# 4 Integration

The theory of infinite series addresses how to sum infinitely many *finite* quantities. Integration, by contrast, is the business of summing infinitely many *infinitesimal* quantities. Attempts to do both have been part of mathematics for well over 2000 years, and the philosophical objections are just as old.[17] The development and increased application of calculus from the late 1600s onward spurred mathematicians to put the theory on a firmer footing, though from Newton and Leibniz it took another 150 years before Bernhard Riemann (1856) provided a thorough development of the integral.

## 4.32 The Riemann Integral

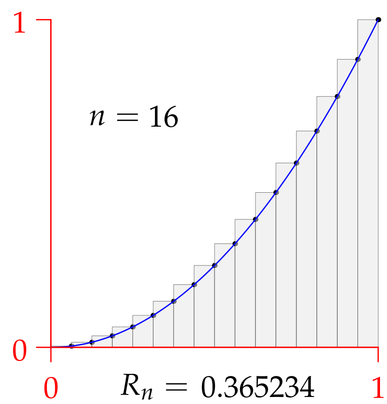
The basic idea behind Riemann integration is to approximate area using a sequence of rectangles whose *width* tends to zero. The following discussion illustrates the essential idea, which should be familiar from elementary calculus.

**Example 4.1.** Suppose $f(x) = x^2$ is defined on $[0, 1]$.

For each $n \in \mathbb{N}$, let $\Delta x = \frac{1}{n}$ and define $x_i = i\Delta x$.

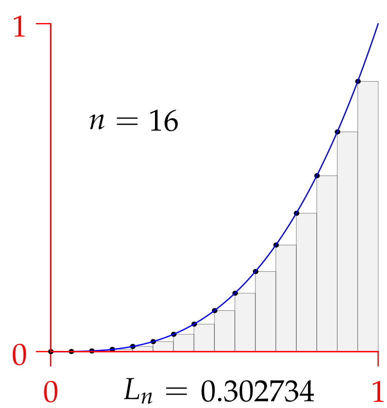
Above each *subinterval* $[x_{i-1}, x_i]$, raise a rectangle of height $f(x_i) = x_i^2$. The sum of the areas of these rectangles is the *Riemann sum with right-endpoints*[18]

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3}$$
$$= \frac{1}{3} + \frac{3n+1}{6n^2}$$




The *Riemann sum with left-endpoints* is defined similarly:

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \frac{1}{3} - \frac{3n-1}{6n^2}$$




Since $f$ is an increasing function, the area $A$ under the curve plainly satisfies

$$L_n \le A \le R_n$$

By the squeeze theorem, we conclude that $A = \frac{1}{3}$.

The example should feel convincing, though perhaps this is due to the simplicity of the function. To apply this approach to more general functions, we need to be significantly more rigorous.

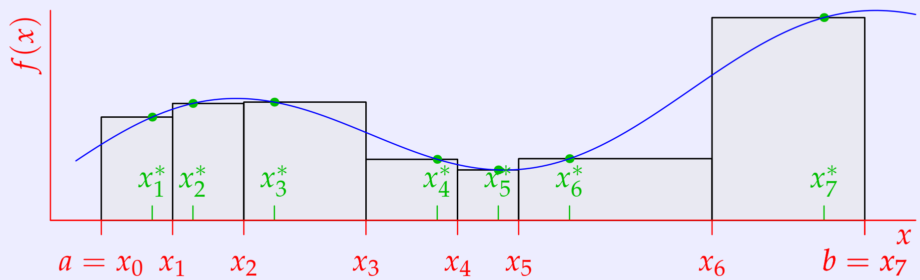
[17] Two of Zeno's ancient paradoxes are relevant here: Achilles and the Tortoise concerns a convergent infinite series, while the Arrow Paradox toys with integration by questioning whether time can be viewed as a sum of instants. Perhaps the most famous contemporary criticism comes from Bishop George Berkeley, who gave his name to the city and first UC campus: in 1734's *The Analyst*, Berkeley savaged the foundations of calculus, describing the infinitesimal increments required in Newton's theory of *fluxions* (derivatives) as merely the "ghosts of departed quantities."

[18] Recall some basic identities: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$

**Definition 4.2.** A *partition* $P = \{x_0, \ldots, x_n\}$ of an interval $[a, b]$ is a finite sequence for which

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

Choosing a *sample point* $x_i^*$ in each *subinterval* $[x_{i-1}, x_i]$ results in a *tagged partition*.
The *mesh* of the partition is $\text{mesh}(P) := \max \Delta x_i$, the width $\Delta x_i = x_i - x_{i-1}$ of the largest subinterval.
If $f : [a, b] \to \mathbb{R}$, the *Riemann sum* $\sum_{i=1}^n f(x_i^*)\,\Delta x_i$ evaluates the area of a family of $n$ rectangles, as pictured. The heights $f(x_i^*)$ and thus areas can be negative or zero.




In elementary calculus, one typically computes Riemann sums for *equally-spaced* partitions with *left*, *right* or *middle* sample points. The flexibility of tagged partitions makes applying Riemann's definition a challenge, so we instead consider two special families of rectangles.

**Definition 4.3.** Given a partition $P$ of $[a, b]$ and a bounded function $f$ on $[a, b]$, define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \qquad U(f, P) = \sum_{i=1}^n M_i\,\Delta x_i$$

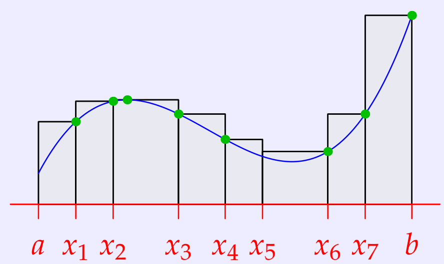
$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad L(f, P) = \sum_{i=1}^n m_i\,\Delta x_i$$

$U(f, P)$ and $L(f, P)$ are the *upper* and *lower Darboux sums* for $f$ with respect to $P$. The *upper* and *lower Darboux integrals* are

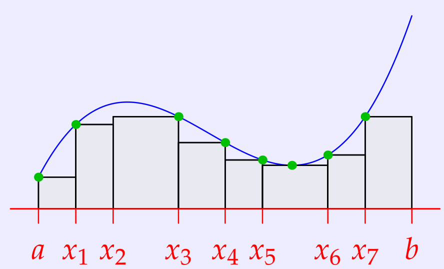
$$U(f) = \inf U(f, P) \qquad L(f) = \sup L(f, P)$$

where the supremum/infimum are taken over all partitions. Necessarily both integrals are *finite.*
We say that $f$ is *(Riemann) integrable* on $[a, b]$ if $U(f) = L(f)$. We denote this value by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,\mathrm{d}x$$



Upper Darboux sum $U(f, P)$



Lower Darboux sum $L(f, P)$

If the interval is understood or irrelevant, one often simply says that $f$ is integrable and writes $\int f$.

Intuitively, $L(f, P)$ is the sum of the areas of rectangles built on $P$ which just fit under the graph of $f$. It is also the infimum of all Riemann sums on $P$. If $f$ is discontinuous, then $L(f, P)$ need not itself be a Riemann sum, as there might not exist suitable sample points!

**Examples 4.4.** 1. We revisit Example 4.1 in this language.

Given a partition $Q = \{x_0, \ldots, x_n\}$ of $[0,1]$ and sample points $x_i^* \in [x_{i-1}, x_i]$, we compute the Riemann sum for $f(x) = x^2$

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \sum_{i=1}^{n} (x_i^*)^2 (x_i - x_{i-1})$$

Since $f$ is increasing, we have $x_{i-1}^2 \le (x_i^*)^2 \le x_i^2$ on each interval, whence

$$L(f,Q) = \sum_{i=1}^{n} (x_{i-1})^2 (x_i - x_{i-1}) \le \sum_{i=1}^{n} (x_i^*)^2 (x_i - x_{i-1}) \le \sum_{i=1}^{n} (x_i)^2 (x_i - x_{i-1}) = U(f,Q)$$

The Darboux sums are therefore the Riemann sums for left- and right-endpoints.

If we take $Q_n$ to be the partition with subintervals of equal width $\Delta x = \frac{1}{n}$, then

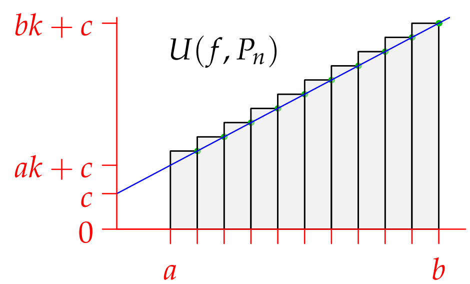
$$U(f) = \inf_P U(f,P) \le U(f,Q_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \Delta x = R_n$$

is the right Riemann sum discussed originally. Similarly $L(f) \ge L_n$. Since $L_n$ and $R_n$ both converge to $\frac{1}{3}$ as $n \to \infty$, the squeeze theorem forces

$$L_n \le L(f) \le U(f) \le R_n \implies L(f) = U(f) = \frac{1}{3}$$

Otherwise said, $f$ is integrable on $[0,1]$ with $\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$.

2. Suppose $f(x) = kx + c$ on $[a,b]$, and that $k > 0$. Take the evenly spaced partition $P_n$ where $x_i = a + \frac{b-a}{n} i$. Since $f$ is increasing, the upper Darboux sum is again the Riemann sum with right-endpoints:




$$\begin{aligned} U(f,P_n) = R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \frac{b-a}{n} \sum_{i=1}^{n} \frac{k(b-a)}{n} i + ak + c \\ &= \frac{b-a}{n} \left[ \frac{k(b-a)}{n} \cdot \frac{1}{2} n(n+1) + (ak+c)n \right] \\ &\xrightarrow[n\to\infty]{} \frac{1}{2} k(b-a)^2 + (b-a)(ak+c) = \frac{k}{2}(b^2 - a^2) + c(b-a) \end{aligned}$$

Similarly, the lower Darboux sum is the Riemann sum with left-endpoints:

$$L(f,P_n) = L_n = \frac{b-a}{n} \left[ \frac{k(b-a)}{n} \cdot \frac{1}{2} n(n-1) + (ak+c)n \right] \xrightarrow[n\to\infty]{} \frac{k}{2}(b^2 - a^2) + c(b-a)$$

As above, $L_n \le L(f) \le U(f) \le R_n$ and the squeeze theorem prove that $f$ is integrable on $[a,b]$ with $\int_a^b f = \frac{k}{2}(b^2 - a^2) + c(b-a)$.

Now we have some examples, a few remarks are in order.

*Riemann versus Darboux* Definition 4.3 is really that of the *Darboux integral.* Here is Riemann's definition: $f : [a, b] \to \mathbb{R}$ being integrable with integral $\int_a^b f$ means

$$\forall \epsilon > 0, \exists \delta \text{ such that } (\forall P, x_i^*) \text{ mesh}(P) < \delta \implies \left| \sum_{i=1}^{n} f(x_i^*)\Delta x_i - \int_a^b f \right| < \epsilon$$

This is significantly more difficult to work with, though it can be shown to be equivalent to the Darboux integral. We won't pursue Riemann's formulation further, except to observe that *if* a function is integrable and $\text{mesh}(P_n) \to 0$, then $\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$: this allows us to approximate integrals using any sample points we choose, hence why *right*-endpoints ($x_i^* = x_i$) are so common in Freshman calculus.

*Monotone Functions* Darboux sums are easy to compute for monotone functions. As in the examples, if $f$ is increasing, then each $M_i = f(x_i)$, from which $U(f, P)$ is the Riemann sum with *right-endpoints.* Similarly, $L(f, P)$ is the Riemann sum with *left-endpoints.*

*Area* If $f$ is positive and continuous,[19] the Riemann integral $\int_a^b f$ serves as a *definition* for the area under the curve $y = f(x)$. This should make intuitive sense:

1. In the second example where we have a straight line, we obtain the same value for the area by computing directly as the sum of a rectangle and a triangle!
2. For any partition $P$, the area under the curve should satisfy the inequalities

   $$L(f, P) \le \text{Area} \le U(f, P)$$

   But these are precisely the same inequalities satisfied by the integral itself!

   $$L(f, P) \le L(f) = \int_a^b f = U(f) \le U(f, P)$$

In the examples we exhibited a sequence of partitions $(P_n)$ where $U(f, P_n)$ and $L(f, P_n)$ converged to the same limit. The remaining results in this section develop some basic properties of partitions and make this limiting process rigorous.

**Definition 4.5.** If $P \subseteq Q$ are both partitions of $[a, b]$, we call $Q$ a *refinement* of $P$.

To refine a partition, we simply throw some more points in!

**Lemma 4.6.** *Suppose $f : [a, b] \to \mathbb{R}$ is bounded.*

1. *If $Q$ is a refinement of $P$ (on $[a, b]$), then*

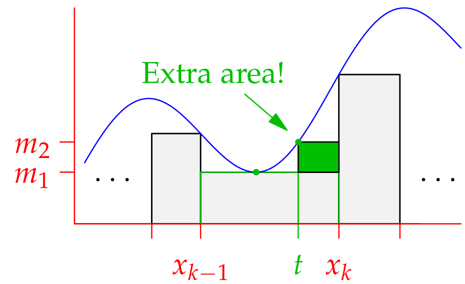
   $$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)$$

2. *For any partitions $P, Q$ of $[a, b]$, we have $L(f, P) \le U(f, Q)$.*
3. $L(f) \le U(f)$

[19] We'll see in Theorem 4.17 that every continuous function is integrable.

*Proof.* 1. We prove inductively. Suppose first that $Q = P \cup \{t\}$ contains exactly one additional point $t \in (x_{k-1}, x_k)$. Write

$$m_1 = \inf\{f(x) : x \in [x_{k-1}, t]\}$$
$$m_2 = \inf\{f(x) : x \in [t, x_{k-1}]\}$$
$$m = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \min\{m_1, m_2\}$$




The Darboux sums $L(f, P)$ and $L(f, Q)$ are identical except for the terms involving $t$. This results in extra area:

$$\begin{aligned} L(f,Q) - L(f,P) &= m_1(t - x_{k-1}) + m_2(x_k - t) - m(x_k - x_{k-1}) \\ &= (m_1 - m)(t - x_{k-1}) + (m_2 - m)(x_k - t) \geq 0 \end{aligned}$$

More generally, since a refinement $Q$ is obtained by adding *finitely many* new points, induction tells us that $P \subseteq Q \Longrightarrow L(f,P) \leq L(f,Q)$.

The argument for $U(f,Q) \leq U(f,P)$ is similar, and the middle inequality is trivial.

2. If $P$ and $Q$ are partitions, then $P \cup Q$ is a refinement of both $P$ and $Q$. By part 1,

$$L(f,P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f,Q) \tag{$*$}$$

3. This is an exercise.

■

**Theorem 4.7.** *Suppose $f : [a,b] \to \mathbb{R}$ is bounded.*

1. *(Cauchy criterion) $f$ is integrable $\iff \forall \epsilon > 0, \exists P$ such that $U(f,P) - L(f,P) < \epsilon$.*
2. *$f$ is integrable $\iff \exists (P_n)_{n \in \mathbb{N}}$ such that $U(f,P_n) - L(f,P_n) \to 0$. In such a situation, both sequences $U(f,P_n)$ and $L(f,P_n)$ converge to $\int_a^b f$.*

Part 1 is termed a 'Cauchy' criterion since it doesn't mention the integral (limit).

*Proof.* We prove the Cauchy criterion, leaving part 2 as an exercise.

($\Rightarrow$) Suppose $f$ is integrable and that $\epsilon > 0$ is given. Since $\inf U(f,Q) = \int f = \sup L(f,R)$, there exist partitions $Q, R$ such that

$$U(f,Q) < \int f + \frac{\epsilon}{2} \quad \text{and} \quad L(f,R) > \int f - \frac{\epsilon}{2}$$

Let $P = Q \cup R$ and apply ($*$): $L(f,R) \leq L(f,P) \leq U(f,P) \leq U(f,Q)$. But then

$$U(f,P) - L(f,P) \leq U(f,Q) - L(f,R) = U(f,Q) - \int f + \int f - L(f,R) < \epsilon$$

($\Leftarrow$) Assume the right hand side. For every partition, $L(f,P) \leq L(f) \leq U(f) \leq U(f,P)$. Thus

$$0 \leq U(f) - L(f) \leq U(f,P) - L(f,P) < \epsilon$$

Since this holds for all $\epsilon > 0$, we see that $U(f) = L(f)$: that is, $f$ is integrable.

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**Examples 4.8.** 1. Consider $f(x) = \sqrt{x}$ on the interval $[0, b]$. We choose a sequence of partitions $(P_n)$ that evaluate nicely when fed to this function:

$$P_n = \{x_0, \ldots, x_n\} \quad \text{where} \quad x_i = \left(\frac{i}{n}\right)^2 b$$
$$\implies \Delta x_i = x_i - x_{i-1} = \frac{b}{n^2}\left(i^2 - (i-1)^2\right) = \frac{(2i-1)b}{n^2}$$

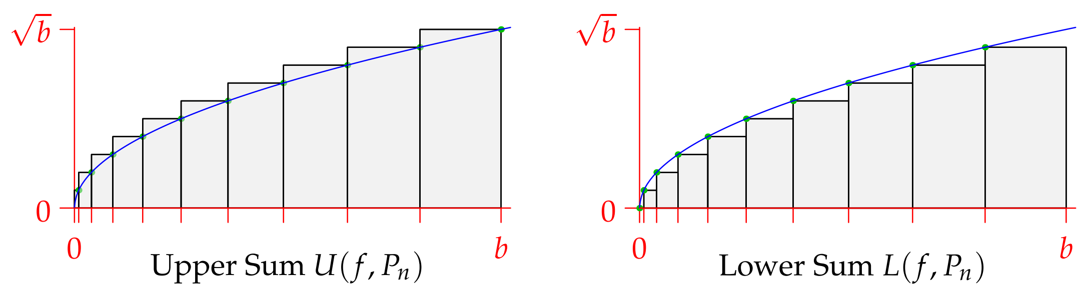
Since $f$ is increasing on $[0, b]$, we see that

$$U(f, P_n) = \sum_{i=1}^n f(x_i)\Delta x_i = \sum_{i=1}^n \frac{i\sqrt{b}}{n} \cdot \frac{(2i-1)b}{n^2} = \frac{b^{3/2}}{n^3}\sum_{i=1}^n 2i^2 - i$$
$$= \frac{b^{3/2}}{n^3}\left[\frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1)\right] \xrightarrow[n\to\infty]{} \frac{2}{3}b^{3/2}$$

Similarly

$$L(f, P_n) = \sum_{i=1}^n f(x_{i-1})\Delta x_i = \sum_{i=1}^n \frac{(i-1)\sqrt{b}}{n} \cdot \frac{(2i-1)b}{n^2} = \frac{b^{3/2}}{n^3}\sum_{i=1}^n 2i^2 - 3i + 1$$
$$= \frac{b^{3/2}}{n^3}\left[\frac{1}{3}n(n+1)(2n+1) - \frac{3}{2}n(n+1) + n\right] \xrightarrow[n\to\infty]{} \frac{2}{3}b^{3/2}$$

Since the limits are equal, we conclude that $f$ is integrable and $\int_0^b \sqrt{x}\,\mathrm{d}x = \frac{2}{3}b^{3/2}$.




2. Here is the classic example of a *non-integrable function*. Let $f : [a, b] \to \mathbb{R}$ to be the indicator function of the irrational numbers,

$$f(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

Suppose $P = \{x_0, \ldots, x_n\}$ is *any* partition of $[a, b]$. Since any interval of positive length contains both rational and irrational numbers, we see that

$$\sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1 \implies U(f, P) = \sum_{i=1}^n (x_i - x_{i-1}) = b - a \implies U(f) = b - a$$
$$\inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0 \implies L(f, P) = 0 \implies L(f) = 0$$

Since the upper and lower Darboux integrals differ, $f$ is not (Riemann) integrable.

As any freshman calculus student can attest, if you can find an anti-derivative, then the fundamental theorem of calculus (Section 4.34) makes evaluating integrals far easier. For instance, you are probably desperate to write

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{3}x^{3/2} = x^{1/2} \implies \int_0^b \sqrt{x}\,\mathrm{d}x = \frac{2}{3}x^{3/2}\Big|_0^b = \frac{2}{3}b^{3/2}$$

rather than computing Riemann/Darboux sums as in the previous example! However, in most practical situations, no easy-to-compute anti-derivative exists; the best we can do is to approximate using Riemann sums for progressively finer partitions. Thankfully computers excel at such tedious work!

**Exercises 4.32.** *Key concepts: Darboux sums/integrals, Partitions, sample points & refinements, Cauchy & sequential criteria for integrability*

1. Use partitions to find the upper and lower Darboux integrals on the interval $[0, b]$. Hence prove that the function is integrable and compute its integral.

   (a) $f(x) = x^3$ (b) $g(x) = \sqrt[3]{x}$

2. Repeat question 1 for the following two functions. You cannot simply compute Riemann sums for left and right endpoints and take limits: why not?

   (a) $h(x) = x(2 - x)$ on $[0, 2]$

   (*Hint: choose a partition with* $2n$ *points such that* $x_n = 1$ *and observe that* $h(2 - x) = h(x)$)

   (b) On the interval $[0, 3]$, let $k(x) = \begin{cases} 2x & \text{if } x \le 1 \\ 5 - x & \text{if } x > 1 \end{cases}$

   (*Hint: this time try a partition with* $3n$ *points*)

3. Let $f(x) = x$ for rational $x$ and $f(x) = 0$ for irrational $x$. Calculate the upper and lower Darboux integrals for $f$ on the interval $[0, b]$. Is $f$ integrable on $[0, b]$?

4. Prove part 3 of Lemma 4.6: $L(f) \le U(f)$.

5. Prove part 2 of Theorem 4.7.

   $$f \text{ is integrable} \iff \exists (P_n)_{n\in\mathbb{N}} \text{ such that } \lim_{n\to\infty} \big(U(f, P_n) - L(f, P_n)\big) = 0$$

   Moreover, prove that both $U(f, P_n)$ and $L(f, P_n)$ converge to $\int f$.

6. (a) Reread Definition 4.3. What happens if we allow $f : [a, b] \to \mathbb{R}$ to be *unbounded*?

   (b) (Hard) Read "*Riemann versus Darboux*" on page 73. Explain why being *Riemann* integrable also forces $f$ to be bounded.

   (c) (Hard) Explain the observation that $L(f, P)$ is the infimum of the set of all Riemann sums on $P$.

7. (If you like coding) Write a short program to estimate $\int_a^b f(x)\,\mathrm{d}x$ using Riemann sums. This can be very simple (equal partitions with right endpoints), or more complex (random partition and sample points given a mesh). Apply your program to estimate $\int_0^5 \sin(x^2 e^{-\sqrt{x}})\,\mathrm{d}x$.

## 4.33 Properties of the Riemann Integral

The rough take-away of this long section is that everything you think is integrable probably is! Examples will be few, since we have not established many explicit values for integrals.

**Theorem 4.9 (Linearity).** *If $f, g$ are integrable and $k, l$ are constant, then $kf + lg$ is integrable and*

$$\int kf + lg = k\int f + l\int g$$

**Example 4.10.** Thanks to examples in the previous section, we can now calculate, e.g.,

$$\int_0^2 5x^3 - 3\sqrt{x}\,\mathrm{d}x = 5\cdot\frac{1}{4}\cdot 2^4 - 3\cdot\frac{2}{3}\cdot 2^{3/2} = 20 - 4\sqrt{2}$$

*Proof.* Suppose $\epsilon > 0$ is given. By the Cauchy criterion (Theorem 4.7, part 1), there exist partitions $R, S$ such that

$$U(f,R) - L(f,R) < \frac{\epsilon}{2} \quad \text{and} \quad U(g,S) - L(g,S) < \frac{\epsilon}{2}$$

If $P = R \cup S$, then both inequalities are satisfied by $P$ (Lemma 4.6). On each subinterval,

$$\inf f(x) + \inf g(x) \le \inf\big(f(x) + g(x)\big) \quad \text{and} \quad \sup\big(f(x) + g(x)\big) \le \sup f(x) + \sup g(x)$$

since the individual suprema/infima could be 'evaluated' at different places. Thus

$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$$

whence $U(f+g,P) - L(f+g,P) < \epsilon$ and $f+g$ is integrable. Moreover,

$$\int (f+g) - \int f - \int g \le \Big(U(f,P) - \int f\Big) + \Big(U(g,P) - \int g\Big) < \epsilon$$

Using lower Darboux integrals similarly obtains the other half of the inequality

$$-\epsilon < \int (f+g) - \int f - \int g < \epsilon$$

Since this holds for all $\epsilon > 0$, we conclude that $\int (f+g) = \int f + \int g$.
That $kf$ is integrable with $\int kf = k\int f$ is an exercise. Put these together for the result. ∎

**Corollary 4.11 (Changing endvalues).** *Suppose $f$ is integrable on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ on $(a,b)$. Then $g$ is also integrable on $[a,b]$ and $\int_a^b g = \int_a^b f$.*

**Definition 4.12 (Integration on an open interval).** A *bounded* function $g : (a,b) \to \mathbb{R}$ is *integrable* if it has an integrable extension $f : [a,b] \to \mathbb{R}$ where $f(x) = g(x)$ on $(a,b)$. In such a case, we define $\int_a^b g := \int_a^b f$.

The Corollary (its proof is an exercise) shows that the choice of extension is irrelevant.

**Theorem 4.13 (Basic integral comparisons).** *Suppose $f$ and $g$ are integrable on $[a,b]$. Then:*

1. $f(x) \le g(x) \implies \int f \le \int g$
2. $m \le f(x) \le M \implies m(b-a) \le \int_a^b f \le M(b-a)$
3. $fg$ *is integrable.*
4. $|f|$ *is integrable and* $\left|\int f\right| \le \int |f|$
5. $\max(f,g)$ *and* $\min(f,g)$ *are both integrable.*

Part 3 is *not* integration by parts since it doesn't tell us how $\int fg$ relates to $\int f$ and $\int g$!

*Proof.* 1. Since $g-f$ is positive and integrable, $L(g-f,P) \ge 0$ for all partitions $P$. But then

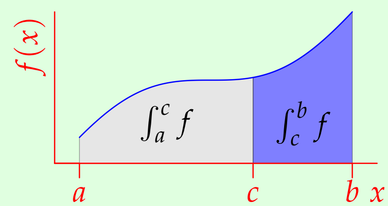
$$0 \le \inf L(g-f,P) = L(g-f) = \int g - f = \int g - \int f$$

2. Apply part 1 twice.
3. This is an exercise.
4. The integrability is an exercise. For the comparison, apply part 1 to $-|f| \le f \le |f|$.
5. Use $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$, etc., together with the previous parts.

■

**Theorem 4.14 (Domain splitting).** *Suppose $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. If $f$ is integrable on both $[a,c]$ and $[c,b]$, then it is integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

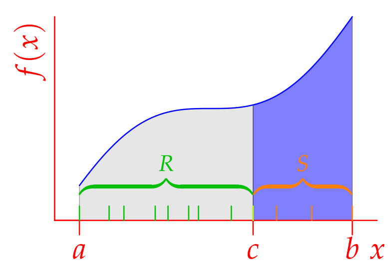



In light of this result, it is conventional to allow integral limits to be reversed: if $a < b$, then

$$\int_b^a f := -\int_a^b f \quad \text{is consistent with} \quad \int_a^a f = 0$$

*Proof.* Let $\epsilon > 0$ be given, then $\exists R, S$ partitions of $[a,c]$, $[c,b]$ such that

$$U(f,R) - L(f,R) < \frac{\epsilon}{2}, \qquad U(f,S) - L(f,S) < \frac{\epsilon}{2}$$




Choose $P = R \cup S$ to partition $[a,b]$, then

$$U(f,P) - L(f,P) = U(f,R) + U(f,S) - L(f,R) - L(f,S) < \epsilon$$

Moreover

$$\int_a^b f - \int_a^c f - \int_c^b f \le U(f,P) - L(f,R) - L(f,S) = U(f,P) - L(f,P) < \epsilon$$

Showing that this expression is greater than $-\epsilon$ is similar.

■

**Example 4.15.** If $f(x) = \sqrt{x}$ on $[0,1]$ and $f(x) = 1$ on $[1,2]$, then

$$\int_0^2 f = \int_0^1 \sqrt{x}\,\mathrm{d}x + \int_1^2 1\,\mathrm{d}x = \frac{2}{3} + 1 = \frac{5}{3}$$

**Monotonic & Continuous Functions** We establish the integrability of two large classes of functions.

**Definition 4.16.** A function $f : [a,b] \to \mathbb{R}$ is:

*Monotonic* if it is either *increasing* ($x < y \implies f(x) \le f(y)$) or *decreasing*.

*Piecewise monotonic* if there is a partition $P = \{x_0, \ldots, x_n\}$ (finite!) of $[a,b]$ such that $f$ is monotonic on each open subinterval $(x_{k-1}, x_k)$.

*Piecewise continuous* if there is a partition such that $f$ is *uniformly continuous* on each $(x_{k-1}, x_k)$.

**Theorem 4.17.** *If $f$ is* monotonic *or* continuous *on $[a,b]$, then it is integrable.*

**Examples 4.18.** 1. Since sine is continuous, we can approximate via a sequence of Riemann sums

$$\int_0^\pi \sin x\,\mathrm{d}x = \frac{\pi}{n} \lim_{n\to\infty} \sum_{i=1}^n \sin\frac{\pi i}{n}$$

*Evaluating* this limit is another matter entirely, one best handled in the next section...

2. Similarly, $e^{\sqrt{x}}$ is integrable and therefore may be approximated via Riemann sums:

$$\int_0^1 e^{\sqrt{x}}\,\mathrm{d}x = \frac{1}{n} \lim_{n\to\infty} \sum_{i=1}^n \exp\sqrt{\frac{i}{n}} = \lim_{n\to\infty} \sum_{j=1}^n \frac{2j-1}{n} \exp\frac{j}{n}$$

Both sums use right endpoints: the first has equal subintervals, while the second is analogous to Example 4.8.1. These limits would typically be estimated using a computer.

*Proof.* Since $[a,b]$ is closed and bounded, a continuous function $f$ is *uniformly* so. Let $\epsilon > 0$ be given:

$$\exists \delta > 0 \text{ such that } \forall x, y \in [a,b],\ |x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$$

Let $P$ be a partition with mesh $P < \delta$. Since $f$ attains its bounds on each $[x_{i-1}, x_i]$,

$$\exists x_i^*, y_i^* \in [x_{i-1}, x_i] \quad \text{such that} \quad M_i - m_i = f(x_i^*) - f(y_i^*) < \frac{\epsilon}{b-a}$$

from which

$$U(f,P) - L(f,P) < \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) = \epsilon$$

The monotonicity argument is an exercise. ∎

Combining the proof with Definition 4.12: every *uniformly continuous* $f : (a,b) \to \mathbb{R}$ is integrable.

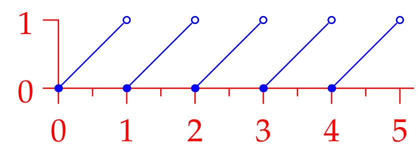
**Corollary 4.19.** *Piecewise continuous and* ***bounded*** *piecewise monotonic functions are integrable.*

*Proof.* If $f$ is piecewise continuous, then the restriction of $f$ to $(x_{k-1}, x_k)$ has a continuous extension $g_k : [x_{k-1}, x_k] \to \mathbb{R}$; this is integrable by Theorem 4.17. By Corollary 4.11, $f$ is integrable on $[x_{k-1}, x_k]$ with $\int_{x_{k-1}}^{x_k} f = \int_{x_{k-1}}^{x_k} g_k$. Theorem 4.14 ($n-1$ times!) finishes things off:

$$\int_a^b f = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f$$

The argument for piecewise monotonicity is similar. ∎

**Example 4.20.** The 'fractional part' function $f(x) = x - \lfloor x \rfloor$ is both piecewise continuous and piecewise monotone on any bounded interval. It is therefore integrable on any such interval.



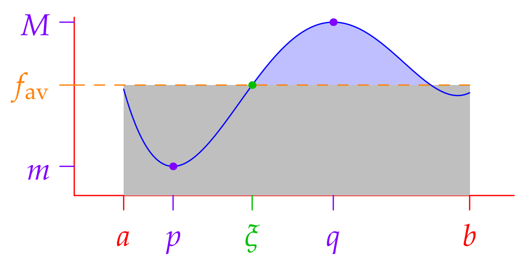

For a final corollary, here is one more incarnation of the intermediate value theorem.

**Corollary 4.21 (IVT for integrals).** *If $f$ is continuous on $[a, b]$, then $\exists \xi \in (a, b)$ for which*

$$f(\xi) = \frac{1}{b-a} \int_a^b f$$

*Proof.* Since $f$ is continuous, it is integrable on $[a, b]$. By the extreme value theorem it is also bounded and attains its bounds: $\exists p, q \in [a, b]$ such that

$$f(p) := \inf_{x \in [a,b]} f(x), \qquad f(q) = \sup_{x \in [a,b]} f(x)$$




Applying Theorem 4.13, part 2, with $m = f(p)$ and $M = f(q)$, we see that

$$(b-a)f(p) \le \int_a^b f \le (b-a)f(q)$$

Divide by $b - a$ and apply the usual intermediate value theorem for $f$ to see that the required $\xi$ exists between $p$ and $q$. ∎

In the picture, when $f$ is positive and continuous, the grey area equals that under the curve; imagine levelling off the blue hill with a bulldozer... The notation $f_{\text{av}} = \frac{1}{b-a} \int_a^b f$ indicates the average value of $f$ on $[a, b]$: to see why this interpretation is sensible, take a sequence of Riemann sums on equally-spaced partitions $P_n$ to see that

$$\frac{1}{b-a} \int_a^b f = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty} \frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

is the limit of a sequence of *averages* of equally-spaced samples $f(x_i^*)$.

**What can/cannot be integrated?**

We now know a great many examples of integrable functions:

- Piecewise continuous & monotonic functions are integrable.
- Linear combinations, products, absolute values, maximums and minimums of (already) integrable functions.

By contrast, we've only seen one non-integrable function (Example 4.8.2). After so many positive integrability conditions, it is reasonable to ask precisely which functions are Riemann integrable. Here is the answer, though it is quite tricky to understand.

**Theorem 4.22 (Lebesgue).** *Suppose $f : [a, b] \to \mathbb{R}$ is bounded. Then*

$$f \text{ is Riemann integrable} \iff \text{it is continuous except on a set of measure zero}$$

Naïvely, the *measure* of a set is the sum of the lengths of its maximal subintervals, though unfortunately this doesn't make for a very useful definition.[20] Any countable subset has measure zero, so Lebesgue's result is almost as if we can extend Corollary 4.19 to allow for infinite sums. For instance, Exercise 1.17.8 describes a function which is continuous only on the irrationals: it is thus Riemann integrable (indeed $\int_a^b f = 0$ for any $a < b$). There are also uncountable sets with measure zero such as Cantor's middle-third set $\mathcal{C}$: the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

is continuous except on $\mathcal{C}$ and therefore Riemann integrable; again $\int_0^1 f(x)\,\mathrm{d}x = 0$.

**Exercises 4.33.** *Key concepts: Linear combinations, products, etc., of integrable functions are integrable, Continuous and monotone functions are integrable, Integrability on open intervals*

1. Explain why $\int_0^{2\pi} x^2 \sin^8(e^x)\,\mathrm{d}x \le \frac{8}{3}\pi^3$
2. If $f$ is integrable on $[a, b]$ prove that it is integrable on any interval $[c, d] \subseteq [a, b]$.
3. We complete the proof of Theorem 4.9 (linearity of integration).
   (a) Suppose $k > 0$, let $A \subseteq \mathbb{R}$ and define $kA := \{kx : x \in A\}$. Prove that $\sup kA = k \sup A$ and $\inf kA = k \inf A$.
   (b) If $k > 0$ prove that $kf$ is integrable on any interval and that $\int kf = k \int f$.
   (c) How should you modify your argument if $k < 0$?

[20]Formally, the *length* of an open interval $(a, b)$ is $b - a$ and a set $A \subseteq \mathbb{R}$ has *measure zero* if

$$\forall \epsilon > 0,\ \exists \text{ open intervals } I_n \text{ such that } A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{i=1}^{\infty} \text{length}(I_n) < \epsilon$$

More generally, the *Lebesgue measure* of a set (subject to a technical condition) is the infimum of the sum of the lengths of any countable collection of open covering intervals. *Measure theory* is properly a matter for graduate study. Surprisingly, there exist sets with positive measure that contain no subintervals, and even sets which are non-measurable!

4. Give an example of an integrable but *discontinuous* function on a closed bounded interval $[a,b]$ for which the conclusion of the Intermediate Value Theorem for Integrals is *false.*

5. Use Darboux sums to compute the value of the integral $\int_{1/2}^{15/2} x - \lfloor x \rfloor \, dx$ (Example 4.20).

6. We prove and extend Corollary 4.11. Suppose $f$ is integrable on $[a,b]$.
   (a) If $g : [a,b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ for all $x \in (a,b)$, prove that $g$ is integrable and $\int_a^b g = \int_a^b f$.
   (*Hint: consider* $h = f - g$ *and show that* $\int h = 0$)
   (b) Now suppose $g : [a,b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ for all $x \in [a,b]$ except at finitely many points. Prove that $g$ is integrable and $\int_a^b g = \int_a^b f$.

7. Show that an increasing function on $[a,b]$ is integrable and thus complete Theorem 4.17.
   (*Hint: Choose a partition with* mesh $P < \frac{\epsilon}{f(b)-f(a)}$)

8. Suppose $f$ and $g$ are integrable on $[a,b]$.
   (a) Define $h(x) = \big(f(x)\big)^2$. We know:
   - $f$ is bounded: $\exists K$ such that $|f(x)| \le K$ on $[a,b]$.
   - Given $\epsilon > 0$, $\exists P$ such that $U(f,P) - L(f,P) < \frac{\epsilon}{2K}$. For each subinterval $[x_{i-1}, x_i]$, let

   $$M_i = \sup f(x), \qquad m_i = \inf f(x), \qquad \overline{M}_i = \sup h(x), \qquad \overline{m}_i = \inf h(x)$$

   Prove that $\overline{M}_i - \overline{m}_i \le 2(M_i - m_i)K$. Hence conclude that $h$ is integrable.
   (b) Prove that $fg$ is integrable.
   (*Hint:* $fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$)
   (c) Prove that $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$ for any partition $P$. Hence conclude that $|f|$ is integrable.

   (*One can extend these arguments to show that if* $j$ *is continuous, then* $j \circ f$ *is integrable. Parts (a) and (c) correspond, respectively, to* $j(x) = x^2$ *and* $j(x) = |x|$.)

9. (Hard) Let $f(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } \sin\frac{1}{x} > 0 \\ -x & \text{if } x \neq 0 \text{ and } \sin\frac{1}{x} < 0 \\ 0 & \text{if } x = 0 \end{cases}$
   (a) Show that $f$ is not piecewise continuous on $[0,1]$.
   (b) Show that $f$ is not piecewise monotonic on $[0,1]$.
   (c) Show that $f$ is integrable on $[0,1]$.
   (*Hint: given* $\epsilon$*, hunt for a suitable partition to make* $U(f,P) - L(f,P) < \epsilon$ *by considering* $[0, x_1]$ *differently to the other subintervals*)
   (d) Make a similar argument which proves that $g = \sin\frac{1}{x}$ is integrable on $(0,1]$.
   (*Hint: Show that* $g$ *has an integrable extension on* $[0,1]$)

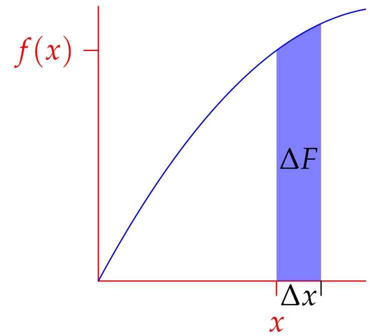
## 4.34 The Fundamental Theorem of Calculus

The key result linking integration and differentiation is usually presented in two parts. While there are significant subtleties, the rough statements are as follows (we follow the traditional numbering):

*Part I* Differentiation reverses integration: $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$

*Part II* Integration reverses differentiation: $\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$

These facts seemed intuitively obvious to early practitioners of calculus. Given a continuous positive function $f$:




- Let $F(x)$ denote the area under $y = f(x)$ between 0 and $x$.
- A small increase $\Delta x$ results in the area increasing by $\Delta F$.
- $\Delta F \approx f(x)\Delta x$ is approximately the area of a rectangle, whence $\frac{\Delta F}{\Delta x} \approx f(x)$. This is part I.
- $F(b) - F(a) \approx \sum \Delta F_i \approx \sum f(x_i)\Delta x_i$. Since $F' = f$, this is part II.

When Leibniz introduced the symbols $\int$ and d in the late 1600s, it was partly to reflect the fundamental theorem.[21] If you're happy with non-rigorous notions of limit, rate of change, area, and (infinite) sums, the above is all you need!

Of course we are very much concerned with the details: What must we assume about $f$ and $F$, and how are these properties used in the proof?

**Theorem 4.23 (FTC, part I).** *Suppose $f$ is integrable on $[a,b]$. For any $x \in [a,b]$, define*

$$F(x) := \int_a^x f(t)\,\mathrm{d}t$$

*Then:*

1. *$F$ is uniformly continuous on $[a,b]$;*
2. *If $f$ is continuous at $c \in [a,b]$, then $F$ is differentiable[22] at $c$ with $F'(c) = f(c)$.*

Compare this with the naïve version above where we assumed $f$ was continuous. We now require only the *integrability* of $f$, and its continuity at *one point* for the full result.

[21] $\int$ is a stylized S for *sum*, while d stands for *difference*. Given a sequence $F = (F_0, F_1, F_2, \ldots, F_n)$, construct a new sequence of *differences*

$$\mathrm{d}F = (F_1 - F_0, F_2 - F_1, \ldots, F_n - F_{n-1})$$

which can then be summed:

$$\int \mathrm{d}F = (F_1 - F_0) + (F_2 - F_1) + \cdots (F_n - F_{n-1}) = F_n - F_0 \qquad (*)$$

Viewing a function as an 'infinite sequence' of values spaced along an interval, d$F$ becomes a sequence of *infinitesimals* and $(*)$ is essentially the fundamental theorem: $\int \mathrm{d}F = F(b) - F(a)$. It is the concept of **function** that is suspect here, not the essential relationship between sums and differences.

[22] Strictly: if $c = a$, then $F$ is *right*-differentiable, etc.

**Examples 4.24.** Examples in every elementary calculus course.

1. Since $f(x) = \sin^2(x^3 - 7)$ is continuous on any bounded interval, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_4^x \sin^2(t^3-7)\,\mathrm{d}t = \sin^2(x^3-7)$$

   If one follows Theorem 4.14 and its conventions, then this is valid for all $x \in \mathbb{R}$.

2. The chain rule permits more complicated examples. For instance: $f(t) = \sin\sqrt{t}$ is continuous on its domain $[0,\infty)$ and $y(x) = x^2+3$ has range $[3,\infty) \subseteq \mathrm{dom}(f)$, whence

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2+3} \sin\sqrt{t}\,\mathrm{d}t = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \sin\sqrt{t}\,\mathrm{d}t = 2x\sin\sqrt{x^2+3}$$

3. For a final positive example, we consider when

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sin x}^{e^x} \tan(t^2)\,\mathrm{d}t = e^x\tan(e^{2x}) - \cos x\tan(\sin^2 x)$$

   Makes sense. To evaluate this, first choose any constant $a$ and write

$$\int_{\sin x}^{e^x} = \int_a^{e^x} + \int_{\sin x}^a = \int_a^{e^x} - \int_a^{\sin x}$$

   before differentiating. This is valid provided $\sin x$, $e^x$ and $a$ all lie in the same subinterval of

$$\mathrm{dom}\tan(t^2) = \mathbb{R}\setminus\{\pm\sqrt{\tfrac{\pi}{2}}, \pm\sqrt{\tfrac{3\pi}{2}}, \pm\sqrt{\tfrac{5\pi}{2}}, \ldots\}$$

   Since $|\sin x| \le 1 < \sqrt{\frac{\pi}{2}}$, this requires

$$\left|e^{2x}\right| < \frac{\pi}{2} \iff x < \frac{1}{2}\ln\frac{\pi}{2}$$

   Choosing $a = 1$ would certainly suffice.

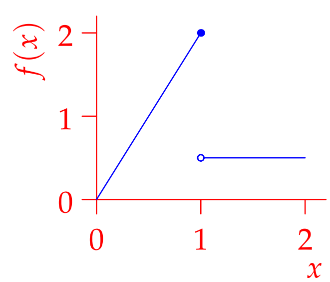
4. Now consider why the theorem requires continuity. The piecewise continuous function

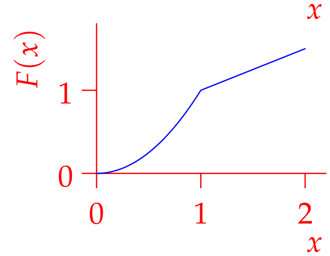
$$f : [0,2] \to \mathbb{R} : x \mapsto \begin{cases} 2x & \text{if } x \le 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases}$$

   has a jump discontinuity at $x = 1$. We can still compute

$$F(x) = \begin{cases} \int_0^x 2t\,\mathrm{d}t = x^2 & \text{if } x \le 1 \\ \int_0^1 2t\,\mathrm{d}t + \int_1^x \frac{1}{2}\mathrm{d}t = \frac{1}{2}(x+1) & \text{if } x > 1 \end{cases}$$

   This is continuous, indeed uniformly so! However the discontinuity of $f$ results in $F$ having a *corner* and thus being *non-differentiable* at $x = 1$. Indeed $F'(x) = f(x)$ whenever $x \neq 1$: that is, at all values of $x$ where $f$ is continuous.



**Proving FTC I** Neither half of the theorem is particularly difficult once you write down what you know and what you need to prove. Here are the key ingredients:

1. Uniform continuity for $F$ means we must control the size of

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \right| = \left| \int_x^y f(t)\,\mathrm{d}t \right| \leq \int_x^y |f(t)|\,\mathrm{d}t$$

   But the boundedness of $f$ allows us to control this last integral...

2. $F'(c) = f(c)$ means showing that $\lim\limits_{x\to c} \frac{F(x)-F(c)}{x-c} = f(c)$, which means controlling the size of

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \left| \frac{1}{x-c} \int_c^x f(t)\,\mathrm{d}t - f(c) \right|$$

   The trick here will is to bring the *constant* $f(c)$ inside the integral as $\frac{1}{x-c}\int_c^x f(c)\,\mathrm{d}t$ so that the above becomes $\frac{1}{|x-c|}\int_c^x |f(t) - f(c)|\,\mathrm{d}t$. This may now be controlled via the continuity of $f$...

*Proof.* 1. Since $f$ is integrable, it is bounded: $\exists M > 0$ such that $|f(x)| \leq M$ for all $x$.

Let $\epsilon > 0$ be given and define $\delta = \frac{\epsilon}{M}$. Then, for any $x, y \in [a, b]$,

$$\begin{aligned} 0 < y - x < \delta \implies |F(y) - F(x)| &= \left| \int_x^y f(t)\,\mathrm{d}t \right| \leq \int_x^y |f(t)|\,\mathrm{d}t && \text{(Theorem 4.13, part 4)} \\ &\leq M(y - x) && \text{(Theorem 4.13, part 2)} \\ &< M\delta = \epsilon \end{aligned}$$

We conclude that $F$ is uniformly continuous on $[a, b]$.

2. Let $\epsilon > 0$ be given. Since $f$ is continuous at $c$, $\exists \delta > 0$ such that, for all $t \in [a, b]$,

$$|t - c| < \delta \implies |f(t) - f(c)| < \frac{\epsilon}{2}$$

Now for all $x \in [a, b]$ (except $c$),

$$\begin{aligned} 0 < |x - c| < \delta \implies \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \frac{1}{x-c} \int_c^x f(t) - f(c)\,\mathrm{d}t \right| && \text{(Theorem 4.9)} \\ &\leq \frac{1}{|x-c|} \int_c^x |f(t) - f(c)|\,\mathrm{d}t && \text{(Theorem 4.13)} \\ &\leq \frac{1}{|x-c|} \frac{\epsilon}{2} |x - c| = \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Clearly $\lim\limits_{x\to c} \frac{F(x)-F(c)}{x-c} = f(c)$. Otherwise said, $F$ is differentiable at $c$ with $F'(c) = f(c)$. ∎

**The Fundamental Theorem, part II** As with part I, the *formulaic* part of the result should be familiar, though we are more interested in the assumptions and where they are needed.

**Theorem 4.25 (FTC, part II).** *Suppose $g$ is continuous on $[a,b]$, differentiable on $(a,b)$, and moreover that $g'$ is integrable on $(a,b)$ (recall Definition 4.12). Then,*

$$\int_a^b g' = g(b) - g(a)$$

Part II is often expressed in terms of *anti-derivatives*: $F$ being an anti-derivative of $f$ if $F' = f$. Combined with FTC, part I, we recover the familiar '$+c$' result and a simpler version of the fundamental theorem often seen in elementary calculus.

**Corollary 4.26.** *Let $f$ be continuous on $[a,b]$.*

- *If $F$ is an anti-derivative of $f$, then $\int_a^b f = F(b) - F(a)$.*
- *Every anti-derivative of $f$ has the form $F(x) = \int_a^x f(t)\,\mathrm{d}t + c$ for some constant $c$.*

**Examples 4.27.** Again, basic examples should be familiar.

1. Plainly $g(x) = x^2 + 2x^{3/2}$ is continuous on $[1,4]$ and differentiable on $(1,4)$ with derivative $g'(x) = 2x + 3\sqrt{x}$; this last is continuous (and thus integrable) on $(1,4)$. We conclude that

$$\int_1^4 2x + 3\sqrt{x}\,\mathrm{d}x = x^2 + 2x^{3/2}\Big|_1^4 = (16+16) - (1+2) = 29$$

2. If $g(x) = \sin(3x^2)$, then $g'(x) = 6x\cos(3x^2)$. Certainly $g$ satisfies the hypotheses of the theorem on any bounded interval $[a,b]$. We conclude

$$\int_a^b 6x\cos(3x^2)\,\mathrm{d}x = \sin(3b^2) - \sin(3a^2)$$

Moreover, every anti-derivative of $f(x) = 6x\cos(3x^2)$ has the form $F(x) = \sin(3x^2) + c$.

3. Recall Example 4.24.4 where the discontinuity of $f$ at $x = 1$ led to the *non-differentiability* of $F(x) = \int_0^x f(t)\,\mathrm{d}t$. The function $F$ therefore fails the *hypotheses* of FTC II on the interval $[0,2]$.

It almost, however, satisfies the *conclusions* of FTC II, though this is somewhat tautological given the definition of $F$: except at $x = 1$, $F$ is certainly an anti-derivative of $f$, and moreover $\int_0^2 f(x)\,\mathrm{d}x = F(2) - F(0)$.

In case you're worried that this makes the theorem trivial, note that other anti-derivatives $\hat{F}$ of $f$ exist (except at $x = 1$) which fail to satisfy the conclusion. For instance

$$\hat{F}(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \frac{1}{2}x & \text{if } x > 1 \end{cases} \implies \hat{F}(2) - \hat{F}(0) = 1 \neq \frac{3}{2} = \int_0^2 f(x)\,\mathrm{d}x$$

**Proving FTC II** Exercise 10 offers a relatively easy proof when $g' = f$ is continuous. For the real McCoy, we can only rely on the *integrability* of $g'$: the trick is to use the mean value theorem to write $g(b) - g(a)$ as a Riemann sum over a suitable partition.

*Proof.* Suppose $\epsilon > 0$ is given. Since $g'$ is integrable, we may choose some partition $P$ satisfying $U(g', P) - L(g', P) < \epsilon$. Since $g$ satisfies the mean value theorem on each subinterval,

$$\exists \xi_i \in (x_{i-1}, x_i) \quad \text{such that} \quad g'(\xi_i) = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}$$

from which

$$g(b) - g(a) = \sum_{i=1}^{n} g(x_i) - g(x_{i-1}) = \sum_{i=1}^{n} g'(\xi_i)(x_i - x_{i-1})$$

This is a Riemann sum for $g'$ associated to the partition $P$. Since the upper and lower Darboux sums are the supremum and infimum of these, we see that

$$L(g', P) \leq g(b) - g(a) \leq U(g', P)$$

However $\int_a^b g'$ satisfies the same inequality: $L(g', P) \leq \int_a^b g' \leq U(g', P)$. Since these inequalities hold for all $\epsilon > 0$, we conclude that $\int_a^b g' = g(b) - g(a)$. ∎

While we certainly used the integrability of $g'$ in the proof, it might seem strange that we assumed it at all: shouldn't every derivative be integrable? Perhaps surprisingly, the answer is no! If you want a challenge, look up the *Volterra function*, which is differentiable everywhere but whose derivative is non-integrable!

### The Rules of Integration

If one wants to *evaluate* an integral, rather than merely show it exists, there are really only two options:

1. Evaluate Riemann sums and take limits. This is often difficult if not impossible to do explicitly.
2. Use FTC II. The problem now becomes the finding of *anti-derivatives*, for which the core method is essentially *guess and differentiate.* To obtain general rules, we can attempt to reverse the rules of differentiation.

**Integration by Parts** Recall the *product rule*: the product $g = uv$ of two differentiable functions is differentiable with $g' = u'v + uv'$. Now apply Theorems 4.9, 4.13 and FTC II.

**Corollary 4.28 (Integration by Parts).** *Suppose $u, v$ are continuous on $[a, b]$, differentiable on $(a, b)$, and that $u', v'$ are integrable on $(a, b)$. Then*

$$\int_a^b u'(x)v(x)\mathrm{d}x = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\mathrm{d}x$$

This is significantly less useful than the product rule since it merely transforms the integral of one product into the integral of another.

**Examples 4.29.** With practice, there is no need to explicitly state $u$ and $v$.

1. Let $u(x) = x$ and $v'(x) = \cos x$. Then $u'(x) = 1$ and $v(x) = \sin x$. These certainly satisfy the hypotheses. We conclude

$$\begin{aligned}\int_0^{\pi/2} x\cos x \,\mathrm{d}x &= [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \,\mathrm{d}x = \frac{\pi}{2}\sin\frac{\pi}{2} - 0 - [-\cos x]_0^{\pi/2} \\ &= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1\end{aligned}$$

2. Let $u(x) = \ln x$ and $v'(x) = 1$. Then $u'(x) = \frac{1}{x}$ and $v(x) = x$, whence

$$\begin{aligned}\int_e^{e^2} \ln x \,\mathrm{d}x &= [x\ln x]_e^{e^2} - \int_e^{e^2} \frac{x}{x}\,\mathrm{d}x = e^2\ln e^2 - e\ln e - [x]_e^{e^2} \\ &= 2e^2 - e - e^2 + e = e^2\end{aligned}$$

**Change of Variables/Substitution** We now turn our attention to the *chain rule*. If $g(x) = F\big(u(x)\big)$, where $F$ and $u$ are differentiable, then $g$ is differentiable with

$$g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = F'\big(u(x)\big)u'(x)$$

Now integrate both sides; the only issue is what assumptions are needed to invoke FTC II.

**Theorem 4.30 (Substitution Rule).** *Suppose $u : [a,b] \to \mathbb{R}$ and $f : \operatorname{range}(u) \to \mathbb{R}$ are continuous. Suppose also that $u$ is differentiable on $(a,b)$ with integrable derivative $u'$. Then*

$$\int_a^b f\big(u(x)\big)\, u'(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u$$

This is the famous '$u$-sub'/change-of-variables formula from elementary calculus.

*Proof.* We leave as an exercise the verification that both integrals exist. By the intermediate and extreme value theorems, $\operatorname{range}(u)$ is a closed bounded interval. Assume $\operatorname{range}(u)$ has positive length for otherwise both integrals are trivially zero.

Choose any $c \in \operatorname{range}(u)$ and define

$$F : \operatorname{range}(u) \to \mathbb{R} \ \text{ by } \ F(v) := \int_c^v f(t)\,\mathrm{d}t$$

Since $f$ is continuous, by FTC I says that $F$ is differentiable with $F'(u) = f(u)$. But now

$$\begin{aligned}\int_a^b f\big(u(x)\big)u'(x)\,\mathrm{d}x &= \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}x}F\big(u(x)\big)\right]\,\mathrm{d}x && \text{(chain rule)} \\ &= F\big(u(b)\big) - F\big(u(a)\big) && \text{(FTC II)} \\ &= \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u\end{aligned}$$

■

**Examples 4.31.** Successfully applying the substitution rule can require significant creativity.[23]

1. To evaluate $\int_0^{\sqrt{\pi}} 2x \sin x^2 \, dx$, we consider the substitution $u(x) = x^2$ defined on $[0, \sqrt{\pi}]$. Certainly $u$ is continuous; moreover its derivative $u'(x) = 2x$ is integrable on $(0, \sqrt{\pi})$. Finally $f(u) = \sin u$ is continuous on $\operatorname{range}(u) = [0, \pi]$. The hypotheses are satisfied, whence

$$\begin{aligned}\int_0^{\sqrt{\pi}} 2x \sin x^2 \, dx &= \int_0^{\sqrt{\pi}} f\big(u(x)\big)u'(x) \, dx = \int_{u(0)}^{u(\pi)} f(u) \, du = \int_0^{\pi} \sin u \, du \\ &= -\cos u \Big|_0^{\pi} = 2\end{aligned}$$

2. For the following integral, a simple factorization suggests the substitution $u(x) = x^2 - 2$. Plainly $u : [\sqrt{2}, \sqrt{3}] \to [0, 1]$ and $u'(x) = 2x$ is integrable. Moreover, $f(u) = \frac{1}{u^2+1}$ is continuous on $\operatorname{range}(u) = [0, 1]$. We conclude

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2x}{x^4 - 4x^2 + 5} \, dx = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2x}{(x^2-2)^2 + 1} \, dx = \int_0^1 \frac{1}{u^2+1} \, du = \arctan u \Big|_0^1 = \frac{\pi}{4}$$

3. The hypotheses on $u$ really are all that's necessary. In particular, $u$ need not be left-/right-differentiable at the endpoints of $[a, b]$. For instance, with $f(u) = u^2$ and $u(x) = \sqrt{x}$ on $[0, 4]$, we easily verify

$$\frac{8}{3} = \int_0^4 \frac{1}{2}\sqrt{x} \, dx = \int_0^4 \frac{x}{2\sqrt{x}} \, dx = \int_0^4 f\big(u(x)\big)u'(x) \, dx = \int_0^2 f(u) \, du = \int_0^2 u^2 \, du = \frac{8}{3}$$

4. Sloppy 'substitutions' might lead to utter nonsense. For instance, $u(x) = x^2$ suggests

$$\int_{-1}^2 \frac{1}{x} \, dx = \int_{-1}^2 \frac{1}{2x^2} 2x \, dx = \int_1^4 \frac{1}{2u} \, du = \frac{1}{2}(\ln 4 - \ln 1) = \ln 2$$

   This is total gibberish: the first integral does not exist since $\frac{1}{x}$ is undefined at $0 \in (-1, 2)$. Thankfully, the hypotheses of the substitution rule prevent this: $f(u) = \frac{1}{2u}$ is not continuous on $\operatorname{range}(u) = [0, 4]$.

   While you are very unlikely to make precisely this mistake, the risk is real in more complicated or abstract situations...

[23]Hence the old adage, "Differentiation is a science, whereas integration is an art." To illustrate by example, consider $f(x) = \tan(e^x \cos(3x^2) + 4x^3)$. The derivative is easily found using the product and chain rules:

$$\frac{df}{dx} = \frac{1}{1 + (e^x \cos(3x^2) + 4x^3)^2} \left(e^x \cos(3x^2) - 6xe^x \sin(3x^2) + 12x^2\right)$$

By contrast, if you want to find an *explicit* anti-derivative of $f(x)$, the integration analogues (parts/substitution) are essentially useless. Similarly, the integral

$$\int_0^1 \tan(e^x \cos(3x^2) + 4x^3) \, dx$$

is likely impossible to evaluate explicitly and can only be approximated, say by using Riemann sums.

**Exercises 4.34.** *Key concepts: Complete statements of FTC parts I & II, Integration by Parts/Substitution*

1. Calculate the following limits:

   (a) $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,\mathrm{d}t$  (b) $\lim_{h\to 0} \frac{1}{h}\int_3^{3+h} e^{t^2}\,\mathrm{d}t$

2. Let $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t \le 1 \\ 4 & \text{if } t > 1 \end{cases}$

   (a) Determine the function $F(x) = \int_0^x f(t)\,\mathrm{d}t$ and sketch it. Where is $F$ continuous?
   (b) Where is $F$ *differentiable*? Calculate $F'$ at the points of differentiability.

3. Let $f$ be continuous on $\mathbb{R}$.

   (a) Define $F(x) = \int_{x-1}^{x+1} f(t)\,\mathrm{d}t$. Carefully show that $F$ is differentiable on $\mathbb{R}$ and compute $F'$.
   (b) Repeat for $G(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t$.

4. Recall Examples 4.24.4 and 4.27.3. Describe *all* anti-derivatives $F$ of $f$ on $[0,1) \cup (1,2]$. Which satisfy $\int_0^2 f(x)\,\mathrm{d}x = F(2) - F(0)$?

5. Suppose $u, v$ satisfy the hypotheses of integration by parts. By FTC I, $\int_a^x u'(t)v(t)\,\mathrm{d}t$ is an anti-derivative of $u'(x)v(x)$: what does integration by parts say is another?

6. Use a substitution to integrate $\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$

7. Use integration by parts and the substitution rule to evaluate $\int_0^b \arcsin x\,\mathrm{d}x$ for any $b < 1$.

8. Use integration by parts to evaluate $\int_0^b x\arctan x\,\mathrm{d}x$ for any $b > 0$

9. If $f$ and $u$ satisfy the hypotheses of the substitution rule, explain why both $(f \circ u)u'$ and $f$ are integrable on the required intervals.

10. We prove a simpler version of the fundamental theorem when $f : [a,b] \to \mathbb{R}$ is *continuous*.

    **Part I** Define $F(x) = \int_a^x f(t)\,\mathrm{d}t$. If $c, x \in [a,b]$ where $c \neq x$, prove that

    $$m \le \frac{F(x) - F(c)}{x - c} \le M$$

    where $m, M$ are the maximum and minimum values of $f(t)$ on the closed interval with endpoints $c, x$; why do $m, M$ exist? Now deduce that $F'(c) = f(c)$.

    **Part II** Now suppose $F$ is *any* anti-derivative of $f$ on $[a,b]$. Use part (a) and the mean value theorem to prove that $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$.

### 4.36 Improper Integrals

The Riemann integral has several limitations. Even allowing for functions to be integrable on open intervals (Definition 4.12), the existence of $\int_a^b f(x)\,\mathrm{d}x$ requires both:

- That $(a, b)$ be a *bounded* interval.
- That $f$ be *bounded* on $(a, b)$.

*Limits* provide a natural way to extend the Riemann integral to unbounded intervals and functions.

**Definition 4.32.** Suppose $f : [a, b) \to \mathbb{R}$ satisfies the following properties:

- $f$ is integrable on every closed bounded subinterval $[a, t] \subseteq [a, b)$.
- If $b$ is finite, then $f$ is unbounded at $b$ ($b$ can be $\infty$!)

The *improper integral* of $f$ on $[a, b)$ is

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x$$

This is *convergent* or *divergent* as is the limit.

If an integral is improper at its lower limit ($f : (a, b] \to \mathbb{R}$, etc.), then $\int_a^b f(x)\,\mathrm{d}x := \lim_{s\to a^+} \int_s^b f(x)\,\mathrm{d}x$.

If an integral is improper at both ends, choose any $c \in (a, b)$ and define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{s\to a^+} \int_s^c f(x)\,\mathrm{d}x + \lim_{t\to b^-} \int_c^t f(x)\,\mathrm{d}x$$

provided *both* one-sided improper integrals exist and the limit sum makes sense.

Theorem 4.14 says that the choice of $c$ for a doubly-improper integral is irrelevant.

Many properties of the Riemann integral transfer naturally to improper integrals, though not everything... For example, part 1 of Theorem 4.13 extends:

**Theorem 4.33.** *If $0 \le f(x) \le g(x)$ on $[a, b)$, then $\int_a^b f \le \int_a^b g$ whenever the integrals exist (standard or improper). In particular:*

- $\int_a^b f = \infty \implies \int_a^b g = \infty$
- $\int_a^b g$ *convergent* $\implies \int_a^b f$ *converges to some value* $\le \int_a^b g$

We leave some of the detail to Exercise 7.

**Examples 4.34.** 1. $\int_0^t x^2 \, \mathrm{d}x = \frac{1}{3}t^3$ for any $t > 0$. Clearly

$$\int_0^\infty x^2 \, \mathrm{d}x = \lim_{t\to\infty} \frac{1}{3}t^3 = \infty$$

More formally, the improper integral $\int_0^\infty x^2 \, \mathrm{d}x$ diverges to infinity.

2. With $f(x) = x^{-4/3}$ defined on $[1, \infty)$,

$$\int_1^\infty x^{-4/3} \, \mathrm{d}x = \lim_{t\to\infty} \int_1^t x^{-4/3} \, \mathrm{d}x = \lim_{t\to\infty} \left[-3x^{-1/3}\right]_1^t = \lim_{t\to\infty} 3 - 3t^{-1/3} = 3$$

3. Consider $f(x) = |x| \, e^{-x^2/2}$ on $(-\infty, \infty)$. On any bounded interval $[0, t]$,

$$\int_0^t f(x) \, \mathrm{d}x = \int_0^t xe^{-x^2/2} \, \mathrm{d}x = \left[-e^{-x^2/2}\right]_0^t = 1 - e^{-t^2/2} \xrightarrow[t\to\infty]{} 1$$

By symmetry,

$$\int_{-\infty}^\infty |x| \, e^{-x^2/2} \, \mathrm{d}x = 1 + 1 = 2$$

This example arises naturally in probability: multiplying by $\frac{1}{\sqrt{2\pi}}$ computes the expectation of $|X|$ when $X$ is a standard normally-distributed random variable

$$\mathbb{E}(|X|) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \, |x| \, e^{-x^2/2} \, \mathrm{d}x = \sqrt{\frac{2}{\pi}}$$

4. Our knowledge of derivatives $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ (or the substitution rule) allows us to evaluate

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \lim_{t\to 1^-} \sin^{-1} t = \frac{\pi}{2}$$

By symmetry, $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \pi$. By comparison, we obtain bounds on another improper integral:

$$\frac{1}{\sqrt{1-x^4}} \le \frac{1}{\sqrt{1-x^2}} \implies \int_{-1}^1 \frac{1}{\sqrt{1-x^4}} \, \mathrm{d}x \le \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \pi$$

5. Improper integrals need not exist. For instance,

$$\lim_{t\to\infty} \int_0^t \sin x \, \mathrm{d}x = \lim_{t\to\infty} 1 - \cos t$$

diverges by oscillation.

**Exercises 4.36.** *Key concepts: Formal definition and careful calculation of Improper Integrals*

1. Use your answers from Section 4.34 to decide whether the improper integrals $\int_0^1 \arcsin x \, \mathrm{d}x$ and $\int_0^\infty x \arctan x \, \mathrm{d}x$ exist. If so, what are their values?

2. Let $p$ be a positive constant. Prove:

$$\int_0^1 \frac{1}{x^p} \, \mathrm{d}x = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases} \qquad \int_1^\infty \frac{1}{x^p} \, \mathrm{d}x = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

(*The first of these justifies the convergence/divergence properties of p-series via the integral test*)

3. Suppose $f$ is integrable on $[a, b]$. Explain why $\int_a^b f(x) \, \mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x) \, \mathrm{d}x$ is still true, even though the integral is not improper.

4. State a version of integration by parts modified for when $\int_a^b u'(x)v(x) \, \mathrm{d}x$ is improper at $b$. Now evaluate $\int_0^\infty xe^{-4x} \, \mathrm{d}x$.

5. What is wrong with the following calculation?

$$\int_{-\infty}^{\infty} x \, \mathrm{d}x = \lim_{t \to \infty} \frac{1}{2}x^2 \Big|_{-t}^{t} = \lim_{t \to \infty} \frac{1}{2}(t^2 - t^2) = \lim_{t \to \infty} 0 = 0$$

6. Prove or disprove: if $\int f$ and $\int g$ are convergent improper integrals, so is $\int fg$.

7. Prove part of Theorem 4.33. Suppose $0 \leq f(x) \leq g(x)$ for all $x \in [a, b)$, and that $\int_a^b g$ is a convergent improper integral. Prove that $\int_a^b f$ converges and that $\int_a^b f \leq \int_a^b g$.

### Extensions of the Riemann Integral (just for fun)

In the 1890s, Thomas Stieltjes [24] offered a generalization of the Riemann integral.

**Definition 4.35.** Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ monotonically increasing. Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, define the sequence of differences

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

The *upper/lower Darboux–Stieltjes sums/integrals* are defined analogously to the pure Riemann case:

$$U(f, P, \alpha) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)\,\Delta\alpha_i \qquad L(f, P, \alpha) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)\,\Delta\alpha_i$$

$$U(f, \alpha) = \inf_P U(f, P, \alpha) \qquad L(f, \alpha) = \sup_P L(f, P, \alpha)$$

If $U(f, \alpha) = L(f, \alpha)$, we say that $f$ is *Riemann–Stieltjes integrable* of class $\mathcal{R}(\alpha)$ and denote its value $\int_a^b f(x)\,\mathrm{d}\alpha$.

The standard Riemann integral corresponds to $\alpha(x) = x$. It is the ability to choose other functions $\alpha$ that makes the Riemann–Stieltjes integral both powerful and applicable.

*Standard Properties* Most results in sections 4.32 and 4.33 hold with suitable modifications, as does the discussion of improper integrals. For instance,

$$f \in \mathcal{R}(\alpha) \iff \exists P \text{ such that } U(f, P, \alpha) - L(f, P, \alpha) < \epsilon$$

The result regarding the piecewise continuity of $f$ is a notable exception: depending on $\alpha$, a piecewise continuous $f$ might not lie in $\mathcal{R}(\alpha)$.

*Weighted integrals* If $\alpha$ is differentiable, we obtain a standard Riemann integral

$$\int_a^b f(x)\,\mathrm{d}\alpha = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x$$

weighted so that $f(x)$ contributes more when $\alpha$ is increasing rapidly.

*Probability* If $\alpha(a) = 0$ and $\alpha(b) = 1$, then $\alpha$ may be viewed as a *probability distribution function* and its derivative $\alpha'$ as the corresponding *probability density function*. For example:

1. The *uniform distribution* on $[a, b]$ has $\alpha = \frac{1}{b-a}(x - a)$ so that

   $$\int_a^b f(x)\,\mathrm{d}\alpha = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

   Since $\alpha'$ is constant, the integrals weigh all values of $x$ *uniformly*.
2. The standard *normal distribution* has $\alpha(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t$. The fact that $\alpha' = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is maximal when $x = 0$ reflects the fact that a normally distributed variable is clustered near its mean.

In all cases, $\int f(x)\,\mathrm{d}\alpha = \mathbb{E}(f(X))$ computes an expectation (see, e.g., Example 4.34.3).

[24] Stieltjes was Dutch; the pronunciation is roughly ‘steelchez.’

*Non-differentiable or continuous α* This provides major flexibility! For example, if $Q = \{s_0, \ldots, s_n\}$ partitions $[a,b]$, and $(c_k)_{k=1}^n$ is a positive sequence, then

$$\alpha(x) = \begin{cases} 0 & \text{if } x = a \\ \sum\limits_{i=1}^{k} c_i & \text{if } x \in (s_{k-1}, s_k] \end{cases}$$

defines an increasing step function, and the Riemann–Stieltjes integral a weighted *sum*

$$\int_a^b f(x)\,\mathrm{d}\alpha = \sum_{i=1}^{n} c_i f(s_i)$$

Taking an infinite increasing sequence $(s_n) \subseteq [a,b]$ results in an *infinite series*, which helps explain why so many results for series and integrals look similar!

This also touches on probability. For example, let $p \in [0,1]$, $n \in \mathbb{N}$, and $s_k = k$ on the interval $[0,n]$. If $c_k = \binom{n}{k} p^k (1-p)^{n-k}$, then

$$\int f(x)\,\mathrm{d}\alpha = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} f(x) = \mathbb{E}(f(X))$$

is the expectation of $f(X)$ when $X \sim B(n,p)$ is binomially distributed.

**Lebesgue Integration: Integrals and Convergence**

Lebesgue's extension essentially uses rectangles whose *heights* tend to zero: cutting up the area under a curve using *horizontal* instead of *vertical* strips. One of its major purposes is to permit a more general interchange of limits and integration in many cases of *pointwise* (non-uniform) convergence. To see the problem, consider the sequence of piecewise continuous functions

$$f_n : [0,1] \to \mathbb{R} : x \mapsto \begin{cases} 1 & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q \le n \\ 0 & \text{otherwise} \end{cases}$$

Each $f_n$ is Riemann integrable with $\int_0^1 f_n(x)\,\mathrm{d}x = 0$. However, the pointwise limit

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is *not* Riemann integrable (compare Example 4.8.2). In the Lebesgue theory, the limit $f$ turns out to be integrable with integral 0, so that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$$

Recall (Theorem 2.19) that the interchange of limits and integrals would be automatic *if* the convergence $f_n \to f$ were *uniform*: of course the convergence isn't uniform here.

Like *measure theory* (recall Theorem 4.22), *Lebesgue integration* is a central topic in graduate analysis.