

4 Continuity

In this chapter we discuss continuous functions. Functions themselves should be familiar. For reference, we begin with a review of some basic concepts and conventions.

We are concerned with functions $f : U \rightarrow V$ where both U, V are subsets of the real numbers \mathbb{R} and f is some *rule* assigning to each real number $x \in U$ a real number $f(x) \in V$. For instance

$$f(x) = \frac{x^2(x-7)}{(x-2)(x^2-9)} \quad \text{assigns to } x = 1 \text{ the value } f(1) = \frac{1(-6)}{(-1)(-8)} = \frac{3}{4}$$

Domain $\text{dom } f = U$ is the set of *inputs* to f . When f is defined by a formula, its *implied domain* is the largest set on which the formula is defined: the above example has implied domain $\text{dom } f = \mathbb{R} \setminus \{2, 3, -3\}$. In examples, the domain is most often a union of intervals of positive length.

Codomain $\text{codom } f = V$ is the set of *possible outputs*. In real analysis, we often take $V = \mathbb{R}$ by default.

Range $\text{range } f = f(U) = \{f(x) : x \in U\}$ is the set of *realized outputs*, and is a subset of $V = \text{codom } f$.

Injectivity f is *injective/one-to-one* if distinct inputs produce distinct outputs. This is usually stated in the contrapositive: $f(x) = f(u) \implies x = u$.

Surjectivity f is *surjective/onto* if every possible output is realized: that is $f(U) = V$.

Inverses f is *bijective/invertible* if it is both injective and surjective. Equivalently, f has an *inverse function* $f^{-1} : V \rightarrow U$ defined as follows:

- Given $y \in V$, f surjective $\implies \exists x \in U$ such that $f(x) = y$.
- Since f is injective, $f(x) = f(u) \implies x = u$, so x is unique. We define $f^{-1}(y) = x$.

Example 4.1. The function defined by $f(x) = \frac{1}{x(x-2)}$ has implied

$$\text{dom } f = \mathbb{R} \setminus \{0, 2\} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

$$\text{range } f = (-\infty, -1] \cup (0, \infty)$$

The function is neither injective (e.g., $f(3) = f(-1)$) nor surjective (e.g., $0 \notin \text{range } f$).

We can remedy both issues by **restricting** the domain and codomain. For instance, the same rule/formula but with

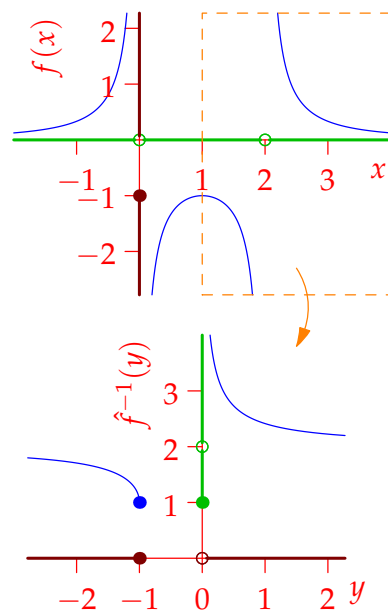
$$\text{dom } \hat{f} = [1, 2) \cup (2, \infty)$$

$$\text{codom } \hat{f} = (-\infty, -1] \cup (0, \infty)$$

defines a bijection with inverse function

$$\hat{f}^{-1}(y) = \begin{cases} 1 + y^{-1}\sqrt{y+1} & \text{if } y > 0 \\ 1 - y^{-1}\sqrt{y+1} & \text{if } y \leq -1 \end{cases}$$

Observe that $\text{dom } \hat{f}^{-1} = \text{codom } \hat{f}$ and $\text{codom } \hat{f}^{-1} = \text{dom } \hat{f}$.



4.17 Continuous Functions

To introduce continuity, consider two common naïve notions.

The graph of f can be drawn without removing one's pen from the page This is intuitive but unusable: *drawn* is poorly defined, so how might we *calculate* or *prove* anything with this concept? It moreover cannot reasonably be extended to other situations or higher dimensions where *drawing a graph* is meaningless.

If x is close to a , then $f(x)$ is close to $f(a)$ This is better and admits generalization. The major issue is the unclear meaning of *close*. Our formal definition of continuity addresses this using *sequences* and *limits*.

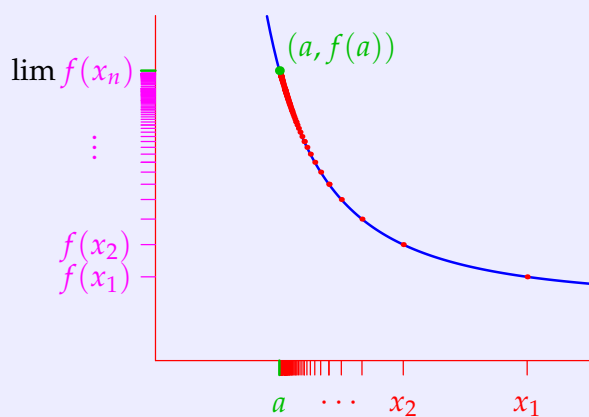
Definition 4.2 (Sequential continuity). A real-valued function $f : U \rightarrow V$ is *continuous at* $a \in U$ if,

$$\forall (x_n) \subseteq U, \lim x_n = a \implies \lim f(x_n) = f(a)$$

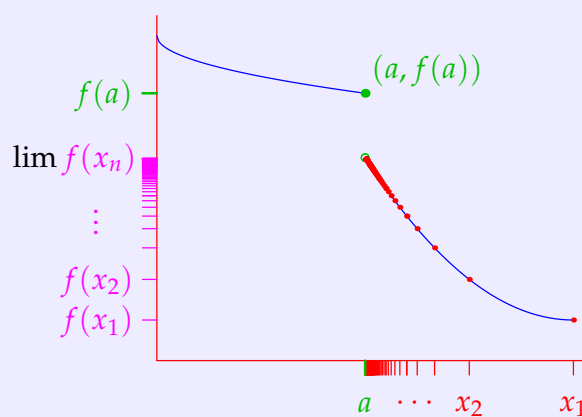
f is *continuous (on U)* if it is continuous at every point $a \in U$.

We say that f is *discontinuous at* $a \in U$ if,

$$\exists (x_n) \subseteq U, \text{ such that } \lim x_n = a \text{ and } (f(x_n)) \text{ does not converge to } f(a)$$



Continuity at a : every sequence with $\lim x_n = a$ has $\lim f(x_n) = f(a)$



Discontinuous at a : at least one sequence with $\lim x_n = a$ has $\lim f(x_n) \neq f(a)$

Examples 4.3. 1. $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2$ is continuous (at every $a \in \mathbb{R}$). To see this, suppose (x_n) converges to a , then, by the limit laws,

$$\lim f(x_n) = \lim x_n^2 = (\lim x_n)^2 = a^2 = f(a)$$

2. The function with $g(x) = 1 + \frac{4}{x^2}$ is continuous. Choose any $a \in \text{dom } g = \mathbb{R} \setminus \{0\}$ and any $(x_n) \subseteq \text{dom } g$ with $\lim x_n = a$. Again, by the limit laws,

$$\lim g(x_n) = \lim \left(1 + \frac{4}{x_n^2} \right) = 1 + \frac{4}{(\lim x_n)^2} = 1 + \frac{4}{a^2} = f(a)$$

This example (with $a = 1$ and $x_n = 1 + \frac{2}{n}$) is the first picture in the above definition.

3. $h : [0, \infty) \rightarrow \mathbb{R} : x \mapsto 3x^{1/4}$ is continuous. Again, everything follows from the limit laws. If $x_n \rightarrow a$ where $x_n \geq 0$ and $a \geq 0$, then

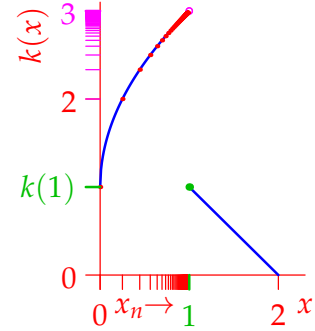
$$\lim h(x_n) = \lim 3x_n^{1/4} = 3(\lim x_n)^{1/4} = 3a^{1/4} = h(a)$$

4. The function defined by

$$k(x) = \begin{cases} 1 + 2\sqrt{x} & \text{if } x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$

is discontinuous at $a = 1$. This seems obvious from the picture, but we need to use the definition. The sequence with $x_n = (1 - \frac{1}{n})^2$ converges to 1 from below, however the limit laws tell us that

$$\lim k(x_n) = \lim \left(1 + 2 \left(1 - \frac{1}{n} \right) \right) = 3 \neq 1 = k(1)$$



Basic Examples and Combinations of Continuous Functions

By appealing to the limit laws for sequences (Theorem 2.15), continuous functions may be combined in natural ways. For instance, if f, g are continuous at a , then

$$\lim x_n = a \implies \lim f(x_n) + g(x_n) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a)$$

whence $f + g$ is continuous at a . Here is a general summary.

Theorem 4.4. 1. Suppose f , and g are continuous and that k is constant. Then the following functions are continuous (on their domains):

$$kf, \quad |f|, \quad f + g, \quad f - g, \quad fg, \quad \frac{f}{g}, \quad \max(f, g), \quad \min(f, g)$$

2. If $n \in \mathbb{N}$ then $f : x \mapsto x^{1/n}$ is continuous on its domain.
3. Compositions of continuous functions are continuous: if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a .
4. Algebraic functions are continuous (includes all polynomials and rational functions).

Proof. Parts 1, 2 are the limit laws; for the maximum and minimum, see Exercise 2. For part 3:

$$\lim x_n = a \xrightarrow{g \text{ cont}} \lim g(x_n) = g(a) \xrightarrow{f \text{ cont}} \lim f(g(x_n)) = f(g(a))$$

Part 4 follows by combining parts 1, 2 and 3. ■

Example 4.5. The following algebraic function is continuous on its domain

$$f : (7, \infty) \rightarrow \mathbb{R} : x \mapsto \sqrt{\frac{3x^{5/2} + 7x^2 + 4}{(x-7)^{1/3}}}$$

Theorem 4.6 (Squeeze theorem). Suppose $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$, that f, h are continuous at a , and that $f(a) = g(a) = h(a)$. Then g is continuous at a .

Proof. This is simply the squeeze theorem (2.12) for sequences: if $\lim x_n = a$, then

$$f(x_n) \leq g(x_n) \leq h(x_n) \implies \lim g(x_n) = g(a)$$

To provide more interesting examples, we state the following without proof.

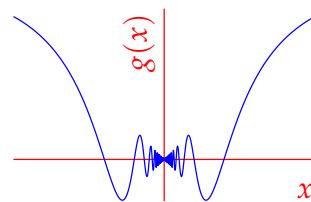
Theorem 4.7. The common trigonometric, exponential and logarithmic functions are continuous.

It is possible, though slow and ugly, to address some of this now. We won't do this since it is cleaner to define these functions using power series,²⁷ which makes their continuity (and differentiability/integrability!) come for free.

Examples 4.8. 1. $f(x) = \frac{\sqrt{x}}{\sin e^x}$ is continuous on its domain $\mathbb{R} \setminus \{\ln(n\pi) : n \in \mathbb{N}_0\}$.

2. If $g(x) = x \sin \frac{1}{x}$ when $x \neq 0$, and $g(0) = 0$, then g is continuous on \mathbb{R} . When $x \neq 0$, this follows from Theorems 4.4 and 4.7, while at $a = 0$ we rely on the squeeze theorem:

$$x \neq 0 \implies -x \leq x \sin \frac{1}{x} \leq x$$



The ϵ - δ Definition of Continuity

The sequential definition of continuity uses limits *twice*. By stating each of these using the ϵ -definition of limit, we can reformulate continuity without mentioning sequences!

To motivate this, consider $f(x) = x^2$ at $a = 2$. By continuity, if (x_n) is a sequence with $\lim x_n = 2$, then $\lim f(x_n) = 4$. We restate each of these using the definition of limit:

$$(a) (\lim x_n = 2) \quad \forall \delta > 0, \exists M \text{ such that } n > M \implies |x_n - 2| < \delta$$

$$(b) (\lim x_n^2 = 4) \quad \forall \epsilon > 0, \exists N \text{ such that } n > N \implies |x_n^2 - 4| < \epsilon$$

Here is a short argument that shows how (a) \implies (b) (we'll revisit this formally in a moment).

Assume (a) and suppose $\epsilon > 0$ is given. Define $\delta = \min(1, \frac{\epsilon}{5})$. Since $\lim x_n = 2$, $\exists M$ such that

$$\begin{aligned} n > M &\implies |x_n^2 - 4| = |x_n - 2| |x_n + 2| < \delta |x_n - 2| + 4 && \text{(by (a))} \\ &\leq \delta (|x_n - 2| + 4) && (\triangle\text{-inequality}) \\ &< \delta(\delta + 4) \leq 5\delta \leq \epsilon && ((a) \text{ again}) \end{aligned}$$

Let $N = M$ to conclude (b).

It turns out not to be very important that (x_n) be a *sequence*. In fact we can dispense with it entirely...

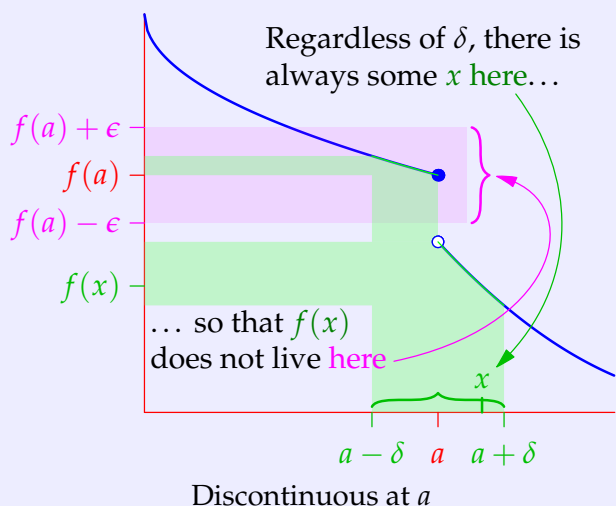
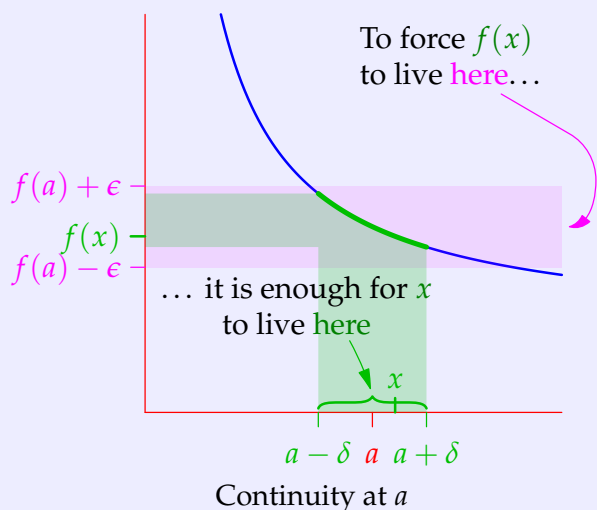
²⁷For instance via Maclaurin series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Definition 4.9 (ϵ - δ continuity). A real-valued function $f : U \rightarrow V$ is continuous at $a \in U$ if²⁸

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } (\forall x \in U) |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \quad (*)$$

We say that f is *discontinuous* at $a \in U$ if,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in U \text{ with } |x - a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon \quad (\dagger)$$



This fits with the intuitive interpretation of continuity: if x is close to a , then $f(x)$ is close to $f(a)$; ϵ and δ are our measures of *closeness*. Many mathematicians consider the ϵ - δ version to be *the* definition of continuity. Thankfully, it doesn't matter which you prefer...

Theorem 4.10. The sequential and ϵ - δ definitions of continuity (4.2 & 4.9) are equivalent.

Examples (4.3, cont). Before seeing a proof, we repeat our earlier examples using the ϵ - δ definition. As with ϵ - N arguments for limits, it is often useful to do some scratch work first.

1. Suppose $f(x) = x^2$ and $a \in \mathbb{R}$. Our goal is to control the size of $|x^2 - a^2|$ whenever $|x - a|$ is small. To keep things simple, assume $|x - a| < 1$, then,

$$\begin{aligned} |x^2 - a^2| &= |x - a| |x + a| = |x - a| |(x - a) + 2a| \\ &\stackrel{\Delta}{\leq} |x - a| (|x - a| + 2|a|) = |x - a| (1 + 2|a|) \end{aligned}$$

Now let $\epsilon > 0$ be given and define $\delta = \min(1, \frac{\epsilon}{1+2|a|})$. Then

$$|x - a| < \delta \implies |f(x) - f(a)| = |x^2 - a^2| < \delta(1 + 2|a|) \leq \epsilon$$

Thus f is continuous at a . This is simply a general version of the argument on page 63 with all mention of sequences removed!

²⁸The bracketed $\forall x \in U$ is often omitted in $(*)$ since the implication requires that x be universally quantified. It is important that $x \in U = \text{dom } f$ rather than merely $x \in \mathbb{R}$! By contrast, the expression $\exists x \in U$ in (\dagger) is *always* written.

2. Let $g(x) = 1 + \frac{4}{x^2}$ and $a \neq 0$. The first challenge is to keep away from zero so that $\frac{1}{x}$ behaves. To do this, we insist that $\delta \leq \frac{|a|}{2}$, so that

$$|x - a| < \delta \implies \frac{|a|}{2} < |x| < \frac{3|a|}{2} \implies \frac{1}{|x|} < \frac{2}{|a|} \quad (*)$$

Now consider the required difference. If $|x - a| < \delta$, then

$$\begin{aligned} |g(x) - g(a)| &= \left| 1 + \frac{4}{x^2} - 1 - \frac{4}{a^2} \right| = \frac{4|a^2 - x^2|}{a^2x^2} = \frac{4|a + x|}{a^2x^2} |x - a| < \frac{4|a + x|}{a^2x^2} \delta \\ &\stackrel{\triangle}{\leq} 4 \left(\frac{1}{|a|x^2} + \frac{1}{a^2|x|} \right) \delta \stackrel{(*)}{<} 4 \left(\frac{4}{|a|^3} + \frac{2}{|a|^3} \right) \delta = \frac{24}{|a|^3} \delta \end{aligned}$$

Given $\epsilon > 0$, it suffices to let $\delta = \min(\frac{1}{2}|a|, \frac{1}{24}|a|^3\epsilon)$. Then $|x - a| < \delta \implies |g(x) - g(a)| < \epsilon$.

3. For $h(x) = 3x^{1/4}$ there are two cases. Suppose $\epsilon > 0$ is given.

- If $a = 0$, let $\delta = (\frac{\epsilon}{3})^4$, then²⁹

$$|x - a| < \delta \implies 0 \leq x < \delta \implies |h(x) - h(a)| = 3x^{1/4} < 3\delta^{1/4} = \epsilon$$

- If $a > 0$, let $\delta = \frac{1}{3}a^{3/4}\epsilon$. Then, if $|x - a| < \delta$,

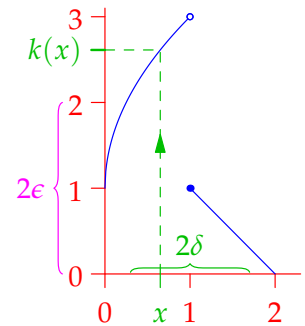
$$|h(x) - h(a)| = 3|x^{1/4} - a^{1/4}| = \frac{3|x - a|}{x^{3/4} + a^{1/4}x^{1/4} + a^{1/2}x^{1/4} + a^{3/4}} \leq \frac{3|x - a|}{a^{3/4}} < \frac{3\delta}{a^{3/4}} = \epsilon$$

4. We could establish the discontinuity statement (+) directly, but it is typically easier to argue by contradiction.

Suppose k is continuous at 1 and let $\epsilon = 1$. Then $\exists \delta > 0$ for which

$$\begin{aligned} |x - 1| < \delta &\implies |k(x) - k(1)| = |k(x) - 1| < 1 \\ &\implies 0 < k(x) < 2 \end{aligned}$$

However, $x = \max(\frac{1}{4}, 1 - \frac{\delta}{2})$ satisfies $|x - 1| \leq \frac{\delta}{2} < \delta$ and $k(x) \geq k(\frac{1}{4}) = 1 + \frac{2}{2} = 2$. Contradiction. Think this last bit through!



The basic rules for combining continuous functions may also be proved using ϵ - δ arguments. E.g.,

ϵ - δ proof of the squeeze theorem. Given $\epsilon > 0$, we know there exist $\delta_1, \delta_2 > 0$ for which

$$|x - a| < \delta_1 \implies |f(x) - f(a)| < \epsilon \quad \text{and} \quad |x - a| < \delta_2 \implies |h(x) - h(a)| < \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$, then

$$|x - a| < \delta \implies |g(x) - g(a)| \leq \max(|f(x) - f(a)|, |h(x) - h(a)|) < \epsilon$$

whence g is continuous at 0. ■

²⁹Remember the hidden quantifier: $|x - a| < \delta$ for all $x \in \text{dom } f = [0, \infty)$, thus $x \geq 0$ for the duration of this example.

Several other arguments are in the exercises. Finally, here is the promised proof of equivalence.

Proof of Theorem 4.10. (sequential $\Rightarrow \epsilon$ - δ) We prove the contrapositive. Suppose a is an ϵ - δ discontinuity (†) and let $\delta = \frac{1}{n}$. Then there exists $x_n \in U$ such that

$$|x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(a)| \geq \epsilon$$

Repeating for all $n \in \mathbb{N}$ plainly produces a sequence (x_n) for which $\lim x_n = a$ and $\lim f(x_n) \neq f(a)$: otherwise said, a is a sequential discontinuity.

(ϵ - $\delta \Rightarrow$ sequential) Assume (*), let $(x_n) \subseteq U$ and suppose $\lim x_n = a$; we must prove that $\lim f(x_n) = f(a)$. Let $\epsilon > 0$ be given so that a suitable δ satisfying (*) exists. Since $\lim x_n = a$,

$$\begin{aligned} \exists N \text{ such that } n > N &\implies |x_n - a| < \delta && \text{(since } x_n \rightarrow a \text{ and } \delta > 0 \text{ is given)} \\ &\implies |f(x_n) - f(a)| < \epsilon && \text{(by (*))} \end{aligned}$$

We conclude that $\lim f(x_n) = f(a)$, as required. ■

Examples 4.11. We finish with a couple of esoteric examples on the same theme.

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the *indicator function* for the rational numbers:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Suppose f is continuous at a and let $\epsilon = 1$. Then $\exists \delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < 1 \quad (\ddagger)$$

There are two cases; both rely on the fact that any interval contains both rational and irrational numbers (Corollary 1.23, etc.).

- (a) If $a \in \mathbb{Q}$, then $f(a) = 1$. There exists an irrational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |0 - 1| = 1 \not< 1$.
- (b) If $a \notin \mathbb{Q}$, then $f(a) = 0$. There exists a rational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |1 - 0| = 1 \not< 1$.

Either way, we have contradicted (‡). We conclude that f is *nowhere continuous*.

2. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since $0 \leq |g(x)| \leq |x|$, the squeeze theorem tells us that g is continuous at $x = 0$.

Now suppose g is continuous at $a \neq 0$ and let $\epsilon = |a|$. Then $\exists \delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < |a|$$

The same two cases as in the previous example provide contradictions. We conclude that g is *continuous at precisely one point!*

Exercises 4.17. Key concepts: Sequential and ϵ - δ continuity definitions/equivalence, ϵ - δ examples

1. Consider the function with $f(x) = \frac{1}{\sqrt{x^2+2x-3}}$.
 - (a) The implied domain of f has the form $\text{dom } f = (-\infty, a) \cup (b, \infty)$. Find a and b .
 - (b) What is the range of f ?
 - (c) Show that $f : (b, \infty) \rightarrow \text{range } f$ is *bijective* and compute its inverse function.
 - (d) Find the inverse function when we instead restrict the domain to $(-\infty, a)$.
 - (e) Briefly explain why f is continuous on its domain.
2. Let f and g be continuous functions at a .
 - (a) Show that $\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$ and deduce that $\max(f, g)$ is continuous at a .
 - (b) How might you show continuity of $\min(f, g)$?
3. Use ϵ - δ arguments to prove the following.
 - (a) $f(x) = x^2 - 3x$ is continuous at $x = 1$.
 - (b) $g(x) = x^3$ is continuous at $x = a$.
 - (c) $h : [0, \infty) \rightarrow \mathbb{R} : x \mapsto \sqrt{x}$ is continuous.
 - (d) $j(x) = 3x^{-1}$ is continuous on $\mathbb{R} \setminus \{0\}$.
4. Rephrase Example 4.3.4's ϵ - δ argument by directly justifying the discontinuity definition (+).
5. Prove that each function is discontinuous at $x = 0$; use *both* sequential and ϵ - δ formulations.
 - (a) $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.
 - (b) $g(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $g(0) = 0$.
6. Suppose f and g are continuous at a . Prove the following using ϵ - δ arguments.
 - (a) $f - g$ is continuous at a .
 - (b) If h is continuous at $f(a)$, then $h \circ f$ is continuous at a .
7. Suppose $f : U \rightarrow V \subseteq \mathbb{R}$ is a function whose domain U contains an *isolated point* a : i.e. $\exists r > 0$ such that $(a - r, a + r) \cap U = \{a\}$. Prove that f is continuous at a .
8. In Example 4.11.2, provide the details of the required contradiction.
9.
 - (a) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function for which $f(x) = 0$ whenever $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
 - (b) Suppose $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all rational x . Prove that $f = g$.
10. (Hard) Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{q} & \text{whenever } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q > 0 \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For instance, $f(1) = f(2) = f(-7) = 1$, and $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \dots = \frac{1}{2}$, etc.

- (a) Prove that f is discontinuous at each rational number r .
- (b) Prove that f is continuous at each irrational number i .
 (Hint: given $\epsilon > 0$, let $q = \lceil \frac{1}{\epsilon} \rceil$, $A = \{r \in \mathbb{Q} : f(r) \geq \frac{1}{q}\}$ and let $\delta = \min_{r \in A} |i - r| \dots$)

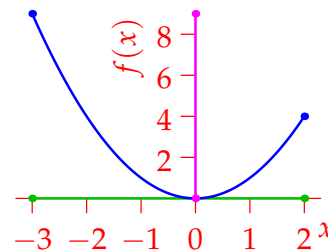
4.18 Properties of Continuous Functions

In this section we consider how continuous functions transform *intervals*.

Example 4.12. $f(x) = x^2$ maps $[-3, 2]$ onto $[0, 9]$. In particular:

- f transforms an **interval** into **another**.
- f transforms a **closed bounded set** into **another**.

Our goal is to see that these are general properties exhibited by *any* continuous function.



First recall a couple of definitions.

Definition 4.13. Suppose $f : U \rightarrow V$ where $U, V \subseteq \mathbb{R}$.

- (a) U is *bounded* if $\exists M$ such that $\forall x \in U, |x| \leq M$.
 (b) f is *bounded* if its range is a bounded set: $\exists M$ such that $\forall x \in U, |f(x)| \leq M$.
- (Definition 2.46) U is *closed* if every convergent sequence in U has its limit in U :

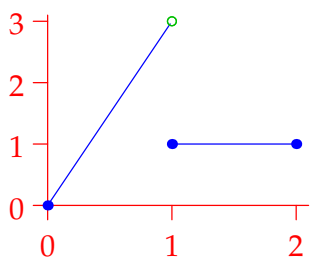
$$\forall (x_n) \subseteq U, \lim x_n = s \ (\in \mathbb{R}) \implies s \in U$$

Theorem 4.14 (Extreme Value Theorem). Suppose $f : U \rightarrow V$ is continuous where U is closed and bounded. Then $f(U)$ is closed and bounded. In particular, f is bounded and attains its bounds:

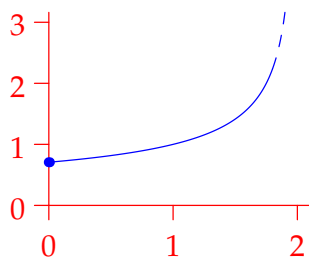
$$\exists s, i \in U \text{ such that } f(s) = \sup f(U) \text{ and } f(i) = \inf f(U)$$

Examples 4.15. 1. (Example 4.12) If $f(x) = x^2$ on $U = [-3, 2]$, then $f(U) = [0, 9]$ is closed and bounded. Moreover, $\sup f(U) = f(-3)$ and $\inf f(U) = f(0)$ (i.e., $s = -3$ and $i = 0$).

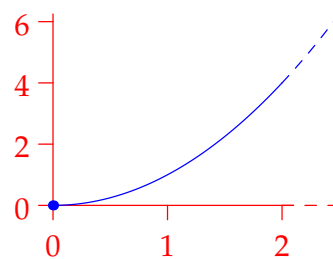
- Before seeing the proof, here are three examples where we weaken one of the hypotheses of the extreme value theorem and see that the conclusion fails.



(a) f discontinuous



(b) U not closed



(c) U not bounded

(a) If $U = [0, 2]$, $f(x) = 3x$ when $x < 1$ and $f(x) = 1$ when $x \geq 1$, then $f(U) = [0, 3)$. In particular, $\sup f(U) = 3$ is not attained.

(b) If $f(x) = \frac{1}{\sqrt{2-x}}$ and $U = [0, 2)$, then $f(U) = [\frac{1}{\sqrt{2}}, \infty)$ is unbounded.

(c) If $f(x) = x^2$ and $U = [0, \infty)$, then $f(U) = [0, \infty)$ is unbounded.

The strategy of the proof is to show that every limit point of $f(U) = \text{range } f$ lies in $f(U)$. We break things into simple steps; observe where each **hypothesis** is used.

- Proof.* 1. Suppose M is a limit point of $f(U)$: that is, $M = \lim f(x_n)$ for some sequence $(x_n) \subseteq U$. *A priori*, M need not be finite, but $M = \sup f(U)$ or $\inf f(U)$ are certainly possible.³⁰
2. Since $(x_n) \subseteq U$ is **bounded**, Bolzano–Weierstraß (Theorem 2.41) says it has a convergent subsequence, $\lim_{k \rightarrow \infty} x_{n_k} = x$.
3. Since U is **closed**, we have $x \in U$. This means $f(x)$ can be evaluated (it is *finite*).
4. Since f is **continuous**, $\lim f(x_{n_k}) = f(x)$.
5. Finally, $M = f(x)$ since all subsequences of a convergent (or divergent to $\pm\infty$) sequence tend to the same limit (Lemma 2.37). It follows that all limit points M are *finite* and lie in $f(U)$: otherwise said, $f(U)$ is closed and bounded.

Choosing $M = \sup f(U)$ yields $x = s \in U$ (similarly $\inf f(U)$ leads to $i \in U$). ■

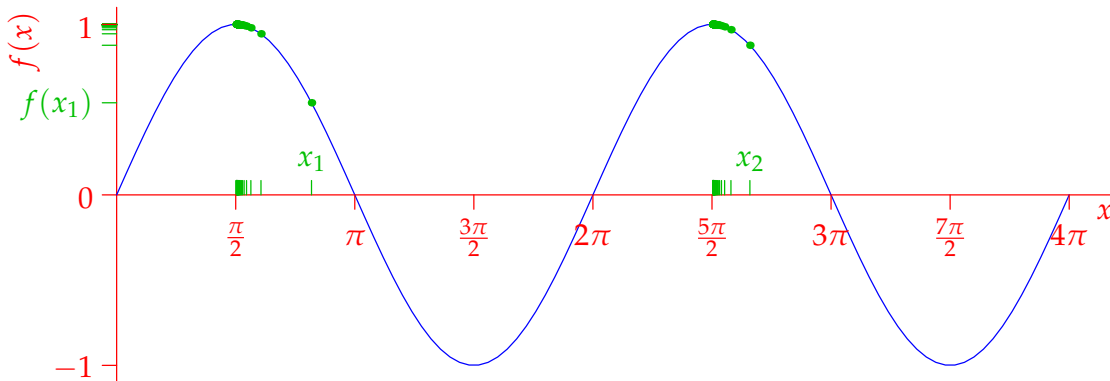
Example 4.16. It is worth considering why we needed a *subsequence* in the proof. The reason is that the bounds of f might be attained multiple times. For example, suppose

$$f : [0, 4\pi] \rightarrow \mathbb{R} : x \mapsto \sin x$$

This satisfies the hypotheses of the extreme value theorem: $U = [0, 4\pi]$ is closed and bounded and f is continuous. Indeed $\max f(U) = 1$ is attained at *both* $x = \frac{\pi}{2}$ and $\frac{5\pi}{2}$. The sequence defined by

$$x_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n} & \text{if } n \text{ is odd} \\ \frac{5\pi}{2} + \frac{1}{n} & \text{if } n \text{ is even} \end{cases} \quad \text{has} \quad f(x_n) = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) \xrightarrow{n \rightarrow \infty} 1 = \sup f(U)$$

and therefore satisfies step 1 of the proof. However, (x_n) itself is *divergent by oscillation*. Bolzano–Weierstraß is used to force the existence of a convergent subsequence; in this case the subsequence of odd terms $(x_{n_k}) = (x_{2k-1})$ satisfies the remaining steps.



³⁰If $M = \sup f(U)$, then a suitable (x_n) might be constructed as follows:

- If $M \in \mathbb{R}$, then for each $n \in \mathbb{N}$, $\exists x_n \in U$ such that $M - \frac{1}{n} < f(x_n) \leq M$ (Lemma 1.20).
- If $M = \infty$, then for each $n \in \mathbb{N}$, $\exists x_n \in U$ such that $f(x_n) \geq n$.

The Intermediate Value Theorem and its Consequences

This result should be familiar from elementary calculus, even if its proof is not. It should also be intuitive: like the Grand Old Duke of York, if you march up a hill, then at some point you must be half-way up...

Theorem 4.17 (Intermediate Value Theorem (IVT)). Suppose f is continuous on $[a, b]$ and that y lies strictly between $f(a)$ and $f(b)$. Then $\exists \xi \in (a, b)$ such that $f(\xi) = y$.

Being an existence result, it should be no surprise that *completeness* is used in the proof.

Proof. WLOG assume $f(a) < y < f(b)$. Let $S = \{x \in [a, b] : f(x) < y\}$ and define $\xi := \sup S$.

Since S is non-empty ($a \in S$) and bounded above (by b), we see that ξ exists and is finite. It remains to prove that $f(\xi) = y$ and $\xi \neq a, b$.

First choose any $(s_n) \subseteq S$ such that $\lim s_n = \xi$. Continuity forces $\lim f(s_n) = f(\xi)$. Moreover

$$f(s_n) < y \implies f(\xi) \leq y$$

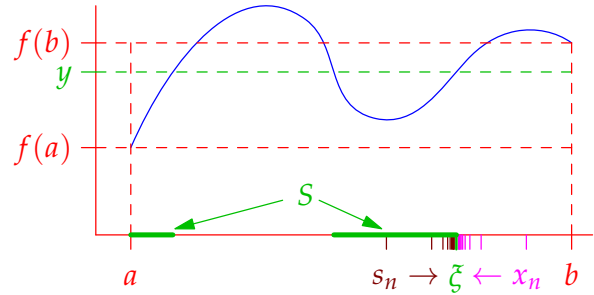
Since $f(b) > y$, this also shows that $\xi \neq b$.

We now play a similar game from the other side: define $x_n := \min(\xi + \frac{1}{n}, b)$, then $\lim x_n = \xi$ and

$$\begin{aligned} x_n > \xi = \sup S &\implies x_n \notin S \implies f(x_n) \geq y \\ &\implies f(\xi) = \lim f(x_n) \geq y \end{aligned}$$

again via the continuity of f and the convergence properties of bounded sequences. Since $y > f(a)$, we also conclude that $\xi \neq a$.

Putting it all together, $f(\xi) = y$ and $\xi \in (a, b)$. ■



Note how the value of ξ in the proof is always the *largest* of potentially several choices.

Examples 4.18. In elementary calculus, the intermediate value theorem is typically applied to demonstrate the existence of solutions to equations.

1. We show that the equation $x^7 + 3x = 1 + 4\cos(\pi x)$ has a solution.

The trick is to express the equation in the form $f(x) = y$ where f is continuous, then choose suitable a, b to fit the theorem. In this case,

$$f(x) = x^7 + 3x - 4\cos(\pi x) \quad \text{and} \quad y = 1$$

are suitable choices. Now observe

$$f(0) = -4 < y \quad \text{and} \quad f(1) = 1 + 3 + 4 = 8 > y \quad (\text{i.e., } a = 0 \text{ and } b = 1)$$

whence $\exists \xi \in (0, 1)$ such that $f(\xi) = y = 1$. Otherwise said, ξ is a solution to the original equation.

The function f is plainly continuous on \mathbb{R} , a much larger interval than $[a, b]$, but no matter.

2. The existence of a root ζ of the (continuous) polynomial

$$f(x) = x^5 - 5x^4 + 150$$

follows from the intermediate value theorem by observing that

$$f(0) = 150 > 0 \quad \text{and} \quad f(4) = -256 + 150 = -106 < 0$$

We conclude that such a root ζ exists satisfying $\zeta \in (0, 4)$.

As the graph suggests, there are other roots (η, ζ), the existence of which may be shown by observing, say,

$$f(-3) = -798 < 0 \quad \text{and} \quad f(5) = 150 > 0$$

With an eye on generalizing, here is an alternative approach. Define sequences $(s_n), (t_n)$ via

$$s_n := \frac{f(-n)}{n^5} = -1 - \frac{5}{n} + \frac{150}{n^5} \quad t_n := \frac{f(n)}{n^5} = 1 - \frac{5}{n} + \frac{150}{n^5}$$

Since $\lim s_n = -1$ and $\lim t_n = 1$, we see that

$$\exists a \text{ such that } s_a < -\frac{1}{2} \implies f(-a) = a^5 s_a < -\frac{1}{2} a^5 < 0$$

$$\exists b \text{ such that } t_b > \frac{1}{2} \implies f(b) = b^5 t_b > \frac{1}{2} b^5 > 0$$

Applying the intermediate value theorem on $[-a, b]$ shows the existence of a root.

The second approach in Example 4.16.2 may be applied to prove a general result.

Corollary 4.19. *A polynomial function of odd degree has at least one real root.*

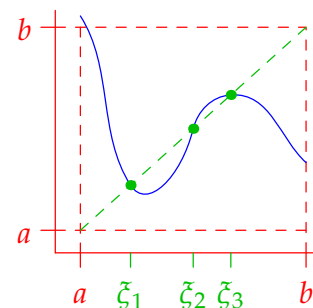
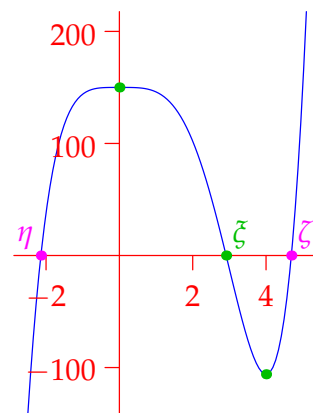
The proof is an exercise. An even simpler exercise shows the existence of a *fixed point* for a particular type of continuous function.

Corollary 4.20 (Fixed Point Theorem). *Suppose a and b are finite and that $f : [a, b] \rightarrow [a, b]$ is continuous. Then f has a fixed point:*

$$\exists \zeta \in [a, b] \text{ such that } f(\zeta) = \zeta$$

As the picture shows, a function could have several fixed points.

This is the most basic fixed-point theorem in analysis: if you continue your studies you'll meet several more. Many important consequences flow from such results, including a common fractal construction and the standard existence/uniqueness result for differential equations.



For a final corollary, first note a straightforward characterization that helps us consider all types of interval simultaneously: $U \subseteq \mathbb{R}$ is an interval precisely when

$$a, b \in U \text{ and } a < y < b \implies y \in U \quad (*)$$

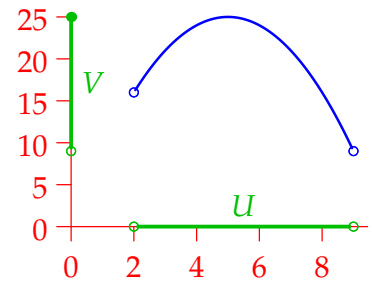
Corollary 4.21 (Preservation of Intervals). Suppose U is an interval of positive length, and that $f : U \rightarrow V$ is continuous and surjective ($V = f(U)$).

1. V is an interval or a point.
2. If f is strictly increasing (decreasing), then:
 - (a) V is an interval of positive length, f is injective, and therefore bijective.
 - (b) The inverse function $f^{-1} : V \rightarrow U$ is also continuous and strictly increasing (decreasing).

Example 4.22. The interval V need not be of the same type as U . For instance, if $f(x) = 10x - x^2$, then f maps the open interval $U = (2, 9)$ to the half-open interval $V = (9, 25]$.

The extreme value theorem, however, guarantees that if U is closed and bounded, then V is also. For instance,

$$f([2, 9]) = [9, 25]$$



Proof. 1. If V is not a point, then $\exists a, b \in U$ such that $f(a) < f(b)$. If y lies between these, IVT says $\exists \xi$ between a and b such that $y = f(\xi)$. That is, $y \in f(U)$. By (*), $V = f(U)$ is an interval.

2. (a) If f is strictly increasing, then $\forall a, b \in U$, $a < b \implies f(a) < f(b)$. Plainly f is injective and V contains at least two points; by part 1 it is an interval of positive length.
- (b) Let $y_1 < y_2$ where both lie in V , and define $x_i = f^{-1}(y_i)$ for $i = 1, 2$. Since f is increasing,

$$x_2 \leq x_1 \implies y_2 = f(x_2) \leq f(x_1) = y_1$$

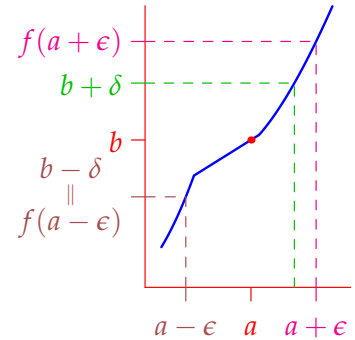
is a contradiction. Thus $x_1 < x_2$ and f^{-1} is also strictly increasing.

If $a \in U$, it remains to show that f^{-1} is continuous at $b = f(a)$. Assume first that a is not an endpoint of U and let $\epsilon > 0$ be given such that $[a - \epsilon, a + \epsilon] \subseteq U$. Now define

$$\delta := \min(b - f(a - \epsilon), f(a + \epsilon) - b)$$

This is positive since f is strictly increasing. But now

$$\begin{aligned} |y - b| < \delta &\implies f(a - \epsilon) - b < y - b < f(a + \epsilon) - b \\ &\implies f(a - \epsilon) < y < f(a + \epsilon) \\ &\implies a - \epsilon < f^{-1}(y) < a + \epsilon \\ &\implies |f^{-1}(y) - f^{-1}(b)| = |f^{-1}(y) - a| < \epsilon \end{aligned}$$



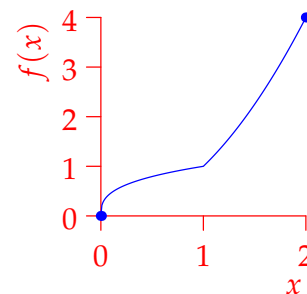
where (\implies) used the fact that f is strictly increasing.

If a is an endpoint of U , instead use $[a - \epsilon, a] \subseteq U$ or $[a, a + \epsilon] \subseteq U$ and only the corresponding half of the expression defining δ . ■

Example 4.23. The function $f : [0, 2] \rightarrow [0, 4]$ defined by

$$f(x) = \begin{cases} \sqrt[3]{x} & \text{if } 0 \leq x \leq 1 \\ x^2 & \text{if } 1 < x \leq 2 \end{cases}$$

is continuous, surjective and strictly increasing. It therefore has a continuous inverse $f^{-1} : [0, 4] \rightarrow [0, 2]$. Compare this with the familiar statement from elementary calculus: $f' > 0 \implies f$ injective. We cannot apply this here since f is not differentiable!



Exercises 4.18. Key concepts: Extreme/Intermediate Value Theorems, Cont functions preserve intervals

- Give an example of a *discontinuous* function $f : [0, 1] \rightarrow \mathbb{R}$ which is *not bounded*.
 - State a *continuous* function with domain $(1, \infty)$ whose range is *bounded but not closed*.
- Let $a < b$ be given. Give examples of *continuous* functions $g, h : (a, b) \rightarrow \mathbb{R}$ such that:
 - g is *not bounded*.
 - h is bounded but *does not attain its bounds*.
- Compute the inverse of the function f in Example 4.23.
- Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to some $x_0 \notin S$. Show that there exists an unbounded continuous function on S .
- Prove that $x = \cos x$ for some $x \in (0, \frac{\pi}{2})$.
- Suppose that f is a real-valued continuous function on \mathbb{R} and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists some x between a, b such that $f(x) = 0$.
- Suppose f is continuous on $[0, 2]$ and that $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.
(Hint: consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$)
- Prove the fixed point theorem (Corollary 4.20).
(Hint: If neither a nor b are fixed points, consider $g(x) = f(x) - x$)
 - Prove Corollary 4.19 for a general odd-degree monic polynomial $f(x) = x^{2m+1} + \sum_{k=0}^{2m} \alpha_k x^k$.
- Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.
 - Explain why f is continuous on any interval U .
 - Suppose $a < 0 < b$ and that $f(a), f(b)$ have opposite signs. If $y = 0$, show that the intermediate value theorem is satisfied by *infinitely many* distinct values ξ .
- Suppose $f : U \rightarrow \mathbb{R}$ is continuous and that $U = \bigcup_{k=1}^n I_k$ is the union of a finite sequence (I_k) of closed bounded intervals. Prove that f is bounded and attains its bounds.
 - Let $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for each $n \in \mathbb{N}$. Give an example of a continuous function $f : U \rightarrow \mathbb{R}$ which is either unbounded or does not attain its bounds. Explain.
(This relates to the idea that *finite* unions of closed sets are closed, but *infinite* unions need not be)

4.19 Uniform Continuity

Suppose $f : U \rightarrow V$ is continuous. By the ϵ - δ definition (4.9),

$$\forall a \in U, \forall \epsilon > 0, \exists \delta(a, \epsilon) > 0 \text{ such that } (\forall x \in U) |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \quad (*)$$

We write $\delta(a, \epsilon)$ to stress that δ can depend both on the *location* a and the *distance* ϵ . The goal of this section is to understand if/when it is possible to choose δ *independently of the location* a .

Example 4.24. We start with an example where our desire cannot be satisfied.

Consider $f(x) = x^2$ with domain $U = [0, \infty)$. Since f is continuous, given $\epsilon > 0$ and $a_1 \in U$, there exists δ such that

$$|x - a_1| < \delta \implies |f(x) - f(a_1)| = |x^2 - a_1^2| < \epsilon$$

On page 64 we saw that $\delta = \min(1, \frac{\epsilon}{1+2a_1})$ was suitable, but this *depends on the location* a_1 . Of course other expressions for δ will also work...

Visualize what happens if we attempt to use the *same constant* δ for different a_i : imagine sliding the fixed-width δ -interval along the x -axis while simultaneously sliding the ϵ -interval vertically. As a_i increases, the *image* of the δ -interval eventually becomes too large for the ϵ -interval to contain: if δ is constant, then

$$\text{length}(f(a_i - \delta, a_i + \delta)) = (a_i + \delta)^2 - (a_i - \delta)^2 = 4a_i\delta$$

increases unboundedly with a_i . For fixed ϵ , as a increases, the *increasing gradient* of f means that we need to choose a *smaller* δ .

By contrast, if $f(x) = x^2$ on a *finite* domain $[0, b]$, then any δ that demonstrates continuity at $x = b$ will also do so everywhere else on $[0, b]$. We'll check this explicitly in a moment.

To obtain a formal definition, we rewrite (*) with the extra assumption that δ may be chosen independently of the location a ; this amounts to moving the quantifier $\forall a \in U$ after δ .

Definition 4.25. A function $f : U \rightarrow V \subseteq \mathbb{R}$ is *uniformly continuous* if

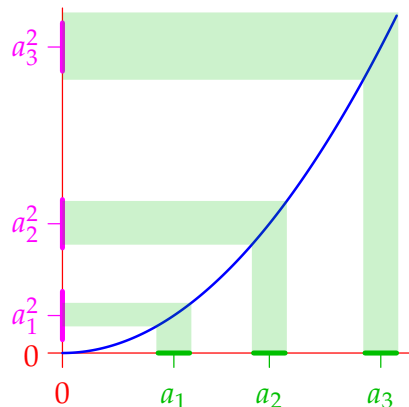
$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } (\forall x, y \in U) |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \quad (\dagger)$$

We use y instead of a for symmetry. Observe how δ , being quantified *before* x, y , now depends only on ϵ . As before, the quantifiers for x, y are usually hidden. Note also how uniform continuity is only relevant on the entire domain U ; it makes no sense to speak of uniform continuity at a single point.

For the sake of tidiness, we make one more observation before seeing some examples.

Lemma 4.26. If f is uniformly continuous on U , then it is continuous on U .

This is trivial: (\dagger) is the ϵ - δ continuity of f at $y \in U$, for *all* y *simultaneously*! The special feature of the definition is that the same δ works for all y .



Examples 4.27. 1. We re-analyze $f(x) = x^2$ in view of the definition. Recall first that

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| |x + y|$$

where $|x - y|$ is easily controlled by δ . We consider the behavior of $|x + y|$ in two cases.

Bounded domain If $U = \text{dom } f \subseteq [-T, T]$ for some $T > 0$, we show that f is uniformly continuous. This will follow because $|x + y| \leq 2T$ is also easily controlled.

Let $\epsilon > 0$ be given and define $\delta = \frac{\epsilon}{2T}$, then

$$|x - y| < \delta \implies |f(x) - f(y)| < \delta \cdot 2T = \epsilon$$

Compare with Example 4.24. Our approach works for *this* function because the gradient (and therefore the discrepancy between $x^2 - y^2$ and $x - y$) is greatest at the endpoints of the interval. The same approach may not work for other functions!

Unbounded domain We show that f is not uniformly continuous when $\text{dom } f = [0, \infty)$.

For contradiction, assume f is uniformly continuous; let $\epsilon = 1$ and suppose $\delta > 0$ satisfies the definition. Taking $x - y = \frac{\delta}{2}$, we see that

$$|x + y| = 2y + \frac{\delta}{2} \implies |f(x) - f(y)| = \frac{\delta}{2} \left(2y + \frac{\delta}{2} \right) = \delta \left(y + \frac{\delta}{4} \right) > \delta y$$

Let $y = \frac{1}{\delta}$ for the contradiction $|f(x) - f(y)| > 1 = \epsilon$ (large y are the problem!).

2. Let $g(x) = \frac{1}{x}$; we again consider two domains.

Uniform continuity on $[a, b]$ whenever $0 < a < b \leq \infty$.

Let $\epsilon > 0$ be given and let $\delta = a^2\epsilon$. Then,

$$\begin{aligned} |x - y| < \delta &\implies |g(x) - g(y)| = \left| \frac{y - x}{xy} \right| \\ &< \frac{\delta}{xy} \leq \frac{\delta}{a^2} = \epsilon \end{aligned}$$

where the last inequality follows because $x, y \geq a$.

Non-uniform continuity on $(0, b)$ whenever $0 < b \leq \infty$.

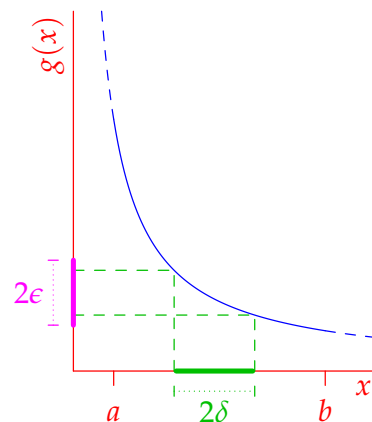
As before, let $\epsilon = 1$ and suppose $\delta > 0$ is given. Let

$$x = \min \left(\delta, 1, \frac{b}{2} \right) \quad \text{and} \quad y = \frac{x}{2}$$

Certainly $x, y \in (0, b)$ and $|x - y| = \frac{x}{2} \leq \frac{\delta}{2} < \delta$. However,

$$|f(x) - f(y)| = \frac{1}{x} \geq 1 = \epsilon$$

Think about how ϵ and δ must relate as one slides the intervals in the picture up/down and left/right. In this case, large values of x, y are not the problem, it's the vertical asymptote at zero that causes trouble.



General Conditions for Uniform Continuity

For the remainder of this section, we develop a few general ideas related to uniform continuity. The first is a little out of order since it depends on differentiation and the mean value theorem.

Theorem 4.28. Suppose f is continuous on an interval U (finite or infinite) and differentiable except perhaps at its endpoints. If f' is bounded, then f is uniformly continuous on U .

Proof. Suppose $|f'(x)| \leq M$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{M}$. Then

$$|x - y| < \delta \implies |f(x) - f(y)| = |f'(\xi)| |x - y| < M\delta = \epsilon$$

for some ξ between x and y . The existence of ξ follows from the mean value theorem.³¹ ■

Examples 4.29. 1. Compare the arguments in the previous exercise. For instance, if $\text{dom } f \subseteq [-T, T]$,

$$f(x) = x^2 \implies f'(x) = 2x \implies |f'(x)| \leq 2T$$

The derivative is bounded, whence f is uniformly continuous on $[-T, T]$.

2. Any polynomial is uniformly continuous on any bounded interval.
3. The function $f(x) = \sin x$ is uniformly continuous on \mathbb{R} since $f'(x) = \cos x$ is bounded (by 1).
4. Consider $f(x) = \frac{1}{x} - \frac{5}{x^2}$ on $(1, \infty)$. We have

$$f'(x) = -\frac{1}{x^2} + \frac{10}{x^3} \implies |f'(x)| \leq 11$$

We conclude that f is uniformly continuous on $(1, \infty)$.

The approach is often useful when you are asked to show *using the definition* that a function is uniformly continuous; provided f' is bounded by M , you may always choose $\delta = \frac{\epsilon}{M}$ to obtain an argument. For instance, with our function:

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{11}$. If $x, y \in (1, \infty)$ and $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} + \frac{5}{y^2} - \frac{5}{x^2} \right| = |x - y| \left| \frac{5(x + y)}{x^2 y^2} - \frac{1}{xy} \right| \\ &= |x - y| \left| \frac{5}{xy^2} + \frac{5}{x^2 y} - \frac{1}{xy} \right| \\ &< 11 |x - y| && (\triangle\text{-inequality, since } x, y > 1) \\ &< 11\delta = \epsilon \end{aligned}$$

As we'll see very shortly, the above result isn't a biconditional: non-differentiable functions and functions with unbounded derivatives can be uniformly continuous.

³¹If $x < y$ then $\exists \xi \in (x, y)$ such that $f'(\xi) = \frac{f(x) - f(y)}{x - y}$.

Our remaining conditions are variations on a theme: uniform continuity on a bounded interval U is roughly the same thing as continuity on its *closure* \bar{U} (Definition 2.46).

Theorem 4.30. *A continuous function on a **closed bounded** domain is uniformly continuous.*

Proof. Assume f is continuous, but not uniformly so, on a closed bounded domain U . Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0, \exists x, y \in U \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon \quad (*)$$

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$ to obtain sequences $(x_n), (y_n) \subseteq U$ satisfying $(*)$.³²

Since $(x_n) \subseteq U$ is bounded, Bolzano–Weierstraß says there exists a convergent subsequence (x_{n_k}) which, since U is closed, converges to some $x_0 \in U$.

Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$, we see that $\lim_{k \rightarrow \infty} y_{n_k} = x_0$. Finally, the continuity of f contradicts $(*)$:

$$\epsilon \leq \lim |f(x_{n_k}) - f(y_{n_k})| = |f(x_0) - f(x_0)| = 0$$

Both **hypotheses** on the domain are crucial: Examples 4.27 provide counter-examples if either is weakened.

Example 4.31. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ since it is already continuous! This **cannot** be concluded from Theorem 4.28, since the derivative $f'(x) = \frac{1}{2}x^{-1/2}$ is unbounded on $(0, 1)$.

We now develop a partial converse, for which we first need a lemma.

Lemma 4.32. *If f is uniformly continuous on U and $(x_n) \subseteq U$ is Cauchy, then $(f(x_n))$ is also Cauchy.*

To apply the result, consider a convergent (Cauchy) sequence in U whose limit is *not* itself in U .

Example (4.27.2, just easier!). Let $f(x) = \frac{1}{x}$ have $U = \text{dom } f = (0, \infty)$ and consider the Cauchy sequence defined by $x_n = \frac{1}{n}$; note crucially that its limit 0 *does not lie in* U . Moreover,

$$\lim f(x_n) = \lim n = \infty$$

Plainly $(f(x_n))$ is not Cauchy, whence f is not uniformly continuous.

Proof. Let $\epsilon > 0$ be given. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ such that } \forall x, y \in U, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Now use this δ in the definition of (x_n) being Cauchy:

$$\exists N \text{ such that } m, n > N \implies |x_n - x_m| < \delta \implies |f(x_n) - f(x_m)| < \epsilon$$

Otherwise said, $(f(x_n))$ is Cauchy.

The Cauchy condition is critical: we cannot apply uniform continuity directly to a convergent sequence $(|x_n - x| < \delta \dots)$ if we do not already know that its limit (x) lies in U !

³²These arguments should feel familiar: compare this line to the proof of Theorem 4.10 and the rest to Theorem 4.14.

We apply the Lemma to show that a continuous function on a *bounded* interval is uniformly continuous if and only if it has a *continuous extension*.

Theorem 4.33. Suppose f is continuous on a bounded interval (a, b) . Define $g : [a, b] \rightarrow \mathbb{R}$ via

$$g(x) := \begin{cases} f(x) & \text{if } x \in (a, b) \\ \lim f(x_n) & \text{whenever } (x_n) \subseteq (a, b) \text{ and } \lim x_n = a \text{ or } b \end{cases}$$

Then f is uniformly continuous if and only if g is well-defined; in such a case g is automatically continuous.

Examples 4.34. 1. $f(x) = x^2 - 3x + 4$ is uniformly continuous on $(-2, 4)$ since it has a continuous extension

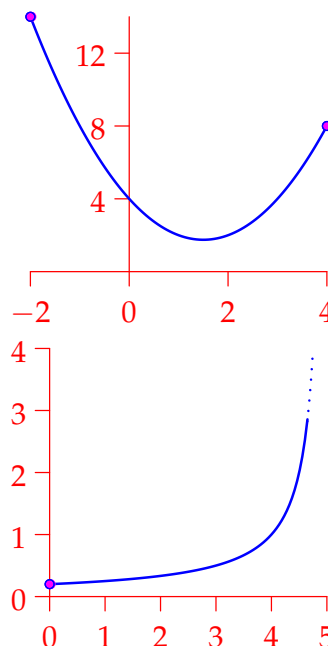
$$g : [-2, 4] \rightarrow \mathbb{R} : x \mapsto x^2 - 3x + 4$$

It should be obvious what is happening from the picture: to create the extension g , we simply **fill in the holes** at the **endpoints** of the graph.

2. The function $f(x) = \frac{1}{5-x}$ is continuous, but not uniformly, on the interval $(0, 5)$. This follows since

$$\lim f\left(5 - \frac{1}{n}\right) = \lim n = \infty$$

means we cannot define $g(5)$ unambiguously. Again the picture is helpful; while we can fill in the hole at the left endpoint ($a = 0$), the vertical asymptote at $b = 5$ means that there is no hole to fill in and thereby extend the function.



Proof. (\Leftarrow) Suppose g is well-defined; we leave the claim that it is continuous as an exercise, but by Theorem 4.30 it is uniformly so. Since $f = g$ on a subset $(a, b) \subseteq \text{dom } g$, the same choice of δ will work for f as it does for g : f is therefore uniformly continuous.

(\Rightarrow) Suppose f is uniformly continuous on (a, b) . Let $(x_n), (y_n) \subseteq (a, b)$ be sequences converging to a . To show that $g(a)$ is unambiguously defined, we must prove that $(f(x_n))$ and $(f(y_n))$ are convergent, and to the same limit.

Define a sequence

$$(u_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \dots)$$

Plainly $\lim u_n = a$ since (x_n) and (y_n) have the same limit. But then (u_n) is Cauchy; by Lemma 4.32, $(f(u_n))$ is also Cauchy and thus convergent. Since $(f(x_n))$ and $(f(y_n))$ are subsequences of a convergent sequence, they also converge to the same (finite!) limit.

The case for $g(b)$ is similar. ■

Examples 4.35. We finish with three related examples of continuous functions $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$; these will appear repeatedly as you continue to study analysis.

1. $f(x) = \sin \frac{1}{x}$ is continuous but *not uniformly so*. To see this, note that $x_n = \frac{1}{(n+\frac{1}{2})\pi}$ defines a Cauchy sequence ($\lim x_n = 0$), and yet

$$f(x_n) = \sin \left(n + \frac{1}{2} \right) \pi = (-1)^n$$

is not Cauchy (it diverges by oscillation). Consequently, we cannot extend f to a continuous function on any interval containing $x = 0$.

2. $f(x) = x \sin \frac{1}{x}$ is *uniformly* continuous. One way to see this is to extend the function to the origin by defining

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

By the squeeze theorem, $\lim x_n = 0 \implies \lim f(x_n) = 0$, so g is well-defined and continuous on \mathbb{R} . By Theorem 4.33, f is uniformly continuous on any bounded interval. Moreover, the derivative

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

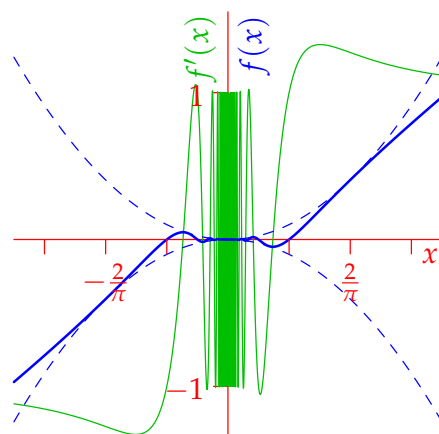
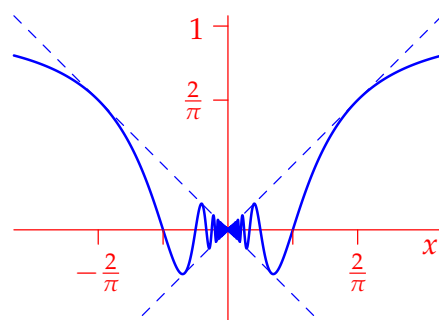
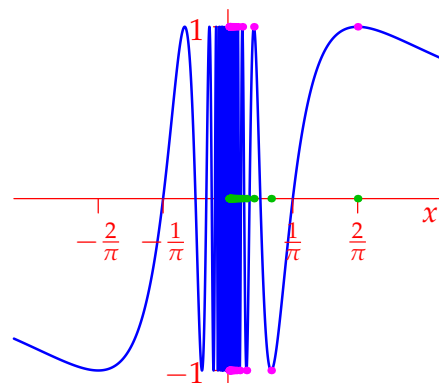
is bounded whenever x is large; together with Exercise 6 we conclude that $f(x)$ is uniformly continuous on $\mathbb{R} \setminus \{0\}$. We cannot use the derivative argument on the whole domain $\mathbb{R} \setminus \{0\}$, since $f'(x)$ is unbounded when x is small ($\lim f'(\frac{1}{2\pi n}) = \lim(-2\pi n) = -\infty$).

3. $f(x) = x^2 \sin \frac{1}{x}$ is also *uniformly* continuous: again extend by $g(0) = 0$. This time however, we could argue that the derivative is bounded

$$|f'(x)| = \left| 2x \sin \frac{1}{x} - \cos \frac{1}{x} \right| \leq 3$$

since $|\sin y| \leq |y|$ and $|\cos y| \leq 1$ for all y .

Something stranger is going on. As you may verify (see Exercise 3), the extended function g is *everywhere differentiable* with $g'(0) = 0$, and yet the derivative $g'(x)$ itself is *discontinuous* at $x = 0$!



Exercises 4.19. Key concepts: Order of quantifiers! Bounded derivative \Rightarrow Unif cont \Leftrightarrow Cont extension

- Decide whether each f is uniformly continuous. Explain your answers.

(a) $f(x) = x^3$ on $[-2, 4]$	(b) $f(x) = x^3$ on $(-2, 4)$
(c) $f(x) = x^{-3}$ on $(0, 4]$	(d) $f(x) = x^{-3}$ on $(1, 4]$
(e) $f(x) = e^x$ on $(-\infty, 100)$	(f) $f(x) = e^x$ on \mathbb{R}
- Prove that each f is uniformly continuous by verifying the ϵ - δ property.

(a) $f(x) = 3x + 11$ on \mathbb{R}	(b) $f(x) = x^2$ on $[0, 3]$
(c) $f(x) = \frac{1}{x^2}$ on $[\frac{1}{2}, \infty)$	(d) $f(x) = \frac{x+2}{x+1}$ on $[0, 1]$
- Verify the claim in Example 4.35.3 that the function $g(x)$ is differentiable at zero³³ but that the derivative $g'(x)$ is discontinuous there.
- (a) If f is uniformly continuous on a bounded set U , prove that f is bounded on U .
 (Hint: for contradiction, assume $\exists(x_n) \subseteq U$ for which $|f(x_n)| \rightarrow \infty \dots$)
 (b) Use (a) to give another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.
 (c) Give an example to show that a uniformly continuous function on an *unbounded* set U could be unbounded.
- Suppose g is defined on U and $a \in U$. Give *very brief* (one line!) arguments for the following.
 - Prove that g is continuous at a provided

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |g(x) - g(a)| < \epsilon$$
 - Prove that g is continuous at a provided

$$\forall (x_n) \subseteq U \setminus \{a\}, \lim x_n = a \implies \lim g(x_n) = g(a)$$
 - Verify that the function g defined in Theorem 4.33 is indeed continuous whenever it is well-defined.
- (a) Suppose f is uniformly continuous on intervals U_1, U_2 for which $U_1 \cap U_2$ is non-empty. Prove that f is uniformly continuous on $U_1 \cup U_2$.
 (Hint: if x, y do not lie in the same interval U_i , choose some $a \in U_1 \cap U_2$ between x and y)
 (b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.
 (c) More generally, prove that any root function $f(x) = x^{1/n}$ ($n \in \mathbb{N}$) is uniformly continuous on its domain (\mathbb{R} if n is odd and $[0, \infty)$ if n is even).
 (d) (Hard) Given $f(x) = x^{1/n}$, show that $\delta = \epsilon^n$ demonstrates uniform continuity when n is even and $\delta = (\frac{\epsilon}{2})^n$ when n is odd.
 (Hint: use the binomial theorem to prove that $0 \leq y < x + \delta \implies y^{1/n} < x^{1/n} + \delta^{1/n}$)

³³Use the definition $g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0}$. Limits of functions are covered formally in the next section (course!), but you should be familiar with the idea from elementary calculus.