STABILIZED HYBRIDIZED FINITE ELEMENT FORMULATIONS: A NEW APPROACH

Cristiane Oliveira de Faria[1]

[1] Rio de Janeiro State University, IME, Mathematical Analysis Department

1 Introduction

In linear elasticity problems by using of usual displacement-based finite element methods, we are able to numerically determine the displacement field directly and the stresses are evaluated by post-processing. It is well known that standard Galerkin finite element approximations degrade when the Poisson's ratio tends to 1/2, corresponding to near incompressible elasticity.

Hybrid methods are characterized by weakly imposing continuity on each edge of the elements through the Lagrange multipliers. In contrast to DG methods, hybrid formulation allows an element-wise assembly process and the elimination of most degrees of freedom at the element level resulting a global system involving only the degrees-of-freedom of the Lagrange multiplier.

Based on hybridization techniques Faria et al [1, 2] propose a primal hybrid finite element method for the displacement field combining the advantages of DG methods with an element based data structure and reduced computational cost. As multiplier was chosen the trace of displacement field. Stabilization and symmetrization terms are added to generate a stable and adjoint consistent formulation allowing greater flexibility in the choice of basis functions of approximation spaces for the displacement field and the Lagrange multiplier.

After this step, stress approximations with observed optimal rates of convergence in $\mathbf{H}(\mathrm{div})$ norm are recovered by a local post-processing of both displacement and stress using the multiplier approximation and residual forms of the constitutive and equilibrium equations at the element level.

2 The Model Problem

Let Ω in $\mathbb{R}^2$ an open bounded domain with boundary $\Gamma = \partial\Omega$ and external force $\mathbf{f} \in [L^2(\Omega)]^2$. The kinematical model of linear elasticity consists in finding a displacement vector field $\mathbf{u}$ satisfying

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega,\\ \boldsymbol{\sigma}(\mathbf{u}) &= \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega,\\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}(\mathbf{u})$ is the symmetric Cauchy stress tensor, $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$ is the linear strain tensor. For linear, homogeneous and isotropic material $\boldsymbol{\sigma}(\mathbf{u})$ is given by $\boldsymbol{\sigma}(\mathbf{u}) = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\mathbb{I}$, where $\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = \operatorname{div}\mathbf{u}$, $\mathbb{I}$ is the identity tensor and λ and μ are the Lamé parameters.

3 Stabilized Hybrid Discontinuous Galerkin Formulation

We now present a Stabilized Hybrid Discontinuous Galerkin (SHDG) formulation for the linear elasticity problem in its primal form with the multiplier $\boldsymbol{\lambda}$ defined as the trace of $\mathbf{u}$: $\boldsymbol{\lambda} = \mathbf{u}|_e$ on each edge $e \in \mathcal{E}_h$.

The Stabilized Hybrid Discontinuous Galerkin (SHDG) method is formulated as:
Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h \times \mathbf{M}_h$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h \times \mathbf{M}_h$

$$\begin{aligned} \sum_{K\in\mathcal{T}_h}\int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h):\boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K)\cdot(\mathbf{v}_h-\boldsymbol{\mu}_h)ds\\ +\theta\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K)\cdot(\mathbf{u}_h-\boldsymbol{\lambda}_h)ds\\ +\sum_{K\in\mathcal{T}_h}2\mu\int_{\partial K}\beta_1(\mathbf{u}_h-\boldsymbol{\lambda}_h)\cdot(\mathbf{v}_h-\boldsymbol{\mu}_h)ds\\ +\sum_{K\in\mathcal{T}_h}\lambda\int_{\partial K}\beta_2((\mathbf{u}_h-\boldsymbol{\lambda}_h)\cdot\mathbf{n}_K)((\mathbf{v}_h-\boldsymbol{\mu}_h)\cdot\mathbf{n}_K)ds\\ = \sum_{K\in\mathcal{T}_h}\int_K \mathbf{f}\cdot\mathbf{v}_h dx. \end{aligned}$$

with $\mathbf{V}_h = \{\mathbf{v} \in [L^2(\Omega)]^2 \;:\; \mathbf{v}|_K \in [S_k(K)]^2 \quad \forall K \in \mathcal{T}_h\}$,

$\mathbf{M}_h = \{\boldsymbol{\lambda} \in [L^2(\mathcal{E}_h)]^2 : \boldsymbol{\lambda}|_e = [p_l(e)]^2,\ \forall e \in \mathcal{E}_h^0,\ \boldsymbol{\lambda}|_e = \mathbf{0},\ \forall e \in \mathcal{E}_h^\partial\}$,

where $S_k(K) = P_k(K)$ (the space of polynomial functions of degree at most k in both variables), and $p_l(e)$ is the discontinuous piecewise polynomial spaces of degree at most l on each edge e. The residual term multiplied by θ has been consistently added according to the following choices: $\theta = -1$, symmetric and adjoint consistent formulation; $\theta = 1$, nonsymmetric and naturally coercive formulation; and for $\theta = 0$, incomplete formulation allowing greater flexibility in the choice of basis functions of the approximation spaces for the displacement field and the Lagrange multiplier. Here, β_1 is a penalty parameter introduced to stabilize the displacement field $\mathbf{u}_h$ and the multiplier $\boldsymbol{\lambda}_h$ and β_2 stabilizes the normal component of both variables. We also define penalty functions β_1 and β_2 as $\beta_1 = \frac{\beta_0}{h}$ and $\beta_2 = \frac{\beta_n - \beta_0}{h}$ $\forall e \in \mathcal{E}_h$ with $\beta_n > \beta_0 > 0$.

3.1 Stress and displacement local post-processing

In most engineering applications stresses are the variables of main interest. Classically, in displacement finite element formulation stresses are computed indirectly using the displacement approximation and the constitutive equation only. With this classical approach, the approximation $\boldsymbol{\sigma}_h = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)$ converges at best with the following rates in $\mathbf{L}^2$ and $\mathbf{H}(\text{div})$ norms: $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2} = Ch^k$, $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\text{div})} = Ch^{k-1}$.

Alternatively, we propose here a local post-processing consisting in solving at each element $K \in \mathcal{T}_h$ the local problem in stress and displacement fields:

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } K, \\ \mathbb{A}\boldsymbol{\sigma}(\mathbf{u}) &= \boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } K, \\ \mathbf{u} &= \boldsymbol{\lambda}_h && \text{on } \partial K, \end{aligned} \tag{2}$$

with $\boldsymbol{\lambda}_h$ given by the solution of the global problem and $\mathbb{A} = \mathbb{D}^{-1}$. Stress and displacement approximations $[\boldsymbol{\sigma}_{PP}, \mathbf{u}_{PP}]$ for $[\boldsymbol{\sigma}, \mathbf{u}]$, solution of (2), are obtained in the finite dimension spaces $\mathbb{W}_h^k(K) = \{\tau_{i,j} \in S_k(K), \quad \tau_{i,j} = \tau_{j,i}, \quad i,j = 1,2\}$, $\mathbf{V}_h^k(K) = \{\mathbf{v}_i \in S_k(K), \ i = 1,2\}$. Considering the following residual form on each element $K \in \mathcal{T}_h$

$$\begin{aligned} &\int_K \mathbb{A}\boldsymbol{\sigma}_{PP} : \boldsymbol{\tau}_h dx + \int_K \mathbf{u}_{PP} \cdot \operatorname{div}\boldsymbol{\tau}_h dx - \int_{\partial K} \boldsymbol{\lambda}_h \cdot \boldsymbol{\tau}_h \mathbf{n}_K ds \\ &+ \int_K \operatorname{div}\boldsymbol{\sigma}_{PP} \cdot \mathbf{v}_h dx + \int_K \mathbf{f} \cdot \mathbf{v}_h dx \\ &+\delta_1 \int_K (\mathbb{A}\boldsymbol{\sigma}_{PP} - \boldsymbol{\varepsilon}(\mathbf{u}_{PP})) : (\boldsymbol{\tau}_h - \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h))dx \\ &+\frac{\delta_2}{2\mu} \int_K (\operatorname{div}\boldsymbol{\sigma}_{PP} + \mathbf{f}) \cdot \operatorname{div}\boldsymbol{\tau}_h dx + 2\mu \int_{\partial K} \beta_1 (\mathbf{u}_{PP} - \boldsymbol{\lambda}_h) \cdot \mathbf{v}_h ds \\ &+\lambda \int_{\partial K} \beta_2 ((\mathbf{u}_{PP} - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K) ds = 0. \end{aligned}$$

For appropriate choices of the stabilization parameters δ_1 and δ_2, we have observed the following convergence rate for the post-processed stress:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{PP}\|_{\mathbf{H}(\text{div})} = Ch^k \tag{3}$$

which is one order higher.

4 Numerical Results

The performance of the method is tested for a plane-strain problem, defined on square domain $\Omega = (0,1) \times (0,1)$ with homogeneous boundary conditions, considering the elasticity modulus $E = 1$ and forcing term:

$$f_1(x,y) = (2\nu(2\mu + \lambda) - (\mu + \lambda)) \sin(\pi x) \ \cos(\pi y) \tag{4}$$

$$f_2(x,y) = (2\nu(2\mu + \lambda) - (3\mu + \lambda)) \sin(\pi y) \ \cos(\pi x) \tag{5}$$

such that the exact solution is given by

$$u_1(x,y) = \frac{\nu}{\pi^2} \sin(\pi x) \cos(\pi y) \tag{6}$$

$$u_2(x,y) = \frac{(\nu - 1)}{\pi^2} \cos(\pi x) \sin(\pi y). \tag{7}$$

Results of a study on the h-convergence for displacement $(\mathbf{u}_h)$ and stresses $(\boldsymbol{\sigma}_h)$ are presented in Figs. 1–2. In these experiments we use uniform partitions of the domain, symmetric formulation $(\theta = -1)$, linear triangular elements with $k = l = 1$, $\beta_0 = 2$ and $\beta_n = 7$. Figure 1 shows optimal rates of convergence for displacement $(\mathbf{u}_h)$ in L^2 norm and H^1 seminorm, respectively with identical accuracy for all approximations when $\nu \to 1/2$. In Figure 2 we have a comparison between the stress recovered by using the constitutive law and the local postprocessing formulation. It is seen that when we use the constitutive law the locking effect appears, but using the proposed formulation optimal rates are obtained.

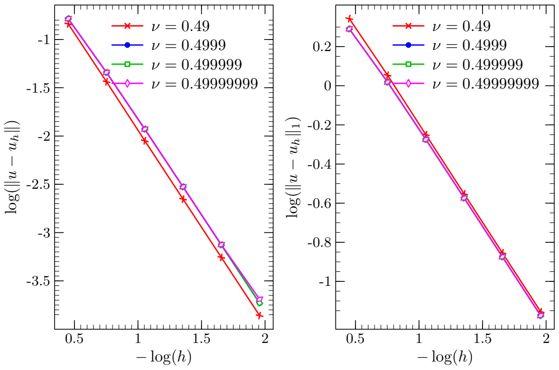

Figure 1: Convergence study for $\mathbf{u}_h$ in (a) $L^2(\Omega)$ norm) and (b) $H^1(\Omega)$ seminorm of SHDG approximations with descontinuous multiplier, $\beta_0 = 2$, $\beta_n = 7$, $\delta_1 = 40$ and $\delta_2 = -1/2$.

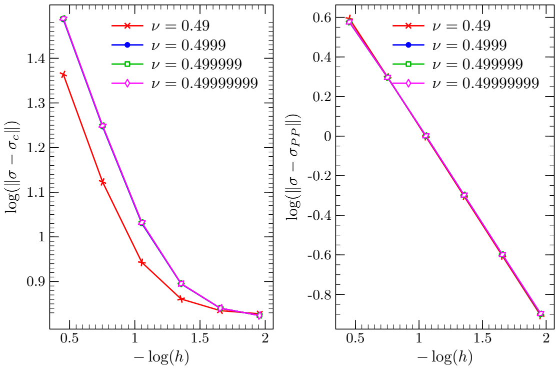

Figure 2: Convergence study for σ in $H(\text{div})$ norm of SHDG approximations with descontinuous multiplier, $\beta_0 = 2$, $\beta_n = 7$, $\delta_1 = 40$ and $\delta_2 = -1/2$.

5 Conclusions

The Hybrid methods preserve the main properties of the DG method but with reduced computational cost. Is easily implemented using the same data structure of continuous Galerkin methods. Numerical results show optimal rates of convergence for the primal variable $\mathbf{u}_h$ and for the Lagrange multiplier $\boldsymbol{\lambda}_h$. A local post-processing based on the multiplier approximation and residual forms of the constitutive and equilibrium equations at the element level is proposed to recover stress approximations with observed optimal rates of convergence in $\mathbf{H}(\text{div})$ norm.

References

[1] C. O. Faria, A. F. D. Loula, and A. J. B. dos Santos. Primal Stabilized Hybrid and DG finite element methods for the linear elasticity problem. *Computers and Mathematics with Applications*, 68:486–507, 2014.

[2] Cristiane O. Faria, Abimael F.D. Loula, and Antônio J.B. dos Santos. Stabilized hybridized finite element method for incompressible and nearly incompressible elasticity. *XXXIV Congresso Ibero Latino Americano de Métodos Computacionais em Engenharia (XXXIV CILAMCE)*, 2013. 10–13 de novembro, Pirenópolis – GO, Brasil.