Solutions to Decimals Exercises

1. If $x = \frac{32}{13}$, we obtain

n	0	1	2	3	4	5	6	$\cdots$
d_n	$\lfloor\frac{32}{13}\rfloor = 2$	$\lfloor\frac{60}{13}\rfloor = 4$	$\lfloor\frac{80}{13}\rfloor = 6$	$\lfloor\frac{20}{13}\rfloor = 1$	$\lfloor\frac{70}{13}\rfloor = 5$	$\lfloor\frac{50}{13}\rfloor = 3$	$\lfloor\frac{110}{13}\rfloor = 8$	$\cdots$
R_n	$\frac{6}{13}$	$\frac{8}{13}$	$\frac{2}{13}$	$\frac{7}{13}$	$\frac{5}{13}$	$\frac{11}{13}$	$\frac{6}{13}$	$\cdots$

Since $R_6 = R_0$, the process repeats and we obtain $D(\frac{32}{13}) = 2.461538461538\cdots$

2. (a) Formally this requires induction. Informally, it should be clear that $0 \le R_0 < 1$, whence $d_1 = \lfloor 10R_0 \rfloor$ is an integer $0 \le d_0 \le 9$.

(b) $D(x)$ converges trivially by the comparison test: $d_n\,10^{-n} \le 9 \cdot 10^{-n}$. Alternatively, the partial sums form a monotone-up sequence bounded above by x.

Following the hint, let $E_n = x - \sum_{k=0}^{n} d_k\,10^{-k}$. We prove by induction that $R_n = 10^n E_n$.

The base case is obvious since $R_0 = x - d_0 = 10^0 E_0$.

For the induction step, fix $n \in \mathbb{N}_0$ and assume $R_n = 10^n E_n$. Then

$$\begin{aligned} R_{n+1} &= 10R_n - d_{n+1} = 10^{n+1}E_n - d_{n+1} \\ &= 10^{n+1}x - \sum_{k=0}^{n} d_k\,10^{n-k} - d_{n+1} \\ &= 10^{n+1}\left(x - \sum_{k=0}^{n+1} d_k\,10^{-k}\right) = 10^{n+1}E_{n+1} \end{aligned}$$

But now $E_n = \frac{1}{10^n}R_n \to 0$, since $0 \le R_n < 1$ for all n.

(c) Following the hint, since $\sum\limits_{l=0}^{\infty} 10^{-rl} = \frac{1}{1-10^{-r}} = \frac{10^r}{10^r-1} \in \mathbb{Q}$, we see that

$$d_0.d_1 \cdots d_m d_{m+1} \cdots d_{m+r} d_{m+1} \cdots d_{m+r} \cdots = \sum_{k=0}^{m} d_k\,10^{-k} + \left(\sum_{j=1}^{r} d_{m+j}\,10^{-m-j}\right)\sum_{l=0}^{\infty} 10^{-rl}$$

is rational.

Conversely, suppose $x = \frac{p}{q}$ is rational in lowest terms. At each stage of the algorithm, one subtracts an integer after multiplying by 10. The denominator of R_n is therefore always a divisor of q, whence

$$R_n = \frac{a}{q} \text{ where } a \in \{0, 1, \ldots, q-1\}$$

For $n \in \{0, 1, \ldots, q\}$ there are only q possible remainders R_n: at least one of these must appear twice; $R_i = R_j$ for some $0 \le i < j \le q$. Writing $r = j - i$, it follows that

$$\forall k \ge i,\ d_{k+r} = d_k$$

whence the decimal is eventually periodic.

(d) That the two series are equal is easy to check via the geometric series formula:

$$9\sum_{n=m+1}^{\infty} 10^{-n} = \frac{9\cdot 10^{-m-1}}{1-1/10} = 10^{-m}$$

Now suppose that two different decimals are equal: that is

$$\sum_{n=0}^{\infty} d_n\, 10^{-n} = \sum_{n=0}^{\infty} c_n\, 10^{-n}$$

Suppose $m \in \mathbb{N}_0$ is minimal such that $c_m \neq d_m$ and assume WLOG that $c_m < d_m$. Then

$$\begin{aligned} (d_m - c_m)10^{-m} + \sum_{n=m+1}^{\infty} d_n\, 10^{-n} &= \sum_{n=m+1}^{\infty} c_n\, 10^{-n} \\ \implies (d_m - c_m) + \sum_{n=1}^{\infty} d_{n+m}\, 10^{-n} &= \sum_{n=1}^{\infty} c_{n+m}\, 10^{-n} \end{aligned}$$

Consider the left and right sides of this equation:

Left Side Since $d_m > c_m$, this is *greater than or equal to* 1 with equality if and only if $d_m = c_m + 1$ and *all* $d_{n+m} = 0$.

Right Side Since $9\sum_{n=1}^{\infty} 10^{-n} = 1$, the right side is *less than or equal to* 1 with equality if and only if *all* $c_{n+m} = 9$.

We conclude that $d_m = c_m + 1$, and that

$$n > m \implies d_n = 0,\ c_n = 9$$

3. (a) $D(x)$ terminates $\implies x = \frac{p}{2^a 5^b}$ is rational in lowest terms. For example, $x = \frac{193}{250} = \frac{193}{2\cdot 5^2}$ has a terminating decimal, namely 0.772. Here is a general proof.

By the Theorem, we know that all possible candidates for a terminating decimal must be rational. Thus assume $x = \frac{p}{q}$ is rational in lowest terms. Observe that

$$R_0 = \frac{p}{q} - d_0 = \frac{p - d_0 q}{q}$$

is a fraction with denominator q. Similarly,

$$R_1 = 10R_0 - \lfloor 10R_0 \rfloor$$

is $10R_0$ *minus an integer*; it is therefore a fraction whose denominator is either q, $\frac{1}{2}q$, $\frac{1}{5}q$ or $\frac{1}{10}q$. Iterating this process, we see that R_n is a fraction with denominator

$$q_n = \frac{1}{2^a 5^b} q \quad \text{where} \quad a, b \in \mathbb{N}_0$$

$D(x)$ terminates if and only if some $q_n = 1$ (then $R_n = \frac{0}{1} = 0$), which happens if and only if x has the form described above.

(b) Think back to the proof. If $x = \frac{p}{q}$ is in lowest terms, then in the first $q+1$ remainders, one remainder must appear at least twice. The seemingly largest period is therefore q (if $R_0 = R_q$). However, if any remainder were ever zero, then the decimal terminates (with 'period' 1). The longest possible period will therefore be $q-1$, which happens if the remainders $R_0, R_1, \ldots, R_{q-2}$ are distinct and non-zero, and $R_{q-1} = R_0$. The example with $\frac{1}{7}$ (period 6) shows this. Similarly,

$$\frac{1}{23} = 0.\textcolor{red}{0434782608695652173913}0434782609\cdots$$

has period 22.

4. (a) $[0.02020202\cdots]_3 = \frac{2}{3^2}[1.010101\cdots]_3 = \frac{2}{9}\sum_{n=0}^{\infty} 3^{-2n} = \frac{2}{9(1-\frac{1}{9})} = \frac{1}{4}$.

Following the algorithm,

n	0	1	2	3	$\cdots$
d_n	$\lfloor \frac{1}{2} \rfloor = 0$	$\lfloor \frac{3}{2} \rfloor = 1$	$\lfloor \frac{3}{2} \rfloor = 1$	$\lfloor \frac{3}{2} \rfloor = 1$	$\cdots$
R_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\cdots$

$$\Longrightarrow \frac{1}{2} = [0.11111\cdots]_3$$

We could also have done this by multiplying the expression for $\frac{1}{4}$ by 2.

Now for $\frac{1}{5}$:

n	0	1	2	3	4	$\cdots$
d_n	$\lfloor \frac{1}{5} \rfloor = 0$	$\lfloor \frac{3}{5} \rfloor = 0$	$\lfloor \frac{9}{5} \rfloor = 1$	$\lfloor \frac{12}{5} \rfloor = 2$	$\lfloor \frac{6}{5} \rfloor = 1$	$\cdots$
R_n	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\cdots$

from which the ternary representation repeats:

$$\frac{1}{5} = [0.012101210121\cdots]_3$$

(b) The theorem goes through almost unchanged: each $t_n \in \{0,1,2\}$ whenever $n \geq 1$, every ternary expansion converges to $T(x) = x$, rational numbers have eventual periodicity, and terminating ternary expansions have another representation: e.g.

$$[1.2012]_3 = [1.20112222222\cdots]_3$$

where the final non-zero term is reduced by 1 and an infinite string of 2's added.

(c) Suppose $n = p_1^{\mu_1} \cdots p_k^{\mu_k}$ is the unique prime factorization of n. The (positive) real numbers x whose n-ary expansion terminates are precisely those rational numbers whose (lowest-term) denominators are divisible by no other primes than $p_1, \ldots, p_n$.
For instance, in base-$60 = 2^2 \cdot 3 \cdot 5$, the expansion of $\frac{1001}{450}$ will terminate, but that of $\frac{1}{7}$ will not (it is 3-periodic though!). In case you are curious:

$$\frac{1001}{450} = [2;13,28]_{60} = 2 + \frac{13}{60} + \frac{28}{60^2}, \qquad \frac{1}{7} = [0;8,34,17,8,34,17,\ldots]_{60}$$