INTRODUCTION TO THE FINITE ELEMENT METHOD: 1D-2D CASES

1st Workshop on Advances in CFD and MD modeling of Interface Dynamics in Capillary Two-Phase Flows

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Lausanne - Switzerland
October 5th, 2016

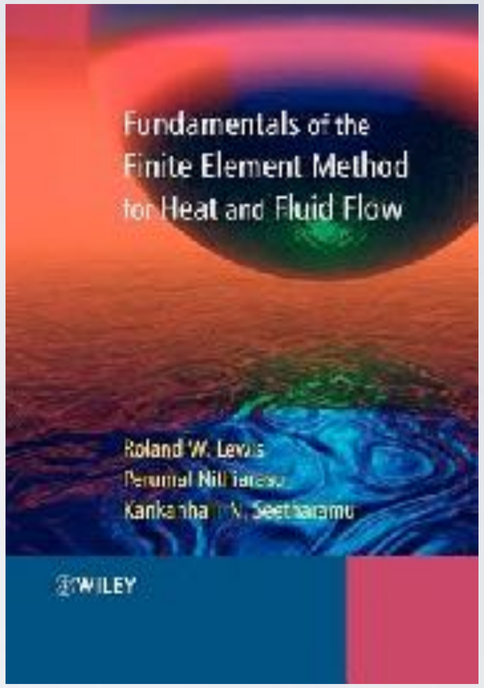
OUTLINE

- Intro to Finite Element Method
- Variational method: the weak form;
- Function approximations: Galerkin method;
- 1D example;
- Tasks: 1D and 2D examples;
- Hands-on: Python scripts for 2D and Axi problems

BIBLIOGRAPHY

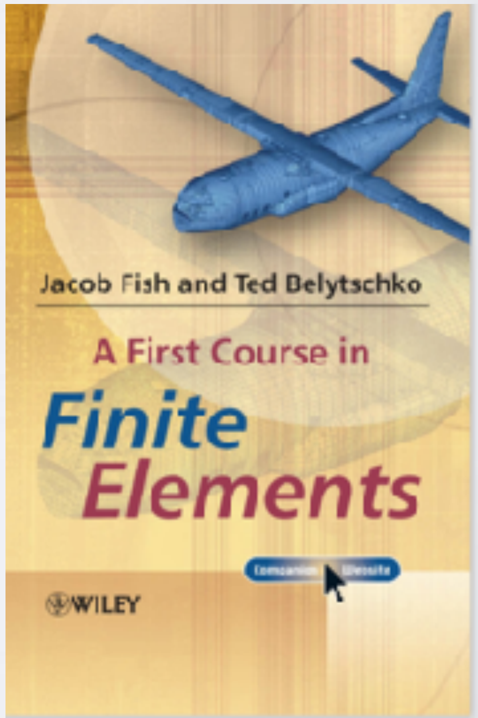
Basic

Fundamentals of the Finite Element Method for Heat and Fluid Flow
Authors: Roland W. Lewis, Perumal Nithiarasu, Kankanhally and N. Seetharamu



Basic

A First Course in Finite Elements
Authors:Jacob Fish and Sand Belytschko



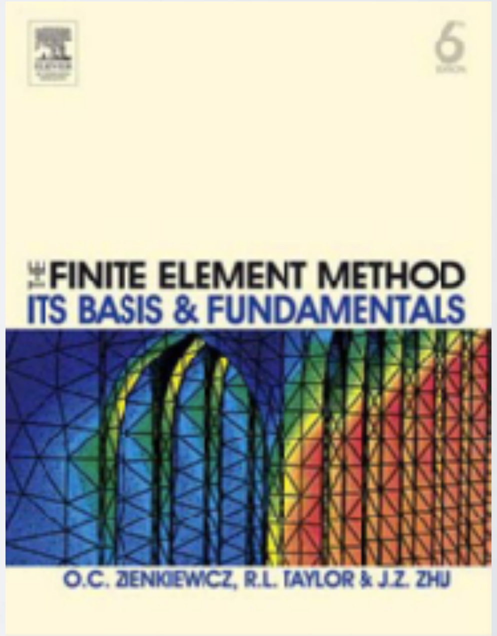
BIBLIOGRAPHY

Basic-advanced

The Finite Element Method - Linear Static and Dynamic Finite Element Analysis
Author: Thomas J.R. Hughes

Advanced

The Finite Element Method - Its Basis & Fundamentals
Authors: O.C. Zienkiewicz, R.L. Taylor & J.Z. Zhu



BRIEF HISTORY OF FEM

- Has been used since 1950's in solid mechanics
- in the 1970's FEM began to be used in CFD
- nowadays FEM is applicably to many engineering problems

Heat transfer, fluid flow, electromagnetic fields, solid mechanics, acoustics, biomechanics etc.

Finite Element Method - FEM	Finite Volume Method - FVM	Finite Difference Method - FDM
• strong math • complex geometry • element geometry • master element • high memory • flexible	• flux formulation • complex geometry • conservative • low memory	• easy math • simple geometries • grid systems • low memory • not flexible

FINITE ELEMENT METHOD

PDE → weak form → ODE → linear system → approximated solution

Governing equations (heat equation, Maxwell equation, Navier-Stokes equation)

Variational form

Set of Ordinary Differential Equations

Ax=b

solution for x

$x=A^{-1}b$

$x=[u,v,w,p,T,c]$

Approximation functions: The Galerkin Method

The approximation functions are combined with the weak form to obtain the discrete finite element equations.

CHOICE OF FUNCTIONS

Finite element function properties:

- the shape functions are 1 at the node and zero elsewhere;
- the sum of all shape function at the element is 1 everywhere, including boundary;
- the weight function is zero at boundary for Dirichlet b.c.

Chart:

function	node, i	node, j	x
Ni	1	0	between 0 e 1
Nj	0	1	between 0 e 1
Ni+Nj	1	1	1

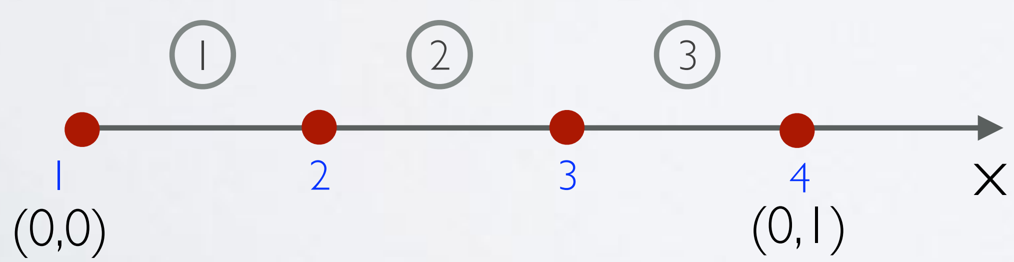
1D PROBLEM - STRONG FORM

Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \left| \begin{array}{l} u(0) = 0 \\ \dfrac{du}{dx}(1) = 1 \end{array} \right.$$

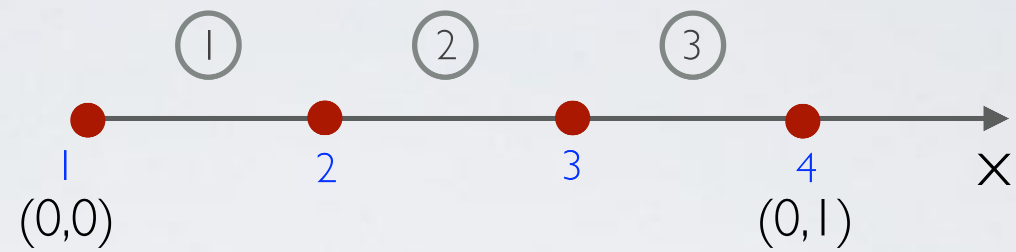
boundary condition

domain: $h_1 = h_2 = h_3 = 1/3$



MESH GENERATION

mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

1D PROBLEM - WEAK FORM

Find u in H^1 with b.c. such that:

$$\int_\Omega w\left(\frac{d^2u}{dx^2}+u+1\right)d\Omega=0$$

weight function

mathematical procedure (integration by parts)

$$\int_0^1 w\frac{d^2u}{dx^2}dx+\int_0^1 wu\,dx+\int_0^1 dx=0$$

$$w\frac{du}{dx}\bigg|_0^1-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wu\,dx+\int_0^1 w\,dx=0$$

$$w\frac{du}{dx}\bigg|_1-w\frac{du}{dx}\bigg|_0-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wu\,dx+\int_0^1 w\,dx=0$$

1D PROBLEM - WEAK FORM

$$w\frac{du}{dx}\bigg|_1 - w\frac{du}{dx}\bigg|_0 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

apply boundary conditions to the equation in the continuous form:

$$w(1)\underbrace{\cancel{\frac{du}{dx}(1)}}_{1} - w(0)\underbrace{\cancel{\frac{du}{dx}(0)}}_{0} - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

$$w(1) - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

GALERKIN METHOD

Approximate functions: $\hat{u} = \sum_{i=1}^{4} N_i(x) u_i \quad \hat{w} = \sum_{j=1}^{4} N_j(x) w_j$

$$w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0) - \sum_{i=j=1}^{4} \int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx +$$

$$+ \sum_{i=j=1}^{4} \int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^{4} \int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}} dx - \int_0^1 \boxed{N_i N_j} dx\right) u_i = \sum_{j=1}^{4} \int_0^1 \boxed{N_j} dx + \boxed{w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0)}$$

stiffness matrix K_{ij} — mass matrix M_{ij} — right hand side. b_i — boundary condition

GALERKIN METHOD

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}dx - \int_0^1 \boxed{N_i N_j} dx\right) u_i = \sum_{j=1}^{4}\int_0^1 \boxed{N_j} dx + \boxed{w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0)}$$

Replacing the shape function to the B.C.

Note that w_j is present at all members and can be removed!

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}dx - \int_0^1 \boxed{N_i N_j} dx\right) u_i = \sum_{j=1}^{4}\int_0^1 \boxed{N_j} dx + \boxed{N_j(1)}$$

boundary condition (evaluated only at x=1)

stiffness matrix K_{ij}

mass matrix M_{ij}

right hand side. b_i

$$(K_{ij} - M_{ij})u_i = b_i + b.c.$$

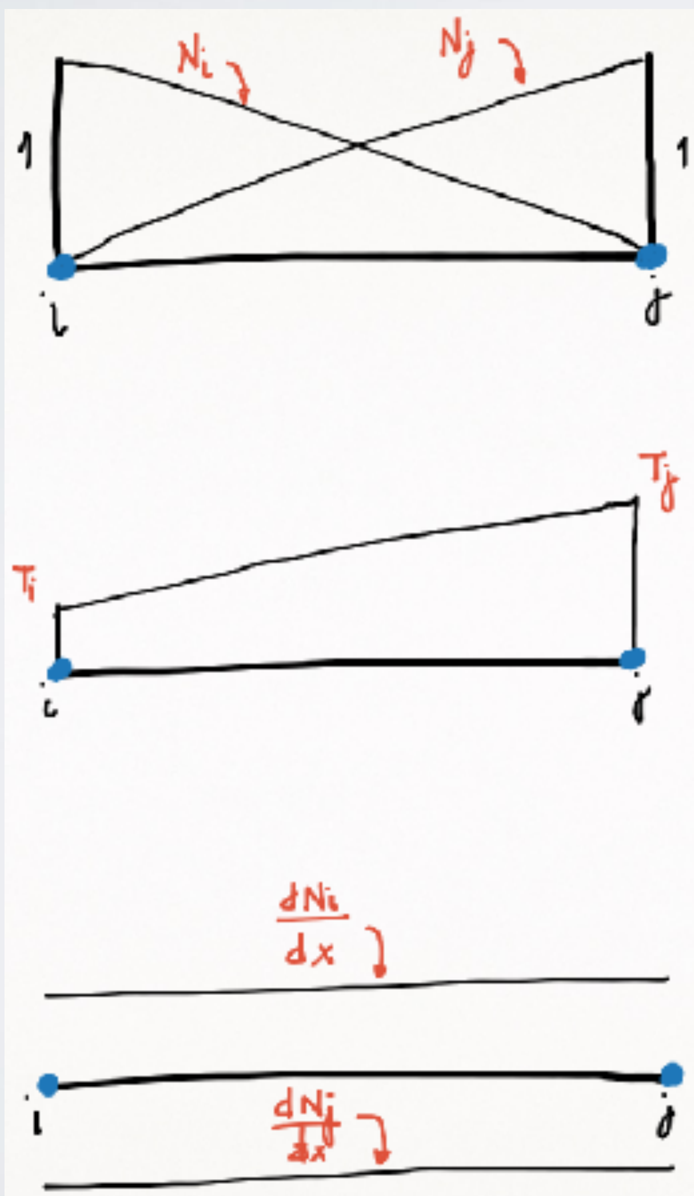
FEM SHAPE FUNCTIONS

1D Problem - linear:

$$T(x) = \alpha_1 + \alpha_2 x$$

1D problem - quadratic:

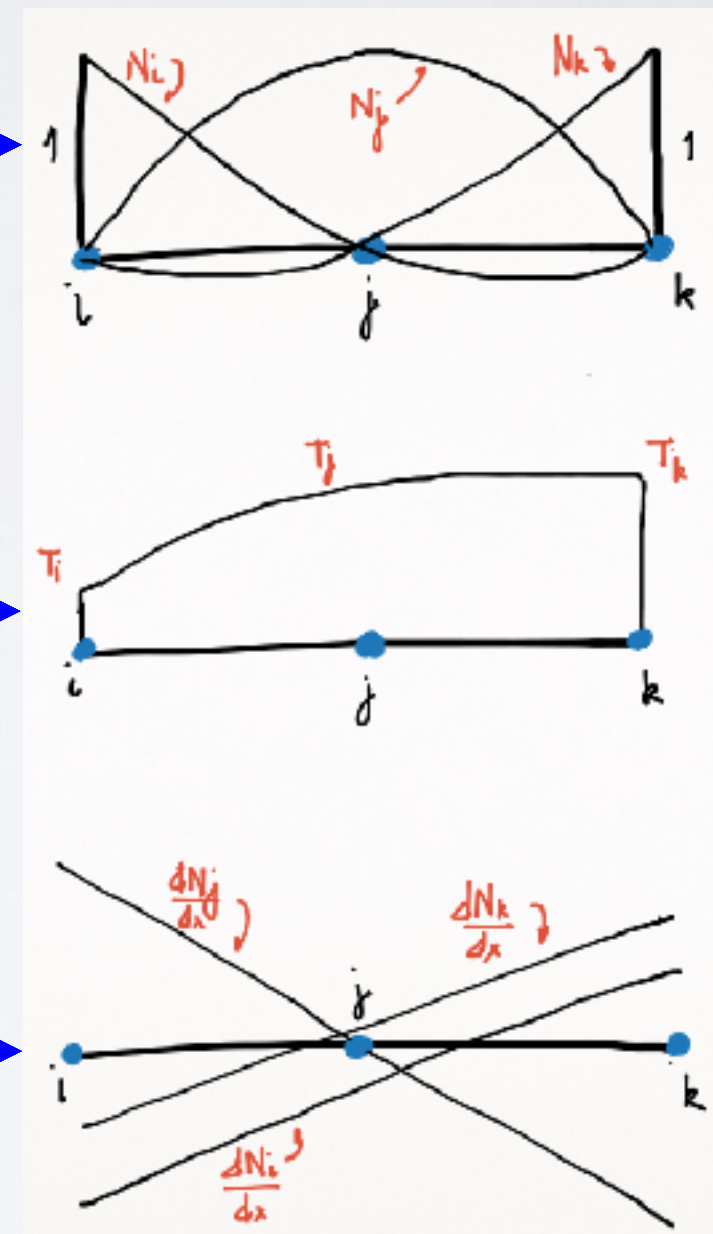
$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



shape function

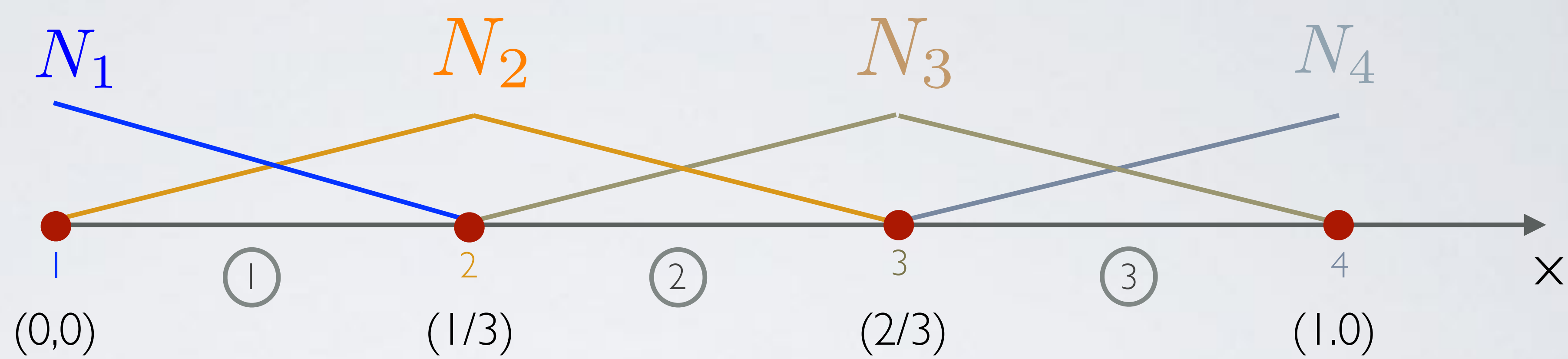
function

shape function derivative



1D PROBLEM - LINEAR

domain and shape functions:



element 1

$\Omega_1^e = [0, 1/3]$

$$N_1 = -3x + 1$$
$$N_2 = 3x$$

element 2

$\Omega_2^e = [1/3, 2/3]$

$$N_2 = -3x + 2$$
$$N_3 = 3x - 1$$

element 3

$\Omega_3^e = [2/3, 1]$

$$N_3 = -3x + 3$$
$$N_4 = 3x - 2$$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element 1

$$\Omega_1^e = [0, 1/3]$$

$$N_1 = -3x + 1$$

$$N_2 = 3x$$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element 2

$$\Omega_2^e = [1/3, 2/3]$$

$$N_2 = -3x + 2$$

$$N_3 = 3x - 1$$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_4 N_4 dx$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element 3

$$\Omega_3^e = [2/3, 1]$$

$$N_3 = -3x + 3$$

$$N_4 = 3x - 2$$

$$b_3 = \int_{2/3}^{1} N_3 dx$$

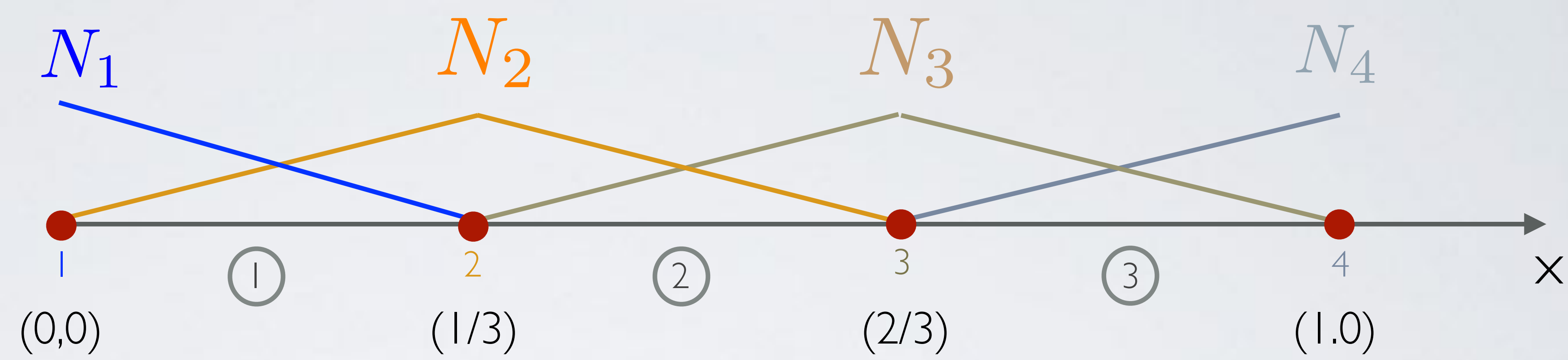
$$b_4 = \int_{2/3}^{1} N_4 dx$$

vector

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 1

$\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

b.c. at $x = 0$

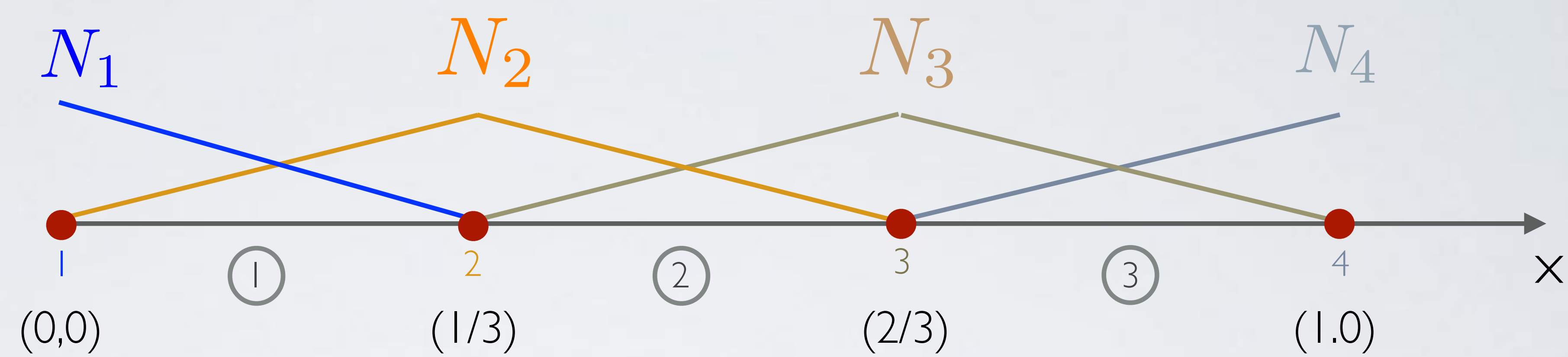
does not exist!

$w(0) = 0$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 2

$\Omega_2^e = [1/3, 2/3]$

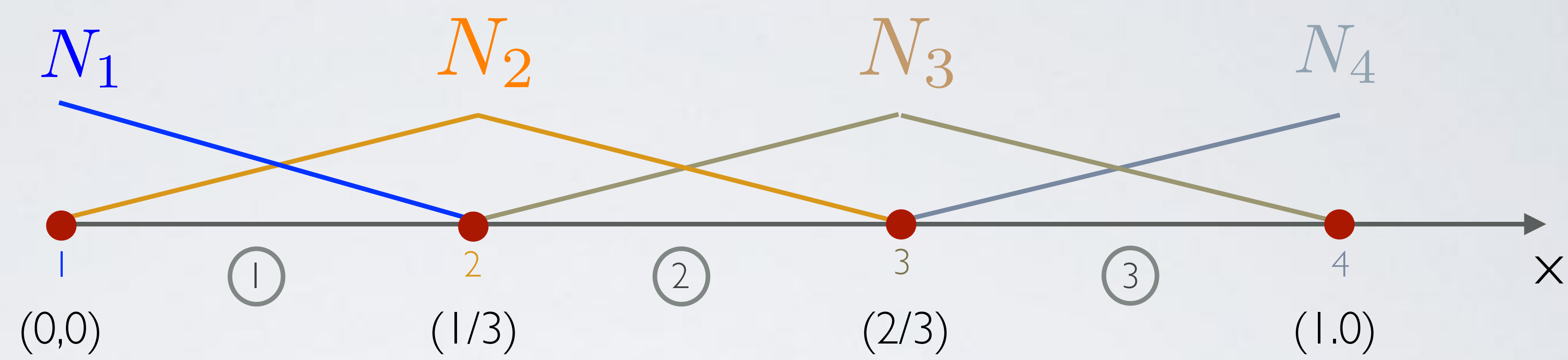
$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 3

$\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

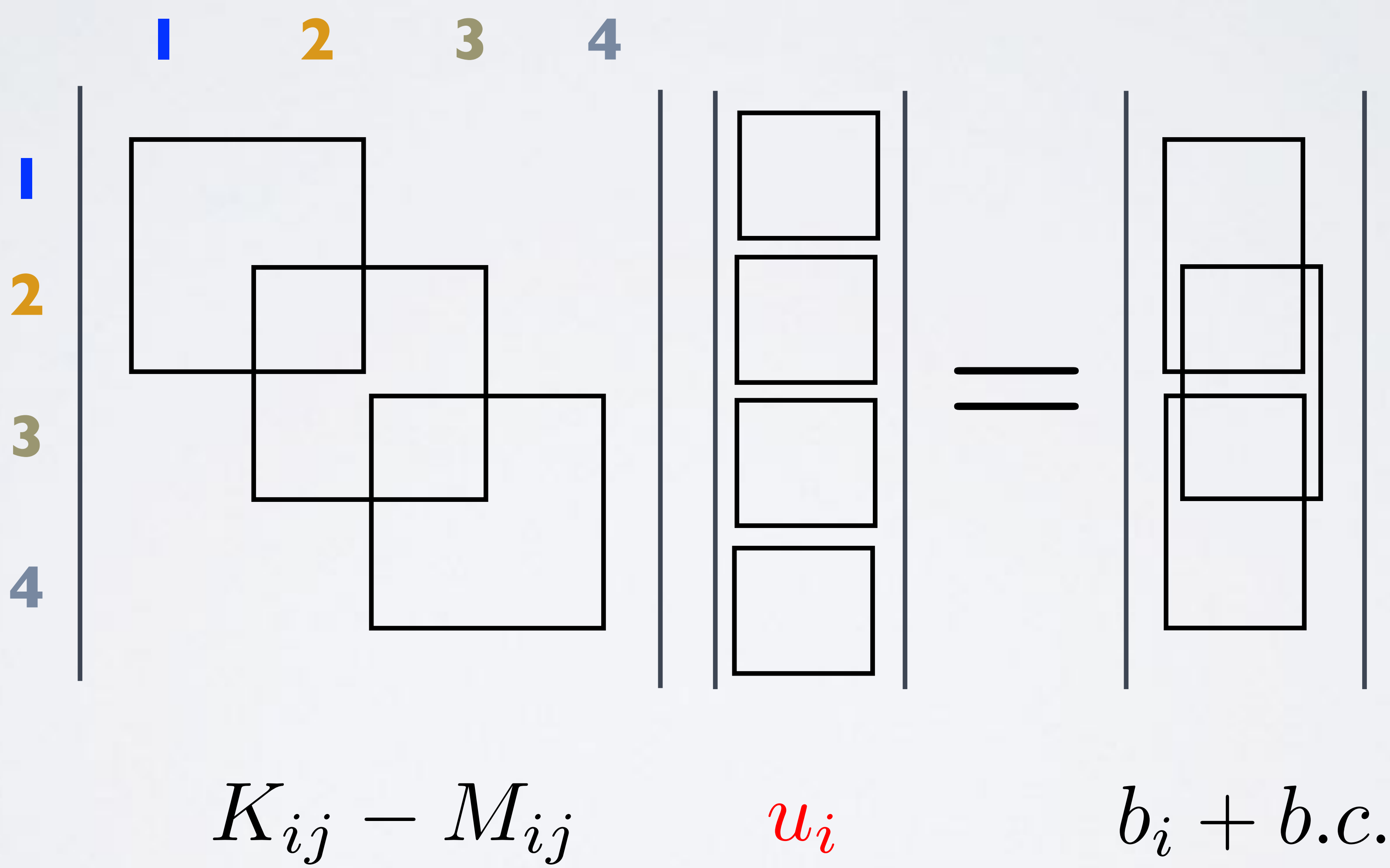
$$\text{b.c. at } x = 1$$

$$\begin{aligned} w(1) &= N_4(1) \\ &= 3.1 - 2 \\ &= 1 \end{aligned}$$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 + \text{b.c. at } x = 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 + 1 \end{bmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$



ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

1: **for** $elem \leftarrow 1, NE$ **do** $\longrightarrow$ NE = Total number of elements
2: **for** $i_{\text{local}} \leftarrow 1, 2$ **do** $\longrightarrow i_{\text{local}} = [1, 2]$
3: $i_{\text{global}} \leftarrow IEN[elem, i_{\text{local}}]$ $\longrightarrow i_{\text{global}} = [v_1, v_2]$
4: **for** $j_{\text{local}} \leftarrow 1, 2$ **do** $\longrightarrow j_{\text{local}} = [1, 2]$
5: $j_{\text{global}} \leftarrow IEN[elem, j_{\text{local}}]$ $\longrightarrow j_{\text{global}} = [v_1, v_2]$
6: $K[i_{\text{global}}, j_{\text{global}}] \leftarrow K[i_{\text{global}}, j_{\text{global}}] + k_{\text{elem}}[i_{\text{local}}, j_{\text{local}}]$
7: **end for**
8: **end for**
9: **end for**

- loop on elements (1)
- loop on neighbors (i=1,2) and loop (j=1,2)
- conversion between local to global node
- assembling of stiffness matrix using SUM

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & KM^1_{11} & KM^1_{12} & & \\
2 & KM^1_{21} & KM^1_{22} & & \\
3 & & & & \\
4 & & & & \\
\end{array}
\left|\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} b^1_1 + \text{b.c.} \\ b^1_2 \\ \\ \\ \end{array}\right|
$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\color{red}{u_i}\color{black} = b_i + b.c.$

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\left|\begin{array}{cccc}
KM^1_{11} & KM^1_{12} & & \\
KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\
 & KM^2_{32} & KM^2_{33} & \\
 & & &
\end{array}\right|
\end{array}
\left|\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 \\ \ \end{array}\right|
$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & KM^1_{11} & KM^1_{12} & & \\
2 & KM^1_{21} & \begin{array}{c} KM^1_{22} \\ + \\ KM^2_{22} \end{array} & KM^2_{23} & \\
3 & & KM^2_{32} & \begin{array}{c} KM^2_{33} \\ + \\ KM^3_{33} \end{array} & KM^3_{43} \\
4 & & & KM^3_{34} & KM^3_{44}
\end{array}
\left|\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} b^1_1 + \mathrm{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 + \mathrm{b.c.} \end{array}\right|
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$u_1(0) = 0$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM_{11}^1 & KM_{12}^1 & & & u_1 & & b_1^1 + \text{b.c.} \\
2 & KM_{21}^1 & KM_{22}^1 + KM_{22}^2 & KM_{23}^2 & & u_2 & = & b_2^1 + b_2^2 \\
3 & & KM_{32}^2 & KM_{33}^2 + KM_{33}^3 & KM_{43}^3 & u_3 & & b_3^2 + b_3^3 \\
4 & & & KM_{34}^3 & KM_{44}^3 & u_4 & & b_3^3 + 1
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

how to remove this line?

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
\hline
1 & KM^1_{11} & KM^1_{12} & & & 0 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & \begin{array}{c} KM^1_{22} \\ + \\ KM^2_{22} \end{array} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & \begin{array}{c} KM^2_{33} \\ + \\ KM^3_{33} \end{array} & KM^3_{43} & u_3 & = & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} & u_4 & & b^3_3 + 1
\end{array}
$$

SETTING B.C.

writing down the equation of line 2

$$KM^1_{21} * u_1 + (KM^1_{22} + KM^2_{22}) * u_2 + KM^2_{23} = b^1_2 + b^2_2$$

subtracting $KM^1_{21} * u_1$ from both sides:

$$(KM^1_{22} + KM^2_{22}) * u_2 + KM^2_{23} = b^1_2 + b^2_2 - KM^1_{21} * u_1$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & 0 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & \begin{array}{c} KM^1_{22} \\ + \\ KM^2_{22} \end{array} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & \begin{array}{c} KM^2_{33} \\ + \\ KM^3_{33} \end{array} & KM^3_{43} & u_3 & = & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} & u_4 & & b^3_3 + 1
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c}
\\ \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{cccc} & 2 & 3 & 4 \end{array} \\
\left|\begin{array}{cccc}
 & & & \\
 & \begin{array}{c} KM_{22}^{1} \\ + \\ KM_{22}^{2} \end{array} & KM_{23}^{2} & \\
 & KM_{32}^{2} & \begin{array}{c} KM_{33}^{2} \\ + \\ KM_{33}^{3} \end{array} & KM_{43}^{3} \\
 & & KM_{34}^{3} & KM_{44}^{3}
\end{array}\right|
\end{array}
\left|\begin{array}{c} \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} \\ b_2^1 + b_2^2 \\ b_3^2 + b_3^3 \\ b_3^3 + 1 \end{array}\right|
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$\begin{array}{cc} & \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{array}{c} 1 \\[2em] 2 \\[2em] 3 \end{array} & \left| \begin{array}{ccc} KM_{22}^{1} + KM_{22}^{2} & KM_{23}^{2} & \\[1em] KM_{32}^{2} & KM_{33}^{2} + KM_{33}^{3} & KM_{43}^{3} \\[1em] & KM_{34}^{3} & KM_{44}^{3} \end{array} \right| \end{array} \left| \begin{array}{c} u_2 \\[1em] u_3 \\[1em] u_4 \end{array} \right| = \left| \begin{array}{c} b_2^1 + b_2^2 - KM_{21}^{1} * u_1 \\[1em] b_3^2 + b_3^3 \\[1em] b_3^3 + 1 \end{array} \right|$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & 26/9 & -55/18 & 0 & 0 & 0 & & 1/6 \\
2 & -55/18 & 52/9 & -55/18 & 0 & u_2 & = & 1/3 \\
3 & 0 & -55/18 & 52/9 & -55/18 & u_3 & & 1/3 \\
4 & 0 & 0 & -55/18 & 26/9 & u_4 & & 1/6 + 1
\end{array}
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})\color{red}{u_i}\color{black} = b_i + b.c.$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\begin{vmatrix} 52/9 & -55/18 & 0 \\ -55/18 & 52/9 & -55/18 \\ 0 & -55/18 & 26/9 \end{vmatrix}
\end{array}
\begin{vmatrix} u_2 \\ u_3 \\ u_4 \end{vmatrix}
=
\begin{vmatrix} 1/3 - \cancelto{0}{KM_{21}^1 * \color{red}{u_1}} \\ 1/3 \\ \color{red}{7/6} \end{vmatrix}
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
1 & 52/9 & -55/18 & 0 \\
2 & -55/18 & 52/9 & -55/18 \\
3 & 0 & -55/18 & 26/9
\end{array}
\begin{vmatrix} u_2 \\ u_3 \\ u_4 \end{vmatrix}
=
\begin{vmatrix} 1/3 \\ 1/3 \\ 7/6 \end{vmatrix}
$$

SOLVING LINEAR SYSTEM

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

solving for u_i :

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})u_i = (K_{ij} - M_{ij})^{-1} + b_i + \mathrm{b.c.}$$

$$u_i = (K_{ij} - M_{ij})^{-1} + b_i + \mathrm{b.c.}$$

how to compute $(K_{ij} - M_{ij})^{-1}$?

direct methods: **not recommended!**

iterative methods: **recommended!**

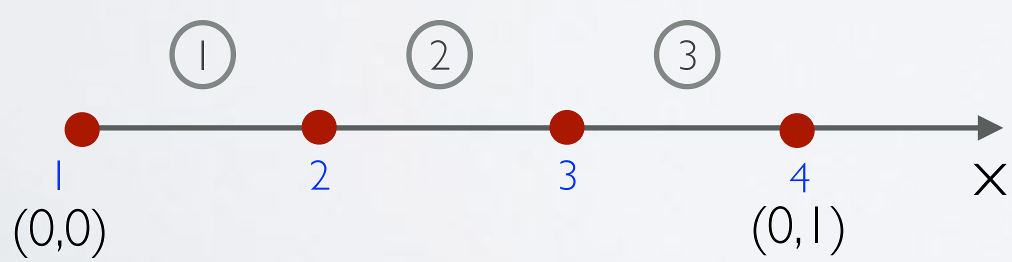
SOLUTION

Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \begin{array}{l} u(0) = 0 \\ \dfrac{du}{dx}(1) = 1 \end{array}$$

boundary condition

domain: $h_1 = h_2 = h_3 = 1/3$



$$u_2 = 1.049$$

$$u_3 = 1.874$$

$$u_4 = 2.386$$

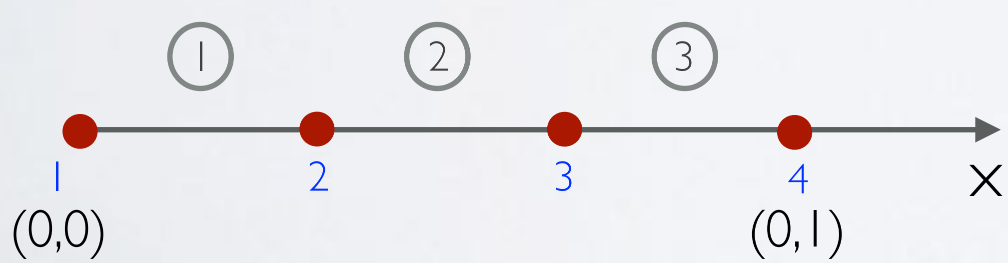
1D PROBLEM

Find u in $\Omega = [0,1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \left|\; \begin{aligned} u(0) &= 0 \\ \frac{du}{dx}(1) &= -u \end{aligned} \right.$$

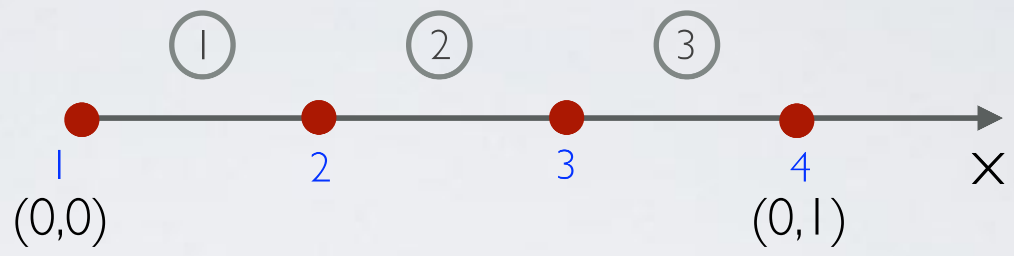
boundary condition

domain: $h_1 = h_2 = h_3 = 1/3$



MESH GENERATION

mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```
1: for elem ← 1, NE do                                          ⟶ NE = Total number of elements
2:     for i_local ← 1, 2 do                                    ⟶ i_local = [1, 2]
3:         i_global ← IEN[elem, i_local]                        ⟶ i_global = [v_1, v_2]
4:         for j_local ← 1, 2 do                                ⟶ j_local = [1, 2]
5:             j_global ← IEN[elem, j_local]                    ⟶ j_global = [v_1, v_2]
6:             K[i_global, j_global] ← K[i_global, j_global] + k_elem[i_local, j_local]
7:         end for
8:     end for
9: end for
```

- loop on elements (1)
- loop on neighbors (i=1,2) and loop (j=1,2)
- conversion between local to global node
- assembling of stiffness matrix using SUM

1D PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_\Omega w\left(\frac{d^2u}{dx^2}+u+1\right)d\Omega = 0$$

weight function

mathematical procedure (integration by parts)

$$\int_0^1 w\frac{d^2u}{dx^2}dx+\int_0^1 wu\,dx+\int_0^1 dx=0$$

$$w\frac{du}{dx}\Big|_0^1-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wu\,dx+\int_0^1 w\,dx=0$$

$$w\frac{du}{dx}\Big|_1-w\frac{du}{dx}\Big|_0-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wu\,dx+\int_0^1 w\,dx=0$$

1D PROBLEM - WEAK FORM

$$w\frac{du}{dx}\bigg|_1 - w\frac{du}{dx}\bigg|_0 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wu\,dx + \int_0^1 w\,dx = 0$$

apply boundary conditions to the equation
in the continuous form:

$$w(1)\cancelto{-u}{\frac{du}{dx}}(1) - \cancelto{0}{w(0)}\frac{du}{dx}(0) - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wu\,dx + \int_0^1 w\,dx = 0$$

$$-w(1)u(1) - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wu\,dx + \int_0^1 w\,dx = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{4} N_i(x) u_i \quad \hat{w} = \sum_{j=1}^{4} N_j(x) w_j$

$$w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0) - \sum_{i=j=1}^{4}\int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx +$$

$$+ \sum_{i=j=1}^{4}\int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^{4}\int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}} dx - \int_0^1 \boxed{N_i N_j} dx\right) u_i = \sum_{j=1}^{4}\int_0^1 \boxed{N_j} dx + \boxed{w(1)\overset{-u}{\cancel{\frac{du}{dx}}}(1) - \overset{0}{\cancel{w}}(0)\frac{du}{dx}(0)}$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

$$(K_{ij} - M_{ij}) u_i = b_i + b.c.$$

GALERKIN METHOD

note that the b.c. has $-u(1)$:

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}dx - \int_0^1 \boxed{N_iN_j}dx\right)u_i = \sum_{j=1}^{4}\int_0^1 \boxed{N_j}dx + \boxed{w(1)\overset{-u}{\cancel{\frac{du}{dx}}}(1) - \overset{0}{\cancel{w}}(0)\frac{du}{dx}(0)}$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx - \int_0^1 N_iN_jdx\right)u_i + w(1)u(1) = \sum_{j=1}^{4}\int_0^1 N_jdx - \overset{0}{\cancel{w}}(0)\frac{du}{dx}(0)$$

Replacing the shape function to the B.C.

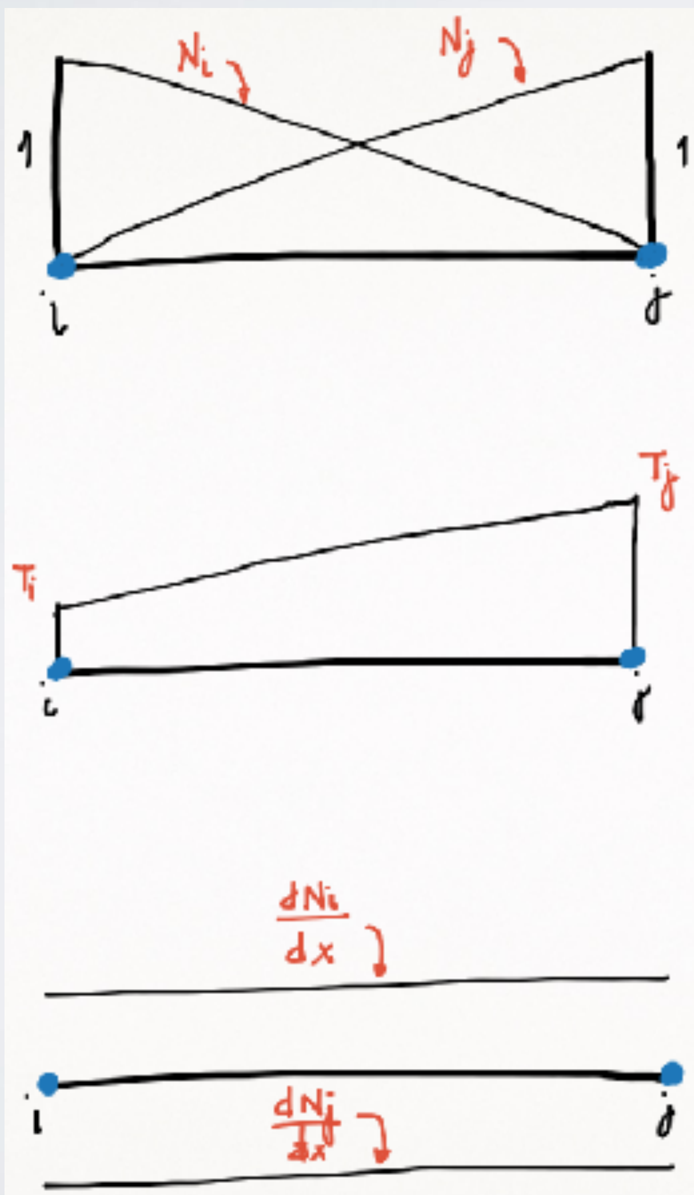
Note that w_j is present at all members and can be removed!

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left[\int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx - \int_0^1 N_iN_jdx + N_i(1)N_j(1)\right]u_i = \sum_{j=1}^{4}\int_0^1 N_jdx$$

FEM SHAPE FUNCTIONS

1D Problem - linear:

$$T(x) = \alpha_1 + \alpha_2 x$$



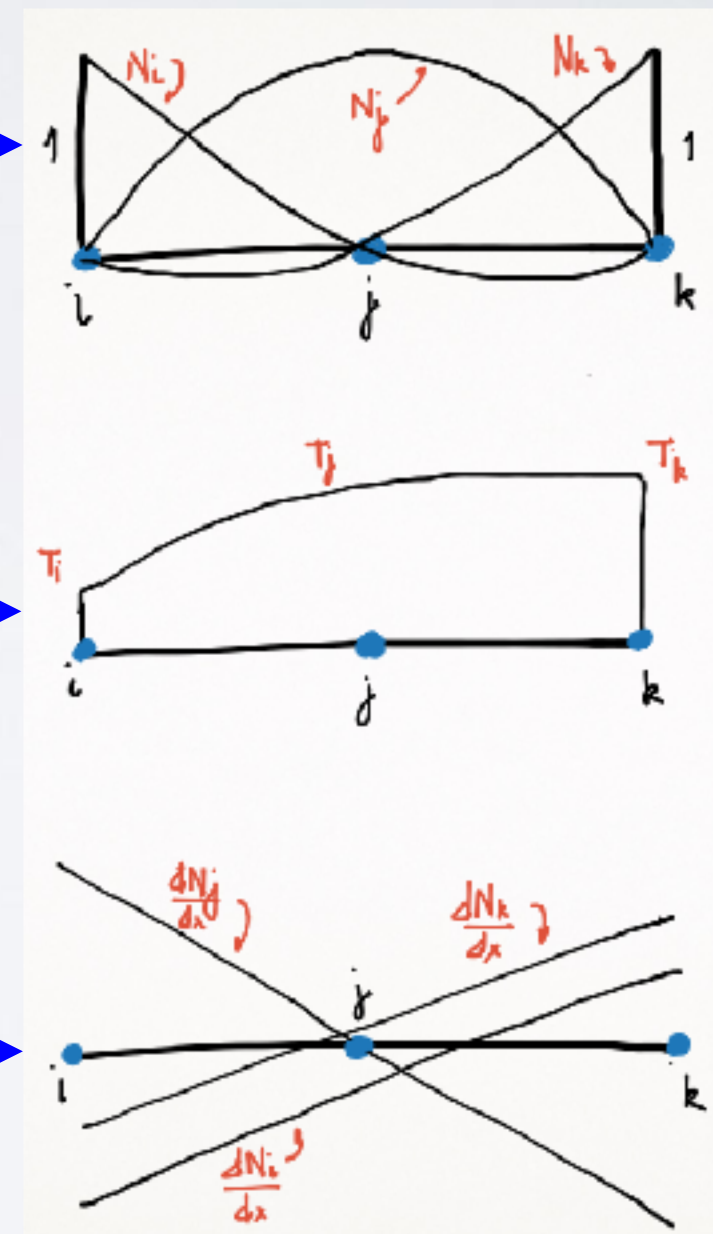
shape function

function

shape function derivative

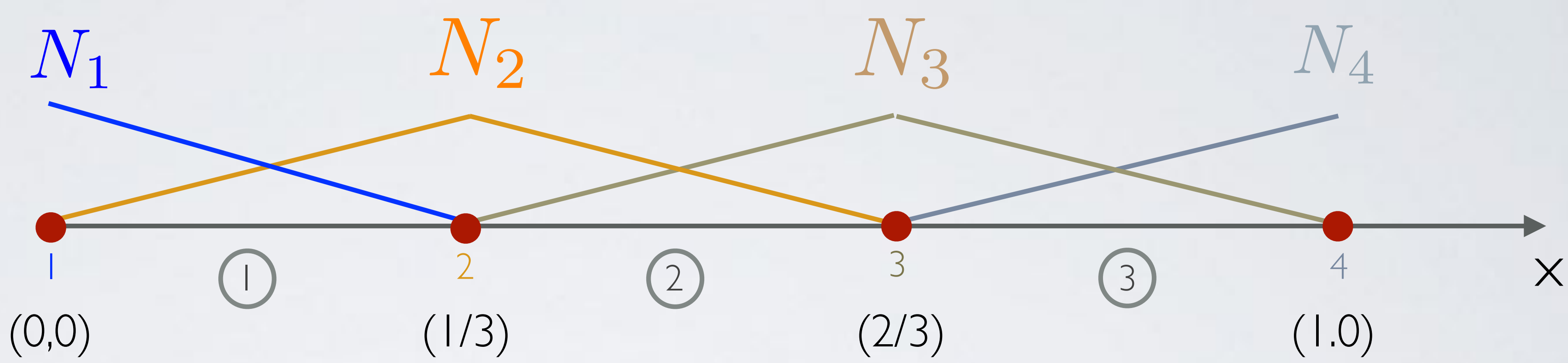
1D problem - quadratic:

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



1D PROBLEM - LINEAR

domain and shape functions:



element 1

$\Omega_1^e = [0, 1/3]$

$$N_1 = -3x + 1$$
$$N_2 = 3x$$

element 2

$\Omega_2^e = [1/3, 2/3]$

$$N_2 = -3x + 2$$
$$N_3 = 3x - 1$$

element 3

$\Omega_3^e = [2/3, 1]$

$$N_3 = -3x + 3$$
$$N_4 = 3x - 2$$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element 1

$\Omega_1^e = [0, 1/3]$

$N_1 = -3x + 1$

$N_2 = 3x$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element 2

$$\Omega_2^e = [1/3, 2/3]$$

$$N_2 = -3x + 2$$

$$N_3 = 3x - 1$$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_4 N_3 dx$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_{44} - M_{44} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_4 N_4 dx + N_4(1)N_4(1)$$

element 3

$$\Omega_3^e = [2/3, 1]$$

$$N_3 = -3x + 3$$

$$N_4 = 3x - 2$$

$$b_3 = \int_{2/3}^{1} N_3 dx$$

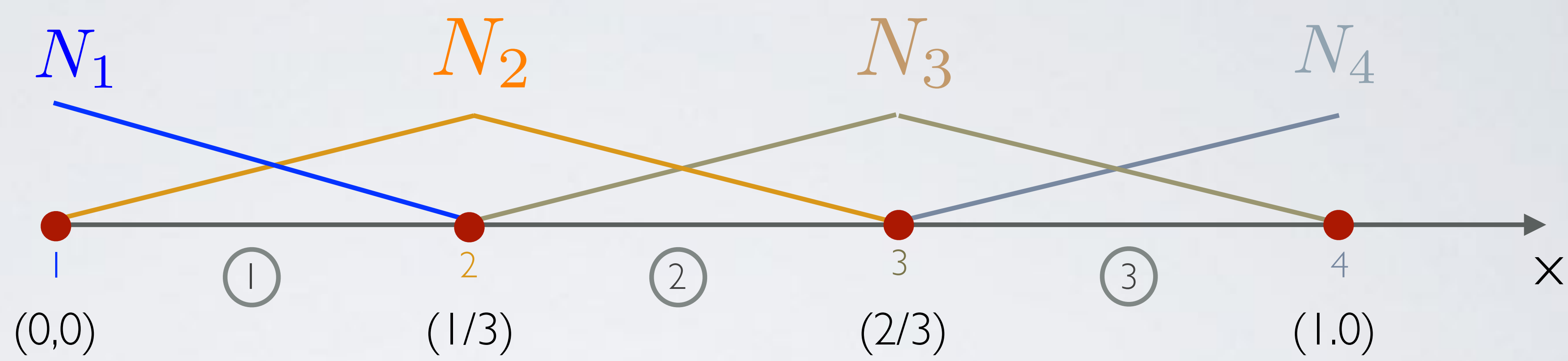
$$b_4 = \int_{2/3}^{1} N_4 dx$$

vector

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 1

$\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

b.c. at $x = 0$

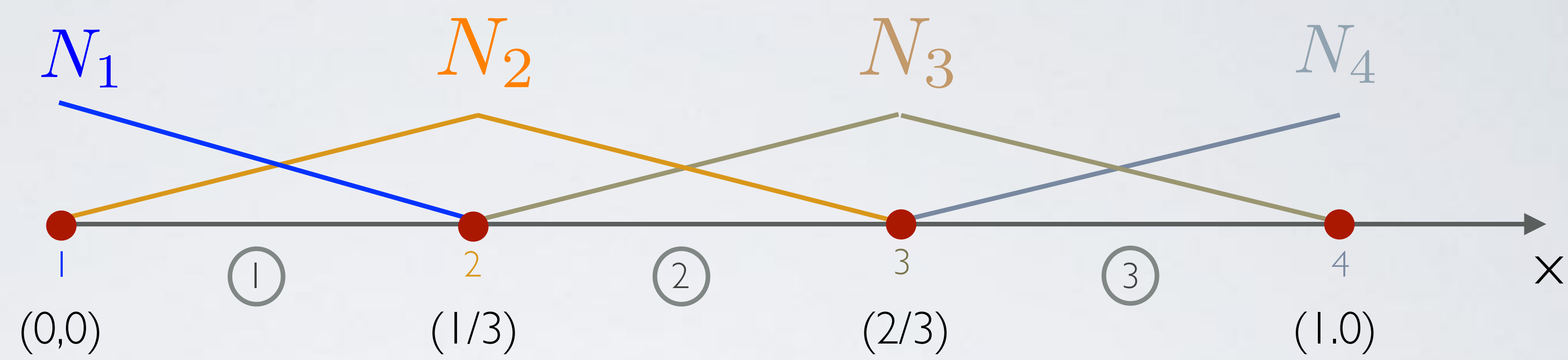
does not exist!

$w(0) = 0$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 2

$\Omega_2^e = [1/3, 2/3]$

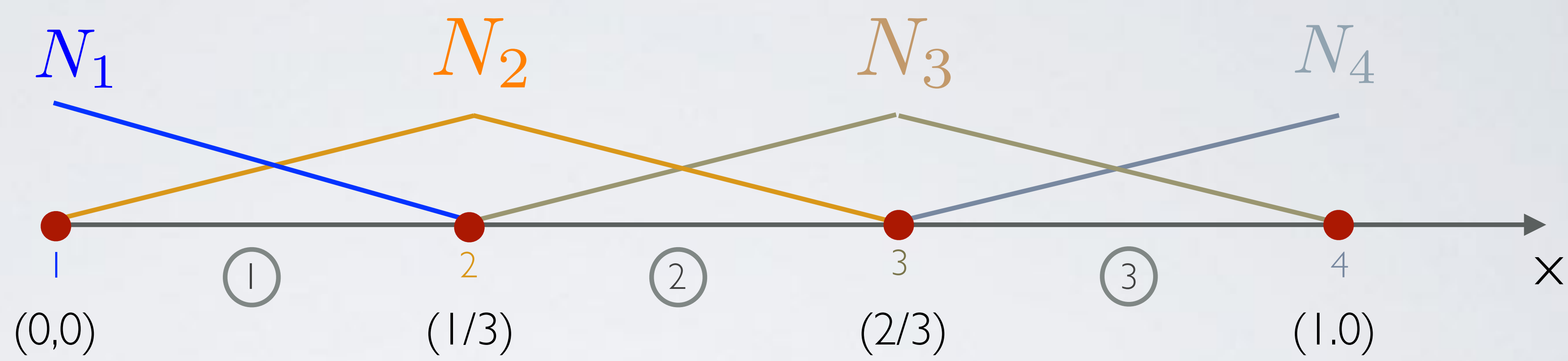
$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 3

$\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} + 1 \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{70}{18} \end{bmatrix}$$

note that:

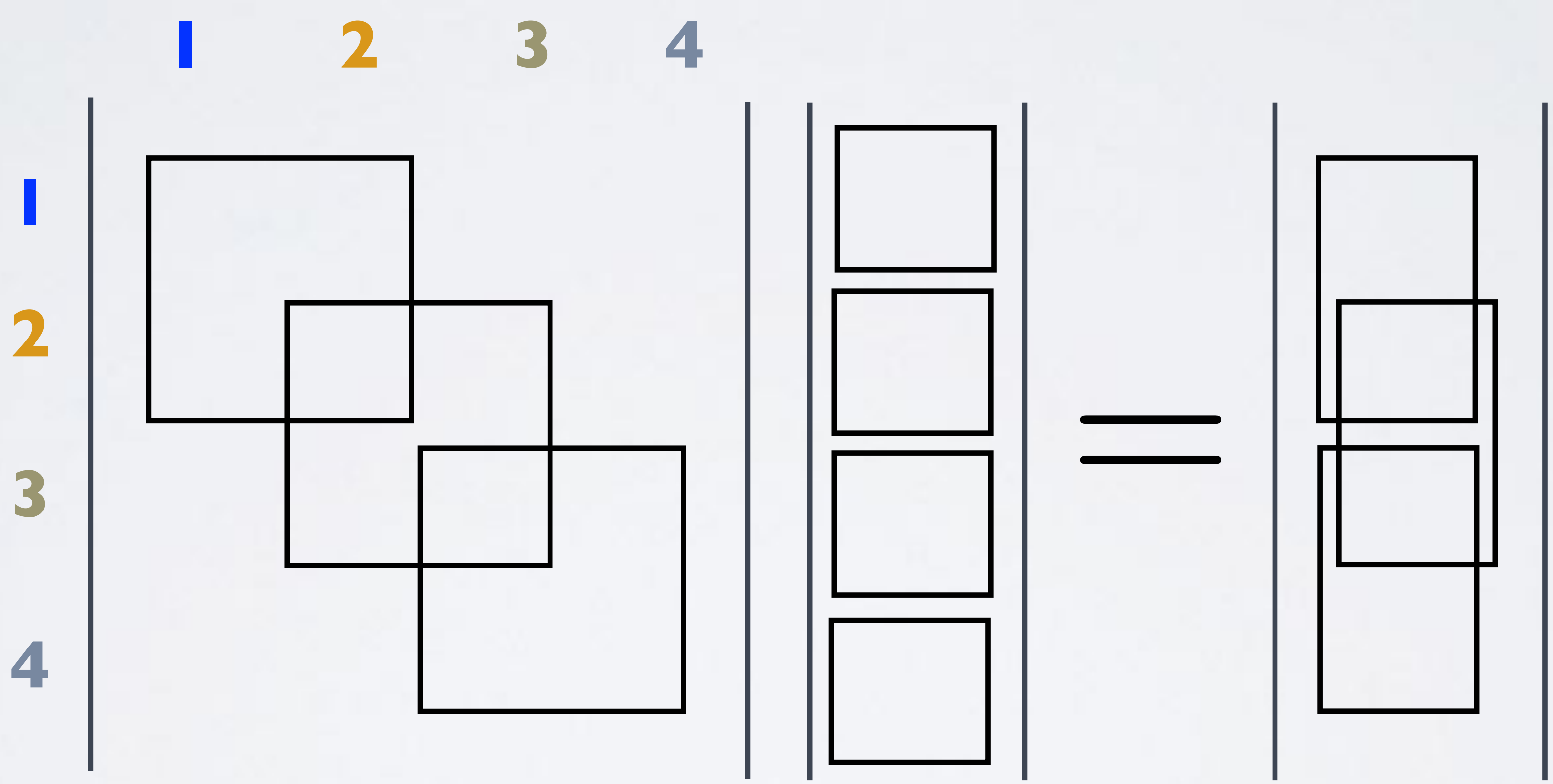
b.c. at $x = 1$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

$$\begin{aligned} w(1)u(1) &= N_4(1) * N_4(1) \\ &= (3.1 - 2)(3.1 - 2) \\ &= 1 \end{aligned}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$



$K_{ij} - M_{ij}$ u_i $b_i + b.c.$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & KM^1_{22} & & & u_2 & = & b^1_2 \\
3 & & & & & u_3 & & \\
4 & & & & & u_4 & &
\end{array}
$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc|c|c|c|c|}
 & 1 & 2 & 3 & 4 & & & & \\
1 & KM_{11}^{1} & KM_{12}^{1} & & & & u_1 & & b_1^1 + \text{b.c.} \\
2 & KM_{21}^{1} & \begin{array}{c} KM_{22}^{1} \\ + \\ KM_{22}^{2} \end{array} & KM_{23}^{2} & & & u_2 & = & b_2^1 + b_2^2 \\
3 & & KM_{32}^{2} & KM_{33}^{2} & & & u_3 & & b_3^2 \\
4 & & & & & & u_4 & & \\
\end{array}
$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} &
\left|\begin{array}{cccc}
KM^1_{11} & KM^1_{12} & & \\
KM^1_{21} & \begin{array}{c} KM^1_{22} \\ + \\ KM^2_{22} \end{array} & KM^2_{23} & \\
 & KM^2_{32} & \begin{array}{c} KM^2_{33} \\ + \\ KM^3_{33} \end{array} & KM^3_{43} \\
 & & KM^3_{34} & KM^3_{44}
\end{array}\right|
\end{array}
\left|\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} b^1_1 + \mathrm{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 \end{array}\right|
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$u_1(0) = 0$$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left|\begin{array}{cccc} KM^1_{11} & KM^1_{12} & & \\ KM^1_{21} & \begin{array}{c} KM^1_{22} \\ + \\ KM^2_{22} \end{array} & KM^2_{23} & \\ & KM^2_{32} & \begin{array}{c} KM^2_{33} \\ + \\ KM^3_{33} \end{array} & KM^3_{43} \\ & & KM^3_{34} & \begin{array}{c} KM^3_{44} \\ + \\ 1 \end{array} \end{array}\right| \end{array} \left|\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\right| = \left|\begin{array}{c} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 \end{array}\right|$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

how to remove this line?

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & 0 & & b^1_1 \\
2 & KM^1_{21} & \begin{array}{c} KM^1_{22} \\ + \\ KM^2_{22} \end{array} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & \begin{array}{c} KM^2_{33} \\ + \\ KM^3_{33} \end{array} & KM^3_{43} & u_3 & = & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & \begin{array}{c} KM^3_{44} \\ + \\ 1 \end{array} & u_4 & & b^3_3
\end{array}
$$

SETTING B.C.

writing down the equation of line 2

$$KM_{21}^{1} * u_1 + (KM_{22}^{1} + KM_{22}^{2}) * u_2 + KM_{23}^{2} = b_2^1 + b_2^2$$

subtracting $KM_{21}^{1} * u_1$ from both sides:

$$(KM_{22}^{1} + KM_{22}^{2}) * u_2 + KM_{23}^{2} = b_2^1 + b_2^2 - KM_{21}^{1} * u_1$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc|c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & 0 & & b^1_1 \\
2 & KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} & u_3 & = & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} + 1 & u_4 & & b^3_3
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})\color{red}{u_i} = b_i + b.c.$

$$
\begin{array}{c|ccc||c|c|c|}
 & 2 & 3 & 4 & & & \\
\hline
 & & & & & & \\
2 & \begin{matrix} KM^1_{22} \\ + \\ KM^2_{22} \end{matrix} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & KM^2_{32} & \begin{matrix} KM^2_{33} \\ + \\ KM^3_{33} \end{matrix} & KM^3_{43} & u_3 & = & b^2_3 + b^3_3 \\
4 & & KM^3_{34} & \begin{matrix} KM^3_{44} \\ + \\ 1 \end{matrix} & u_4 & & b^3_3
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$
\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
\hline
1 & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\
2 & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\
3 & & KM^3_{34} & KM^3_{44} + 1
\end{array}
\begin{vmatrix} u_2 \\ u_3 \\ u_4 \end{vmatrix}
=
\begin{vmatrix} b^1_2 + b^2_2 - KM^1_{21} * \cancelto{0}{{\color{red}u_1}} \\ b^2_3 + b^3_3 \\ b^3_3 \end{vmatrix}
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 26/9 & -55/18 & 0 & 0 \\
2 & -55/18 & 52/9 & -55/18 & 0 \\
3 & 0 & -55/18 & 52/9 & -55/18 \\
4 & 0 & 0 & -55/18 & 26/9+1
\end{array}
\left|\begin{array}{c} 0 \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{array}\right|
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})\color{red}{u_i}\color{black} = b_i + b.c.$

$$
\begin{array}{c c}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\left|\begin{array}{ccc}
52/9 & -55/18 & 0 \\
-55/18 & 52/9 & -55/18 \\
0 & -55/18 & \color{red}{35/9}
\end{array}\right|
\end{array}
\left|\begin{array}{c}
u_2 \\
u_3 \\
u_4
\end{array}\right|
=
\left|\begin{array}{c}
1/3 - \cancelto{0}{KM^1_{21} * \color{red}{u_1}} \\
1/3 \\
1/6
\end{array}\right|
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\begin{vmatrix} 52/9 & -55/18 & 0 \\ -55/18 & 52/9 & -55/18 \\ 0 & -55/18 & \color{red}{35/9} \end{vmatrix}
\end{array}
\begin{vmatrix} u_2 \\ u_3 \\ u_4 \end{vmatrix} = \begin{vmatrix} 1/3 \\ 1/3 \\ 1/6 \end{vmatrix}
$$

SOLVING LINEAR SYSTEM

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

solving for u_i :

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})u_i = (K_{ij} - M_{ij})^{-1} + b_i + \mathrm{b.c.}$$

$$u_i = (K_{ij} - M_{ij})^{-1} + b_i + \mathrm{b.c.}$$

how to compute $(K_{ij} - M_{ij})^{-1}$?

direct methods: **slow and high memory consumption!**

iterative methods: **faster!**

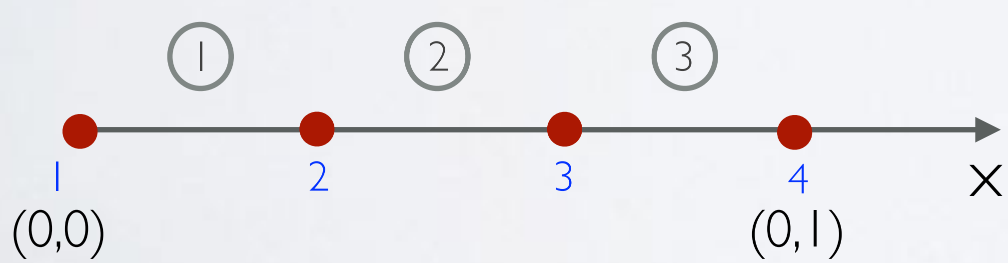
SOLUTION

Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \left| \begin{array}{l} u(0) = 0 \\ \dfrac{du}{dx}(1) = -u \end{array} \right.$$

boundary condition

domain: $h_1 = h_2 = h_3 = 1/3$



$$u_2 = 0.251$$

$$u_3 = 0.363$$

$$u_4 = 0.328$$

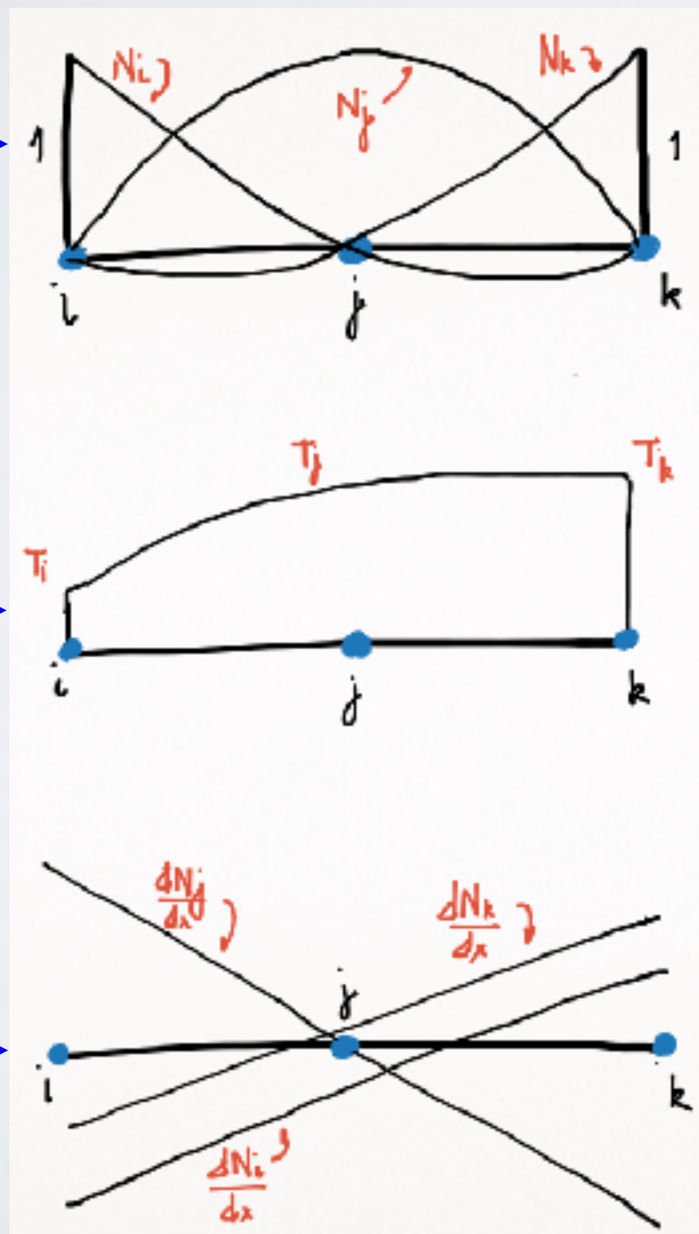
EXERCISE

Repeat exercise 1 and 2 for quadratic elements.

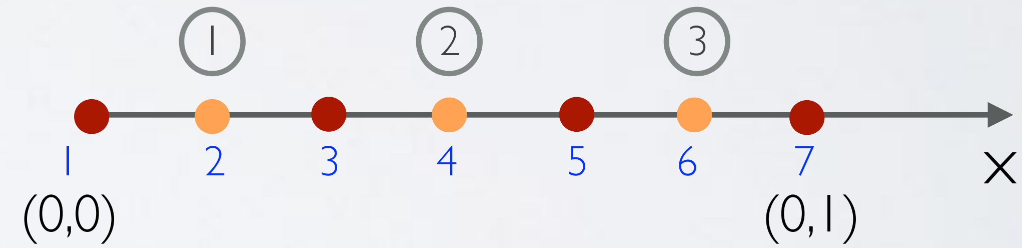
shape function

function

shape function derivative

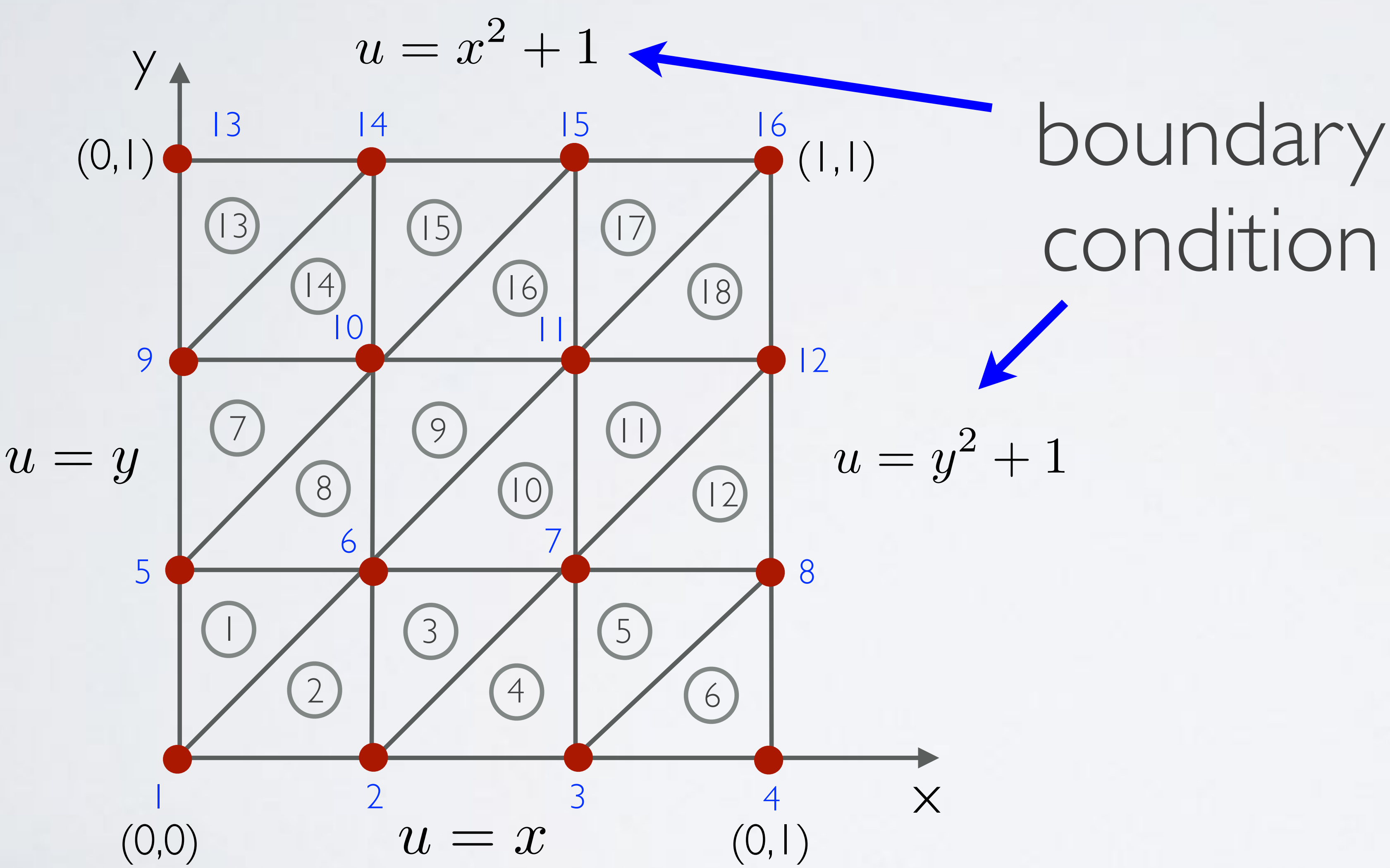


domain: $h_1 = h_2 = h_3 = 1/3$



2D PROBLEM

Find u in $\Omega = [0,1] \times [0,1]$ such that:



Equation:

$$\nabla^2 u = 0$$

MESH GENERATION

element	node 1	node 2	node 3
1	1	6	5
2	1	2	6
3	2	7	6
4	2	3	7
5	3	8	7
6	3	4	8
7	5	10	9
8	5	6	10
9	6	11	10
10	6	7	11
11	7	12	11
12	7	8	12
13	9	14	13
14	9	10	14
15	10	15	14
16	10	11	15
17	11	16	15
18	11	12	16

node	not b.c.
1	0
2	0
3	0
4	0
5	0
6	1
7	2
8	0
9	0
10	3
11	4
12	0
13	0
14	0
15	0
16	0

node	X	Y
1	0	0
2	1/3	0
3	2/3	0
4	1	0
5	0	1/3
6	1/3	1/3
7	2/3	1/3
8	1	1/3
9	0	2/3
10	1/3	2/3
11	2/3	2/3
12	1	2/3
13	0	1
14	1/3	1
15	2/3	1
16	1	1

node	b.c. value
1	0
2	1/3
3	2/3
4	1
5	1/3
6	-
7	-
8	10/9
9	2/3
10	-
11	-
12	13/9
13	1
14	10/9
15	13/9
16	2

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

1:	**for** $elem \leftarrow 1, NE$ **do**	$\longrightarrow$ NE = Total number of elements
2:	**for** $i_{\text{local}} \leftarrow 1, 3$ **do**	$\longrightarrow i_{\text{local}} = [1, 2, 3]$
3:	$i_{\text{global}} \leftarrow IEN[elem, i_{\text{local}}]$	$\longrightarrow i_{\text{global}} = [v_1, v_2, v_3]$
4:	**for** $j_{\text{local}} \leftarrow 1, 3$ **do**	$\longrightarrow j_{\text{local}} = [1, 2, 3]$
5:	$j_{\text{global}} \leftarrow IEN[elem, j_{\text{local}}]$	$\longrightarrow j_{\text{global}} = [v_1, v_2, v_3]$
6:	$K[i_{\text{global}}, j_{\text{global}}] \leftarrow K[i_{\text{global}}, j_{\text{global}}] + k_{\text{elem}}[i_{\text{local}}, j_{\text{local}}]$	
7:	**end for**	
8:	**end for**	
9:	**end for**	

- loop on elements (1)
- loop on neighbors (i=1,2,3) and loop (j=1,2,3)
- conversion between local to global node
- assembling of stiffness matrix using SUM

2D PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_\Omega w\left(\nabla^2 u\right) d\Omega = 0$$

weight function

→ mathematical procedure (Green theorem)

$$\int_\Omega w\nabla^2 u d\Omega = \int_\Omega w\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) d\Omega = 0$$

$$\nabla u\, w\Big|_\Gamma - \int_\Omega \nabla w \cdot \nabla u d\Omega = 0$$

2D PROBLEM - WEAK FORM

$$\nabla u\, w\Big|_{\Gamma} - \int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

apply boundary conditions to the equation
in the continuous form:

$$\cancel{\nabla u\, w\Big|_{\Gamma}}^{\,0} - \int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

$$\int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{16} N_i(x,y)u_i \quad \hat{w} = \sum_{j=1}^{16} N_j(x,y)w_j$

$$\boxed{K^{\mathrm{e}} = \int_{\Omega} B^T B d\Omega}$$

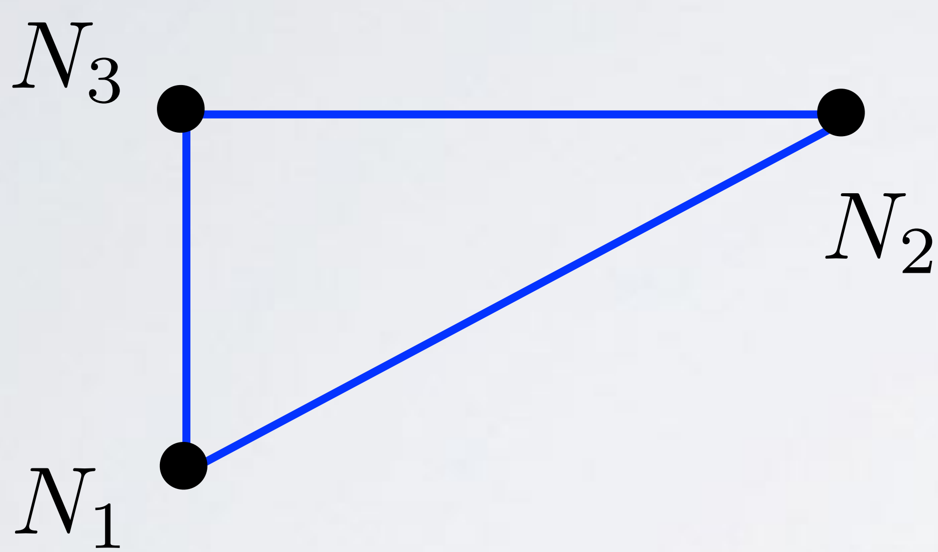
formula

where:

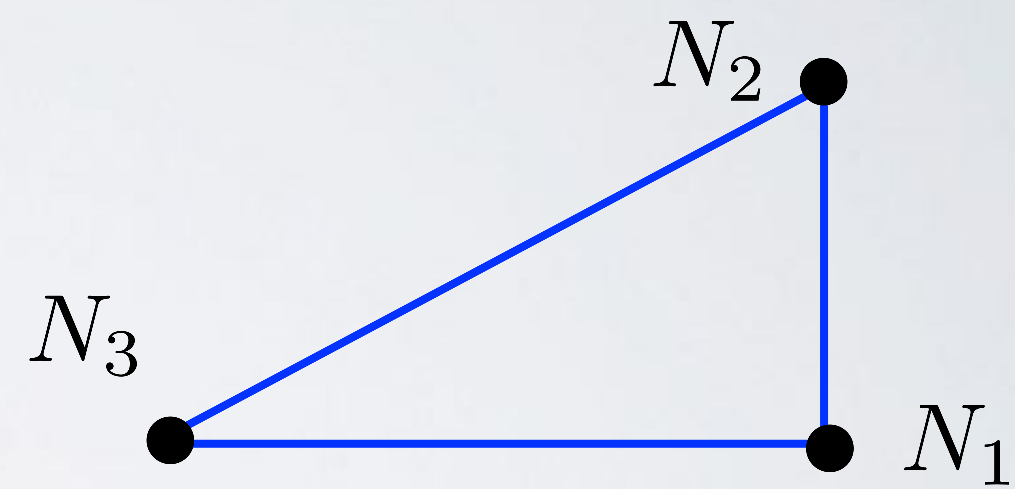
$$B = \begin{bmatrix} \frac{\partial N(x,y)}{\partial x} \\ \frac{\partial N(x,y)}{\partial y} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial x} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial y} \end{bmatrix}$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{16} N_i(x,y)u_i \quad \hat{w} = \sum_{j=1}^{16} N_j(x,y)w_j$

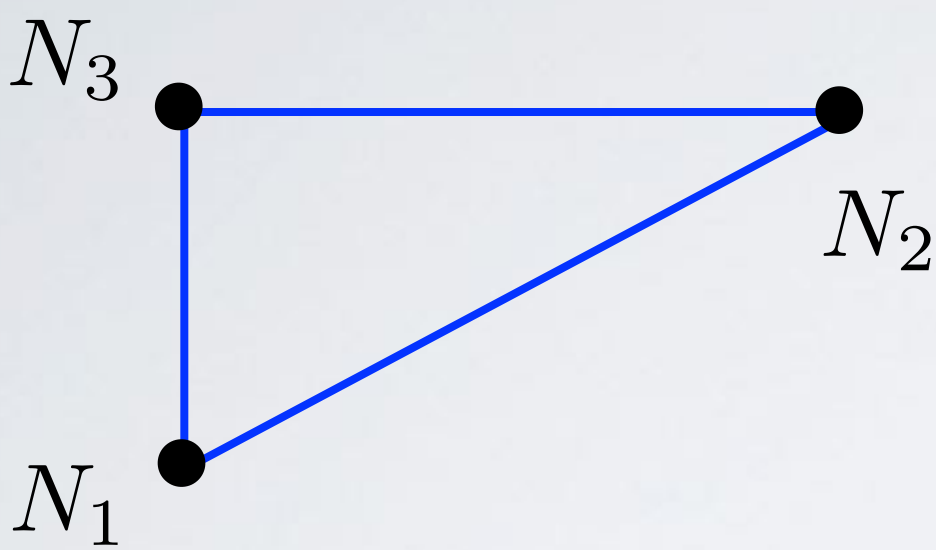


clockwise
or
counter
clockwise

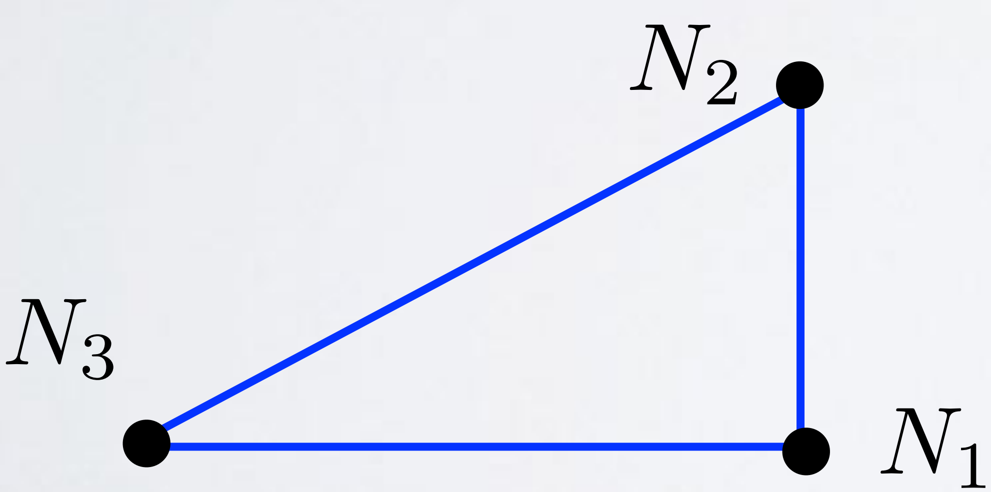


$$B = \frac{1}{2A}\begin{bmatrix} (y_j - y_k) & (y_k - y_i) & (y_i - y_j) \\ (x_j - x_j) & (x_i - x_k) & (x_j - x_i) \end{bmatrix}$$

ELEMENT MATRIX

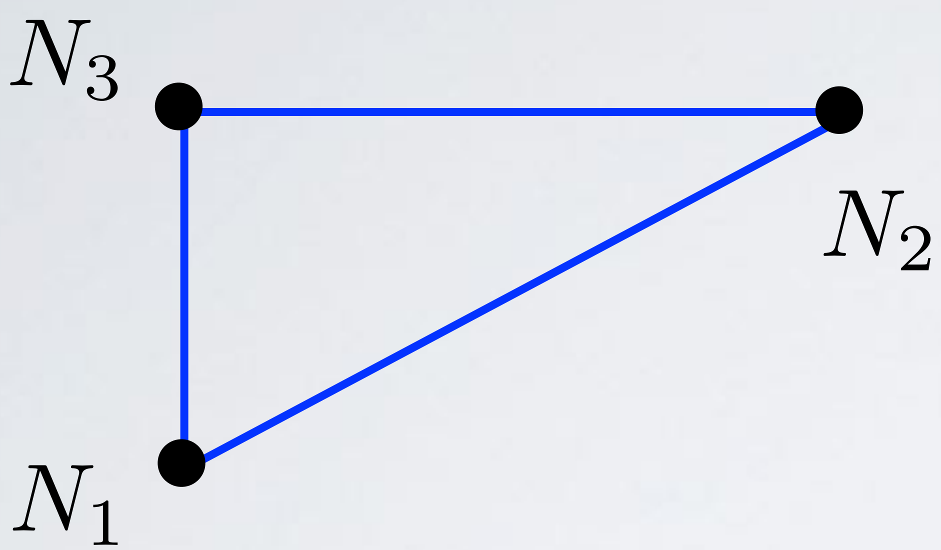


$$B = \frac{1}{2A}\begin{bmatrix} 0 & 1/3 & -1/3 \\ -1/3 & 0 & 1/3 \end{bmatrix}$$

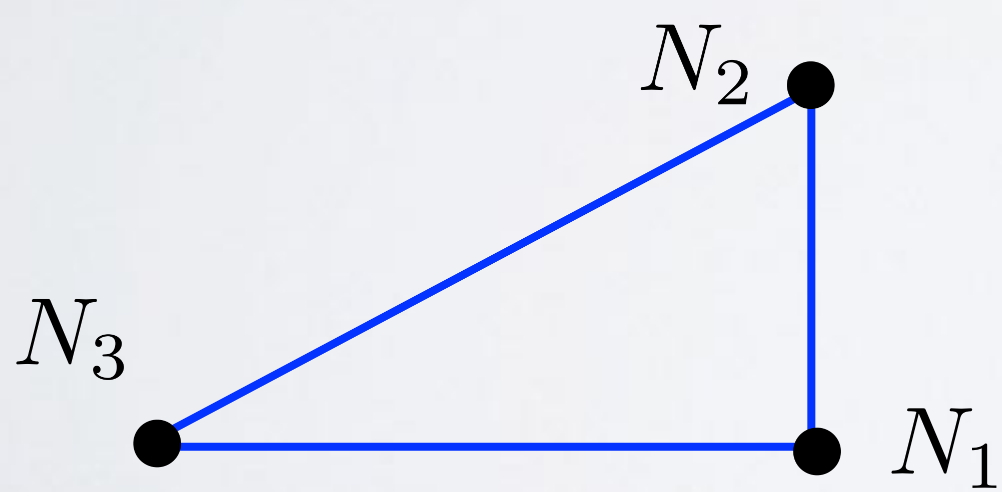


$$B = \frac{1}{2A}\begin{bmatrix} 1/3 & 0 & -1/3 \\ -1/3 & 1/3 & 0 \end{bmatrix}$$

ELEMENT MATRIX



$$B^T B = \frac{1}{4A} \begin{bmatrix} 1/9 & 0 & -1/9 \\ 0 & 1/9 & -1/9 \\ -1/9 & -1/9 & 2/9 \end{bmatrix}$$



$$B^T B = \frac{1}{4A} \begin{bmatrix} 2/9 & -1/9 & -1/9 \\ -1/9 & 1/9 & 0 \\ -1/9 & 0 & 1/9 \end{bmatrix}$$

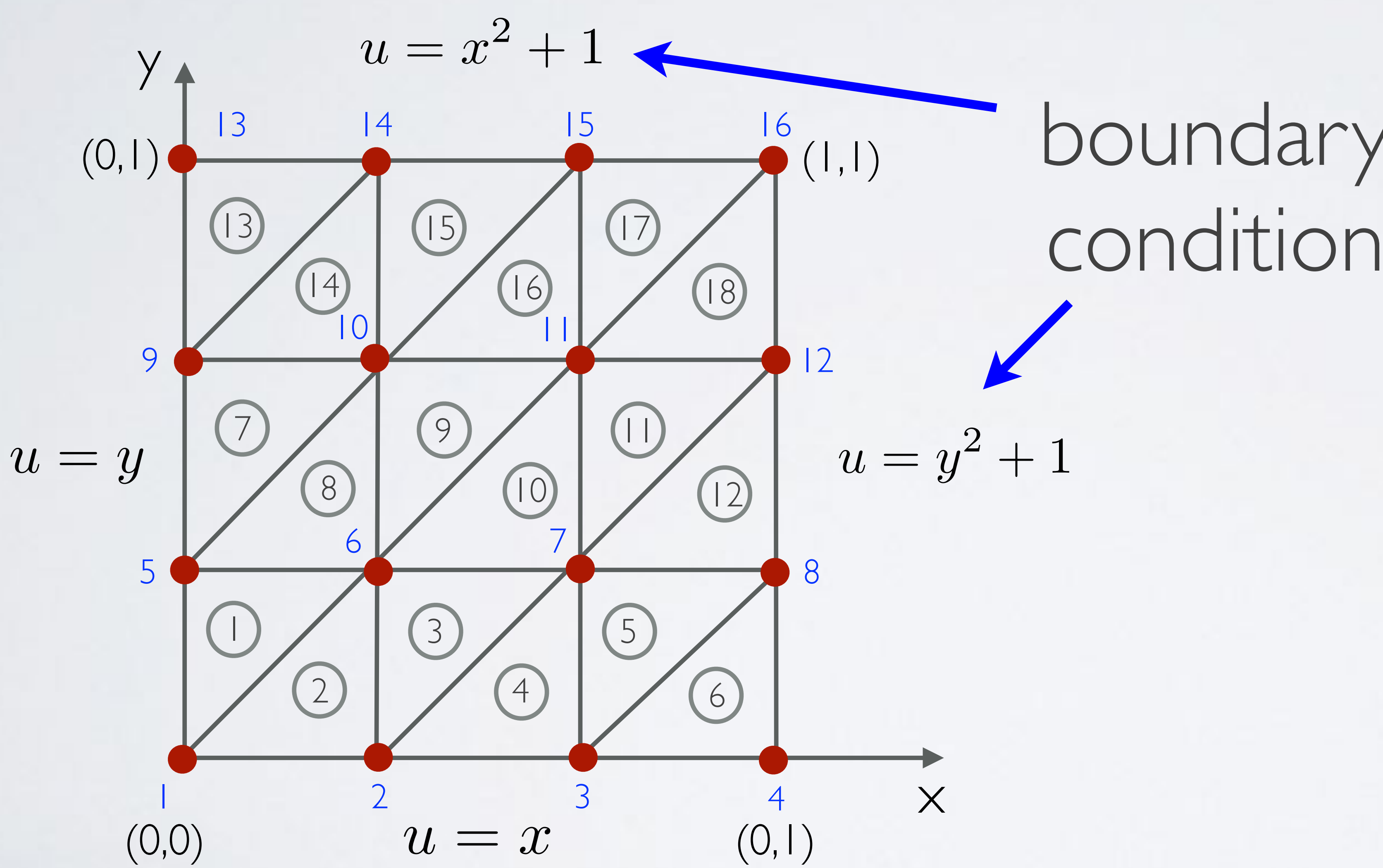
ASSEMBLING

linear system of equations: $K_{ij}u_i = b_i + \text{b.c.}$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & 4 & -1 & -1 & 0 & u_6 & & 2/3 \\
2 & -1 & 4 & 0 & -1 & u_7 & = & 16/9 \\
3 & -1 & 0 & 4 & -1 & u_{10} & & 16/9 \\
4 & 0 & -1 & -1 & 4 & u_{11} & & 26/9
\end{array}
$$

SOLUTION

Find u in $\Omega = [0,1] \times [0,1]$ such that:



Equation:

$$\nabla^2 u = 0$$

$$u_6 = 0.611$$
$$u_7 = 0.889$$
$$u_{10} = 0.889$$
$$u_{11} = 1.167$$