

Fourier Transform: part 2

SCC0251/5830 – Image Processing

Prof. Moacir Ponti

Instituto de Ciências Matemáticas e de Computação – USP

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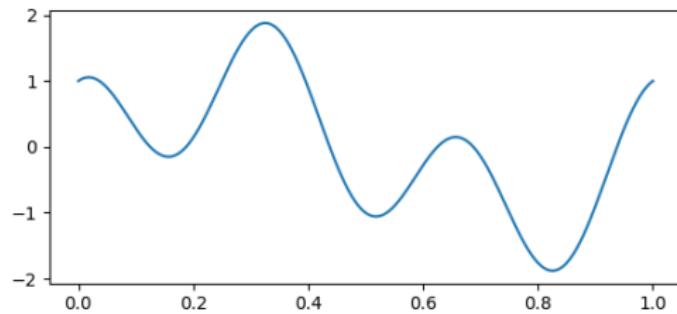
Agenda

- 1 Big Picture
- 2 Discrete Fourier Transform
 - Convolution Theorem
- 3 Filtering in frequency domain
- 4 Fast Fourier Transform (FFT)

Fourier Transform

It allows us to see the frequency content of a given signal.

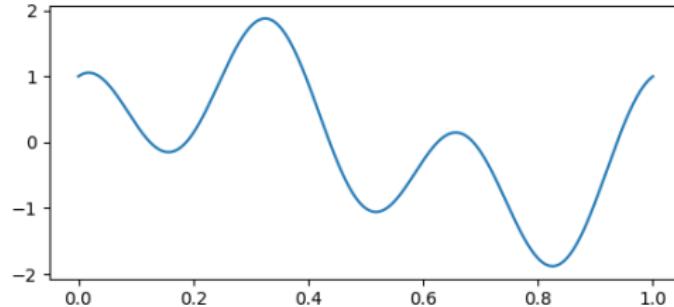
$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi t\omega} dt \\ &= \int_{-\infty}^{\infty} f(t) \cos(2\pi t\omega) dt + j \int_{-\infty}^{\infty} f(t) \sin(2\pi t\omega) dt \end{aligned}$$



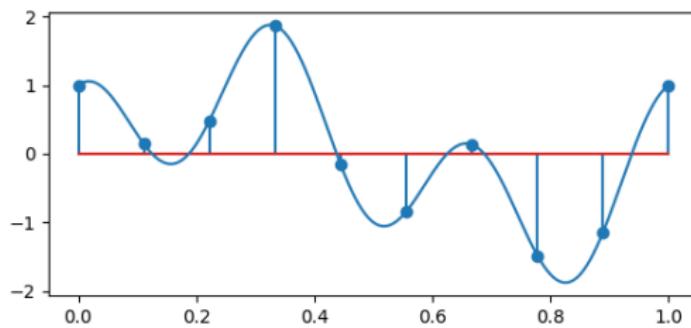
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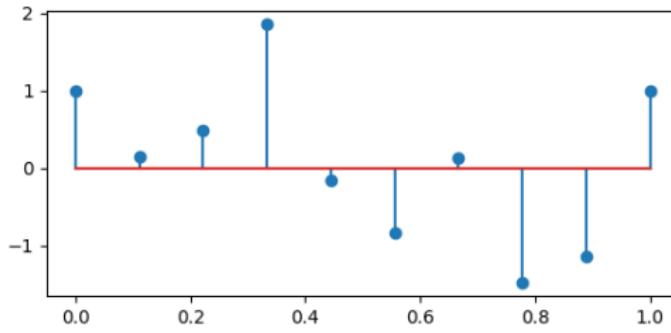
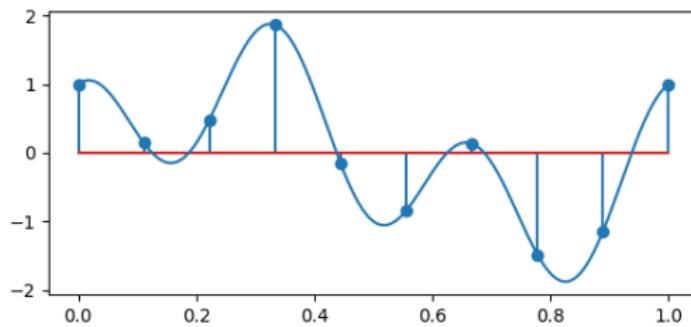
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Fourier Transform



Fourier Transform



Discrete Fourier Transform

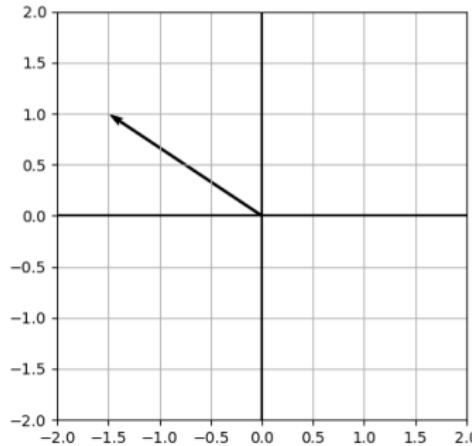
for N samples:

$$\begin{aligned} F[k] &= \sum_{n=0}^{N-1} f[n] e^{-j2\pi n \frac{k}{N}} \\ &= A[k] + j B[k] \end{aligned}$$

Fourier Transform

For each frequency a complex exponential with:

- the relative amplitude of the cosine (real part) and of the sine (imaginary part) as a function of different frequencies,
- the representation of the signal in the **frequency domain**:
 - $A[k] = \text{Re}(F[k])$
 - $B[k] = \text{Im}(F[k])$

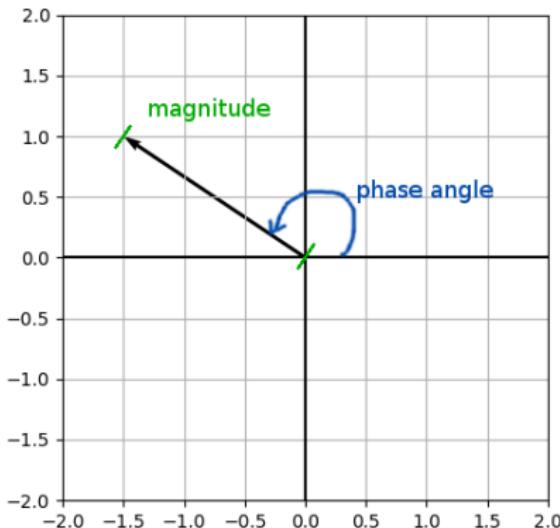


Fourier Transform

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-j2\pi n \frac{k}{N}}$$

- evaluating $F[k]$ on different k , we obtain the **amplitudes of** cosines (real part) and sines (imaginary part) so that we can reconstruct $f[n]$ if needed.

Fourier Transform



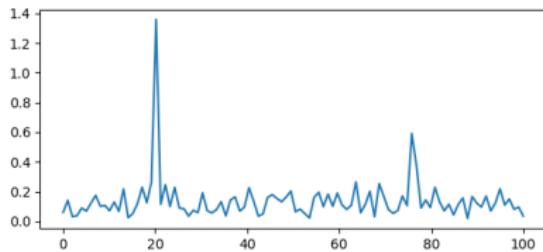
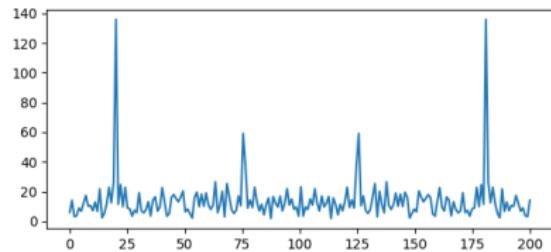
$$|F[k]| = \sqrt{A[k]^2 + B[k]^2}$$

$$\phi(F[k]) = \arctan\left(\frac{A[k]}{B[k]}\right)$$

Fourier Transform: practical considerations

- Cosine is an odd (anti-symmetric) function
- Sine is an even (symmetric) function

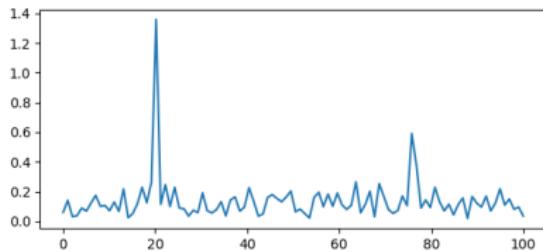
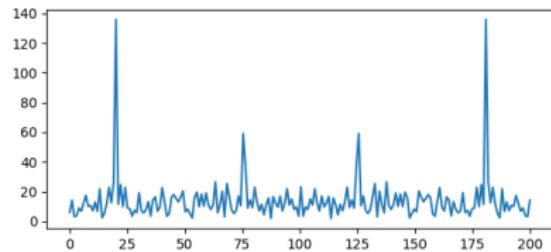
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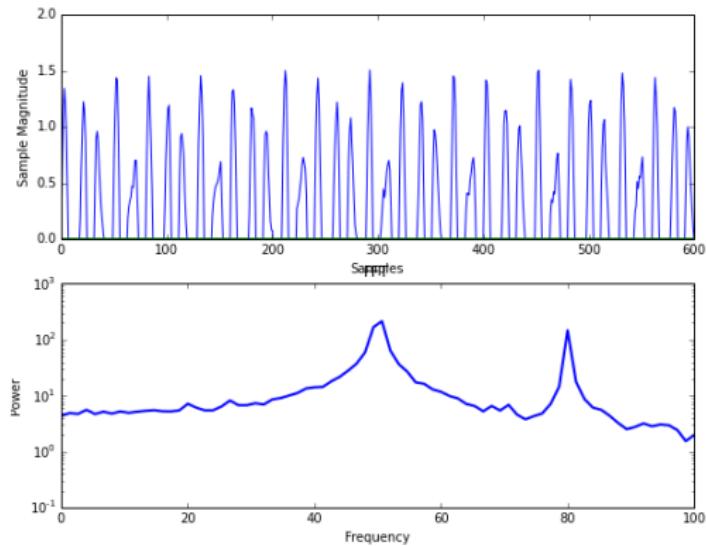
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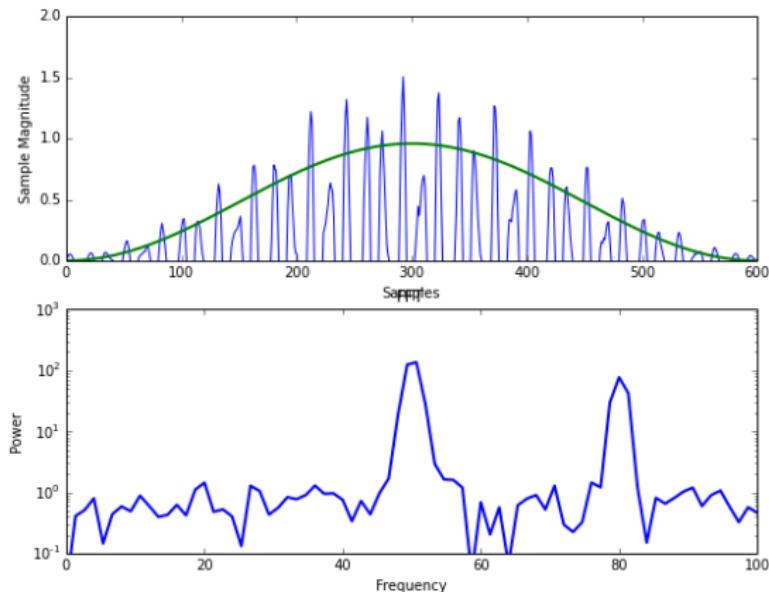
The maximum frequency $N/2$ has an important relationship with the Nyquist-Shannon Theorem of sampling.

Fourier Transform: noise and abrupt signals

- Because we start to measure at $k = 0$ and stop at $k = N - 1$, this can be considered high frequency content, hampering the analysis
- To prevent that, it is possible to use a windowing function, to obtain a signal that increases its amplitude more slowly



Fourier Transform: noise and abrupt signals



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Discrete Fourier Transform 2D (DFT)

$$F(u, v) = \frac{1}{\sqrt{mn}} \sum_{x=0}^{n-1} \sum_{y=0}^{m-1} f(x, y) e^{-j2\pi(ux/n + vy/m)},$$

$f(x, y)$ is a function representing an image with size $n \times m$.

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- considering the 2-D case: x, y are coordinates, u, v are frequencies in each direction.
- as in 1-D, we evaluate it for a range of u and v :
 - $u = 0, 1, \dots, n - 1$ and $v = 0, 1, \dots, m - 1$

Inverse Discrete Fourier Transform 2D (IDFT)

$$f(x, y) = \frac{1}{\sqrt{nm}} \sum_{u=0}^{n-1} \sum_{v=0}^{m-1} F(u, v) e^{j2\pi(ux/n+vy/m)},$$

for $x = 0, 1, \dots, n - 1$ and $y = 0, 1, \dots, m - 1$

Spectrum and phase

- Writing the DFT-2D in the polar form

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- the *power spectrum* is:

$$P(u, v) = |F(u, v)|^2$$

Properties

- From Fourier Transform

$$F(0, 0) = \frac{1}{\sqrt{nm}} \sum_{x=0}^{n-1} \sum_{y=0}^{m-1} f(x, y),$$

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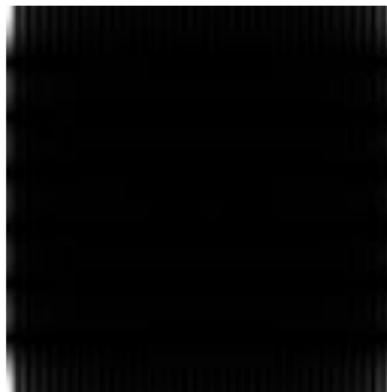
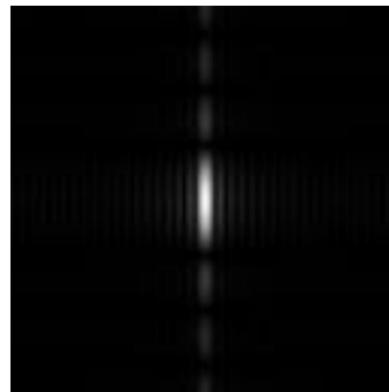
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- the phase angle is odd (anti-symmetric):

$$\phi(u, v) = -\phi(-u, -v)$$

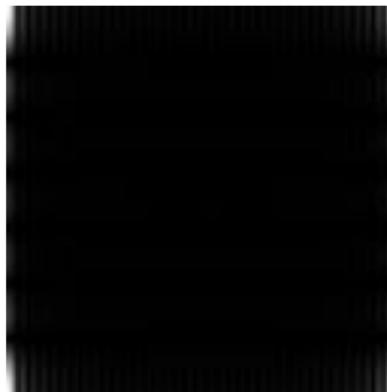
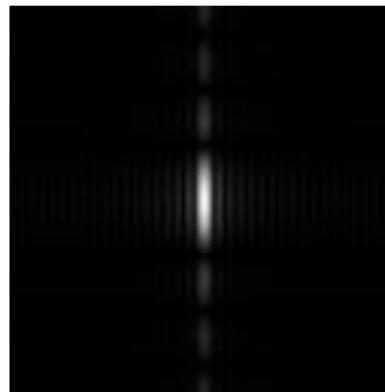
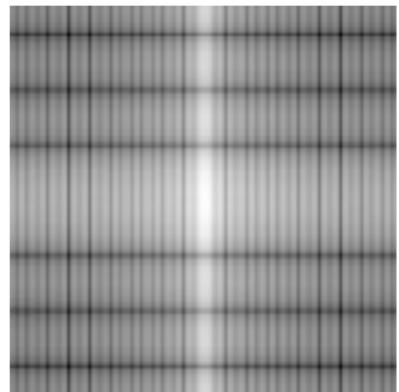
Plotting the Fourier spectrum

- Re-positioning of the quadrants (*FFT shift*) by using coordinates $(-1)^{x+y}$

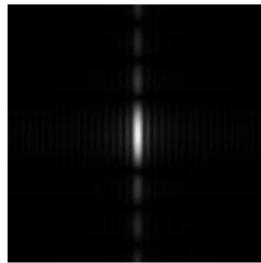
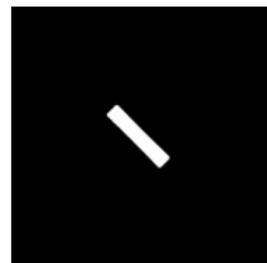
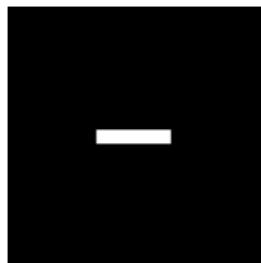
 $|F(u, v)|$  $Shift(|F(u, v)|)$

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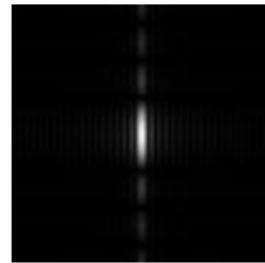
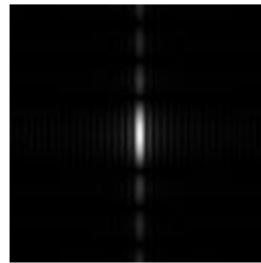
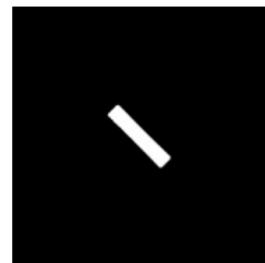
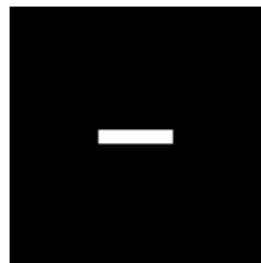
- Re-positioning of the quadrants (*FFT shift*) by using coordinates $(-1)^{x+y}$
- Applying the logarithm: $(\log(1 + |F(u, v)|))$ to increase the dynamic range of frequencies with small amplitude.

 $|F(u, v)|$  $Shift(|F(u, v)|)$  $\log(1 + Shift(|F(u, v)|))$

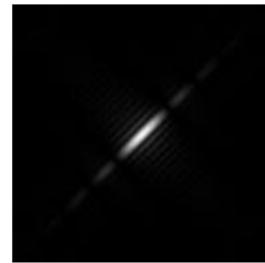
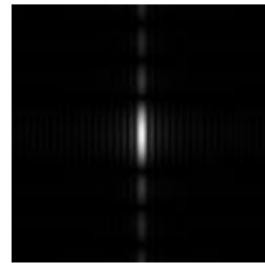
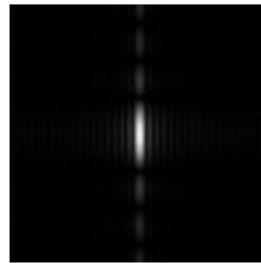
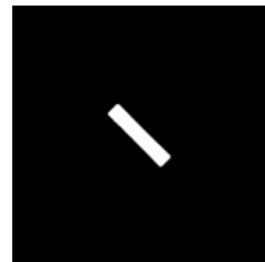
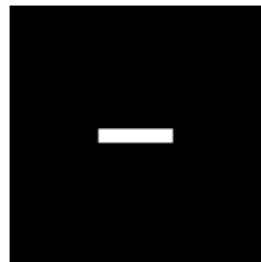
Examples



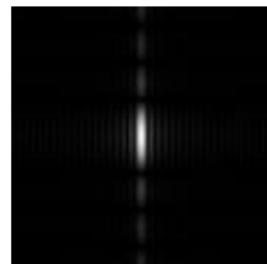
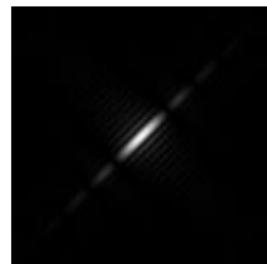
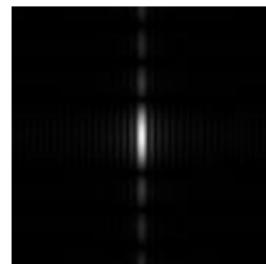
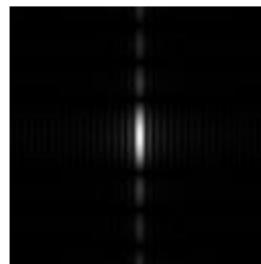
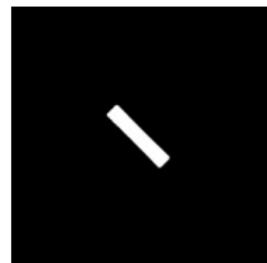
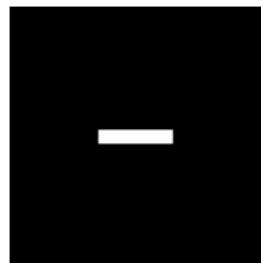
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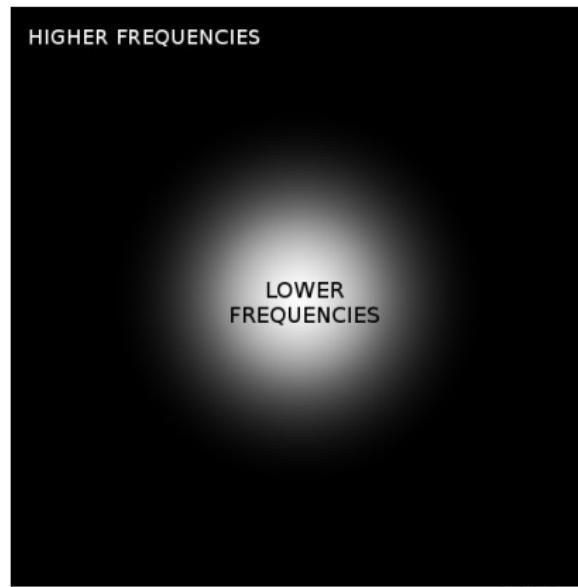
Examples



Visualizing the Fourier Spectrum

After FFT-Shift:

- In the central point of the image we have lower frequencies (with respect to u, v)
- Higher frequencies are further from the centre



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Convolution Theorem

- A circular convolution in 2D is given by:

$$f(x, y) * h(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(i, j)h(x - i, y - j),$$

Convolution Theorem

- The 2D convolution theorem has the following result :

$$f(x, y) * h(x, y) \Leftrightarrow F(u, v) \cdot H(u, v),$$

$$f(x, y) \cdot h(x, y) \Leftrightarrow F(u, v) * H(u, v).$$

F and H are the Fourier Transforms of f and h , respectively

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Filtering in frequency domain

- Modify the Fourier transform of an image and calculate its inverse:

$$g(x, y) = \mathfrak{F}^{-1} [F(u, v)H(u, v)]$$

Filtering in frequency domain: steps

- ① Input: $f(x, y)$ with size $m \times n$

Filtering in frequency domain: steps

- ① Input: $f(x, y)$ with size $m \times n$
- ② Obtain optimal values to perform the transform: P and Q . Options:
 - use the next power of two, or
 - use $P = 2n$ and $Q = 2m$.

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- ④ Compute DFT $F(u, v) = \mathfrak{F}[f_p(x, y)]$,
- ⑤ Use a symmetric filter $H(u, v)$ of size $P \times Q$ centred in $(P/2, Q/2)$,

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- ⑥ Compute processed image: $g_p(x, y) = \text{real}(\mathfrak{F}^{-1}[F(u, v)H(u, v)])$,

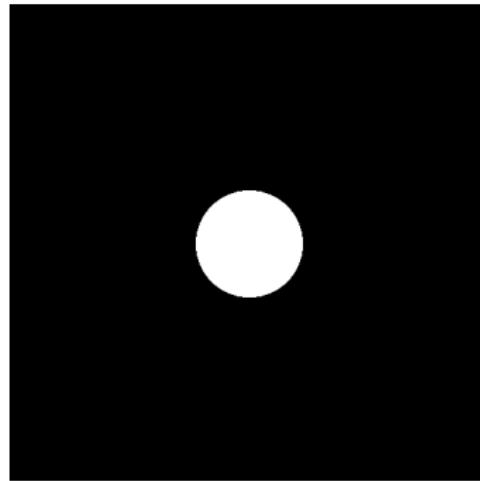
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- ⑦ Output: $g(x, y)$ size $n \times m$, after removing the zero-padded region.

Filtering in frequency domain

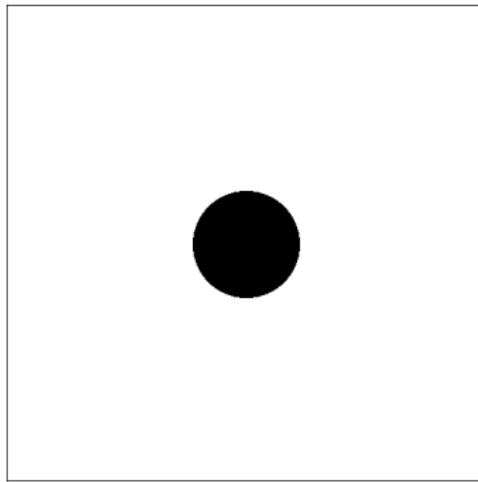
- (*Low-Pass filter*):

- Allow the low frequencies to “pass”, retaining ~~higher~~ lower frequencies
- An “ideal” filter removes all frequencies beyond a given threshold



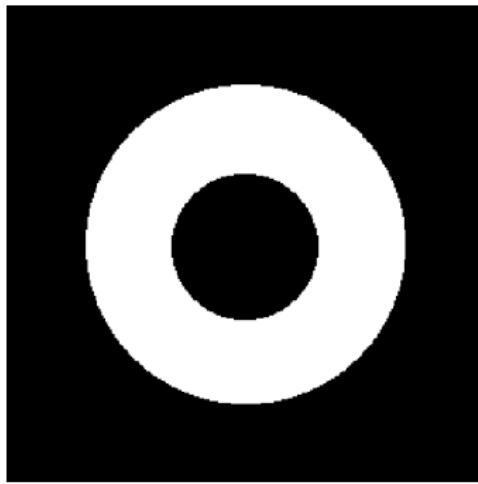
Filtering in frequency domain

- (*High-Pass filter*):
 - remove lower frequencies, while allowing to “pass” higher frequencies
 - An “ideal” high-pass filter removes all frequencies below a given threshold



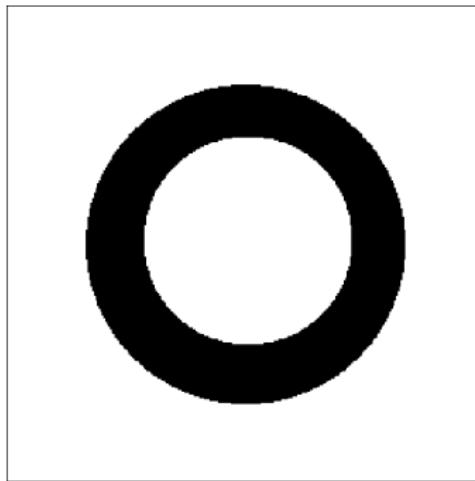
Filtering in frequency domain

- (*Bandpass filter*):
 - select a range of frequencies to be kept

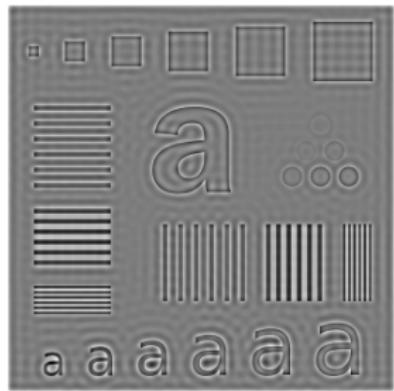
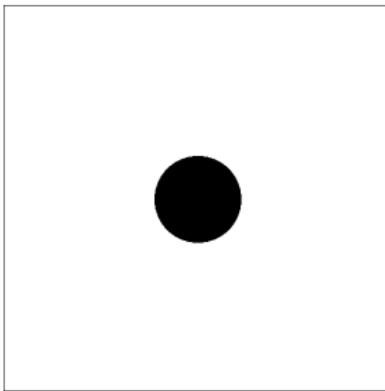


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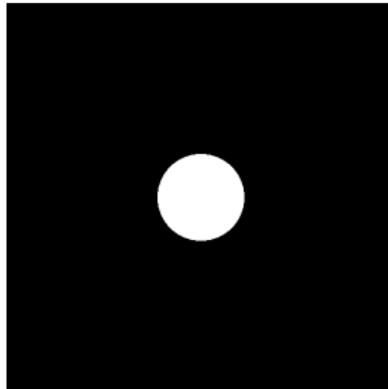
- (*Band-stop filter*):
 - select a range of frequencies to be removed



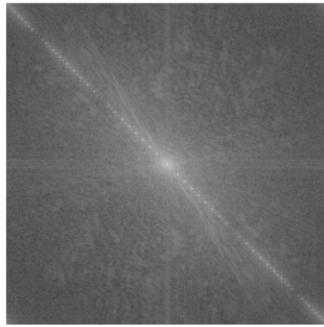
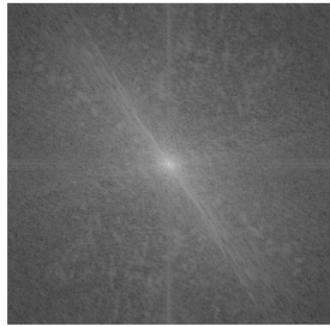
High-pass filter



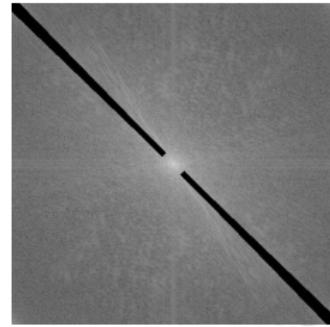
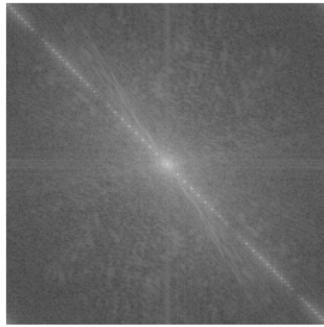
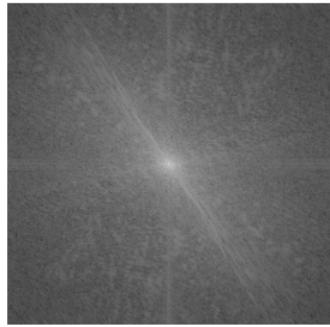
Low-pass filter



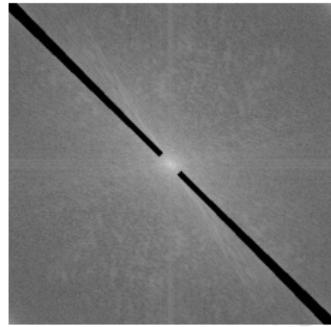
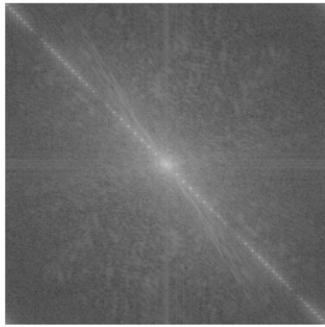
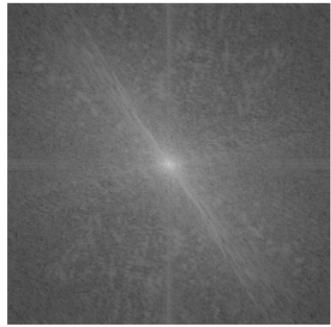
Band-stop filter



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Band-stop filter



Agenda

- 1 Big Picture
- 2 Discrete Fourier Transform
 - Convolution Theorem
- 3 Filtering in frequency domain
- 4 Fast Fourier Transform (FFT)

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- The FFT Algorithm was published by Cooley and Tukey, 1965.

Fast Fourier Transform (FFT)



- James W. Cooley (1926–2016):
 - BSc., MSc., PhD. Mathematics;
 - Computer programmer with von Neumann at U.Princeton (1953-1956);
 - Researcher at IBM until 1991, his last position was professor at University of Rhode Island.



- John Tukey (1915–2000):
 - BSc., MSc. in Chemistry; PhD. Mathematics;
 - Professor at U.Princeton;
 - Researcher at AT&T Bell Labs;
 - Also known for developing (*boxplot*) and the *post-hoc* Tukey HSD test.

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 - note that the DFT is a sequence of products and sums:

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-j \frac{2\pi u}{N} x},$$

- however, the term $e^{-j2\pi}$ is always the same, with different powers of $(u/N)x$

Fast Fourier Transform (FFT)

- Re-writing:

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- ② Periodicity in x, u :

$$W_N^{ux} = W_N^{u(x+N)} = W_N^{(u+N)x} \quad (2)$$

Fast Fourier Transform (FFT)

- Decimate: obtain DFT using smaller parts assuming $N = 2^m$
 - decompose $f(x)$ into indexes:
even (index $x = 2r$) and **odd** (index $x = 2r + 1$):

$$F(u) = \sum_{r=0}^{(N/2)-1} f(2r) W_N^{u2r} + \sum_{r=0}^{(N/2)-1} f(2r+1) W_N^{u(2r+1)},$$

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 &= \sum_{r=0}^{(N/2)-1} f(2r)(W_N^2)^{ux} + W_N^u \sum_{r=0}^{(N/2)-1} f(2r+1)(W_N^2)^{ux},
 \end{aligned}$$

Fast Fourier Transform (FFT)

- Due to the periodicity of DFT:

$$F\left(u + \frac{N}{2}\right) = F(u)$$

- We can write the transform as:

$$F(u) = \begin{cases} F_e(u) + W_N^{ux} F_o(u), & \text{for } 0 \leq u < N/2 \\ F_e(u - N/2) + W_N^{ux} F_o(u - N/2), & \text{for } N/2 \leq u < N \end{cases}$$

Fast Fourier Transform (FFT)

- By using the relationship between the terms W :

$$W_N^2 = e^{-j\frac{2\pi}{N}2} = e^{-j\frac{4\pi}{N}} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

- Split the DFT into a sum of 2 DFTs with even and odd indices:

$$F(u) = \sum_{r=0}^{(N/2)-1} f(2r)W_{N/2}^{ux} + W_N^u \sum_{r=0}^{(N/2)-1} f(2r+1)W_{N/2}^{ux},$$
$$F(u) = F_e(u) + W_N^u \cdot F_o(u)$$

Fast Fourier Transform (FFT) – running time complexity

- For each $N/2$ we now have $2 \cdot N/4$
- Repeating the procedure

$$1 : \frac{N}{2} \rightarrow 2 \left(\frac{N}{2} \right)^2 + N = \frac{N^2}{2} + N$$

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$$P : \frac{N}{2^P} \rightarrow \frac{N^2}{2^P} + pN = N + N \log_2 N$$

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