4 Continuity

In this chapter we discuss continuous functions. Functions themselves should be familiar. For reference, we begin with a review of some basic concepts and conventions.

We are concerned with functions $f : U \to V$ where both U, V are subsets of the real numbers $\mathbb{R}$ and f is some *rule* assigning to each real number $x \in U$ a real number $f(x) \in V$. For instance

$$f(x) = \frac{x^2(x-7)}{(x-2)(x^2-9)} \quad \text{assigns to } x = 1 \text{ the value} \quad f(1) = \frac{1(-6)}{(-1)(-8)} = \frac{3}{4}$$

Domain $\operatorname{dom} f = U$ is the set of *inputs* to f. When f is defined by a formula, its *implied domain* is the largest set on which the formula is defined: the above example has implied domain $\operatorname{dom} f = \mathbb{R} \setminus \{2, 3, -3\}$. In examples, the domain is most often a union of intervals of positive length.

Codomain $\operatorname{codom} f = V$ is the set of *possible outputs*. In real analysis, we often take $V = \mathbb{R}$ by default.

Range $\operatorname{range} f = f(U) = \{f(x) : x \in U\}$ is the set of *realized outputs*, and is a subset of $V = \operatorname{codom} f$.

Injectivity f is *injective/one-to-one* if distinct inputs produce distinct outputs. This is usually stated in the contrapositive: $f(x) = f(u) \Longrightarrow x = u$.

Surjectivity f is *surjective/onto* if every possible output is realized: that is $f(U) = V$.

Inverses f is *bijective/invertible* if it is both injective and surjective. Equivalently, f has an *inverse function* $f^{-1} : V \to U$ defined as follows:

- Given $y \in V$, f surjective $\Longrightarrow$ $\exists x \in U$ such that $f(x) = y$.
- Since f is injective, $f(x) = f(u) \Longrightarrow x = u$, so x is unique. We *define* $f^{-1}(y) = x$.

Example 4.1. The function defined by $f(x) = \frac{1}{x(x-2)}$ has implied

$$\operatorname{dom} f = \mathbb{R} \setminus \{0, 2\} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

$$\operatorname{range} f = (-\infty, -1] \cup (0, \infty)$$

The function is neither injective (e.g., $f(3) = f(-1)$) nor surjective (e.g., $0 \notin \operatorname{range} f$).

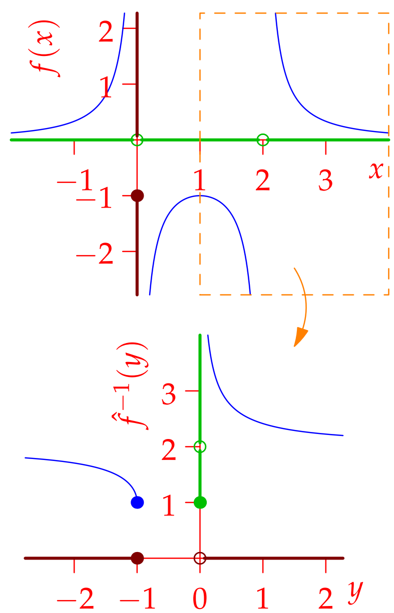
We can remedy both issues by restricting the domain and codomain. For instance, the same rule/formula but with

$$\operatorname{dom} \hat{f} = [1, 2) \cup (2, \infty)$$
$$\operatorname{codom} \hat{f} = (-\infty, -1] \cup (0, \infty)$$

defines a bijection with inverse function

$$\hat{f}^{-1}(y) = \begin{cases} 1 + y^{-1}\sqrt{y+1} & \text{if } y > 0 \\ 1 - y^{-1}\sqrt{y+1} & \text{if } y \le -1 \end{cases}$$

Observe that $\operatorname{dom} \hat{f}^{-1} = \operatorname{codom} \hat{f}$ and $\operatorname{codom} \hat{f}^{-1} = \operatorname{dom} \hat{f}$.



4.17 Continuous Functions

To introduce continuity, consider two common naïve notions.

The graph of f can be drawn without removing one's pen from the page This is intuitive but unusable: *drawn* is poorly defined, so how might we *calculate* or *prove* anything with this concept? It moreover cannot reasonably be extended to other situations or higher dimensions where *drawing a graph* is meaningless.

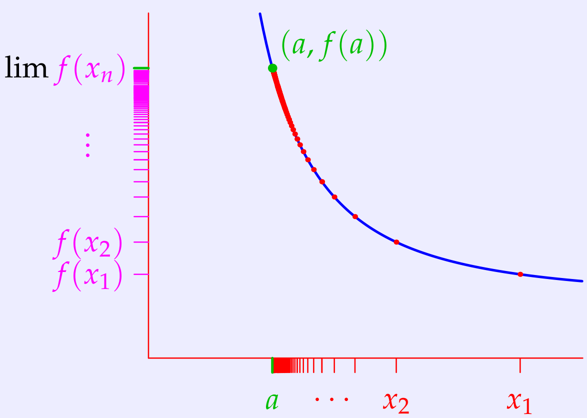
If x is close to a, then $f(x)$ is close to $f(a)$ This is better and admits generalization. The major issue is the unclear meaning of *close*. Our formal definition of continuity addresses this using *sequences* and *limits*.

Definition 4.2 (Sequential continuity). A real-valued function $f : U \to V$ is *continuous at* $a \in U$ if,

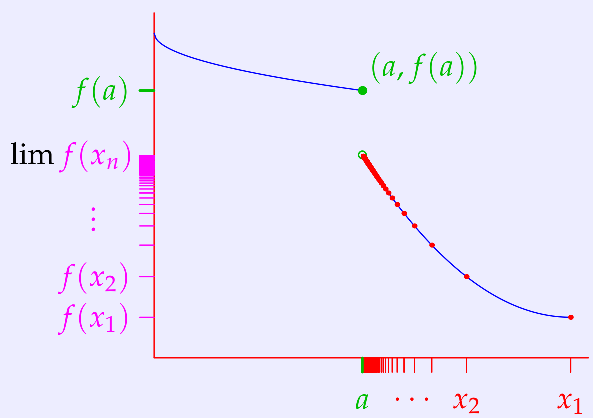
$$\forall (x_n) \subseteq U,\ \lim x_n = a \implies \lim f(x_n) = f(a)$$

f is *continuous (on U)* if it is continuous at every point $a \in U$.
We say that f is *discontinuous* at $a \in U$ if,

$$\exists (x_n) \subseteq U, \text{ such that } \lim x_n = a \text{ and } \big(f(x_n)\big) \text{ does not converge to } f(a)$$



Continuity at a: *every sequence* with $\lim x_n = a$ has $\lim f(x_n) = f(a)$



Discontinuous at a: *at least one sequence* with $\lim x_n = a$ has $\lim f(x_n) \neq f(a)$

Examples 4.3. 1. $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is continuous (at every $a \in \mathbb{R}$). To see this, suppose (x_n) converges to a, then, by the limit laws,

$$\lim f(x_n) = \lim x_n^2 = (\lim x_n)^2 = a^2 = f(a)$$

2. The function with $g(x) = 1 + \frac{4}{x^2}$ is continuous. Choose any $a \in \operatorname{dom} g = \mathbb{R} \setminus \{0\}$ and any $(x_n) \subseteq \operatorname{dom} g$ with $\lim x_n = a$. Again, by the limit laws,

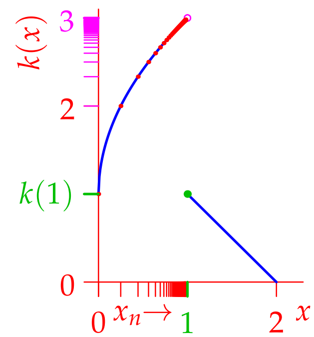
$$\lim g(x_n) = \lim \left(1 + \frac{4}{x_n^2}\right) = 1 + \frac{4}{(\lim x_n)^2} = 1 + \frac{4}{a^2} = f(a)$$

This example (with $a = 1$ and $x_n = 1 + \frac{2}{n}$) is the first picture in the above definition.

3. $h : [0,\infty) \to \mathbb{R} : x \mapsto 3x^{1/4}$ is continuous. Again, everything follows from the limit laws. If $x_n \to a$ where $x_n \geq 0$ and $a \geq 0$, then

$$\lim h(x_n) = \lim 3x_n^{1/4} = 3(\lim x_n)^{1/4} = 3a^{1/4} = h(a)$$

4. The function defined by



$$k(x) = \begin{cases} 1 + 2\sqrt{x} & \text{if } x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$

is discontinuous at $a = 1$. This seems obvious from the picture, but we need to use the definition. The sequence with $x_n = (1 - \frac{1}{n})^2$ converges to 1 from below, however the limit laws tells us that

$$\lim k(x_n) = \lim \left(1 + 2\left(1 - \frac{1}{n}\right)\right) = 3 \neq 1 = k(1)$$

Basic Examples and Combinations of Continuous Functions

By appealing to the limit laws for sequences (Theorem 2.15), continuous functions may be combined in natural ways. For instance, if f, g are continuous at a, then

$$\lim x_n = a \implies \lim f(x_n) + g(x_n) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a)$$

whence $f + g$ is continuous at a. Here is a general summary.

Theorem 4.4. 1. *Suppose f, and g are continuous and that k is constant. Then the following functions are continuous (on their domains):*

$$kf, \quad |f|, \quad f+g, \quad f-g, \quad fg, \quad \frac{f}{g}, \quad \max(f,g), \quad \min(f,g)$$

2. *If $n \in \mathbb{N}$ then $f : x \mapsto x^{1/n}$ is continuous on its domain.*

3. *Compositions of continuous functions are continuous: if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

4. *Algebraic functions are continuous (includes all polynomials and rational functions).*

Proof. Parts 1, 2 are the limit laws; for the maximum and minimum, see Exercise 2. For part 3:

$$\lim x_n = a \overset{g\text{ cont}}{\implies} \lim g(x_n) = g(a) \overset{f\text{ cont}}{\implies} \lim f(g(x_n)) = f(g(a))$$

Part 4 follows by combining parts 1, 2 and 3. ∎

Example 4.5. The following algebraic function is continuous on its domain

$$f : (7,\infty) \to \mathbb{R} : x \mapsto \sqrt{\frac{3x^{5/2} + 7x^2 + 4}{(x-7)^{1/3}}}$$

Theorem 4.6 (Squeeze theorem). *Suppose $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$, that f, h are continuous at a, and that $f(a) = g(a) = h(a)$. Then g is continuous at a.*

Proof. This is simply the squeeze theorem (2.12) for sequences: if $\lim x_n = a$, then

$$f(x_n) \leq g(x_n) \leq h(x_n) \implies \lim g(x_n) = g(a)$$

∎

To provide more interesting examples, we state the following without proof.

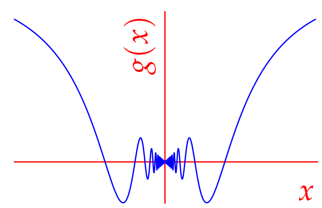
Theorem 4.7. *The common trigonometric, exponential and logarithmic functions are continuous.*

It is possible, though slow and ugly, to address some of this now. We won't do this since it is cleaner to define these functions using power series,[27] which makes their continuity (and differentiability/integrability!) come for free.

Examples 4.8. 1. $f(x) = \frac{\sqrt{x}}{\sin e^x}$ is continuous on its domain $\mathbb{R} \setminus \{\ln(n\pi) : n \in \mathbb{N}_0\}$.

2. If $g(x) = x \sin \frac{1}{x}$ when $x \neq 0$, and $g(0) = 0$, then g is continuous on $\mathbb{R}$. When $x \neq 0$, this follows from Theorems 4.4 and 4.7, while at $a = 0$ we rely on the squeeze theorem:

$$x \neq 0 \implies -x \leq x \sin \frac{1}{x} \leq x$$



The ϵ-δ Definition of Continuity

The sequential definition of continuity uses limits *twice*. By stating each of these using the ϵ-definition of limit, we can reformulate continuity without mentioning sequences!

To motivate this, consider $f(x) = x^2$ at $a = 2$. By continuity, if (x_n) is a sequence with $\lim x_n = 2$, then $\lim f(x_n) = 4$. We restate each of these using the definition of limit:

(a) $(\lim x_n = 2)$ $\quad \forall \delta > 0, \exists M$ such that $n > M \implies |x_n - 2| < \delta$

(b) $(\lim x_n^2 = 4)$ $\quad \forall \epsilon > 0, \exists N$ such that $n > N \implies |x_n^2 - 4| < \epsilon$

Here is a short argument that shows how (a) $\Rightarrow$ (b) (we'll revisit this formally in a moment).

Assume (a) and suppose $\epsilon > 0$ is given. *Define* $\delta = \min(1, \frac{\epsilon}{5})$. Since $\lim x_n = 2$, $\exists M$ such that

$$\begin{aligned} n > M \implies |x_n^2 - 4| &= |x_n - 2|\,|x_n + 2| < \delta\,|(x_n - 2) + 4| && \text{(by (a))} \\ &\leq \delta\big(|x_n - 2| + 4\big) && (\triangle\text{-inequality}) \\ &< \delta(\delta + 4) \leq 5\delta \leq \epsilon && \text{((a) again)} \end{aligned}$$

Let $N = M$ to conclude (b).

It turns out not to be very important that (x_n) be a *sequence*. In fact we can dispense with it entirely…

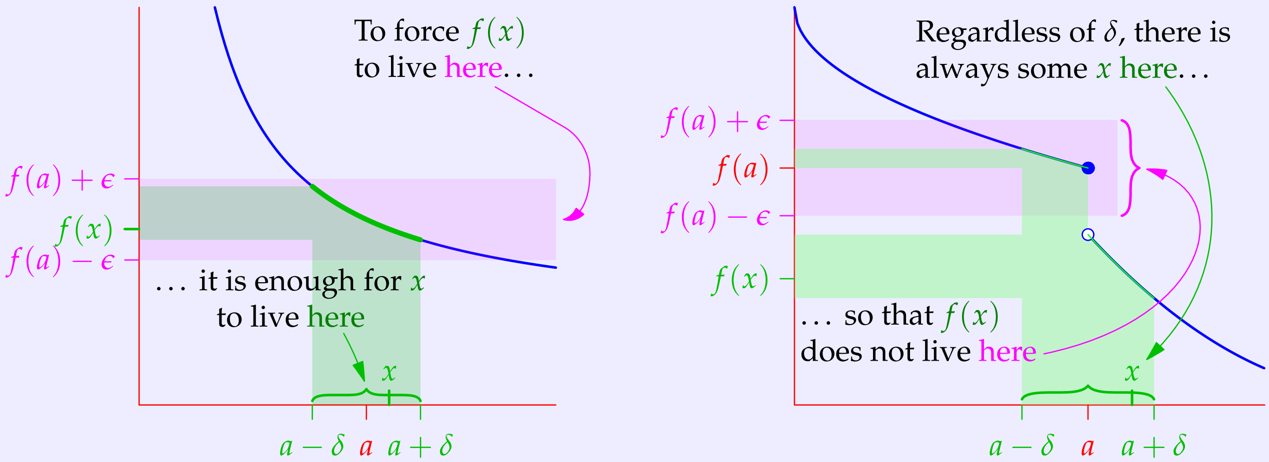
[27]For instance via Maclaurin series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Definition 4.9 (ϵ-δ continuity). A real-valued function $f : U \to V$ is continuous at $a \in U$ if[28]

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x \in U)\ |x-a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{$*$}$$

We say that f is *discontinuous* at $a \in U$ if,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x \in U \text{ with } |x-a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon \tag{$\dagger$}$$



Continuity at a — Discontinuous at a

This fits with the intuitive interpretation of continuity: if x is close to a, then $f(x)$ is close to $f(a)$; ϵ and δ are our measures of *closeness*. Many mathematicians consider the ϵ-δ version to be *the* definition of continuity. Thankfully, it doesn't matter which you prefer...

Theorem 4.10. *The sequential and ϵ-δ definitions of continuity (4.2 & 4.9) are equivalent.*

Examples (4.3, cont). Before seeing a proof, we repeat our earlier examples using the ϵ-δ definition. As with ϵ-N arguments for limits, it is often useful to do some scratch work first.

1. Suppose $f(x) = x^2$ and $a \in \mathbb{R}$. Our goal is to control the size of $\left|x^2 - a^2\right|$ whenever $|x-a|$ is small. To keep things simple, assume $|x-a| < 1$, then,

$$\begin{aligned}\left|x^2 - a^2\right| &= |x-a|\,|x+a| = |x-a|\,\left|(x-a) + 2a\right| \\ &\overset{\triangle}{\leq} |x-a|\left(|x-a| + 2\,|a|\right) = |x-a|\left(1 + 2\,|a|\right)\end{aligned}$$

Now let $\epsilon > 0$ be given and define $\delta = \min(1, \frac{\epsilon}{1+2|a|})$. Then

$$|x-a| < \delta \implies |f(x) - f(a)| = \left|x^2 - a^2\right| < \delta\left(1 + 2\,|a|\right) \leq \epsilon$$

Thus f is continuous at a. This is simply a general version of the argument on page 63 with all mention of sequences removed!

[28]The bracketed $\forall x \in U$ is often omitted in $(*)$ since the implication requires that x be universally quantified. It is important that $x \in U = \operatorname{dom} f$ rather than merely $x \in \mathbb{R}$! By contrast, the expression $\exists x \in U$ in $(\dagger)$ is *always* written.

2. Let $g(x) = 1 + \frac{4}{x^2}$ and $a \neq 0$. The first challenge is to keep away from zero so that $\frac{1}{x}$ behaves. To do this, we insist that $\delta \le \frac{|a|}{2}$, so that

$$|x-a| < \delta \implies \frac{|a|}{2} < |x| < \frac{3\,|a|}{2} \implies \frac{1}{|x|} < \frac{2}{|a|} \tag{$*$}$$

Now consider the required difference. If $|x-a| < \delta$, then

$$\begin{aligned} |g(x)-g(a)| &= \left|1 + \frac{4}{x^2} - 1 - \frac{4}{a^2}\right| = \frac{4\,|a^2-x^2|}{a^2x^2} = \frac{4\,|a+x|}{a^2x^2}\,|x-a| < \frac{4\,|a+x|}{a^2x^2}\delta \\ &\overset{\triangle}{\le} 4\left(\frac{1}{|a|\,x^2} + \frac{1}{a^2\,|x|}\right)\delta \overset{(*)}{<} 4\left(\frac{4}{|a|^3} + \frac{2}{|a|^3}\right)\delta = \frac{24}{|a|^3}\delta \end{aligned}$$

Given $\epsilon > 0$, it suffices to let $\delta = \min(\frac{1}{2}\,|a|\,, \frac{1}{24}|a|^3\epsilon)$. Then $|x-a| < \delta \implies |g(x)-g(a)| < \epsilon$.

3. For $h(x) = 3x^{1/4}$ there are two cases. Suppose $\epsilon > 0$ is given.

   - If $a = 0$, let $\delta = \left(\frac{\epsilon}{3}\right)^4$, then[29]

     $$|x-a| < \delta \implies 0 \le x < \delta \implies |h(x)-h(a)| = 3x^{1/4} < 3\delta^{1/4} = \epsilon$$

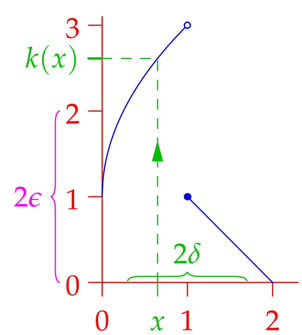
   - If $a > 0$, let $\delta = \frac{1}{3}a^{3/4}\epsilon$. Then, if $|x-a| < \delta$,

     $$|h(x)-h(a)| = 3\left|x^{1/4} - a^{1/4}\right| = \frac{3\,|x-a|}{x^{\frac{3}{4}} + a^{\frac{1}{4}}x^{\frac{2}{4}} + a^{\frac{2}{4}}x^{\frac{1}{4}} + a^{\frac{3}{4}}} \le \frac{3\,|x-a|}{a^{3/4}} < \frac{3\delta}{a^{3/4}} = \epsilon$$

4. We could establish the discontinuity statement (†) directly, but it is typically easier to argue by contradiction.
Suppose k is continuous at 1 and let $\epsilon = 1$. Then $\exists\delta > 0$ for which

   $$\begin{aligned} |x-1| < \delta &\implies |k(x)-k(1)| = |k(x)-1| < 1 \\ &\implies 0 < k(x) < 2 \end{aligned}$$

   However, $x = \max(\frac{1}{4}, 1-\frac{\delta}{2})$ satisfies $|x-1| \le \frac{\delta}{2} < \delta$ and $k(x) \ge k(\frac{1}{4}) = 1 + \frac{2}{2} = 2$. Contradiction. Think this last bit through!



The basic rules for combining continuous functions may also be proved using ϵ-δ arguments. E.g.,

ϵ-δ proof of the squeeze theorem. Given $\epsilon > 0$, we know there exist $\delta_1, \delta_2 > 0$ for which

$$|x-a| < \delta_1 \implies |f(x)-f(a)| < \epsilon \quad \text{and} \quad |x-a| < \delta_2 \implies |h(x)-h(a)| < \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$, then

$$|x-a| < \delta \implies |g(x)-g(a)| \le \max\big(|f(x)-f(a)|\,, |h(x)-h(a)|\big) < \epsilon$$

whence g is continuous at 0. ■

[29] Remember the hidden quantifier: $|x-a| < \delta$ *for all* $x \in \operatorname{dom} f = [0,\infty)$, thus $x \ge 0$ for the duration of this example.

Several other arguments are in the exercises. Finally, here is the promised proof of equivalence.

Proof of Theorem 4.10. (sequential $\Rightarrow$ ϵ–δ) We prove the contrapositive. Suppose a is an ϵ-δ discontinuity (†) and let $\delta = \frac{1}{n}$. Then there exists $x_n \in U$ such that

$$|x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(a)| \geq \epsilon$$

Repeating for all $n \in \mathbb{N}$ plainly produces a sequence (x_n) for which $\lim x_n = a$ and $\lim f(x_n) \neq f(a)$: otherwise said, a is a sequential discontinuity.

(ϵ-δ $\Rightarrow$ sequential) Assume $(*)$, let $(x_n) \subseteq U$ and suppose $\lim x_n = a$; we must prove that $\lim f(x_n) = f(a)$. Let $\epsilon > 0$ be given so that a suitable δ satisfying $(*)$ exists. Since $\lim x_n = a$,

$$\begin{aligned} \exists N \text{ such that } n > N &\implies |x_n - a| < \delta && \text{(since } x_n \to a \text{ and } \delta > 0 \text{ is given)} \\ &\implies |f(x_n) - f(a)| < \epsilon && \text{(by } (*)) \end{aligned}$$

We conclude that $\lim f(x_n) = f(a)$, as required. ∎

Examples 4.11. We finish with a couple of esoteric examples on the same theme.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the *indicator function* for the rational numbers:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Suppose f is continuous at a and let $\epsilon = 1$. Then $\exists \delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < 1 \tag{‡}$$

There are two cases; both rely on the fact that any interval contains both rational and irrational numbers (Corollary 1.23, etc.).

(a) If $a \in \mathbb{Q}$, then $f(a) = 1$. There exists an irrational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |0 - 1| = 1 \not< 1$.

(b) If $a \notin \mathbb{Q}$, then $f(a) = 0$. There exists a rational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |1 - 0| = 1 \not< 1$.

Either way, we have contradicted (‡). We conclude that f is *nowhere continuous.*

2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since $0 \leq |g(x)| \leq |x|$, the squeeze theorem tells us that g is continuous at $x = 0$.

Now suppose g is continuous at $a \neq 0$ and let $\epsilon = |a|$. Then $\exists \delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < |a|$$

The same two cases as in the previous example provide contradictions. We conclude that g *is continuous at precisely one point!*

Exercises 4.17. *Key concepts: Sequential and ϵ-δ continuity definitions/equivalence, ϵ-δ examples*

1. Consider the function with $f(x) = \frac{1}{\sqrt{x^2+2x-3}}$.
   (a) The implied domain of f has the form $\operatorname{dom} f = (-\infty, a) \cup (b, \infty)$. Find a and b.
   (b) What is the range of f?
   (c) Show that $f : (b, \infty) \to \operatorname{range} f$ is *bijective* and compute its inverse function.
   (d) Find the inverse function when we instead restrict the domain to $(-\infty, a)$.
   (e) Briefly explain why f is continuous on its domain.

2. Let f and g be continuous functions at a.
   (a) Show that $\max(f, g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and deduce that $\max(f, g)$ is continuous at a.
   (b) How might you show continuity of $\min(f, g)$?

3. Use ϵ-δ arguments to prove the following.
   (a) $f(x) = x^2 - 3x$ is continuous at $x = 1$.
   (b) $g(x) = x^3$ is continuous at $x = a$.
   (c) $h : [0, \infty) \to \mathbb{R} : x \mapsto \sqrt{x}$ is continuous.
   (d) $j(x) = 3x^{-1}$ is continuous on $\mathbb{R} \setminus \{0\}$.

4. Rephrase Example 4.3.4's ϵ-δ argument by directly justifying the discontinuity definition (†).

5. Prove that each function is discontinuous at $x = 0$; use *both* sequential and ϵ-δ formulations.
   (a) $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.
   (b) $g(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $g(0) = 0$.

6. Suppose f and g are continuous at a. Prove the following using ϵ-δ arguments.
   (a) $f - g$ is continuous at a.
   (b) If h is continuous at $f(a)$, then $h \circ f$ is continuous at a.

7. Suppose $f : U \to V \subseteq \mathbb{R}$ is a function whose domain U contains an *isolated point* a: i.e. $\exists r > 0$ such that $(a-r, a+r) \cap U = \{a\}$. Prove that f is continuous at a.

8. In Example 4.11.2, provide the details of the required contradiction.

9. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which $f(x) = 0$ whenever $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
   (b) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all rational x. Prove that $f = g$.

10. (Hard) Consider $f : \mathbb{R} \to \mathbb{R}$ where
$$f(x) = \begin{cases} \frac{1}{q} & \text{whenever } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q > 0 \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$
    For instance, $f(1) = f(2) = f(-7) = 1$, and $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, etc.
    (a) Prove that f is discontinuous at each rational number r.
    (b) Prove that f is continuous at each irrational number i.
    (*Hint: given $\epsilon > 0$, let $q = \lceil \frac{1}{\epsilon} \rceil$, $A = \left\{ r \in \mathbb{Q} : f(r) \geq \frac{1}{q} \right\}$ and let $\delta = \min_{r \in A} |i - r| \ldots$*)

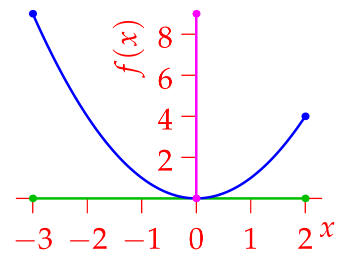
4.18 Properties of Continuous Functions

In this section we consider how continuous functions transform *intervals.*

Example 4.12. $f(x) = x^2$ maps $[-3, 2]$ onto $[0, 9]$. In particular:

- f transforms an interval into another.
- f transforms a closed bounded set into another.

Our goal is to see that these are general properties exhibited by *any* continuous function.



First recall a couple of definitions.

Definition 4.13. Suppose $f : U \to V$ where $U, V \subseteq \mathbb{R}$.

1. (a) U is *bounded* if $\exists M$ such that $\forall x \in U,\ |x| \leq M$.
   (b) f is *bounded* if its range is a bounded set: $\exists M$ such that $\forall x \in U,\ |f(x)| \leq M$.
2. (Definition 2.46) U is *closed* if every convergent sequence in U has its limit in U:

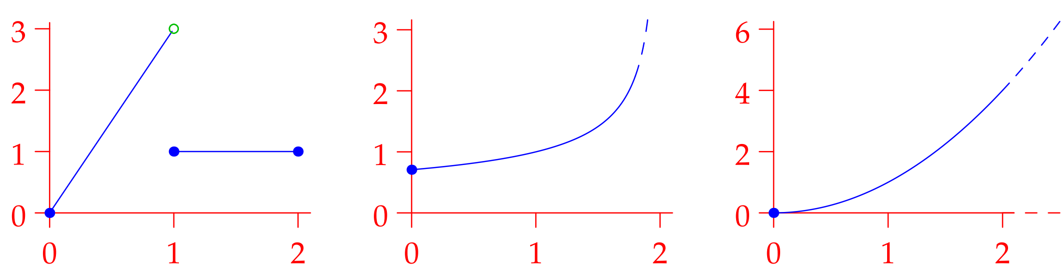
$$\forall (x_n) \subseteq U,\ \lim x_n = s\ (\in \mathbb{R}) \implies s \in U$$

Theorem 4.14 (Extreme Value Theorem). *Suppose $f : U \to V$ is continuous where U is closed and bounded. Then $f(U)$ is closed and bounded. In particular, f is bounded and attains its bounds:*

$$\exists s, i \in U \quad \text{such that} \quad f(s) = \sup f(U) \quad \text{and} \quad f(i) = \inf f(U)$$

Examples 4.15. 1. (Example 4.12) If $f(x) = x^2$ on $U = [-3, 2]$, then $f(U) = [0, 9]$ is closed and bounded. Moreover, $\sup f(U) = f(-3)$ and $\inf f(U) = f(0)$ (i.e., $s = -3$ and $i = 0$).

2. Before seeing the proof, here are three examples where we weaken one of the hypotheses of the extreme value theorem and see that the conclusion fails.



(a) f discontinuous (b) U not closed (c) U not bounded

(a) If $U = [0, 2]$, $f(x) = 3x$ when $x < 1$ and $f(x) = 1$ when $x \geq 1$, then $f(U) = [0, 3)$. In particular, $\sup f(U) = 3$ is not attained.

(b) If $f(x) = \frac{1}{\sqrt{2-x}}$ and $U = [0, 2)$, then $f(U) = [\frac{1}{\sqrt{2}}, \infty)$ is unbounded.

(c) If $f(x) = x^2$ and $U = [0, \infty)$, then $f(U) = [0, \infty)$ is unbounded.

The strategy of the proof is to show that every limit point of $f(U) = \operatorname{range} f$ lies in $f(U)$. We break things into simple steps; observe where each **hypothesis** is used.

Proof. 1. Suppose M is a limit point of $f(U)$: that is, $M = \lim f(x_n)$ for some sequence $(x_n) \subseteq U$. *A priori*, M need not be finite, but $M = \sup f(U)$ or $\inf f(U)$ are certainly possible.[30]

2. Since $(x_n) \subseteq U$ is **bounded**, Bolzano–Weierstraß (Theorem 2.41) says it has a convergent subsequence, $\lim\limits_{k\to\infty} x_{n_k} = x$.

3. Since U is **closed**, we have $x \in U$. This means $f(x)$ can be evaluated (it is *finite*).

4. Since f is **continuous**, $\lim f(x_{n_k}) = f(x)$.

5. Finally, $M = f(x)$ since all subsequences of a convergent (or divergent to $\pm\infty$) sequence tend to the same limit (Lemma 2.37). It follows that all limit points M are *finite* and lie in $f(U)$: otherwise said, $f(U)$ is closed and bounded.

Choosing $M = \sup f(U)$ yields $x = s \in U$ (similarly $\inf f(U)$ leads to $i \in U$). ∎

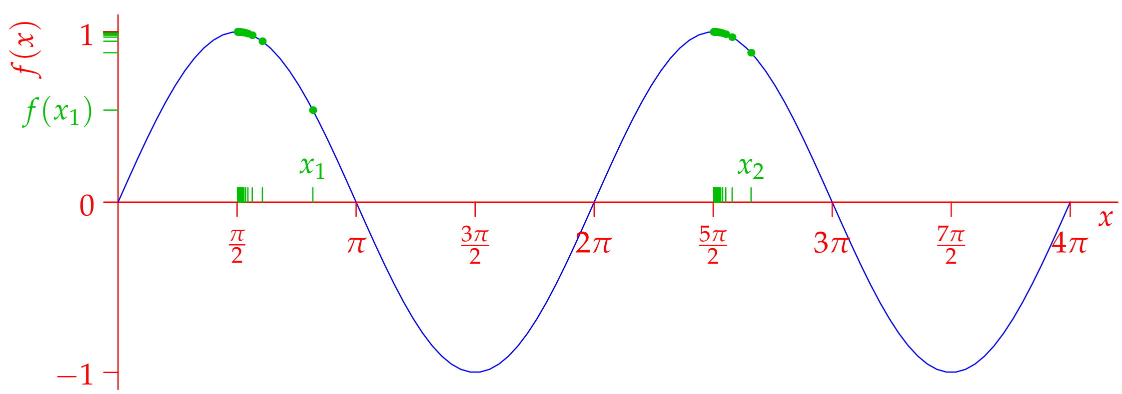
Example 4.16. It is worth considering why we needed a *subsequence* in the proof. The reason is that the bounds of f might be attained multiple times. For example, suppose

$$f : [0, 4\pi] \to \mathbb{R} : x \mapsto \sin x$$

This satisfies the hypotheses of the extreme value theorem: $U = [0, 4\pi]$ is closed and bounded and f is continuous. Indeed $\max f(U) = 1$ is attained at *both* $x = \frac{\pi}{2}$ and $\frac{5\pi}{2}$. The sequence defined by

$$x_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n} & \text{if } n \text{ is odd} \\ \frac{5\pi}{2} + \frac{1}{n} & \text{if } n \text{ is even} \end{cases} \quad \text{has} \quad f(x_n) = \sin\left(\tfrac{\pi}{2} + \tfrac{1}{n}\right) \xrightarrow[n\to\infty]{} 1 = \sup f(U)$$

and therefore satisfies step 1 of the proof. However, (x_n) itself is *divergent by oscillation*. Bolzano–Weierstraß is used to force the existence of a convergent subsequence; in this case the subsequence of odd terms $(x_{n_k}) = (x_{2k-1})$ satisfies the remaining steps.



[30] If $M = \sup f(U)$, then a suitable (x_n) might be constructed as follows:

- If $M \in \mathbb{R}$, then for each $n \in \mathbb{N}$, $\exists x_n \in U$ such that $M - \frac{1}{n} < f(x_n) \le M$ (Lemma 1.20).
- If $M = \infty$, then for each $n \in \mathbb{N}$, $\exists x_n \in U$ such that $f(x_n) \ge n$.

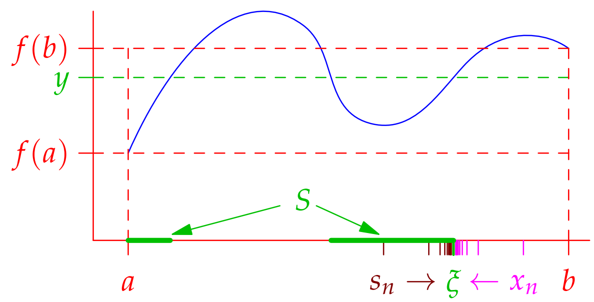
The Intermediate Value Theorem and its Consequences

This result should be familiar from elementary calculus, even if its proof is not. It should also be intuitive: like the Grand Old Duke of York, if you march up a hill, then at some point you must be half-way up...

Theorem 4.17 (Intermediate Value Theorem (IVT)). *Suppose f is continuous on $[a,b]$ and that y lies strictly between $f(a)$ and $f(b)$. Then $\exists \xi \in (a,b)$ such that $f(\xi) = y$.*

Being an existence result, it should be no surprise that *completeness* is used in the proof.

Proof. WLOG assume $f(a) < y < f(b)$. Let $S = \{x \in [a,b] : f(x) < y\}$ and *define* $\xi := \sup S$.



Since S is non-empty ($a \in S$) and bounded above (by b), we see that ξ exists and is finite. It remains to prove that $f(\xi) = y$ and $\xi \neq a, b$.

First choose any $(s_n) \subseteq S$ such that $\lim s_n = \xi$. Continuity forces $\lim f(s_n) = f(\xi)$. Moreover

$$f(s_n) < y \implies f(\xi) \le y$$

Since $f(b) > y$, this also shows that $\xi \neq b$.

We now play a similar game from the other side: define $x_n := \min(\xi + \frac{1}{n}, b)$, then $\lim x_n = \xi$ and

$$\begin{aligned} x_n > \xi = \sup S &\implies x_n \notin S \implies f(x_n) \ge y \\ &\implies f(\xi) = \lim f(x_n) \ge y \end{aligned}$$

again via the continuity of f and the convergence properties of bounded sequences. Since $y > f(a)$, we also conclude that $\xi \neq a$.

Putting it all together, $f(\xi) = y$ and $\xi \in (a,b)$. ∎

Note how the value of ξ in the proof is always the *largest* of potentially several choices.

Examples 4.18. In elementary calculus, the intermediate value theorem is typically applied to demonstrate the existence of solutions to equations.

1. We show that the equation $x^7 + 3x = 1 + 4\cos(\pi x)$ has a solution.

   The trick is to express the equation in the form $f(x) = y$ where f is continuous, then choose suitable a, b to fit the theorem. In this case,

   $$f(x) = x^7 + 3x - 4\cos(\pi x) \quad \text{and} \quad y = 1$$

   are suitable choices. Now observe

   $$f(0) = -4 < y \quad \text{and} \quad f(1) = 1 + 3 + 4 = 8 > y \qquad \text{(i.e., } a = 0 \text{ and } b = 1\text{)}$$

   whence $\exists \xi \in (0,1)$ such that $f(\xi) = y = 1$. Otherwise said, ξ is a solution to the original equation.

   The function f is plainly continuous on $\mathbb{R}$, a much larger interval than $[a,b]$, but no matter.

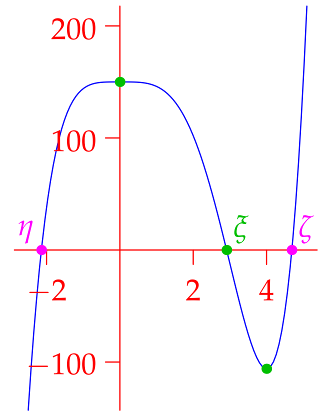
2. The existence of a root ξ of the (continuous) polynomial

$$f(x) = x^5 - 5x^4 + 150$$

follows from the intermediate value theorem by observing that

$$f(0) = 150 > 0 \quad \text{and} \quad f(4) = -256 + 150 = -106 < 0$$

We conclude that such a root ξ exists satisfying $\xi \in (0, 4)$.
As the graph suggests, there are other roots (η, ζ), the existence of which may be shown by observing, say,



$$f(-3) = -798 < 0 \quad \text{and} \quad f(5) = 150 > 0$$

With an eye on generalizing, here is an alternative approach. Define sequences (s_n), (t_n) via

$$s_n := \frac{f(-n)}{n^5} = -1 - \frac{5}{n} + \frac{150}{n^5} \qquad t_n := \frac{f(n)}{n^5} = 1 - \frac{5}{n} + \frac{150}{n^5}$$

Since $\lim s_n = -1$ and $\lim t_n = 1$, we see that

$$\exists a \text{ such that } s_a < -\frac{1}{2} \implies f(-a) = a^5 s_a < -\frac{1}{2}a^5 < 0$$

$$\exists b \text{ such that } t_b > \frac{1}{2} \implies f(b) = b^5 t_b > \frac{1}{2}b^5 > 0$$

Applying the intermediate value theorem on $[-a, b]$ shows the existence of a root.

The second approach in Example 4.16.2 may be applied to prove a general result.

Corollary 4.19. *A polynomial function of odd degree has at least one real root.*

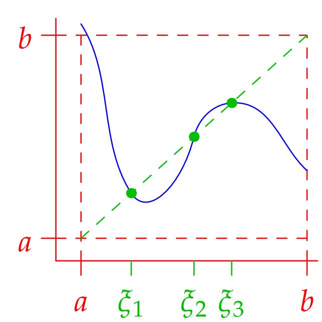
The proof is an exercise. An even simpler exercise shows the existence of a *fixed point* for a particular type of continuous function.

Corollary 4.20 (Fixed Point Theorem). *Suppose a and b are finite and that $f : [a, b] \to [a, b]$ is continuous. Then f has a fixed point:*

$$\exists \xi \in [a, b] \text{ such that } f(\xi) = \xi$$

As the picture shows, a function could have several fixed points.

This is the most basic fixed-point theorem in analysis: if you continue your studies you'll meet several more. Many important consequences flow from such results, including a common fractal construction and the standard existence/uniqueness result for differential equations.



For a final corollary, first note a straightforward characterization that helps us consider all types of interval simultaneously: $U \subseteq \mathbb{R}$ is an interval precisely when

$$a, b \in U \text{ and } a < y < b \implies y \in U \qquad (*)$$

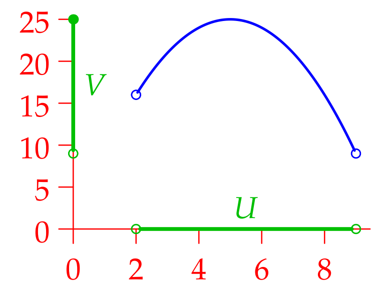
Corollary 4.21 (Preservation of Intervals). *Suppose U is an interval of positive length, and that $f : U \to V$ is continuous and surjective ($V = f(U)$).*

1. *V is an interval or a point.*
2. *If f is strictly increasing (decreasing), then:*
   (a) *V is an interval of positive length.*
   (b) *f is* injective*, and therefore bijective.*
   (c) *The inverse function $f^{-1} : V \to U$ is also continuous and strictly increasing (decreasing).*

Example 4.22. The interval V *need not be of the same type* as U. For instance, if $f(x) = 10x - x^2$, then f maps the *open* interval $U = (2, 9)$ to the *half-open* interval $V = (9, 25]$.

The extreme value theorem, however, guarantees that if U is closed and bounded, then V is also. For instance,

$$f([2, 9]) = [9, 25]$$



Proof. 1. If V is not a point, then $\exists a, b \in U$ such that $f(a) < f(b)$. If y lies between these, IVT says $\exists \xi$ between a and b such that $y = f(\xi)$. That is, $y \in f(U)$. By $(*)$, $V = f(U)$ is an interval.

2. (a,b) If f is strictly increasing, then $\forall a, b \in U,\ a < b \implies f(a) < f(b)$. Plainly f is injective and V contains at least two points; by part 1 it is an interval of positive length.

(c) Let $y_1 < y_2$ where both lie in V, and define $x_i = f^{-1}(y_i)$ for $i = 1, 2$. Since f is increasing,

$$x_2 \le x_1 \implies y_2 = f(x_2) \le f(x_1) = y_1$$

is a contradiction. Thus $x_1 < x_2$ and f^{-1} is also strictly increasing.

If $a \in U$, it remains to show that f^{-1} is continuous at $b = f(a)$. Assume first that a is not an endpoint of U and let $\epsilon > 0$ be given such that $[a - \epsilon, a + \epsilon] \subseteq U$,. Now define

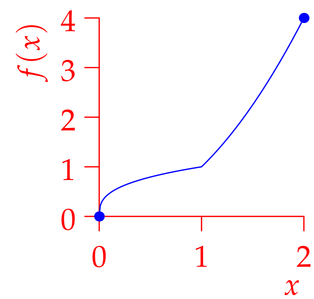
$$\delta := \min\big(b - f(a - \epsilon), f(a + \epsilon) - b\big)$$

This is *positive* since f is strictly increasing. Now observe that

$$\begin{aligned} |y - b| < \delta &\implies f(a-\epsilon) - b < y - b < f(a+\epsilon) - b \implies f(a-\epsilon) < y < f(a+\epsilon) \\ &\implies a - \epsilon < f^{-1}(y) < a + \epsilon \qquad (f \text{ strictly increasing}) \\ &\implies \left| f^{-1}(y) - f^{-1}(b) \right| = \left| f^{-1}(y) - a \right| < \epsilon \end{aligned}$$

If a is an endpoint of U, instead use $[a - \epsilon, a] \subseteq U$ or $[a, a + \epsilon] \subseteq U$ and only the corresponding half of the expression defining δ. ■

Example 4.23. The function $f : [0,2] \to [0,4]$ defined by



$$f(x) = \begin{cases} \sqrt[3]{x} & \text{if } 0 \le x \le 1 \\ x^2 & \text{if } 1 < x \le 2 \end{cases}$$

is continuous, surjective and strictly increasing. It therefore has a continuous inverse $f^{-1} : [0,4] \to [0,2]$. Compare this with the familiar statement from elementary calculus: $f' > 0 \implies f$ injective. We cannot apply this here since f is not differentiable!

Exercises 4.18. *Key concepts: Extreme/Intermediate Value Theorems, Cont functions preserve intervals*

1. (a) Give an example of a *discontinuous* function $f : [0,1] \to \mathbb{R}$ which is *not bounded.*
   (b) State a *continuous* function with domain $(1, \infty)$ whose range is *bounded* but *not closed.*

2. Let $a < b$ be given. Give examples of *continuous* functions $g, h : (a,b) \to \mathbb{R}$ such that:
   (a) g is *not bounded.*
   (b) h is bounded but *does not attain its bounds.*

3. Compute the inverse of the function f in Example 4.23.

4. Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to some $x_0 \notin S$. Show that there exists an unbounded continuous function on S.

5. Prove that $x = \cos x$ for some $x \in (0, \frac{\pi}{2})$.

6. Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists some x between a, b such that $f(x) = 0$.

7. Suppose f is continuous on $[0,2]$ and that $f(0) = f(2)$. Prove that there exist $x, y \in [0,2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.
   (*Hint: consider* $g(x) = f(x+1) - f(x)$ *on* $[0,1]$)

8. (a) Prove the fixed point theorem (Corollary 4.20).
   (*Hint: If neither a nor b are fixed points, consider* $g(x) = f(x) - x$)
   (b) Prove Corollary 4.19 for a general odd-degree monic polynomial $f(x) = x^{2m+1} + \sum_{k=0}^{2m} \alpha_k x^k$.

9. Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.
   (a) Explain why f is continuous on any interval U.
   (b) Suppose $a < 0 < b$ and that $f(a), f(b)$ have opposite signs. If $y = 0$, show that the intermediate value theorem is satisfied by *infinitely many* distinct values ξ.

10. (a) Suppose $f : U \to \mathbb{R}$ is continuous and that $U = \bigcup_{k=1}^{n} I_k$ is the union of a finite sequence (I_k) of closed bounded intervals. Prove that f is bounded and attains its bounds.
    (b) Let $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for each $n \in \mathbb{N}$. Give an example of a continuous function $f : U \to \mathbb{R}$ which is either unbounded or does not attain its bounds. Explain.

(*This relate to the idea that **finite** unions of closed sets are closed, but **infinite** unions need not be*)

4.19 Uniform Continuity

Suppose $f : U \to V$ is continuous. By the ϵ-δ definition (4.9),

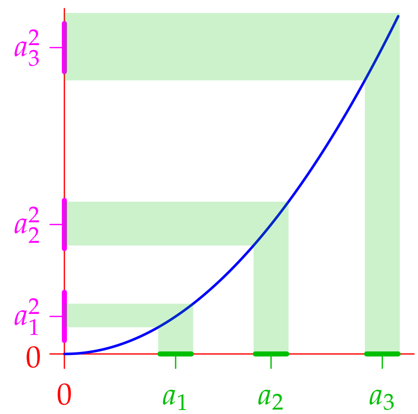
$$\forall a \in U,\ \forall \epsilon > 0,\ \exists \delta(a,\epsilon) > 0 \text{ such that } (\forall x \in U)\ |x-a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{$*$}$$

We write $\delta(a,\epsilon)$ to stress that δ can depend both on the *location* a and the *distance* ϵ. The goal of this section is to understand if/when it is possible to choose δ *independently of the location* a.

Example 4.24. We start with an example where our desire cannot be satisfied.

Consider $f(x) = x^2$ with domain $U = [0,\infty)$. Since f is continuous, given $\epsilon > 0$ and $a_1 \in U$, there exists δ such that

$$|x - a_1| < \delta \implies |f(x) - f(a_1)| = |x^2 - a_1^2| < \epsilon$$



On page 64 we saw that $\delta = \min(1, \frac{\epsilon}{1+2a_1})$ was suitable, but this *depends on the location* a_1. Of course other expressions for δ will also work...
Visualize what happens if we attempt to use the *same* ***constant*** δ for different a_i: imagine sliding the fixed-width δ-interval along the x-axis while simultaneously sliding the ϵ-interval vertically. As a_i increases, the image of the δ-interval eventually becomes too large for the ϵ-interval to contain: if δ is constant, then

$$\text{length}\big(f(a_i - \delta, a_i + \delta)\big) = (a_i + \delta)^2 - (a_i - \delta)^2 = 4a_i\delta$$

increases unboundedly with a_i. For fixed ϵ, as a increases, the *increasing gradient* of f means that we need to choose a *smaller* δ.

By contrast, if $f(x) = x^2$ on a *finite* domain $[0,b]$, then any δ that demonstrates continuity at $x = b$ will also do so everywhere else on $[0,b]$. We'll check this explicitly in a moment.

To obtain a formal definition, we rewrite $(*)$ with the extra assumption that δ may be chosen independently of the location a; this amounts to moving the quantifier $\forall a \in U$ after δ.

Definition 4.25. A function $f : U \to V \subseteq \mathbb{R}$ is *uniformly continuous* if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x, y \in U)\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \tag{$\dagger$}$$

We use y instead of a for symmetry. Observe how δ, being quantified *before* x, y, now depends only on ϵ. As before, the quantifiers for x, y are usually hidden. Note also how uniform continuity is only relevant on the entire domain U; it makes no sense to speak of uniform continuity at a single point.

For the sake of tidiness, we make one more observation before seeing some examples.

Lemma 4.26. *If f is uniformly continuous on U, then it is continuous on U.*

This is trivial: $(\dagger)$ is the ϵ-δ continuity of f at $y \in U$, *for all y simultaneously*! The special feature of the definition is that the same δ works for all y.

Examples 4.27. 1. We re-analyze $f(x) = x^2$ in view of the definition. Recall first that

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\,|x + y|$$

where $|x - y|$ is easily controlled by δ. We consider the behavior of $|x + y|$ in two cases.

Bounded domain If $U = \operatorname{dom} f \subseteq [-T, T]$ for some $T > 0$, we show that f is uniformly continuous. This will follow because $|x + y| \le 2T$ is also easily controlled.
Let $\epsilon > 0$ be given and define $\delta = \frac{\epsilon}{2T}$, then

$$|x - y| < \delta \implies |f(x) - f(y)| < \delta \cdot 2T = \epsilon$$

Compare with Example 4.24. Our approach works for *this* function because the gradient (and therefore the discrepancy between $x^2 - y^2$ and $x - y$) is greatest at the endpoints of the interval. The same approach may not work for other functions!

Unbounded domain We show that f is not uniformly continuous when $\operatorname{dom} f = [0, \infty)$.
For contradiction, assume f is uniformly continuous; let $\epsilon = 1$ and suppose $\delta > 0$ satisfies the definition. Taking $x - y = \frac{\delta}{2}$, we see that

$$|x + y| = 2y + \frac{\delta}{2} \implies |f(x) - f(y)| = \frac{\delta}{2}\left(2y + \frac{\delta}{2}\right) = \delta\left(y + \frac{\delta}{4}\right) > \delta y$$

Let $y = \frac{1}{\delta}$ for the contradiction $|f(x) - f(y)| > 1 = \epsilon$ (large y are the problem!).

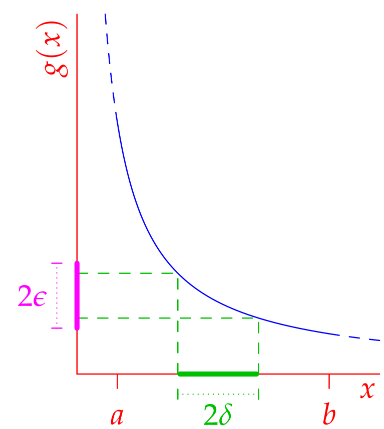
2. Let $g(x) = \frac{1}{x}$; we again consider two domains.

Uniform continuity on $[a, b)$ whenever $0 < a < b \le \infty$.

Let $\epsilon > 0$ be given and let $\delta = a^2\epsilon$. Then,

$$\begin{aligned} |x - y| < \delta \implies |g(x) - g(y)| &= \left|\frac{y - x}{xy}\right| \\ &< \frac{\delta}{xy} \le \frac{\delta}{a^2} = \epsilon \end{aligned}$$

where the last inequality follows because $x, y \ge a$.



Non-uniform continuity on $(0, b)$ whenever $0 < b \le \infty$.

As before, let $\epsilon = 1$ and suppose $\delta > 0$ is given. Let

$$x = \min\left(\delta, 1, \frac{b}{2}\right) \quad \text{and} \quad y = \frac{x}{2}$$

Certainly $x, y \in (0, b)$ and $|x - y| = \frac{x}{2} \le \frac{\delta}{2} < \delta$. However,

$$|f(x) - f(y)| = \frac{1}{x} \ge 1 = \epsilon$$

Think about how ϵ and δ must relate as one slides the intervals in the picture up/down and left/right. In this case, large values of x, y are not the problem, it's the vertical asymptote at zero that causes trouble.

General Conditions for Uniform Continuity

For the remainder of this section, we develop a few general ideas related to uniform continuity. The first is a little out of order since it depends on differentiation and the mean value theorem.

Theorem 4.28. *Suppose f is continuous on an interval U (finite or infinite) and differentiable except perhaps at its endpoints. If f' is bounded, then f is uniformly continuous on U.*

Proof. Suppose $|f'(x)| \leq M$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{M}$. Then

$$|x-y| < \delta \implies |f(x) - f(y)| = \left|f'(\xi)\right| |x-y| < M\delta = \epsilon$$

for some ξ between x and y. The existence of ξ follows from the mean value theorem.[31] ∎

Examples 4.29. 1. Compare the arguments in the previous exercise. For instance, if $\operatorname{dom} f \subseteq [-T, T]$,

$$f(x) = x^2 \implies f'(x) = 2x \implies \left|f'(x)\right| \leq 2T$$

The derivative is bounded, whence f is uniformly continuous on $[-T, T]$.

2. Any polynomial is uniformly continuous on any bounded interval.

3. The function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$ since $f'(x) = \cos x$ is bounded (by 1).

4. Consider $f(x) = \frac{1}{x} - \frac{5}{x^2}$ on $(1, \infty)$. We have

$$f'(x) = -\frac{1}{x^2} + \frac{10}{x^3} \implies \left|f'(x)\right| \leq 11$$

We conclude that f is uniformly continuous on $(1, \infty)$.

The approach is often useful when you are asked to show *using the definition* that a function is uniformly continuous; provided f' is bounded by M, you may always choose $\delta = \frac{\epsilon}{M}$ to obtain an argument. For instance, with our function:

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{11}$. If $x, y \in (1, \infty)$ and $|x-y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y} + \frac{5}{y^2} - \frac{5}{x^2}\right| = |x-y| \left|\frac{5(x+y)}{x^2y^2} - \frac{1}{xy}\right| \\ &= |x-y| \left|\frac{5}{xy^2} + \frac{5}{x^2y} - \frac{1}{xy}\right| \\ &< 11\,|x-y| && (\triangle\text{-inequality, since } x, y > 1) \\ &< 11\delta = \epsilon \end{aligned}$$

As we'll see very shortly, the above result isn't a biconditional: non-differentiable functions and functions with unbounded derivatives can be uniformly continuous.

[31] If $x < y$ then $\exists \xi \in (x, y)$ such that $f'(\xi) = \dfrac{f(x) - f(y)}{x - y}$.

Our remaining conditions are variations on a theme: uniform continuity on a bounded interval U is roughly the same thing as continuity on its *closure* $\overline{U}$ (Definition 2.46).

Theorem 4.30. *A continuous function on a closed bounded domain is uniformly continuous.*

Proof. Assume f is continuous, but not uniformly so, on a closed bounded domain U. Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x, y \in U \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon \qquad (*)$$

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$ to obtain sequences $(x_n), (y_n) \subseteq U$ satisfying $(*)$.[32]
Since $(x_n) \subseteq U$ is bounded, Bolzano–Weierstraß says there exists a convergent subsequence (x_{n_k}) which, since U is closed, converges to some $x_0 \in U$.
Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$, we see that $\lim\limits_{k\to\infty} y_{n_k} = x_0$. Finally, the continuity of f contradicts $(*)$:

$$\epsilon \leq \lim |f(x_{n_k}) - f(y_{n_k})| = |f(x_0) - f(x_0)| = 0$$

■

Both hypotheses on the domain are crucial: Examples 4.27 provide counter-examples if either is weakened.

Example 4.31. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ since it is already continuous! This **cannot** be concluded from Theorem 4.28, since the derivative $f'(x) = \frac{1}{2}x^{-1/2}$ is unbounded on $(0, 1)$.

We now develop a partial converse, for which we first need a lemma.

Lemma 4.32. *If f is uniformly continuous on U and $(x_n) \subseteq U$ is Cauchy, then $\big(f(x_n)\big)$ is also Cauchy.*

To apply the result, consider a convergent (Cauchy) sequence in U whose limit is *not* itself in U.

Example (4.27.2, just easier!). Let $f(x) = \frac{1}{x}$ have $U = \operatorname{dom} f = (0, \infty)$ and consider the Cauchy sequence defined by $x_n = \frac{1}{n}$; note crucially that its limit 0 *does not lie in U.* Moreover,

$$\lim f(x_n) = \lim n = \infty$$

Plainly $\big(f(x_n)\big)$ is not Cauchy, whence f is not uniformly continuous.

Proof. Let $\epsilon > 0$ be given. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ such that } \forall x, y \in U,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Now use this δ in the definition of (x_n) being Cauchy:

$$\exists N \text{ such that } m, n > N \implies |x_n - x_m| < \delta \implies |f(x_n) - f(x_m)| < \epsilon$$

Otherwise said, $\big(f(x_n)\big)$ is Cauchy.

■

The Cauchy condition is critical: we cannot apply uniform continuity directly to a convergent sequence ($|x_n - x| < \delta \ldots$) if we do not already know that its limit (x) lies in U!

[32]These arguments should feel familiar: compare this line to the proof of Theorem 4.10 and the rest to Theorem 4.14.

We apply the Lemma to show that a continuous function on a *bounded* interval is uniformly continuous if and only if has a *continuous extension.*

Theorem 4.33. *Suppose f is continuous on a bounded interval (a,b). Define $g:[a,b]\to\mathbb{R}$ via*

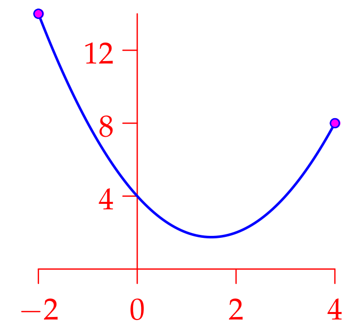
$$g(x) := \begin{cases} f(x) & \text{if } x\in(a,b) \\ \lim f(x_n) & \text{whenever } (x_n)\subseteq(a,b) \text{ and } \lim x_n = a \text{ or } b \end{cases}$$

Then f is uniformly continuous if and only if g is well-defined; in such a case g is automatically continuous.

Examples 4.34. 1. $f(x) = x^2-3x+4$ is uniformly continuous on $(-2,4)$ since it has a continuous extension

$$g:[-2,4]\to\mathbb{R}: x\mapsto x^2-3x+4$$

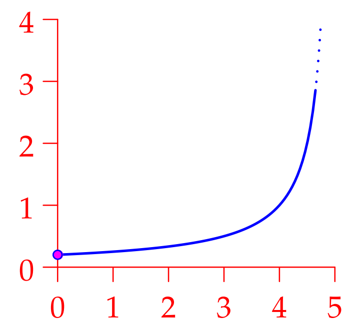
It should be obvious what is happening from the picture: to create the extension g, we simply fill in the holes at the endpoints of the graph.



2. The function $f(x)=\frac{1}{5-x}$ is continuous, but not uniformly, on the interval $(0,5)$. This follows since

$$\lim f\left(5-\frac{1}{n}\right) = \lim n = \infty$$

means we cannot define $g(5)$ unambiguously. Again the picture is helpful; while we can fill in the hole at the left endpoint ($a=0$), the vertical asymptote at $b=5$ means that there is no hole to fill in and thereby extend the function.



Proof. ($\Leftarrow$) Suppose g is well-defined; we leave the claim that it is continuous as an exercise, but by Theorem 4.30 it is uniformly so. Since $f=g$ on a subset $(a,b)\subseteq \operatorname{dom} g$, the same choice of δ will work for f as it does for g: f is therefore uniformly continuous.

($\Rightarrow$) Suppose f is uniformly continuous on (a,b). Let $(x_n),(y_n)\subseteq(a,b)$ be sequences converging to a. To show that $g(a)$ is unambiguously defined, we must prove that $\big(f(x_n)\big)$ and $\big(f(y_n)\big)$ are convergent, and to the same limit.

Define a sequence

$$(u_n) = (x_1,y_1,x_2,y_2,x_3,y_3,\ldots)$$

Plainly $\lim u_n = a$ since (x_n) and (y_n) have the same limit. But then (u_n) is Cauchy; by Lemma 4.32, $\big(f(u_n)\big)$ is also Cauchy and thus convergent. Since $\big(f(x_n)\big)$ and $\big(f(y_n)\big)$ are subsequences of a convergent sequence, they also converge to the same (finite!) limit.

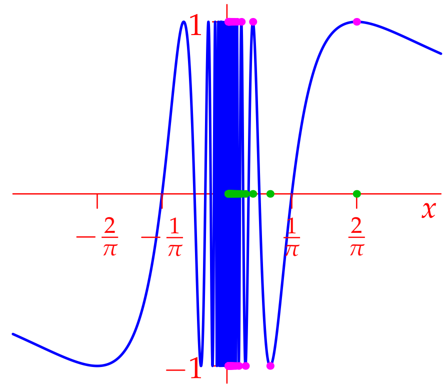
The case for $g(b)$ is similar. ∎

Examples 4.35. We finish with three related examples of continuous functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$; these will appear repeatedly as you continue to study analysis.

1. $f(x) = \sin \frac{1}{x}$ is continuous but *not uniformly so.* To see this, note that $x_n = \frac{1}{(n+\frac{1}{2})\pi}$ defines a Cauchy sequence ($\lim x_n = 0$), and yet

$$f(x_n) = \sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$$

is not Cauchy (it diverges by oscillation). Consequently, we cannot extend f to a continuous function on any interval containing $x = 0$.



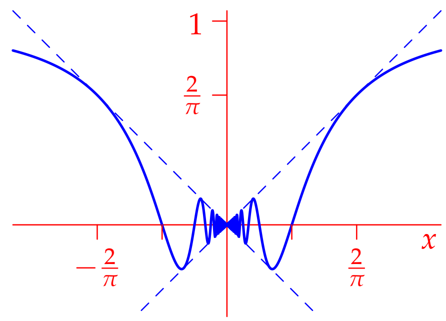
2. $f(x) = x \sin \frac{1}{x}$ is *uniformly* continuous. One way to see this is to extend the function to the origin by defining

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

By the squeeze theorem, $\lim x_n = 0 \implies \lim f(x_n) = 0$, so g is well-defined and continuous on $\mathbb{R}$. By Theorem 4.33, f is uniformly continuous on any bounded interval. Moreover, the derivative

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

is bounded whenever x is large; together with Exercise 6 we conclude that $f(x)$ is uniformly continuous on $\mathbb{R} \setminus \{0\}$. We cannot use the derivative argument on the whole domain $\mathbb{R} \setminus \{0\}$, since $f'(x)$ is unbounded when x is small ($\lim f'\left(\frac{1}{2\pi n}\right) = \lim(-2\pi n) = -\infty$).

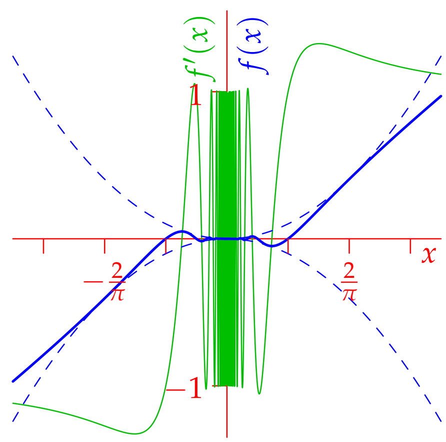


3. $f(x) = x^2 \sin \frac{1}{x}$ is also *uniformly* continuous: again extend by $g(0) = 0$. This time however, we could argue that the derivative is bounded

$$\left|f'(x)\right| = \left|2x \sin \frac{1}{x} - \cos \frac{1}{x}\right| \leq 3$$

since $|\sin y| \leq |y|$ and $|\cos y| \leq 1$ for all y.

Something stranger is going on. As you may verify (see Exercise 3), the extended function g is *everywhere differentiable* with $g'(0) = 0$, and yet the derivative $g'(x)$ itself is *discontinuous* at $x = 0$!



Exercises 4.19. *Key concepts: Order of quantifiers! Bounded derivative $\Rightarrow$ Unif cont $\Leftrightarrow$ Cont extension*

1. Decide whether each f is uniformly continuous. Explain your answers.

   (a) $f(x) = x^3$ on $[-2, 4]$ (b) $f(x) = x^3$ on $(-2, 4)$
   (c) $f(x) = x^{-3}$ on $(0, 4]$ (d) $f(x) = x^{-3}$ on $(1, 4]$
   (e) $f(x) = e^x$ on $(-\infty, 100)$ (f) $f(x) = e^x$ on $\mathbb{R}$

2. Prove that each f is uniformly continuous by verifying the ϵ-δ property.

   (a) $f(x) = 3x + 11$ on $\mathbb{R}$ (b) $f(x) = x^2$ on $[0, 3]$
   (c) $f(x) = \frac{1}{x^2}$ on $[\frac{1}{2}, \infty)$ (d) $f(x) = \frac{x+2}{x+1}$ on $[0, 1]$

3. Verify the claim in Example 4.35.3 that the function $g(x)$ is differentiable at zero[33] but that the derivative $g'(x)$ is discontinuous there.

4. (a) If f is uniformly continuous on a bounded set U, prove that f is bounded on U.
   (*Hint: for contradiction, assume* $\exists (x_n) \subseteq U$ *for which* $|f(x_n)| \to \infty \ldots$)
   (b) Use (a) to give another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.
   (c) Give an example to show that a uniformly continuous function on an *unbounded* set U could be unbounded.

5. Suppose g is defined on U and $a \in U$. Give *very brief* (one line!) arguments for the following.

   (a) Prove that g is continuous at a provided

   $$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |g(x) - g(a)| < \epsilon$$

   (b) Prove that g is continuous at a provided

   $$\forall (x_n) \subseteq U \setminus \{a\},\ \lim x_n = a \implies \lim g(x_n) = g(a)$$

   (c) Verify that the function g defined in Theorem 4.33 is indeed continuous whenever it is well-defined.

6. (a) Suppose f is uniformly continuous on intervals U_1, U_2 for which $U_1 \cap U_2$ is non-empty. Prove that f is uniformly continuous on $U_1 \cup U_2$.
   (*Hint: if* x, y *do not lie in the same interval* U_i*, choose some* $a \in U_1 \cap U_2$ *between* x *and* y)
   (b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.
   (c) More generally, prove that any root function $f(x) = x^{1/n}$ ($n \in \mathbb{N}$) is uniformly continuous on its domain ($\mathbb{R}$ if n is odd and $[0, \infty)$ if n is even).
   (d) (Hard) Given $f(x) = x^{1/n}$, show that $\delta = \epsilon^n$ demonstrates uniform continuity when n is even and $\delta = \left(\frac{\epsilon}{2}\right)^n$ when n is odd.
   (*Hint: use the binomial theorem to prove that* $0 \le y < x + \delta \implies y^{1/n} < x^{1/n} + \delta^{1/n}$)

[33] Use the definition $g'(0) = \lim\limits_{x \to 0} \frac{g(x) - g(0)}{x - 0}$. Limits of functions are covered formally in the next section (course!), but you should be familiar with the idea from elementary calculus.