SPECTRAL THEORY IN HILBERT SPACES

Author: Alex Farah Pereira[1] *alexpereira@id.uff.br*

[1] Fluminense Federal University

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Introduction

In Linear Algebra courses, we have learned that any linear transformation between spaces of finite dimension can be represented by a matrix. The reciprocal is also true, that is, a matrix defines a linear transformation between finite-dimensional spaces. Besides that, we know that if V is a finite-dimensional Euclidean space, then a linear operator $T : V \longrightarrow V$ is self-adjoint if, and only if, there exists an orthonormal basis of V formed by eigenvectors of T. In this case, the matrix of T in such a base is a diagonal matrix. However, in the case of V is a space of infinite dimension, we can not represent T through a matrix, which makes it difficult to study the operator T. The objective of this mini-article is to present the main results of this theory in infinite-dimensional spaces, in particular, the theorem of spectral decomposition for compact and self-adjoint operators.

Hilbert Spaces

Let E be a vector space of $\mathbb{K}$ (real or complex). The inner product in E is a function

$$\langle \cdot , \cdot \rangle : E \times E \longrightarrow \mathbb{K}$$

that satisfies:

(P1) $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle \;\; \forall x_1, x_2, y \in E$

(P2) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle \;\; \forall x, y \in E\, ,\, \lambda \in \mathbb{C}$

(P3) $\langle x, y\rangle = \overline{\langle y, x\rangle} \;\; \forall x, y \in E$

(P4) $\langle x, x\rangle > 0 \;\; \forall x \neq 0$

The pair $(E, \langle \cdot , \cdot \rangle)$ is called Euclidean space. A Hilbert space is a Banach sapce with induced norm by the inner product.

For instance, $\mathbb{C}^n$ is an Hilbert space (of finite dimension) with

$$\langle x, y\rangle = \sum_{j=0}^{n} x_j \overline{y_j}$$

where $x, y \in \mathbb{C}^n$ and $\ell_2 = \{(x_n)_n \in \mathbb{C}\, ;\, \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$ is an Hilbert space (of infinite dimension) with

$$\langle x, y\rangle = \sum_{n=0}^{\infty} x_j \overline{y_j}$$

where $x = (x_n)_n, y = (y_n)_n \in \ell_2$.

Let H be a Hilbert space. A subset S of H is a complete orthonormal system when S is an orthonormal set (that is, they are all unit vetor and orthogonal to each other) of H and $S^\perp = \{0\}$, where $S^\perp$ denotes the orthogonal complement of S. It is possible to prove that a set $S = \{x_i\, ;\, i \in I\}$ is a complete orthonormal system in a Hilbert space H if, and only if, satisfies the so-called Parseval's Identity, that is, for all $x \in H$ we have

$$\|x\|^2 = \sum_{i \in I} |\langle x, x_i\rangle|^2.$$

It is important to note that the Gram-Schmidt orthogonalization process remains valid in Hilbert spaces of infinite dimension. In that way, it is always possible to obtain orthonormal bases in these spaces. What we can guarantee is even more: all Hilbert space admits a complete orthonormal system formed by eigenvectors of an operator $T : H \longrightarrow H$, where T satisfies some properties.

Spectral Theory

If V is a finite-dimensional vector space and T is an operator linear in V, then $\lambda \in \mathbb{C}$ is not an eigenvalue of T if, and only if, $(T - \lambda I)^{-1}$ exists. This comes from the fact that a linear operator T over a finite-dimensional space is injective if, and only if, it is surjective. In this case, or T is bijective or T is neither injective and surjective. In addition, T is a continuous operator. On an infinite dimension, the following questions arise: Let E be a normed space and $T : E \longrightarrow E$ a continuous operator in E. If λ is not eigenvalue, then we can state that $(T - \lambda I)$ is surjective? If so, can we say that $(T - \lambda I)^{-1}$ is continuous? We can not always get positivly answers from both questions. In this way we define:

- λ is a regular value of T when $(T - \lambda I)$ is bijective and its inverse is continuous.
- $\rho(T)$ is the set of regular values of T called resolvent set of T.

- $\sigma(T) = \mathbb{K} - \rho(T)$ is called spectrum of T.

If E is a Banach space, the Open Mapping Theorem guarantees that

$$\rho(T) = \{\lambda \in \mathbb{K}\,;\, (T - \lambda I)\, \text{is bijective}\}.$$

In this case, we must consider the concept of compact operators. An operator $T : E \longrightarrow E$ is said to be compact for every bounded sequence $(x_n)_n$ in E, a sequence $(T(x_n))_n$ has convergent subsequence in E. For example, integral operators are compact. Compact operators are very resemble those operators in finite-dimensional spaces since all operator is continuous and, moreover, the following result is worth: If E is a Banach space, $T : E \longrightarrow E$ is a compact operator and $\lambda \neq 0$, then $(T - \lambda I)$ is injective if, and only if, it is surjective. For spaces with infinite dimension, the diagonalization of an operator is presented by the;

Spectral Decomposition Theorem
for Compact and Self-Adjoint Operators:

Let H be a Hilbert space and $T : H \longrightarrow H$ a compact operator and self-adjoint. Then H admits a complete orthonormal system formed by eigenvectors of T. Moreover, there are sequences (finite or infinite) eigenvalues $(\lambda_n)_n$ of T and vectors $(v_n)_n$ such that each v_n is eigenvector associated with λ_n and

$$T(x) = \sum_n \lambda_n \langle x, v_n \rangle v_n.$$

The proof of this result requires exactly the study of the spectrum of these operators and can be found in any of the books cited in Bibliography of this mini-article.

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