SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS BY PROJECTION METHODS USING THE INTEGRAL TRANSFORM TECHNIQUE

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1 Introduction

The major difficulty for the numerical simulation of incompressible flows is that the velocity and pressure are coupled by the incompressibility constraint. To overcome this difficulty in time dependent viscous incompressible flows, fractional step methods, which are also referred in the literature as projection methods, were developed. The major computational time associated with these schemes is the solution of the Poisson equation at each step, which consumes large amounts of computer time. One reason for this is that the convergence rate of iterative methods that are commonly used for this purpose, such as the Jacobi and Gauss-Seidel, rapidly decreases as the mesh is refined [7].

On the other hand, in the realm of analytical methods, the Integral Transform Technique [4] has been playing a big role. It deals with expansions of the sought solution in terms of infinite orthogonal basis of eigenfunctions, keeping the solution process always within a continuous domain. The resulting system is generally composed of a set of uncoupled differential equations which can be solved analytically. However, a truncation error is involved since the infinite series must be truncated to obtain numerical results. This error decreases as the number of summation terms (truncation order) is increased, and the solution converges to a final value. Due to the series representation nature of the Integral Transform Technique, the estimated error can be easily obtained, which results in better global error control of the solution. The disadvantage associated with this approach is the need for more elaborate analytical manipulation. This effort can be greatly minimized with the use of symbolical computation.

Recent works of Chalhub et al. [2, 1] introduced the idea of a semi-analytical solution for the poisson equation arising from the incompressible Navier-Stokes equations.

2 Classical Projection Method

In order to solve an in compressible flow problem, one needs to solve the incompressible Navier-Stokes equations. A more efficient way to solve these equations was introduced by Chorin [3] as Projection Methods. The key advantage of the projection method is that the computations of the velocity and pressure fields are decoupled. The algorithm of projection method is based on the Helmholtz-Hodge decomposition[1] [8] of any vector field into a solenoidal part and an irrotational part.

3 Solution of the Pressure Poisson Equation

The general Poisson equation for pressure in two-dimensions, using cartesian coordinates and compatible boundary conditions can be written in the following form:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = Q(x, y, t) \tag{1}$$

where Q is the source term.

Due to the Navier-Stokes nature, Neumann boundary conditions are used for the pressure.

3.1 Filtering Scheme for Integral Transformation

Before continuing to the solution via integral transformation, it is proposed a separation of the pressure on the following form:

$$p(x, y, t) = p^*(x, y, t) + p_f(x, y) \tag{2}$$

where p^* is the filtered pressure and p_f is a known filter function.

4 Classical Integral Transform Technique: Single Transformation

In this work the Classical Integral Transform Technique [4] is used for the purpose of solving the filtered Poisson equation. This is an analytical technique that uses expansions of the sought solution in terms of an infinite orthogonal basis of eigenfunctions, keeping the

[1] Also known as Helmholtz decomposition and theorem of Ladyzhenskaya

solution process always within a continuous domain. In order to establish the transformation pair, the pressure field is written as function of an orthogonal eigenfunctions obtained from an auxiliary eigenvalue problem known as the Helmholtz classic problem in cartesian coordinates [4].

5 Results

In this chapter the results obtained from the computational code developed using the new formulation will be presented. The solution of the incompressible flow in a test case simulated by the proposed method will be presented. The codes were implemented in FORTRAN 90 using GFortran, they were compiled in serial computation and using the -O3 optimization flag.

The solution of the classic Lid-Driven Cavity problem is shown to validate the proposed formulation and evaluate its performance for more demanding problems. Three methods are computed and evaluated: CITT using single transformation with filtering scheme (CITT-ST-F), CITT using single transformation without filtering scheme (CITT-ST) and a code developed using Finite Volumes Methods (FVM) [6] together with a Gauss-Seidel linear system solver [5].

In figure 1 presents the total computational time required to achieve steady state versus mesh size. It can be seen that CITT-ST without filter requires a lot more computational time compared to the other methods. For very coarse meshes, CITT-ST-F and FVM require approximately the same computational time, however, when the mesh is refined, CITT-F overcomes FVM performance.

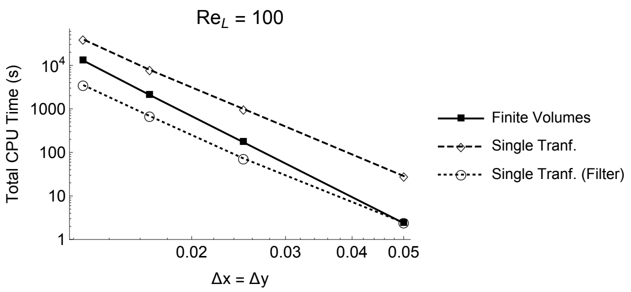

Figure 1: Lid-Driven Cavity: Total computational time consumed to achieve steady state.

For the computation, all results were calculated with prescribed relative precision of 10^{-6} for the Gauss-Seidel iterative solver used in FVM and also for the CITT summation series convergence. In order to guarantee this precision in the truncated summations, an automatic truncation procedure was developed and implemented. The projection method was used and the time-steps Δt utilized were the same for all methods. The results computed in the current work were carried out using only uniform meshes and also considering $\Delta x = \Delta y$ and the Neumann boundary conditions for pressure on all impermeable/no slip walls were approximated to zero without violating the Poisson-Neumann compatibility condition.

6 Conclusions

The present work developed a numerical method for solving the unsteady incompressible Navier-Stokes equations with primitive variables in two dimensions, although it can be easily extended to tree dimensions. The novel methodology is based on projection methods schemes using a mixed approach through the Integral Transform Technique. The Lid-Driven Cavity problem was analysed. Results showed a very similar qualitative behavior. One could see that for very poorly refined meshes CITT-ST-F and FVM had similar performances. However for more refined meshes, CITT-ST-F had a dramatically better performance and CITT-ST (without filter) had the worst performance overall. Although more investigations are needed, CITT-ST-F has a great potential of being a good substitute for the methodologies currently used to solve the pressure Poisson equation.

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