



INTRODUCTION TO THE FINITE ELEMENT METHOD: 1D-2D CASES

1st Workshop on Advances in CFD and MD modeling of Interface Dynamics in Capillary Two-Phase Flows

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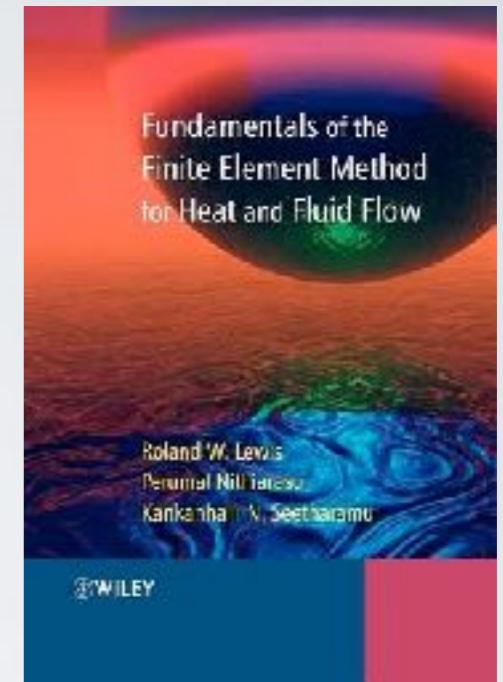
OUTLINE

- Intro to Finite Element Method
- Variational method: the weak form;
- Function approximations: Galerkin method;
- 1D example;
- Tasks: 1D and 2D examples;
- Hands-on: Python scripts for 2D and Axi problems

BIBLIOGRAPHY

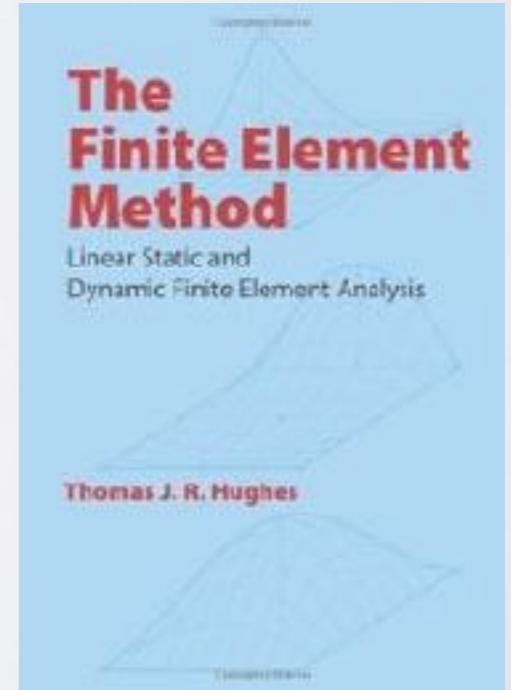
Basic

Fundamentals of the Finite Element Method for Heat and Fluid Flow
Authors: Roland W. Lewis, Perumal Nithiarasu, Kankanhally e N. Seetharamu



Basic-advanced

The Finite Element Method - Linear Static and Dynamic Finite Element Analysis
Authors: Thomas J.R. Hughes



BRIEF HISTORY OF FEM

- Has been used since 1950's in solid mechanics
- in the 1970's FEM began to be used in CFD
- nowadays FEM is applicable to many engineering problems

Heat transfer, fluid flow, electromagnetic fields, solid mechanics, acoustics, biomechanics etc.

Finite Element Method - FEM

- strong math
- complex geometry
- element geometry
- master element
- high memory
- flexible

Finite Volume Method - FVM

- flux formulation
- complex geometry
- conservative
- low memory

Finite Difference Method - FDM

- easy math
- simple geometries
- grid systems
- low memory
- not flexible

CHOOSING ELEMENT

Finite element function properties:

- the shape functions are 1 at the node and zero everywhere else;
- the sum of all shape function at the element is 1 everywhere, including boundary;
- the weight function is zero at boundary for Dirichlet b.c.

Chart:

function	node, i	node, j	x
N_i	1	0	between 0 e 1
N_j	0	1	between 0 e 1
$N_i + N_j$	1	1	1

ID PROBLEM - STRONG FORM

Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0$$

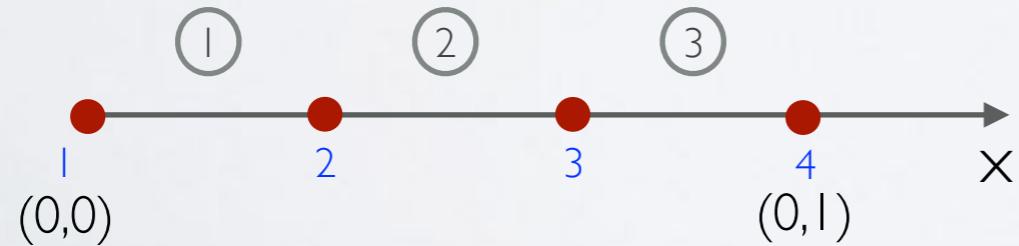
$$\left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = 1 \end{array} \right.$$



boundary
condition

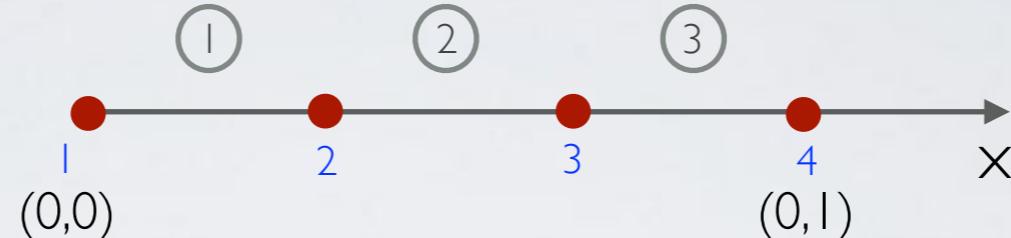
domain: $h_1 = h_2 = h_3 = 1/3$

Answer: $u_2 = 1.049$;
 $u_3 = 1.874$;
 $u_4 = 2.386$



MESH GENERATION

mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```

1: for  $elem \leftarrow 1, NE$  do                                     → NE = Total number of elements
2:   for  $i_{local} \leftarrow 1, 2$  do                         →  $i_{local} = [1, 2]$ 
3:      $i_{global} \leftarrow IEN[elem, i_{local}]$            →  $i_{global} = [v_1, v_2]$ 
4:     for  $j_{local} \leftarrow 1, 2$  do                         →  $j_{local} = [1, 2]$ 
5:        $j_{global} \leftarrow IEN[elem, j_{local}]$            →  $j_{global} = [v_1, v_2]$ 
6:        $K[i_{global}, j_{global}] \leftarrow K[i_{global}, j_{global}] + k_{elem}[i_{local}, j_{local}]$ 
7:     end for
8:   end for
9: end for

```

- loop on elements (l)
- loop on neighbors (i=1,2) and loop (j=1,2)
- conversion between local to global node
- assembling of stiffness matrix using SUM

ID PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_{\Omega} w \left(\frac{d^2 u}{dx^2} + u + 1 \right) d\Omega = 0$$

weight function

→ mathematical procedure (integration by parts)

$$\int_0^1 w \frac{d^2 u}{dx^2} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_0^1 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

ID PROBLEM - WEAK FORM

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

apply boundary conditions to the equation
in the continuous form:

$$\cancel{w(1) \frac{du}{dx}(1)} - \cancel{w(0) \frac{du}{dx}(0)} - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w(1) - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^4 N_i(x)u_i$ $\hat{w} = \sum_{j=1}^4 N_j(x)w_j$

$$w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0) - \sum_{i=j=1}^4 \int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx + \\ + \sum_{i=j=1}^4 \int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^4 \int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \mathbf{u}_i = \sum_{j=1}^4 \int_0^1 N_j dx + \boxed{w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0)}$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

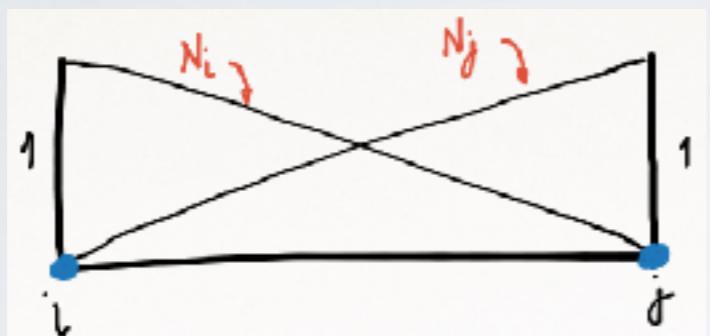
$$(K_{ij} - M_{ij}) \mathbf{u}_i = b_i + b.c.$$



FEM SHAPE FUNCTIONS

ID Problem - linear:

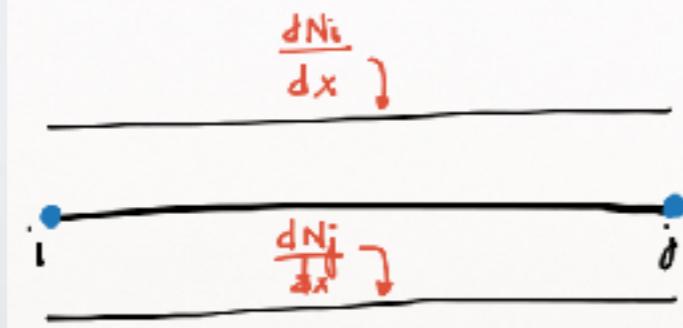
$$T(x) = \alpha_1 + \alpha_2 x$$



shape
function



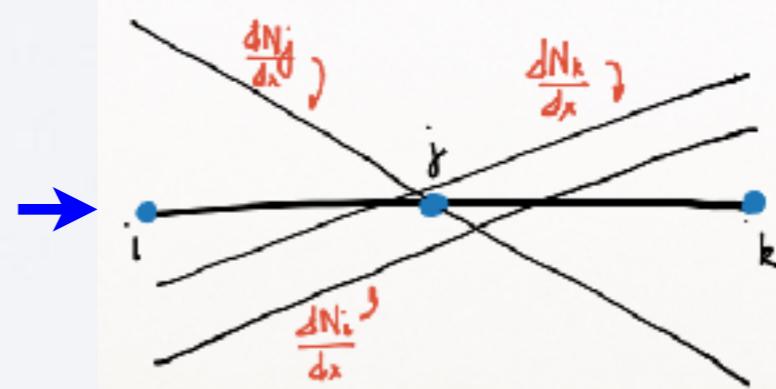
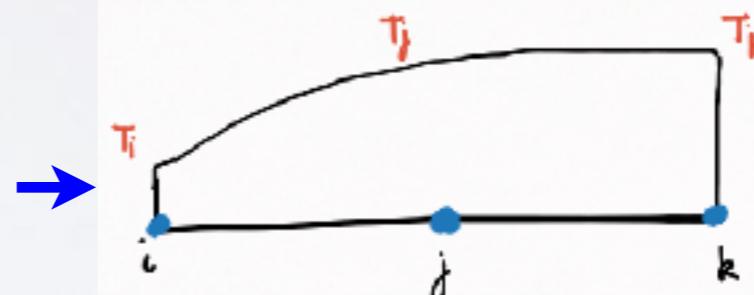
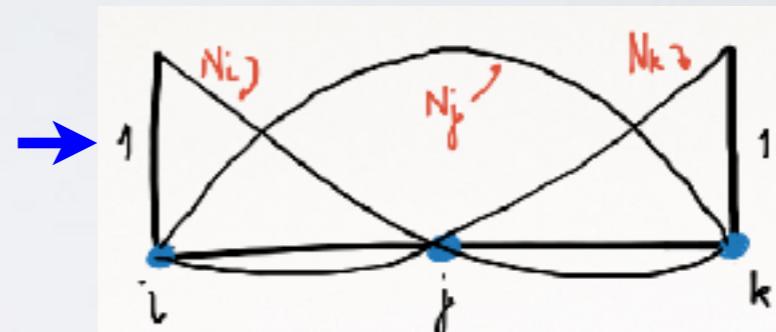
function



shape
function
derivative

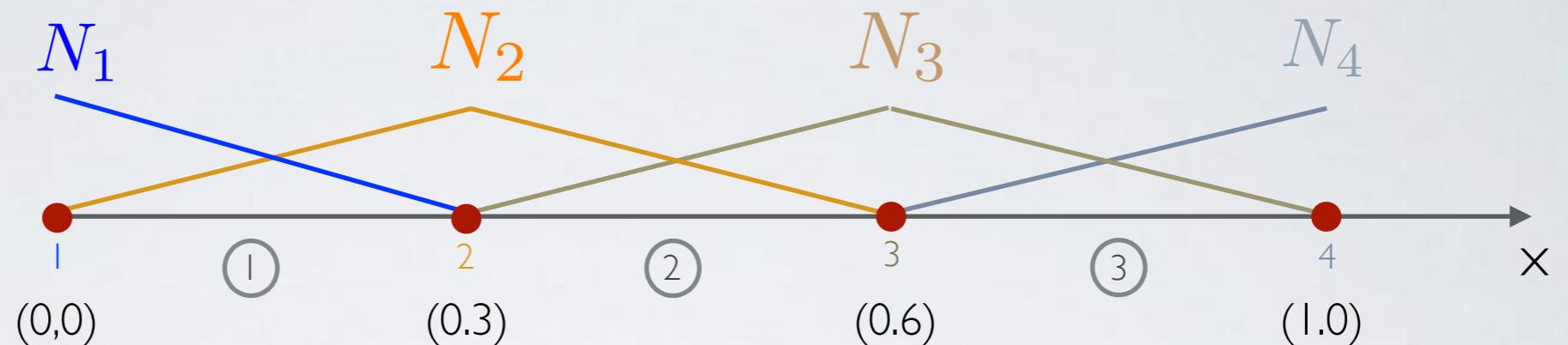
ID problem - quadratic:

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



ID PROBLEM - LINEAR

domain
and shape
functions:



element ① $N_1 = -3x + 1$
 $\Omega_1^e = [0, 1/3]$ $N_2 = 3x$

element ② $N_2 = -3x + 2$
 $\Omega_2^e = [1/3, 2/3]$ $N_3 = 3x - 1$

element ③ $N_3 = -3x + 3$
 $\Omega_3^e = [2/3, 1]$ $N_4 = 3x - 2$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element ①

$$\Omega_1^e = [0, 1/3]$$

$$N_1 = -3x + 1$$

$$N_2 = 3x$$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element ②

$$\Omega_2^e = [1/3, 2/3]$$

$$N_2 = -3x + 2$$

$$N_3 = 3x - 1$$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_4 N_4 dx$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element ③

$$\Omega_3^e = [2/3, 1]$$

$$N_3 = -3x + 3$$

$$N_4 = 3x - 2$$

$$b_3 = \int_{2/3}^1 N_3 dx$$

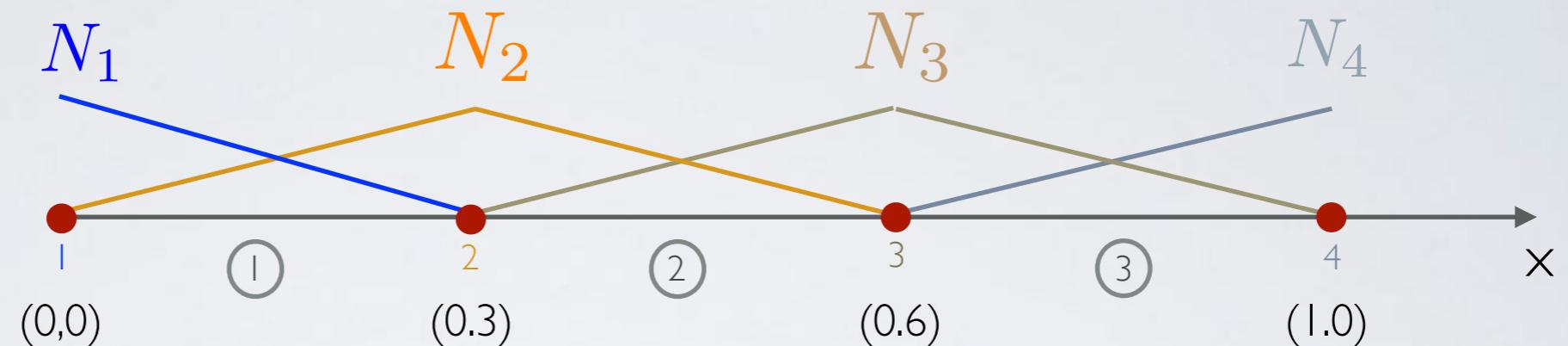
$$b_4 = \int_{2/3}^1 N_4 dx$$

vector

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

MATRIX FORM

domain
and shape
functions:



element $\textcircled{1}$
 $\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

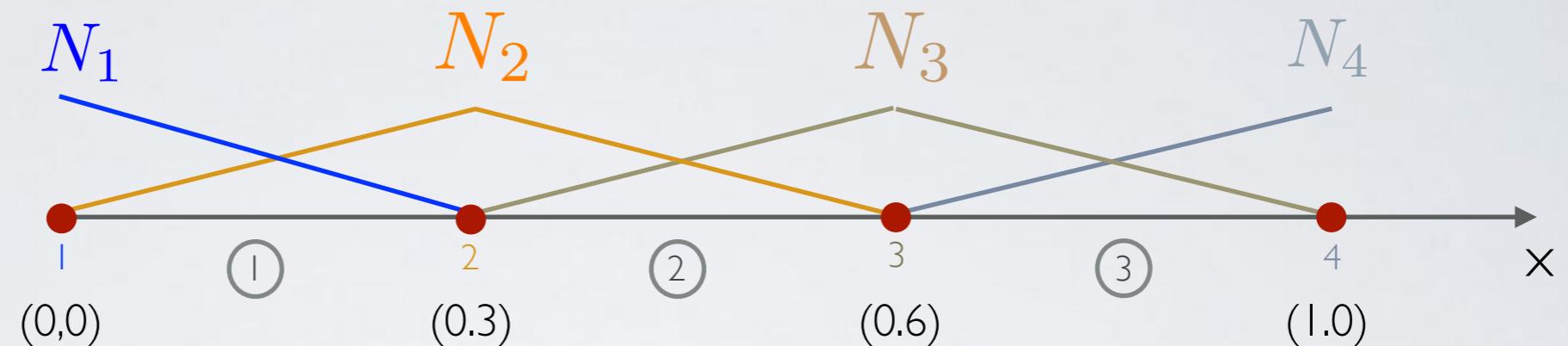
b.c. at $x = 0$
does not exist!

$$w(0) = 0$$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain
and shape
functions:



element ②
 $\Omega_2^e = [1/3, 2/3]$

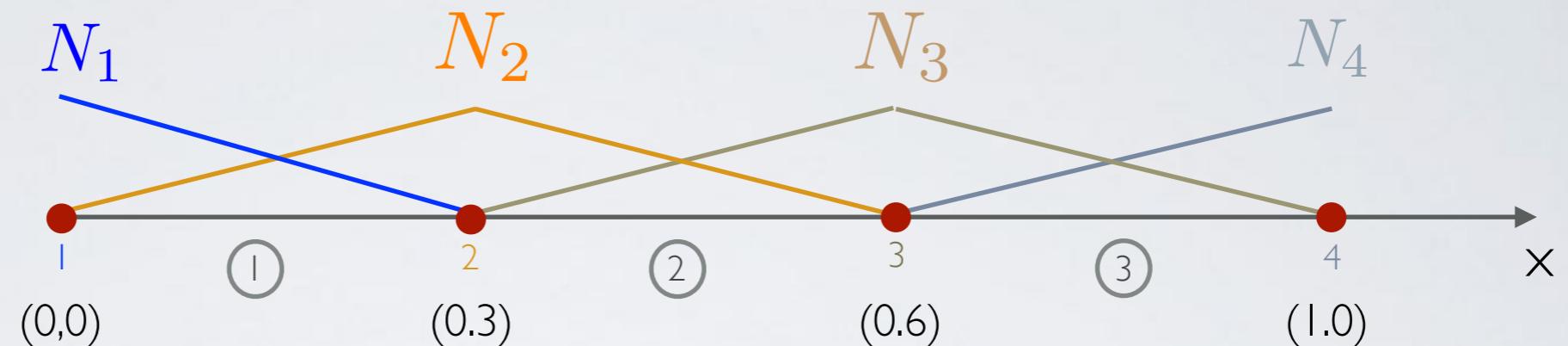
$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain
and shape
functions:



element ③
 $\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

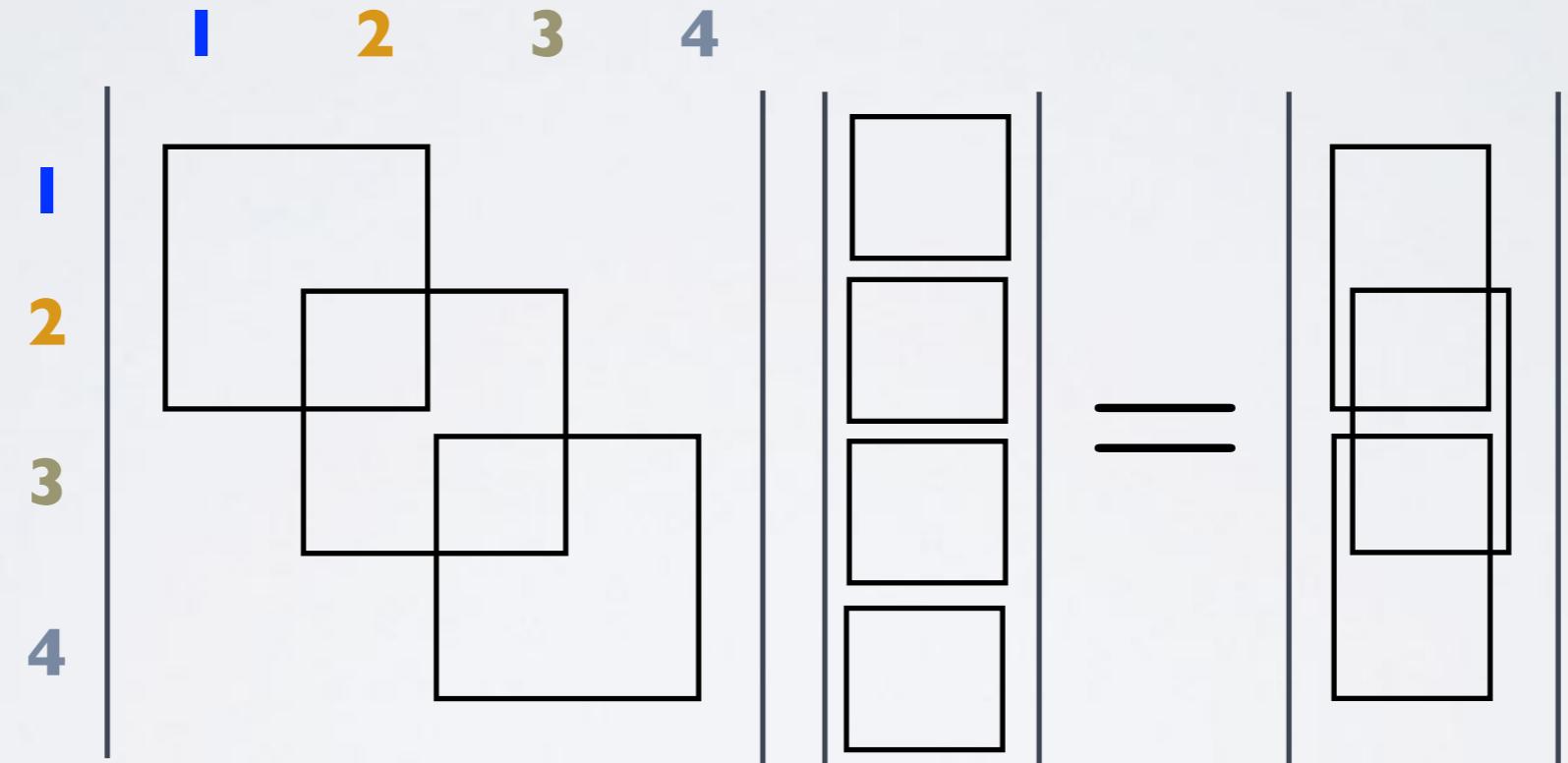
b.c. at $x = 1$

$$\begin{aligned} w(1) &= N_4(1) \\ &= 3 \cdot 1 - 2 \\ &= 1 \end{aligned}$$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 + \text{b.c. at } x = 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 + 1 \end{bmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$



$$K_{ij} - M_{ij}$$

$$\textcolor{red}{u}_i$$

$$b_i + b.c.$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

$$\begin{array}{c|cc|c|c}
 & 1 & 2 & 3 & 4 \\ \hline
 1 & KM_{11}^1 & KM_{12}^1 & & u_1 \\
 2 & KM_{21}^1 & KM_{22}^1 & & u_2 \\
 3 & & & & u_3 \\
 4 & & & & u_4 \\
 \end{array} = \begin{array}{c|c}
 & b_1^1 + \text{b.c.} \\
 & b_2^1 \\
 \hline
 & \parallel \\
 & \parallel
 \end{array}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4			
1		KM_{11}^1	KM_{12}^1		u_1		$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1	$+ KM_{22}^2$	u_2		$b_2^1 + b_2^2$
3			KM_{32}^2	KM_{33}^2	u_3	$=$	b_3^2
4					u_4		

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4			
1		KM_{11}^1	KM_{12}^1		u_1		$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1	KM_{23}^2	u_2		$b_2^1 + b_2^2$
3		KM_{32}^2	KM_{33}^2	KM_{43}^3	u_3	$=$	$b_3^2 + b_3^3$
4			KM_{34}^3	KM_{44}^3	u_4		$b_3^3 + \text{b.c.}$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4	$u_1(0) = 0$	
1		KM_{11}^1	KM_{12}^1			$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1 + KM_{22}^2	KM_{23}^2	u_1	$b_2^1 + b_2^2$
3			KM_{32}^2 + KM_{33}^3	KM_{43}^3	u_2	$b_3^2 + b_3^3$
4				KM_{34}^3	u_3	$b_3^3 + \text{b.c.}$
					u_4	

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4		how to remove this line?
1		KM_{11}^1	KM_{12}^1		0	$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1 + KM_{22}^2	KM_{23}^2	u_2	$b_2^1 + b_2^2$
3		KM_{32}^2	KM_{33}^2 + KM_{33}^3	KM_{43}^3	u_3	$b_3^2 + b_3^3$
4			KM_{34}^3	KM_{44}^3	u_4	$b_3^3 + \text{b.c.}$

ASSEMBLING

writing down the equation of line 2

$$KM_{21}^1 * \textcolor{red}{u_1} + (KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2$$

subtracting $KM_{21}^1 * \textcolor{red}{u_1}$ from both sides:

$$(KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2 - KM_{21}^1 * \textcolor{red}{u_1}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4		
1	KM_{11}^1	KM_{12}^1			0	$b_1^1 + \text{b.c.}$
2	KM_{21}^1	KM_{22}^1 $+ KM_{22}^2$	KM_{23}^2		u_2	$b_2^1 + b_2^2$
3		KM_{32}^2	KM_{33}^2 $+ KM_{33}^3$	KM_{43}^3	u_3	\equiv $b_3^2 + b_3^3$
4			KM_{34}^3	KM_{44}^3	u_4	$b_3^3 + \text{b.c.}$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	2	3	4		
2				u_2	$=$ $b_2^1 + b_2^2$
3				u_3	$=$ $b_3^2 + b_3^3$
4				u_4	$b_3^3 + \text{b.c.}$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

$$\begin{array}{c|ccc|c|c}
 & \textcolor{orange}{1} & \textcolor{brown}{2} & \textcolor{blue}{3} & & \\
 \textcolor{orange}{1} & KM_{22}^1 & KM_{23}^2 & & u_2 & b_2^1 + b_2^2 - KM_{21}^1 * \textcolor{red}{u}_1 \\
 \textcolor{brown}{2} & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 & b_3^2 + b_3^3 \\
 \textcolor{blue}{3} & & KM_{34}^3 & KM_{44}^3 & u_4 & b_3^3 + \text{b.c.}
 \end{array}$$

SOLVING LINEAR SYSTEM

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

solving for $\textcolor{red}{u}_i$:

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})\textcolor{red}{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + \text{b.c.}$$

$$\textcolor{red}{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + \text{b.c.}$$

how to compute $(K_{ij} - M_{ij})^{-1}$?

direct methods: **not recommended!**

iterative methods: **recommended!**

ID PROBLEM

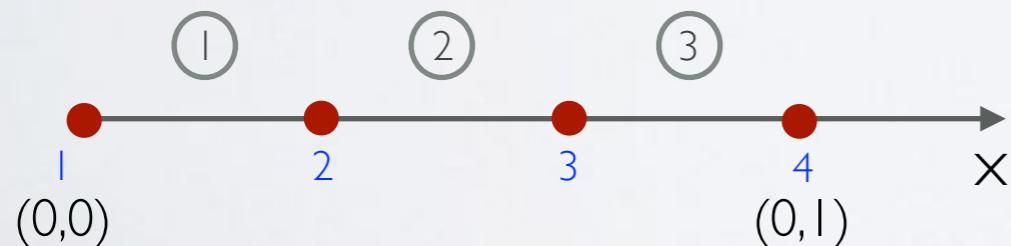
Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0$$

$$\left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = -u \end{array} \right.$$

boundary condition

domain: $h_1 = h_2 = h_3 = 1/3$



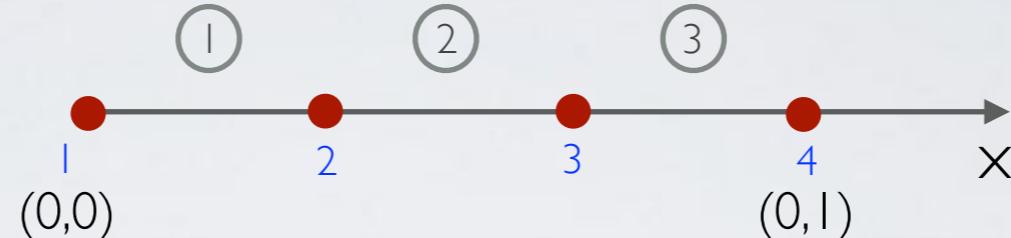
Answer: $u_2=0.251$;

$u_3=0.363$;

$u_4=0.328$

MESH GENERATION

mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```

1: for  $elem \leftarrow 1, NE$  do                                     → NE = Total number of elements
2:   for  $i_{local} \leftarrow 1, 2$  do                         →  $i_{local} = [1, 2]$ 
3:      $i_{global} \leftarrow IEN[elem, i_{local}]$            →  $i_{global} = [v_1, v_2]$ 
4:     for  $j_{local} \leftarrow 1, 2$  do                         →  $j_{local} = [1, 2]$ 
5:        $j_{global} \leftarrow IEN[elem, j_{local}]$            →  $j_{global} = [v_1, v_2]$ 
6:        $K[i_{global}, j_{global}] \leftarrow K[i_{global}, j_{global}] + k_{elem}[i_{local}, j_{local}]$ 
7:     end for
8:   end for
9: end for

```

- loop on elements (l)
- loop on neighbors (i=1,2) and loop (j=1,2)
- conversion between local to global node
- assembling of stiffness matrix using SUM

ID PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_{\Omega} w \left(\frac{d^2 u}{dx^2} + u + 1 \right) d\Omega = 0$$

weight function

→ mathematical procedure (integration by parts)

$$\int_0^1 w \frac{d^2 u}{dx^2} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_0^1 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

1D PROBLEM - WEAK FORM

$$w \frac{du}{dx} \Big|_1 - w \frac{du}{dx} \Big|_0 - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

apply boundary conditions to the equation
in the continuous form:

$$\cancel{w(1) \frac{du}{dx}(1)}^{\rightarrow -u} - \cancel{w(0) \frac{du}{dx}(0)}^{\rightarrow 0} - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

$$-w(1)u(1) - \int_0^1 \frac{du}{dx} \frac{dw}{dx} dx + \int_0^1 w u dx + \int_0^1 w dx = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^4 N_i(x)u_i$ $\hat{w} = \sum_{j=1}^4 N_j(x)w_j$

$$w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0) - \sum_{i=j=1}^4 \int_0^1 \frac{dN_i}{dx} u_i \frac{dN_j}{dx} w_j dx + \\ + \sum_{i=j=1}^4 \int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^4 \int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \mathbf{u}_i = \sum_{j=1}^4 \int_0^1 N_j dx + \boxed{w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0)}$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

$$(K_{ij} - M_{ij}) \mathbf{u}_i = b_i + b.c.$$

GALERKIN METHOD

note that the b.c. has $-u(1)$:

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \mathbf{u}_i = \sum_{j=1}^4 \int_0^1 N_j dx + \boxed{w(1) \frac{du}{dx}(1) - w(0) \frac{du}{dx}(0)}$$

stiffness matrix K_{ij}
mass matrix M_{ij}
right hand side. b_i
boundary condition

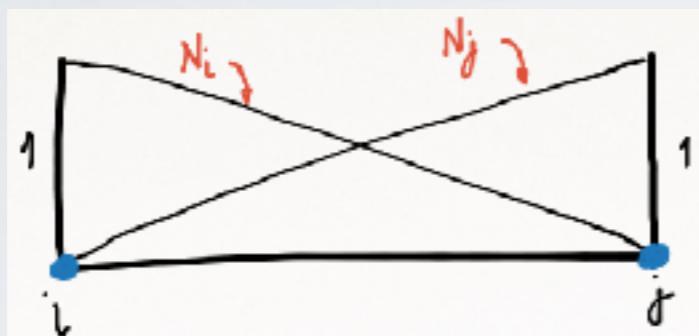
$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \mathbf{u}_i = \sum_{j=1}^4 \int_0^1 N_j dx - w(1) \cancel{u(1)} - w(0) \frac{du}{dx}(0)$$

$$\sum_{i=1}^4 \sum_{j=1}^4 \left(\int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx} dx - \int_0^1 N_i N_j dx \right) \mathbf{u}_i + w(1) \cancel{u(1)} = \sum_{j=1}^4 \int_0^1 N_j dx - w(0) \frac{du}{dx}(0)$$

FEM SHAPE FUNCTIONS

ID Problem - linear:

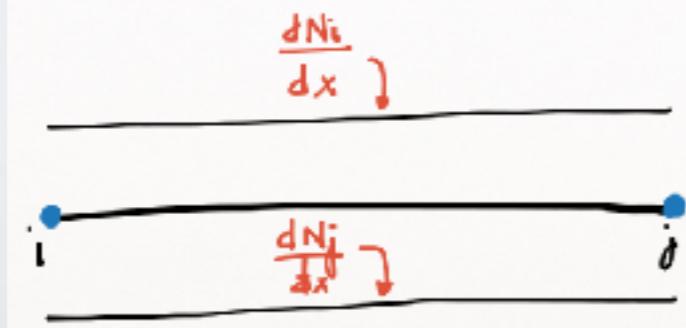
$$T(x) = \alpha_1 + \alpha_2 x$$



shape
function



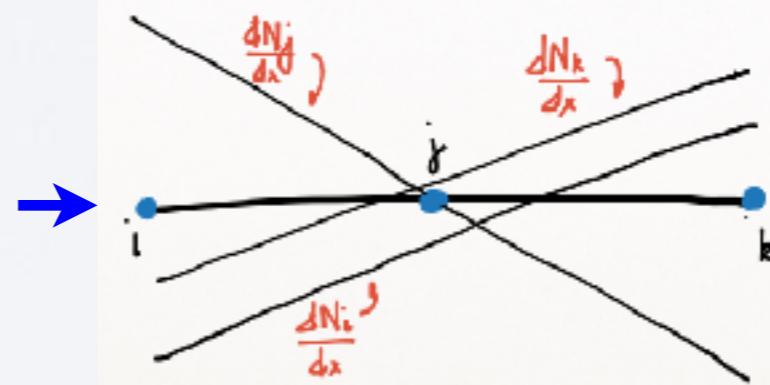
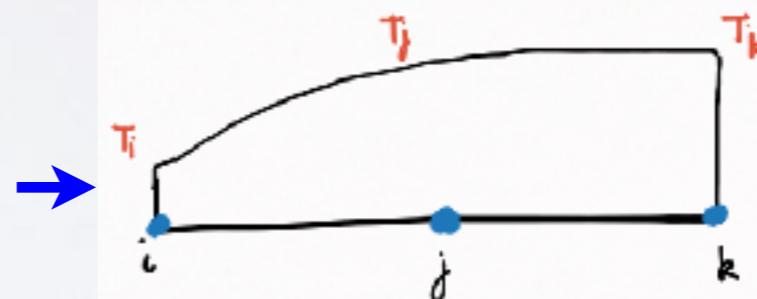
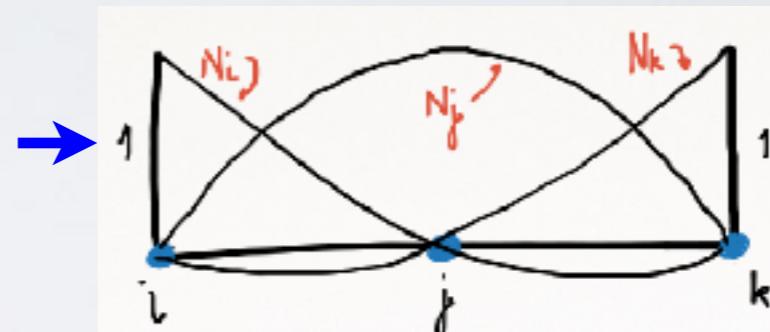
function



shape
function
derivative

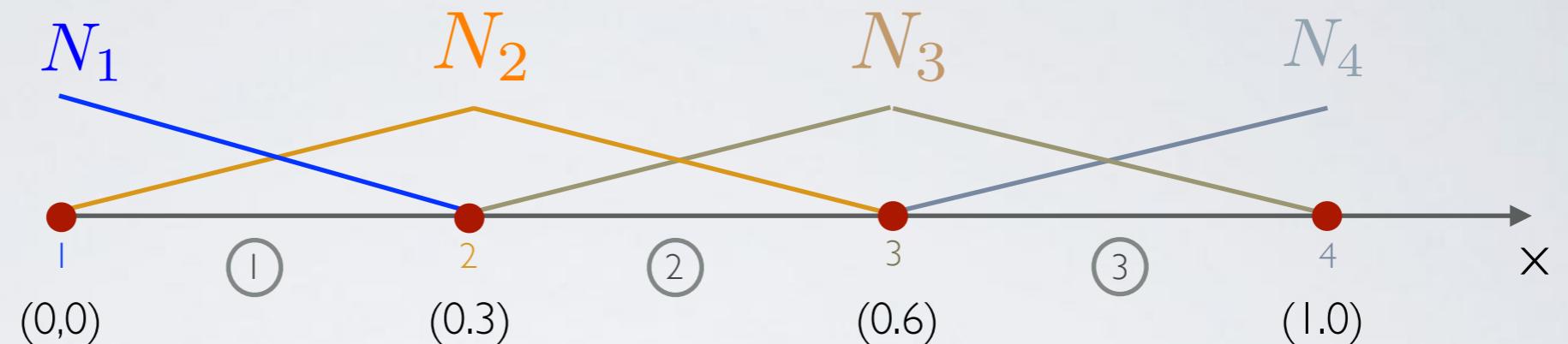
ID problem - quadratic:

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



ID PROBLEM - LINEAR

domain
and shape
functions:



element ① $N_1 = -3x + 1$
 $\Omega_1^e = [0, 1/3]$ $N_2 = 3x$

element ② $N_2 = -3x + 2$
 $\Omega_2^e = [1/3, 2/3]$ $N_3 = 3x - 1$

element ③ $N_3 = -3x + 3$
 $\Omega_3^e = [2/3, 1]$ $N_4 = 3x - 2$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_1}{dx} dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element ①

$$\Omega_1^e = [0, 1/3]$$

$$N_1 = -3x + 1$$

$$N_2 = 3x$$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_2}{dx} dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element ②

$$\Omega_2^e = [1/3, 2/3]$$

$$N_2 = -3x + 2$$

$$N_3 = 3x - 1$$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^1 \frac{dN_3}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_3}{dx} dx - \int_{2/3}^1 N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^1 \frac{dN_4}{dx} \frac{dN_4}{dx} dx - \int_{2/3}^1 N_4 N_4 dx + N_4(1)N_4(1)$$

element ③

$$\Omega_3^e = [2/3, 1]$$

$$N_3 = -3x + 3$$

$$N_4 = 3x - 2$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

vector

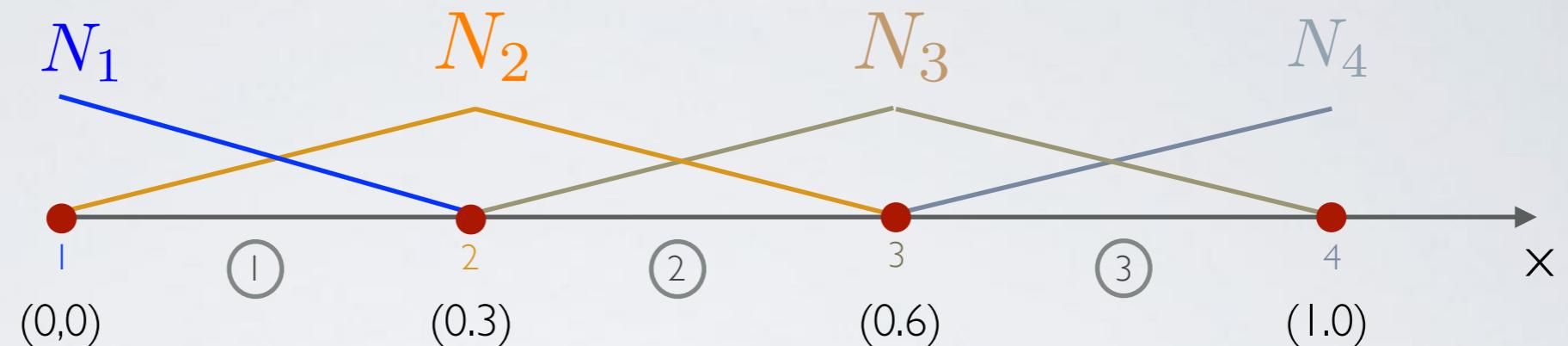
$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

$$b_3 = \int_{2/3}^1 N_3 dx$$

$$b_4 = \int_{2/3}^1 N_4 dx$$

MATRIX FORM

domain
and shape
functions:



element $\textcircled{1}$
 $\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

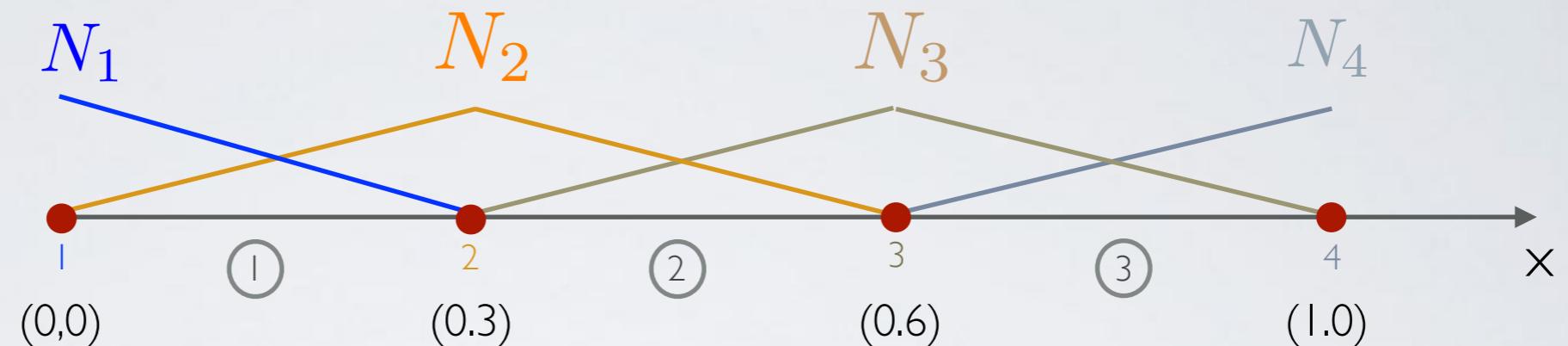
b.c. at $x = 0$
does not exist!

$$w(0) = 0$$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain
and shape
functions:



element ②
 $\Omega_2^e = [1/3, 2/3]$

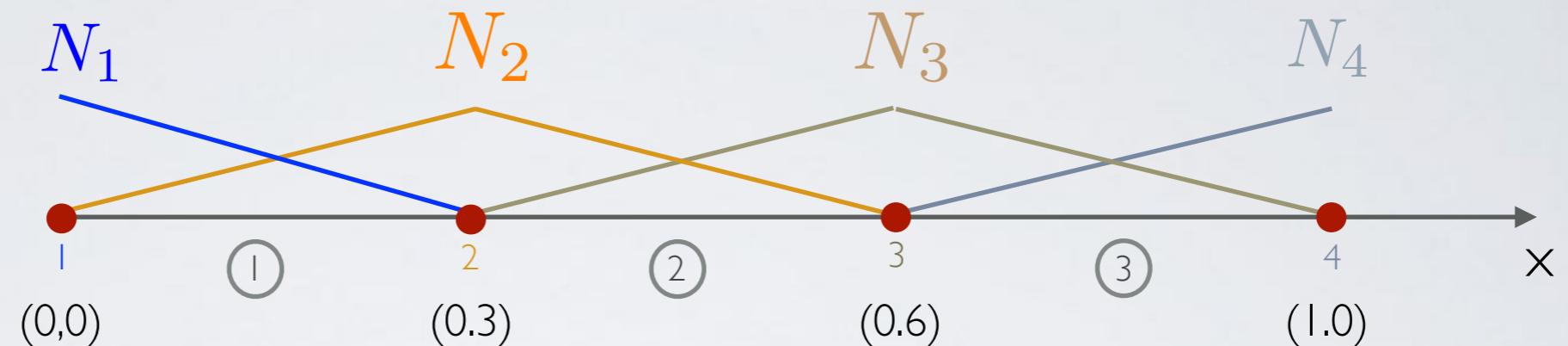
$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain
and shape
functions:



element $\textcircled{3}$
 $\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} + 1 \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{70}{18} \end{bmatrix}$$

note that:

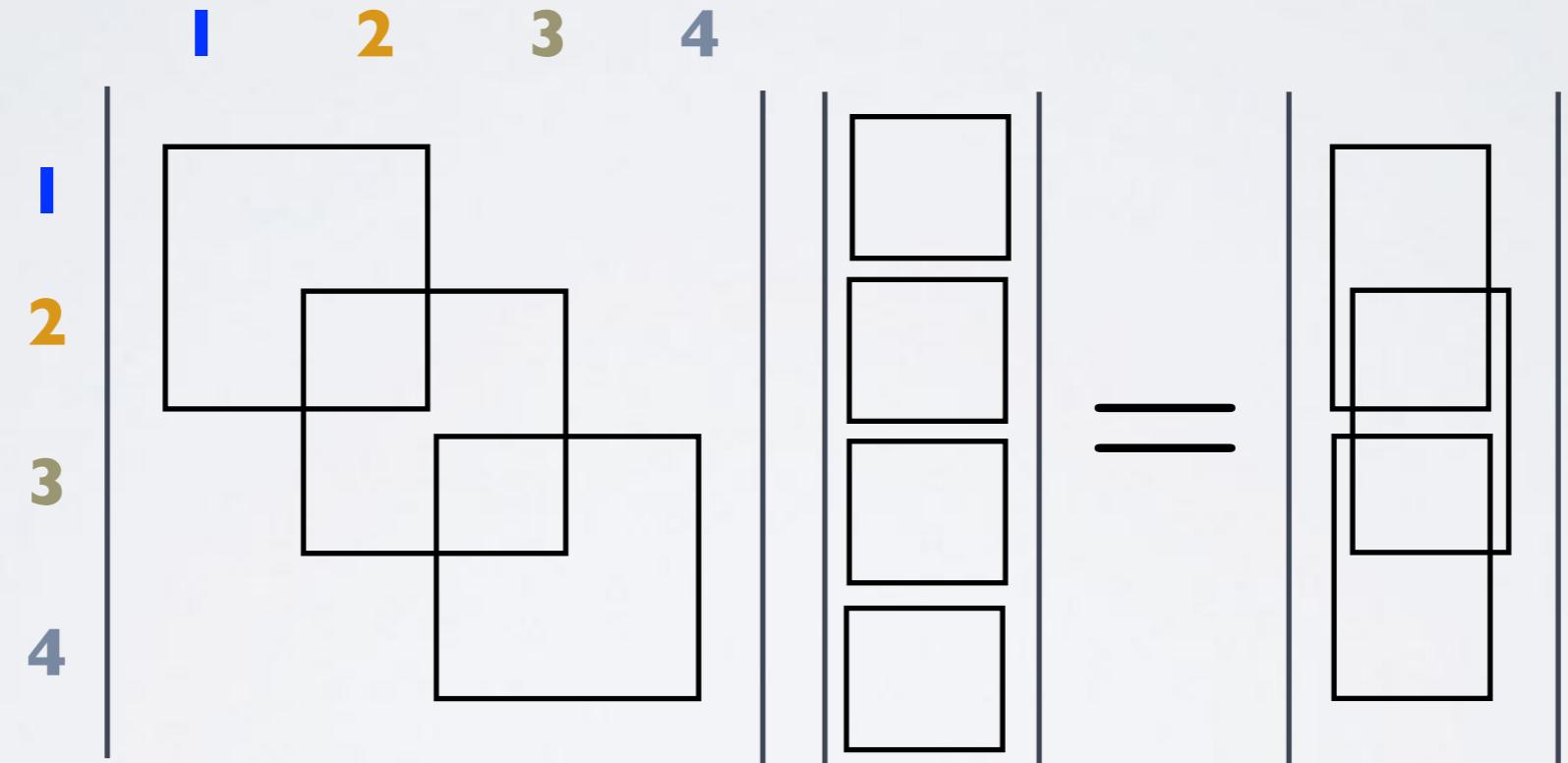
b.c. at $x = 1$

$$-w(1)u(1) = -N_4(1) * N_4(1)$$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$



$$K_{ij} - M_{ij}$$

$$\textcolor{red}{u}_i$$

$$b_i + b.c.$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

$$\begin{array}{c|cc|c|c}
 & 1 & 2 & 3 & 4 \\ \hline
 1 & KM_{11}^1 & KM_{12}^1 & & u_1 \\
 2 & KM_{21}^1 & KM_{22}^1 & & u_2 \\
 3 & & & & u_3 \\
 4 & & & & u_4 \\
 \end{array} = \begin{array}{c|c}
 & b_1^1 + \text{b.c.} \\
 & b_2^1 \\
 \hline
 & \parallel \\
 & \parallel
 \end{array}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4			
1		KM_{11}^1	KM_{12}^1		u_1		$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1	$+ KM_{22}^2$	u_2		$b_2^1 + b_2^2$
3			KM_{32}^2	KM_{33}^2	u_3	$=$	b_3^2
4					u_4		

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

$$\begin{array}{c|ccccc}
 & \textcolor{blue}{1} & \textcolor{orange}{2} & \textcolor{brown}{3} & \textcolor{teal}{4} & \\
 \hline
 \textcolor{blue}{1} & KM_{11}^1 & KM_{12}^1 & & & u_1 \\
 \textcolor{orange}{2} & KM_{21}^1 & KM_{22}^1 & KM_{23}^2 & & u_2 \\
 \textcolor{brown}{3} & & KM_{32}^2 & KM_{33}^2 & KM_{43}^3 & u_3 \\
 \textcolor{teal}{4} & & & KM_{34}^3 & KM_{44}^3 & u_4 \\
 \hline
 & & & & & b_1^1 + \text{b.c.} \\
 & & & & & b_2^1 + b_2^2 \\
 & & & & \hline \hline & b_3^2 + b_3^3 \\
 & & & & & b_3^3
 \end{array}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4	$u_1(0) = 0$	
1		KM_{11}^1	KM_{12}^1			$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1 + KM_{22}^2	KM_{23}^2	u_2	$b_2^1 + b_2^2$
3			KM_{32}^2 + KM_{33}^3	KM_{43}^3	u_3	$b_3^2 + b_3^3$
4				KM_{34}^3	u_4	b_3^3

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4		how to remove this line?
1		KM_{11}^1	KM_{12}^1		0	$b_1^1 + \text{b.c.}$
2		KM_{21}^1	KM_{22}^1 + KM_{22}^2	KM_{23}^2	u_2	$b_2^1 + b_2^2$
3		KM_{32}^2	KM_{33}^2 + KM_{33}^3	KM_{43}^3	u_3	$b_3^2 + b_3^3$
4			KM_{34}^3	KM_{44}^3	u_4	b_3^3

ASSEMBLING

writing down the equation of line 2

$$KM_{21}^1 * \textcolor{red}{u_1} + (KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2$$

subtracting $KM_{21}^1 * \textcolor{red}{u_1}$ from both sides:

$$(KM_{22}^1 + KM_{22}^2) * u_2 + KM_{23}^2 = b_2^1 + b_2^2 - KM_{21}^1 * \textcolor{red}{u_1}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

	1	2	3	4		
1	KM_{11}^1	KM_{12}^1			0	$b_1^1 + \text{b.c.}$
2	KM_{21}^1	KM_{22}^1 $+ KM_{22}^2$	KM_{23}^2		u_2	$b_2^1 + b_2^2$
3		KM_{32}^2	KM_{33}^2 $+ KM_{33}^3$	KM_{43}^3	u_3	$b_3^2 + b_3^3$
4			KM_{34}^3	KM_{44}^3	u_4	b_3^3

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$

	2	3	4		
2				u_2	$=$ $b_2^1 + b_2^2$
3				u_3	$=$ $b_3^2 + b_3^3$
4				u_4	b_3^3

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + b.c.$

$$\begin{array}{c|ccc|c|c}
 & \textcolor{brown}{1} & \textcolor{brown}{2} & \textcolor{brown}{3} & & \\
 \textcolor{brown}{1} & KM_{22}^1 & KM_{23}^2 & & u_2 & b_2^1 + b_2^2 - KM_{21}^1 * \textcolor{red}{u}_1 \\
 \textcolor{brown}{2} & KM_{22}^2 & KM_{33}^2 & KM_{43}^3 & u_3 & b_3^2 + b_3^3 \\
 \textcolor{brown}{3} & KM_{32}^2 & KM_{33}^3 & KM_{43}^3 & u_4 & b_3^3 \\
 & KM_{34}^3 & KM_{44}^3 & & &
 \end{array}$$

SOLVING LINEAR SYSTEM

linear system of equations: $(K_{ij} - M_{ij})\textcolor{red}{u}_i = b_i + \text{b.c.}$

solving for $\textcolor{red}{u}_i$:

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})\textcolor{red}{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + \text{b.c.}$$

$$\textcolor{red}{u}_i = (K_{ij} - M_{ij})^{-1} + b_i + \text{b.c.}$$

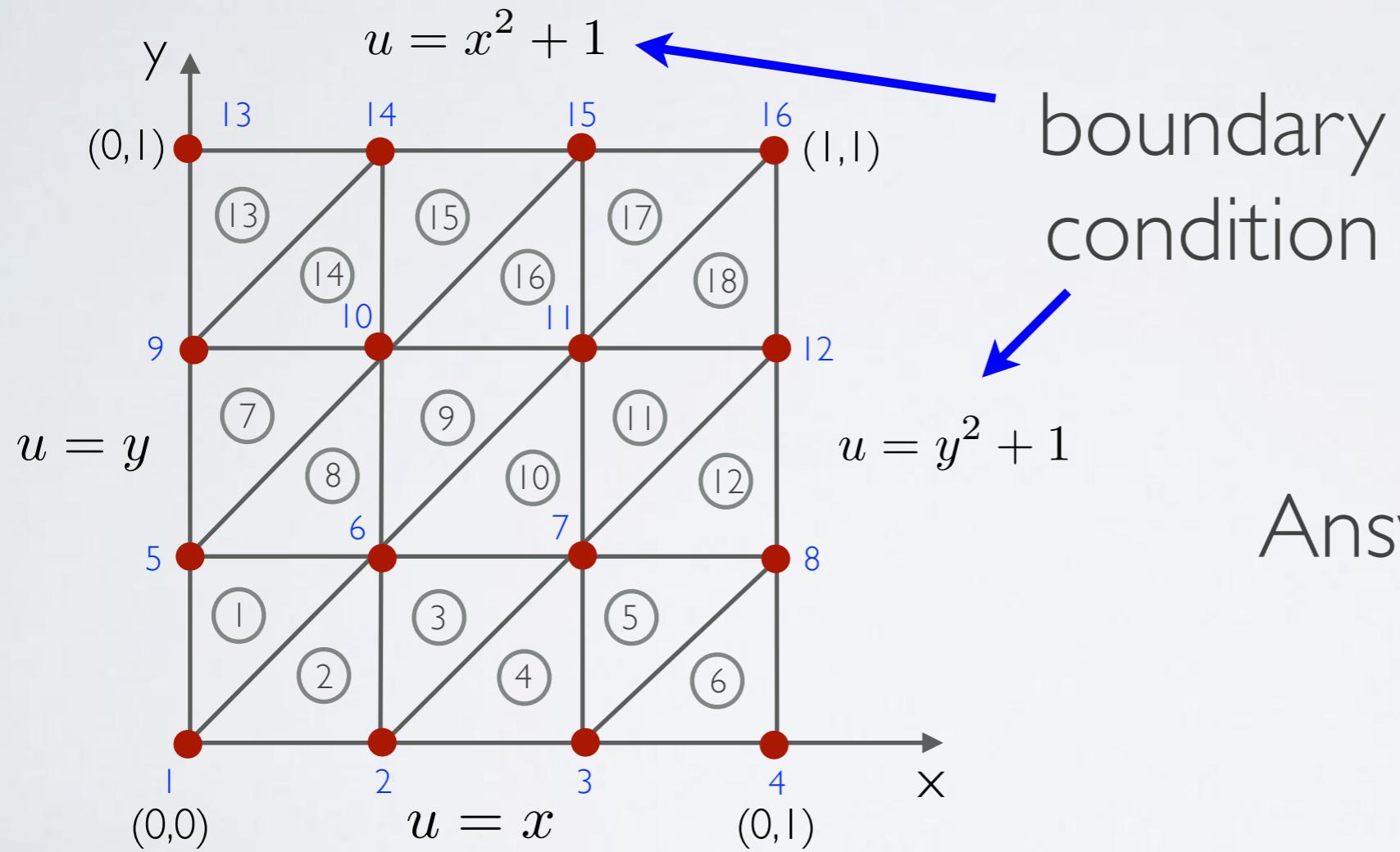
how to compute $(K_{ij} - M_{ij})^{-1}$?

direct methods: **slow and high memory consumption!**

iterative methods: **faster!**

2D PROBLEM

Find u in $\Omega = [0, 1] \times [0, 1]$ such that:



Equation:

$$\nabla^2 u = 0$$

Answer: $u_6=0.611$;
 $u_7=0.889$;
 $u_{10}=0.889$;
 $u_{11}=1.167$

MESH GENERATION

element	node 1	node 2	node 3
1	1	6	5
2	1	2	6
3	2	7	6
4	2	3	7
5	3	8	7
6	3	4	8
7	5	10	9
8	5	6	10
9	6	11	10
10	6	7	11
11	7	12	11
12	7	8	12
13	9	14	13
14	9	10	14
15	10	15	14
16	10	11	15
17	11	16	15
18	11	12	16

node	not b.c.
1	0
2	0
3	0
4	0
5	0
6	1
7	2
8	0
9	0
10	3
11	4
12	0
13	0
14	0
15	0
16	0

node	X	Y
1	0	0
2	1/3	0
3	2/3	0
4	1	0
5	0	1/3
6	1/3	1/3
7	2/3	1/3
8	1	1/3
9	0	2/3
10	1/3	2/3
11	2/3	2/3
12	1	2/3
13	0	1
14	1/3	1
15	2/3	1
16	1	1

node	b.c. value
1	0
2	1/3
3	2/3
4	1
5	1/3
6	-
7	-
8	10/9
9	2/3
10	-
11	-
12	13/9
13	1
14	10/9
15	13/9
16	2

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```

1: for  $elem \leftarrow 1, NE$  do                                     →  $NE =$  Total number of elements
2:   for  $i_{local} \leftarrow 1, 3$  do                         →  $i_{local} = [1, 2, 3]$ 
3:      $i_{global} \leftarrow IEN[elem, i_{local}]$            →  $i_{global} = [v_1, v_2, v_3]$ 
4:     for  $j_{local} \leftarrow 1, 3$  do                         →  $j_{local} = [1, 2, 3]$ 
5:        $j_{global} \leftarrow IEN[elem, j_{local}]$            →  $j_{global} = [v_1, v_2, v_3]$ 
6:        $K[i_{global}, j_{global}] \leftarrow K[i_{global}, j_{global}] + k_{elem}[i_{local}, j_{local}]$ 
7:     end for
8:   end for
9: end for

```

- loop on elements (l)
- loop on neighbors (i=1,2,3) and loop (j=1,2,3)
- conversion between local to global node
- assembling of stiffness matrix using SUM

2D PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_{\Omega} w \left(\nabla^2 u \right) d\Omega = 0$$

↗ weight function

→ mathematical procedure (Green theorem)

$$\int_{\Omega} w \nabla^2 u d\Omega = \int_{\Omega} w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega = 0$$

$$\left. \nabla u w \right|_{\Gamma} - \int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

2D PROBLEM - WEAK FORM

$$\left. \nabla u w \right|_{\Gamma} - \int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

apply boundary conditions to the equation
in the continuous form:

$$\left. \nabla u w \right|_{\Gamma}^0 - \int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

$$\int_{\Omega} \nabla w \cdot \nabla u d\Omega = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{16} N_i(x, y) u_i$ $\hat{w} = \sum_{j=1}^{16} N_j(x, y) w_j$

$$K^e = \int_{\Omega} B^T B d\Omega$$

formula

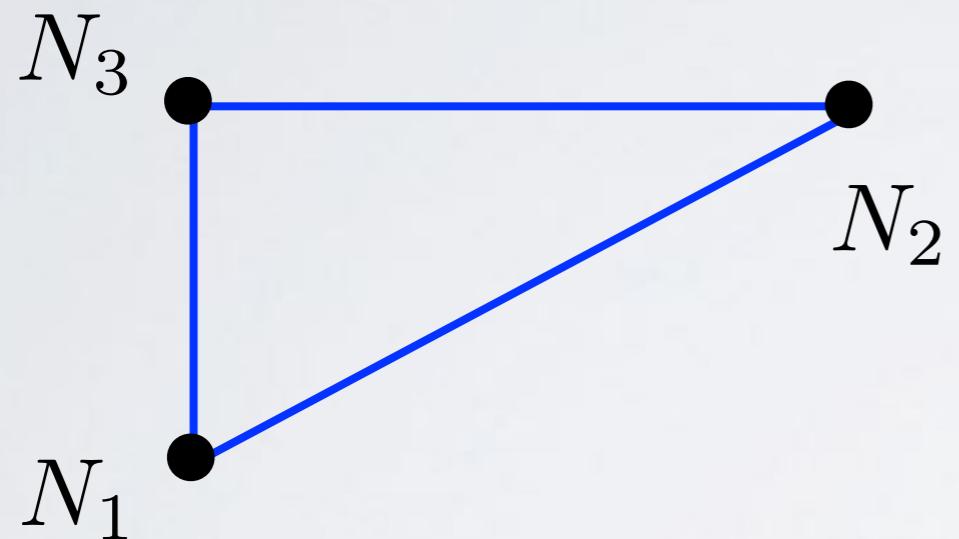
where:

$$B = \begin{bmatrix} \frac{\partial N(x,y)}{\partial x} \\ \frac{\partial N(x,y)}{\partial y} \end{bmatrix}$$

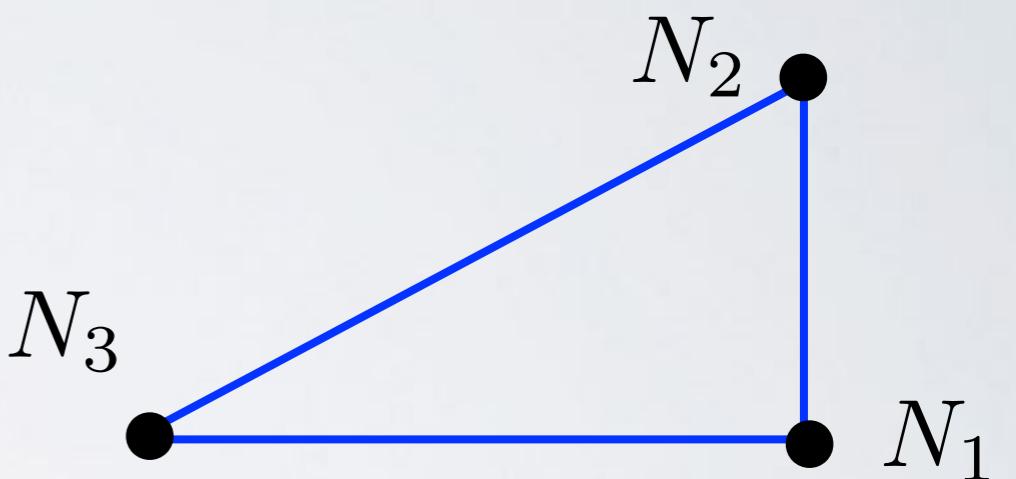
$$B = \begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial x} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial y} \end{bmatrix}$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{16} N_i(x, y) u_i$ $\hat{w} = \sum_{j=1}^{16} N_j(x, y) w_j$

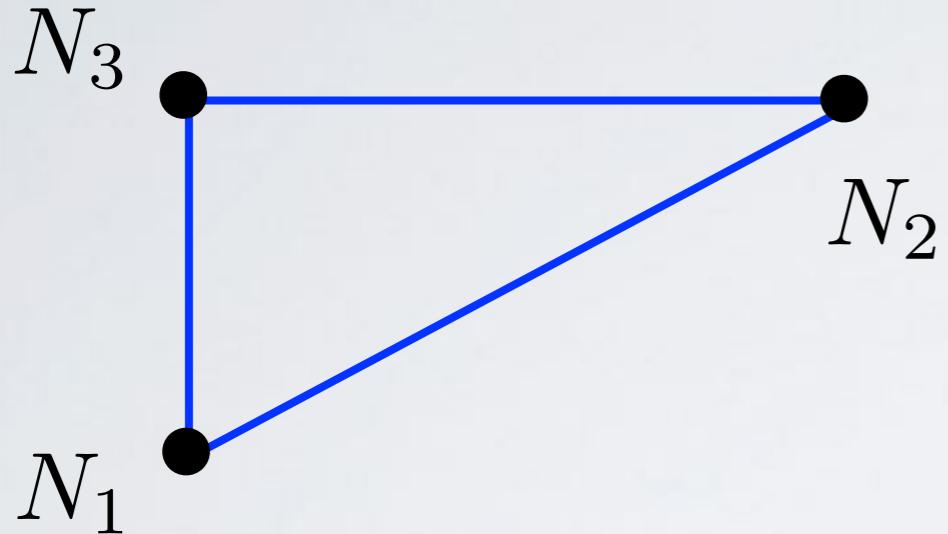


clockwise
or
counter
clockwise

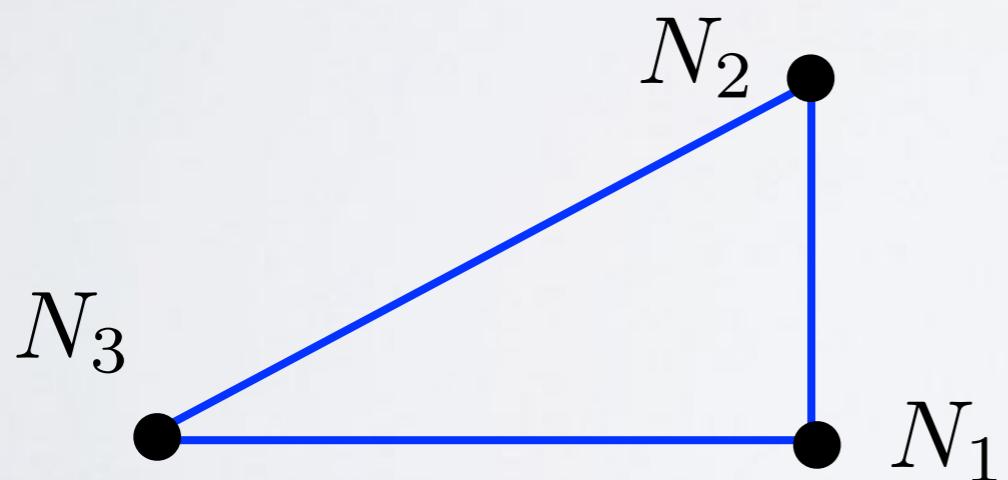


$$B = \frac{1}{2A} \begin{bmatrix} (y_j - y_k) & (y_k - y_i) & (y_i - y_j) \\ (x_j - x_k) & (x_i - x_k) & (x_j - x_i) \end{bmatrix}$$

ELEMENT MATRIX

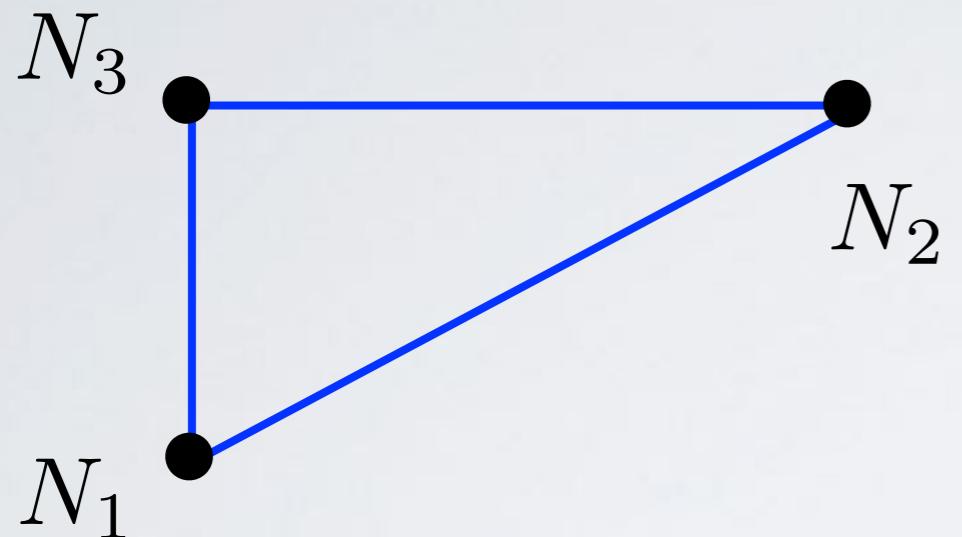


$$B = \frac{1}{2A} \begin{bmatrix} 0 & 1/3 & -1/3 \\ -1/3 & 0 & 1/3 \end{bmatrix}$$

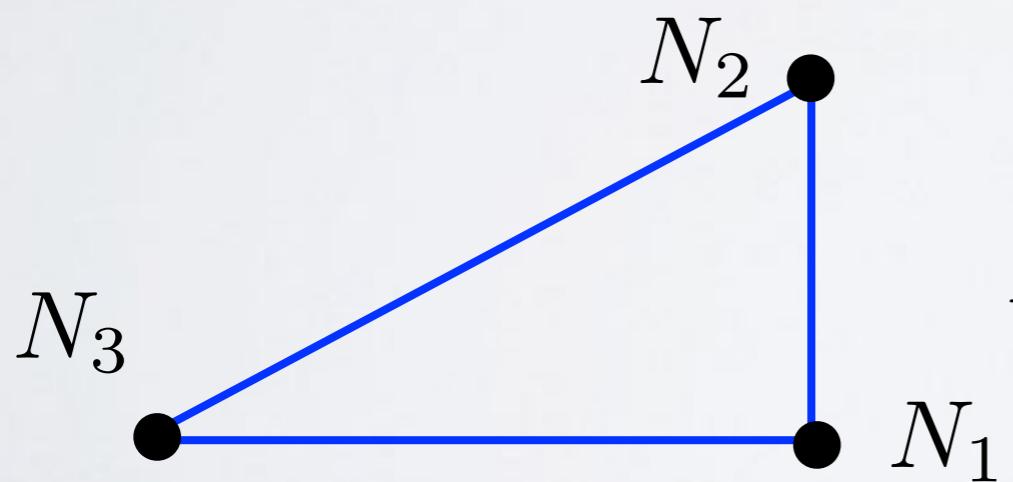


$$B = \frac{1}{2A} \begin{bmatrix} 1/3 & 0 & -1/3 \\ -1/3 & 1/3 & 0 \end{bmatrix}$$

ELEMENT MATRIX



$$B^T B = \frac{1}{4A} \begin{bmatrix} 1/9 & 0 & -1/9 \\ 0 & 1/9 & -1/9 \\ -1/9 & -1/9 & 2/9 \end{bmatrix}$$



$$B^T B = \frac{1}{4A} \begin{bmatrix} 2/9 & -1/9 & -1/9 \\ -1/9 & 1/9 & 0 \\ -1/9 & 0 & 1/9 \end{bmatrix}$$

ASSEMBLING

linear system of equations:

$$K_{ij} \mathbf{u}_i = b_i + \text{b.c.}$$

	1	2	3	4			
1	4	-1	-1	0	u_6		$\frac{2}{3}$
2	-1	4	0	-1	u_7		$\frac{16}{9}$
3	-1	0	4	-1	u_{10}	$=$	$\frac{16}{9}$
4	0	-1	-1	4	u_{11}	$=$	$\frac{26}{9}$