GESAR

The Group for Environmental Studies in Reservoirs

MOVING MESH-MOVING BOUNDARY METHOD FOR TWO-PHASE FLOWS WITH PHASE CHANGE

Tutorial 11-4-2: Two Phase Boiling Computational Modelling Challenges

Gustavo R. ANJOS

http://gustavo.rabello.org http://www.gesar.uerj.br

San Francisco - CA, InterPACK2015

July 6th, 2015

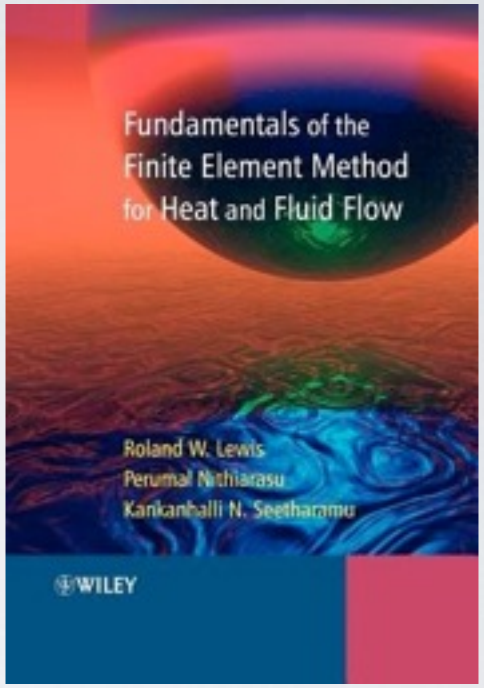
OUTLINE

- Intro to Finite Element Method
- Variational method: the weak form;
- Function approximations: Galerkin method;
- 1D example;
- Tasks: 1D and 2D examples;

BIBLIOGRAPHY

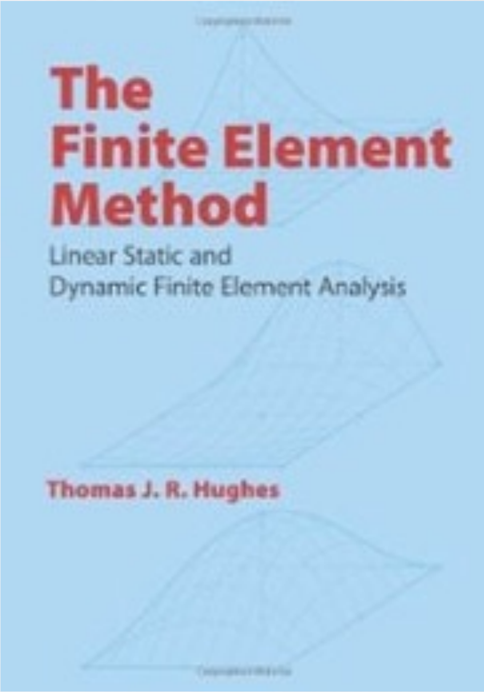
Basic

Fundamentals of the Finite Element Method for Heat and Fluid Flow
Authors: Roland W. Lewis, Perumal Nithiarasu, Kankanhally e N. Seetharamu



Basic-advanced

The Finite Element Method - Linear Static and Dynamic Finite Element Analysis
Authors: Thomas J.R. Hughues



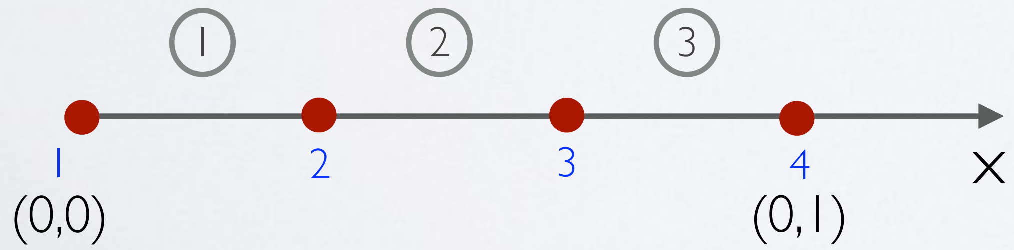
1D PROBLEM - STRONG FORM

Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \begin{array}{|l} u(0) = 0 \\ \dfrac{du}{dx}(1) = 1 \end{array}$$

boundary condition

domain: $h_1 = h_2 = h_3 = 1/3$



Answer: u_2=1.049;
u_3=1,874;
u_4=2,386

1D PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_\Omega w\left(\frac{d^2u}{dx^2} + u + 1\right) d\Omega = 0$$

weight function

→ mathematical procedure (integration by parts)

$$\int_0^1 w\frac{d^2u}{dx^2}dx + \int_0^1 wudx + \int_0^1 dx = 0$$

$$w\frac{du}{dx}\bigg|_0^1 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

$$w\frac{du}{dx}\bigg|_1 - w\frac{du}{dx}\bigg|_0 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

CHOOSING ELEMENT

Finite element properties:

- the shape functions are 1 at the node and zero everywhere else;
- the sum of all shape function at the element is 1 everywhere, including boundary.

Chart:

function	node, i	node, j	x
Ni	1	0	between 0 e 1
Nj	0	1	between 0 e 1
Ni+Nj	1	1	1

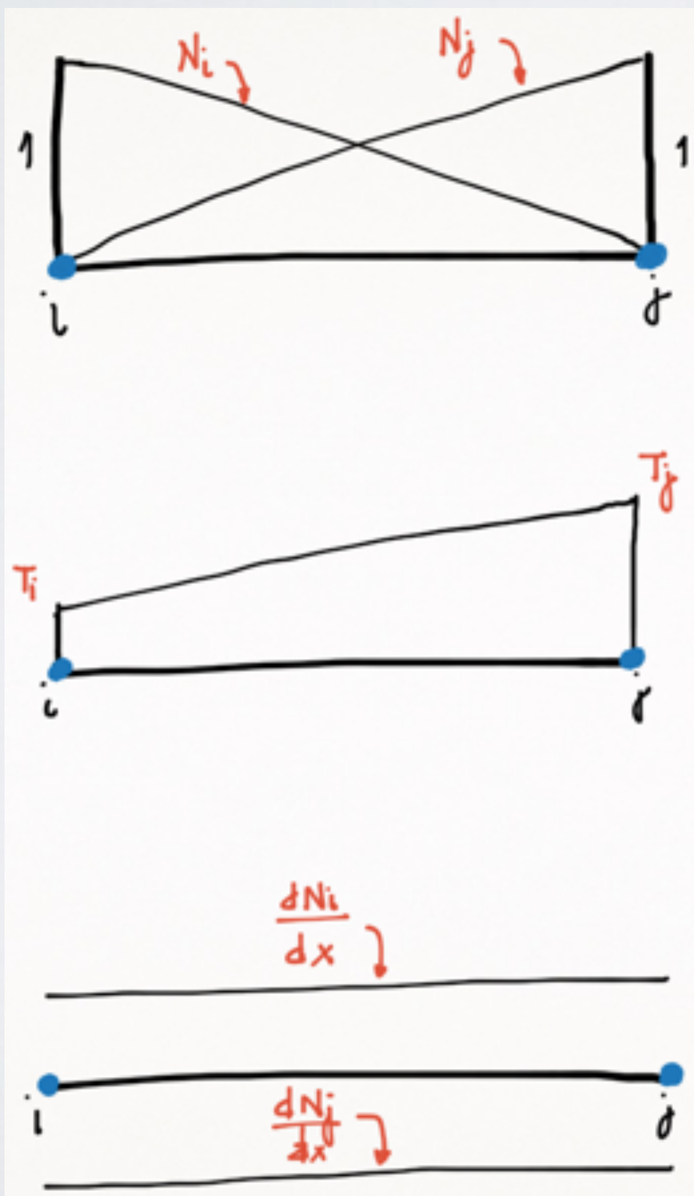
FEM SHAPE FUNCTIONS

1D Problem - linear:

$$T(x) = \alpha_1 + \alpha_2 x$$

1D problem - quadratic:

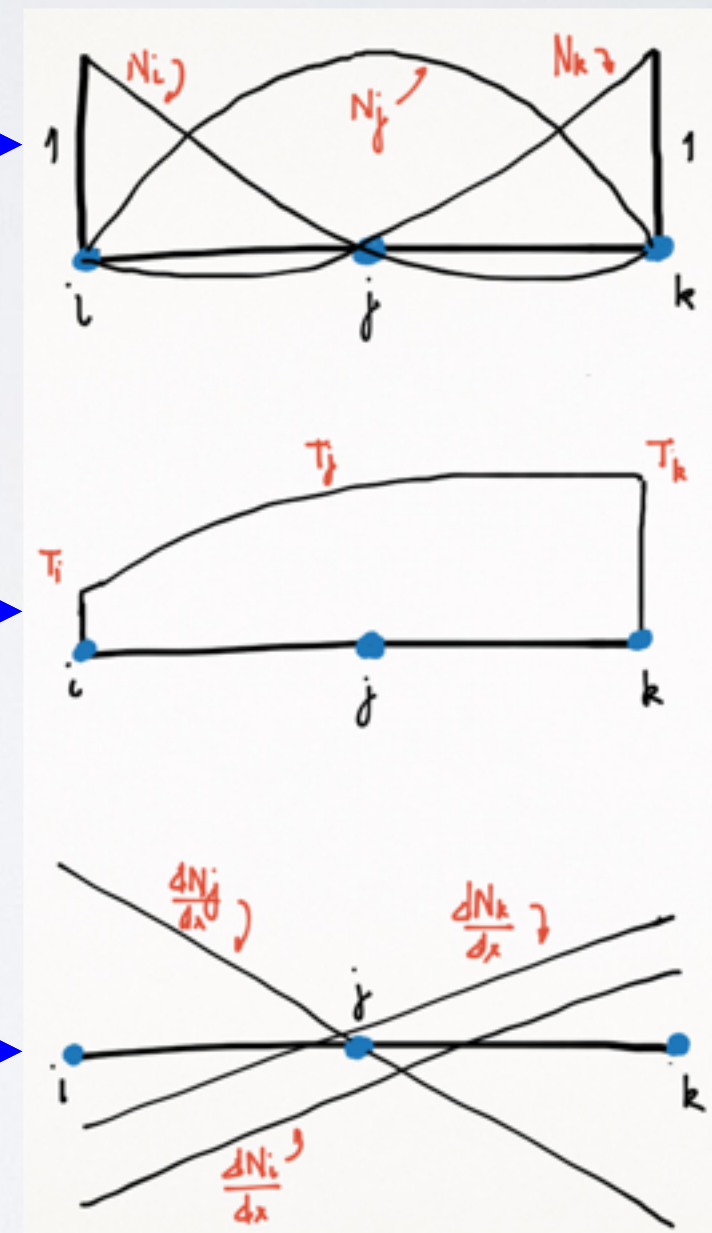
$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



shape function

function

shape function derivative



GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{4} N_i u_i \qquad \hat{w} = \sum_{j=1}^{4} N_j w_j$

$$w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0) - \sum_{i=j=1}^{4}\int_0^1 \frac{dN_i}{dx}u_i\frac{dN_j}{dx}w_j dx +$$

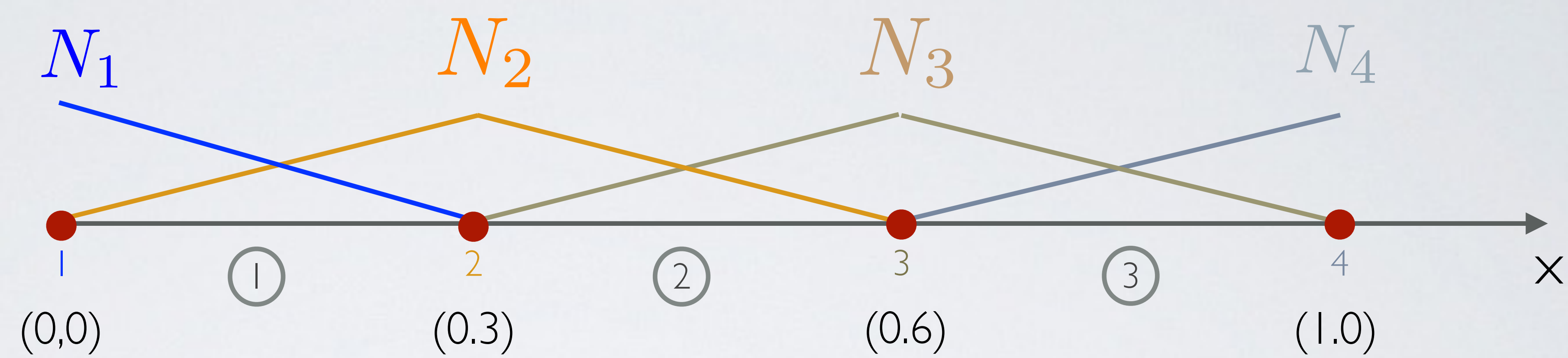
$$+\sum_{i=j=1}^{4}\int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^{4}\int_0^1 N_j w_j dx = 0$$

$$\sum_{i=j=1}^{4}\left(\underbrace{\boxed{\int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx}}_{\text{stiffness matrix } K_{ij}} - \underbrace{\boxed{\int_0^1 N_i N_j dx}}_{\text{mass matrix } M_{ij}}\right)u_i = \sum_{j=1}^{4}\underbrace{\boxed{\int_0^1 N_j dx}}_{\text{right hand side } b_i} + \underbrace{\boxed{\frac{du}{dx}(1) - \frac{du}{dx}(0)}}_{\text{boundary condition}}$$

$$(K_{ij} - M_{ij})u_i = b_i + b.c.$$

1D PROBLEM - LINEAR

domain and shape functions:



element 1

$\Omega_1^e = [0, 1/3]$

$$N_1 = -3x + 1$$
$$N_2 = 3x$$

element 2

$\Omega_2^e = [1/3, 2/3]$

$$N_2 = -3x + 2$$
$$N_3 = 3x - 1$$

element 3

$\Omega_3^e = [2/3, 1]$

$$N_3 = -3x + 3$$
$$N_4 = 3x - 2$$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element 1

$\Omega_1^e = [0, 1/3]$

$N_1 = -3x + 1$

$N_2 = 3x$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element 2

$\Omega_2^e = [1/3, 2/3]$

$N_2 = -3x + 2$

$N_3 = 3x - 1$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_4 N_4 dx$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element 3

$$\Omega_3^e = [2/3, 1]$$

$$N_3 = -3x + 3$$

$$N_4 = 3x - 2$$

$$b_3 = \int_{2/3}^{1} N_3 dx$$

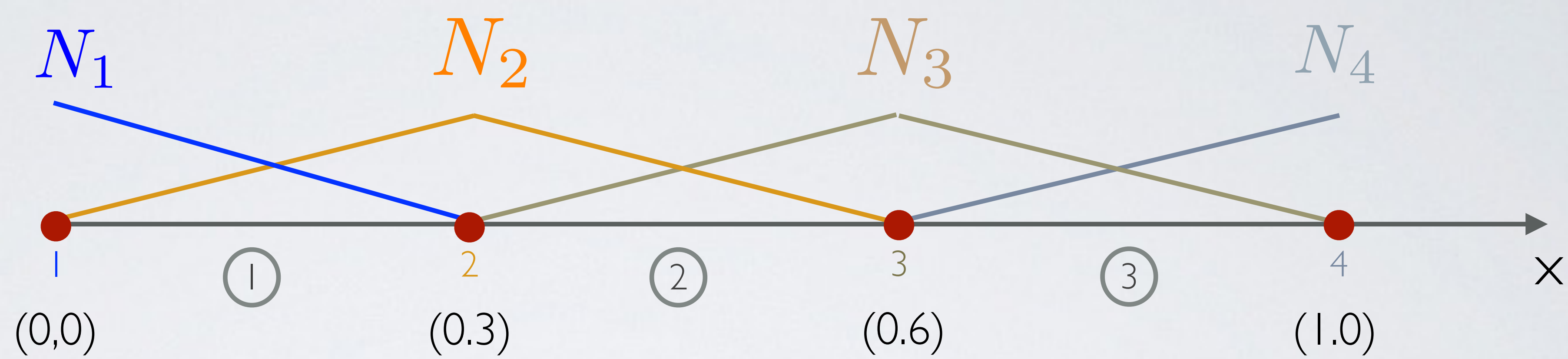
$$b_4 = \int_{2/3}^{1} N_4 dx$$

vector

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 1

$\Omega_1^e = [0, 1/3]$

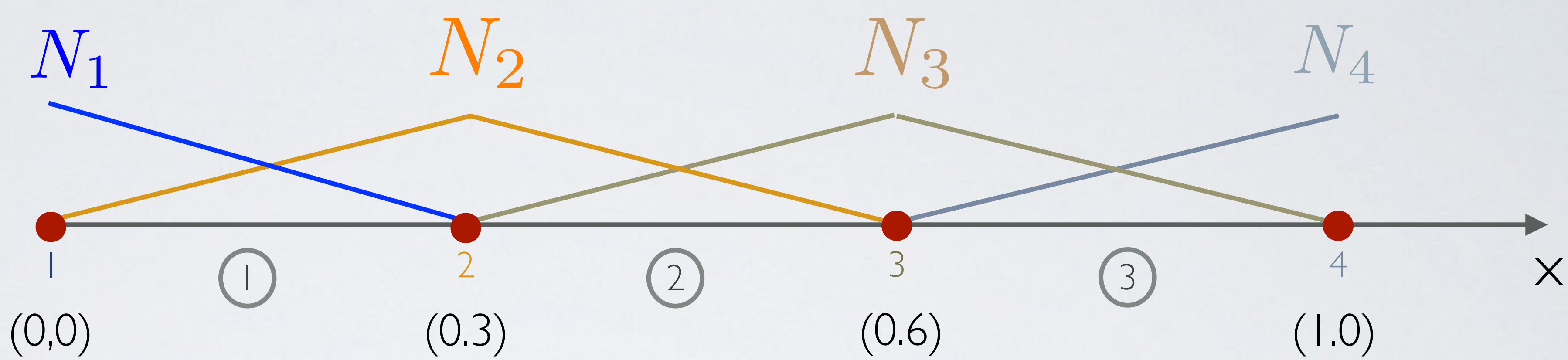
$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 2

$\Omega_2^e = [1/3, 2/3]$

$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

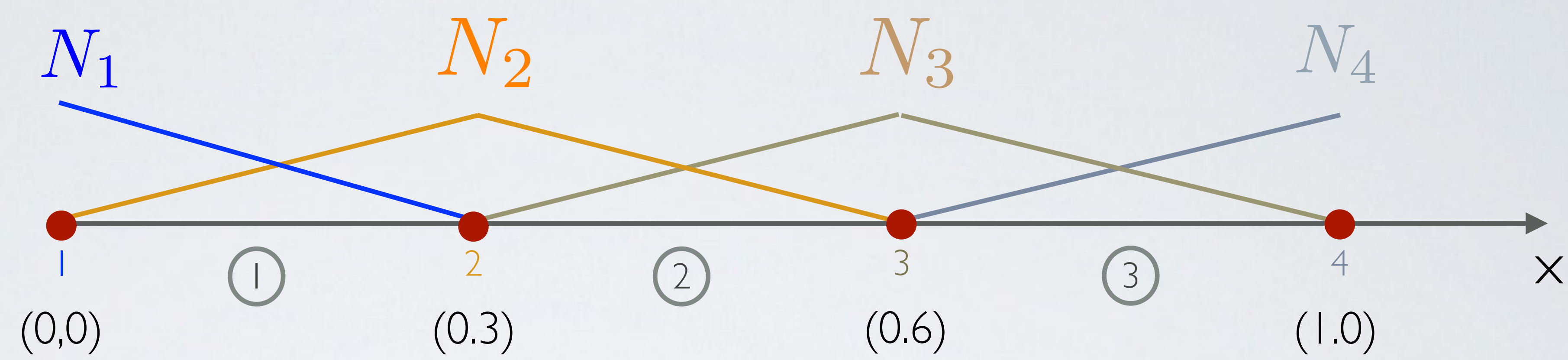
$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

GESAR

domain and shape functions:



element 3

$\Omega_3^e = [2/3, 1]$

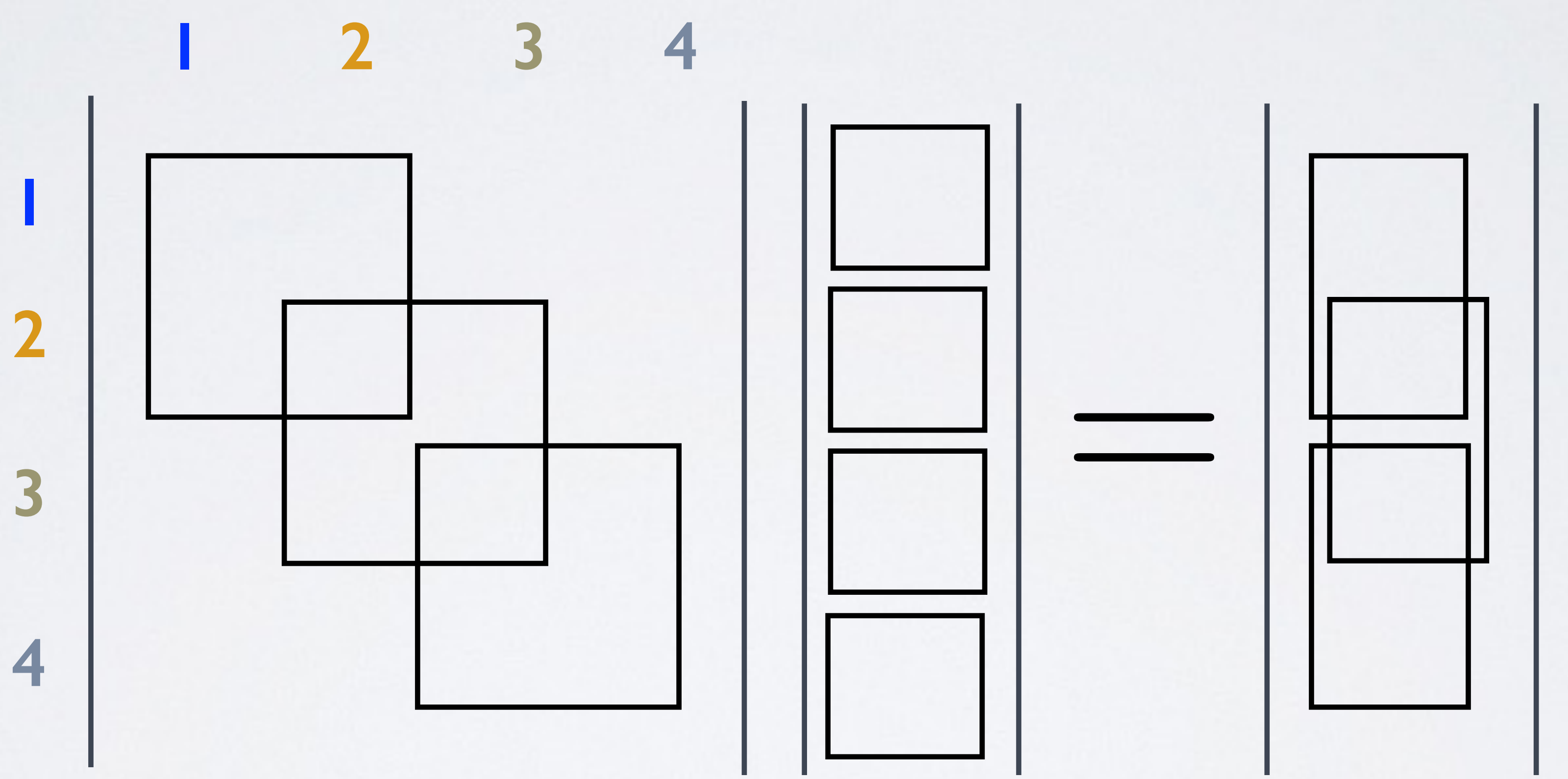
$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 + \text{b.c. at } x = 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 + 1 \end{bmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\color{red}{u_i}\color{black} = b_i + b.c.$



$K_{ij} - M_{ij}$ $\quad$ $\color{red}{u_i}$ $\quad$ $b_i + b.c.$

1D PROBLEM

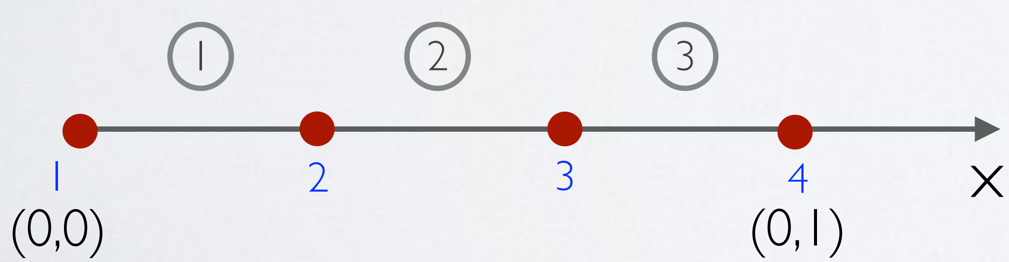
Find u in $\Omega = [0,1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0$$

$$\begin{array}{|l} u(0) = 0 \\ \dfrac{du}{dx}(1) = -u \end{array}$$

← boundary condition

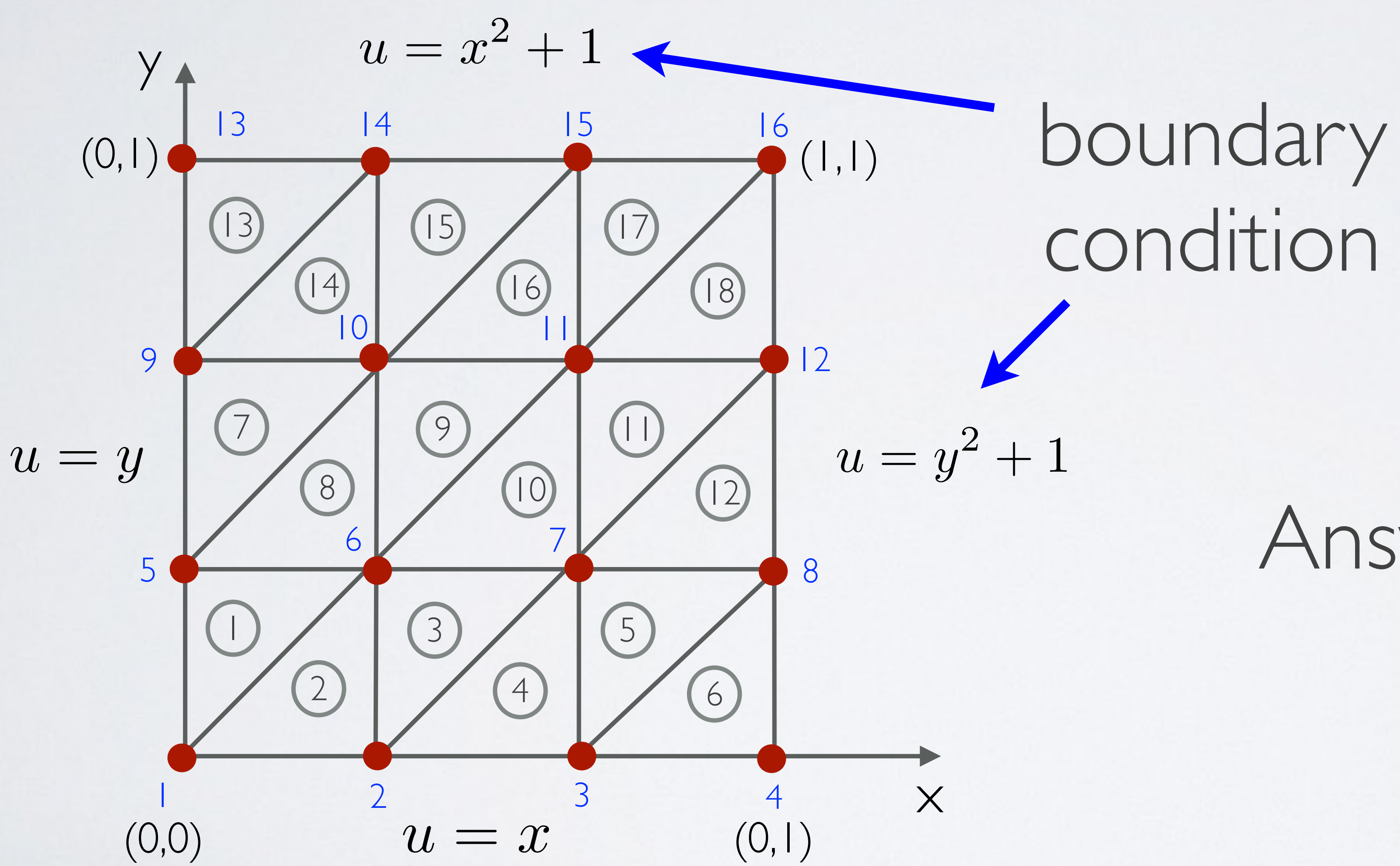
domain: $h_1 = h_2 = h_3 = 1/3$



Answer: u_2=0.251; u_3=0.363; u_4=0.328

2D PROBLEM

Find u in $\Omega = [0,1] \times [0,1]$ such that:



Equation:

$$\nabla^2 u = 0$$

Answer: $u_6 = 0.611$; $u_7 = 0.889$; $u_{10} = 0.889$; $u_{11} = 1.167$