

INTRODUCTION TO THE FINITE ELEMENT METHOD: 1D-2D CASES

1st Workshop on Advances in CFD and MD modeling of Interface Dynamics in Capillary Two-Phase Flows

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OUTLINE

- Intro to Finite Element Method
- Variational method: the weak form;
- Function approximations: The Galerkin method;
- 1D example;
- Tasks: 1D and 2D examples;
- Hands-on: Python scripts for 2D and Axi problems

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BIBLIOGRAPHY

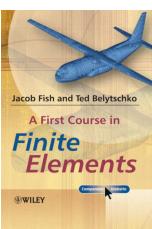
Basic

Fundamentals of the Finite Element Method for Heat and Fluid Flow
Authors: Roland W. Lewis, Perumal Nithiarasu, Kankanhally and N. Seetharamu



Basic

A First Course in Finite Elements
Authors: Jacob Fish and Ted Belytschko



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BIBLIOGRAPHY

Basic-advanced

The Finite Element Method - Linear Static and Dynamic Finite Element Analysis
Author: Thomas J.R. Hughes



Advanced

The Finite Element Method - Its Basis & Fundamentals
Authors: O.C. Zienkiewicz, R.L. Taylor & J.Z. Zhu



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BRIEF HISTORY OF FEM



- Has been used since 1950's in solid mechanics
- in the 1970's FEM began to be used in CFD
- nowadays FEM is applicable to many engineering problems
Heat transfer, fluid flow, electromagnetic fields, solid mechanics, acoustics, biomechanics etc.

Finite Element Method - FEM
strong math
complex geometry
element geometry
master element
high memory
flexible

Finite Volume Method - FVM
flux formulation
complex geometry
conservative
low memory

Finite Difference Method - FDM
easy math
simple geometries
grid systems
low memory
not flexible

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CHOICE OF FUNCTIONS



Finite element function properties:

- the shape functions are 1 at the node and zero elsewhere;
- the sum of all shape functions at the element is 1 everywhere, including boundary;
- the weight function is zero at boundary for Dirichlet b.c.

Chart:

function	node, i	node, j	x
N_i	1	0	between 0 e 1
N_j	0	1	between 0 e 1
$N_i + N_j$	1	1	1

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FINITE ELEMENT METHOD



PDE

weak form

ODE

linear system

approximated solution

Governing equations
(heat equation,
Maxwell equation,
Navier-Stokes equation)

Variational form
Set of Ordinary Differential Equations

Ax=b

solution for x
 $x = A^{-1}b$
 $x = [u, v, w, p, T, c]$

Approximation functions: The Galerkin Method

The approximation functions are combined with the weak form to obtain the discrete finite element equations.

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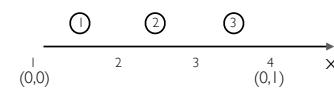
ID PROBLEM - STRONG FORM



Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \quad \left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = 1 \end{array} \right. \quad \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$

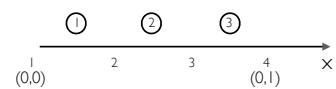


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MESH GENERATION



mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector
boundary vector

node	X
1	0
2	1/3
3	2/3
4	1

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ID PROBLEM - WEAK FORM



Find u in H^1 with b.c. such that:

$$\int_{\Omega} w \left(\frac{d^2 u}{dx^2} + u + 1 \right) d\Omega = 0$$

weight function

mathematical procedure (integration by parts)

$$\int_0^1 w \frac{d^2 u}{dx^2} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_0^1 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

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ID PROBLEM - WEAK FORM



$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

apply boundary conditions to the equation
in the continuous form:

$$w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0) - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w(1) - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

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GALERKIN METHOD



Approximate functions: $\hat{u} = \sum_{i=1}^4 N_i(x) u_i$ $\hat{w} = \sum_{j=1}^4 N_j(x) w_j$

$$w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0) - \sum_{i=j=1}^4 \int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx + \\ + \sum_{i=j=1}^4 \int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^4 \int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \textcolor{red}{u_i} = \sum_{j=1}^4 \int_0^1 N_j dx + w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0)$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side: b_i boundary condition

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GALERKIN METHOD



$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) u_i = \sum_{j=1}^4 \int_0^1 N_j dx + w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0)$$

Replacing the shape function to the B.C.

Note that w_j is present at all members and can be removed!

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) u_i = \sum_{j=1}^4 \int_0^1 N_j dx + N_j(1)$$

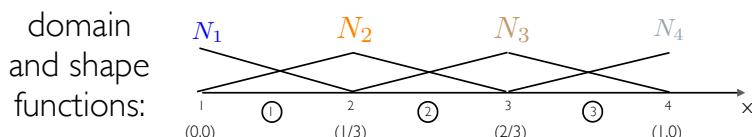
stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i

boundary condition (evaluated only at $x=1$)

$$(K_{ij} - M_{ij}) u_i = b_i + b.c.$$

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ID PROBLEM - LINEAR



element ① $N_1 = -3x + 1$
 $\Omega_1^e = [0, 1/3]$ $N_2 = 3x$

element ② $N_2 = -3x + 2$
 $\Omega_2^e = [1/3, 2/3]$ $N_3 = 3x - 1$

element ③ $N_3 = -3x + 3$
 $\Omega_3^e = [2/3, 1]$ $N_4 = 3x - 2$

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FEM SHAPE FUNCTIONS

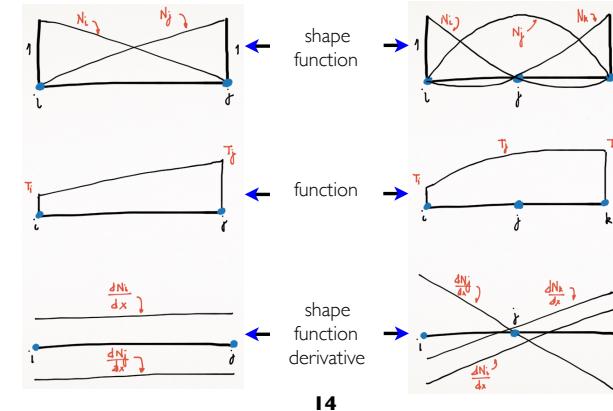
ID Problem - linear:

$$T(x) = \alpha_1 + \alpha_2 x$$



ID problem - quadratic:

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



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MATRIX FORM



$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element ①
 $\Omega_1^e = [0, 1/3]$
 $N_1 = -3x + 1$
 $N_2 = 3x$

$b_1 = \int_0^{1/3} N_1 dx$
 $b_2 = \int_0^{1/3} N_2 dx$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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MATRIX FORM



$$\begin{aligned} K_{22} - M_{22} &= \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_2 N_2 dx \\ K_{23} - M_{23} &= \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_2 N_3 dx \\ K_{32} - M_{32} &= \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_3 N_2 dx \\ K_{33} - M_{33} &= \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_3 N_3 dx \end{aligned}$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element ②
 $\Omega_2^e = [1/3, 2/3]$
 $N_2 = -3x + 2$
 $N_3 = 3x - 1$

$$\begin{aligned} b_2 &= \int_{1/3}^{2/3} N_2 dx \\ b_3 &= \int_{1/3}^{2/3} N_3 dx \end{aligned}$$

vector

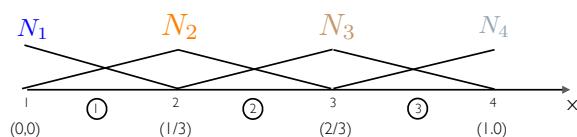
$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

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MATRIX FORM



domain
and shape
functions:



element ①
 $\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

b.c. at $x = 0$
does not exist!
 $w(0) = 0$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x=0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

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MATRIX FORM



$$\begin{aligned} K_{33} - M_{33} &= \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_3 N_3 dx \\ K_{34} - M_{34} &= \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_3 N_4 dx \\ K_{43} - M_{43} &= \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_4 N_3 dx \\ K_{44} - M_{44} &= \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_4 N_4 dx \end{aligned}$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element ③
 $\Omega_3^e = [2/3, 1]$
 $N_3 = -3x + 3$
 $N_4 = 3x - 2$

$$\begin{aligned} b_3 &= \int_{2/3}^1 N_3 dx \\ b_4 &= \int_{2/3}^1 N_4 dx \end{aligned}$$

vector

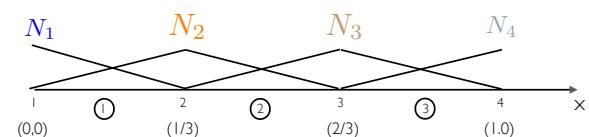
$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

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MATRIX FORM



domain
and shape
functions:



element ②
 $\Omega_2^e = [1/3, 2/3]$

$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

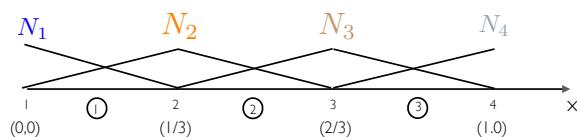
$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

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MATRIX FORM

domain
and shape
functions:



element ③
 $\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

b.c. at $x = 1$
 $w(1) = N_4(1)$
 $= 3.1 - 2$
 $= 1$

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ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```

1: for elem ← 1, NE do
2:   for ilocal ← 1, 2 do
3:     iglobal ← IEN[elem, ilocal] → ilocal = [1, 2]
4:     for jlocal ← 1, 2 do
5:       jglobal ← IEN[elem, jlocal] → iglobal = [v1, v2]
6:       K[iglobal, jglobal] ← K[iglobal, jglobal] + kelem[ilocal, jlocal] → jglobal = [v1, v2]
7:     end for
8:   end for
9: end for

```

- loop on elements (1)
- loop on neighbors ($i=1,2$) and loop ($j=1,2$)
- conversion between local to global node
- assembling of stiffness matrix using SUM

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ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & & \\ \mathbf{1} & & \square & & & & \\ \mathbf{2} & & & \square & & & \\ \mathbf{3} & & & & \square & & \\ \mathbf{4} & & & & & \square & \\ \hline & K_{ij} - M_{ij} & \mathbf{u}_i & & & b_i + b.c. & \end{array}$$

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ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & & \\ \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & u_1 & b_1^1 + b.c. \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & & & u_2 & b_2^1 \\ \mathbf{3} & & & & & u_3 & \\ \mathbf{4} & & & & & u_4 & \\ \hline & & & & & & \end{array}$$

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ASSEMBLING



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & & \\ \mathbf{4} & & & & & \\ \hline & u_1 & u_2 & u_3 & u_4 & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array} \quad \begin{array}{c|c} & b_1^1 + b.c. \\ \hline & b_2^1 + b_2^2 \\ & b_3^2 \\ & b_4^3 \end{array}$$

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ASSEMBLING



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & \\ \mathbf{4} & & & KM_{34}^3 & KM_{44}^3 & \\ \hline & u_1 & u_2 & u_3 & u_4 & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array} \quad \begin{array}{c|c} & b_1^1 + b.c. \\ \hline & b_2^1 + b_2^2 \\ & b_3^2 + b_3^3 \\ & b_4^3 + b.c. \end{array}$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & \\ \mathbf{4} & & & KM_{34}^3 & KM_{44}^3 & \\ \hline & u_1 & u_2 & u_3 & u_4 & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array} \quad \begin{array}{c|c} u_1(0) = 0 & b_1^1 + b.c. \\ \hline & b_2^1 + b_2^2 \\ & b_3^2 + b_3^3 \\ & b_4^3 + 1 \end{array}$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & \\ \mathbf{4} & & & KM_{34}^3 & KM_{44}^3 & \\ \hline & 0 & u_2 & u_3 & u_4 & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array} \quad \begin{array}{c|c} b_1^1 + b.c. & \\ \hline & b_2^1 + b_2^2 \\ & b_3^2 + b_3^3 \\ & b_4^3 + 1 \end{array}$$

how to remove this line?

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SETTING B.C.



writing down the equation of line 2

$$KM_{21}^1 * \textcolor{red}{u}_1 + (KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2$$

subtracting $KM_{21}^1 * \textcolor{red}{u}_1$ from both sides:

$$(KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2 - KM_{21}^1 * \textcolor{red}{u}_1$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$

	2	3	4		
2	KM_{22}^1 +	KM_{22}^2	KM_{23}^2	u_2	$b_2^1 + b_2^2$
3	KM_{32}^2	KM_{33}^2 +	KM_{43}^3	u_3	\equiv $b_3^2 + b_3^3$
4		KM_{34}^3	KM_{44}^3	u_4	$b_3^3 + 1$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$

	1	2	3	4		
1	KM_{11}^1	KM_{12}^1		0		$b_1^1 + b.c.$
2	KM_{21}^1	KM_{22}^1 +	KM_{23}^2	u_2		$b_2^1 + b_2^2$
3		KM_{32}^2 +	KM_{33}^2 +	KM_{43}^3	u_3	\equiv $b_3^2 + b_3^3$
4			KM_{34}^3	KM_{44}^3	u_4	$b_3^3 + 1$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$

	1	2	3		
1	KM_{22}^1 +	KM_{22}^2	KM_{23}^2	u_2	$b_2^1 + b_2^2 - KM_{21}^1 * \textcolor{red}{u}_1$
2	KM_{32}^2	KM_{33}^2 +	KM_{43}^3	u_3	\equiv $b_3^2 + b_3^3$
3		KM_{34}^3	KM_{44}^3	u_4	$b_3^3 + 1$

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SUBSTITUTING VALUES



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & & \\ \mathbf{1} & 26/9 & -55/18 & 0 & 0 & 0 & 1/6 \\ \mathbf{2} & -55/18 & 52/9 & -55/18 & 0 & u_2 & 1/3 \\ \mathbf{3} & 0 & -55/18 & 52/9 & -55/18 & u_3 & 1/3 \\ \mathbf{4} & 0 & 0 & -55/18 & 26/9 & u_4 & 1/6 + 1 \end{array}$$

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SUBSTITUTING VALUES



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & & & 0 \\ \mathbf{1} & 52/9 & -55/18 & 0 & u_2 & 1/3 - KM_{21}^1 * \cancel{\mathbf{u}_1} & \\ \mathbf{2} & -55/18 & 52/9 & -55/18 & u_3 & & 1/3 \\ \mathbf{3} & 0 & -55/18 & 26/9 & u_4 & & 7/6 \end{array}$$

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SUBSTITUTING VALUES



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{ccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \\ \mathbf{1} & 52/9 & -55/18 & 0 & u_2 & 1/3 \\ \mathbf{2} & -55/18 & 52/9 & -55/18 & u_3 & 1/3 \\ \mathbf{3} & 0 & -55/18 & 26/9 & u_4 & 7/6 \end{array}$$

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SOLVING LINEAR SYSTEM



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

solving for \mathbf{u}_i :

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})\mathbf{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + b.c.$$

$$\mathbf{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + b.c.$$

how to compute $(K_{ij} - M_{ij})^{-1}$?
 direct methods: **not recommended!**
 iterative methods: **recommended!**

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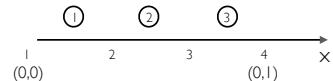
SOLUTION



Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \quad \left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = 1 \end{array} \right. \quad \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



$$u_2 = 1.049$$

$$u_3 = 1.874$$

$$u_4 = 2.386$$

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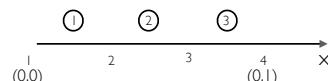
ID PROBLEM



Find u in $\Omega = [0, 1]$ such that:

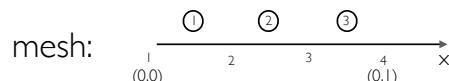
$$\frac{d^2u}{dx^2} + u + 1 = 0 \quad \left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = -u \end{array} \right. \quad \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



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MESH GENERATION



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

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ASSEMBLING



Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```

1: for elem ← 1, NE do                                → NE = Total number of elements
2:   for ilocal ← 1, 2 do                                → ilocal = [1, 2]
3:     iglobal ← IEN[elem, ilocal]                      → iglobal = [v1, v2]
4:     for jlocal ← 1, 2 do                                → jlocal = [1, 2]
5:       jglobal ← IEN[elem, jlocal]                      → jglobal = [v1, v2]
6:       K[iglobal, jglobal] ← K[iglobal, jglobal] + kelem[ilocal, jlocal]
7:     end for
8:   end for
9: end for

```

- loop on elements (1)
- loop on neighbors ($i=1,2$) and loop ($j=1,2$)
- conversion between local to global node
- assembling of stiffness matrix using SUM

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ID PROBLEM - WEAK FORM



Find u in H^1 such that:

$$\int_{\Omega} w \left(\frac{d^2 u}{dx^2} + u + 1 \right) d\Omega = 0$$

weight function

→ mathematical procedure (integration by parts)

$$\int_0^1 w \frac{d^2 u}{dx^2} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_0^1 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

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ID PROBLEM - WEAK FORM



$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

apply boundary conditions to the equation
in the continuous form:

$$w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0) - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$-w(1)u(1) - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

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GALERKIN METHOD



Approximated functions: $\hat{u} = \sum_{i=1}^4 N_i(x) u_i$ $\hat{w} = \sum_{j=1}^4 N_j(x) w_j$

$$w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0) - \sum_{i=j=1}^4 \int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx + \\ + \sum_{i=j=1}^4 \int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^4 \int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \underline{\underline{u}}_i = \sum_{j=1}^4 \int_0^1 N_j dx + w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0)$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

$$(K_{ij} - M_{ij}) \underline{\underline{u}}_i = b_i + b.c.$$

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GALERKIN METHOD



note that the b.c. has $-u(1)$:

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \underline{\underline{u}}_i = \sum_{j=1}^4 \int_0^1 N_j dx + w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0)$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \underline{\underline{u}}_i + w(1) \underline{\underline{u}}(1) = \sum_{j=1}^4 \int_0^1 N_j dx - w(0) \frac{du}{dx}(0)$$

Replacing the shape function to the B.C.
Note that w_j is present at all members and can be removed!

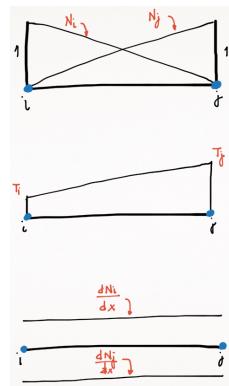
$$\sum_{i=1}^4 \sum_{j=1}^4 \left[\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx + N_i(1)N_j(1) \right] \underline{\underline{u}}_i = \sum_{j=1}^4 \int_0^1 N_j dx$$

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FEM SHAPE FUNCTIONS

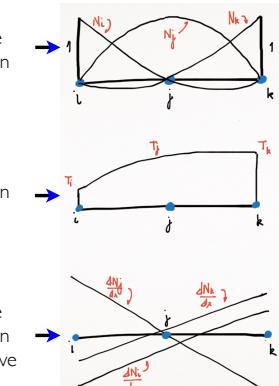
ID Problem - linear:

$$T(x) = \alpha_1 + \alpha_2 x$$



ID problem - quadratic:

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



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shape
function

function

shape
function
derivative

vector

element ①
 $\Omega_1^e = [0, 1/3]$
 $N_1 = -3x + 1$
 $N_2 = 3x$

$$b_1 = \int_0^{1/3} N_1 dx$$

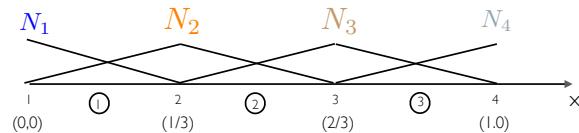
$$b_2 = \int_0^{1/3} N_2 dx$$

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ID PROBLEM - LINEAR



domain
and shape
functions:



element ① $N_1 = -3x + 1$
 $\Omega_1^e = [0, 1/3]$ $N_2 = 3x$

element ② $N_2 = -3x + 2$
 $\Omega_2^e = [1/3, 2/3]$ $N_3 = 3x - 1$

element ③ $N_3 = -3x + 3$
 $\Omega_3^e = [2/3, 1]$ $N_4 = 3x - 2$

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MATRIX FORM



$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_1 N_1 dx \quad \text{matrix}$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_2 N_2 dx$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element ①
 $\Omega_1^e = [0, 1/3]$
 $N_1 = -3x + 1$
 $N_2 = 3x$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_2 N_2 dx \quad \text{matrix}$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

element ②
 $\Omega_2^e = [1/3, 2/3]$
 $N_2 = -3x + 2$
 $N_3 = 3x - 1$

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

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MATRIX FORM



$$K_{33} - M_{33} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_4 N_4 dx + N_4(1)N_4(1)$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element ③
 $\Omega_3^e = [2/3, 1]$
 $N_3 = -3x + 3$
 $N_4 = 3x - 2$

$$b_3 = \int_{2/3}^1 N_3 dx$$

$$b_4 = \int_{2/3}^1 N_4 dx$$

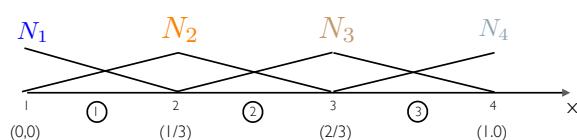
$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

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MATRIX FORM



domain
and shape
functions:



element ②
 $\Omega_2^e = [1/3, 2/3]$

$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

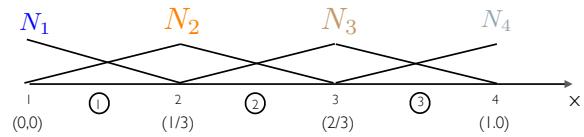
$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

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MATRIX FORM



domain
and shape
functions:



element ①
 $\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

b.c. at $x = 0$
does not exist!

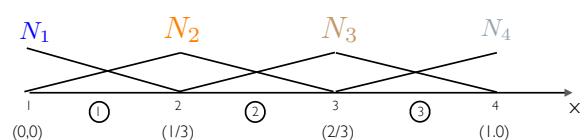
$$w(0) = 0$$

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MATRIX FORM



domain
and shape
functions:



element ③
 $\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} + 1 \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{70}{18} \end{bmatrix}$$

note that:

b.c. at $x = 1$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

$$w(1)u(1) = N_4(1) * N_4(1)$$

$$= (3.1 - 2)(3.1 - 2)$$

$$= 1$$

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ASSEMBLING



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|cccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & & \square & & \\ \mathbf{2} & & \square & \square & \\ \mathbf{3} & & \square & \square & \square \\ \mathbf{4} & & \square & & \end{array} = \begin{array}{c|c} & \square \\ & \square \\ & \square \\ & \square \end{array} = \begin{array}{c|c} & \square \\ & \square \\ & \square \\ & \square \end{array}$$

$K_{ij} - M_{ij}$ \mathbf{u}_i $b_i + b.c.$

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ASSEMBLING



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & & & \\ \mathbf{3} & & & & & \\ \mathbf{4} & & & & & \end{array} = \begin{array}{c|c} u_1 & b_1^1 + b.c. \\ u_2 & b_2^1 \\ u_3 & b_3^1 \\ u_4 & b_4^1 \end{array} = \begin{array}{c|c} & b_1^1 + b.c. \\ & b_2^1 \\ & b_3^1 \\ & b_4^1 \end{array}$$

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ASSEMBLING



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|cccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & \\ & & KM_{22}^2 & & \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & \\ \mathbf{4} & & & & \end{array} = \begin{array}{c|c} u_1 & b_1^1 + b.c. \\ u_2 & b_2^1 + b_2^2 \\ u_3 & b_3^2 \\ u_4 & b_4^1 \end{array} = \begin{array}{c|c} & b_1^1 + b.c. \\ & b_2^1 + b_2^2 \\ & b_3^2 \\ & b_4^1 \end{array}$$

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ASSEMBLING



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{c|ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & \\ & & KM_{22}^2 & KM_{23}^2 & & \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & \\ & & KM_{33}^3 & & KM_{43}^3 & \\ \mathbf{4} & & & KM_{34}^3 & KM_{44}^3 & \end{array} = \begin{array}{c|c} u_1 & b_1^1 + b.c. \\ u_2 & b_2^1 + b_2^2 \\ u_3 & b_3^2 + b_3^3 \\ u_4 & b_4^3 \end{array} = \begin{array}{c|c} & b_1^1 + b.c. \\ & b_2^1 + b_2^2 \\ & b_3^2 + b_3^3 \\ & b_4^3 \end{array}$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{l|llll|l|l|l} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & u_1(0) = 0 \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & u_1 & & b_1^1 + b.c. \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & u_2 & & b_2^1 + b_2^2 \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 & & b_3^2 + b_3^3 \\ \mathbf{4} & & KM_{34}^3 & KM_{44}^3 & 1 & u_4 & & b_3^3 \end{array}$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{l|llll|l|l|l} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \text{how to remove this line?} \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & 0 & & b_1^1 \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & u_2 & & b_2^1 + b_2^2 \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 & & b_3^2 + b_3^3 \\ \mathbf{4} & & KM_{34}^3 & KM_{44}^3 & 1 & u_4 & & b_3^3 \end{array}$$

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SETTING B.C.



writing down the equation of line 2

$$KM_{21}^1 * \mathbf{u}_1 + (KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2$$

subtracting $KM_{21}^1 * \mathbf{u}_1$ from both sides:

$$(KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2 - KM_{21}^1 * \mathbf{u}_1$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{l|llll|l|l|l} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \\ \hline \mathbf{1} & KM_{11}^1 & KM_{12}^1 & & & 0 & & b_1^1 \\ \mathbf{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & u_2 & & b_2^1 + b_2^2 \\ \mathbf{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 & & b_3^2 + b_3^3 \\ \mathbf{4} & & KM_{34}^3 & KM_{44}^3 & 1 & u_4 & & b_3^3 \end{array}$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{cccc|c|c} & \mathbf{2} & \mathbf{3} & \mathbf{4} & & & \\ \mathbf{2} & \left[\begin{array}{ccc|c} & & & \\ KM_{22}^1 & KM_{23}^2 & & u_2 \\ KM_{22}^2 & & & \\ \hline KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 \\ KM_{32}^3 & KM_{33}^3 & & \\ \hline KM_{34}^3 & KM_{44}^3 & 1 & u_4 \end{array} \right] & & & & b_2^1 + b_2^2 & \\ & & & & & b_3^2 + b_3^3 & \\ & & & & & b_3^3 & \\ \hline & & & & & & \end{array}$$

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SETTING B.C.



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & & & \\ \mathbf{1} & \left[\begin{array}{ccc|c} KM_{22}^1 & KM_{23}^2 & & u_2 \\ KM_{22}^2 & & & \\ \hline KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 \\ KM_{32}^3 & KM_{33}^3 & & \\ \hline KM_{34}^3 & KM_{44}^3 & 1 & u_4 \end{array} \right] & & & & b_2^1 + b_2^2 - KM_{21}^1 * \cancel{\mathbf{u}_1} & 0 \\ & & & & & b_3^2 + b_3^3 & \\ & & & & & b_3^3 & \\ \hline & & & & & & \end{array}$$

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SUBSTITUTING VALUES



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & & \\ \mathbf{1} & 26/9 & -55/18 & 0 & 0 & 0 & 1/6 \\ \mathbf{2} & -55/18 & 52/9 & -55/18 & 0 & u_2 & 1/3 \\ \mathbf{3} & 0 & -55/18 & 52/9 & -55/18 & u_3 & 1/3 \\ \mathbf{4} & 0 & 0 & -55/18 & 26/9 & u_4 & 1/6 \\ \hline & & & & & & \end{array}$$

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SUBSTITUTING VALUES



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{cccc|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & & & \\ \mathbf{1} & 52/9 & -55/18 & 0 & & u_2 & 1/3 - KM_{21}^1 * \cancel{\mathbf{u}_1} \\ \mathbf{2} & -55/18 & 52/9 & -55/18 & & u_3 & 1/3 \\ \mathbf{3} & 0 & -55/18 & 35/9 & & u_4 & 1/6 \\ \hline & & & & & & \end{array}$$

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SUBSTITUTING VALUES



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

$$\begin{array}{ccc|c|c|c} & \mathbf{1} & \mathbf{2} & \mathbf{3} & & \\ \mathbf{1} & 52/9 & -55/18 & 0 & u_2 & = & 1/3 \\ \mathbf{2} & -55/18 & 52/9 & -55/18 & u_3 & = & 1/3 \\ \mathbf{3} & 0 & -55/18 & 35/9 & u_4 & = & 1/6 \end{array}$$

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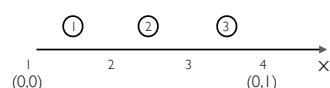
SOLUTION



Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \quad \left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = -u \end{array} \right. \quad \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



$$u_2 = 0.251$$

$$u_3 = 0.363$$

$$u_4 = 0.328$$

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SOLVING LINEAR SYSTEM



linear system of equations: $(K_{ij} - M_{ij})\mathbf{u}_i = b_i + b.c.$

solving for \mathbf{u}_i :

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})\mathbf{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + b.c.$$

$$\mathbf{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + b.c.$$

how to compute $(K_{ij} - M_{ij})^{-1}$?

direct methods: **slow and high memory consumption!**

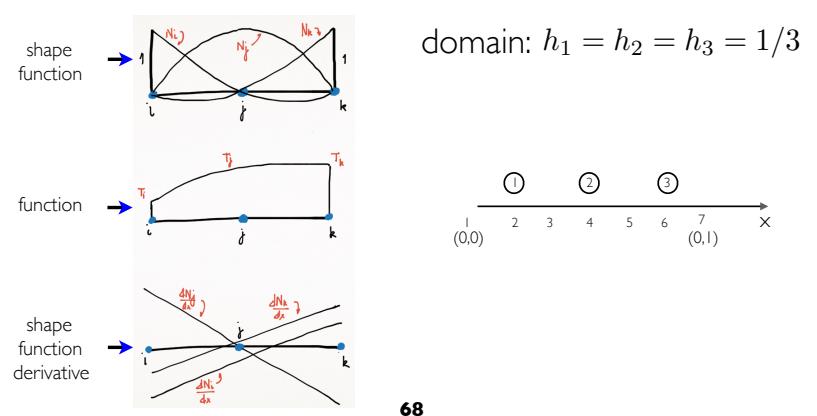
iterative methods: **faster!**

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EXERCISE



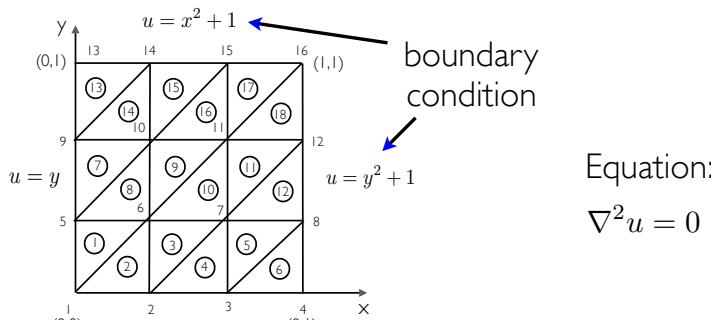
Repeat exercise 1 and 2 for quadratic elements.



2D PROBLEM



Find u in $\Omega = [0, 1] \times [0, 1]$ such that:



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MESH GENERATION



element	node 1	node 2	node 3
1	1	6	5
2	1	2	6
3	2	7	6
4	2	3	7
5	3	8	7
6	3	4	8
7	5	10	9
8	5	6	10
9	6	11	10
10	6	7	11
11	7	12	11
12	7	8	12
13	9	14	13
14	9	10	14
15	10	15	14
16	10	11	15
17	11	16	15
18	11	12	16

node	not b.c.
1	0
2	0
3	0
4	0
5	0
6	1
7	2
8	0
9	0
10	3
11	4
12	0
13	0
14	0
15	0
16	0

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node	b.c. value
1	0
2	1/3
3	2/3
4	1
5	1/3
6	-
7	-
8	10/9
9	2/3
10	-
11	-
12	13/9
13	1
14	10/9
15	13/9
16	2

ASSEMBLING



Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```

1: for elem ← 1, NE do
2:   for ilocal ← 1, 3 do
3:     iglobal ← IEN[elem, ilocal] → ilocal = [1, 2, 3]
4:     for jlocal ← 1, 3 do → iglobal = [v1, v2, v3]
5:       jglobal ← IEN[elem, jlocal] → jlocal = [1, 2, 3]
6:       K[iglobal, jglobal] ← K[iglobal, jglobal] + kelem[ilocal, jlocal] → jglobal = [v1, v2, v3]
7:     end for
8:   end for
9: end for

```

- loop on elements (1)
- loop on neighbors ($i=1,2,3$) and loop ($j=1,2,3$)
- conversion between local to global node
- assembling of stiffness matrix using SUM

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2D PROBLEM - WEAK FORM



Find u in H^1 such that:

$$\int_{\Omega} w (\nabla^2 u) d\Omega = 0$$

weight function

→ mathematical procedure (Green theorem)

$$\int_{\Omega} w \nabla^2 u d\Omega = \int_{\Omega} w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega = 0$$

$$\oint_{\Gamma} w \nabla u \cdot d\Gamma - \int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

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2D PROBLEM - WEAK FORM



$$\oint_{\Gamma} w \nabla u \cdot d\Gamma - \int_{\Omega} \nabla w \cdot \nabla u \, d\Omega = 0$$

apply boundary conditions to the equation
in the continuous form:

$$\oint_{\Gamma} w \cancel{\nabla u} \, d\Gamma - \int_{\Omega} \nabla w \cdot \nabla u \, d\Omega = 0$$

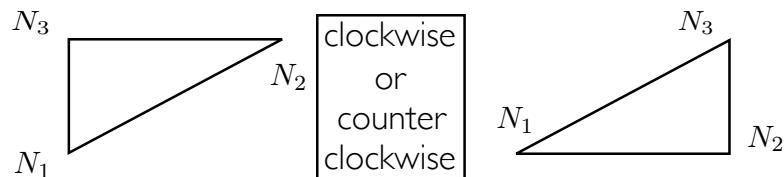
$$\int_{\Omega} \nabla w \cdot \nabla u \, d\Omega = 0$$

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GALERKIN METHOD



$$\text{Approximated functions: } \hat{u} = \sum_{i=1}^{16} N_i(x, y) u_i \quad \hat{w} = \sum_{j=1}^{16} N_j(x, y) w_j$$



$$B = \frac{1}{2A} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

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GALERKIN METHOD



$$\text{Approximated functions: } \hat{u} = \sum_{i=1}^{16} N_i(x, y) u_i \quad \hat{w} = \sum_{j=1}^{16} N_j(x, y) w_j$$

$$K^e = \int_{\Omega} B^T D B \, d\Omega \quad \rightarrow \text{formula}$$

where:

$$B = \begin{bmatrix} \frac{\partial N(x,y)}{\partial x} \\ \frac{\partial N(x,y)}{\partial y} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial x} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial y} \end{bmatrix} \quad D = k \mathbf{I}$$

if B and k are constants: $K^e = k \underbrace{AB^T B}_{\text{area}} \downarrow \text{coefficient}$

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ELEMENT MATRIX



$$N_3 \quad N_2 \quad N_1 \quad B = \frac{1}{2A} \begin{bmatrix} 0 & 1/3 & -1/3 \\ -1/3 & 0 & 1/3 \end{bmatrix}$$

$$N_3 \quad N_2 \quad N_1 \quad B = \frac{1}{2A} \begin{bmatrix} -1/3 & 1/3 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

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ELEMENT MATRIX



N_3 N_2 N_1

$$B^T B = \frac{1}{4A} \begin{bmatrix} 1/9 & 0 & -1/9 \\ 0 & 1/9 & -1/9 \\ -1/9 & -1/9 & 2/9 \end{bmatrix}$$

N_3 N_1 N_2

$$B^T B = \frac{1}{4A} \begin{bmatrix} 1/9 & -1/9 & 0 \\ -1/9 & 2/9 & -1/9 \\ 0 & -1/9 & 1/9 \end{bmatrix}$$

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ASSEMBLING



linear system of equations: $K_{ij} \mathbf{u}_i = b_i + \text{b.c.}$

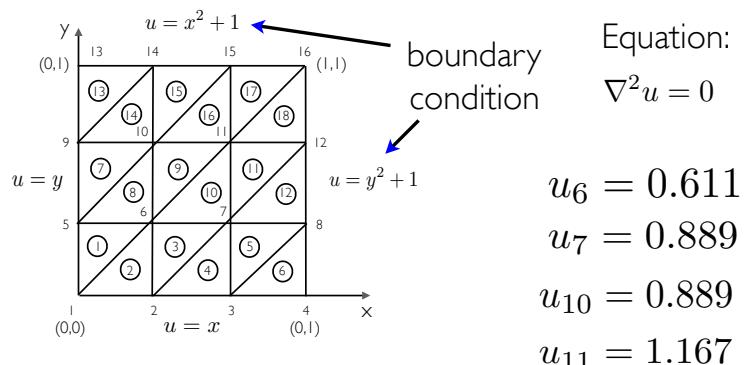
$$\begin{array}{c|ccccc|c|c} & \mathbf{5} & \mathbf{6} & \mathbf{9} & \mathbf{10} & & & \\ \hline \mathbf{5} & 4 & -1 & -1 & 0 & u_6 & & 2/3 \\ \mathbf{6} & -1 & 4 & 0 & -1 & u_7 & = & 16/9 \\ \mathbf{9} & -1 & 0 & 4 & -1 & u_{10} & & 16/9 \\ \mathbf{10} & 0 & -1 & -1 & 4 & u_{11} & & 26/9 \end{array}$$

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SOLUTION



Find u in $\Omega = [0, 1] \times [0, 1]$ such that:



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