Solution of extended Graetz problem by Integral transform with orthotropic duct

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Introduction

Motivation and application

- Industry, ducts
- Advantages
  - safety
  - financial alternative (cost)
  - optimization

GITT History

- Koshlyakov(1936)
- Grimnberg(1948)
- Ozisik and Tranter
- Europe and USA
- Ozisik and Murray(1974)

- Applications
  - Numerical calculation
  - Functions presentations
  - Equations resolution
  - Graphics plot
  - Advantages
    - Fast calculation and graphics presentation
    - Simple language

Steps

- Energy equation
- Dimensionless groups
- Bessel equation
- Transformation pair
- Mathematica
- Convergence analysis

Problem formulation

General heat conduction equation

$$\rho\, c_p \,\frac{DT}{Dt} = \nabla.\,(k\nabla T) + \mu\phi + \beta_e T\,\frac{Dp}{Dt} + \dot{g}^*$$

Assumptions

$$u\frac{\partial T}{\partial z} + v\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial T^2}{\partial^2 z} + \frac{\partial T^2}{\partial^2 r}\right)$$

- Permanent regime;
- Newtonian fluid, incompressible laminar flow ($\beta_e = 0$);
- ($\dot{g}^* = 0$ and $\phi = 0$);
- constant thermophysical properties;
- hydrodynamically developed flow, but thermally developing;
- axysimmetric flow and orthotropic duct.

Specific formulation

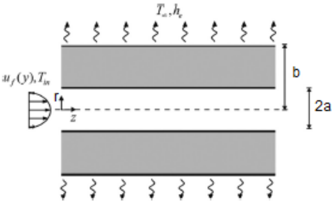

Figure: axysimmetric duct

$$u(r)\, w_f \frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(k_r(r)\, r \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_z(r)\frac{\partial T}{\partial z}\right) \quad \text{for} \quad 0 \leq z \leq \infty \quad \text{and} \quad 0 \leq r \leq b$$

$$T = T_{\text{in}} \qquad \text{for} \qquad z = 0 \qquad \text{and} \qquad 0 \leq r \leq b$$

$$\frac{\partial T}{\partial z} = 0 \qquad \text{for} \qquad z \to \infty \qquad \text{and} \qquad 0 \leq r \leq b$$

$$\frac{\partial T}{\partial r} = 0 \qquad \text{for} \qquad r = 0 \qquad \text{and} \qquad z \geq 0$$

$$-k_r(r)\frac{\partial T}{\partial r} = h\left(T - T_f\right) \qquad \text{for} \qquad r = b \qquad \text{and} \qquad z \geq 0$$

Dimensionless groups

The following dimensionless groups are defined:

$$K_r(\eta) = \frac{k_r(r)}{k_f}; \quad K_z(\eta) = \frac{k_z(r)}{k_f}; \quad \tilde{k}_r = \frac{k_{sr}}{k_f}; \quad \tilde{k}_z = \frac{k_{sz}}{k_f};$$

$$\xi = \frac{z\,k_f}{b^2\,\bar{u}\,w_f}; \quad \eta = \frac{r}{b}; \quad \beta = \frac{a}{b}; \quad \theta = \frac{T - T_f}{T_{\text{in}} - T_f}; \quad u^*(r) = \frac{u(r)}{\bar{u}};$$

$$\mathrm{Pe} = \frac{(2a)\bar{u}\,k_f}{w_f}; \quad \mathrm{Bi} = \frac{hb}{k_{sr}}; \quad w^*(\eta) = \frac{w(r)}{w_f}; \quad \tilde{w} = \frac{w_s}{w_f}$$

$$u^*(\eta) = \begin{cases} 2(1 - \frac{\eta^2}{\beta^2}) & \text{if } \eta \le \beta \\ 0 & \text{if } \eta > \beta \end{cases}$$

$$K_r(\eta) = \begin{cases} 1 & \text{if} \quad \eta \le \beta \\ \tilde{k}_r & \text{if} \quad \eta > \beta \end{cases}$$

$$K_z(\eta) = \begin{cases} 1 & \text{if} \quad \eta \le \beta \\ \tilde{k}_z & \text{if} \quad \eta > \beta \end{cases}$$

$$w^*(\eta) = \begin{cases} 1 & \text{se } \eta \le \beta \\ \tilde{w} & \text{se } \eta > \beta \end{cases}$$

Nusselt number

$$u^*(\eta)w^*(\eta)\frac{\partial\theta}{\partial\xi} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(K_r(\eta)\eta\frac{\partial\theta}{\partial\eta}\right) + \frac{4K_z(\eta)\beta^2}{\mathrm{Pe}^2}\frac{\partial^2\theta}{\partial\xi^2} \quad \text{for} \quad 0 \le \xi \le \infty \quad \text{and} \quad 0 \le \eta \le 1$$

$$\theta(0,\eta) = 1;$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0} = 0$$

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1} = -\mathrm{Bi}\,\theta(\xi,1);$$

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=\infty} = 0;$$

$$\mathrm{Nu}(\xi) = \frac{2\beta(\partial\theta/\partial\eta)_{\eta=\beta}}{\theta(\xi,\beta) - \frac{2}{\beta^2}\int_0^\beta u^*\theta\eta\,\mathrm{d}\eta}$$

Equations

- Bessel eigenvalue problem

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}R_n}{\mathrm{d}\eta}\right)+\lambda_n^2 R_n=0$$

- Transformation Pair

$$\text{Transformation} \quad \Rightarrow \quad \bar{\theta}_n(\xi)=\int_0^1 \theta(\xi,\eta)R_n(\eta)\eta\,\mathrm{d}\eta$$

$$\text{Inversion} \quad \Rightarrow \quad \theta(\xi,\eta)=\sum_{n=1}^{\infty}\frac{\bar{\theta}_n(\xi)R_n(\eta)}{N_n}$$

Transformed equation

$$\sum_{m=1}^{\infty} B_{n,m}\bar{\theta}''_m(\xi) = \sum_{m=1}^{\infty} A_{n,m}\bar{\theta}'_m(\xi) + \sum_{m=1}^{\infty} C_{n,m}\bar{\theta}_m(\xi)$$

$$A_{n,m} = -\frac{1}{N_m}\int_0^1 u^* R_m R_n \eta \, \mathrm{d}\eta$$

$$B_{n,m} = \frac{4\,\beta^2}{\mathrm{Pe}^2 N_m}\int_0^1 K_z R_m R_n \eta \, \mathrm{d}\eta$$

$$C_{n,m} = \frac{1}{N_m}\left[\mathrm{Bi}\,\tilde{k}_{sr}\, R_n(1)\, R_m(1) + \int_0^1 K_r R'_m R'_n \eta \, \mathrm{d}\eta\right]$$

$$\bar{\theta}_n\left(0\right) = \int_0^1 R_n\left(\eta\right)\eta \, \mathrm{d}\eta = b_n$$

$$\bar{\theta}_n\left(\xi \to \infty\right) = 0$$

Proposed analytical solution

$$\boldsymbol{y}(\xi) = \left(\bar{\theta}_1(\xi), \bar{\theta}_2(\xi), \bar{\theta}_3(\xi), \ldots, \bar{\theta}_{n_{\max}}(\xi), \bar{\theta}'_1(\xi), \bar{\theta}'_2(\xi), \bar{\theta}'_3(\xi), \ldots, \bar{\theta}'_{n_{\max}}(\xi)\right)$$

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\xi} = \mathbf{M}\boldsymbol{y}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{E} & \mathbf{D} \end{pmatrix}$$

$$y_n(\xi) = \sum_{m=1}^{2n_{\max}} G_{n,m} c_m \exp(\omega_m \xi), \quad \text{for} \quad n = 1, 2, 3, \ldots, 2n_{\max}$$

$$c_n = 0 \qquad \text{if} \qquad \omega_n > 0 \qquad \text{for} \qquad n = 1, 2, 3, \ldots, 2n_{\max}$$

$$\sum_{m=1}^{2n_{\max}} G_{n,m} c_m = b_n \qquad \text{for} \qquad n = 1, 2, 3, \ldots, n_{\max}$$

$$\mathrm{Nu}(\xi) = \frac{2\beta \sum_{n=1}^{\infty} \frac{\bar{\theta}'}{N_n K_r(\beta)\beta} \int_0^{\beta} \eta u^* w^* R_n(\beta) \mathrm{d}\eta - \sum_{n=1}^{\infty} \frac{4\ \beta\ \bar{\theta}''}{\mathrm{Pe}^2 N_n K_r(\beta)} \int_0^{\beta} \eta K_z(\beta) R_n(\beta) \mathrm{d}\eta}{\theta(\xi,\beta) - \frac{2}{\beta^2} \int_0^{\beta} u^* \theta \eta \,\mathrm{d}\eta} \tag{1}$$

Finite Differences Method

$$\theta_{i-1,j}\left(\frac{\gamma_j}{2\Delta\xi}+\frac{\nu_j}{\Delta\xi^2}\right)+\theta_{i+1,j}\left(\frac{\nu_j}{\Delta\xi^2}-\frac{\gamma_j}{2\Delta\xi}\right)+\theta_{i,j}\left(-\frac{2\phi_j}{\Delta\eta^2}-\frac{2\nu_j}{\Delta\xi^2}\right)+\theta_{i,j-1}\left(\frac{\phi_j}{\Delta\eta^2}-\frac{\sigma_j}{2\Delta\eta}\right)+$$
$$+\,\theta_{i,j+1}\left(\frac{\phi_j}{\Delta\eta^2}+\frac{\sigma_j}{2\Delta\eta}\right)=0 \quad \text{for} \quad 2\le i\le i_{\text{max}}-1 \quad \text{and} \quad 2\le j\le j_{\text{max}}-1 \qquad (2)$$

where:

$$\gamma_j = \eta_j u^*(\eta_j)$$
$$\nu_j = \frac{4\beta^2\eta_j K_x(\eta_j)}{\text{Pe}}$$
$$\sigma_j = \frac{\eta_j\left[K_r(\eta_{j+1})-K_r(\eta_{j-1})\right]}{2\Delta\eta}+K_r(\eta_j)$$
$$\phi_j = \eta_j K_r(\eta_j)$$

Table: Nusselt convergence solved by GITT for $\mathrm{Pe}=1$, $\beta=0.8$, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	24.9386	8.73310	4.59526	4.07013	59.3387	12.7663	8.09697	7.34972
20	16.6396	8.86891	4.60797	4.07462	52.9510	10.0700	8.07524	7.34203
30	13.0605	8.93626	4.60793	4.07225	38.9362	9.51772	8.03717	7.31200
40	11.4494	8.96457	4.60399	4.06759	29.9504	9.35841	7.99681	7.27766
50	10.6344	8.97535	4.59823	4.06179	24.2268	9.28162	7.95492	7.24100
60	10.1871	8.97664	4.59131	4.05521	20.3801	9.22285	7.91115	7.20213
70	9.92637	8.97182	4.58341	4.04790	17.6561	9.16620	7.86468	7.16052
80	9.76651	8.96220	4.57443	4.03971	15.6269	9.10620	7.81402	7.11493
90	9.66558	8.94766	4.56385	4.03018	14.0046	9.03792	7.75585	7.06237
100	9.66424	8.92294	4.54882	4.01676	11.9168	8.93935	7.67148	6.98587
	Bi=10							
10	38.3267	11.1952	4.38735	3.76267	60.3647	18.9176	8.60404	7.22643
20	33.5521	11.3415	4.37869	3.75031	103.777	13.6426	8.61157	7.25189
30	24.1883	11.4465	4.37421	3.74462	95.0136	12.0450	8.57293	7.22679
40	19.1150	11.4958	4.36862	3.73889	76.3690	11.6154	8.52947	7.19407
50	16.4139	11.5182	4.36216	3.73279	60.7474	11.4569	8.48415	7.15827
60	14.9104	11.5259	4.35497	3.72625	49.0647	11.3640	8.43690	7.12003
70	14.0381	11.5241	4.34705	3.71919	40.3727	11.2860	8.38686	7.07897
80	13.5166	11.5150	4.33821	3.71143	33.7381	11.2077	8.33244	7.03393
90	13.2079	11.4989	4.32793	3.70247	28.3613	11.1207	8.27005	6.98195
100	13.2781	11.4689	4.31342	3.68994	21.2508	10.9969	8.17962	6.90615

Table: Nusselt convergence solved by GITT for $\mathrm{Pe} = 10$, $\beta = 0.8$, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	8.79019	4.24575	3.44971	3.44971	13.5674	7.80217	6.28704	6.28701
20	8.86173	4.23896	3.44112	3.44112	11.6492	7.83561	6.31029	6.31026
30	8.89617	4.23253	3.43481	3.43481	11.2722	7.81772	6.29521	6.29519
40	8.90681	4.22568	3.42871	3.42871	11.1662	7.78809	6.27113	6.27110
50	8.90678	4.21846	3.42252	3.42251	11.1092	7.75311	6.24285	6.24282
60	8.90083	4.21082	3.41610	3.41610	11.0577	7.71432	6.21155	6.21152
70	8.89086	4.20266	3.40932	3.40931	11.0020	7.67176	6.17726	6.17723
80	8.87744	4.19373	3.40196	3.40195	10.9388	7.62442	6.13912	6.13909
90	8.86003	4.18350	3.39357	3.39357	10.8635	7.56924	6.09469	6.09466
100	8.83319	4.16932	3.38200	3.38199	10.7501	7.48813	6.02939	6.02937
	Bi=10							
10	10.7146	4.02746	3.26900	3.26900	17.3373	7.75672	6.01095	6.01095
20	10.7943	4.00833	3.25229	3.25229	13.5830	7.81416	6.05585	6.05584
30	10.8635	4.00033	3.24520	3.24520	12.5117	7.79792	6.04412	6.04411
40	10.8937	3.99326	3.23913	3.23913	12.2288	7.76808	6.02157	6.02156
50	10.9047	3.98617	3.23317	3.23317	12.1203	7.73270	5.99454	5.99453
60	10.9051	3.97881	3.22706	3.22706	12.0490	7.69358	5.96449	5.96448
70	10.8984	3.97101	3.22063	3.22063	11.9820	7.65077	5.93152	5.93151
80	10.8861	3.96251	3.21366	3.21366	11.9097	7.60325	5.89484	5.89484
90	10.8680	3.95280	3.20573	3.20573	11.8253	7.54796	5.85212	5.85211
100	10.8375	3.93930	3.19474	3.19474	11.6999	7.46670	5.78924	5.78923

Table: Nusselt convergence solved by GITT for $\mathrm{Pe}=1$, $\beta=0.9$, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	-24.9032	11.8685	4.77098	4.00867	292.464	9.05233	8.71448	7.50106
20	25.8121	12.3238	4.80417	4.02972	72.0341	14.4669	8.63079	7.44034
30	8.58828	12.4807	4.81216	4.03386	-38.2978	13.0112	8.56437	7.38832
40	18.3809	12.5577	4.81225	4.03262	44.7887	13.2422	8.50752	7.34213
50	12.9224	12.5965	4.80881	4.02895	-11.3675	13.0378	8.45427	7.29795
60	16.3312	12.6146	4.80333	4.02383	30.9638	12.9724	8.40169	7.25377
70	14.2496	12.6197	4.79635	4.01760	-1.48503	12.8691	8.34772	7.20805
80	15.6137	12.6152	4.78794	4.01026	23.7744	12.7753	8.29012	7.15898
90	14.7446	12.6018	4.77769	4.00146	3.15032	12.6673	8.22471	7.10302
100	15.4236	12.5733	4.76291	3.98893	18.3959	12.5194	8.13165	7.02309
	Bi=10							
10	815.415	15.7255	4.50087	3.62432	86.3093	8.72404	9.03581	7.07751
20	45.6724	16.2721	4.49508	3.61661	112.879	19.8168	9.03687	7.08931
30	3.72534	16.4895	4.49511	3.61519	-152.590	16.6098	8.98032	7.05089
40	30.2469	16.6048	4.49228	3.61212	93.2234	17.3239	8.92428	7.01032
50	15.3625	16.6663	4.48753	3.60782	-50.6483	16.9232	8.86976	6.96968
60	24.7440	16.6979	4.48147	3.60262	63.4281	16.8717	8.81517	6.92830
70	18.9663	16.7102	4.47431	3.59662	-22.1934	16.7200	8.75882	6.88511
80	22.7733	16.7086	4.46598	3.58975	45.5964	16.5981	8.69848	6.83853
90	20.3730	16.6943	4.45604	3.58163	-9.26427	16.4531	8.62983	6.78522
100	22.3424	16.6586	4.44187	3.57014	31.5850	16.2578	8.53171	6.70865

Table: Nusselt convergence solved by GITT for $\mathrm{Pe}= 10$, $\beta= 0.9$, and different Biot numbers.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
	Bi=1							
10	11.5356	4.45525	3.39984	3.39981	10.4834	8.35705	6.38527	6.38515
20	11.9049	4.47013	3.40800	3.40797	15.1630	8.32587	6.35740	6.35728
30	12.0262	4.47165	3.40791	3.40788	13.9792	8.27996	6.32234	6.32222
40	12.0836	4.46871	3.40502	3.40499	14.1941	8.23433	6.28772	6.28760
50	12.1106	4.46369	3.40080	3.40078	14.0253	8.18845	6.25288	6.25277
60	12.1209	4.45738	3.39574	3.39571	13.9671	8.14131	6.21704	6.21693
70	12.1207	4.45002	3.38995	3.38992	13.8715	8.09172	6.17931	6.17920
80	12.1125	4.44156	3.38336	3.38333	13.7799	8.03792	6.13833	6.13822
90	12.0968	4.43155	3.37563	3.37560	13.6716	7.97607	6.09120	6.09109
100	12.0671	4.41747	3.36482	3.36479	13.5181	7.88706	6.02335	6.02324
	Bi=10							
10	14.8838	4.20730	3.19321	3.19320	9.93832	8.20466	5.96452	5.96448
20	15.3400	4.19445	3.18350	3.18350	18.6137	8.24153	5.98720	5.98715
30	15.5207	4.19169	3.18104	3.18104	16.1234	8.20474	5.96074	5.96069
40	15.6155	4.18761	3.17771	3.17770	16.6997	8.16138	5.92971	5.92967
50	15.6648	4.18231	3.17353	3.17353	16.3874	8.11634	5.89738	5.89733
60	15.6886	4.17607	3.16870	3.16869	16.3423	8.06965	5.86375	5.86370
70	15.6960	4.16897	3.16323	3.16323	16.2137	8.02041	5.82820	5.82815
80	15.6913	4.16090	3.15705	3.15705	16.1045	7.96693	5.78953	5.78948
90	15.6754	4.15140	3.14980	3.14979	15.9725	7.90542	5.74499	5.74495
100	15.6400	4.13802	3.13962	3.13961	15.7892	7.81664	5.68065	5.68061

Table: Nusselt convergence solved by GITT for $\mathrm{Pe} = 10^3$, β= 1 and $\mathrm{Bi} = 10^6$.

	$\tilde{k}_z = 0.5$ and $\tilde{k}_r = 1.5$				$\tilde{k}_z = 1.5$ and $\tilde{k}_r = 0.5$			
$n_{\max}$	ξ=0.01	ξ=0.1	ξ=1	ξ=10	ξ=0.01	ξ=0.1	ξ=1	ξ=10
10	7.49530	4.00603	3.65744	3.65744	7.49543	4.00611	3.65751	3.65751
20	7.48211	4.00499	3.65682	3.65682	7.48239	4.00514	3.65696	3.65696
30	7.48065	4.00484	3.65672	3.65672	7.48108	4.00508	3.65693	3.65693
40	7.48023	4.00478	3.65667	3.65667	7.48081	4.00509	3.65695	3.65695
50	7.48004	4.00473	3.65662	3.65662	7.48077	4.00512	3.65698	3.65698
60	7.47991	4.00468	3.65659	3.65659	7.48079	4.00516	3.65702	3.65702
70	7.47980	4.00464	3.65655	3.65655	7.48083	4.00520	3.65705	3.65705
80	7.47973	4.00460	3.65651	3.65651	7.48087	4.00523	3.65709	3.65709
90	7.47969	4.00456	3.65647	3.65647	7.48091	4.00527	3.65713	3.65713
100	7.47965	4.00453	3.65644	3.65644	7.48094	4.00530	3.65716	3.65716

Schedule

- Introduction and abstract - 01/2016
- Literature review - 03/2016
- GITT code at Mathematica - 06/2016
- Results for GITT 06/2016
- ENCIT publication - 07/2016
- FDM code - 08/2016
- Results for FDM - 08/2016
- Final convergence analysis and comparison between the methods - 11/2016
- Final presentation - 12/2016