SOLUTION OF EXTENDED GRAETZ PROBLEM BY INTEGRAL TRANSFORM WITH ORTHOTROPIC DUCT

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1 Introduction

There is a growing interest for applications of heat and mass transfer in orthotropic ducts. In the realm of simulation, studies for heat transfer in composite ducts, this paper proposes a hybrid solution strategies for solving the conjugated heat transfer in an axisymmetric duct made of an orthotropic material, which the conductivity vary in three main directions[2]. Since the wall material is anisotropic, the axial diffusion is also considered in this formulation. The final result will be the analysis of temperature distribution for this case. The formulation to be employed is the Generalized Integral Transform Technique (GITT) which in the realm of hybrid analytical-numerical methods, has been playing a big role[3].The Integral Transform Technique is employed as the main solution methodology. The presented results can serve as guidance for choosing an optimum solution methodology for this type of problem.

Keywords:: Axysimmetric duct, Orthotropic material, Conjugate Heat Transfer, Generalized Integral Transform Technique.

2 Methodology

In order to solve the conjugate heat-transfer problem, one needs to solve the energy equation. The flow is assumed to be hydrodynamically developed but thermally developing, with negligible viscous dissipation and temperature independent of physical properties[1]. The duct is considered to be axyssimetric, and the fluid flows with a known fully developed laminar velocity profile. The duct solid wall ($a \leq r \leq b$) has anisotropic conductivity. The inlet temperature is prescribed and there is a fluid flowing outside the duct, resulting in a Robin (third kind) boundary condition at $r = b$. The general formulation of conjugated problem in classical two-dimensional cylindrical coordinates is given by:

$$u(r)\, w_f \frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(k_r(r)\, r \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_z(r)\frac{\partial T}{\partial z}\right) \tag{1}$$

$$T = T_{\text{in}} \quad \text{for} \quad z = 0 \quad \text{and} \quad 0 \leq r \leq b \tag{2}$$

$$\frac{\partial T}{\partial z} = 0 \quad \text{for} \quad z \to \infty \quad \text{and} \quad 0 \leq r \leq b \tag{3}$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{for} \quad r = 0 \quad \text{and} \quad z \geq 0 \tag{4}$$

$$-k_r(r)\frac{\partial T}{\partial r} = h\,(T - T_f) \;\text{for}\; r = b \quad \text{and} \quad z \geq 0 \tag{5}$$

where $u(r)$ is the parabolic fully developed velocity profile, that flows between ($0 \leq r \leq a$), w_f is the inner fluid heat capacity, $k_r(r)$ and $k_z(r)$ are the thermal conductivities in direction r and z respectively. T_f is the external environment temperature, T_{in} is the temperature at entrance of channel and h is the heat transfer coefficient.

The following dimensionless groups are defined:

$$K_r(\eta) = \frac{k_r(r)}{k_f}; \quad K_z(\eta) = \frac{k_z(r)}{k_f}; \quad \tilde{k}_r = \frac{k_{sr}}{k_f}; \quad \tilde{k}_z = \frac{k_{sz}}{k_f}; \tag{6}$$

$$\xi = \frac{z\, k_f}{b^2\, \bar{u}\, w_f}; \quad \eta = \frac{r}{b}; \quad \beta = \frac{a}{b}; \quad \theta = \frac{T - T_f}{T_{\text{in}} - T_f}; \tag{7}$$

$$u^*(\eta) = \begin{cases} 2(1 - \frac{\eta^2}{\beta^2}) & \text{if } \eta \leq \beta \\ 0 & \text{if } \eta > \beta \end{cases} \tag{8}$$

$$K_r(\eta) = \begin{cases} 1 & \text{if} \quad \eta \leq \beta \\ \tilde{k}_r & \text{if} \quad \eta > \beta \end{cases} \tag{9}$$

$$K_z(\eta) = \begin{cases} 1 & \text{if} \quad \eta \leq \beta \\ \tilde{k}_z & \text{if} \quad \eta > \beta \end{cases} \tag{10}$$

Where k_f is the fluid thermal conductivity, k_{sr} is the solid thermal conductivity in r-direction, k_{sz} is the solid thermal conductivity in z-direction, $\bar{u}$ is the fluid average velocity ($0 \leq r \leq a$), a is the inner radius, and b is the outer radius, Bi is the Biot number, Pe is the Péclet number, $K_r(\eta)$, $K_z(\eta)$, θ, ξ, η and $u^*(\eta)$ are nondimensional versions of $k_r(r)$, $k_z(r)$, T, z, r and $u(r)$ respectively and β is the aspect ratio.

In order to solve this system of equations, an analytical solution is proposed. One can obtain the solution of the modified system by integrating analytically. The eigenvalues and eigenvectors of $\mathbf{M}$ are calculated so that the solution of the components of $\boldsymbol{y}$ can be written in the following form:

$$y_n(\xi) = \sum_{m=1}^{2n_{\max}} G_{n,m} c_m \exp(\omega_m \xi) \tag{11}$$

The solution of transformed potential is obtained, and then the temperature field is calculated by applying the inversion formula and the Nusselt number can be obtained directly from the transformed temperatures using the following expression:

$$\mathrm{Nu}(\xi) = \frac{\sum_{n=1}^{\infty} \bar{\theta}_n \frac{2\beta}{N_n} R'(\beta)}{\sum_{n=1}^{\infty} \frac{\bar{\theta}_n}{N_n} \left[R_n(\beta) - \frac{2}{\beta^2} \int_0^\beta u^* R_n \eta \, \mathrm{d}\eta \right]} \tag{12}$$

3 Results

It is analyzed the local Nusselt number convergence at different axial positions, ranging from 0.01 up to 10, for different Biot numbers and thermal conductivities. Nusselt values were presented for different truncation orders, so that it is possible to analyze the convergence behavior. Converged three digits are noticed for position $\xi = 0.1$, in which k_z=0.5 and k_r=1.5, among 50 and 70 terms in the series, for Bi=1. Meanwhile for position $\xi = 1$, we have three digits converged among 20 and 40 terms in the series. On the second case, it's realized three converged digits for Bi=10, in which k_z=0.5 and k_r=1.5, for position ξ=0.01 among 70 and 100 terms in the series, and at position ξ=0.1, also three digits converged behavior among 10 and 30 terms, and for 60 and 70 terms as well. In other case, for Bi=1 is possible to see that convergence occurs for positions ranging from ξ= 0.1 until ξ=10 just for 2 meaningful digits, except for position ξ=0.1, which we have four converged digits, among 60 and 80 terms. When Bi is increased to 10, there is no convergence of four digits.

In the thrid case, the convergence of 2 digits is not obtained at position ξ= 0.01 among 20 and 100 terms in the summation. But this occurs for Bi=10. And also 2 digits from 10 until 100 terms at position ξ=0.1, that does not happen for when Bi=10. For both tables it was not still noticed fully-converged six digits.

This effect of range of convergence rate seems to be strongly dependent on both the Peclet number and the Biot number. For larger Pe values, notably better convergence rates are seen and it also seems to improve for larger Bi values. In second and in the fourth case, in spite of larger Peclet Number, a low convergence rate occurs in the entrance of channel in which in which k_z=1.5 and k_r=0.5. In all cases we can see a worse convergence rate for Pe=1 due to greater influence of the axial diffusion term. On the other hand when Péclet number increases for 10, the convergence rate improves considerably. For all tables, at the entrance of channel at position ξ= 0.01, is verified the worst convergence rate due to the boundary condition discontinuity.

Results presented have been analyzed mainly for different Biot and Peclet numbers, according the axial position (ξ), truncation orders ($n_{\max}$) and two combinations of $\tilde{k}_z$ and $\tilde{k}_r$. The Nusselt number is chosen for the convergence analysis due to its importance in this kind of heat transfer problem. We will consider axial diffusion as reference only. One can confirm what the literature claims about Nusselt local number to be equal to 3.66 for larger ξ, Peclet numbers, Biot Numbers and negligible duct wall. One should note that for larger Bi the boundary condition approaches to the prescribed temperature in $\eta = 1$. The result is that neglecting the axial diffusion, due the high value of Peclet number, the local Nusselt number converges to 3.66.

4 Conclusions

This paper presented a semi-analytical solution for the conjugate heat transfer problem in a duct with anisotropic wall material considering axial diffusion. It was shown the convergence behavior of local Nusselt. In this work, the axial diffusion has been considered and the solution methodology was based on the Generalized Integral Transform Technique, and a simple eigenvalue problem with analytical solution was employed for the transformation. Although a coupled ODE system was obtained, the equations could be solved analytically by rewriting this system in a modified form and employing a matrix method. The results were verified by comparisons with the datas from the literature and it achieved good agreement. A convergence analysis of the solution showed that very good converge rates are seen for larger Peclet, Biot and aspect ratio values. On the other hand, a worse convergence behavior was seen for the beginning of the duct length. This is due to the boundary condition discontinuity at the entrance of the channel. A worse convergence behavior was also observed for smaller values of Peclet and Biot. Regarding the convergence analysis, illustrative results were presented, showing the variation of the local Nusselt number with axial positions for different values of Peclet, Biot, β and thermal conductivities.

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