5^o **SEMINÁRIO DO PPG-EM**

Stabilized Hybridized Finite Element Formulations: A New Approach

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Currículo

- (1996) Bacharelado em Física pelo IF-UFG
- (2000) Mestrado em física pelo IFGW-UNICAMP
- (2003) Bacharelado em Matemática Aplicada e Computação Científica pelo ICMC-USP São Carlos
  Ênfase em Dinâmica dos Fluidos
- (2010) Doutorado em Modelagem Computacional pelo LNCC
  Tema: Formulação Mista Estabilizada de Elementos Finitos para um Fluido de Bingham
- (2010-2014) Pós-Doutorado no LNCC
  Tema: Formulações Híbridas Estabilizadas de Elementos Finitos
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Currículo

Formulações hibridizadas propostas para os seguintes problemas:

- Problema elíptico
- Elasticidade Linear
- Escoamento em meios porosos homogêneos e heterogêneos (problema de Darcy)
- Escoamento de Stokes
- Problema acoplado Stokes-Darcy
- Problema do Calor (Problema Parabólico)

Principais Colaboradores:

- Abimael Loula e Sandra Malta (LNCC)
- A.J. Boness dos Santos (UFPB)
- Sônia Gomes e Philippe Devloo (UNICAMP)

Co-orientações de Doutorado no LNCC: Yoisell Núñez (2014), Iury Igreja.

Orientação de Iniciação Científica (UERJ): Luís Carnevale

Linear Elasticity Problem

Outline

1. Introduction
   - Model Problem
   - Approaches found in literature
2. Stabilized Hybrid Formulation
3. Solver Strategies
4. Local post-processing
5. Numerical Results

The kinematical model of linear elasticity

Find the displacement vector field **u** such that:

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) &= \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma \end{aligned} \tag{1}$$

- $\boldsymbol{\sigma}(\mathbf{u})$ is the symmetric Cauchy stress tensor;
- $\varepsilon(\mathbf{u})$ is the linear strain tensor with $\varepsilon(\mathbf{u}) = \frac{1}{2}(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T)$.
- For linear, homogeneous and isotropic material

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbb{D}\varepsilon(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda(\operatorname{tr}\varepsilon(\mathbf{u}))\mathbf{I} \tag{2}$$

where λ and μ are called the Lamé parameters.

- For plane strain

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2(1+\nu)}.$$

where E is the Young's modulus and ν is Poisson's ratio.

Displacement-based method

Classical displacement-based methods determine the displacement field DIRECTLY and evaluate the stresses by POST-PROCESSING.

ADVANTAGES: Introduction of additional unknowns and related difficulties are avoided.

DISVANTAGES: The poor accuracy of the recovered stress approximations given by standard post-processing.

- Galerkin finite element approximations degrade when the Poisson's ratio tends to $1/2$, corresponding to near incompressible elasticity when low-order are used.
- Nonrobustness of FEM is termed "locking".

Mixed Methods in stress and displacement fields

- Appears to be a natural choice.
- The pair forms a unique saddle point of the Hellinger-Reissner functional.
- Due to the symmetry constraint on the stress tensor, it is difficult to construct stable finite element spaces which satisfy Brezzi's stability condition.

  COMPATIBILITY CONDITION BETWEEN THE SPACES

- Examples:

Mixed Methods in stress and displacement fields

- Stable mixed finite elements with weakly imposed symmetry:
  - D. N. Arnold, F. Brezzi, J. Douglas Jr., *PEERS: A new mixed finite element for plane elasticity*, Japan Journal of Applied Mathematics **1** (2) (1984) 347–367.
  - D. N. Arnold, R. S. Falk, R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*, Mathematics of Computation **76** (260) (2007) 1699–1723.
  - W. Qiu, L. Demkowicz, *Mixed hp-finite element method for linear elasticity with weakly imposed symmetry*, Computer Methods in Applied Mechanics and Engineering **198** (47–48) (2009) 3682–3701.
  - G. Awanou, *Rectangular mixed elements for elasticity with weakly imposed symmetry condition*, Advances in Computational Mathematics **38** (2) (2013) 351–367.

Mixed Methods in stress and displacement fields

- Stabilized formulations:
  - T. Hughes, L. Franca, *A mixed finite element formulation for Reissner–Mindlin plate theory: Uniform convergence of all higher-order spaces*, Computer Methods in Applied Mechanics and Engineering **67** (2) (1988) 223–240.
  - L. P. Franca, T. J. R. Hughes, A. F. D. Loula, I. Miranda, *A new family of stable elements for nearly incompressible elasticity based on a mixed Petrov-Galerkin finite element formulation*, Numerische Mathematik **53** (1988) 123–141.
  - L. P. Franca, R. Stenberg, *Error analysis of some Galerkin least squares methods for the elasticity equations*, SIAM J. Numer. Anal. **28** (6) (1991) 1680–1697.

Discontinuous Galerkin (DG) Methods

ADVANTAGES:

- Finite element spaces consisting of discontinuous piecewise polynomials.
- Polynomials of arbitrary degree can be used on each element.
- It can handle nonconforming meshes.
- It possible to use complex implementation and high cost computational.

Robustness and flexibility for implementing *hp*-adaptivity

DISVANTAGES:

- DG methods has been limited by their more complex formulation, computational implementation and much larger number of degrees-of-freedom.

- Examples:

Discontinuous Galerkin (DG) Methods

- Interior penalty DG methods:
  - B. Rivière, M. F. Wheeler, *Optimal error estimates for discontiuous Galerkin methods applied to linear elasticity problems*, Comput. Math. Appl **46** (2000) 141–163.
  - T. P. Wihler, *Locking-free adaptive discontinuous Galerkin FEM for linear elasticity problems*, Mathematics of Computation **75** (255) (2006) 1087–1102.
  - P. Hansbo, M. G. Larson, *Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitscheś method*, Comput. Methods Appl. Mech. Engrg. **191** (17–18) (2002) 1895–1908.
  - P. Hansbo, M. G. Larson, *Discontinuous Galerkin and the Crouzeix-Raviart element: application to elasticity*, Mathematical Modelling and Numerical Analysis **37** (1) (2003) 63–72.

Discontinuous Galerkin (DG) Methods

- Mixed DG methods:
    - Y. Chen, J. Huang, X. Huang, Y. Xu, *On the Local Discontinuous Galerkin method for linear elasticity*, Mathematical Problems in Engineering **2010** (2010) 20 pages.
    - B. Cockburn, D. Schötzau, J. Wang, *Discontinuous Galerkin methods for incompressible elastic materials*, Comput. Methods Appl. Mech. Engrg. **195** (2006) 3184–3204.
    - R. Bustinza, *A note on the local discontinuous Galerkin method for linear problems in elasticity*, Scientia Series **A** **13** (2006) 72–83.

Proposed Formulation

Goal:

Determine the displacement field DIRECTLY and evaluate the stresses by POST-PROCESSING.

- Advantages of DG methods:
  - Flexibility for implementing *hp*-adaptivity.
  - Robustness when the Poisson's ratio tends to $1/2$.
- Element based data structure.
- Reduced computational cost.
- Stress approximations with improved rates of convergence in H(div) norm.

Proposed Formulation

Primal Stabilized Hybrid Finite Element Method for Linear Elasticity Problem

$+$

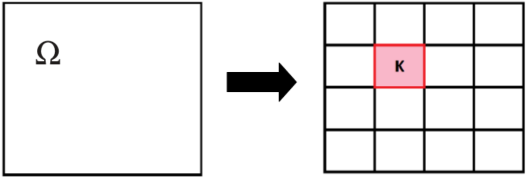
A local post-processing technique by recovering stress field

The Hybrid Method

Let $\mathcal{T}_h = \cup_{r=1}^{R} \mathcal{K}_r$ a decomposition of Ω into $\mathcal{K}_r$ subdomain, such as:

(i) $\mathcal{K}_r$ is a subdomain of Ω with a lipschitzian boundary, $\partial \mathcal{K}_r$, where $1 \leq r \leq R$;

(ii) $\mathcal{K}_r \cap \mathcal{K}_s = \varnothing$ for $r \neq s$.



$$\text{GLOBAL PROBLEM} \Leftrightarrow \sum_{\mathcal{K}} \text{LOCAL PROBLEMS}$$

The Local problem

By Raviart and Thomas[1], a function $\mathbf{u} \in \mathbf{L}^2(\Omega)$ belongs the space $\mathbf{H}^1(\Omega)$, if

a) the restriction $\mathbf{u}_r$ of $\mathbf{u}$ in $\mathcal{K}$ belongs the space $\mathbf{H}^1(\mathcal{K})$;
b) the traces of $\mathbf{u}_r$ and $\mathbf{u}_s$ are the same on $\partial\mathcal{K}_r \cap \partial\mathcal{K}_s$;

To relax the condition b) it has introduced the follow space[2]:

$$V = \{\mathbf{u} \in \mathbf{L}^2(\Omega); \mathbf{u}_r \in \mathbf{H}^1(\mathcal{K}_r), 1 \leq r \leq R\} \approx \prod_{r=1}^{R} \mathbf{H}^1(\mathcal{K}_r), \tag{3}$$

with the norm

$$\|\mathbf{u}\|_V = \left(\sum_{r=1}^{R} \|\mathbf{u}_r\|^2_{1,\Omega} \right)^{1/2}. \tag{4}$$

[1] P.A. Raviart and J.M. Thomas, *Primal hybrid finite element method for second order elliptic equations*, Mathematics of Computation, **31**(138), 391–413, (1977).

[2] $L^2(\Omega)$ is the space of all square-integrable-valued functions equipped with the usual inner product $(p, q) = \int_\Omega p\, q dx$ and usual norm $\|\cdot\|_{0,\Omega}$.
$H^1(\Omega) = \{q \in L^2(\Omega), \nabla q \in L^2(\Omega)\}$, with inner product $(p, q)_1 = (p, q) + (\nabla p, \nabla q)$ and norm $\|q\|_1^2 = \|q\|^2 + \|\nabla q\|^2$. $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^2$ and $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^2$.

The Local problem

For each $\mathcal{K}_r = \mathcal{K}$, we have the follow local problem

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \mathcal{K}, \\ \boldsymbol{\sigma}(\mathbf{u}) &= \mathbb{D}\varepsilon(\mathbf{u}) && \text{in } \mathcal{K}, \\ \mathbf{u} &= \bar{\mathbf{u}} && \text{on } \partial\mathcal{K} \end{aligned} \tag{5}$$

To introduce the hybrid formulation, consider the local problem

$$\int_{\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx - \underbrace{\int_{\partial\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u})\mathbf{n} \cdot \mathbf{v} ds}_{(*)} = \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v} dx, \ \forall \mathbf{v} \in \mathbf{H}^1(\mathcal{K}).$$

$(*)$ arises naturally from an integration by parts, ensures the consistency of the method.

The Local problem

For each $\mathcal{K}_r = \mathcal{K}$, we have the follow local problem

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To introduce the hybrid formulation, consider the local problem

$$\int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) dx - \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u})\mathbf{n} \cdot \mathbf{v} ds + \underbrace{\int_{\partial\mathcal{K}} \frac{\beta_0}{h}(\mathbf{u} - \bar{\mathbf{u}}) \cdot \mathbf{v} ds}_{(**)} = \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v} dx, \ \forall \mathbf{v} \in \mathbf{H}^1(\mathcal{K}).$$

$(**)$ is added to ensure stability [3] where β_0 is a constant independent of h.

[3] I. Babuska, The finite element method with penalty, Math. Comp.,**27**, 221–228, (1973).

The Local problem

For each $\mathcal{K}_r = \mathcal{K}$, we have the follow local problem

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To make the problem symmetric and hence ensures the property of Adjoint consistency[3] we add

$$\int_{\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx - \int_{\partial\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u})\mathbf{n} \cdot \mathbf{v} ds - \int_{\partial\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{v})\mathbf{n} \cdot (\mathbf{u} - \bar{\mathbf{u}}) ds$$
$$+ \int_{\partial\mathcal{K}} \frac{\beta_0}{h}(\mathbf{u} - \bar{\mathbf{u}}) \cdot \mathbf{v} ds = \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v} dx, \ \forall \mathbf{v} \in \mathbf{H}^1(\mathcal{K}).$$

[3] J.A. Nitsche, “Uber ein Variationsprinzip zur Losung Dirichlet-Problemen bei Verwendung von Teilraumen, die keinen Randbedingungen unteworfen sind”, Abh. Math. Sem. Univ. Hamburg, **36**, 9–15, (1971).

The Local problem

For each $\mathcal{K}_r = \mathcal{K}$, we have the follow local problem

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If we add the same term, but with sinal changed (like Baumann[3]) we have the coercivity property.

$$\int_{\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) dx - \int_{\partial\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u})\mathbf{n} \cdot \mathbf{v} ds + \int_{\partial\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{v})\mathbf{n} \cdot (\mathbf{u} - \bar{\mathbf{u}}) ds$$
$$+ \int_{\partial\mathcal{K}} \frac{\beta_0}{h} (\mathbf{u} - \bar{\mathbf{u}}) \cdot \mathbf{v} ds = \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v} dx, \ \forall \mathbf{v} \in \mathbf{H}^1(\mathcal{K}).$$

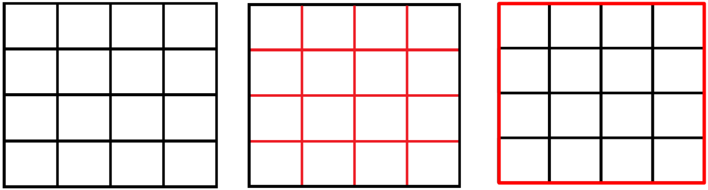
[3] C.E. Baumann and J.T. Oden, "A discontinuous hp finite element method for the Euler and Navier-Stokes equations", Comput. Methods Appl. Mech. Engrg., **175**, 311–341, (1999).

Definitions

But $\bar{\mathbf{u}}$ is unknown in $\partial\mathcal{K}$ and the continuity property must be satisfy. So

The Multiplier $\boldsymbol{\lambda}$

defined as the trace of $\mathbf{u}$: $\boldsymbol{\lambda} = \mathbf{u}|_e$ on each edge $e \in \mathcal{E}_h$.



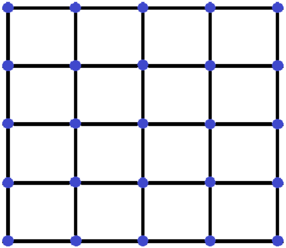
(a) $\mathcal{E}_h$ (b) $\mathcal{E}_h^0$ (c) $\mathcal{E}_h^\partial$

Definitions

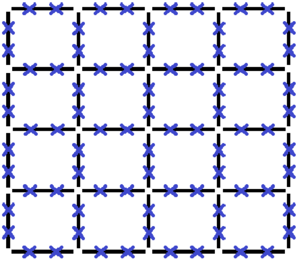
But $\bar{\mathbf{u}}$ is unknown in $\partial\mathcal{K}$ and the continuity property must be satisfy. So

The Multiplier λ

defined as the trace of $\mathbf{u}$: $\boldsymbol{\lambda} = \mathbf{u}|_e$ on each edge $e \in \mathcal{E}_h$.



(d) Continuous multipliers



(e) Discontinuous multipliers

Definitions

Spaces

- Displacement field

$$\mathbf{V}_h^k = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \ : \ \mathbf{v}_h|_{\mathcal{K}} \in [S_k(\mathcal{K})]^2 \quad \forall \mathcal{K} \in \mathcal{T}_h\}$$

- Discontinuous multiplier

$$\mathbf{M}_h^l = \{\boldsymbol{\lambda} \in \mathbf{L}^2(\mathcal{E}_h) : \boldsymbol{\lambda}|_e = [p_l(e)]^2, \ \forall e \in \mathcal{E}_h^0\}$$

- Continuous multiplier

$$\mathbf{M}_h^l = \{\boldsymbol{\lambda} \in \mathbf{C}^0(\mathcal{E}_h) : \boldsymbol{\lambda}|_e = [p_l(e)]^2, \ \forall e \in \mathcal{E}_h^0\}$$

where

- $S_k(\mathcal{K}) = P_k(\mathcal{K})$ (Triangular elements) or
- $S_k(\mathcal{K}) = Q_k(\mathcal{K})$ (Quadrilateral elements),
- $p_l(e)$ is the space of of polynomials of degree at most l on each edge e.

Definitions

Boundary Condition

is weakly imposed using the Nitsche's approach

- the boundary condition $\mathbf{u} = \mathbf{g}$ on Γ is weakly imposed using the Nitsche's approach.

The Stabilized Hybrid Discontinuous Galerkin method[4]

Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) dx - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \mathbf{v}_h ds + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \boldsymbol{\mu}_h ds$$
$$+\theta \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_{\mathcal{K}} \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h) ds + 2\mu \sum_{\mathcal{K}\in\mathcal{T}_h} \beta_1 \int_{\partial\mathcal{K}} (\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) ds$$
$$+ \lambda \sum_{\mathcal{K}\in\mathcal{T}_h} \beta_2 \int_{\partial\mathcal{K}} ((\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_{\mathcal{K}})((\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_{\mathcal{K}}) ds = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v}_h dx,$$

where:

- It arises naturally from an integration by parts, ensures the consistency of the method.

[4] C. O. Faria, A. F. D. Loula, and A. J. B. dos Santos (2014). "Primal Stabilized Hybrid and DG finite element methods for the linear elasticity problem". In: *Computers and Mathematics with Applications* 68, pages 486–507.

The Stabilized Hybrid Discontinuous Galerkin method

Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) dx - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \mathbf{v}_h ds + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \boldsymbol{\mu}_h ds$$

$$+\theta \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_{\mathcal{K}} \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h) ds + 2\mu \sum_{\mathcal{K}\in\mathcal{T}_h} \beta_1 \int_{\partial\mathcal{K}} (\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) ds$$

$$+ \lambda \sum_{\mathcal{K}\in\mathcal{T}_h} \beta_2 \int_{\partial\mathcal{K}} ((\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_{\mathcal{K}})((\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_{\mathcal{K}}) ds = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v}_h dx,$$

where:

- $\theta = 0$: It renders an incomplete method.
- $\theta = -1$: It renders symmetric problem and ensures the property of Adjoint consistency.
- $\theta = 1$: It renders nonsymmetric problem and ensures the property of Coercivity.

The Stabilized Hybrid Discontinuous Galerkin method

Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h):\boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}}\cdot\mathbf{v}_h ds + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}}\cdot\boldsymbol{\mu}_h ds$$
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$$+\lambda \sum_{\mathcal{K}\in\mathcal{T}_h}\beta_2\int_{\partial\mathcal{K}}((\mathbf{u}_h-\boldsymbol{\lambda}_h)\cdot\mathbf{n}_{\mathcal{K}})((\mathbf{v}_h-\boldsymbol{\mu}_h)\cdot\mathbf{n}_{\mathcal{K}})ds = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\mathbf{f}\cdot\mathbf{v}_h dx,$$

where:

- It imposes weakly the continuity of the normal component of the symmetric Cauchy stress tensor.

The Stabilized Hybrid Discontinuous Galerkin method

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$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) dx - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \mathbf{v}_h ds + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \boldsymbol{\mu}_h ds$$
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where:

- β_1 and β_2 are the stabilization parameters dependent on h and can be defined as

$$\beta_1 = \frac{\beta_0}{h} \text{ and } \beta_2 = \frac{\beta_n - \beta_0}{h} \quad \forall e \in \mathcal{E}_h \text{ with } \beta_n > \beta_0 > 0. \tag{6}$$

The Stabilized Hybrid Discontinuous Galerkin method

Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}}\cdot\mathbf{v}_h ds + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}}\cdot\boldsymbol{\mu}_h ds$$

$$+\theta \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_{\mathcal{K}}\cdot(\mathbf{u}_h-\boldsymbol{\lambda}_h)ds + 2\mu \sum_{\mathcal{K}\in\mathcal{T}_h}\beta_1\int_{\partial\mathcal{K}}(\mathbf{u}_h-\boldsymbol{\lambda}_h)\cdot(\mathbf{v}_h-\boldsymbol{\mu}_h)ds$$

$$+\lambda \sum_{\mathcal{K}\in\mathcal{T}_h}\beta_2\int_{\partial\mathcal{K}}((\mathbf{u}_h-\boldsymbol{\lambda}_h)\cdot\mathbf{n}_{\mathcal{K}})((\mathbf{v}_h-\boldsymbol{\mu}_h)\cdot\mathbf{n}_{\mathcal{K}})ds = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\mathbf{f}\cdot\mathbf{v}_h dx,$$

where:

- the unknown $\boldsymbol{\lambda}_h$ is restricted to $\mathbf{M}_h$ and on each $e \in \mathcal{E}_h^{\partial}$ we set $\boldsymbol{\lambda}_h = \mathbf{0}$.
- The boundary condition $\mathbf{u} = \mathbf{g}$ on Γ is weakly imposed using the Nitsche's approach.

Average and Jump Operators

Definition

Let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$, $\mathbf{x}$ be an arbitrary point of the set $e = \partial K^+ \cap \partial K^-$, $\mathbf{n}^+$ and $\mathbf{n}^-$ be the corresponding outward unit normals at that point. For a scalar-valued function, q, a vector-valued function, $\mathbf{v}$, or a matrix-valued function, $\boldsymbol{\tau}$, the averages at $\mathbf{x} \in e$ are as follows:

$$\{\!\{q\}\!\} = \frac{1}{2}(q^+ + q^-), \quad \{\!\{\mathbf{v}\}\!\} = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \quad \{\!\{\boldsymbol{\tau}\}\!\} = \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-) \text{ on } e \in \mathcal{E}_h^0.$$

and the jumps at $\mathbf{x} \in e$ on $e \in \mathcal{E}_h^0$ are given by

$$[\![p]\!] = p^+\mathbf{n}^+ + p^-\mathbf{n}^-, \quad [\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-, \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^-.$$

If $\mathbf{x}$ is on an edge e lying on the boundary $\partial\Omega$, i.e., $e \in \mathcal{E}_h^\partial$, the above average and jump operators are defined by

$$\begin{aligned} \{\!\{p\}\!\} &= p, & \{\!\{\mathbf{v}\}\!\} &= \mathbf{v}, & \{\!\{\boldsymbol{\tau}\}\!\} &= \boldsymbol{\tau}, \\ [\![p]\!] &= p\mathbf{n}, & [\![\mathbf{v}]\!] &= \mathbf{v} \cdot \mathbf{n}, & [\![\boldsymbol{\tau}]\!] &= \boldsymbol{\tau}\mathbf{n}. \end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$.

Average and Jump Operators

Definition

We define a matrix-valued jump $[\![\cdot]\!]$ of a vector $\mathbf{v}$ as in Chen et al.[a] if $\mathbf{x} \in e \in \mathcal{E}_h^0$:

$$[\![\mathbf{v}]\!] = \frac{1}{2}(\mathbf{v}^+ \otimes \mathbf{n}^+ + \mathbf{n}^+ \otimes \mathbf{v}^+ + \mathbf{v}^- \otimes \mathbf{n}^- + \mathbf{n}^- \otimes \mathbf{v}^-),$$

if$\mathbf{x} \in e \in \mathcal{E}_h^\partial$:

$$[\![\mathbf{v}]\!] = \frac{1}{2}(\mathbf{v} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{v}).$$

[a] Y. Chen, J. Huang, X. Huang, and Y. Xu, *On the Local Discontinuous Galerkin method for linear elasticity*, Mathematical Problems in Engineering,**2010**,(2010).

Identity

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{w} ds = \int_{\mathcal{E}_h^0} [\![\boldsymbol{\tau}]\!] \cdot \{\!\{\mathbf{w}\}\!\} \, ds + \int_{\mathcal{E}_h} \{\!\{\boldsymbol{\tau}\}\!\} : [\![\mathbf{w}]\!] ds, \quad (6)$$

The SHDG method

Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that

$$A_{SH}([\mathbf{u}_h, \boldsymbol{\lambda}_h], [\mathbf{v}_h, \boldsymbol{\mu}_h]) = F([\mathbf{v}_h, \boldsymbol{\mu}_h]) \quad \text{for all } [\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l, \tag{7}$$

with $F([\mathbf{v}_h, \boldsymbol{\mu}_h]) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v}_h dx$,

$$\begin{aligned} A_{SH}([\mathbf{u}_h, \boldsymbol{\lambda}_h], [\mathbf{v}_h, \boldsymbol{\mu}_h]) = & \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u}_h) : \varepsilon(\mathbf{v}_h) dx \\ & + \sum_{e\in\mathcal{E}_h} \int_e \left(\theta \,\{\!\{\mathbb{D}\varepsilon(\mathbf{v}_h)\}\!\} : [\![\mathbf{u}_h]\!] - \{\!\{\mathbb{D}\varepsilon(\mathbf{u}_h)\}\!\} : [\![\mathbf{v}_h]\!]\right) ds \\ & + 2\mu \sum_{e\in\mathcal{E}_h} \frac{\beta_0}{2h} \int_e [\![\mathbf{u}_h]\!] : [\![\mathbf{v}_h]\!] ds + \lambda \sum_{e\in\mathcal{E}_h} \frac{\beta_n - \beta_0}{2h} \int_e [\![\mathbf{u}_h]\!][\![\mathbf{v}_h]\!] ds \\ & + \sum_{e\in\mathcal{E}_h^0} \int_e \left(\theta \,[\![\mathbb{D}\varepsilon(\mathbf{v}_h)]\!] \cdot (\{\!\{\mathbf{u}_h\}\!\} - \boldsymbol{\lambda}_h) - [\![\mathbb{D}\varepsilon(\mathbf{u}_h)]\!] \cdot (\{\!\{\mathbf{v}_h\}\!\} - \boldsymbol{\mu}_h)\right) ds \\ & + 2\mu \sum_{e\in\mathcal{E}_h} \frac{2\beta_0}{h} \int_e (\{\!\{\mathbf{u}_h\}\!\} - \boldsymbol{\lambda}_h) \cdot (\{\!\{\mathbf{v}_h\}\!\} - \boldsymbol{\mu}_h) \, ds \\ & + \lambda \sum_{e\in\mathcal{E}_h} \frac{2(\beta_n - \beta_0)}{h} \int_e ((\{\!\{\mathbf{u}_h\}\!\} - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_e)((\{\!\{\mathbf{v}_h\}\!\} - \boldsymbol{\mu}_h) \cdot \mathbf{n}_e) \, ds. \end{aligned} \tag{8}$$

The SHDG method

Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that

$$A_{SH}([\mathbf{u}_h, \boldsymbol{\lambda}_h], [\mathbf{v}_h, \boldsymbol{\mu}_h]) = F([\mathbf{v}_h, \boldsymbol{\mu}_h]) \quad \text{for all } [\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l, \tag{7}$$

with $F([\mathbf{v}_h, \boldsymbol{\mu}_h]) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v}_h dx$,

$$\begin{aligned} A_{SH}([\mathbf{u}_h, \boldsymbol{\lambda}_h], [\mathbf{v}_h, \boldsymbol{\mu}_h]) = & \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbb{D}\varepsilon(\mathbf{u}_h) : \varepsilon(\mathbf{v}_h) dx \\ & + \sum_{e\in\mathcal{E}_h} \int_e \left(\theta \{\!\{\mathbb{D}\varepsilon(\mathbf{v}_h)\}\!\} : [\![\mathbf{u}_h]\!] - \{\!\{\mathbb{D}\varepsilon(\mathbf{u}_h)\}\!\} : [\![\mathbf{v}_h]\!]\right) ds \\ & + 2\mu \sum_{e\in\mathcal{E}_h} \frac{\beta_0}{2h} \int_e [\![\mathbf{u}_h]\!] : [\![\mathbf{v}_h]\!] ds + \lambda \sum_{e\in\mathcal{E}_h} \frac{\beta_n - \beta_0}{2h} \int_e [\![\mathbf{u}_h]\!][\![\mathbf{v}_h]\!] ds \\ & + \sum_{e\in\mathcal{E}_h^0} \int_e \left(\theta\, [\![\mathbb{D}\varepsilon(\mathbf{v}_h)]\!] \cdot (\{\!\{\mathbf{u}_h\}\!\} - \boldsymbol{\lambda}_h) - [\![\mathbb{D}\varepsilon(\mathbf{u}_h)]\!] \cdot (\{\!\{\mathbf{v}_h\}\!\} - \boldsymbol{\mu}_h)\right) ds \\ & + 2\mu \sum_{e\in\mathcal{E}_h} \frac{2\beta_0}{h} \int_e (\{\!\{\mathbf{u}_h\}\!\} - \boldsymbol{\lambda}_h) \cdot (\{\!\{\mathbf{v}_h\}\!\} - \boldsymbol{\mu}_h)\, ds \\ & + \lambda \sum_{e\in\mathcal{E}_h} \frac{2(\beta_n - \beta_0)}{h} \int_e ((\{\!\{\mathbf{u}_h\}\!\} - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_e)((\{\!\{\mathbf{v}_h\}\!\} - \boldsymbol{\mu}_h) \cdot \mathbf{n}_e)\, ds. \end{aligned} \tag{8}$$

Numerical Analysis

- Compressible case[4]
- Incompressible case[5]

[4] C. O. Faria, A. F. D. Loula, and A. J. B. dos Santos (2014). "Primal Stabilized Hybrid and DG finite element methods for the linear elasticity problem". In: *Computers and Mathematics with Applications* 68, pages 486–507.

[5] Cristiane O. Faria, Abimael F.D. Loula, and Antônio J.B. dos Santos (2013). "STABILIZED HYBRIDIZED FINITE ELEMENT METHOD FOR INCOMPRESSIBLE AND NEARLY INCOMPRESSIBLE ELASTICITY". In: *XXXIV Congresso Ibero Latino Americano de Métodos Computacionais em Engenharia (XXXIV CILAMCE)*. 10–13 de novembro, Pirenópolis – GO, Brasil.

Decoupling

SHDG formulation: Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$
\begin{aligned}
&\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \mathbf{v}_h ds + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_{\mathcal{K}} \cdot \boldsymbol{\mu}_h ds \\
&\quad + \theta \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_{\mathcal{K}} \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h)ds + 2\mu \sum_{\mathcal{K}\in\mathcal{T}_h} \beta_1 \int_{\partial\mathcal{K}} (\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)ds \\
&\quad + \lambda \sum_{\mathcal{K}\in\mathcal{T}_h} \beta_2 \int_{\partial\mathcal{K}} ((\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_{\mathcal{K}})((\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_{\mathcal{K}})ds = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \mathbf{f} \cdot \mathbf{v}_h dx,
\end{aligned}
$$

Decoupling

Local problems: Find $\mathbf{u}_h|_K \in \mathbf{V}_h^k(K) = \mathbf{V}_h^k|_K$, such that, for all $\mathbf{v}_h|_K \in \mathbf{V}_h^k(K)$,

$$\int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \mathbf{v}_h ds + \theta \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h)ds$$
$$+ 2\mu \int_{\partial K} \beta_1(\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{v}_h ds + \lambda \int_{\partial K} \beta_2((\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K)ds = \int_K \mathbf{f} \cdot \mathbf{v}_h dx, \tag{9}$$

Global Problem: Find $\boldsymbol{\lambda}_h \in \mathbf{M}_h^l$, such that, for all $\boldsymbol{\mu}_h \in \mathbf{M}_h^l$,

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \boldsymbol{\mu}_h ds - \sum_{K\in\mathcal{T}_h} 2\mu \int_{\partial K} \beta_1(\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \boldsymbol{\mu}_h ds$$
$$- \sum_{K\in\mathcal{T}_h} \lambda \int_{\partial K} \beta_2((\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_K)(\boldsymbol{\mu}_h \cdot \mathbf{n}_K)ds = 0. \tag{10}$$

A modified DG formulation

Adopting discontinuous interpolations for the multiplier, $\boldsymbol{\lambda}_h$ can be eliminated at the level of an edge e on each element $K \in \mathcal{T}_h$ to obtain a DG method in the primal variable $\mathbf{u}_h$ only.

For $l \geq k$ and $\beta_2 = 0$, the Global problem can be easily solved exactly, obtaining

$$\boldsymbol{\lambda}_h = \{\mathbf{u}_h\} - \frac{1}{2\beta_1 \, 2\mu}[\![\mathbb{D}\varepsilon(\mathbf{u}_h)]\!] \text{ on each interior edge } e \in \mathcal{E}_h^0. \quad (11)$$

Replacing (11) in the Local Problem , we obtain the following modified DG formulation: Find $\mathbf{u}_h \in \mathbf{V}_h^k$ such that

$$a_{MDG}(\mathbf{u}_h, \mathbf{v}_h) = f(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h^k \quad (12)$$

with

$$a_{MDG}(\mathbf{u}_h, \mathbf{v}_h) = a_{DG}(\mathbf{u}_h, \mathbf{v}_h) + \frac{\theta}{2\mu} \sum_{e \in \mathcal{E}_h} \int_e \frac{1}{2\beta_1} [\![\mathbb{D}\varepsilon(\mathbf{u}_h)]\!] \cdot [\![\mathbb{D}\varepsilon(\mathbf{v}_h)]\!] ds. \quad (13)$$

A modified DG formulation

$$\frac{\theta}{2\mu}\sum_{e\in\mathcal{E}_h^0}\int_e \frac{1}{2\beta_1}[\![\mathbb{D}\varepsilon(\mathbf{u}_h)]\!]\cdot[\![\mathbb{D}\varepsilon(\mathbf{v}_h)]\!]ds$$

- does not affect the consistency
- for sufficiently large β_0, stability and continuity of $a_{SH}(\cdot,\cdot)$ is proved using the same arguments adopted for $a_{DG}(\cdot,\cdot)$.
- the error estimates are applicable to the stabilized DG formulation.

Hybridization

Find $\mathbf{u}_h|_K \in \mathbf{V}_k(K)$, for each $K \in \mathcal{T}_h$, and $\boldsymbol{\lambda}_h \in M_h$ such that

$$a_K(\mathbf{u}_h, \mathbf{v}_h) + b_K(\boldsymbol{\lambda}_h, \mathbf{v}_h) = f_K(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_k(K), \tag{11}$$

$$\sum_{K \in \mathcal{T}_h} c_K(\mathbf{u}_h, \boldsymbol{\mu}_h) + \sum_{K \in \mathcal{T}_h} d_K(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) = \mathbf{0}, \quad \forall \boldsymbol{\mu}_h \in \mathbf{M}_h, \tag{12}$$

where

$$\begin{aligned}
a_K(\mathbf{u}_h, \mathbf{v}_h) \quad &:= \quad \int_K \mathbb{D}\varepsilon(\mathbf{u}_h) : \varepsilon(\mathbf{v}_h)dx - \int_{\partial K} \mathbb{D}\varepsilon(\mathbf{u}_h)\mathbf{n}_K \cdot \mathbf{v}_h ds + \theta \int_{\partial K} \mathbb{D}\varepsilon(\mathbf{v}_h)\mathbf{n}_K \cdot \mathbf{u}_h ds \\
&+ \quad 2\mu \int_{\partial K} \beta_1 \, \mathbf{u}_h \cdot \mathbf{v}_h ds + \lambda \int_{\partial K} \beta_2 \, (\mathbf{u}_h \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K) ds, \\
b_K(\boldsymbol{\lambda}_h, \mathbf{v}_h) \quad &:= \quad -\theta \int_{\partial K} \mathbb{D}\varepsilon(\mathbf{v}_h)\mathbf{n} \cdot \boldsymbol{\lambda}_h ds - 2\mu \int_{\partial K} \beta_1 \boldsymbol{\lambda}_h \cdot \mathbf{v}_h ds \\
&- \quad \lambda \int_{\partial K} \beta_2 (\boldsymbol{\lambda}_h \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K) ds,
\end{aligned}$$

and the linear functional

$$f_K(\mathbf{v}_h) = \int_K \mathbf{f} \cdot \mathbf{v}_h dx,$$

Hybridization

Find $\mathbf{u}_h|_K \in \mathbf{V}_k(K)$, for each $K \in \mathcal{T}_h$, and $\boldsymbol{\lambda}_h \in M_h$ such that

$$a_K(\mathbf{u}_h, \mathbf{v}_h) + b_K(\boldsymbol{\lambda}_h, \mathbf{v}_h) = f_K(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_k(K), \tag{11}$$

$$\sum_{K\in\mathcal{T}_h} c_K(\mathbf{u}_h, \boldsymbol{\mu}_h) + \sum_{K\in\mathcal{T}_h} d_K(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) = \mathbf{0}, \quad \forall \boldsymbol{\mu}_h \in \mathbf{M}_h, \tag{12}$$

where

$$\begin{aligned}
c_K(\mathbf{u}_h, \boldsymbol{\mu}_h) \quad &:= \quad \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \boldsymbol{\mu}_h ds \\
&- \quad 2\mu \int_{\partial K} \beta_1\, \mathbf{u}_h \cdot \boldsymbol{\mu}_h ds - \lambda \int_{\partial K} \beta_2\, (\mathbf{u}_h \cdot \mathbf{n}_K)(\boldsymbol{\mu}_h \cdot \mathbf{n}_K) ds, \\
d_K(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) \quad &:= \quad 2\mu \int_{\partial K} \beta_1 \boldsymbol{\lambda}_h \cdot \boldsymbol{\mu}_h ds + \lambda \int_{\partial K} \beta_2 (\boldsymbol{\lambda}_h \cdot \mathbf{n}_K)(\boldsymbol{\mu}_h \cdot \mathbf{n}_K) ds.
\end{aligned}$$

and the linear functional

$$f_K(\mathbf{v}_h) = \int_K \mathbf{f} \cdot \mathbf{v}_h dx,$$

Hybridization

or in matrix form:

$$\mathbf{A}_K \mathbf{U} + \mathbf{B}_K \mathbf{\Lambda} = \mathbf{F}_K. \tag{13}$$

$$\sum_{K \in \mathcal{T}_h} \mathbf{C}_K \mathbf{U} + \sum_{K \in \mathcal{T}_h} \mathbf{D}_K \mathbf{\Lambda} = \mathbf{0}. \tag{14}$$

Inverting the local matrix $\mathbf{A}_K$ we have

$$\mathbf{U} = \mathbf{A}_K^{-1}(\mathbf{F}_K - \mathbf{B}_K \mathbf{\Lambda}). \tag{15}$$

Replacing (15) in (14), we obtain the global system in the multiplier only:

$$\sum_{K \in \mathcal{T}_h} (\mathbf{D}_K - \mathbf{C}_K \mathbf{A}_K^{-1} \mathbf{B}_K) \mathbf{\Lambda} = \sum_{K \in \mathcal{T}_h} -\mathbf{C}_K^T \mathbf{A}_K^{-1} \mathbf{F}_K. \tag{16}$$

After solving the global (16), the vector $\mathbf{U}$ is obtained from (15) with $\mathbf{\Lambda}$ given by (16).

Stress and displacement local post-processing

Classically, stresses are computed indirectly using the displacement approximation and the constitutive equation only.

$$\boldsymbol{\sigma}_h = \mathbb{D}\varepsilon(\mathbf{u}_h) \tag{17}$$

converges at best with the following rates in $\mathbf{L}^2$ and $\mathbf{H}(\mathrm{div})$ norms:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2} = \|\mathbb{D}\varepsilon(\mathbf{u}) - \mathbb{D}\varepsilon(\mathbf{u}_h)\|_{\mathbf{L}^2} = Ch^k, \tag{18}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div})} = Ch^{k-1}. \tag{19}$$

Stress and displacement local post-processing

By solving at each element $K \in \mathcal{T}_h$ the local problem in stress and displacement fields:

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } K, \\ \mathbb{A}\boldsymbol{\sigma}(\mathbf{u}) &= \boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } K, \\ \mathbf{u} &= \boldsymbol{\lambda}_h && \text{on } \partial K, \end{aligned} \tag{20}$$

with $\boldsymbol{\lambda}_h$ given by the solution of the global problem (16) and $\mathbb{A} = \mathbb{D}^{-1}$.
Stress and displacement approximations $[\boldsymbol{\sigma}_{PP}, \mathbf{u}_{PP}]$ for $[\boldsymbol{\sigma}, \mathbf{u}]$, solution of (20), are obtained in the finite dimension spaces

$$\mathbb{W}_h^k(K) = \{\tau_{i,j} \in S_k(K),\ \ \tau_{i,j} = \tau_{j,i},\ \ i,j = 1,2\} \text{ and } \mathbf{V}_h^k(K) = \{\mathbf{v}_i \in S_k(K),\ \ i = 1,2\}. \tag{21}$$

Considering the following residual form on each element $K \in \mathcal{T}_h$

$$\begin{aligned} &\int_K \mathbb{A}\boldsymbol{\sigma}_{PP} : \boldsymbol{\tau}_h dx + \int_K \mathbf{u}_{PP} \cdot \operatorname{div}\boldsymbol{\tau}_h dx - \int_{\partial K} \boldsymbol{\lambda}_h \cdot \boldsymbol{\tau}_h \mathbf{n}_K ds + \int_K \operatorname{div}\boldsymbol{\sigma}_{PP} \cdot \mathbf{v}_h dx + \int_K \mathbf{f} \cdot \mathbf{v}_h dx \\ &+ \delta_1 \int_K (\mathbb{A}\boldsymbol{\sigma}_{PP} - \varepsilon(\mathbf{u}_{PP})) : (\boldsymbol{\tau}_h - \mathbb{D}\varepsilon(\mathbf{v}_h))dx + \frac{\delta_2}{2\mu} \int_K (\operatorname{div}\boldsymbol{\sigma}_{PP} + \mathbf{f}) \cdot \operatorname{div}\boldsymbol{\tau}_h dx \\ &+ 2\mu \int_{\partial K} \beta_1 (\mathbf{u}_{PP} - \boldsymbol{\lambda}_h) \cdot \mathbf{v}_h ds + \lambda \int_{\partial K} \beta_2 ((\mathbf{u}_{PP} - \boldsymbol{\lambda}_h) \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K) ds = 0. \end{aligned} \tag{22}$$

Stress and displacement local post-processing

Given $\boldsymbol{\lambda}_h$, find $[\boldsymbol{\sigma}_{PP}|_K, \mathbf{u}_{PP}|_K] \in \mathbb{W}_h^k(K) \times \mathbf{V}_h^k(K)$, such that

$$a_{PP}([\boldsymbol{\sigma}_{PP}, \mathbf{u}_{PP}], [\boldsymbol{\tau}_h, \mathbf{v}_h]) = f_{PP}([\boldsymbol{\tau}_h, \mathbf{v}_h]) \; \forall \; [\boldsymbol{\tau}_h|_K, \mathbf{v}_h|_K] \in \mathbb{W}_h^k(K) \times \mathbf{V}_h^k(K), \tag{23}$$

with

$$\begin{aligned} a_{PP}([\boldsymbol{\sigma}_{PP}, \mathbf{u}_{PP}], [\boldsymbol{\tau}_h, \mathbf{v}_h]) &= \int_K \mathbb{A}\boldsymbol{\sigma}_{PP} : \boldsymbol{\tau}_h dx + \int_K \mathbf{u}_{PP} \cdot \mathrm{div}\boldsymbol{\tau}_h dx + \int_K \mathrm{div}\boldsymbol{\sigma}_{PP} \cdot \mathbf{v}_h dx \\ &+ \delta_1 \int_K (\mathbb{A}\boldsymbol{\sigma}_{PP} - \boldsymbol{\varepsilon}(\mathbf{u}_{PP})) : (\boldsymbol{\tau}_h - \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h))dx + \frac{\delta_2}{2\mu} \int_K \mathrm{div}\boldsymbol{\sigma}_{PP} \cdot \mathrm{div}\boldsymbol{\tau}_h dx \\ &+ 2\mu \int_{\partial K} \beta_1 \mathbf{u}_{PP} \cdot \mathbf{v}_h ds + \lambda \int_{\partial K} \beta_2 (\mathbf{u}_{PP} \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K) ds \end{aligned} \tag{24}$$

$$\begin{aligned} f_{PP}([\boldsymbol{\tau}_h, \mathbf{v}_h]) = &\int_{\partial K} \boldsymbol{\lambda}_h \cdot \boldsymbol{\tau}_h \mathbf{n}_K ds - \frac{\delta_2}{2\mu} \int_K \mathbf{f} \cdot \mathrm{div}\boldsymbol{\tau}_h dx - \int_K \mathbf{f} \cdot \mathbf{v}_h dx \\ &+ 2\mu \int_{\partial K} \beta_1 \boldsymbol{\lambda}_h \cdot \mathbf{v}_h ds + \lambda \int_{\partial K} \beta_2 (\boldsymbol{\lambda}_h \cdot \mathbf{n}_K)(\mathbf{v}_h \cdot \mathbf{n}_K) ds. \end{aligned} \tag{25}$$

For appropriate choices of the stabilization parameters δ_1 and δ_2, we have observed the following convergence rate for the post-processed stress:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{PP}\|_{\mathbf{H}(\mathrm{div})} = Ch^k \tag{26}$$

which is one order higher than that observed for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div})}$.

Convergence studies - Compressible case

- For plane-strain problem defined on square domain $\Omega = (0,1) \times (0,1)$ with homogeneous boundary conditions,
- Elasticity modulus $E = 1$, Poisson ratio $\nu = 0.3$,
- Forcing term:

$$f_1(x,y) = \mu\ \cos(\pi x - \pi y) - 2\mu\ \cos(\pi x + \pi y) - \lambda\ \cos(\pi x + \pi y) \tag{27}$$

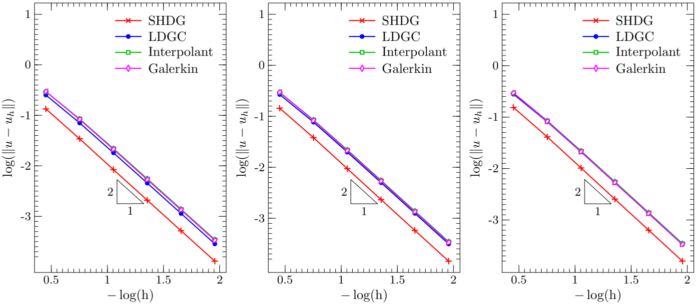
$$f_2(x,y) = \mu\ \cos(\pi x - \pi y) - 2\mu\ \cos(\pi x + \pi y) - \lambda\ \cos(\pi x + \pi y) \tag{28}$$

- The exact solution is given by

$$u_1(x,y) = \frac{1}{\pi^2}\sin(\pi x)\sin(\pi y) \tag{29}$$

$$u_2(x,y) = \frac{1}{\pi^2}\sin(\pi x)\sin(\pi y). \tag{30}$$

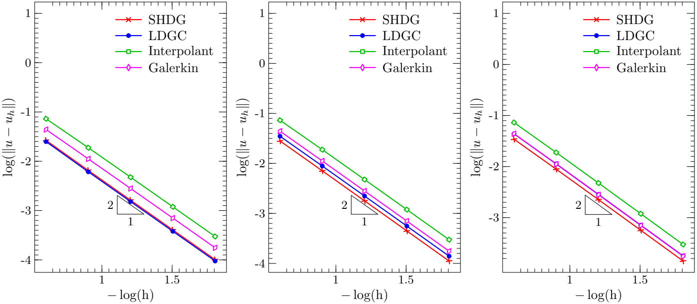
Convergence studies - Compressible case



(f) $P_1 - p_1$ $(\theta = -1)$ (g) $P_1 - p_1$ $(\theta = 0)$ (h) $P_1 - p_1$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{L}^2$ norm for $\mathbf{u}_h$ of LDGC and SHDG approximations compared to Galerkin and interpolant with $\beta_0 = \beta_n = 8$.

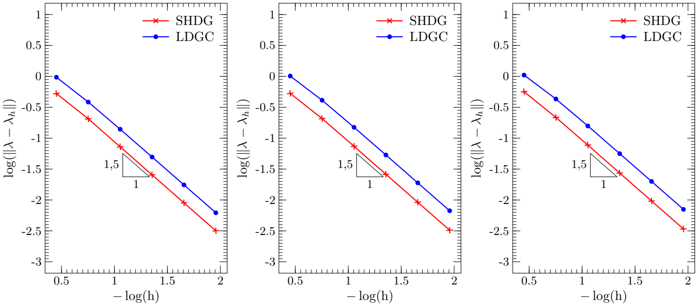
Convergence studies - Compressible case



(a) $Q_1 - p_1$ $(\theta = -1)$ (b) $Q_1 - p_1$ $(\theta = 0)$ (c) $Q_1 - p_1$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{L}^2$ norm for $\mathbf{u}_h$ of LDGC and SHDG approximations compared to Galerkin and interpolant with $\beta_0 = \beta_n = 8$.

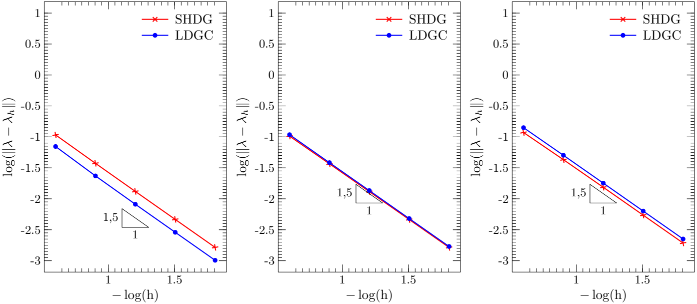
Convergence studies - Compressible case



(a) $P_1 - p_1$ $(\theta = -1)$ (b) $P_1 - p_1$ $(\theta = 0)$ (c) $P_1 - p_1$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{L}^2$ norm for λ_h of LDGC and SHDG approximations with $\beta_0 = \beta_n = 8$.

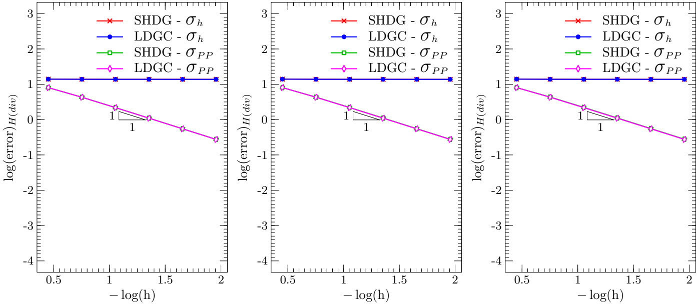
Convergence studies - Compressible case



(a) $Q_1 - p_1$ $(\theta = -1)$ (b) $Q_1 - p_1$ $(\theta = 0)$ (c) $Q_1 - p_1$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{L}^2$ norm for λ_h of LDGC and SHDG approximations with $\beta_0 = \beta_n = 8$.

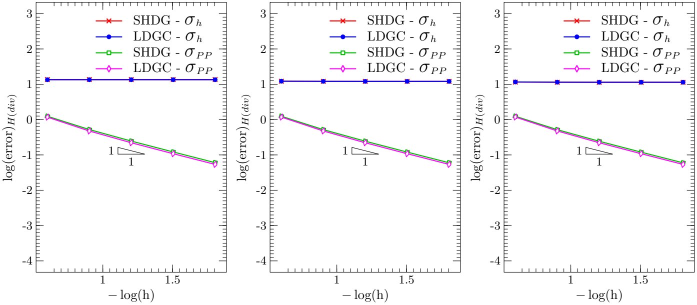
Convergence studies - Compressible case



(a) $P_1 - p_1(P1)$ $(\theta = -1)$ (b) $P_1 - p_1(P1)$ $(\theta = 0)$ (c) $P_1 - p_1(P1)$ $(\theta = 1)$

Figure : h-convergence study in $H(\mathrm{div})$ norm for σ_h of LDGC and SHDG approximations using the constitutive equation and a post-processing technique with $\beta_0 = \beta_n = 8, \delta_1 = 1$ and $\delta_2 = -1/2$.

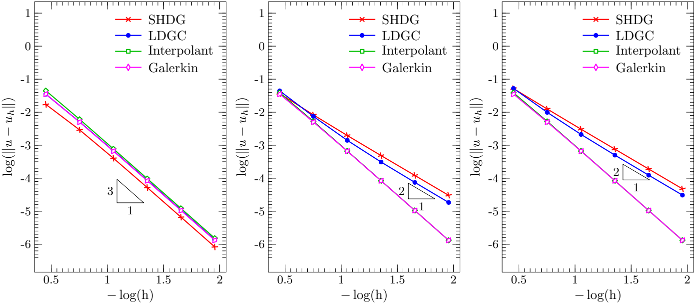
Convergence studies - Compressible case



(a) $Q_1 - p_1(Q1)$ $(\theta = -1)$ (b) $Q_1 - p_1(Q1)$ $(\theta = 0)$ (c) $Q_1 - p_1(Q1)$ $(\theta = 1)$

Figure : h-convergence study in $H(\mathrm{div})$ norm for σ_h of LDGC and SHDG approximations using the constitutive equation and a post-processing technique with $\beta_0 = \beta_n = 8, \delta_1 = 1$ and $\delta_2 = -1/2$.

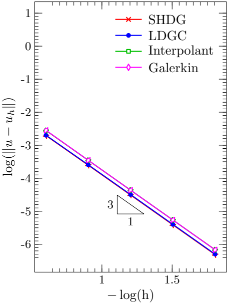
Convergence studies - Compressible case



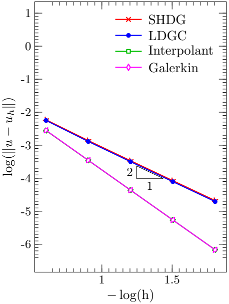
(a) $P_2 - p_2(P2)$ $(\theta = -1)$ (b) $P_2 - p_2(P2)$ $(\theta = 0)$ (c) $P_2 - p_2(P2)$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{L}^2$ norm for $\mathbf{u}_h$ of LDGC and SHDG approximations compared to Galerkin and interpolant with $\beta_0 = \beta_n = 18$.

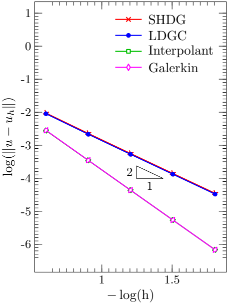
Convergence studies - Compressible case



(a) $Q_2 - p_2(Q2)$ ($\theta = -1$)



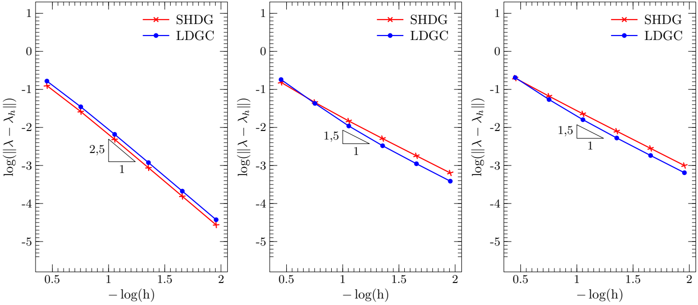
(b) $Q_2 - p_2(Q2)$ ($\theta = 0$)



(c) $Q_2 - p_2(Q2)$ ($\theta = 1$)

Figure : h-convergence study in $\mathbf{L}^2$ norm for $\mathbf{u}_h$ of LDGC and SHDG approximations compared to Galerkin and interpolant with $\beta_0 = \beta_n = 18$.

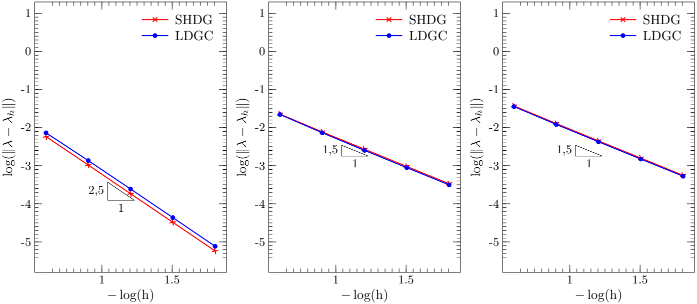
Convergence studies - Compressible case



(a) $P_2 - p_2(P2)$ $(\theta = -1)$ (b) $P_2 - p_2(P2)$ $(\theta = 0)$ (c) $P_2 - p_2(P2)$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{L}^2$ norm for λ_h of LDGC and SHDG approximations with $\beta_0 = \beta_n = 18$.

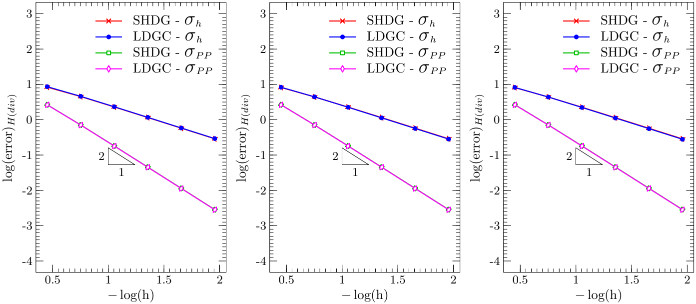
Convergence studies - Compressible case



(a) $Q_2 - p_2(Q2)$ $(\theta = -1)$ (b) $Q_2 - p_2(Q2)$ $(\theta = 0)$ (c) $Q_2 - p_2(Q2)$ $(\theta = 1)$

Figure : *h*-convergence study in $\mathbf{L}^2$ norm for λ_h of LDGC and SHDG approximations with $\beta_0 = \beta_n = 18$.

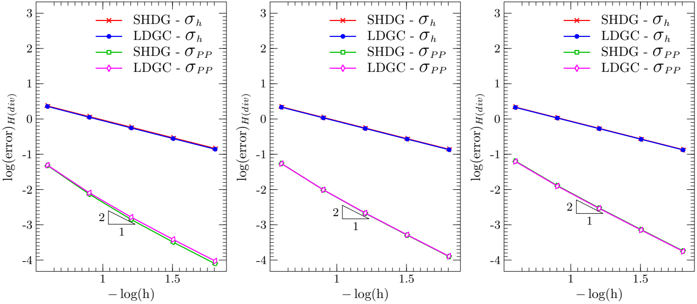
Convergence studies - Compressible case



(a) $P_2 - p_2(P2)$ $(\theta = -1)$ (b) $P_2 - p_2(P2)$ $(\theta = 0)$ (c) $P_2 - p_2(P2)$ $(\theta = 1)$

Figure : h-convergence study in $\mathbf{H}(\mathrm{div})$ norm for the stress of LDGC and SHDG approximations using the constitutive equation σ_h and a post-processing technique $\sigma_P P$ with $\beta_0 = \beta_n = 18, \delta_1 = 1$ and $\delta_2 = -1/2$.

Convergence studies - Compressible case



(a) $Q_2 - p_2(Q2)$ ($\theta = -1$) (b) $Q_2 - p_2(Q2)$ ($\theta = 0$) (c) $Q_2 - p_2(Q2)$ ($\theta = 1$)

Figure : h-convergence study in $\mathbf{H}(\mathrm{div})$ norm for the stress of LDGC and SHDG approximations using the constitutive equation σ_h and a post-processing technique $\sigma_P P$ with $\beta_0 = \beta_n = 18, \delta_1 = 1$ and $\delta_2 = -1/2$.

Convergence studies - Incompressible case

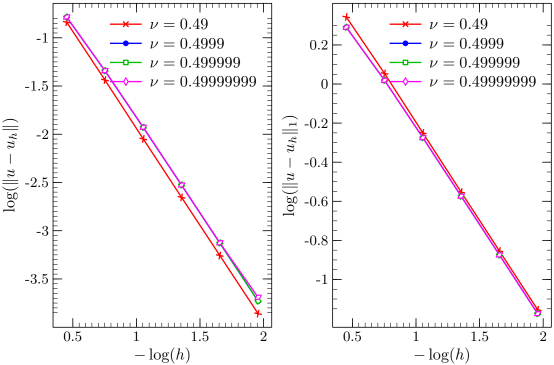
- For plane-strain problem defined on square domain $\Omega = (0,1) \times (0,1)$ with homogeneous boundary conditions,
- Elasticity modulus $E = 1$,
- Forcing term:

$$f_1(x,y) = (2\nu(2\mu+\lambda) - (\mu+\lambda))\sin(\pi x)\ \cos(\pi y) \tag{31}$$
$$f_2(x,y) = (2\nu(2\mu+\lambda) - (3\mu+\lambda))\sin(\pi y)\ \cos(\pi x) \tag{32}$$

- The exact solution is given by

$$u_1(x,y) = \frac{\nu}{\pi^2}\sin(\pi x)\cos(\pi y) \tag{33}$$
$$u_2(x,y) = \frac{(\nu-1)}{\pi^2}\cos(\pi x)\sin(\pi y). \tag{34}$$

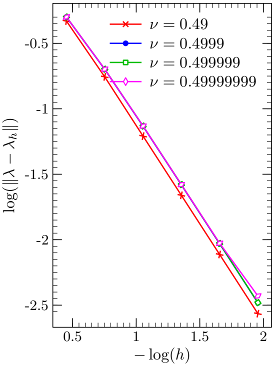
Convergence studies - Incompressible case



(a) $L^2(\Omega)$ ($O(h^2)$) (b) $H^1(\Omega)$ ($O(h)$)

Figure : Convergence study for $\mathbf{u}_h$ in (a) $L^2(\Omega)$ norm) and (b) $H^1(\Omega)$ seminorm of SHDG approximations with descontinuous multiplier, $\beta_0 = 2$, $\beta_n = 7$, $\delta_1 = 40$ and $\delta_2 = -1/2$.

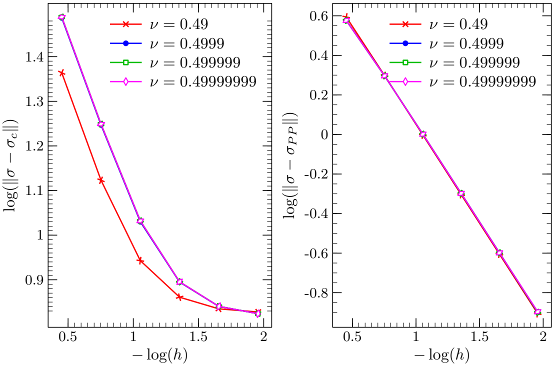
Convergence studies - Incompressible case



(a) λ_h ($O(h^{1.5})$)

Figure : Convergence study for λ_h in $L^2(\mathcal{E}_h)$ norm of SHDG approximations with descontinuous multiplier, $\beta_0 = 2$, $\beta_n = 7$, $\delta_1 = 40$ and $\delta_2 = -1/2$.

Convergence studies - Incompressible case



(a) σ_h (b) σ_{PP} $(O(h^1))$

Figure : Convergence study for σ in $H(\mathrm{div})$ norm of SHDG approximations with descontinuous multiplier, $\beta_0 = 2$, $\beta_n = 7$, $\delta_1 = 40$ and $\delta_2 = -1/2$.

Conclusions

- The Hybrid methods preserve the main properties of the DG method but with reduced computational cost.
- Is easily implemented using the same data structure of continuous Galerkin methods.
- Numerical results show optimal rates of convergence for the primal variable $\mathbf{u}_h$ and for the Lagrange multiplier $\boldsymbol{\lambda}_h$.
- A local post-processing based on the multiplier approximation and residual forms of the constitutive and equilibrium equations at the element level is proposed to recover stress approximations with observed optimal rates of convergence in $\mathbf{H}(\mathrm{div})$ norm.

Next Stages

- Numerical analysis for the local post-processing.
- Numerical studies using diferent orders for displacement, Lagrange multiplier and Stress tension
- Numerical studies using irregular meshes

THANK YOU!!!