-\mathbf{x}_1)\cdot\mathbf{x}_2 & \cdots & (\mathbf{x}_2-\mathbf{x}_1)\cdot\mathbf{x}_{n-1} & (\mathbf{x}_2-\mathbf{x}_1)\cdot\mathbf{x}_n \\ (\mathbf{x}_3-\mathbf{x}_1)\cdot\mathbf{x}_1 & (\mathbf{x}_3-\mathbf{x}_1)\cdot\mathbf{x}_2 & \cdots & (\mathbf{x}_3-\mathbf{x}_1)\cdot\mathbf{x}_{n-1} & (\mathbf{x}_3-\mathbf{x}_1)\cdot\mathbf{x}_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (\mathbf{x}_n-\mathbf{x}_1)\cdot\mathbf{x}_1 & (\mathbf{x}_n-\mathbf{x}_1)\cdot\mathbf{x}_2 & \cdots & (\mathbf{x}_n-\mathbf{x}_1)\cdot\mathbf{x}_{n-1} & (\mathbf{x}_n-\mathbf{x}_1)\cdot\mathbf{x}_n \end{pmatrix} \lambda = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

The third is that intuitively, the minimal distance point on a simplex is either orthogonal to the feature, or on a vertex of a feature, and never anything else. This means that, for some feature of the simplex, the system will have a solution. So, the sub-algorithm terminates as soon as it finds such a feature, starting with the dimensionally lower ones. The equation above asks the following question: What is the affine combination of the tip of a vector perpendicular to the feature. If the affine combination is actually convex (all λ are positive), we are done.

Not that for the 0-simplex (point), orthogonality is trivially satisfied, and so, the sub-algorithm checks the signs of some selected λ values of related 1-simplices; the same procedure is repeated for higher simplices, until the largest simplex is reached where the orthogonality conditions can only be satisfied by the origin (see following paragraph). The idea selected λ values of the higher dimensional simplices is geometrically related to voronoi regions, but in the original paper, it is treated in a full algebraic way. The details can be probably found in Johnson's elusive PhD paper[6], nevertheless, it is nice to be able to fully derive the algebraic proof (outlilned in the GJK paper's appendix) as it is actually quite interesting how this can be done without explicit recourse to any geometry. The history of GJK seems to come from that fact that Johnson originally invented a simplex based algorithm (the distance sub-algorithm) and used in practice it by dividing shapes (robotic manipulators) into simplices. Later, this evolved into GJK where dividing shapes into simplices is not anymore needed, instead, convex shapes can be used directly. Concave shapes are still elusive.

The sub-algorithm is made performant in the original paper, with explanation in [5], by removing any duplicate or avoidable work (notice the many 1s and 0s in the formula). Caching is also an option.

Notice that when we reach the largest simplex, the orthogonality conditions can only be satisfied by the origin, since the largest simplex affinely spans the whole space. So at that point, we always get the origin as a solution, expressed as a convex combination of the simplex's vertices.
We only get to this stage if all smaller simplices failed, and this is intuitive; If the origin is outside the largest simplex, it must be on one of its features, and it will be found before reaching the largest simplex.

Note. In the above quote, the linear equation is formally justified by

> *The $\lambda^2, \ldots, \lambda^r$ in R result from the unconstrained minimization of*
> $$f(\lambda^2, \ldots, \lambda^r) = | x_1 + \sum^r_{i=2} \lambda^i (x_i - x_1) |^2.$$
> *Since f is convex, the necessary and sufficient conditions for optimality are $\partial f(\lambda^2, \ldots, \lambda^r) / \partial \lambda^i = 0$, $i=2,..,r$.*

It would be nice to finally have a formal understanding of a proof of this.

Note. I currently stop at this point for lack of time to return later. Fully deriving and internalizing all the proofs of the paper is very important.

Preliminaries

Polytope

A generalization of the two dimensional polygon and three dimensional polyhedron.

Convex Set

A set S in a vector space over R is called a convex set if the line segment joining any pair of points of S lies entirely in S.

Convex Geometry

Convex geometry is the branch of geometry studying convex sets, mainly in Euclidean space.

Convex geometry is a relatively young mathematical discipline. Although the first known contributions to convex geometry date back to antiquity and can be traced in the works of Euclid and Archimedes, it became an independent branch of mathematics at the turn of the 19th century, mainly due to the works of Hermann Brunn and Hermann Minkowski in dimensions two and three. A big part of their results was soon generalized to spaces of higher dimensions, and in 1934 T. Bonnesen and W. Fenchel gave a comprehensive survey of convex geometry in Euclidean space Rn. Further development of convex geometry in the 20th century and its relations to numerous mathematical disciplines are summarized in the Handbook of convex geometry edited by P. M. Gruber and J. M. Wills.

The Supporting Hyperplane Theorem

Supporting hyperplane is a concept in geometry. A hyperplane divides a space into two half-spaces. A hyperplane is said to support a set S in Euclidean space Rn if it meets both of the following:

S is entirely contained in one of the two closed half-spaces determined by the hyperplane

S has at least one point on the hyperplane.

Here, a closed half-space is the half-space that includes the hyperplane.

Supporting hyperplane theorem
A convex set can have more than one supporting hyperplane at a given point on its boundary.

This theorem states that if S is a closed convex set in a topological vector space X, and x_0 is a point on the boundary of S, then there exists a supporting hyperplane containing x_0. If $x^* \in X^* \backslash \{0\}$ (the dual space of X) such that $x^*\left(x_0\right) \geq x^*(x)$ for all $x \in S$, then

$$H = \{x \in X: x^*(x) = x^*\left(x_0\right)\}$$

defines a supporting hyperplane.[1]

Conversely, if S is a closed set with nonempty interior such that every point on the boundary has a supporting hyperplane, then S is a convex set.[1]

The hyperplane in the theorem may not be unique, as noticed in the second picture on the right. If the closed set S is not convex, the statement of the theorem is not true at all points on the boundary of S, as illustrated in the third picture on the right.

The Hyperplane Separation Theorem

In geometry, the hyperplane separation theorem is either of two theorems about disjoint convex sets in n-dimensional Euclidean space. In the first version of the theorem, if both these sets are closed and at least one of them is compact, then there is a hyperplane in between them and even two parallel hyperplanes in between them separated by a gap. In the second version, if both disjoint convex sets are open, then there is a hyperplane in between them, but not necessarily any gap. An axis which is orthogonal to a separating hyperplane is a separating axis, because the orthogonal projections of the convex bodies onto the axis are disjoint.

The hyperplane separation theorem is due to Hermann Minkowski. The Hahn–Banach separation theorem generalizes the result to topological vector spaces.

Affine Geometry

> *In mathematics, affine geometry is the study of parallel lines. Its use of Playfair's axiom is fundamental since comparative measures of angle size are foreign to affine geometry so that Euclid's parallel postulate is beyond the scope of pure affine geometry. In affine geometry, the relation of parallelism may be adapted so as to be an equivalence relation. Comparisons of figures in affine geometry are made with affinities which are mappings comprising the affine group A. Since A lies between the Euclidean group E and the group of projectivities P, affine geometry is sometimes mentioned[1] in connection with the Erlangen program, which is concerned with group inclusions such as* $E \subset A \subset P$. *Affine geometry can be developed on the basis of linear algebra. One can define an affine space as a set of points equipped with a set of transformations, the translations, which forms (the additive group of) a vector space (over a given field), and such that for any given ordered pair of points there is a unique translation sending the first point to the second. In more concrete terms, this amounts to having an operation that associates to any two points a vector and another operation that allows translation of a point by a vector to give another point; these operations are required to satisfy a number of axioms (notably that two successive translations have the effect of translation by the sum vector). By choosing any point as "origin", the points are in one-to-one correspondence with the vectors, but there is no preferred choice for the origin; thus this approach can be characterized as obtaining the affine space from its associated vector space by "forgetting" the origin (zero vector).*

Affine Hull, Convex Hull

> *In mathematics, the affine hull of a set S in Euclidean space Rn is the smallest affine set containing S, or equivalently, the intersection of all affine sets containing S. Here, an affine set may be defined as the translation of a vector subspace. The affine hull aff(S) of S is the set of all affine combinations of elements of S, that is,*
>
> *TODO: formula (see end of note)*
>
> ** The affine hull of a set of two different points is the line through them.*
> ** The affine hull of a set of three points not on one line is the plane going through them.*
> ** The affine hull of a set of four points not in a plane in R^3 is the entire space R^3.*
>
> *If instead of an affine combination one uses a convex combination, that is one requires in the formula above that all \alpha_i be non-negative, one obtains the convex hull of S, which cannot be larger than the affine hull of S as more restrictions are involved.*
>
> *The notion of conical combination gives rise to the notion of the conical hull*
>
> *If however one puts no restrictions at all on the numbers \alpha_i, instead of an affine combination one has a linear combination, and the resulting set is the linear span of S, which contains the affine hull of S.*
>
> *The convex hull of a finite point set S is the set of all convex combinations of its points. In a convex combination, each point x_i in S is assigned a weight or coefficient \alpha_i in such a way that the coefficients are all positive and sum to one, and these weights are used to compute a weighted average of the points. For each choice of coefficients, the resulting convex combination is a point in the convex hull, and the whole convex hull can be formed by choosing coefficients in all possible ways. Expressing this as a single formula, the convex hull is the set:*
>
> *TODO: formula (see end of note)*
>
> *A set of points is defined to be convex if it contains the line segments connecting each pair of its points. The convex hull of a given set X may be defined as*
> ** The (unique) minimal convex set containing*
> ** The intersection of all convex sets containing*
> ** The set of all convex combinations of points in*
> ** The union of all simplices with vertices in X.*

This ties in nicely to the linear algebraic linear span of S, and we can think of both the convex hull and the affine hull as restrictions. It is indeed very interesting (because new to me) that we can, for two vectors, find the set of vectors that defines the line between them in such a way, or the set forming the convex hull. The linear algebra descriptions of these two hulls turns out to be beautifully simple.

Note that both formulas above can be easily described. Starting from the linear span of a set of vectors, if we restrict all the scalar coefficients such that for each linear combination, the sum of the coefficients is one, we get the affine hull. If in addition, we require that all coefficients are positive, we get the convex hull. Notice how this relates to the inside of a triangle in combination with barycentric coordinates, or Plucker coordinates.

Note. For the requirement that the sum of coefficients be one, the (beautiful) explanation comes from affine spaces [2]. The addition and subtraction of points in an affine space makes no sense in general (this is one essence of the difference between an affine and a linear space), therefore, linear combinations of points do not make sense either in general. However, in the particular case where the sum of coefficients is one, linear combinations do make sense. In an affine space, a point x is represented by an origin O and a translation v=(x-O), that is

$$x=O+(x-O)=O+\mathrm{sum}(x_k\, v_k),$$

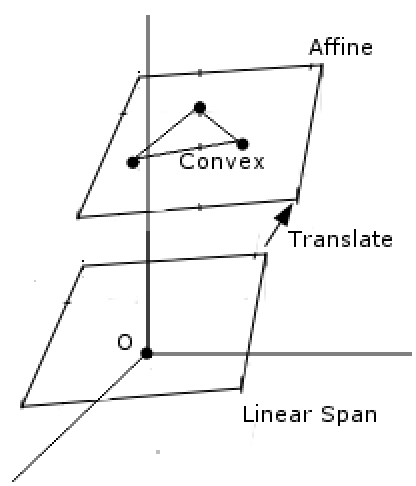
where x_k are the coordinates of the translation in the vector space, given the choice of basis $\langle v_1,\ldots,v_k \rangle$. It is easy to see that a linear combination $\mathrm{sum}(a_k\, p_k)$ where a_i are coefficients and p_i are points, can be reduced to the useful form above since then

$$\mathrm{sum}(a_k\, p_k) = O + \mathrm{sum}(a_k\, (p_k - O))$$

where $(p_k - O)$ are vectors, and their sum is a vector, that can be added to the origin.
Better yet, it turns out that for any O and O', we have

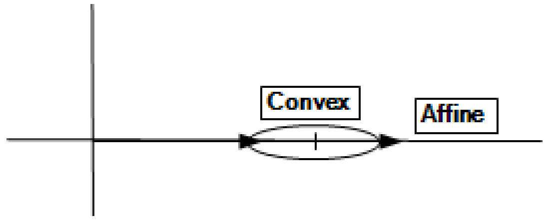
$$\mathrm{sum}(a_k\, p_k) = O + \mathrm{sum}(a_k\, (p_k - O)) = O' + \mathrm{sum}(a_k\, (p_k - O'))$$

and so the linear combination in question produces a resulting point that is independent of the choice of origin.



In a sense, an affine combination of points is a translation of some linear combination of 'vectors', 'translated', and from this we can immediately see that an affine combination must be a hyperplane, 'translated'.

To keep the affine and convex hulls in mind, and why the numbers summing to one, or being positive work out as they do, thinking in two dimension about linear combinations of two vectors (that both lie on the horizontal axis per example), resulting in the whole line or the segment between the two points (including the linear combination of half each, which results in the vector whose tip is in the middle of the tips) is helpful.



Note. Lengths and ratios of lengths in an Affine Space

> *Unless the vector space of translations is equipped with an inner product there is no notion of lengths in an affine space. But for points on a line the ratio of lengths makes sense. Let x1,x2,y1,y2 be four points on a line and choose a non zero vector v on the line, e.g., the difference between two of the given points. Then there exist numbers t1,t2∈ R such that we have y1−x1=t1v and y2−x2=t2v.The ratio between the line segments x_1 y_2 and x_2 y2is now defined as t1 t2 . If we had chosen another vector w then v=αw and yk−xk=tkαw and the ratio αt1 / αt2 = t1 / t2 is the same. Observe that we even have a well defined signed ratio. [2]*

This reminds us of the Greek rational numbers.

Affine Space

> *Even though we often identify our surrounding space with R^3 and we can add elements of R^3 it does obviously not make sense to add two points in space. The identification with R^3 depends on the choice of coordinate system, and the result of adding the coordinates of two points depends on the choice of coordinate system, see Fig.2.5.*

What does make sense in the usual two dimensional plane and three dimensional space is the notion of translation along a vector v. It is often written as adding a vector to a point,x→ x+v. An abstract affine space is a space where the notation of translation is defined and where this set of translations forms a vector space. Formally it can be defined as follows.

> *Definition 2.24*
> *An affine space is a set X that admits a free transitive action of a vector space V. That is, there is a map X×V→X: (x,v)→ x+v, called translation by the vector v, such that*
> *1. Addition of vectors corresponds to composition of translations, i.e., for all x∈X and u,v∈V, x+(u+v)=(x+u)+v.*
> *2. The zero vector acts as the identity, i.e., for all x∈X, x+0=x.*
> *3. The action is free, i.e., if there for a given vector v∈V exists a point x∈X such that x+v=x then v=0.*
> *4. The action is transitive, i.e., for all points x,y∈X exists a vector v∈V such that y=x+v. The dimension of X is the dimension of the vector space of translations,V.[2]*

Note the difference and relation to a Vector Space. In fact, most of our usual thinking about points in space is related to an affine space and not a vector space. Indeed, *In mathematics, an affine space is a geometric structure that generalizes the affine properties of Euclidean space.*

Affine Geometry

In mathematics, affine geometry is the study of parallel lines. Its use of Playfair's axiom is fundamental since comparative measures of angle size are foreign to affine geometry so that Euclid's parallel postulate is beyond the scope of pure affine geometry. In affine geometry, the relation of parallelism may be adapted so as to be an equivalence relation. Comparisons of figures in affine geometry are made with affinities which are mappings comprising the affine group A. Since A lies between the Euclidean group E and the group of projectivities P, affine geometry is sometimes mentioned[1] in connection with the Erlangen program, which is concerned with group inclusions such as $E \subset A \subset P$.
Affine geometry can be developed on the basis of linear algebra. One can define an affine space as a set of points equipped with a set of transformations, the translations, which forms (the additive group of) a vector space (over a given field), and such that for any given ordered pair of points there is a unique translation sending the first point to the second. In more concrete terms, this amounts to having an operation that associates to any two points a vector and another operation that allows translation of a point by a vector to give another point; these operations are required to satisfy a number of axioms (notably that two successive translations have the effect of translation by the sum vector). By choosing any point as "origin", the points are in one-to-one correspondence with the vectors, but there is no preferred choice for the origin; thus this approach can be characterized as obtaining the affine space from its associated vector space by "forgetting" the origin (zero vector).

Several **axiomatic approaches to affine geometry** have been put forward:

Pappus' law
Pappus law: if the red lines are parallel and the blue lines are parallel, then the dotted black lines must be parallel. As affine geometry deals with parallel lines, one of the properties of parallels noted by Pappus of Alexandria has been taken as a premise:[10][11]
If A, B, C are on one line and A', B', C' on another, then (AB' \parallel A'B \ \and \ BC' \parallel B'C) \Rightarrow CA' \parallel C'A.
The full axiom system proposed has point, line, and line containing point as primitive notions:

Two points are contained in just one line.
For any line l and any point P, not on l, there is just one line containing P and not containing any point of l. This line is said to be parallel to l.
Every line contains at least two points.
There are at least three points not belonging to one line.

According to H. S. M. Coxeter: The interest of these **five axioms** is enhanced by the fact that they can be developed into a vast body of propositions, holding not only in Euclidean geometry but also in Minkowski's geometry of time and space (in the simple case of 1 + 1 dimensions, whereas the special theory of relativity needs 1 + 3). The extension to either Euclidean or Minkowskian geometry is achieved by adding various further axioms of orthogonality, etc[12] The various types of affine geometry correspond to what interpretation is taken for rotation. Euclidean geometry corresponds to the ordinary idea of rotation, while Minkowski's geometry corresponds to hyperbolic rotation. With respect to perpendicular lines, they remain perpendicular when the plane is subjected to ordinary rotation. In the Minkowski geometry, lines that are hyperbolic-orthogonal remain in that relation when the plane is subjected to hyperbolic rotation.

Ordered structure

An axiomatic treatment of plane affine geometry can be built from the axioms of ordered geometry by the addition of two additional axioms:[13] (Affine axiom of parallelism) Given a point A and a line r, not through A, there is at most one line through A which does not meet r. (Desargues) Given seven distinct points A, A', B, B', C, C', O, such that AA', BB', and CC' are distinct lines through O and AB is parallel to A'B' and BC is parallel to B'C', then AC is parallel to A'C'. The affine concept of parallelism forms an equivalence relation on lines. Since the axioms of ordered geometry as presented here include properties that imply the structure of the real numbers, those properties carry over here so that this is an axiomatization of affine geometry over the field of real numbers.

Ternary rings

Main article: planar ternary ring The first non-Desarguesian plane was noted by David Hilbert in his Foundations of Geometry.[14] The Moulton plane is a standard illustration. In order to provide a context for such geometry as well as those where Desargues theorem is valid, the concept of a ternary ring has been developed. Rudimentary affine planes are constructed from ordered pairs taken from a ternary ring. A plane is said to have the "minor affine Desargues property" when two triangles in parallel perspective, having two parallel sides, must also have the third sides parallel. If this property holds in the rudimentary affine plane defined by a ternary ring, then there is an equivalence relation between "vectors" defined by pairs of points from the plane.[15] Furthermore, the vectors form an abelian group under addition, the ternary ring is linear, and satisfies right distributivity: (a + b) c = ac + bc.

Carathéodory's theorem (convex hull)

The theorem asserts is that if a point P is in a convex hull H of m points in R^n, then there is a n-simplex (n+1 points) S (obviously convex), such that P is in S. Intuitively, this is obvious, since in H with m>n+1 we can join any two non-neighboring points by a segment, which splits H into two convex hulls H_1 and H_2 such that P is in one (or both) of them. Determining which of them contains P is easy by checking on which side of the segment it is. This can be repeated until some convex set H_k containing P has n+1 points.

There is a formal proof that relies on the linear algebraic fact that for the convex hull, the set of m-1 vectors between the first and other points, if m>n+1 must be linearly dependent, and the fact that P, is a convex combination of the hull's points. From these two facts, the number of points in the convex combination can be methodically reduced by one recursively until it only uses n+1 points.

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