EXPLICIT RESOLUTION BY LINEAR FINITE ELEMENTS OF A SYSTEM WHICH DESCRIBES EVOLUTIVE VISCOELASTIC FLOWS

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1 Introduction

A three-field finite element scheme designed for solving systems of partial differential equations governing stationary incompressible flows is presented. It is based on the simulation of a time-dependent behavior. Once a classical time-discretization is performed, the resulting three-field system of equations allows for a stable approximation of velocity, pressure and extra stress tensor, by means of continuous piecewise linear finite elements, in both two- and three-dimension space. The main advantage of this formulation is the fact that it implicitly provides an algorithm for the iterative resolution of system non-linearities. We show that it can be employed with advantages, to the case of newtonian or quasi-newtonian fluids.

2 Generalized Stokes System

We introduce our methodology in the context of the following generalized Stokes system, derived from the linearization of the equations that govern the flow of a Maxwell viscolelastic liquid, assuming moderate velocities and velocity gradients, the non linear terms may be neglected, namely:

From a given state at time $t = 0$ defined by a given solenoidal velocity $\mathbf{u}^0$ and an extra stress tensor σ^0, for $t > 0$ find $p, \mathbf{u}, \sigma$ that solve the following system, with $\mathbf{u} = \mathbf{g}$ on $\partial\Omega \times (0, \infty)$:

$$\left.\begin{array}{rcl} \dfrac{\partial \mathbf{u}}{\partial t} - \nabla\cdot\sigma + \nabla p &=& \mathbf{f} \\ \nabla\cdot\mathbf{u} &=& 0 \\ \sigma + \lambda\dfrac{\partial\sigma}{\partial t} &=& 2\eta D(\mathbf{u}) \end{array}\right\} \text{in } \Omega\times(0,\infty). \quad (1)$$

3 Time Discretization and Splitting Algorithm

We present an algorithm for solving both newtonian and non newtonian flow equations, in the $\mathbf{u}, p, \sigma$ formulation. Although this algorithm and the underlying variational formulation are described here only in the context of problem (1), its adaption to more general cases is straightforward, including for instance the Navier-Stokes equations, or yet turbulent flow with turbulent stress models. Indeed in the latter cases it suffices to take $\lambda = 0$, before incorporating non linear expressions or terms. It seems however that in the context of viscolastic flow the new approach appears to be the most promising, since in this case the use of a three-field formulation is mandatory.

We have mainly dealt with an explicit splitting algorithm for the time integration or the iterative solution of system (1). However before presenting it we consider the underlying implicit discretization in time of (1).

Let $\Delta t > 0$ be a given time step. Then starting from $\mathbf{u}^0$ and σ^0, for $n = 1, 2, \ldots$, and prescribing $\mathbf{u}^n = \mathbf{g}$ on $\partial\Omega$ for every n, we determine approximations of $p(n\Delta t)$, $\mathbf{u}(n\Delta t)$ and $\sigma(n\Delta t)$, denoted by p^n, $\mathbf{u}^n$ and σ^n respectively, as the solution of the following problem:

$$\left.\begin{array}{rcl} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} - \nabla\cdot\sigma^n + \nabla p^n &=& \mathbf{f} \\ \nabla\cdot\mathbf{u}^n &=& 0 \\ \sigma^n + \lambda\left(\dfrac{\sigma^n - \sigma^{n-1}}{\Delta t}\right) &=& 2\eta D(\mathbf{u}^n) \end{array}\right\} \text{in } \Omega. \quad (2)$$

4 Space Discretization

Once we held the demonstration of equivalence between the time discretized system and the chosen variational formulation, we proceed then to the space discretization of the system. This was performed with linear finite element for the three fields u, p, σ and regularity assumptions in respect of the domain and the spaces (cf. [1]). Once we have obtained the fully discretized system, we consider a lumped mass version of system, denoted by $()_h$, which diagonalizes weighted mass matrices. Next we consider the internal iteration algorithm to explicitly solve the system in every time step, with respect to each velocity and extra stress tensor components. As can be observed in the system (3), s is a index of internal iteration. It is intended to perform an approximation as closer as possible between the solution of the explicit scheme and the solution of the implicit scheme when s tends to infinity.

$$
\begin{cases}
\Delta t^2(\nabla p_h^{n,s}, \nabla q) = \Delta t^2[(\mathbf{f}_h^n, \nabla q) + (\nabla\cdot\sigma_h^{n,s-1}, \nabla q)] \\
+\Delta t(\mathbf{u}_h^{n-1}, \nabla q) - \Delta t < \mathbf{g}_h^n, q\nu >_{1/2,\Gamma} \quad \forall q \in Q_h \\
\\
(\mathbf{u}_h^{n,s}, \mathbf{v})_h = \Delta t(\mathbf{f}_h^n + \nabla\cdot\sigma_h^{n,s-1} - \nabla p_h^{n,s}, \mathbf{v}) + \\
(\mathbf{u}_h^{n-1}, \mathbf{v})_h \quad \forall \mathbf{v} \in V_h \\
\\
\dfrac{\lambda+\Delta t}{2\eta}(\sigma_h^{n,s}, \tau)_h = \dfrac{\lambda}{2\eta}(\sigma_h^{n,s-1}, \tau)_h - \Delta t^2(\mathbf{f}_h^n \\
+\nabla\cdot\sigma_h^{n,s-1} - \nabla p_h^{n,s}, \nabla\cdot\tau) - \Delta t\left[(\mathbf{u}_h^{n-1}, \nabla\cdot\tau)\right. \\
\left. - < \mathbf{g}_h^n, \tau\nu >_{1/2,\Gamma}\right] \ \forall \tau \in \Sigma_h
\end{cases}
\tag{3}
$$

5 Numerical Results

In order to check the accuracy of our method (3), we performed error estimates to some three-dimensional problems. We present one particular case of (1) with known exact solution presented in section 5.3 in [1]. More specifically, we solved the system of equations (3) in the domain $\Omega \times (0,T)$, Ω being the unit cube $(0,1)^3$ and $T = 1$, subject to volumetric force $\mathbf{f}$ and defined initial and boundary conditions. We solved this problem with uniform tetrahedral meshes obtained by first subdividing into M^3 equal cubes with edge length $h = 1/M$, each one of them being in turn subdivided into six tetrahedra in a classical manner. The figure 1 displays approximate relative errors for velocity, pressure and extra stress tensor in the standard L^2-norm for different values of M with $t = 1$, $\lambda = 10$, $\eta = 1$, and Δt taken equal to $h/50$.

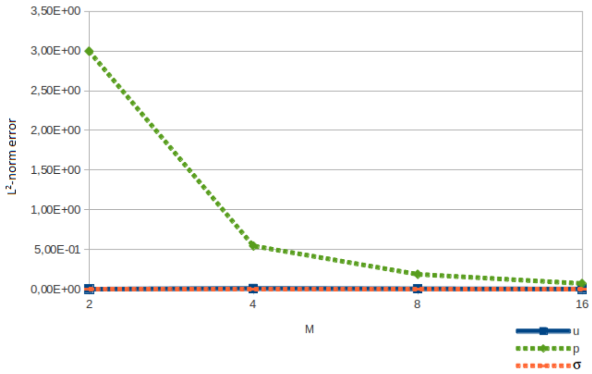

Figure 1: L^2-norm relative errors for $\lambda = 10$, $\eta = 1$ and $t = 1.0$

6 Conclusions

- Widespread simplifying assumption of Oldroyd-like fluid avoided;
- Convergence results derived even for the pressure;
- In practice explicit stable scheme for $\Delta t = O(h)$;
- Code adaption to treat Oldroyd fluids;
- Application to thixotropic models for jelly-like fluids;
- Extending convergence analysis to complete non-linear system;

References

[1] P. Gomes. *Resolução Explícita por Elementos Finitos do Sistema que Descreve Escoamentos Evolutivos Viscoelásticos.* Phd thesis, UFF, Rio de Janeiro - RJ, 2015.