Decimal Expansions of Real Numbers

We are typically introduced to decimals in elementary mathematics; for many in grade-school they become a working *definition* of the real numbers. Strictly speaking, the decimals must be given a formal meaning in terms of the real numbers.

Definition. A *decimal* $d_0.d_1d_2d_3\cdots$ is an infinite series of the form

$$\sum_{n=0}^{\infty} d_n\, 10^{-n} \text{ where } d_0 \in \mathbb{Z} \text{ and } \forall n \in \mathbb{N},\ d_n \in \{0,1,2,\ldots,9\}$$

Let x be a non-negative real number. Define sequences $(d_n)_{n=0}^{\infty}$ and $(R_n)_{n=0}^{\infty}$ as follows:[1]

$$d_0 = \lfloor x \rfloor \qquad R_0 = x - d_0$$
$$\forall n \in \mathbb{N}_0 : \quad d_{n+1} = \lfloor 10R_n \rfloor, \qquad R_{n+1} = 10R_n - d_{n+1}$$

The *decimal expansion* of x is the decimal $D(x) := d_0.d_1d_2d_3\cdots$.
If $x < 0$, first find the decimal expansion of $|x| = -x$, then change the sign of d_0.

Examples

1. Let $x = \frac{27}{20}$. We compute:

n	0	1	2	3	4	$\cdots$
d_n	$\lfloor\frac{27}{20}\rfloor = 1$	$\lfloor\frac{70}{20}\rfloor = 3$	$\lfloor\frac{10}{2}\rfloor = 5$	$\lfloor 0\rfloor = 0$	0	$\cdots$
R_n	$\frac{7}{20}$	$\frac{1}{2}$	0	0	0	$\cdots$

   Both sequences continue with zeros forever: we obtain the terminating decimal $D(\frac{27}{20}) = 1.35$.

2. Let $x = \frac{1}{3}$. We have

n	0	1	2	3	$\cdots$
d_n	$\lfloor\frac{1}{3}\rfloor = 0$	$\lfloor\frac{10}{3}\rfloor = 3$	$\lfloor\frac{10}{3}\rfloor = 3$	$\lfloor\frac{10}{3}\rfloor = 3$	$\cdots$
R_n	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\cdots$

   By induction, all $R_n = \frac{1}{3}$ and we recover the periodic decimal $D(\frac{1}{3}) = 0.33333\cdots$.

3. If $x = \frac{1}{7}$, we obtain

n	0	1	2	3	4	5	6	7	$\cdots$
d_n	0	$\lfloor\frac{10}{7}\rfloor = 1$	$\lfloor\frac{30}{7}\rfloor = 4$	$\lfloor\frac{20}{7}\rfloor = 2$	$\lfloor\frac{60}{7}\rfloor = 8$	$\lfloor\frac{40}{7}\rfloor = 5$	$\lfloor\frac{50}{7}\rfloor = 7$	1	$\cdots$
R_n	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\cdots$

   Since $R_6 = R_0$, both sequences will now repeat: $R_{n+6} = R_n$ and $d_{n+6} = d_n$. We recover the *period-six* decimal $D(\frac{1}{7}) = 0.142857142857\cdots$.

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[1] Recall the *floor* function: $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

In the main result, we check that the decimal expansion is well-defined and that it behaves as expected. We also give two well-known properties of decimal representations.

Theorem. *Let $x \in \mathbb{R}$ have decimal expansion $D(x) = \sum_{n=0}^{\infty} d_n\, 10^{-n}$.*

(a) Every decimal (infinite series) converges.

(b) Each $d_n \in \{0, 1, 2, \ldots, 9\}$ whenever $n \geq 1$.

(c) $x = D(x)$.

(d) The sequence (d_n) is eventually periodic *if and only if $x \in \mathbb{Q}$.*

(e) x equals a unique decimal, except when $D(x)$ is terminating, in which case there are exactly two decimal representations:

$$x = D(x) = \sum_{n=0}^{m} d_n\, 10^{-n} = \sum_{n=0}^{m-1} d_n\, 10^{-n} + (d_m - 1)10^{-m} + 9 \sum_{n=m+1}^{\infty} 10^{-n}$$

Otherwise said, we subtract 1 from the final term of (d_n) and insert an infinite string of 9's.

Examples

1. Part (d) explains why so many people consider memorizing the digits (decimal expansion) of π to be interesting: since π is irrational, the pattern never repeats.

2. We explicitly evaluate a *period-three* decimal:

$$\begin{aligned} 3.1279279279279\cdots &= \frac{31}{10} + \frac{279}{10000} \sum_{n=0}^{\infty} 1000^{-n} = \frac{31}{10} + \frac{279}{10000} \cdot \frac{1}{1 - \frac{1}{1000}} \\ &= \frac{31}{10} + \frac{279}{9990} = \frac{1736}{555} \end{aligned}$$

3. Here are two examples of part (e):

$$1 = 0.99999\cdots \qquad 27.164 = 27.1639999\cdots$$

Questions

1. Compute the decimal expansions of $\frac{32}{13}$.

2. Prove all parts of the Theorem. Here are some hints:

   (a) Use a series test…

   (b) Prove by induction: it should be obvious that $0 \leq R_0 < 1$, etc.

   (c) Let $E_n = x - \sum_{k=0}^{n} d_k\, 10^{-k}$. Prove by induction that $R_n = 10^n E_n$ and conclude that $E_n \to 0$.

(d) A decimal is eventually periodic with period r if

$$d_0.d_1 \cdots d_m d_{m+1} \cdots d_{m+r} d_{m+1} \cdots d_{m+r} \cdots = \sum_{k=0}^{m} d_k \, 10^{-k} + \left(\sum_{j=1}^{r} d_{m+j} \, 10^{-m-j} \right) \sum_{l=0}^{\infty} 10^{-rl}$$

Convince yourself this is a rational number. For the converse, suppose that $x = \frac{p}{q}$ is a rational number in lowest terms where $q \in \mathbb{N}$, and observe that there are only *finitely many* possible values for the remainders:

$$R_n = \frac{a}{q} \text{ where } a \in \{0, 1, \ldots, q-1\}$$

(e) Suppose that $d_0.d_1d_2 \cdots = c_0.c_1c_2 \cdots$ but where $(d_n) \neq (c_n)$. WLOG there is a minimal m such that $c_m < d_m \ldots$

3. (a) Can you find a simple way to describe all the real numbers x for which $D(x)$ is terminating? Prove your assertion.
   (*Hint: what form can the denominator of R_n take if $x = \frac{p}{q}$?*)

   (b) Given a rational number $x = \frac{p}{q}$ in lowest terms and with $q \in \mathbb{N}$, what is the *largest* possible period of $D(x)$? Explain.

4. Similar analyses can be done for other representations of real numbers. For instance, by replacing 10 with 3 in the definition, one could consider the *ternary* expansion of a real number

$$T(x) = [t_0.t_1t_2 \cdots]_3 = \sum_{n=0}^{\infty} t_n \, 3^{-n} \text{ where } t_n \in \{0, 1, 2\} \text{ whenever } n \geq 1$$

For example, $\frac{1}{3} = [0.1]_3$ and $[0.12]_3 = \frac{1}{3} + \frac{2}{3^2} = \frac{5}{9}$.

   (a) Compute $[0.02020202 \cdots]_3$ and find the ternary representation of $\frac{1}{2}$ and $\frac{1}{5}$.

   (b) Read over the Theorem. How can we modify its statements for ternary representations?

   (c) Describe all real numbers x whose ternary representation is terminating. More generally, describe all real numbers x whose n-ary (base n) representation is terminating.

The major take-away from this discussion is that there is *nothing special* about representing real numbers using decimals. Computers typically use base 2, 8 or 16; the ancient Babylonians used base 60. We only like decimals because they're familiar, and because we're blessed with 10 fingers…

Solutions

1. If $x = \frac{32}{13}$, we obtain

n	0	1	2	3	4	5	6	7	$\cdots$
d_n	$\lfloor\frac{32}{13}\rfloor = 2$	$\lfloor\frac{60}{13}\rfloor = 4$	$\lfloor\frac{80}{13}\rfloor = 6$	$\lfloor\frac{20}{13}\rfloor = 1$	$\lfloor\frac{70}{13}\rfloor = 5$	$\lfloor\frac{50}{13}\rfloor = 3$	$\lfloor\frac{110}{13}\rfloor = 8$	$\lfloor\frac{60}{13}\rfloor = 4$	$\cdots$
R_n	$\frac{6}{13}$	$\frac{8}{13}$	$\frac{2}{13}$	$\frac{7}{13}$	$\frac{5}{13}$	$\frac{11}{13}$	$\frac{6}{13}$	$\frac{8}{13}$	$\cdots$

The process repeats and we obtain $D(\frac{32}{13}) = 2.461538461538\cdots$

2. (a) This is trivial by the comparison test, since $d_n\, 10^{-n} \leq 9 \cdot 10^{-n}$.

(b) Formally this requires induction. Informally, it is clear that $0 \leq R_0 < 1$, whence $d_1 = \lfloor 10R_0 \rfloor$ is an integer $0 \leq d_0 \leq 9$. Now iterate.

(c) Following the hint, let $E_n = x - \sum\limits_{k=0}^{n} d_k\, 10^{-k}$. We prove by induction that $R_n = 10^n E_n$.

The base case is obvious, since $R_0 = x - d_0 = 10^0 E_0$.
For the induction step, fix $n \in \mathbb{N}_0$ and assume $R_n = 10^n E_n$. Then

$$\begin{aligned} R_{n+1} &= 10R_n - d_{n+1} = 10^{n+1} E_n - d_{n+1} \\ &= 10^{n+1} x - \sum_{k=0}^{n} d_k\, 10^{n-k} - d_{n+1} \\ &= 10^{n+1} \left(x - \sum_{k=0}^{n+1} d_k\, 10^{-k} \right) = 10^{n+1} E_{n+1} \end{aligned}$$

But now $E_n = \frac{1}{10^n} R_n \to 0$, since $0 \leq R_n < 1$ for all n.

(d) Following the hint, since $\sum\limits_{l=0}^{\infty} 10^{-rl} = \frac{1}{1-10^{-r}} \in \mathbb{Q}$, we see that

$$d_0.d_1 \cdots d_m d_{m+1} \cdots d_{m+r} d_{m+1} \cdots d_{m+r} \cdots = \sum_{k=0}^{m} d_k\, 10^{-k} + \left(\sum_{j=1}^{r} d_{m+j}\, 10^{-m-j} \right) \sum_{l=0}^{\infty} 10^{-rl}$$

is rational.
Conversely, suppose that $x = \frac{p}{q}$ is a rational number in lowest terms where $q \in \mathbb{N}$. At each stage of the algorithm, one is subtracting an integer after multiplying by 10. The result is that the denominator of R_n is always a divisor of q. It follows that

$$R_n = \frac{a}{q} \text{ where } a \in \{0, 1, \ldots, q-1\}$$

For $n \in \{0, 1, \ldots, q\}$ there are only q possible remainders: at least one remainder must appear twice; $R_i = R_j$ where $0 \leq i < j \leq q$. Suppose $r = j - i$. It follows that

$$\forall k \geq i,\ d_{k+r} = d_k$$

whence the decimal is eventually periodic.

(e) That the two series are equal is easy to check via the geometric series formula:

$$9\sum_{n=m+1}^{\infty} 10^{-n} = \frac{9\cdot 10^{-m-1}}{1-1/10} = 10^{-m}$$

Now suppose that two different decimals are equal: that is

$$\sum_{n=0}^{\infty} d_n\, 10^{-n} = \sum_{n=0}^{\infty} c_n\, 10^{-n}$$

Suppose $m \in \mathbb{N}_0$ is minimal such that $c_m \neq d_m$ and assume WLOG that $c_m < d_m$. Then

$$\begin{aligned}
(d_m - c_m)10^{-m} + \sum_{n=m+1}^{\infty} d_n\, 10^{-n} &= \sum_{n=m+1}^{\infty} c_n\, 10^{-n} \\
\implies (d_m - c_m) + \sum_{n=1}^{\infty} d_{n+m}\, 10^{-n} &= \sum_{n=1}^{\infty} c_{n+m}\, 10^{-n}
\end{aligned}$$

Consider the left and right sides of this equation:

Left Side Since $d_m > c_m$, this is *greater than or equal to* 1 with equality if and only if *all* $d_{n+m} = 0$.

Right Side Since $9\sum_{n=1}^{\infty} 10^{-n} = 1$, the right side is *less than or equal to* 1 with equality if and only if *all* $c_{n+m} = 9$.

We conclude that $d_m = c_m + 1$, and that

$$\forall n > m, \quad d_n = 0,\ c_n = 9$$

3. (a) $D(x)$ terminates if and only if $x = \frac{p}{2^a 5^b}$ is rational in lowest terms. Thus, for example, $x = \frac{193}{250} = \frac{193}{2\cdot 5^2}$ has a terminating decimal, namely 0.772. Here is a proof.

By the Theorem, we know that all possible candidates for a terminating decimal have to be rational. Thus we assume $x = \frac{p}{q}$ is rational in lowest terms. Observe that

$$R_0 = \frac{p}{q} - d_0 = \frac{p - d_0 q}{q}$$

is a fraction with denominator q. Similarly,

$$R_1 = 10R_0 - \lfloor 10R_0 \rfloor$$

is $10R_0$ *minus an integer*: it is therefore a fraction whose denominator is either q, $\frac{1}{2}q$, $\frac{1}{5}q$ or $\frac{1}{10}q$. Repeating this process, we see that R_n is a fraction with denominator

$$q_n = \frac{1}{2^a 5^b} q \quad \text{where} \quad a, b \in \mathbb{N}_0$$

$D(x)$ terminates if and only if some $q_n = 1$ (then $R_n = \frac{0}{1} = 0$). This clearly happens if and only if x has the form described above.

(b) Think back to the proof. If $x = \frac{p}{q}$ is in lowest terms, then in the first $q+1$ remainders, one remainder has to be repeated, so the seemingly largest period is q (if $R_0 = R_q$). However, if any remainder is ever zero, the period is 1. Thus the longest possible period will be $q-1$, which will happen if the remainders $R_0, R_1, \ldots, R_{q-2}$ are distinct and non-zero, and we have $R_{q-1} = R_0$. The example with $\frac{1}{7}$ (period 6) shows this. Similarly,

$$\frac{1}{23} = 0.\textcolor{red}{0434782608695652173913}0434782609\cdots$$

has period 22.

4. (a) $[0.02020202\cdots]_3 = \dfrac{2}{3^2}[1.010101\cdots]_3 = \dfrac{2}{9}\displaystyle\sum_{n=0}^{\infty} 3^{-2n} = \dfrac{2}{9(1-\frac{1}{9})} = \dfrac{1}{4}.$

Following the algorithm,

n	0	1	2	3	$\cdots$
d_n	$\lfloor \frac{1}{2} \rfloor = 0$	$\lfloor \frac{3}{2} \rfloor = 1$	$\lfloor \frac{3}{2} \rfloor = 1$	$\lfloor \frac{3}{2} \rfloor = 1$	$\cdots$
R_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\cdots$

$$\implies \frac{1}{2} = [0.11111\cdots]_3$$

We could also have done this by multiplying the expression for $\frac{1}{4}$ by 2...

Now for $\frac{1}{5}$:

n	0	1	2	3	4	$\cdots$
d_n	$\lfloor \frac{1}{5} \rfloor = 0$	$\lfloor \frac{3}{5} \rfloor = 0$	$\lfloor \frac{9}{5} \rfloor = 1$	$\lfloor \frac{12}{5} \rfloor = 2$	$\lfloor \frac{6}{5} \rfloor = 1$	$\cdots$
R_n	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\cdots$

from which we see that the ternary representation repeats:

$$\frac{1}{5} = [0.012101210121\cdots]_3$$

(b) The theorem goes through almost unchanged: every ternary expression converges, each $t_n \in \{0,1,2\}$ whenever $n \geq 1$, $x = T(x)$, rational number have eventual periodicity of (t_n). Finally, terminating ternary expressions have a secondary representation: e.g.

$$[1.2012]_3 = [1.20112222222\cdots]_3$$

where the final non-zero term is reduced by 1 and an infinite string of 2's added.

(c) Suppose $n = p_1^{\mu_1} \cdots p_k^{\mu_k}$ is the unique prime factorization of n. The real numbers x whose n-ary represenation terminates are precisely those rational numbers whose (lowest-term) denominators are divisible by no other primes than $p_1, \ldots, p_n$. For example, base $60 = 2^2 \cdot 3 \cdot 5$, the representation of $\frac{1001}{450}$ will terminate, but $\frac{1}{7}$ will not. In case you are curious...

$$\frac{1001}{450} = [2;13,28]_{60} = 2 + \frac{13}{60} + \frac{28}{60^2}, \qquad \frac{1}{7} = [0;8,34,17,8,34,17,\ldots]_{60}$$