APPROXIMATE POSITIONING OF THE FINAL STATE OF THERMAL SYSTEMS DESCRIBED BY THE LINEAR HEAT EQUATION

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1 Introduction

Control problems for systems described by *linear evolution equations* (roughly speaking equations involving partial derivatives related to time and space coordinates) have received considerable atention in recent literature, see for example, [1] and its references.

In particular, the basic goal of achieving a desired end state from a given initial state has given rise to various problems of optimal control (in *open loop*) to parabolic equations in general and in particular for the heat equation. Such problems may include different types of boundary conditions (Dirichlet, Neumann and Robin) and the control can be acting on the border of the domain space considered or as a source term in the interior of the domain. Usually, these problems aim to obtain a *control function* defined (in a general sense) in both, a prespecified time interval as on the spatial domain in which the equation is defined, *i.e.* each time the control "signal" assumes a "value" of a function defined throughout the refered space domain.

An *open loop* control system consists of applying a control signal, $\boldsymbol{u}(\boldsymbol{x},t)$, that is not calculated from a measurement of the output signal (*no feedback*), to a system o process, $\theta(\boldsymbol{x},t)$, and it is expected that the output of the controlled variable, at the end of a determined time, t_F, reaches a certain specified value, $\theta_{\text{Cont.}}(\boldsymbol{x},t_F)$.

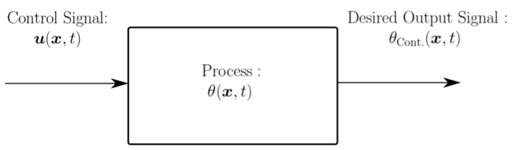


Figure 1: A General Open loop control system

Moreover, in view of potential applications, it is interesting to consider the case of control functions depending only on time, (each time the control signal assumes as a "value" a point in $\mathbb{R}^n$ for some n fixed *a priori*), whose action in space is defined by "actuators" used. In this work, it will be considered a quadratic optimal control problem for the linear heat equation in rectangular areas with a Dirichlet boundary condition type and in which the control function (solely dependent on time) is a source term. Also, an example of the $1-$D case with a restriction on the maximum value of $\boldsymbol{u}_K$ and an example of the $2-$D case without this restriction will be presented.

1.1 Optimal Control Problem

In order to see the formulation of an optimal control problem applied to the heat equation, for clarity, we take the $1-$D problem which is:
Find $\boldsymbol{u}$ such that

$$\min_{\boldsymbol{u}\in L_2(0,t_F)^m} \mathcal{J}(\boldsymbol{u}) \tag{1}$$

subject to:

$$\begin{aligned}
\frac{\partial\theta}{\partial t}-\alpha\frac{\partial^2\theta}{\partial x^2} &= f_{\boldsymbol{u}}(x,t) && \forall x\in(0,L_x),\ \forall t\in(0,t_F)\\
\theta(x,t) &= 0 && \text{for } x=\{0,L_x\},\ \forall t\in(0,t_F)\\
\theta(x,0) &= g(x) && \forall x\in(0,L_x)\\
\|\boldsymbol{u}\|_\infty &\le \mu_{\boldsymbol{u}}, && \mu_{\boldsymbol{u}}\in\mathbb{R}_+,
\end{aligned}$$

where $m\in\mathbb{N}$, α,t_F,L_x and $\rho_F\in\mathbb{R}_+$, $\mathcal{J}(\boldsymbol{u}) = \|\boldsymbol{u}\|^2_{L_2(0,t_F)^m}+\rho_F\,\|\theta_o(t_F;\boldsymbol{u})-\underline{\theta}_r(x,t_F)\|^2_{L_2(0,L_x)}$ and $f_{\boldsymbol{u}}(x,t) = f_S(x,t)+\boldsymbol{\beta}_{\boldsymbol{S}}^{\mathrm{T}}(x)\,\boldsymbol{u}(t)$. The function $\theta_o(t_F;\boldsymbol{u})$ *controlled function*, $\underline{\theta}_r(x,t_F)$ is the *desired final state function* and $\boldsymbol{\beta}_{\boldsymbol{S}}$ is the *spatial distribution* function on which the control function acts in order to take the system to the desired state and $\|\boldsymbol{u}\|_\infty=\max|\boldsymbol{u}|$.

2 Approximate Positioning of the Final State

Since our main interests lies in the field of applications, it will suffice to find a procedure that gives us a realiable approximation, $\boldsymbol{u}_K$, to the analytical solution, $\boldsymbol{u}$, of our previous problem. Along these lines we present a suitable approximation. For complete details on how the following results were obtained cf. [2].

2.1 Matrix Form of the Approximate Control Function

The approximate control function that satisfies (1) subject to the given conditions, in matrix form is given by

$$\boldsymbol{u}_K(\tau) = \overline{\boldsymbol{\beta}}_{\boldsymbol{S}K}^{\mathrm{T}} \boldsymbol{\Phi}_K(t_F - \tau)\,\overline{\boldsymbol{\alpha}}_K \in \mathbb{R}^m \;\forall \tau \in [0, t_F], \tag{2}$$

where m is the number of signals in the vector $\boldsymbol{\beta}_{\boldsymbol{S}j} = [\langle \boldsymbol{\beta}_{\boldsymbol{S}1}, \phi_j\rangle, \ldots, \langle \boldsymbol{\beta}_{\boldsymbol{S}m}, \phi_j\rangle]^{\mathrm{T}}, j = 1, \ldots, K$, where $K \leq K_a$ is the number of terms used to calculate the control signal and K_a is the number of terms chosen to calculate the "weak solution" of the heat equation, $\overline{\boldsymbol{\beta}}_{\boldsymbol{S}K}^{\mathrm{T}} = [\langle \boldsymbol{\beta}_{\boldsymbol{S}}, \phi_1\rangle, \ldots, \langle \boldsymbol{\beta}_{\boldsymbol{S}}, \phi_K\rangle] \in \mathbb{R}^{m\times K}$, where $\langle \boldsymbol{\beta}_{\boldsymbol{S}}, \phi_i\rangle = [\langle \boldsymbol{\beta}_{\boldsymbol{S}1}, \phi_i\rangle, \ldots, \langle \boldsymbol{\beta}_{\boldsymbol{S}m}, \phi_i\rangle]^{\mathrm{T}}$ for $i = 1, \ldots, K$, $\boldsymbol{\Phi}_K(t_F - \tau) = \exp[\mathbf{A}_K(t_F - \tau)] \in \mathbb{R}^{K\times K}$, $\overline{\boldsymbol{\alpha}}_K$ comes from the solution of a *Sylvester* matrix equation. The matrix $\mathbf{A}_K = diag(-\alpha(k\pi/L_x)^2)$ and $\{\phi_k\}_{k=1}^K = \{\sqrt{2/L_x}\sin((k\pi x)/L_x)\}_{k=1}^K$.

2.2 Approximately Controlled Function

Since we obtained an approximate control function, $\boldsymbol{u}_K$, then it is possible to define an approximately controlled function at time t_F, $\theta_{\text{cont.}}$, that also approximates to the analytical controlled function $\theta_o(t_F; \boldsymbol{u})$. This function is represented explicitly as

$$\theta_{\text{cont.}}(x, t_F) = \Sigma_{k=1}^{K_a} \widehat{C}_{\theta k}(t_F)\,\phi_k(x), \tag{3}$$

where $\widehat{C}_{\theta k}(t_F) = \widehat{C}_{Sk}(t_F) + \widehat{C}_{\boldsymbol{u}_K k}(t_F; \boldsymbol{u}_K(\tau))$. Explicitly we have $\widehat{C}_{Sk}(t_F) = g_k \exp[-\alpha(k\pi/L_x)^2 t_F] + \int_0^{t_F} \exp[-\alpha(k\pi/L_x)^2(t_F - \tau)] f_{Sk}(\tau)d\tau$, where $g_k = \langle g(x), \phi_k\rangle$ and $f_{Sk}(\tau) = \langle f_S(x,\tau), \phi_k\rangle$. The term $\widehat{C}_{\boldsymbol{u}_K k}(t_F; \boldsymbol{u}_K(\tau))$ comes from solving a *Sylvester* matrix equation involving the approximate controllable function $\boldsymbol{u}_K$.

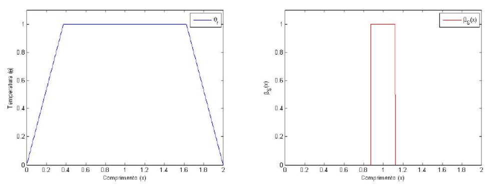
2.3 Restriction on the Maximum Value of the Control Function

The restriction $\|\boldsymbol{u}\|_\infty \leq \mu_{\boldsymbol{u}}$ is distributed throughout the system by means of quadratic programming optimization over the approximate functional $\mathcal{J}(\boldsymbol{u}_{\boldsymbol{K}})$ with the given constraint $\mu_{\boldsymbol{u}}$.

3 Examples

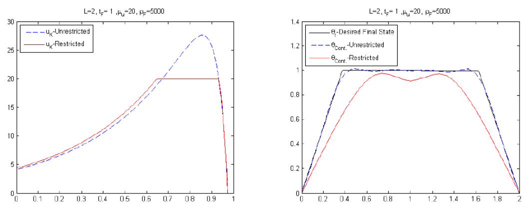
3.1 Example for the 1−D Case with Energy Restriction

Approximate control of the linear heat equation for the 1−D case with an desired final state function given by $\theta_r(x) = \begin{cases} \frac{8}{3}x & \text{if } 0 \leq x < \frac{3}{8}, \\ 1 & \text{if } \frac{3}{8} \leq x < \frac{13}{8}, \\ -\frac{8}{3}x + \frac{16}{3} & \text{if } \frac{13}{8} \leq x \leq 2, \end{cases}$ and using as spacial distribution function $\boldsymbol{\beta}_{\boldsymbol{S}}(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{7}{8}, \\ 1 & \text{if } \frac{7}{8} \leq x < \frac{9}{8}, \\ 0 & \text{if } \frac{9}{8} \leq x \leq 2, \end{cases}$ with the restriction of $\|\boldsymbol{u}\|_\infty \leq 20$ for $t_F = 1$.



(a) θ_r (b) β_S

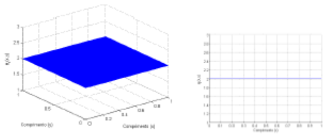
Results for the 1−D case with restriction.



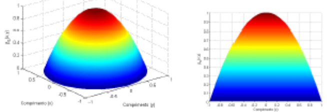
(a) u_K (b) $\theta_{\text{Cont.}}$

3.2 Example for the 2−D Case without Energy Restriction

Approximation of the desired final state $\theta_r(x, y) = 2$ using the spacial distribution function $\beta_S(x, y) = -[(x - 1/2)^2 + (y - 1/2)^2 - 1]$ in the domain defined by $[0, 1] \times [0, 1]$.

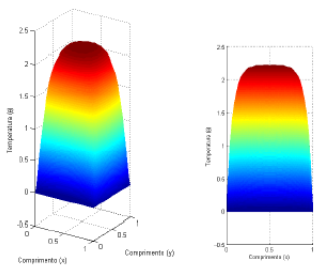


(a) $\theta_r(x, y)$ (b) $\theta_r(x, y)$ vista no plano $X\theta$



(c) $\beta_S(x, y)$ (d) $\beta_S(x, y)$ vista no plano $X\theta$

Results for the 2−D case without restriction.



(a) $\rho_F = 5000, \theta_{cont.}(x, y)$ (b) ρ_F 5000, $\theta_{cont.}(x, y)$. Vista plano $X\theta$

4 Future Work

This model will be extended to cover more general boundary conditions, such as the Robin boundary condition.

References

[1] E. Bünsch and P. Benner. *Constrained Optimization and Optimal Control for Partial Differential Equations.* Birkhäuser, 2012.

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