UNCERTAINTIES IN PHYSICAL SYSTEMS: WHY TO QUANTIFY AND HOW TO MODEL?

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1 Introduction

Computational models has been increasingly used in sciences and engineering for design and analysis of complex physical systems. This increase has taken place due to the versatility and low cost of a numerical simulation compared to an approach based on experimental analyzes on a test rig. However, any computational model is subject to a series of uncertainties, due to variabilities on its parameters and, mainly, because of assumptions made in the model conception that may not be in agreement with reality [4]. The first source of uncertainty is inherent limitations in measurement processes, manufacturing, etc., while the second source is essentially due to lack of knowledge about the phenomena observed in the physical system. Also, an increasingly frequent requirement in several projects of engineering is the robust design of a component, i.e., with low sensitivity to the variation of a certain parameter, and this requires the quantification of model uncertainties. In this short work it will exposed some fundamental notions related to the quantification of uncertainties in physical systems, and it will be illustrated the construction of a probabilistic model for uncertainties description in a simplistic mechanical system.

2 Uncertainties, variabilities and errors

To fix ideas, consider a designed system, which will give rise to a real system through a manufacturing process. This manufacturing process is subject to a series of variabilities (due to differences in the geometric dimensions of the components, variations in operating conditions, etc) that result in some differences in the parameters (geometrical dimensions, physical properties, etc) of two or more real systems manufactured. The inaccuracies on these parameters is known as *data uncertainty* [4, 5].

In order to make predictions about the behavior of the physical system, a computational model should be used. In the conception this model mathematical hypotheses are made. These considerations may be or not in agreement with the reality and should introduce additional inaccuracies in the model, known as *model uncertainty*. This source of uncertainty is essentially due to lack of knowledge about the phenomenon of interest and, usually, is the largest source of inaccuracy in model response [4, 5].

A schematic representation of the conceptual process which show how uncertainties of a physical system are introduced into a computational model is shown in Figure 1.

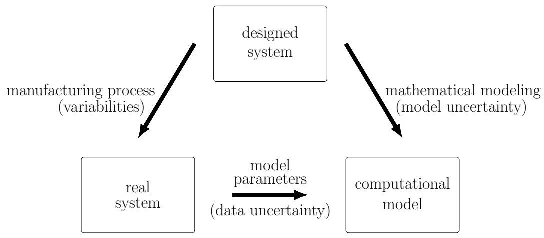

Figure 1: Representation showing how uncertainties are introduced into a computational model.

Uncertainties affect the response of a computational model, but should not be considered errors because they are physical in nature. Errors in the model response are due to the discretization process of the equations, to the use of finite precision arithmetic to perform the calculations, and possible bugs during the computer code implementation

Therefore, unlike the uncertainties, that have physical origin, errors are purely mathematical in nature, and can be controlled if the numerical methods and algorithms used are well known by the analyst.

3 Frameworks for uncertainties description

Being uncertainties in the physical system the focus of stochastic modeling, two approaches are found in the scientific literature for the treatment of uncertainties: (i) *non-probabilistic*, and (ii) *probabilistic*.

The non-probabilistic approach uses techniques such as interval and fuzzy finite elements; imprecise probabilities; evidence theory; probability bounds analysis; fuzzy probabilities; etc, and is generally applied only when the probabilistic approach can not be used [1].

The probabilistic approach uses probability theory to model the uncertainties of the physical system as random mathematical objects. This approach has a more well-developed and consistent mathematical framework, and, for this reason, there is a consensus among the experts that it is preferable whenever possible to use it [2, 5].

4 A simplistic stochastic mechanical system

Consider the simplistic mechanical system shown in Figure 2, which the spring extreme displacement is given by $u = k^{-1} f$.

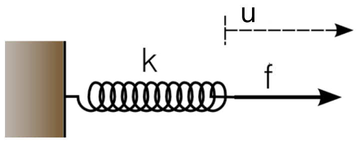

Figure 2: A spring subjected to a force.

If the spring stiffness is uncertain the system response is also subject to uncertainties. Representing the stiffness the random variable K, the spring displacement is now the random variable $U = K^{-1} f$.

Note that, to compute any statistical information of U, such as the mean value

$$\mathbb{E}\left\{U\right\} = \int_{\mathbb{R}} k^{-1}\, f\, p_K(k)\, dk, \qquad (1)$$

it is necessary to know the *probability density function* of K, denoted by $p_K(k)$.

5 Specifying probability distributions

In order to obtain a consistent stochastic model, one cannot arbitrarily choose the probability distribution of a random parameters, under the penalty of violating some physical principle and/or obtain an inconsistent mathematical model. It is a consensus that all information available about these parameters must be taken into account before define their distributions, i.e., specify their PDFs [4].

Accordingly, the *maximum entropy principle* can be used to obtain a desired PDF.

Among all the probability distributions, consistent with the current known information of a given random parameter, the one which best represents your knowledge about this random parameter is the one which maximizes its entropy:

$$\mathrm{S}\left(p_K\right) = - \int_{\mathbb{R}} p_K(k)\, \ln p_K(k)\, dk. \qquad (2)$$

Assuming the following information is known about K:

- $\mathsf{Supp}\, p_K \subset (0, +\infty) \Longrightarrow K > 0 \quad a.s.$
- $\mathbb{E}\left\{K^2\right\} < +\infty$
- $\mathbb{E}\left\{K\right\} = m$ is known
- $\mathbb{E}\left\{K^{-2}\right\} < +\infty$,

it can be shown that the *gamma distribution*

$$p_K(k) = \mathbb{1}_{(0,+\infty)}(k) \frac{1}{m} \frac{\delta^{-2\delta^{-2}}}{\Gamma(\delta^{-2})} \left(\frac{k}{m}\right)^{\delta^{-2}-1} \exp\left\{-\frac{k/m}{\delta^2}\right\},$$

with mean m and dispersion parameter δ, is the probability distribution that maximizes the entropy, respecting the known statistical information on the parameter K [3].

To the best of the authors knowledge, the distribution obtained this way is the one that most accurately describes the current knowledge about the random parameter K.

6 Conclusions

In this short paper are presented some fundamental notions related to the quantification of uncertainties in physical systems and it is illustrated the construction of a probabilistic model for uncertainties description in a simplistic mechanical system.

7 Acknowledgments

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