2 Sequences and Series of Functions

If (f_n) is a sequence of functions, what should we mean by $\lim f_n$? This question is of huge relevance to the history of calculus: Issac Newton's work in the late 1600's made great use of *power series*, which are naturally constructed as limits of sequences of polynomials.

For instance, for each $n \in \mathbb{N}_0$, we might consider the polynomial function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n$$

This is easily differentiated and integrated using the power law. What, however, are we to make of the *series*

$$f(x) := \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \ ?$$

Does this make sense as a function? What is its domain? Does it equal the limit of the sequence (f_n) in any meaningful way? Is it continuous, differentiable, integrable? If so, can we compute its derivative or integral term-by-term: for instance, is it legitimate to write

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots \ ?$$

To many in Newton's time, such technical questions were less important than the application of calculus to the natural sciences. For the 18^{th} and 19^{th} century mathematicians who followed, however, the widespread application of calculus only increased the imperative to rigorously address these issues.

2.23 Power Series

First we review some of the important definitions, examples and results concerning infinite series.

Definition 2.1. Let $(b_n)_{n=m}^{\infty}$ be a sequence of real numbers. The *(infinite) series* $\sum b_n$ is the limit of the sequence (s_n) of *partial sums*,

$$s_n = \sum_{k=m}^{n} b_n = b_m + b_{m+1} + \cdots + b_n, \qquad \sum_{n=m}^{\infty} b_n = \lim_{n\to\infty} s_n$$

The series $\sum b_n$ is said to *converge*, *diverge to infinity* or *diverge by oscillation*[6] as does (s_n).

$\sum b_n$ is *absolutely convergent* if $\sum |b_n|$ converges. A convergent series that is not absolutely convergent is *conditionally convergent*.

[6] Recall that every sequence (s_n) has subsequences tending to each of

$$\limsup s_n = \lim_{N\to\infty} \sup\{x_n : n > N\} \quad \text{and} \quad \liminf s_n = \lim_{N\to\infty} \inf\{x_n : n > N\}$$

If (s_n) converges, or diverges to $\pm\infty$, then $\lim s_n = \limsup s_n = \liminf s_n$. The remaining case, divergence by oscillation, is when $\liminf s_n \neq \limsup s_n$: there exist (at least) two subsequences tending to different limits.

Examples 2.2. These examples form the standard reference dictionary for analysis of more complicated series. Make sure they are familiar![7]

1. (Geometric series) If r is constant, then $s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$. It follows that

$$\sum_{n=0}^{\infty} r^n \quad \begin{cases} \text{converges (absolutely) to } \frac{1}{1-r} & \text{if } -1 < r < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \\ \text{diverges by oscillation} & \text{if } r \leq -1 \end{cases}$$

2. (Telescoping series) If $b_n = \frac{1}{n(n+1)}$, then $s_n = \sum_{k=1}^{n} b_n = 1 - \frac{1}{n+1} \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is (absolutely) convergent. In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, though checking this explicitly is tricky.
4. (Harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent to ∞.
5. (Alternating harmonic series) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 2.3 (Root Test). *Given a series $\sum b_n$, let $\beta = \limsup |b_n|^{1/n}$,*

- *If $\beta < 1$ then the series converges absolutely.*
- *If $\beta > 1$ then the series diverges.*

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[7] We give sketch proofs or refer to a standard 'test.' Review these if you are unfamiliar.

1. $s_n - rs_n = 1 + r + \cdots + r^n - (r + \cdots + r^n + r^{n+1}) = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$.
2. By partial fractions, $b_n = \frac{1}{n} - \frac{1}{n+1} \implies s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.
3. Use the comparison or integral tests. Alternatively: For each $n \geq 2$, we have $\frac{1}{n^2} < \frac{1}{n(n-1)}$. By part 2,

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=1}^{n} \frac{1}{k(k-1)} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n-1)} = 2$$

   Since (s_n) is monotone-up and bounded above by 2, we conclude that $\sum \frac{1}{n^2}$ is convergent.
4. Use the integral test. Alternatively, observe that

$$s_{2^{n+1}} - s_{2^n} = \sum_{k=2^n-1}^{2^{n+1}} \frac{1}{k} \geq \frac{2^n}{2^{n+1}} = \frac{1}{2} \implies s_{2^n} \geq \frac{n}{2} \xrightarrow[n\to\infty]{} \infty$$

   Since $s_n = \sum_{k=1}^{n} \frac{1}{k}$ is monotone-up, we conclude that $s_n \to \infty$.
5. Use the alternating series test, or explicitly check that both the even and odd partial sums (s_{2n}) and (s_{2n+1}) are convergent (monotone and bounded) to the same limit (essentially the proof of the alternating series test).

Root Test: $\beta < 1 \implies \exists \epsilon > 0$ such that $|b_n|^{1/n} \leq 1 - \epsilon$ (for large n) $\implies \sum |b_n|$ converges by comparison with $\sum (1-\epsilon)^n$.
$\beta > 1 \implies$ some subsequence of $(|b_n|^{1/n})$ converges to $\beta > 1 \implies b_n \not\to 0 \implies \sum b_n$ diverges (n^{th}-term test).

The root test is inconclusive if $\beta = 1$. Some simple inequalities[8] yield a test that is often easier to apply.

Corollary 2.4 (Ratio Test). *Given a series $\sum b_n$:*

- *If* $\limsup \left|\frac{b_{n+1}}{b_n}\right| < 1$ *then $\sum b_n$ converges absolutely.*
- *If* $\liminf \left|\frac{b_{n+1}}{b_n}\right| > 1$ *then $\sum b_n$ diverges.*

We are now ready to properly define and analyze our main objects of interest.

Definition 2.5. A *power series centered at $c \in \mathbb{R}$* with *coefficients $a_n \in \mathbb{R}$* is a formal expression

$$\sum_{n=m}^{\infty} a_n (x-c)^n$$

where $x \in \mathbb{R}$ is considered a variable. A power series is a *function* whose implied domain is the set of x for which the resulting infinite series converges.

It is common to refer simply to a *series*, and modify by infinite/power only when clarity requires. Almost always $m = 0$ or 1, and it is common for examples to be centered at $c = 0$.

Example 2.6. By the geometric series formula,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-4)^n = \frac{1}{1 - \frac{-(x-4)}{2}} = \frac{2}{x-2} \quad \text{whenever} \quad \left|-\frac{x-4}{2}\right| < 1 \iff 2 < x < 6$$

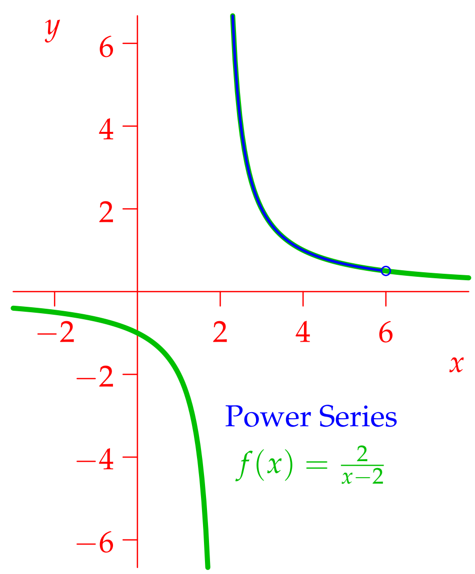
The series is valid (converges) only on the subinterval $(2,6)$ of the implied domain of the function $x \mapsto \frac{2}{x-2}$.
The behavior as $x \to 2^+$ is unsurprising, since evaluating the power series results in the divergent infinite series

$$\sum 1 = +\infty$$

By contrast, as $x \to 6^-$ we see that limits and infinite series do not interact as we might expect,

$$\lim_{x\to 6^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-4)^n = \lim_{x\to 6^-} \frac{2}{x-2} = \frac{1}{2}$$
$$\sum_{n=0}^{\infty} \lim_{x\to 6^-} \frac{(-1)^n}{2^n}(x-4)^n = \sum (-1)^n = \text{DNE}$$

with the last series being divergent by oscillation.



The example shows that we cannot blindly take limits inside an infinite sum; understanding precisely when this is possible is one of our primary goals.

[8] You should have encountered these previously: $\liminf \left|\frac{b_{n+1}}{b_n}\right| \le \liminf |b_n|^{1/n} \le \limsup |b_n|^{1/n} \le \limsup \left|\frac{b_{n+1}}{b_n}\right|$

Radius and Interval of Convergence

The implied domain of the series in Example 2.6 turned out to be an *interval* $(2,6)$. Somewhat amazingly, the root test (Theorem 2.3) shows that the same is true for *every* power series!

Theorem 2.7 (Root Test for Power Series). *Given a power series $\sum a_n(x-c)^n$, define*[9]

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

The precisely one of the following statements holds:

$R \in (0,\infty)$	*the series converges absolutely when $\lvert x-c\rvert < R$ and diverges when $\lvert x-c\rvert > R$*
$R = \infty$	*the series converges absolutely for all $x \in \mathbb{R}$*
$R = 0$	*the series converges only at the center $x = c$*

Proof. For each fixed $x \in \mathbb{R}$, let $b_n = a_n(x-c)^n$ and apply the root test to $\sum b_n$, noting that

$$\limsup |b_n|^{1/n} = \begin{cases} \limsup |a_n|^{1/n} |x-c| = \frac{1}{R}|x-c| & \text{if } R \in (0,\infty) \\ 0 & \text{if } R = \infty \text{ or } x = c \\ \infty & \text{if } R = 0 \text{ and } x \neq c \end{cases}$$

In the first situation, $\limsup |b_n|^{1/n} < 1 \iff |x-c| < R$, etc. ∎

Definition 2.8. The *radius of convergence* is the value R defined in Theorem 2.7. The *interval of convergence* is the set of $x \in \mathbb{R}$ for which the series converges; its implied domain.

Radius of convergence	Interval of convergence
$R \neq 0, \infty$	$(c-R, c+R)$, $(c-R, c+R]$, $[c-R, c+R)$ or $[c-R, c+R]$
∞	$\mathbb{R} = (-\infty, \infty)$
0	$\{c\}$

In the first case, convergence/divergence at the endpoints of the interval of convergence must be tested for separately.

The ratio test (Corollary 2.4) provides a more user-friendly version.

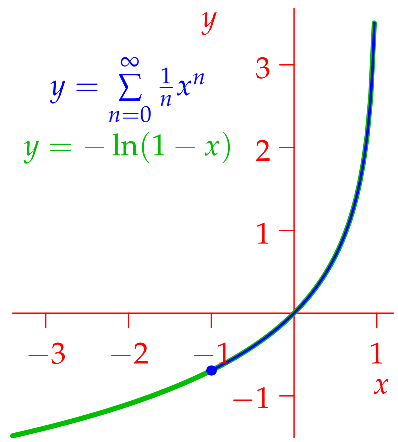
Corollary 2.9 (Ratio Test for Power Series). *If the limit exists, $R = \lim\limits_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$.*

The ratio test is *weaker* than the root test: as Example 2.10.5 shows, there exist series for which the ratio test is be inconclusive.

[9] Since $|a_n| \geq 0$, we here adopt the conventions $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$. With similar caveats, one can write $R = \liminf |a_n|^{-1/n}$. Since every sequence has a limit superior, this really is a *definition*. Whether one can easily *compute* R is another matter...

Examples 2.10. 1. The series $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ is centered at 0. The ratio test tells us that

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$



Test the endpoints of the interval of convergence separately:

$$\begin{aligned} x &= 1 & \sum \tfrac{1}{n} &= \infty \text{ diverges} \\ x &= -1 & \sum \tfrac{(-1)^n}{n} & \text{ converges (conditionally)} \end{aligned}$$

We conclude that the interval of convergence is $[-1, 1)$.

It can be seen (later) that the series converges to $-\ln(1-x)$ on its interval of convergence. As in Example 2.6, this function has a larger domain $(-\infty, -1)$, than that of the series.

2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ similarly has

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$$

Since $\sum \frac{1}{n^2}$ is absolutely convergent, we conclude that the power series also converges absolutely at $x = \pm 1$; the interval of convergence is $[-1, 1]$.

3. The series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges absolutely for all $x \in \mathbb{R}$, since

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$$

You should recall from elementary calculus that this series converges to the natural exponential function $\exp(x) = e^x$ everywhere on $\mathbb{R}$; indeed this is one of the common *definitions* of the exponential function.

4. The series $\sum_{n=0}^{\infty} n! x^n$ has $R = \lim \frac{n!}{(n+1)!} = 0$. It therefore converges only at its center $x = 0$.

5. Let $a_n = \left(\frac{2}{3}\right)^n$ if n is even and $\left(\frac{3}{2}\right)^n$ if n is odd. If we try to apply the ratio test to the series $\sum_{n=0}^{\infty} a_n x^n$, we see that

$$\left| \frac{a_n}{a_{n+1}} \right| = \begin{cases} \left(\frac{2}{3}\right)^{2n+1} & \text{if } n \text{ even} \\ \left(\frac{3}{2}\right)^{2n+1} & \text{if } n \text{ odd} \end{cases} \implies \limsup \left| \frac{a_n}{a_{n+1}} \right| = \infty \neq 0 = \liminf \left| \frac{a_n}{a_{n+1}} \right|$$

The ratio test is therefore inconclusive. However, by the root test,

$$|a_n|^{1/n} = \begin{cases} \frac{2}{3} & \text{if } n \text{ even} \\ \frac{3}{2} & \text{if } n \text{ odd} \end{cases} \implies R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{3/2} = \frac{2}{3}$$

It is easy to check that the series diverges at $x = \pm\frac{2}{3}$; the interval of convergence is $(-\frac{2}{3}, \frac{2}{3})$.

With the help of the root test the domain of a power series is fully understood. Limits, continuity, differentiability and integrability are more delicate. We will return to these once we've developed some of the ideas around *convergence* for *sequences of functions.*

Exercises 2.23. *Key concepts: Power Series, Radius/interval of convergence* $R = \frac{1}{\limsup |a_n|^{1/n}}$

1. For each power series, find the radius and interval of convergence:

   (a) $\sum \frac{(-1)^n}{n^2 4^n} x^n$ (b) $\sum \frac{(n+1)^2}{n^3}(x-3)^n$ (c) $\sum \sqrt{n} x^n$

   (d) $\sum \frac{1}{n\sqrt{n}}(x+7)^n$ (e) $\sum (x-\pi)^{n!}$ (f) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

2. For each $n \in \mathbb{N}$ let $a_n = \left(\frac{4+2(-1)^n}{5}\right)^n$

   (a) Find $\limsup |a_n|^{1/n}$, $\liminf |a_n|^{1/n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.

   (b) Does the series $\sum a_n$ converge? What about $\sum (-1)^n a_n$? Why?

   (c) Find the interval of convergence of the power series $\sum a_n x^n$.

3. Suppose that $\sum a_n x^n$ has radius of convergence R. If $\limsup |a_n| > 0$, prove that $R \le 1$.

4. On the interval $(-\frac{2}{3}, \frac{2}{3})$, express the series in Example 2.10.5 as a simple function.

   (*Hints: Use geometric series formulæ and the fact that the value of an absolutely convergent series is independent of rearrangements*)

5. Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{3^n n}(x-7)^{5n+1} = \frac{1}{3}(x-7) + \frac{1}{18}(x-7)^6 + \frac{1}{81}(x-7)^{11} + \cdots$$

   Since only one in five of the terms are non-zero, it is a little tricky to analyze using a naïve application of our standard tests.

   (a) Explain why the ratio test for power series (Corollary 2.9) does not apply.

   (b) Writing the series as $\sum a_m (x-7)^m$, observe that

$$a_m = \begin{cases} \frac{5}{3^{\frac{m-1}{5}}(m-1)} & \text{if } m \equiv 1 \mod 5 \\ 0 & \text{otherwise} \end{cases}$$

   Use the root test (Theorem 2.7) and your understanding of elementary limits to directly compute the radius of convergence.

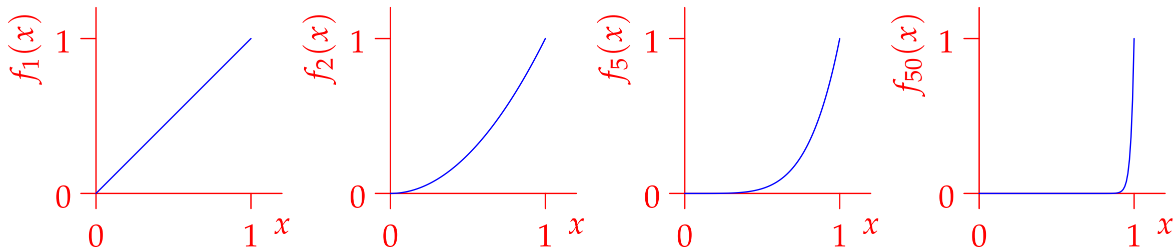
   (c) Alternatively, write $\sum \frac{1}{3^n n}(x-7)^{5n+1} = \sum b_n$. Apply the ratio test for *infinite* series (Corollary 2.4): what do you observe? Use your observation to compute the radius of convergence of the original series in a simpler manner than part (a).

   (d) Finally, check the endpoints to determine the interval of convergence.

2.24 Uniform Convergence

In this section we consider sequences (f_n) of *functions* $f_n : U \to \mathbb{R}$ and their limits.

Example 2.11. For each $n \in \mathbb{N}$, define $f_n : (0,1) \to \mathbb{R} : x \mapsto x^n$. Several examples are graphed.



There are several useful notions of convergence for sequences of functions. The simplest is where, for each input x, $(f_n(x))$ is treated as a distinct sequence of real numbers.

Definition 2.12. Suppose a function f and a sequence of functions (f_n) are given, all with domain U. We say that (f_n) *converges pointwise to f on U* if,

$$\forall x \in U, \lim_{n\to\infty} f_n(x) = f(x)$$

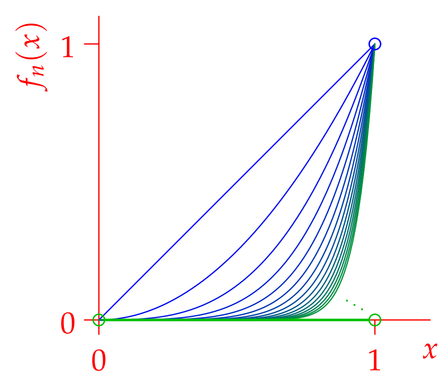
It is common to write '$f_n \to f$ pointwise.' For reference, here are two equivalent rephrasings:

1. $\forall x \in U, \lim_{n\to\infty} |f_n(x) - f(x)| = 0$;
2. $\forall x \in U, \forall \epsilon > 0, \exists N$ such that $n > N \implies |f_n(x) - f(x)| < \epsilon$.

As we'll see shortly, the relative position of the quantifiers ($\forall x$, $\exists N$) is crucial: in this definition, the value of N is permitted to depend on x as well as ϵ.

Example (2.11, mk. II). The sequence (f_n) converges pointwise on the domain $U = (0,1)$ to

$$f : (0,1) \to \mathbb{R} : x \mapsto 0$$



As a sanity check, we prove this explicitly. First observe that

$$|f_n(x) - f(x)| = x^n$$

Suppose $x \in (0,1)$, that $\epsilon > 0$ is given, and let $N = \frac{\ln \epsilon}{\ln x}$. Then

$$n > N \implies n \ln x < \ln \epsilon \implies x^n < \epsilon$$

where the inequality switches sign since $\ln x < 0$.

The example is nice in that a sequence of continuous functions converges pointwise to a continuous function. Unfortunately, this desirable situation is not universal...

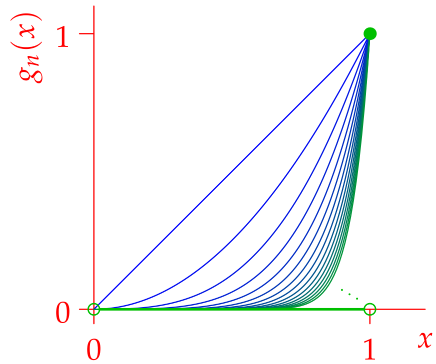
Example (2.11, mk. III). Define

$$g_n : (0,1] \to \mathbb{R} : x \mapsto x^n$$

Each g_n is a continuous function, however its pointwise limit

$$g(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

has a *jump discontinuity* at $x = 1$.

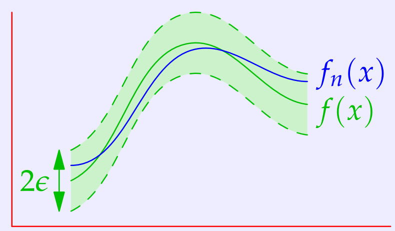


We'd like the limit of a sequence of continuous functions to itself be continuous. With this goal in mind, we make a tighter definition.

Definition 2.13. (f_n) *converges uniformly* to f on U if either

1. $\displaystyle\sup_{x\in U} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$, or,
2. $\forall \epsilon > 0$, $\exists N$ such that $\forall x \in U$, $n > N \implies |f_n(x) - f(x)| < \epsilon$

A common notation is $f_n \rightrightarrows f$, though we won't use it.



As pictured, whenever $n > N$, the graph of $f_n(x)$ must lie between $f(x) \pm \epsilon$.

We'll show that statements 1 and 2 are equivalent momentarily. For the present, compare with the corresponding statements for pointwise convergence:

- As with *continuity* versus *uniform continuity*, the distinction comes in the *order of the quantifiers*: in uniform convergence, x is quantified *after* N and so *the same N works for all locations x.*
- Uniform convergence implies pointwise convergence.

Example (2.11, mk. IV). For the final time we revisit our main example. If $f_n(x) = x^n$ and $f(x) = 0$ are defined on $U = (0,1)$, then $f_n \to f$ *non-uniformly.* We show this using both criteria.

1. For *every* n,

$$\sup_{x\in(0,1)} |f_n(x) - f(x)| = \sup\{x^n : 0 < x < 1\} = 1 \not\longrightarrow 0$$

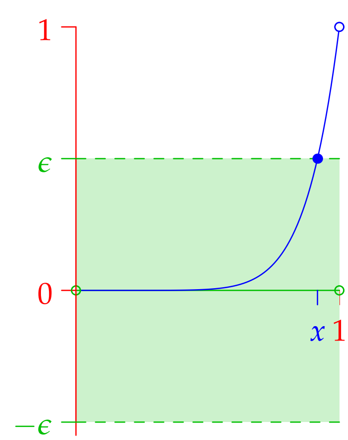
which plainly fails to converge to zero.

2. Suppose the convergence were uniform and let $\epsilon = \frac{1}{2}$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall x \in (0,1),\ n > N \implies x^n < \frac{1}{2}$$

Since $N \in \mathbb{N}$, a simple choice results in a contradiction;

$$x = \left(\frac{1}{2}\right)^{\frac{1}{N+1}} \in (0,1) \implies x^{N+1} = \frac{1}{2}$$



Theorem 2.14. *The criteria for uniform convergence in Definition 2.13 are equivalent.*

Proof. (1 ⇒ 2) This follows immediately from the fact that

$$\forall x \in U,\ |f_n(x) - f(x)| \leq \sup_{x \in U} |f_n(x) - f(x)|$$

(2 ⇒ 1) Suppose $\epsilon > 0$ is given. Then

$$\exists N \in \mathbb{R} \text{ such that } \forall x \in U,\ n > N \implies |f_n(x) - f(x)| < \frac{\epsilon}{2}$$

But then

$$n > N \implies \sup_{x \in U} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$$

■

Amazingly, this subtle change of definition is enough to preserve continuity.

Theorem 2.15. *Suppose (f_n) is a sequence of continuous functions. If $f_n \to f$ uniformly, then f is continuous.*

Proof. We demonstrate the continuity of f at $a \in U$. Let $\epsilon > 0$ be given.

- Since $f_n \to f$ uniformly,

$$\exists N \text{ such that } \forall x \in U,\ n > N \implies |f(x) - f_n(x)| < \frac{\epsilon}{3}$$

- Choose any $n > N$. Since f_n is continuous at a,

$$\exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3} \tag{†}$$

Put these together with the triangle inequality to see that

$$\begin{aligned} |x - a| < \delta \implies |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

■

We need not have fixed a at the start of the proof. Rewriting (†) to become

$$\exists \delta > 0 \text{ such that } \forall x, a \in U,\ |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

proves a related result.

Corollary 2.16. *Suppose (f_n) is a sequence of* ***uniformly*** *continuous functions. If $f_n \to f$ uniformly, then f is also* ***uniformly*** *continuous.*

Examples 2.17. 1. Let $f_n(x) = x + \frac{1}{n}x^2$. This is continuous on $\mathbb{R}$ for all x, and converges pointwise to the continuous function $f : x \mapsto x$.

(a) On any bounded interval $[-M, M]$ the convergence $f_n \to f$ is uniform,

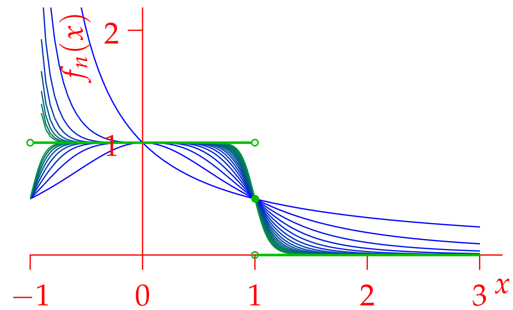
$$\sup_{x\in[-M,M]} |f_n(x) - f(x)| = \sup\left\{\frac{1}{n}x^2 : x \in [-M, M]\right\} = \frac{M^2}{n} \xrightarrow[n\to\infty]{} 0$$

(b) On any unbounded interval, $\mathbb{R}$ say, the convergence is non-uniform,

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup\left\{\frac{1}{n}x^2 : x \in \mathbb{R}\right\} = \infty$$

2. Consider $f_n(x) = \frac{1}{1+x^n}$; this is continuous on $(-1, \infty)$ and converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } -1 < x < 1 \end{cases}$$



We consider the convergence $f_n \to f$ on several intervals.

(a) On $[2, \infty)$, the pointwise limit is continuous. Moreover, $f_n(x)$ is decreasing, whence

$$\sup_{x\in[2,\infty)} |f_n(x) - 0| = \frac{1}{1+2^n} \xrightarrow[n\to\infty]{} 0$$

and the convergence is uniform. Alternatively; if $\epsilon \in (0, 1)$, let $N = \log_2(\epsilon^{-1} - 1)$, then

$$\forall x \ge 2,\ n > N \implies |f_n(x) - 0| = \frac{1}{1+x^n} \le \frac{1}{1+2^n} < \frac{1}{1+2^N} = \epsilon$$

The same argument shows that $f_n \to f$ uniformly on any interval $[a, \infty)$ where $a > 1$.

(b) On $[1, \infty)$ the convergence is not uniform, since the pointwise limit is discontinuous,

$$f(x) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$$

(c) The convergence is not even uniform on the open interval $(1, \infty)$,

$$\sup_{x\in[1,\infty)} |f_n(x) - f(x)| = \sup\left\{\frac{1}{1+x^n} : x > 1\right\} = \frac{1}{2} \underset{n\to\infty}{\not\longrightarrow} 0$$

(d) Similarly, for any $a \in (0, 1)$, the convergence $f_n \to f$ is uniform on $[0, a]$, this time to the (continuous) constant function $f(x) = 1$,

$$\sup_{x\in[0,a]} |f_n(x) - 1| = \left|1 - \frac{1}{1+a^n}\right| = \frac{a^n}{1+a^n} \xrightarrow[n\to\infty]{} 0$$

(e) Finally, on $(-1, 1)$ the convergence is not uniform,

$$\sup_{x\in[0,1)} |f_n(x) - f(x)| = \sup\left\{\frac{x^n}{1+x^n} : x \in [0, 1)\right\} = \frac{1}{2} \underset{n\to\infty}{\not\longrightarrow} 0$$

Exercises 2.24. *Key concepts: Pointwise & Uniform Convergence, Uniform conv preserves continuity*

1. For each sequence of functions defined on $[0, \infty)$:

   (i) Find the pointwise limit $f(x)$ as $n \to \infty$.
   (ii) Determine whether $f_n \to f$ uniformly on $[0, 1]$.
   (iii) Determine whether $f_n \to f$ uniformly on $[1, \infty)$.

   (a) $f_n(x) = \dfrac{x}{n}$ (b) $f_n(x) = \dfrac{x^n}{1+x^n}$ (c) $f_n(x) = \dfrac{x^n}{n+x^n}$

   (d) $f_n(x) = \dfrac{x}{1+nx^2}$ (e) $f_n(x) = \dfrac{nx}{1+nx^2}$

2. Let $f_n(x) = \left(x - \frac{1}{n}\right)^2$. If $f(x) = x^2$, we clearly have $f_n \to f$ pointwise on any domain.

   (a) Prove that the convergence is uniform on $[-1, 1]$.
   (b) Prove that the convergence is non-uniform on $\mathbb{R}$.

3. For each sequence, find the pointwise limit and decide if the convergence is uniform.

   (a) $f_n(x) = \frac{1+2\cos^2(nx)}{\sqrt{n}}$ for $x \in \mathbb{R}$.
   (b) $f_n(x) = \cos^n(x)$ on $[-\pi/2, \pi/2]$.

4. For each $n \in \mathbb{N}$, consider the continuous function

   $$f_n : [0, 1] \to \mathbb{R} : x \mapsto nx^n(1-x)$$

   (a) Given $0 \le x < 1$, let $a \in (x, 1)$. Explain why $\exists N$ such that

   $$n > N \implies |f_{n+1}(x)| \le a\,|f_n(x)|$$

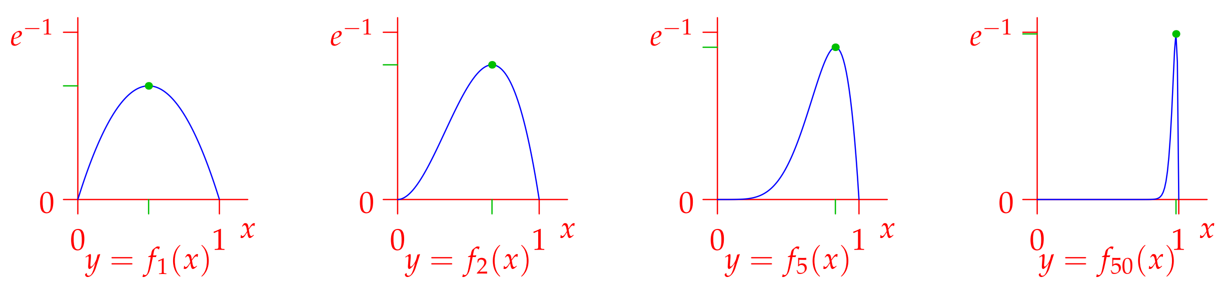
   Hence conclude that the pointwise limit of (f_n) is the zero function.

   (b) Use elementary calculus ($f_n'(x) = 0 \iff \ldots$) to prove that the maximum value of f_n is located at $x_n = \frac{n}{1+n}$. Hence compute

   $$\sup_{x \in [0,1]} |f_n(x) - f(x)|$$

   and use it to show that the convergence $f_n \to 0$ is non-uniform.

   This shows that the converse to Theorem 2.15 is false, even on a bounded interval: the continuous sequence (f_n) converges non-uniformly *to a continuous function. Sketches of several f_n are below.*



5. Explain where the proof of Theorem 2.15 fails if $f_n \to f$ *non-uniformly.*

2.25 More on Uniform Convergence

While we haven't yet developed calculus, our familiarity with basic differentiation and integration makes it natural to pause to consider the interaction of these concepts with sequences of functions.

We also consider a Cauchy-criterion for uniform convergence, which leads to the useful Weierstraß M-test.

Example 2.18. Recall that $f_n(x) = x^n$ converges uniformly to $f(x) = 0$ on any interval $[0,a]$ where $a < 1$. We easily check that

$$\int_0^a f_n(x)\,\mathrm{d}x = \frac{1}{n+1}a^{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^a f(x)\,\mathrm{d}x$$

In fact the sequence of derivatives converge here also

$$\frac{\mathrm{d}}{\mathrm{d}x}f_n(x) = nx^{n-1} \xrightarrow[n\to\infty]{} 0 = f'(x)$$

It is perhaps surprising that integration interacts more nicely with uniform limits than does differentiation. We therefore consider integration first.

Theorem 2.19. *Let $f_n \to f$ uniformly on $[a,b]$ where the functions f_n are integrable. Then f is integrable on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

Proof. Given $\epsilon > 0$, note that $\int_a^b \frac{\epsilon}{2(b-a)}\,\mathrm{d}x = \frac{\epsilon}{2}$. Since $f_n \to f$ uniformly, $\exists N$ such that[10]

$$\begin{aligned}
\forall x \in [a,b],\ n > N \implies& |f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)} \\
\implies& f_n(x) - \frac{\epsilon}{2(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{2(b-a)} \\
\implies& \int_a^b f_n(x)\,\mathrm{d}x - \frac{\epsilon}{2} \le \int_a^b f(x)\,\mathrm{d}x \le \int_a^b f_n(x)\,\mathrm{d}x + \frac{\epsilon}{2} \\
\implies& \left|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\right| \le \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

■

The appearance of uniform convergence in the proof is subtle. If $N = N(\epsilon)$ were allowed to depend on x, then the integral $\int_a^b f_n(x)\,\mathrm{d}x$ would be meaningless: Which n would we consider? Larger than $N(x,\epsilon)$ for *which* x? Taking n 'larger' than *all* the $N(x,\epsilon)$ might produce the absurdity $n = \infty$!

[10]This assumes f is already integrable. Once we've properly defined (Riemann/Darboux) integrability at the end of the course, we can insert the following

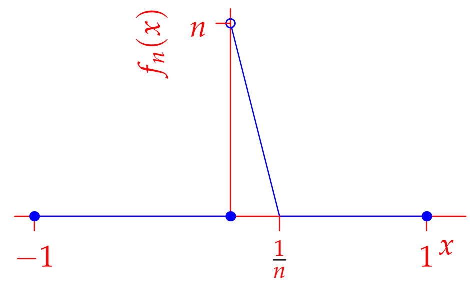
$$\int_a^b f_n(x)\,\mathrm{d}x - \frac{\epsilon}{2} \le L(f) \le U(f) \le \int_a^b f_n(x)\,\mathrm{d}x + \frac{\epsilon}{2} \implies 0 \le U(f) - L(f) \le \epsilon \implies U(f) = L(f)$$

where $U(f)$ and $L(f)$ are the upper and lower Darboux integrals of f; equality shows that f is integrable on $[a,b]$.

Examples 2.20. 1. Uniform convergence is not required for the integrals to converge as we'd like. For instance, recall that extending the previous example to the domain $[0,1]$ results in non-uniform convergence; however, we still have

$$\int_0^1 f_n(x)\,dx = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^1 f(x)\,dx$$

2. To obtain a sequence of functions $f_n \to f$ for which $\int f_n \not\to \int f$ requires a bit of creativity. Consider the sequence



$$f_n : [-1,1] \to \mathbb{R} : x \mapsto \begin{cases} n - n^2 x & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

If $0 < x < 1$, then for large $n \in \mathbb{N}$ we have

$$x \geq \frac{1}{n} \implies f_n(x) = 0$$

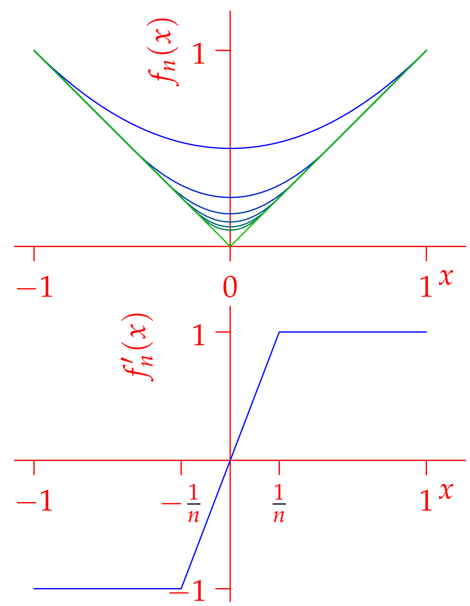
We conclude that $f_n \to 0$ pointwise. Since the area under f_n is a triangle with base $\frac{1}{n}$ and height n, the integral is constant and *non-zero*;

$$\int_{-1}^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_{-1}^1 f(x)\,dx$$

It should be obvious that the convergence $f_n \to 0$ is non-uniform; why?

Derivatives and Uniform Limits We've already seen that a uniform limit of differentiable functions *might* be differentiable (Example 2.18). As the next example shows, this should not be expected in general, since even uniform limits of differentiable functions can have corners.

Example 2.21. For each $n \in \mathbb{N}$, consider the function



$$f_n : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} |x| & \text{if } |x| \geq \frac{1}{n} \\ \frac{n}{2}x^2 + \frac{1}{2n} & \text{if } |x| < \frac{1}{n} \end{cases}$$

- Each f_n is differentiable: $f_n'(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ nx & \text{if } |x| < \frac{1}{n} \\ -1 & \text{if } x \leq -\frac{1}{n} \end{cases}$
- f_n converges pointwise to $f(x) = |x|$, which is *non-differentiable* at $x = 0$.
- $f_n \to f$ uniformly since

$$\sup_{x\in[-1,1]} |f_n(x) - f(x)| = f_n(0) = \frac{1}{2n} \to 0$$

If our goal is to transfer differentiability to the limit of a sequence of functions, then we have some work to do.

Theorem 2.22. *Suppose (f_n) is a sequence and f, g functions, all with domain $[a, b]$. Suppose also:*

- *$f_n \to f$ pointwise;*
- *Each f_n is differentiable with continuous derivative;*[11]
- *$f_n' \to g$ uniformly.*

Then $f_n \to f$ uniformly on $[a, b]$ and f is differentiable with derivative g.

The issue in the previous example is that the *pointwise limit* of the derived sequence (f_n') is discontinuous at $x = 0$ and therefore $f_n' \to g$ isn't uniform!

Proof. For any $x \in [a, b]$, the fundamental theorem of calculus (part II) tells us that

$$\int_a^x f_n'(t)\, \mathrm{d}t = f_n(x) - f_n(a)$$

As $n \to \infty$, Theorem 2.19 says the left side converges to $\int_a^x g(t)\, \mathrm{d}t$ and the right to $f(x) - f(a)$ (both pointwise). Since $f_n' \to g$ uniformly, we see that g is continuous and can apply the fundamental theorem (part I): $\int_a^x g(t)\, \mathrm{d}t = f(x) - f(a)$ is differentiable with derivative g.

The uniformity of the convergence $f_n \to f$ follows from Exercise 10. ∎

Uniformly Cauchy Sequences and the Weierstraß M-Test

Recall that one may use Cauchy sequences to demonstrate convergence *without knowing the limit in advance.* An analogous discussion is available for sequences of functions.

Definition 2.23. A sequence of functions (f_n) is *uniformly Cauchy* on U if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall x \in U,\ m, n > N \implies |f_n(x) - f_m(x)| < \epsilon$$

Example 2.24. Let $f_n(x) = \sum\limits_{k=1}^{n} \frac{1}{k^2} \sin k^2 x$ be defined on $\mathbb{R}$. Given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$, then

$$\begin{aligned} m > n > N \implies |f_m(x) - f_n(x)| &= \left| \sum_{k=n+1}^{m} \frac{1}{k^2} \sin k^2 x \right| \le \sum_{k=n+1}^{m} \frac{1}{k^2} \le \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} = \frac{1}{n} - \frac{1}{m} < \frac{1}{N} = \epsilon \end{aligned}$$

whence (f_n) is uniformly Cauchy.

[11] Without this continuity assumption, the fundamental theorem of calculus doesn't apply and the proof requires an alternative approach. One can also weaken the hypotheses: if $f_n' \to g$ uniformly and $(f_n(x))$ converges for *at least one* $x \in [a, b]$, then there exists f such that $f_n \to f$ is uniform and $f' = g$.

As with sequences of real numbers, uniformly Cauchy sequences converge; in fact uniformly!

Theorem 2.25. *A sequence (f_n) is uniformly Cauchy on U if and only if it converges uniformly to some $f : U \to \mathbb{R}$.*

Proof. $(\Rightarrow)$ Let (f_n) be uniformly Cauchy on U. For each $x \in U$, the sequence $(f_n(x)) \subseteq \mathbb{R}$ is Cauchy and thus convergent. Define $f : U \to \mathbb{R}$ to be the pointwise limit:

$$f(x) := \lim_{n\to\infty} f_n(x)$$

We claim that $f_n \to f$ uniformly. Let $\epsilon > 0$ be given, then $\exists N \in \mathbb{N}$ such that

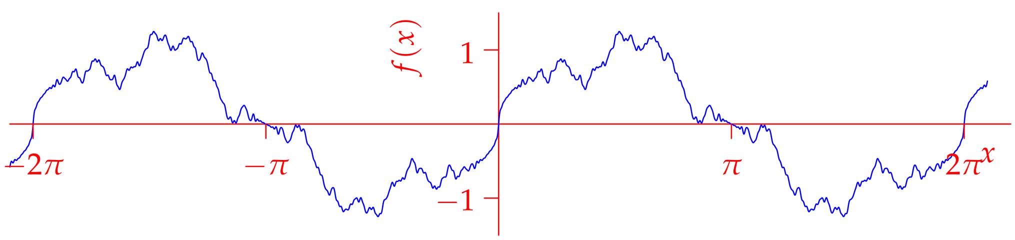
$$\begin{aligned} m > n > N &\implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2} \\ &\implies f_n(x) - \frac{\epsilon}{2} < f_m(x) < f_n(x) + \frac{\epsilon}{2} \\ &\implies f_n(x) - \frac{\epsilon}{2} \le f(x) \le f_n(x) + \frac{\epsilon}{2} && \text{(take limits as } m \to \infty) \\ &\implies |f_n(x) - f(x)| \le \frac{\epsilon}{2} < \epsilon \end{aligned}$$

$(\Leftarrow)$ This is Exercise 2. ■

Example (2.24, mk. II). Since (f_n) is uniformly Cauchy on $\mathbb{R}$, it converges uniformly to some $f : \mathbb{R} \to \mathbb{R}$. It seems reasonable to write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 x$$

The graph of this function looks somewhat bizarre:



Since each f_n is (uniformly) continuous, Theorem 2.15 says that f is also (uniformly) continuous. By Theorem 2.19, $f(x)$ is integrable, indeed

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} -k^{-4} \cos k^2 x \Big|_a^b = \sum_{n=1}^{\infty} \frac{1}{n^4}(\cos n^2 a - \cos n^2 b)$$

which converges (comparison test) for all a, b. By contrast, the derived sequence

$$f_n'(x) = \sum_{k=1}^{n} \cos k^2 x$$

does not converge for *any* x since $\lim_{n\to\infty} \cos n^2 x \neq 0$. We should therefore expect (though we offer no proof) that f is nowhere differentiable.

The example generalizes. Suppose (g_k) is a sequence of functions on U and define the *series* $\sum g_k(x)$ as the pointwise limit of the sequence (f_n) of partial sums

$$\sum_{k=k_0}^{\infty} g_k(x) := \lim_{n\to\infty} f_n(x) \quad \text{where} \quad f_n(x) = \sum_{k=k_0}^{n} g_k(x)$$

whenever the limit exists. The series is said to converge uniformly whenever (f_n) does so. Theorems 2.15, 2.19 and 2.22 immediately translate.

Corollary 2.26. *Let $\sum g_k$ be a series of functions converging uniformly on U. Then:*

1. *If each g_k is (uniformly) continuous then $\sum g_k$ is (uniformly) continuous.*
2. *If each g_k is integrable, then $\int \sum g_k(x)\,\mathrm{d}x = \sum \int g_k(x)\,\mathrm{d}x$.*
3. *If each g_k is continuously differentiable, and the sequence of derived partial sums f_n' converges uniformly, then $\sum g_k$ is differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}\sum g_k(x) = \sum g_k'(x)$.*

As an application of the uniform Cauchy criterion, we obtain an easy test for uniform convergence.

Theorem 2.27 (Weierstraß M-test). *Suppose (g_k) is a sequence of functions on U. Moreover assume:*

1. *(M_k) is a non-negative sequence such that $\sum M_k$ converges.*
2. *Each g_k is bounded by M_k; that is $|g_k(x)| \le M_k$.*

Then $\sum g_k(x)$ converges uniformly on U.

Proof. Let $f_n(x) = \sum_{k=k_0}^{n} g_k(x)$ define the sequence of partial sums. Since $\sum M_k$ converges, its sequence of partial sums is Cauchy (the *Cauchy criterion* for infinite series); given $\epsilon > 0$,

$$\exists N \text{ such that } m > n > N \implies \sum_{k=n+1}^{m} M_k < \epsilon$$

However, by assumption,

$$m > n > N \implies |f_m(x) - f_n(x)| = \left|\sum_{k=n+1}^{m} g_k(x)\right| \le \sum_{k=n+1}^{m} |g_k(x)| \le \sum_{k=n+1}^{m} M_k < \epsilon$$

The sequence of partial sums is uniformly Cauchy and thus uniformly convergent. ∎

Example 2.28. Given the series $\sum_{n=1}^{\infty} \frac{1+\cos^2(nx)}{n^2}\sin(nx)$, we clearly have

$$\left|\frac{1+\cos^2(nx)}{n^2}\sin(nx)\right| \le \frac{2}{n^2} \text{ for all } x \in \mathbb{R}$$

Since $\sum \frac{2}{n^2}$ converges, the M-test shows that the original series converges uniformly on $\mathbb{R}$.

Exercises 2.25. *Key concepts: Uniform convergence preseves integration, Uniform Cauchyness, M-test*

1. For each $n \in \mathbb{N}$, let $f_n(x) = nx^n$ when $x \in [0,1)$ and $f_n(1) = 0$.
   (a) Prove that $f_n \to 0$ pointwise on $[0,1]$.
   (*Hint: recall Exercise 2.24.4 if you're not sure how to prove this*)
   (b) By considering the integrals $\int_0^1 f_n(x)\,\mathrm{d}x$ show that $f_n \to 0$ is not uniform.
2. Prove that if $f_n \to f$ uniformly, then the sequence (f_n) is uniformly Cauchy.
3. (a) Suppose (f_n) is a sequence of bounded functions on U and suppose that $f_n \to f$ converges uniformly on U. Prove that f is bounded on U.
   (b) Give an example of a sequence of bounded functions (f_n) converging pointwise to f on $[0,\infty)$, but for which f is *unbounded.*
4. The sequence defined by $f_n(x) = \frac{nx}{1+nx^2}$ (Exercise 2.24.1) converges uniformly on any closed interval $[a,b]$ where $0 < a < b$.
   (a) Check explicitly that $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, where $f = \lim f_n$.
   (b) Is the same thing true for derivatives?
5. Let $f_n(x) = n^{-1}\sin n^2x$ be defined on $\mathbb{R}$.
   (a) Prove that f_n converges uniformly on $\mathbb{R}$.
   (b) Check that $\int_0^x f_n(t)\,\mathrm{d}t$ converges for any $x \in \mathbb{R}$.
   (c) Does the derived sequence (f_n') converge? Explain.
6. Use the M-test to prove that $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$ defines a continuous function on $[-1,1]$.
7. Prove that $\sum\limits_{n=1}^{\infty} \frac{x^n \sin x}{(n+1)^3 2^n}$ converges uniformly to a continuous function on the interval $[-2,2]$.
8. Prove that if $\sum g_k$ converges uniformly on a set U and if h is a bounded function on U, then $\sum hg_k$ converges uniformly on U.
   (*Warning: you cannot simply write* $\sum hg_k = h\sum g_k$)
9. Consider Example 2.20.2.
   (a) Check explicitly that the convergence isn't uniform by computing $\sup\limits_{x\in[-1,1]} |f_n(x) - f(x)|$
   (b) Prove that $f_n \to 0$ pointwise on $(0,1]$ using the ϵ–N definition of convergence: that is, given $\epsilon > 0$ and $x \in (0,1]$, find an explicit $N(x,\epsilon)$ such that
   $$n > N \implies |f(x)| < \epsilon$$
   What happens to your choice of $N(x,\epsilon)$ as $x \to 0^+$?
10. Suppose (f_n') converges uniformly on $[a,b]$ and that each f_n' is continuous.
   (a) Use the fact that (f_n') is uniformly Cauchy to prove that (f_n) is uniformly Cauchy and thus converges uniformly to some function f.
   (*Hint:* $|f_n(x) - f_m(x)| = \left|\int_a^x f_n'(t) - f_m'(t)\,\mathrm{d}t\right| \ldots$)
   (b) Explain why we need not have assumed the existence of f in Theorem 2.22.

2.26 Differentiation and Integration of Power Series

We now specialize our recent results to power series. While everything will be stated for series centered at $x = 0$, all are easily translated to arbitrary centers.

Theorem 2.29. *Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$ and let $T \in (0, R)$. Then:*

1. *The series converges uniformly on $[-T, T]$.*
2. *The series is uniformly continuous on $[-T, T]$ and continuous on $(-R, R)$.*

Proof. This is a straightforward application of the Weierstraß M-test (Theorem 2.27). For each k, define $M_k = |a_k| T^k$, and observe that

$$T < R \implies \sum a_n T^n \text{ converges absolutely } \implies \sum M_k \text{ converges}$$

By the M-test and Corollary 2.26, the power series converges uniformly on $[-T, T]$ to a uniformly continuous function.

Finally, every $x \in (-R, R)$ lies in some such interval (take $T = |x|$), whence the power series is continuous on $(-R, R)$. ■

Example 2.30. On its interval of convergence $(-1, 1)$, the geometric series $\sum_{n=0}^{\infty} x^n$ converges pointwise to $\frac{1}{1-x}$; convergence is uniform on any interval $[-T, T] \subseteq (-1, 1)$.
We needn't use the Theorem for this is simple to verify directly: writing f, f_n for the series and its partial sums,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{1 - x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \left| \frac{x^{n+1}}{1-x} \right| \\ \implies \sup_{x \in [-T,T]} |f_n(x) - f(x)| &= \frac{T^{n+1}}{1-T} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

By contrast, the convergence is non-uniform on $(-1, 1)$, since

$$\sup_{x \in (-1,1)} |f_n(x) - f(x)| = \infty$$

Theorem 2.31. *Suppose a power series $\sum a_n x^n$ has radius of convergence $R > 0$. Then the series is integrable and differentiable term-by-term on the interval $(-R, R)$. Indeed for any $x \in (-R, R)$,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int_0^x \sum_{n=0}^{\infty} a_n t^n \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

where both series also have radius of convergence R.

Proof. Let $f(x) = \sum a_n x^n$ have radius of convergence R, and observe that

$$\limsup |na_n|^{1/n} = \lim n^{1/n} \limsup |a_n|^{1/n} = \frac{1}{R}$$

whence $\sum na_n x^n$ also has radius of convergence R. At any given non-zero $x \in (-R, R)$, we may write

$$\sum_{n=1}^{\infty} na_n x^{n-1} = x^{-1} \sum_{n=1}^{\infty} na_n x^n$$

to see that the derived series also has radius of convergence R. On any interval $[-T, T] \subseteq (-R, R)$, the derived series converges uniformly (Theorem 2.29). Since each $a_n x^n$ is continuously differentiable, Corollary 2.26 says that f is differentiable on $[-T, T]$ and that

$$f'(x) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} a_n x^n = \sum_{n=1}^{\infty} na_n x^{n-1}$$

Since any $x \in (-R, R)$ lies in some such interval $[-T, T]$, we are done.

Exercise 7 discusses the corresponding result for integration. ∎

We postpone the canonical examples until after the next result.

Continuity at Endpoints?

There is one small hole in our analysis. A series $\sum a_n x^n$ with radius of convergence R converges and is continuous on $(-R, R)$. But what if it also converges at $x = \pm R$? Is the series continuous at the endpoints? The answer is yes, though demonstrating this small benefit requires a lot of work!

Theorem 2.32 (Abel's Theorem). *Power series are continuous on their full interval of convergence.*

Examples 2.33. 1. Apply our results to the geometric series;

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\ln(1-x) = -\int_0^x \frac{1}{1-t}\,\mathrm{d}t = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)$$

where both are valid on $(-1, 1)$. In fact the first series has exactly this interval of convergence, whereas the second has $[-1, 1)$. By Abel's Theorem and the fact that logarithms are continuous, we have equality at $x = -1$ and recover the famous identity

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This also shows that while integrated and differentiated series have the same radius of convergence as the original, convergence at the endpoints need not be the same.

2. Substitute $x \mapsto -x^2$ in the geometric series and integrate term-by-term: if $|x| < 1$, then

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \implies \arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

In fact the arctangent series also converges at $x = \pm 1$; Abel's Theorem says it is continuous on $[-1,1]$. Since arctangent is continuous (on $\mathbb{R}$!) we recover another famous identity

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

As with the identity for $\ln 2$, this is a very slowly converging alternating series and therefore doesn't provide an efficient method for approximating π.

3. The series $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ has radius of convergence ∞. Differentiate to obtain

$$f'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}x^{2n+1}$$

This series is also valid for all $x \in \mathbb{R}$. Differentiating again,

$$f''(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}x^{2n} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = -f(x)$$

Recalling that $f(x) = \cos x$ is the unique solution to the initial value problem

$$\begin{cases} f''(x) = -f(x) \\ f(0) = 1,\ f'(0) = 0 \end{cases}$$

We conclude that, $\forall x \in \mathbb{R}$,

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \qquad \sin x = -f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

These last expressions can be taken as the *definitions* of sine and cosine. As promised earlier, continuity and differentiability of these functions now come for free! The only real downside of this definition is believing that it has anything to do with right-triangles!

We can similarly define other common transcendental functions using power series: for instance

$$\exp(x) = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

Example 2.33.1 could also be taken as a definition of the logarithm on the interval $(0,2]$,

$$\ln x = \ln(1-(1-x)) = -\sum_{n=1}^{\infty}\frac{1}{n}(1-x)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$$

though this is unnecessary since it is more natural to define ln as the inverse of the exponential.

Proof of Abel's Theorem (non-examinable)

This requires a lot of work, so feel free to omit on a first reading!

First observe that there is nothing to check unless $0 < R < \infty$. By the change of variable $x \mapsto \pm\frac{x}{R}$, it is enough for us to prove the following:

$$\sum_{n=0}^{\infty} a_n \text{ convergent and } f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ on } (-1,1) \implies \lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$$

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ and write $s = \lim s_n = \sum a_n$. It is an easy exercise to check that

$$\sum_{k=0}^{n} a_k x^k = s_n x^n + (1-x)\sum_{k=0}^{n-1} s_k x^k$$

If $|x| < 1$, then (since $s_n \to s$) $\lim s_n x^n = 0$, whence we obtain

$$\forall x \in (-1,1),\ f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$$

Let $\epsilon \in (0,1)$ be given and fix $x \in (0,1)$. Then

$$\exists N \in \mathbb{N} \text{ such that } n > N \implies |s_n - s| < \frac{\epsilon}{2} \qquad (*)$$

Use the geometric series formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and write $h(x) = (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n\right|$ to observe

$$\begin{aligned}
|f(x)-s| &= \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s\right| = \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s(1-x)\sum_{n=0}^{\infty} x^n\right| \\
&= \left|(1-x)\sum_{n=0}^{\infty}(s_n - s)x^n\right| = (1-x)\left|\sum_{n=0}^{N}(s_n-s)x^n + \sum_{n=N+1}^{\infty}(s_n-s)x^n\right| \\
&\le (1-x)\left|\sum_{n=0}^{N}(s_n-s)x^n\right| + (1-x)\left|\sum_{n=N+1}^{\infty}(s_n-s)x^n\right| && (\triangle\text{-inequality}) \\
&< h(x) + \frac{\epsilon}{2}(1-x)\left|\sum_{n=N+1}^{\infty} x^n\right| && (\text{by } (*)) \\
&\le h(x) + \frac{\epsilon}{2}
\end{aligned}$$

Since $h > 0$ is continuous and $h(1) = 0$, $\exists\delta > 0$ such that $x \in (1-\delta, 1) \implies h(x) < \frac{\epsilon}{2}$ (the computation of a suitable δ is another exercise).

We conclude that $\lim_{x\to 1^-} f(x) = s$. ∎

Exercises 2.26. *Key concepts:* *Power series continuous on $(-R, R)$, uniformly on any $[-T, T] \subset (-R, R)$, Power series differentiable and integrable term-by-term on $(-R, R)$*

1. (a) Prove that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.
   (b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$, $\sum_{n=1}^{\infty} \frac{n}{4^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$
2. (a) Starting with a power series centered at $x = 0$, evaluate the integral $\int_0^{1/2} \frac{1}{1+x^4}\,\mathrm{d}x$ as an infinite series.
   (b) (Harder) Repeat part (a) but for $\int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x$. What extra ingredients do you need?
3. The probability that a standard normally distributed random variable X lies in the interval $[a, b]$ is given by the integral

   $$\mathbb{P}(a \le X \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x$$

   Find $\mathbb{P}(-1 \le X \le 1)$ as an infinite series.
4. If $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ is defined as in Example 2.33.3, prove that $\left(f(x)\right)^2 + \left(f'(x)\right)^2 = 1$. What does (the converse of) Pythagoras' Theorem say about $f(x)$, at least when both it and $f'(x)$ are positive?

   (*Hint: Differentiate and evaluate at zero!*)
5. Define $c(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $s(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.
   (a) Prove that $c'(x) = s(x)$ and that $s'(x) = c(x)$.
   (b) Prove that $c(x)^2 - s(x)^2 = 1$ for all $x \in \mathbb{R}$.

   (*These functions are the hyperbolic sine and cosine:* $s(x) = \sinh x$ *and* $c(x) = \cosh x$)
6. Let $a, b \in (-1, 1)$. Extending Example 2.30, show that the convergence $\sum x^n = \frac{1}{1-x}$ is non-uniform on any interval of the form $(-1, a)$ or $(b, 1)$.
7. Prove the integration part of Theorem 2.31.
8. Prove or disprove: If a series converges absolutely at the *endpoints* of its interval of convergence then its convergence is uniform on the entire interval.
9. Complete the proof of Abel's Theorem:
   (a) Let $s_n = \sum_{k=0}^{n} a_k$ be the partial sum of the series $\sum a_n$. For each n, prove that,

   $$\sum_{k=0}^{n} a_k x^k = s_n x^n + (1-x) \sum_{k=0}^{n-1} s_k x^k$$

   (b) Suppose $x > 0$. Let $S = \max\{|s_n - s| : n \le N\}$ and prove that $h(x) \le S(1 - x^{N+1})$. Hence find an explicit δ that completes the final step.

2.27 The Weierstraß Approximation Theorem

A major theme of analysis is *approximation*; for instance power series are an example of (uniform) approximation by polynomials. It is reasonable to ask whether any function can be so approximated. In 1885, Weierstraß answered a specific case in the affirmative.

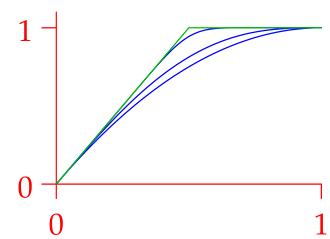
Theorem 2.34 (Weierstraß). *If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists a sequence of polynomials converging uniformly to f on $[a, b]$.*

Suitable polynomials can be defined in various ways. By scaling the domain, it is enough to do this on $[a, b] = [0, 1]$ where perhaps the simplest approach is via the *Bernstein Polynomials*,

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \qquad \left(\binom{n}{k} = \tfrac{n!}{k!(n-k)!} \text{ is the binomial coefficient}\right)$$

We omit the proof due to length; Weierstraß' original argument was completely different. Instead we compute a couple of examples and give an important interpretation/application.

Examples 2.35. 1. Suppose $f(x) = 2x$ if $x < \frac{1}{2}$ and $f(x) = 1$ otherwise.



$$B_1 f(x) = f(0)(1-x)f(0) + f(1)x = x$$

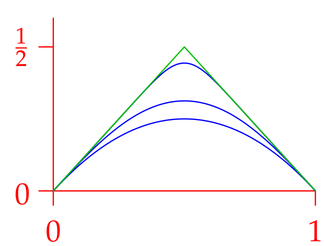
$$\begin{aligned} B_2 f(x) &= f(0)(1-x)^2 + 2f(\tfrac{1}{2})x(1-x) + f(1)x^2 \\ &= 2x(1-x) + x^2 \\ &= x(2-x) \end{aligned}$$

$$\begin{aligned} B_3 f(x) &= f(0)(1-x)^3 + 3f(\tfrac{1}{3})x(1-x)^2 + 3f(\tfrac{2}{3})x^2(1-x) + f(1)x^3 \\ &= 0(1-x)^3 + 2x(1-x)^2 + 3x^2(1-x) + x^3 \\ &= x(2-x) = B_2 f(x) \end{aligned}$$

$$\begin{aligned} B_4 f(x) &= 0(1-x)^4 + 2x(1-x)^3 + 6x^2(1-x)^2 + 4x^3(1-x) + x^4 \\ &= x(x^3 - 2x^2 + 2) \end{aligned}$$

The Bernstein polynomials $B_2 f(x), B_4 f(x)$ and $B_{50} f(x)$ are drawn.

2. Now assume $f(x) = x$ if $x < \frac{1}{2}$ and $f(x) = 1 - x$ otherwise.



$$B_1 f(x) = f(0)(1-x) + f(1)x = 0$$

$$B_2 f(x) = x(1-x)$$

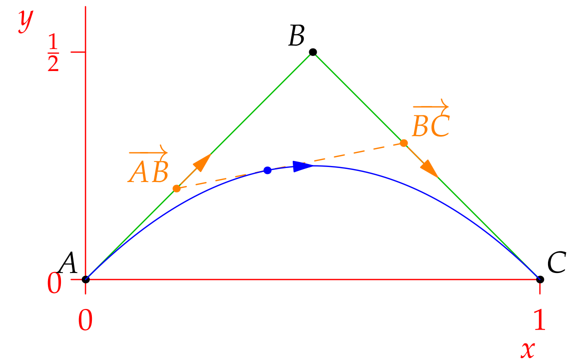
$$\begin{aligned} B_3 f(x) &= 0(1-x)^3 + x(1-x)^2 + x^2(1-x) + 0x^3 \\ &= x(1-x) = B_2 f(x) \end{aligned}$$

$$\begin{aligned} B_4 f(x) &= f(0)(1-x)^4 + f(\tfrac{1}{4}) \cdot 4x(1-x)^3 + f(\tfrac{1}{2}) \cdot 6x^2(1-x)^2 + f(\tfrac{3}{4}) \cdot 4x^3(1-x) + f(1)x^4 \\ &= x(1-x)^3 + 3x^2(1-x)^2 + x^3(1-x) \\ &= x(1-x)(1+x-x^2) \end{aligned}$$

Bézier curves (just for fun!)

The Bernstein polynomials arise naturally when considering *Bézier curves.* These have many applications, particularly in computer graphics. Given three points A, B, C, define points on the line segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$ for each $t \in [0,1]$, via

$$\overrightarrow{AB}(t) = (1-t)A + tB \qquad \overrightarrow{BC}(t) = (1-t)B + tC$$



These points move at a constant speed along the corresponding segments. Now consider a point on the *moving* segment between the points defined above:

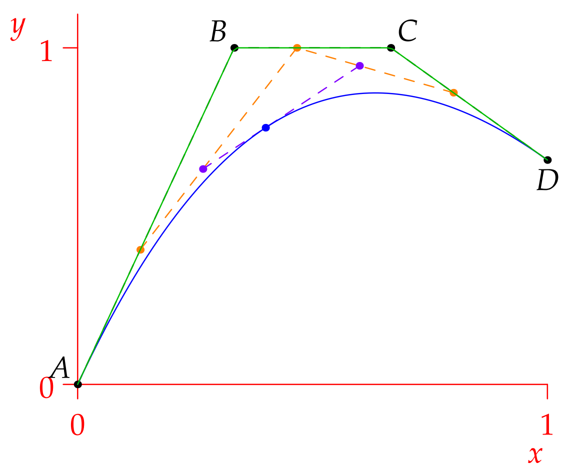
$$R(t) := (1-t)\overrightarrow{AB}(t) + t\overrightarrow{BC}(t) = (1-t)^2A + 2t(1-t)B + t^2C$$

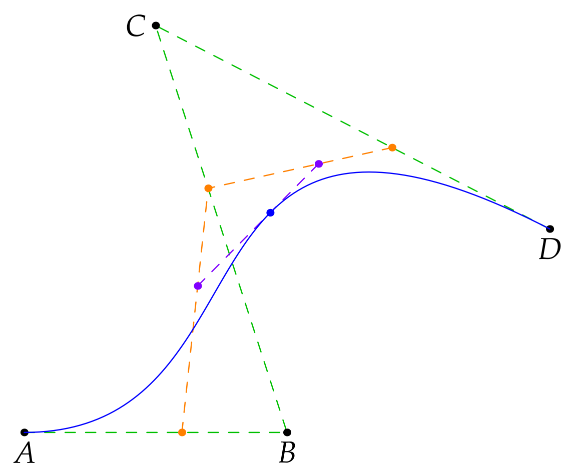
This is the *quadratic Bézier curve with control points A, B, C.* The 2^{nd} Bernstein polynomial for a function f is simply the quadratic Bézier curve with control points $(0, f(0))$, $\left(\frac{1}{2}, f(\frac{1}{2})\right)$ and $(1, f(1))$. The picture[12] above shows $B_2f(x)$ for the above example.

We can repeat the construction with more control points: with four points A, B, C, D, one constructs $\overrightarrow{AB}(t)$, $\overrightarrow{BC}(t)$, $\overrightarrow{CD}(t)$, then the second-order points between these, and finally the cubic Bézier curve

$$\begin{aligned} R(t) :&= (1-t)\left((1-t)\overrightarrow{AB}(t) + t\overrightarrow{BC}(t)\right) + t\left((1-t)\overrightarrow{BC}(t) + t\overrightarrow{CD}(t)\right) \\ &= (1-t)^3A + 3t(1-t)^2B + 3t^2(1-t)C + t^3D \end{aligned}$$

where we now recognize the relationship to the 3^{rd} Bernstein polynomial.





The pictures show cubic Bézier curves: the first is the graph of the Bernstein polynomial

$$B_3f(x) = 0(1-x)^3 + 3x(1-x)^2 + 3x^2(1-x) + \tfrac{2}{3}x^3$$

while the second is for the four given control points A, B, C, D.

[12]Click on any of the pictures to see all of them move.

Exercises 2.27. *Key concepts: Every continuous function is the uniform limit of a polynomial sequence*

1. Show that the closed bounded interval assumption in the approximation theorem is required by giving an example of a continuous function $f : (-1,1) \to \mathbb{R}$ which is *not* the uniform limit of a sequence of polynomials.

2. If $g : [a,b] \to \mathbb{R}$ is continuous, then $f(x) := g\big((b-a)x + a\big)$ is continuous on $[0,1]$. If $P_n \to f$ uniformly on $[0,1]$, prove that $Q_n \to g$ uniformly on $[a,b]$, where

$$Q_n(x) = P_n\left(\frac{x-a}{b-a}\right)$$

3. Use the binomial theorem to check that every Bernstein polynomial for $f(x) = x$ is $B_n f(x) = x$ itself!

4. Find a parametrization of the cubic Bézier curve with control points $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Now sketch the curve.

   (*Use a computer algebra package if you like!*)

5. (Hard) Show that the Bernstein polynomials for $f(x) = x^2$ are given by

$$B_n f(x) = \frac{1}{n}x + \frac{n-1}{n}x^2$$

   and thus verify explicitly that $B_n f \to f$ uniformly.