Math 147 — Complex Analysis

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Complex Analysis and Applications, James Ward Brown & Ruel V. Churchill, 9th Ed 2014, McGraw Hill.

1 Complex Numbers

1.1 Basic Algebraic Properties (roughly § 1–6 in textbook)

There are many ways to introduce the complex numbers, though essentially all center on the seemingly absurd equation $x^2 = -1$. Positing a solution to this equation and playing with the result provided the first 'modern' discussion by the Italian mathematician Rafael Bombelli[1] in the 1500's.

Definition 1.1. The *complex numbers* $\mathbb{C}$ comprises the vector space $\mathbb{R}^2$ together with a new operation (*complex multiplication*.. Specifically, if $z = (x, y)$ and $w = (u, v)$ are complex numbers ($x, y, u, v \in \mathbb{R}$), we *define*

Addition $z + w := (x + u, y + v)$

Multiplication $zw := (xu - yv, xv + yu)$

The co-ordinates of a complex number $z = (x, y)$ are termed its *real* and *imaginary parts*:

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y$$

The vector space notation is unwieldy: instead we write

$$z = x + iy$$

by introducing the symbol i as a short-hand for the point $(0, 1)$: this is termed the *imaginary unit.* In this notation, the complex number 1 is really the point $(1, 0)$.

It is the strange properties of i under multiplication that make $\mathbb{C}$ different (and more interesting!) than $\mathbb{R}^2$. Before considering this, we remind ourselves of several simple properties of $\mathbb{R}^2$ and how these provide interpretations of the complex numbers.

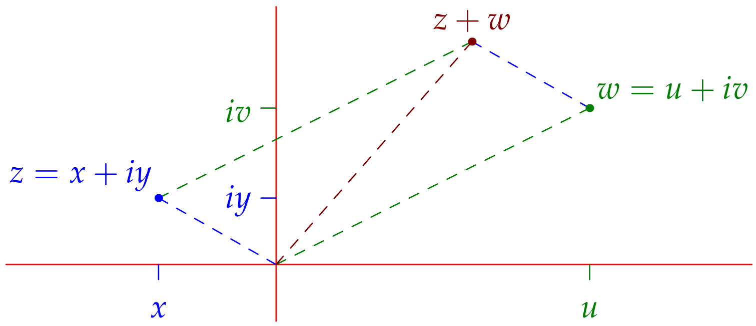
[1]While Bombelli was happy to calculate and discuss the relationships between solutions of various quadratic equations, he always considered them entirely 'fictitious.'

Vector Addition Since complex addition is merely the addition of real vectors, we immediately observe that addition of complex numbers is:

Associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

Commutative: $z + w = w + z$

Indeed commutativity is the familiar *parallelogram law* from vector spaces:



These can be proved algebraically, though it is tedious. Instead here's an easy example of addition in the new language: observe how we simply sum the real and imaginary parts:

Example 1.2. If $z = 3 + 4i$ and $w = 2 - 7i$, then

$$z - w = (3 + 4i) - (2 - 7i) = (3 - 2) + (4 - (-7))i = 1 + 11i$$

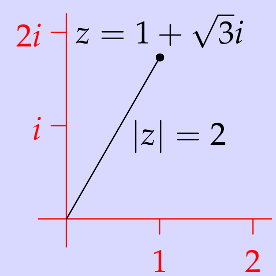
When representing the complex numbers, the plane is often known as the *Argand diagram*. The horizontal axis is known as the *real axis* while the vertical is the *imaginary axis*. As we've done in the picture, it is common to label entries on the imaginary axis using i's.

The Modulus of a Complex Number & the Triangle Inequality

Definition 1.3. The *modulus* of a complex number $z = x + iy$ is the Euclidean *distance* of the point (x, y) from the origin:

$$|z| := \sqrt{x^2 + y^2}$$

In the picture, $z = 1 + \sqrt{3}i$ has modulus $|z| = \sqrt{1+3} = 2$.



Considering the picture at the top of the page, we immediately see the *triangle inequality*:

$$|z + w| \le |z| + |w|$$

Unlike in real analysis, this follows from an honest triangle! By induction, it holds generally

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + \cdots + |z_n|$$

In the exercises below, you'll be asked to prove a useful generalization:

$$|z + w| \ge \big||z| - |w|\big|$$

One can use the modulus to efficiently describe various curves and regions in the plane:

Examples 1.4.
1. $|z| = 4$ describes the circle centered at the origin of radius 4.
2. $|z - 3i| \leq 2$ describes the *disk* centered at $3i$ with radius 2.
3. $|z| + |z - 1| = 3$ describes an *ellipse* with foci 0 and 1 (the sum of the distances from two fixed points is constant).

You can try writing these using square-roots and multiplying out to obtain the 'usual' algebraic equations: be careful with example 3!

Complex multiplication

The link with the equation $x^2 = -1$ comes immediately from the definition of complex multiplication. In what follows, it is important to note that -1, *as a complex number* is really the *point* $(-1, 0) \in \mathbb{R}^2$!

Lemma 1.5. $i^2 = -1$.

Proof. $i^2 = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1.$ ∎

Now that we've established the fundamental property of i, we no longer have any need for the vector space notation: from now on, we will always denote complex numbers in the form $z = x + iy$.

The basic algebraic properties of multiplication are also straightforward: check one or two by multiplying out!

Lemma 1.6. *For any complex numbers z_1, z_2, z_3, we have*

Associativity: $z_1(z_2z_3) = (z_1z_2)z_3$

Commutativity: $z_1z_2 = z_2z_1$

Distributivity: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

The upshot of all this is that we can treat complex addition, subtraction and multiplication as if we are working with *linear polynomials* in the abstract variable i: we simply need to remember to replace i^2 with -1 whenever necessary.

Example 1.7. As above, if $z = 3 + 4i$ and $w = 2 - 7i$,

$$\begin{aligned} zw &= (3 + 4i)(2 - 7i) = 3 \cdot 2 + 4i \cdot 2 - 3 \cdot 7i - 4i \cdot 7i = 6 + 8i - 21i - 28i^2 \\ &= 6 + 8i - 21i + 28 = 34 - 13i \end{aligned}$$

Division of Complex Numbers and the Complex Conjugate

Division is simply multiplication by an inverse: $\frac{w}{z} = wz^{-1}$. The inverse of a complex number $z = x + iy$ should be some $z^{-1} = u + iv$ satisfying $zz^{-1} = 1$. Let us unpack this:

$$zz^{-1} = xu - yv + i(xv + yu) = 1 \iff \begin{cases} xu - yv = 1 \\ xv + yu = 0 \end{cases}$$

It is a little tedious, but we can work through this in cases:

- If $x \neq 0$, then the second equation require $v = -\frac{y}{x}u$. Substituting into the first equation forces

$$1 = xu + \frac{y^2}{x}u \implies u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2} \tag{$*$}$$

- If $x = 0$, we're forced to have $u = 0$ and $v = -\frac{1}{y}$, which fits the same pattern as $(*)$!

In particular, *every* non-zero complex number $z = x + iy$ has a unique multiplicative inverse:

$$z^{-1} = \frac{x - iy}{x^2 + y^2}$$

This is a critical part of the statement that the complex numbers form a *field.* Note how the numerator looks very like z itself, except for a negative sign: also observe how the denominator is simply $|z|^2$.

Definition 1.8. The *conjugate* of a complex number $z = x + iy$ is the complex number $\overline{z} = x - iy$, obtained by reflecting z in the real axis.

As observed above, $z\overline{z} = |z|^2$ and, provided $z \neq 0$,

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

The conjugate helps to facilitate the computation of division:

Example 1.9. If $z = 3 + 4i$ and $w = 2 - 7i$, then we compute[a]

$$\begin{aligned} \frac{w}{z} &= \frac{2 - 7i}{3 + 4i} = wz^{-1} = \frac{w\overline{z}}{|z|^2} = \frac{(2 - 7i)(3 - 4i)}{|3 + 4i|^2} = \frac{2 \cdot 3 - 2 \cdot 4i - 7i \cdot 3 + 7i \cdot 4i}{3^2 + 4^2} \\ &= \frac{6 - 8i - 21i + 28i^2}{25} = \frac{-22 - 29i}{25} \end{aligned}$$

[a]This can be viewed as multiplying the top and bottom by the conjugate of the denominator:

$$\frac{2 - 7i}{3 + 4i} = \frac{2 - 7i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \cdots$$

Exercises. 1.1.1. Prove that $\operatorname{Re}(iz) = -\operatorname{Im} z$ and that $\operatorname{Im}(iz) = \operatorname{Re} z$.

1.1.2. (a) Check explicitly that $z = 2 + 3i$ and its conjugate $\overline{z} = 2 - 3i$ solve the quadratic equation $z^2 - 4z + 13 = 0$.

(b) Suppose $a, b, c \in \mathbb{R}$ where $\omega := 4ac - b^2 > 0$. Check that $z = \frac{-b+i\sqrt{\omega}}{2a}$ and its conjugate $\overline{z}$ both solve the quadratic equation $az^2 + bz + c = 0$.
(*Since $i^2 = -1$, it makes sense to write $\sqrt{-\omega} = i\sqrt{\omega}$: we see that the quadratic formula now applies to all real quadratics*)

1.1.3. Explicitly prove the commutativity of complex multiplication (Lemma 1.6) using the vector definition of $\mathbb{C}$ (Definition 1.1).

1.1.4. Evaluate the following in the form $x + iy$:

(a) $\dfrac{2-i}{3-5i}$ (b) $(1+i)^4$ (c) $(2+3i)^{-2} - (2-3i)^{-2}$

1.1.5. Prove the following: you should write $z = x + iy$ rather than using the vector definition.

(a) $\overline{\overline{z}} = z$

(b) $(z^{-1})^{-1} = z$

(c) $\overline{zw} = \overline{z} \cdot \overline{w}$

1.1.6. (a) For any z, w, use the triangle inequality to prove that $|z + w| \geq \big||z| - |w|\big|$.

(b) What relationship between z, w corresponds to *equality* here? Draw a picture to try to figure it out!

1.1.7. Suppose that $|z| \geq 2$ and consider the polynomial $P(z) = z^3 + 3z - 1$.

(a) Prove that $\left|\frac{3z-1}{z^3}\right| \leq \frac{7}{8}$

(b) Write $|P(z)| = \left|z^3 + 3z - 1\right| = \left|z^3\right| \left|1 + \frac{3z-1}{z^3}\right|$ and use the result of Exercise 1.1.6. (a) to prove that $|P(z)| \geq 1$.
(*This shows that all zeros of $P(z)$ must lie inside the circle $|z| < 2$.*)

1.1.8. By considering the inequality $(|x| - |y|)^2 \geq 0$, prove that

$$\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$$

1.1.9. Prove that the hyperbola $x^2 - y^2 = 1$ can be written in the form $z^2 + \overline{z}^2 = 2$.

1.1.10. Draw a picture of the ellipse satisfying the equation $|z| + |z - 4i| = 6$. Find the equation of the curve in Cartesian coordinates: $\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1$ where (c, d) is the center of the ellipse and a, b are the semi-axes.
(*Hint: write $|z - 4i| = 6 - |z|$, square both sides, cancel x^2, y^2 terms and repeat…*)

1.2 The Exponential or Polar Form of a Complex Number (§7–9)

Recall Definition 1.3 of the *modulus* of a complex number. We extend this to also consider the *angle.*

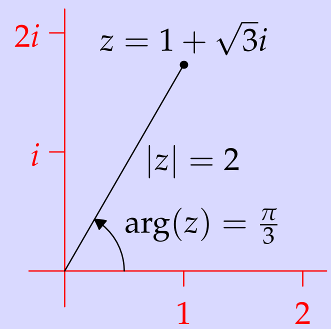
Definition 1.10. A complex number can be written in polar co-ordinates:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Clearly $r = |z|$ is the modulus of z. The *argument* $\arg z = \theta$ is the angle (in radians) measured counter-clockwise from the positive real axis.

Its *principal value* $\operatorname{Arg} z$ is chosen such that $-\pi < \operatorname{Arg} z \le \pi$.

In the picture, $z = 1 + \sqrt{3}i$ has argument $\arg z = \frac{\pi}{3}$.



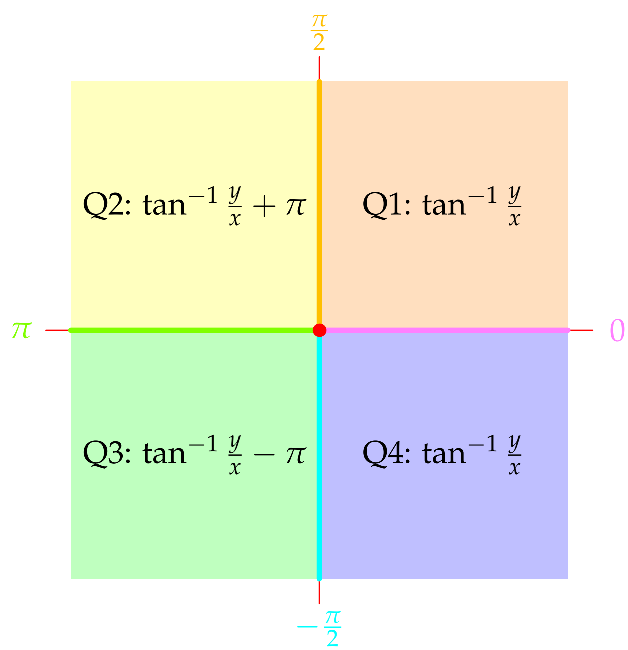
As with standard polar co-ordinates, the argument can be awkward to calculate: it may seem that the following is all we need

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x} \implies \arg z = \tan^{-1}\frac{y}{x}$$

However, this only makes sense if $x \neq 0$.
Moreover, $\tan^{-1}$ has range $(-\frac{\pi}{2}, \frac{\pi}{2})$ and so an addition or subtraction of π may be required to get the correct value.
Here is a picture to remind you, with the principal value $\operatorname{Arg} z$ indicated: for instance, $z = -3 - 3i$ lies in the third quadrant, so

$$\operatorname{Arg} z = \tan^{-1}\frac{-3}{-3} - \pi = -\frac{3\pi}{4}$$

Since the argument itself is unconstrained, it is perfectly acceptable, for instance, to say that $\arg z = \frac{5\pi}{4}$.
Note particularly that 0 does not have an argument!



Definition 1.11. If $\theta \in \mathbb{R}$ then the *exponential* $e^{i\theta}$ is defined by *Euler's formula*

$$e^{i\theta} := \cos\theta + i\sin\theta$$

The *polar form* of a complex number can now be written $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$.
If $w = u + iv$ is complex, then its exponential is $e^w = e^u e^{iv} = e^u(\cos v + i\sin v)$.

Why should Euler's formula be true? There are several reasons why Euler's formula provides a sensible definition of $e^{i\theta}$.

1. $e^{k\theta}$ is defined as the solution to the initial value problem $y' = ky$ with $y(0) = 1$. If we assume differentiation still works when $k = i$, we see that Euler's formula satisfies this criterion.

2. Evaluate the Maclaurin series for $\exp z$ when $z = i\theta$: the real and imaginary parts are easily seen to be the Maclaurin series for $\cos\theta$ and $\sin\theta$.

3. The multiple-angle formulæ for sine and cosine confirm the exponential law

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi} \tag{$\dagger$}$$

Just as with standard polar co-ordinates, the polar form of a complex number is well-suited to describing *circles.* For example, the circle centered at $z_0 = 3+4i$ with radius $R = 2$ may be parametrized by

$$z = z_0 + Re^{i\theta} = 3+4i+2e^{i\theta} = 3+2\cos\theta + i(4+2\sin\theta)$$

We thus have a curve parametrized by θ.

Exponential Laws As observed in (†), the complex exponential follows the usual exponential law. In particular we observe that if $z = re^{i\theta}$ and $w = se^{i\psi}$, then

$$zw = rse^{i(\theta+\psi)} \implies |zw| = |z|\,|w| \text{ and } \arg(zw) = \arg z + \arg w$$

The *principal value* might not behave so nicely since $\operatorname{Arg} z = \arg z + 2\pi n$ for some $n \in \mathbb{Z}$.

Example 1.12. Find the modulus and argument of zw given $z = -7+i$ and $w = -4+3i$. We can proceed two ways:

1. Find the polar forms of z, w

$$z = 5\sqrt{2}e^{i\theta} \text{ and } w = 5e^{i\psi} \text{ where } \theta = \pi - \tan^{-1}\frac{1}{7} \text{ and } \psi = \pi - \tan^{-1}\frac{3}{4}$$

   from which

$$|zw| = |z|\,|w| = 25\sqrt{2}, \qquad \arg(zw) = \arg z + \arg w = 2\pi - \tan^{-1}\frac{1}{7} - \tan^{-1}\frac{3}{4}$$

2. First find $zw = (-7+i)(-4+3i) = 25-25i$ then compute its polar form:

$$zw = 25\sqrt{2}e^{-\pi i/4} \implies |zw| = 25\sqrt{2}, \qquad \arg(zw) = -\frac{\pi}{4}$$

The discrepancy[a] in the argument comes from the extra copy of 2π: in fact $\operatorname{Arg} z = -\frac{\pi}{4}$.

[a] $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{3}{4} = \frac{\pi}{4}$ can be checked using the multiple-angle formula for tangent: $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

The polar form also works easily with division and (integer) exponentiation:

$$z = re^{i\theta}, \quad w = se^{i\psi} \implies \begin{cases} \frac{z}{w} = \frac{r}{s}e^{i(\theta-\psi)} \\ z^n = r^n e^{in\theta},\ n \in \mathbb{Z} \end{cases}$$

Examples 1.13. 1. The identity $(e^{i\theta})^n = e^{in\theta}$, equivalently $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is known as de Moivre's formula; it yields some interesting trig identities. For instance, by taking real parts,

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$
$$\implies \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$$

2. We compute z^{10} when $z = \sqrt{3} - i$. First observe that $z = 2e^{-\pi i/6}$, from which

$$z^{10} = 2^{10}e^{-5\pi i/3} = 1024e^{\pi i/3} = 512(1 + \sqrt{3}i)$$

Exercises. 1.2.1. Use induction to prove that for any $n \in \mathbb{N}_{\geq 2}$ we have

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)}$$

1.2.2. Find the principal argument of $(1+i)^{2020}$.

1.2.3. Prove that $\left|e^{i\theta}\right| = 1$ and that $\overline{e^{i\theta}} = e^{-i\theta}$.

1.2.4. Show that if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, then $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$.

1.2.5. Use de Moivre's formula to establish the identity

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

1.2.6. With reference to Example 1.12, if $\alpha = \tan^{-1}\frac{3}{4}$ and $\beta = \tan^{-1}\frac{1}{7}$, show that

$$\cos\alpha = \frac{4}{5}, \qquad \sin\alpha = \frac{3}{5}, \qquad \cos\beta = \frac{7}{\sqrt{50}}, \qquad \sin\beta = \frac{1}{\sqrt{50}}$$

Now use the cosine multiple-angle formula to check that $\alpha + \beta = \frac{\pi}{4}$. For a challenge, generalize your approach to prove the general multiple-angle formula for tangent, at least when α, β are acute angles.

1.3 Roots of Complex Numbers (§10–11)

Taking roots is a difficult proposition in $\mathbb{C}$. For instance, consider squaring $c = 2 + 3i$, which has

$$z := c^2 = -5 + 12i$$

If we were given $z = -5 + 12i$, how would we go about finding a square-root c? Naively, you might try hacking at it:

$$-5 + 12i = (x + iy)^2 = x^2 - y^2 + 2ixy \iff \begin{cases} x^2 - y^2 = -5 \\ xy = 6 \end{cases}$$

Substituting $y = 6x^{-1}$ into the first equation yields a quadratic in x^2:

$$x^4 + 5x^2 - 36 = (x^2 - 4)(x^2 + 9)$$

from which we'd conclude $x = \pm 2$ and obtain the square-roots $\pm c = \pm(2 + 3i)$. This is messy and hard to generalize; imagine if we were working with cube, or higher, roots!

This is where the polar form comes in. Suppose $n \in \mathbb{N}$ and that c, z satisfy $z = c^n$. In polar form

$$z = re^{i\theta}, \quad c = se^{i\psi} \implies re^{i\theta} = s^n e^{in\psi}$$

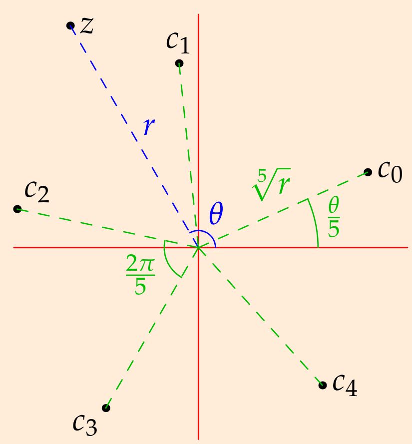
By equating moduli and arguments, we conclude that

$$r = s^n \qquad n\psi = \theta + 2\pi k \tag{$*$}$$

where k is any integer. We can put this together to obtain a proper definition; since there are some conventions to follow however, we first do an example:

Example 1.14. $z = 1.5e^{\pi i/3}$ has fifth roots $c_0, \ldots, c_4$:

$$\begin{aligned}
c_0 &= \sqrt[5]{1.5}e^{\pi i/15} \\
c_1 &= \sqrt[5]{1.5}e^{\pi i/15 + 2\pi/5} = \sqrt[5]{1.5}e^{7\pi i/15} \\
c_2 &= \sqrt[5]{1.5}e^{\pi i/15 + 4\pi/5} = \sqrt[5]{1.5}e^{13\pi i/15} \\
c_3 &= \sqrt[5]{1.5}e^{\pi i/15 + 6\pi/5} = \sqrt[5]{1.5}e^{19\pi i/15} \\
c_4 &= \sqrt[5]{1.5}e^{\pi i/15 + 8\pi/5} = \sqrt[5]{1.5}e^{25\pi i/15}
\end{aligned}$$



We've chosen arguments in the range $[0, 2\pi)$ rather than principal arguments. Note how the roots form the vertices of a regular pentagon, equally spaced around the circle of radius $\sqrt[5]{1.5}$. Note also how there are precisely five 5^{th} roots: once $k \geq n$ in $(*)$, the roots start repeating.

Definition 1.15. Given $z = re^{i\theta}$ and a positive integer n, the n^{th} *roots* of z are the n complex numbers

$$c_k = \sqrt[n]{r} \exp \frac{(\theta + 2k\pi)i}{n} \qquad k = 0, 1, \ldots, n-1$$

The n^{th} roots form the vertices of a regular n-gon, equally spaced around the circle of radius $\sqrt[n]{r}$.

There are some conventions on how to refer to n^{th} roots: suppose $\theta = \operatorname{Arg} z$ is the *principal argument*. We write:

$\sqrt[n]{z} = \sqrt[n]{r} e^{i\theta/n}$: this is the *principal n^{th} root* of z.

$z^{1/n} = \{c_0, \ldots, c_{n-1}\}$: this is the *set* of n^{th} roots of z.

The *n^{th} roots of unity* are commonly encountered: these are the set of n^{th} roots of $1 \in \mathbb{C}$:

$$1^{1/n} = \{e^{2\pi k i/n} : k = 0, \ldots, n-1\}$$

The notation $\omega_n = e^{2\pi i/n}$ is often seen, so that the roots of unity are the set $\{\omega^k : k = 0, \ldots, n-1\}$.

Compare this with what happens in the real numbers. If $r = 16$, we might write

$\sqrt[4]{16} = 2$ for the principal fourth root.

$16^{1/4} = \pm 2$ for all (real) fourth roots. In complex analysis, there are *four* fourth roots, and we'd write $16^{1/4} = \{2, 2i, -2, -2i\}$.

Depending on the format (rectangular versus polar), it can be very hard to *explicitly* compute n^{th} roots of a complex number, particularly if $\arg z$ is only known in $\tan^{-1}$ format. Various trig identities can be used, but in practice this might require more effort than it is worth.

Example 1.16. Find the fourth roots of $z = 1 + i$.

First we write in polar form: $z = \sqrt{2} e^{\pi i/4}$. Since $\operatorname{Arg} z = \frac{\pi}{4}$, we see that the principal fourth root is $\sqrt[4]{1+i} = \sqrt[8]{2} e^{\pi i/16}$. To evaluate this in rectangular form requires us to compute cosine and sine of $\frac{\pi}{16}$. For this, the half-angle formulas are useful:

$$\cos^2 \frac{\alpha}{2} = \frac{1}{2}(1 + \cos \alpha) \implies \cos \frac{\pi}{8} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{\sqrt{2}+1}{4}} = \frac{1}{2}\sqrt{1 + \sqrt{2}}$$

Continuing this, and applying similar expressions for sine, we obtain:

$$\sqrt[4]{1+i} = \sqrt[8]{2} \cos \frac{\pi}{16} + i \sqrt[8]{2} \sin \frac{\pi}{16} = \frac{\sqrt[2]{8}}{2}\sqrt{2 + \sqrt{1 + \sqrt{2}}} + i \frac{\sqrt[2]{8}}{2}\sqrt{2 - \sqrt{1 + \sqrt{2}}}$$

This is definitely not attractive! The set of fourth roots is then the product of this with each of the fourth roots of unity: $1^{1/4} = \{1, \omega_4, \omega_4^2, \omega_4^3\} = \{1, i, -1, -i\}$. In this example, it is probably not worth the effort of converting to rectangular form.

Exercises. 1.3.1. Find the square roots of $-\sqrt{3}+i$ and express them in rectangular co-ordinates. (*Hint: you may find it useful that* $(\sqrt{3}-1)^2 = 4-2\sqrt{3}\ldots$)

1.3.2. Find the sixth roots of i in polar co-ordinates. Which is the principal root?

1.3.3. Use the fact that the cube roots of unity are $1, \omega_3 = \frac{-1+\sqrt{3}i}{2}$ and $\omega_3^2 = \frac{-1-\sqrt{3}i}{2}$ to evaluate the cube roots of -27 in rectangular co-ordinates.

1.3.4. We previously found the fourth roots of 16. Use these to find the fourth roots of -16. Hence factorize the equation $z^4+16=0$ as a product of two quadratic equations with real coefficients.

1.3.5. If ω is any n^{th} root of unity other than 1, prove that $\sum\limits_{k=0}^{n-1} \omega^k = 0$.

(*Hint: recall geometric series*)

1.3.6. (a) Suppose that $a,b,c \in \mathbb{C}$ with $a \neq 0$ and suppose that z satisfies the quadratic equation $az^2+bz+c=0$. Prove the quadratic formula:

$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$$

Note that $(b^2-4ac)^{1/2}$ is the *set* of square roots of b^2-4ac, so that this provides *two* solutions whenever $b^2-4ac \neq 0$.

(b) Find the roots of the equation $iz^2+(1+i)z+3=0$ in rectangular form.

2 Analytic Functions

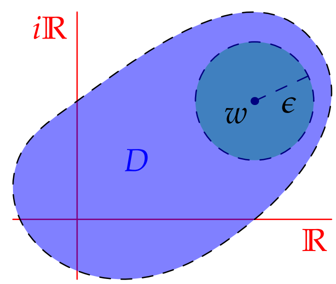
The purpose of this chapter is to discuss functions of a complex variable and what it means for such functions to be *differentiable.* This turns out to be much more subtle and restrictive than the story for functions of a real variable.

2.1 Functions of a Complex Variable (§12–14)

Let $D \subseteq \mathbb{C}$ be a region of the complex plane. We can define a function $f : D \to \mathbb{C}$ in the obvious manner, using a *rule*: for example $f(z) = z^2$. The first challenge is that you cannot easily *graph* such a function—the picture of a parabola that likely sneaked into your mind is unhelpful! Since $\mathbb{C}$ is itself a copy of $\mathbb{R}^2$ (albeit with extra structure), the graph of a function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is necessarily a subset of the *four (real) dimensional space* $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$. That said, there is nothing stopping us from computing with f: for example

$$f(3+i) = (3+i)^2 = 8+6i$$

Before going further, we standardize notation a little. The *domain* $D \subseteq \mathbb{C}$ of a function $f : D \to \mathbb{C}$ can be any set, however there are some general conventions;



- Typically a domain is *open*: this means that every point $w \in D$ is *interior* to D. More precisely,

$$\forall w \in D,\ \exists \epsilon > 0 \text{ such that } |z-w| < \epsilon \implies z \in D$$

- Typically a domain is also *connected*: this means that any two points in the domain may be joined by a curve which never leaves the domain.

- *Implied domain*: If the domain is not specified and only a rule is given, D is assumed to be the largest possible set. For example, $f(z) = \frac{1}{z^2+9}$ is assumed to have domain $D = \mathbb{C} \setminus \{\pm 3i\}$: note that this is itself a connected open set.

Ways to represent complex functions Being, essentially, multi-variable functions, there are several ways to display complex formulae.

1. z and $\overline{z}$ form: for example $f(z) = z^2 - 3\overline{z}$

2. Real and imaginary parts: write $f(z) = u(x,y) + iv(x,y)$ where $u, v : D \to \mathbb{R}$. For our previous example we'd have

$$f(z) = (x+iy)^2 - 3(x-iy) = x^2 - y^2 - 3x + i(2xy+3y)$$

   thus $u(x,y) = x^2 - y^2 - 3x$ and $v(x,y) = 2xy + 3y$.

3. Polar form: write $z = re^{i\theta}$. For our example,

$$f(z) = r^2 e^{2i\theta} - 3re^{-i\theta}$$

Depending on the function, each of these approaches might have certain advantages or disadvantages. You might also wish to combine the approaches: for instance writing u, v as functions of r, θ.

Basic 2D geometry with complex numbers

Complex functions can easily describe the primary geometric transformations of the plane. These functions are often known by their geometric effect.

Translation For some fixed $w \in \mathbb{C}$, the function $f(z) = z + w$ translates the complex plane, shifting the origin to w.

Scaling Given a constant $R \in \mathbb{R}$, the function $f(z) = Rz$ scales the complex plane.

Rotation Given ϕ, the function $f(z) = e^{i\phi}z$ rotates the complex plane ϕ radians counter-clockwise around the origin. The easy way to see this is to write the function in polar form itself:

$$f(z) = f(re^{i\theta}) = e^{i\phi}re^{i\theta} = re^{i(\theta+\phi)}$$

Reflection Complex conjugation $f(z) = \overline{z} = x - iy$ reflects the complex plane in the horizontal axis. Combining this with rotation, we can produce the reflection in any line through the origin. To reflect in the line through the origin and any complex number with $\operatorname{Arg} w = \phi$;

1. Rotate the plane by $-\phi$: $z \mapsto e^{-i\phi}z$
2. Reflect in the real axis: $z \mapsto \overline{e^{-i\phi}z} = e^{i\phi}\overline{z}$
3. Rotate the plane back by ϕ: the result is the function $f(z) = e^{2i\phi}\overline{z}$

Combining these allows one to rotate around any point and reflect across any line.

Examples 2.1. (a) The function $f(z) = e^{i\pi/3}(z - 2i) + 2i$ rotates the complex plane $\frac{\pi}{3}$ radians around the point $2i$.

(b) We find the function that reflects across the line joining the points $\alpha = 2 + i$ and $\beta = 4 + 3i$.

Since $\beta - \alpha = 2 + 2i$ has argument $\phi = \frac{\pi}{4}$, we combine reflection across the line making ϕ through the origin with translation by α (translate by $-\alpha$, reflect, translate back by α):

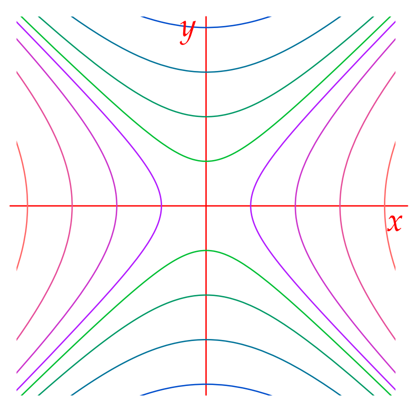
$$f(z) = e^{2i\phi}(\overline{z - \alpha}) + \alpha = e^{i\pi/2}(\overline{z - 2 - i}) + 2 + i = i(\overline{z} - 2 + i) + 2 + i = i\overline{z} + 1 - i$$

The function $f(z) = z^2$

Using our three types of expression for a complex-valued function,

$$f(z) = z^2 = x^2 - y^2 + 2ixy = r^2e^{2i\theta}$$

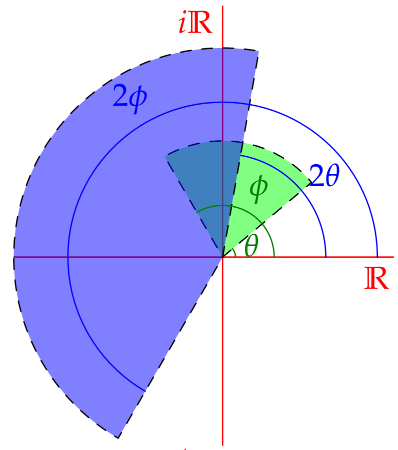
The real and imaginary parts $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ can be visualized as graphs of functions $\mathbb{R}^2 \to \mathbb{R}$: both are *saddle surfaces* and can be analyzed using the standard tools from multivariable calculus: indeed the graphs of u and v are related by rotation 45° around the vertical axis. It is easier to think about *level curves*: the curves u and v constant are *hyperbolae.*



Level curves of $u = x^2 - y^2$

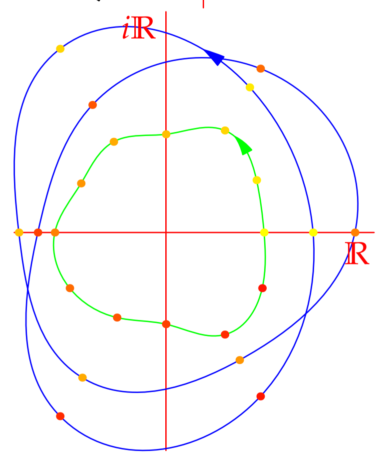
Now consider the polar form of the function and what it means for the argument:

$$f(z) = r^2 e^{2i\theta} \implies \arg(z^2) = 2\arg z$$



This is easily visualized by considering sectors of the plane: a sector between arguments θ and ϕ is *doubled* in angle to a sector between 2θ and 2ϕ.
Indeed the fact that $f(\pm z) = z^2$ says that, away from the origin, f is a *two-to-one* function: f maps the sector with $\arg z \in [0, \pi)$ to the entire plane.

It can be helpful to visualize this pathwise: if z traces a path around the origin, z^2 will trace a path round the origin *twice*! The colored dots on the two paths correspond under $z \mapsto f(z)$.



A similar trick will be seen with higher powers. For instance, $z \mapsto z^3$ will map a single loop around the origin to a triple loop; away from the origin we have a *three-to-one* map. The function $z \mapsto z^n$ is an *n-to-one* map.

By contrast, the square-root function $g(z) = \sqrt{z} = \sqrt{z}e^{i\theta/2}$ *halves* argument of z: this can only be made explicit by insisting that $\theta = \operatorname{Arg} z$ is the *principal argument* of z: then $\arg(\sqrt{z}) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Written using the fractional-exponent notation, we see that

$$z \mapsto z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$$

is *multi-valued* (in this case *two-valued*). There are ways to make this watertight as a *function*, but not for a while :)

Exercises. 2.1.1. For each function, describe its implied domain (page 12).

(a) $f(z) = \frac{1}{4+z^2}$ (b) $f(z) = \frac{z-1}{e^z-1}$ (c) $f(z) = \frac{z^2+z+1}{z^4-1}$

2.1.2. Write the function in terms of its real and imaginary parts: $f(z) = u(x,y) + iv(x,y)$.

(a) $f(z) = z^3 - 4z^2 + 2$ (b) $f(z) = \frac{z^2}{1-\overline{z}}$ (c) $f(z) = e^{\overline{z}}$

2.1.3. Write the function $f(z) = \frac{1}{|z|^2}\overline{z}$ in polar form.

2.1.4. Find an expression for the function which reflects across the vertical line through the point $\alpha = -1$.

2.1.5. For Example 2.1(b) evaluate the function $g(z) = e^{2i\phi}(\overline{z-\beta}) + \beta$. Why are you not surprised by the result?

2.1.6. Let $\phi = \tan^{-1}\frac{3}{4}$. Find the result (in rectangular co-ordinates) of rotating $z = -2+i$ counter-clockwise by ϕ radians around the origin.
(*Hint: consider a 3:4:5 triangle!*)

2.1.7. Prove, using the expressions on page 13, that the composition of two reflections is a rotation, and that the composition of a rotation and a reflection is a reflection.

2.2 Limits & Continuity (§15–18)

As in real analysis, before we consider derivatives we first require continuity and the notion of *limits*. The definition is identical to that in real analysis:

Definition 2.2. Let $f : D \to \mathbb{C}$ be a function, and $w_0, z_0 \in \mathbb{C}$ be such that D contains a punctured neighborhood[a] of z_0. We say that w_0 is the *limit of f as z approaches z_0*, and write $\lim_{z \to z_0} f(z) = w_0$, if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

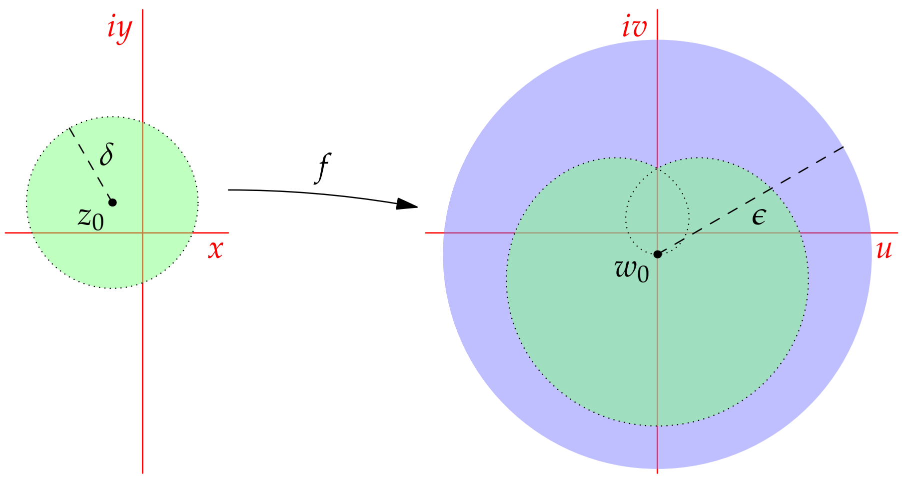
We say that f is *continuous* at z_0 if $w_0 = f(z_0)$ in the above definition: specifically

$$\lim_{z \to z_0} f(z) \text{ exists}, \quad f(z_0) \text{ exists and } \quad \lim_{z \to z_0} f(z) = f(z_0)$$

More generally, f is *continuous* if it is continuous at every point of its domain D.

[a] f is defined on a disk centered at z_0, but not necessarily *at* z_0 itself: $\exists r > 0$ such that $0 < |z - z_0| < r \implies z \in D$.

In the picture, given an ϵ-ball centered at w_0, we can choose $\delta > 0$ such that the δ-ball centered at z_0 is mapped to a region *inside* the original ϵ-ball.



Example 2.3. The picture above is that of the function $f : \mathbb{C} \to \mathbb{C} : z \mapsto z^2$, where

$$z_0 = \frac{1}{2} e^{3\pi i/4} = \frac{1}{2\sqrt{2}}(-1 + i), \qquad w_0 = -\frac{i}{4}, \qquad \text{and} \quad \epsilon = \frac{5}{2}$$

are given. It can be checked that $\delta = 1$ suffices for the definition: indeed

$$\begin{aligned} |z - z_0| < 1 &\implies |z| < 1 + |z_0| = \frac{3}{2} \\ &\implies |z^2 - w_0| = \left| z^2 + \frac{i}{4} \right| \le |z|^2 + \frac{1}{4} < \frac{9+1}{4} = \epsilon \end{aligned}$$

where we used the triangle-inequality. Since $w_0 = z_0^2 = f(z_0)$, we see that f is continuous at z_0.

Just as in real-analysis, we can prove limits and continuity generally: indeed the same proofs often work…

Example (cont). We show that $f(z) = z^2$ is continuous at every $z_0 \in \mathbb{C}$.

Let z_0 and $\epsilon > 0$ be given and define $\delta = \min\{1, \frac{\epsilon}{1+2|z_0|}\}$. We again use the triangle-inequality:

$$|z - z_0| < \delta \implies |z + z_0| = |z - z_0 + 2z_0| \leq |z - z_0| + 2|z_0| < \delta + 2|z_0| \leq 1 + 2|z_0|$$

from which

$$|z - z_0| < \delta \implies \left|z^2 - z_0^2\right| = |z - z_0| |z + z_0| < \delta(1 + 2|z_0|) \leq \epsilon$$

Basic Theorem on Limits

Most of the simple theorems on limits are unchanged from real analysis (certainly from *multi-variable* real analysis). We state some here for reference: try to prove as many as you can.

Theorem 2.4. *Throughout, let $f, g : D \to \mathbb{C}$ be functions and $z_0 = x_0 + iy_0$ be a point satisfying the assumptions of Definition 2.2.*

1. *Limits are unique: if w_0 and $\hat{w}_0$ satisfy Definition 2.2, then $\hat{w}_0 = w_0$.*
2. *If $f(z) = u(x,y) + iv(x,y)$ and $w_0 = u_0 + iv_0$, then*

$$\lim_{z \to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

3. *Suppose $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1$:*
   - (a) *For any $a, b \in \mathbb{C}$, $\lim_{z \to z_0} (af(z) + bg(z)) = aw_0 + bw_1$*
   - (b) $\lim_{z \to z_0} (f(z)g(z)) = w_0 w_1$
   - (c) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$*, provided $w_1 \neq 0$ and $g(z) \neq 0$ on a neighbourhood of z_0.*
   - (d) *If $h : E \to \mathbb{C}$ is a function such that $\lim_{z \to w_0} h(z) = w_2$, then $\lim_{z \to z_0} h(f(z)) = w_2$*
4. $\lim_{z \to z_0} |f(z)| = |w_0|$
5. *For any $n \in \mathbb{N}$, $\lim_{z \to z_0} z^n = w_0^n$*

Parts 2–5 can be easily restated in the language of continuous *functions.*

In particular, it follows that every polynomial function is continuous on $\mathbb{C}$, and that every rational function is continuous everywhere that $q(z) \neq 0$. Since cosine and sine are continuous on $\mathbb{R}$, we see that the exponential function $f(z) = e^z = e^x e^{iy}$ is also continuous on $\mathbb{C}$.

Continuity and compactness

Just as for real-valued functions, continuous functions on closed bounded regions (*compact sets*) must be bounded. First recall the definition of an *open set* on page 12: we say that a set is *closed* if its complement is open. You possibly saw the following in a previous analysis class (for subsets of $\mathbb{R}$ and $\mathbb{R}^2$ rather than $\mathbb{C}$):

Lemma 2.5. *A set $K \subseteq \mathbb{C}$ is closed if and only if every convergent sequence $(z_n) \subseteq K$ has its limit in K.*

Theorem 2.6. *A continuous function on a compact (closed bounded) set is bounded.*

Proof. Let the function be $f : K \to \mathbb{C}$ where K is compact.

- Let $g(z) = |f(z)|$ and let $M = \sup\{g(z) : z \in D\}$. Since g is a real-valued function, $\exists (z_n) \subseteq K$ such that $g(z_n) \to M$.
- K is bounded; the Bolzano–Weierstraß theorem says (z_n) has a convergent subsequence (z_{n_k}) with limit z_0.
- K is closed, whence $z_0 \in K$ and $g(z_0)$ is defined.
- g is continuous, whence $g(z_{n_k}) \to g(z_0)$: necessarily $M = g(z_0)$ is finite. ∎

The point at infinity and the Riemann Sphere[2]

Essentially everything regarding limits and continuity has thusfar been identical to the situation in real analysis. The treatment of infinity provides a major difference. In two-dimensional real analysis, which includes complex analysis, there is a choice to be made: one infinity, or many? To put it another way, we need to decide whether the sequences defined by $z_n = n$ and $w_n = in$ 'diverge' to the 'same' infinity, or whether we should consider ∞ and $i\infty$ as separate 'objects.' The convention in complex analysis is to have a single infinity.

Definition 2.7. The *extended complex plane* or *Riemann sphere* is the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where the symbol ∞ denotes an extra point with the property that the notion of limit (Definition 2.2) extends to neighborhoods of ∞, which include any set of the form $\{\infty\} \cup \{z \in \mathbb{C} : |z| > M\}$. In particular:

- $\lim\limits_{z \to z_0} f(z) = \infty$ means: $\forall M > 0, \exists \delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z)| > M$
- $\lim\limits_{z \to \infty} f(z) = w_0$ means: $\forall \epsilon > 0, \exists N > 0$ such that $|z| > N \implies |f(z) - w_0| < \epsilon$
- $\lim\limits_{z \to \infty} f(z) = \infty$ means: $\forall M > 0, \exists N > 0$ such that $|z| > N \implies |f(z)| > M$

The *Riemann sphere* gets its name because the extended complex plane can be easily visualized as a sphere with ∞ playing the role of the north pole. The remaining ('finite') points on the sphere are identified with the points in the equatorial plane via the *stereographic projection*, a natural bijection of

[2] In honor of the German mathematician Bernhard Riemann.

the sphere with the complex plane $S^2 \to \overline{\mathbb{C}}$: given a point $P \in S^2 \setminus \{N\}$, its image is the intersection of the equatorial plane with the line through P and N. This interpretation is not critical to us, though it is incredibly fun and interesting! More important are the intuitive relationships between functions, limits, zero and infinity.

Theorem 2.8. *Provided all limits make sense:*

- $\lim\limits_{z\to z_0} f(z) = \infty \iff \lim\limits_{z\to z_0} \frac{1}{f(z)} = 0$
- $\lim\limits_{z\to\infty} f(z) = w_0 \iff \lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = w_0$
- $\lim\limits_{z\to\infty} f(z) = \infty \iff \lim\limits_{z\to 0} \frac{1}{f(1/z)} = 0$

In essence this feels very much like the dubious $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$! As a memory aid, this is more effective than in real analysis, where $\pm\infty$ make life more complicated. Provided continuity is satisfied (following the above limits), it is common to extend the definition of functions to the extended complex plane.

Examples 2.9. 1. $f(z) = \frac{1}{z}$ is a continuous bijection $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying $f(0) = \infty$ and $f(\infty) = 0$.

2. Consider $f(z) = \frac{5iz+1}{3z-2i}$. Since

$$\lim_{z\to\frac{2}{3}i} f(z) = \infty \quad \text{and} \quad \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{5i + 1/z}{3 - 2i/z} = \frac{5}{3}i$$

we can consider f as a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying $f\left(\frac{2}{3}i\right) = \infty$ and $f(\infty) = \frac{5}{3}i$. Viewed this way, f also turns out to be a bijection (exercise)!

Unless indicated otherwise, we will assume that all sets are subsets of the (*finite*) complex plane.

Exercises. 2.2.1. Use the ϵ–δ definition (2.2) to prove the following.

(a) $\lim\limits_{z\to z_0} \overline{z} = \overline{z_0}$ (b) $\lim\limits_{z\to 0} \frac{\overline{z}^2}{z} = 0$ (c) $\lim\limits_{z\to 2} \frac{1}{z-i} = \frac{1}{2-i}$

2.2.2. Show that the function $f(z) = \left(\frac{z}{\overline{z}}\right)^2$ has value 1 at all non-zero points on the real and imaginary axes, but that it has the value -1 at all non-zero points on the line $y = x$. Hence explain why $\lim\limits_{z\to 0} f(z)$ does not exist.

2.2.3. Use Definition 2.7 to prove part of Theorem 2.8: $\lim\limits_{z\to z_0} f(z) = \infty \implies \lim\limits_{z\to z_0} \frac{1}{f(z)} = 0$.

2.2.4. (a) Show that $f(z) = \frac{5iz+1}{3z-2i}$ defines a bijection of the Riemann sphere.
(*Hint: let $w = f(z)$ and solve for z . . .*)

(b) More generally, for any complex numbers $\alpha, \beta, \gamma, \delta$, consider the function $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Prove that this defines a bijection of the Riemann sphere if and only if $\alpha\delta - \beta\gamma \neq 0$. How does this discussion relate to the 2×2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$?

2.3 Derivatives & the Cauchy–Riemann Equations (§19-23)

We finally come to the topic where complex analysis starts to show some serious differences from real analysis. The definition of derivative is the same as you are used to.

Definition 2.10. Let $f : D \to \mathbb{C}$ be a complex function and $z_0 \in D$ (as usual we assume D is open so that z_0 is interior to D). We say that f is *differentiable at* z_0 if the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. In such a case, we call the limit the *derivative* of f at z_0, and denote it by $f'(z_0)$.
If $f'(z_0)$ is defined on the domain D, we say that f is *differentiable on* D (or is just *differentiable)*). It is common to write $f'(z)$ or $\frac{\mathrm{d}f}{\mathrm{d}z}$ for this function.

Thusfar, this looks to be uncontroversial. We try a few examples to see what happens. When we compute a limit as $z \to z_0$, it is often useful to write $z = x + iy$, $z_0 = x_0 + iy_0$, $\Delta x = x - x_0$ and $\Delta y = y - y_0$: we see that

$$z \to z_0 \iff (\Delta x, \Delta y) \to (0, 0)$$

Examples 2.11.

1. First a very familiar function. Let $f(z) = z^2$. Then

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{z^2-z_0^2}{z-z_0} = \frac{(z-z_0)(z+z_0)}{z-z_0}$$

   When taking limits, we only compute on on a punctured disk $0 < |z - z_0| < \delta$, whence

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} z + z_0 = 2z_0$$

   It follows that f is differentiable everywhere in $\mathbb{C}$, with derivative $f(z) = 2z$.

2. Now for something a little different. Let $f(z) = \overline{z}$. Then

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{x-iy-x_0+iy_0}{x+iy-x_0-iy_0} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

   Observe that this quotient takes different values depending on $\Delta x, \Delta y$:

$$\Delta y = 0 \implies \frac{f(z)-f(z_0)}{z-z_0} = 1 \qquad \Delta x = 0 \implies \frac{f(z)-f(z_0)}{z-z_0} = -1$$

   For the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ to exist, we need to obtain the same value *regardless of how* $(\Delta x, \Delta y) \to (0, 0)$. We conclude that $f'(z_0)$ *does not exist*. Indeed the function $f(z) = \overline{z}$ is not differentiable anywhere!

The second example should be surprising. There is barely a simpler complex-valued function than the complex conjugate, and yet this is not differentiable!

Rules for Differentiation

The basic rule for differentiation are identical as for real-valued functions, and can be proved the same way. We state them for reference, without proof.

Theorem 2.12. *Suppose f and g are differentiable (either at a point z_0 or as functions).*

1. *(Linearity)* *For any constants $a, b \in \mathbb{C}$, $\frac{\mathrm{d}}{\mathrm{d}z}(af(z) + bg(z)) = af'(z) + bg'(z)$*
2. *(Power Law)* *For any $n \in \mathbb{N}_0$, $\frac{\mathrm{d}}{\mathrm{d}z}z^n = nz^{n-1}$*
3. *(Product rule)* $\frac{\mathrm{d}}{\mathrm{d}z}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z)$
4. *(Quotient Rule)* *If $g(z) \neq 0$, then $\frac{\mathrm{d}}{\mathrm{d}z}\frac{f(z)}{g(z)} = \frac{f'(z)g(z)-f(z)g'(z)}{[g(z)]^2}$*
5. *(Chain Rule)* *If h is differentiable at $g(z_0)$, then $h \circ g$ is differentiable at z_0 and*

$$(h \circ g)'(z_0) = h'\big(g(z_0)\big)g'(z_0)$$

We immediately see that every polynomial and rational function is differentiable. Indeed we can easily compute familiar examples without caring whether we are in $\mathbb{R}$ or $\mathbb{C}$!

Example 2.13. $$\frac{\mathrm{d}}{\mathrm{d}z}\frac{3(z^2-2)^5 + z^2}{z^3+1} = \frac{[30z(z^2-2)^4 + 2z)](z^3+1) - 3z^2[3(z^2-2)^5 + z^2]}{(z^3+1)^2}$$

The Cauchy–Riemann Equations

It is now time to see the deeper reason behind why the complex conjugate function proved impossible to differentiate.
Let $f(z) = u(x,y) + iv(x,y)$ be written in terms of its real and imaginary parts. As before, we write $z = x + iy$, $z_0 = x_0 + iy_0$ and denote the differences by

$$\Delta z = z - z_0 = \Delta x + i\Delta y = (x - x_0) + i(y - y_0)$$

We attempt to evaluate the limit of the difference quotient

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x,y) - u(x_0,y_0) + i(v(x,y) - v(x_0,y_0))}{\Delta x + i\Delta y}$$

as $z \to z_0$. If this limit is to exist, then we *must* have the same result when evaluated along the curves approaching z_0 both horizontally and vertically:

Horizontally We have $\Delta y = 0$, and must evaluate

$$\lim_{\Delta x \to 0} \frac{u(x,y_0) - u(x_0,y_0) + i(v(x,y_0) - v(x_0,y_0))}{\Delta x} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x_0,y_0)$$

where we see the *partial derivatives* of the functions u and v.

Vertically We have $\Delta x = 0$, and must evaluate

$$\lim_{\Delta y \to 0} \frac{u(x_0, y) - u(x_0, y_0) + i(v(x_0, y) - v(x_0, y_0))}{i \Delta y} = \frac{1}{i} \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) (x_0, y_0)$$

If $f'(z)$ exists, then these limits must both equal $f'(z)$. Indeed we have proved:

Theorem 2.14. *If $f(z)$ is complex-differentiable, then the real and imaginary parts of f satisfy the* Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or equivalently $u_x = v_y$, $u_y = -v_x$. Moreover, we can always write

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Examples 2.15. 1. $f(z) = \overline{z} = x - iy$ has $u(x, y) = x$ and $v(x, y) = -y$. We quickly see that

$$u_x = 1 \neq -1 = v_y, \qquad u_y = 0 = v_x$$

Since u, v do not satisfy the Cauchy–Riemann equations anywhere (at least not both of them!), f fails to be differentiable at all $z \in \mathbb{C}$.

2. $f(z) = z^2 = x^2 - y^2 + 2ixy$ has $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We check

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x$$

As expected, u, v satisfy the Cauchy–Riemann equations. Moreover,

$$f'(z) = 2z = 2x + 2iy = u_x + iv_x = v_y - iu_y$$

3. $f(z) = \frac{z}{2+|z|^2} = \frac{x}{2+x^2+y^2} + \frac{iy}{2+x^2+y^2}$. We compute

$$u_x = \frac{2 - x^2 + y^2}{(2 + x^2 + y^2)^2} \qquad v_y = \frac{2 + x^2 - y^2}{(2 + x^2 + y^2)^2}$$
$$u_y = \frac{-2xy}{(2 + x^2 + y^2)^2} \qquad -v_x = \frac{2xy}{(2 + x^2 + y^2)^2}$$

The second Cauchy–Riemann equation is satisfied if and only if $xy = 0$. When $x = 0$, the first requires $y = 0$, and vice versa. We conclude that f is not differentiable at any non-zero $z \in \mathbb{C}$.

Notice that the Cauchy–Riemann equations only provide a *necessary* condition for differentiability. We do not (yet!) have a sufficient condition. However, we can easily check that this example is differentiable at $z = 0$, straight from the definition and the continuity of $|z|$:

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{1}{2 + |z|^2} = \frac{1}{2}$$

A Sufficient Condition for Differentiability?

Theorem 2.14 only provides a necessary condition for differentiability, which is therefore most useful in the contrapositive form:

$$\text{Cauchy–Riemann not satisfied} \implies f \text{ not differentiable}$$

We'd like this to be an if and only if, so that the Cauchy–Riemann equations become a positive condition for differentiability. To some extent we can.

Suppose a complex function $f(z) = u(x, y) + iv(x, y)$ has partial derivatives on an open neighborhood D of a point $z_0 = x_0 + iy_0$. Also suppose f satisfies the Cauchy–Riemann equations and that the partial derivatives $u_x = v_y$ and $u_y = -v_x$ are *continuous* at z_0. Write $\Delta z = z - z_0 = \Delta x + i\Delta y$. The rough idea here is that, for $z \in D$

$$\begin{aligned}
f(z) - f(z_0) &\approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \\
&= (u_x(x_0, y_0) + iv_x(x_0, y_0))\Delta x + (u_y(x_0, y_0) + iv_y(x_0, y_0))\Delta y \\
&= (u_x(x_0, y_0) + iv_x(x_0, y_0))\Delta x + (-v_x(x_0, y_0) + iu_x(x_0, y_0))\Delta y \\
&= (u_x(x_0, y_0) + iv_x(x_0, y_0))(\Delta x + i\Delta y) \\
\implies \frac{f(z) - f(z_0)}{z - z_0} &\approx u_x(x_0, y_0) + iv_x(x_0, y_0)
\end{aligned}$$

The continuity of the partial derivatives at (x_0, y_0) means that the approximation approaches equality as $\Delta z \to 0$. We don't give a complete proof of this since there are too many details. The upshot however is a near converse of Theorem 2.14.

Theorem 2.16. *Let u, v have partial derivatives on a neighbourhood of $z_0 = x_0 + iy_0$. Assume $f = u + iv$ satisfies the Cauchy–Riemann equations and has continuous partial derivatives at z_0. Then f is differentiable at z_0 and*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, v_0)$$

Examples 2.17.

1. Revisiting Example 2.15.3, we see that $f(z) = \frac{z}{2+|z|^2}$ has partial derivatives everywhere; these are continuous and satisfy Cauchy–Riemann at $z = 0$ and so f is differentiable there. Moreover

$$f'(0) = u_x(0) + iv_x(0) = \frac{1}{2}$$

2. The exponential function is now easily seen to be differentiable everywhere. Indeed

$$f(z) = e^z = e^x \cos y + ie^x \sin y$$

satisfies

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin x = -v_x$$

where these are certainly continuous on $\mathbb{C}$. Moreover, as expected,

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$$

Exercises. 2.3.1. Use Theorem 2.12 to find the derivatives of the following functions:

(a) $f(z) = \frac{1}{z^2+2z}$ (b) $f(z) = (z^3 + 2iz + 1)^7$ (c) $f(z) = \frac{(3z^2-i)^3}{(iz^3+4)^2}$

2.3.2. Use the limit definition of the derivative to compute the derivative of the functions:

(a) $f(z) = 3z^3 - iz^2$ (b) $f(z) = \frac{1}{z^2}$

2.3.3. Give a proof of the quotient rule, directly using the definition of the derivative.

2.3.4. Use the quotient rule to prove the power law for negative integer exponents: that is

$$\forall n \in \mathbb{N}, \quad \frac{\mathrm{d}}{\mathrm{d}z} z^{-n} = -nz^{-n-1}$$

2.3.5. Suppose $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exists, where $g'(z_0) \neq 0$. Use the definition of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

2.3.6. Prove that the functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not differentiable anywhere.

2.3.7. Exactly as in Example 2.17.2, prove that $\frac{\mathrm{d}}{\mathrm{d}z} e^{kz} = ke^{kz}$ for any complex constant k.

2.3.8. Consider the Cauchy–Riemann equations for the following functions: what can you conclude, if anything?

(a) $f(z) = \frac{z}{\overline{z}-i}$ (b) $f(z) = z^3 - \frac{2}{z}$ (c) $f(z) = (|z|^2 + z)^2$

2.3.9. Consider $f(z) = f(z, \overline{z})$ is a complex function written as a function of z and $\overline{z}$. For example,

$$f(z) = |z|^2 = z\overline{z}$$

Noting that $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$, use the chain rule

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}}$$

to prove that f satisfies the Cauchy–Riemann equations if and only if $\frac{\partial f}{\partial \overline{z}} = 0$.

Hence give a quick proof that $f(z) = z\overline{z}^2$ is not differentiable when $z \neq 0$.

2.4 Analytic and Harmonic Functions (§24–28)

As in real analysis, we tend to be most interested in functions which are differentiable at more than just one point.

Definition 2.18. Let D be an open subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *analytic* (on D) or (*holomorphic*) if it is differentiable at every point of D.

A function is *analytic at a point* z_0 if it is analytic on some open set containing z_0.

An *entire function* is an analytic function $f : \mathbb{C} \to \mathbb{C}$ (domain $= \mathbb{C}$!).

If f is not analytic at z_0, it is said to have a *singular point* at z_0.

Examples 2.19.
1. The function $f(z) = \frac{1}{z^2+4}$ is analytic except at its singularities $z = \pm 2i$: note that these are not in the domain of f, but are still singularities.
2. The function $f(z) = e^{4z}$ is entire, as is every polynomial.

Analytic functions have many properties in common with simple differentiable functions from real analysis. For instance.

Theorem 2.20. *If $f'(z) = 0$ on a (connected open) domain D, then $f(z)$ is constant.*

Proof. We can join any two points in D using a zig-zag path of finitely many horizontal and vertical segments.[a] Since $f'(z) = u_x + iv_x = u_y - iv_y = 0$, we see that all four partial derivatives are zero. For a horizontal segment ($x_1 \le x \le x_2$, y fixed), the mean value theorem says $\exists \hat{x} \in (x_1, x_2)$ with

$$u(x_2, y) - u(x_1, y) = (x_2 - x_1)u_x(\hat{x}, y) = 0$$

whence u is constant along any horizontal segment. The same holds for v, and similarly for both along any vertical segment. We conclude that u, v take the same values at any two points in D. ∎

[a] If you've done enough topology... Choose any path and select an open box around each point lying within D (openness). The path is closed and bounded, whence (compactness) finitely many boxes suffice to cover the path. It is now easy to describe a suitable zig-zag path within these boxes.

There is also significant restriction and many surprises when it comes to analytic functions.

Corollary 2.21. *If $|f(z)|$ is constant and f is analytic, then f is constant.*

Proof. Clearly $|f(z)|^2 = f(z)\overline{f(z)} = k$ is constant. If $k = 0$, we are done. Otherwise, $\overline{f(z)} = \frac{k}{f(z)}$ is analytic. Write $f = u + iv$, whence $\overline{f} = u - iv$, and consider the Cauchy–Riemann equations for *both*:

$$u_x = v_y, \qquad u_y = -v_x, \qquad u_x = -v_y, \qquad u_y = v_x$$

We conclude that all partial derivatives are zero, whence f is constant. ∎

Harmonic Functions

Perhaps the biggest surprise relating to analytic functions is the following:

Theorem 2.22. *If $f(z) = u + iv$ is analytic, then f is* infinitely differentiable. *Otherwise said:*

- $f^{(n)}(z)$ *exists and is continuous for all* $n \in \mathbb{N}$.
- u *and* v *have continuous partial derivatives of all orders.*

We will return to this important result (much) later in the course
.
If you think about it, however, this maybe isn't such a surprise. How does one cook up an example of a real function that is once differentiable but not twice? Use the absolute value: for example

$$f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \qquad \Longrightarrow \quad f'(x) = 2\,|x|$$

Corresponding functions of the complex numbers (using modulus), aren't differentiable at all!

For the present, it is worth seeing what happens to the Cauchy–Riemann equations of an analytic function once differentiated.

$$u_{xx} = \frac{\partial}{\partial x} u_x \overset{CR1}{=} \frac{\partial}{\partial x} v_y = v_{yx} \overset{(*)}{=} v_{xy} = \frac{\partial}{\partial y} v_x \overset{CR2}{=} -\frac{\partial}{\partial y} u_y = -u_{yy}$$

The equality $(*)$ of mixed partial derivatives follows because all derivatives are continuous. The same equation holds for v. We conclude:

Theorem 2.23. *If $f = u + iv$ is analytic in D, then u and v are both* harmonic functions*: that is, they are both solutions to* Laplace's Equation*:*

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Laplace's equation is one of the most important partial differential equations, and is widely used throughout mathematics and physics.

Example 2.24. $f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$ is analytic on $\mathbb{C} \setminus \{0\}$: its real and imaginary parts are therefore harmonic away from the origin. Indeed,

$$\begin{aligned} u_{xx} + u_{yy} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{x}{x^2 + y^2} = \frac{\partial}{\partial x} \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{\partial}{\partial y} \frac{2xy}{(x^2 + y^2)^2} \\ &= \frac{-2x(x^2 + y^2) - 4x(y^2 - x^2)}{(x^2 + y^2)^3} - \frac{2x(x^2 + y^2) - 8xy^2}{(x^2 + y^2)^3} = 0 \end{aligned}$$

Analytic Continuations

Though we cannot yet prove it, it is worth mentioning one last surprising property of analytic functions.

Theorem 2.25. *Suppose $f, g : D \to \mathbb{C}$ are analytic functions on an open connected domain D and that $f(z) = g(z)$ on any line segment contained in D. Then $f = g$ throughout D.*

This result should seem highly counter-intuitive since any equivalent would be palpably false in $\mathbb{R}$: in essence it says that you only need know the values of an analytic function on a line segment to know the full function on its whole (connected) domain! This gives rise to a new concept:

Definition 2.26. Let $f : D \to \mathbb{C}$ and $g : E \to \mathbb{C}$ be analytic functions on open connected domains such that $D \subseteq E$ and $f = g$ on D. We call g the *analytic continuation* of f to E.

By the theorem above, the analytic continuation must be unique (certainly $g = f$ on D). It is not guaranteed that a given analytic function has an analytic continuation to any larger domain. Here is the classic example.

Example 2.27. Consider $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$ with domain $D = \{z : \operatorname{Re} z, \operatorname{Im} z > 0\}$ being the first quadrant. In Exercise 2.4.4. below, we see that f is differentiable on D. We consider two analytic continuations of f:

(a) Let $G = \mathbb{C} \setminus \{-x : x \in \mathbb{R}_0^+\}$ omit zero and the *negative* real axis. Define

$$g : G \to \mathbb{C} : z \mapsto \sqrt{z} = \sqrt{r}e^{i\theta/2}$$

where $\theta = \operatorname{Arg} z \in (-\pi, \pi)$ is the principal argument of z. The codomain of g is the *right* half-plane $\{z : \operatorname{Re} z > 0\}$.

(b) Let $H = \mathbb{C} \setminus \{x : x \in \mathbb{R}_0^+\}$ omit zero and the *positive* real axis. Define

$$h : H \to \mathbb{C} : z \mapsto \sqrt{r}e^{i\theta/2}$$

where $\theta = \arg z$ is assumed to lie in the interval $(0, 2\pi)$. This codomain of h is the *upper* half-plane $\{z : \operatorname{Im} z > 0\}$.

These are both analytic continuations of f, however the functions do not agree on the entirety of the overlap of their domains. Indeed, choosing a point $z = 4e^{-3\pi/4} = 4e^{5\pi/4}$ in the third quadrant, we see that

(a) $g(z) = g(4e^{-3\pi/4}) = 2e^{-3\pi/8}$ lies in the *fourth quadrant*,

(b) $h(z) = h(4e^{5\pi/4}) = 2e^{5\pi/8}$ lies in the *second quadrant*.

Indeed $h(z) = -g(z)$ for *any* point z in the third quadrant!

It fact there is no analytic continuation of f to any domain which loops completely around the origin. We shall return to this topic later…

Exercises. 2.4.1. Suppose $g'(z) = h'(z)$ on an open connected domain D. Prove that $h(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

2.4.2. Check that $u = e^{nx} \cos ny$ and $v = e^{nx} \sin ny$ are harmonic functions for any $n \in \mathbb{Z}$.

2.4.3. Suppose $f(z) = u + iv$ and write $z = x + iy = re^{i\theta}$ in polar form.

(a) Use the chain rule applied to the polar co-ordinate relations

$$x = r\cos\theta, \qquad y = r\sin\theta$$

to compute the partial derivatives u_r, u_θ, v_r and v_θ.

(b) Deduce the polar form of the Cauchy–Riemann equations:

$$ru_r = v_\theta \qquad u_\theta = -rv_r, \qquad f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{-i}{z}(u_\theta + iv_\theta)$$

2.4.4. Use the previous exercise to prove that $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$ is differentiable on the first quadrant, and find $f'(z)$.

2.4.5. In Example 2.27, we saw that $g : E \to \mathbb{C} : z \mapsto \sqrt{z}$ is an analytic function on the domain E consisting of the complex plane without the non-positive real axis. Prove that there exists no analytic continuation of g to any set larger than E.
(*Hint: every differentiable function is continuous...*)

2.4.6. Prove the polar form of Laplace's equation:

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0$$

2.4.7. Show that $u = r^n \cos n\theta$ is a harmonic function for any $n \in \mathbb{N}$: find *two* ways to show that this is true!

3 Elementary Functions

We already know a great deal about polynomials and rational functions: these are analytic on their entire domains. We have thought a little about the square-root function and seen some difficulties. The remaining elementary functions are the exponential, logarithmic and trigonometric functions.

3.1 The Exponential and Logarithmic Functions (§30–32, 34)

We have already defined the exponential function $\exp : \mathbb{C} \to \mathbb{C} : z \mapsto e^z$ using Euler's formula

$$e^z := e^x \cos y + ie^x \sin y \tag{$*$}$$

and seen that its real and imaginary parts satisfy the Cauchy–Riemann equations on $\mathbb{C}$, whence exp is entire (analytic on $\mathbb{C}$). Indeed recall that $\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$. We have also seen several of the basic properties of the exponential function, we state these and several others for reference.

Lemma 3.1. *Throughout let $z, w \in \mathbb{C}$.*

1. $e^z \neq 0$.
2. $e^{z+w} = e^z e^w$ *and* $e^{z-w} = \dfrac{e^z}{e^w}$
3. *For all* $n \in \mathbb{Z}$, $(e^z)^n = e^{nz}$.
4. e^z *is* periodic *with period* $2\pi i$. *Indeed more is true:*

$$e^z = e^w \iff z - w = 2\pi in \text{ for some } n \in \mathbb{Z}$$

Proof. Part 1 follows trivially from $(*)$. To prove 2, recall the multiple-angle formulae for cosine and sine. Part 3 requires an induction using part 2 with $z = w$.

Part 4 is more interesting: certainly $e^{w+2\pi in} = e^w$ by the periodicity of sine and cosine. Now suppose $e^z = e^w$ where $z = x + iy$ and $w = u + iv$. Then, by considering the modulus and argument,

$$e^x e^{iy} = e^u e^{iv} \implies \begin{cases} e^x = e^u \\ y = v + 2\pi in \text{ for some } n \in \mathbb{Z} \end{cases}$$

We conclude that $x = u$ and so $z - w = i(y - v) = 2\pi in$. ∎

Example 3.2. Find all $z \in \mathbb{C}$ such that $e^z = 5(-1 + i)$.

Following the Lemma, write $z = x + iy$ and take the polar form of $5(-1 + i)$ to see that

$$e^z = 5(-1+i) \iff e^x e^{iy} = 5\sqrt{2}e^{3\pi i/4} \iff \begin{cases} x = \ln(5\sqrt{2}) \\ y = \frac{3\pi}{4} + 2\pi n \quad \text{for some } n \in \mathbb{Z} \end{cases}$$

$$\iff z = \ln(5\sqrt{2}) + \left(\frac{3\pi}{4} + 2\pi n\right) i \quad \text{for some} \quad n \in \mathbb{Z}$$

We see that there are *infinitely many* suitable z!

Duplicate Notation Warning! When $n \in \mathbb{N}$, the expression $e^{1/n}$ can now mean two things. For instance $e^{1/3}$ can mean:

1. The *set* of cube roots of e, namely $\{\sqrt[3]{e}, \sqrt[3]{e}e^{2\pi i/3}, \sqrt[3]{e}e^{-2\pi i/3}\}$;
2. The *real value* $\sqrt[3]{e} \in \mathbb{R}^+$.

Given that e^z is such a common function, in both real and complex analysis, we default to the second meaning: if you mean the set of n^{th} roots, say so!

The periodicity of the exponential leads to the far more interesting notion of the complex *logarithm.*

Definition 3.3. Let $z = re^{i\theta}$ be a non-zero complex number with principal argument $\theta = \operatorname{Arg} z$. The *principal logarithm* of z is the value

$$\operatorname{Log} z := \ln r + i\theta = \ln|z| + i \operatorname{Arg} z$$

where ln is the usual natural logarithm. The *logarithm* of z is any (and all!) of the values[a]

$$\log z = \ln|z| + i \arg z = \ln r + i(\theta + 2\pi n) : n \in \mathbb{Z}$$

[a]This is identical to how we use $\arg z$, which, depending on context, means either the *set* $\{\operatorname{Arg} z + 2\pi n i\}$ or some particular value from this set. We'll more formally discuss such *multi-valued* functions in the next section.

Examples 3.4. 1. Since $-4 = 4e^{\pi i}$, we see that

$$\operatorname{Log}(-4) = \ln 4 + \pi i \quad \text{and} \quad \log(-4) = \ln 4 + (1+2n)\pi i$$

2. Again write in polar form to compute:

$$\operatorname{Log}(\sqrt{3} - i) = \operatorname{Log}(2e^{-\pi i/6}) = \ln 2 - \frac{\pi}{6} i \quad \text{and} \quad \log(\sqrt{3} - i) = \ln 2 - \frac{\pi}{6} i + 2\pi n i$$

These examples involve solving equations of the form $e^w = z$: writing $z = re^{i\theta} = e^{\ln r + i\theta}$ as above, and appealing to part 4 of Lemma 3.1, we instantly see that

$$e^w = z \iff w = \log z$$

Read this carefully, remembering that the logarithm is multi-valued and the exponential periodic:

$$e^{\log z} = z \qquad \text{and} \qquad \log(e^w) = w + 2\pi n i \quad \text{where } n \in \mathbb{Z}$$

Before moving on, we clear up some of the basic properties of the principal logarithm function. All parts of this should be clear from Definition 3.3.

Lemma 3.5. *Throughout, z and w are complex numbers with $z \neq 0$, and $n \in \mathbb{Z}$.*

- $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \{w \in \mathbb{C} : \operatorname{Im} w \in (-\pi, \pi]\}$ *is a bijection with inverse* exp.
- $\operatorname{Log}(e^w) = w + 2\pi ni$ *where $n \in \mathbb{Z}$ is chosen such that* $\operatorname{Im} w + 2\pi n \in (-\pi, \pi]$.
- *If $z \in \mathbb{R}^+$, then* $\operatorname{Log} z = \ln z$ *is the usual* natural *logarithm.*

The Logarithm Laws

Just as the standard rules for exponentiation (Lemma 3.1 parts 2 and 3) apply to the complex exponential, something similar works for the log laws. However, the multi-valued nature of the logarithm makes this a little more subtle.
Suppose non-zero z, w are given: since $|zw| = |z|\,|w|$ and $\arg zw = \arg z + \arg w$, we conclude that

$$\begin{aligned}\log zw &= \ln |z|\,|w| + i(\arg z + \arg w) = \ln |z| + i \arg z + \ln |w| + i \arg w \\ &= \log z + \log w\end{aligned}$$

Be very careful with this expression, since it is *not* an identity of *functions.* Indeed it means two things:

- We have *set equality*: in particular, the following sets are identical:

$$\begin{aligned}\log z + \log w &= \{\alpha + \beta : \alpha \in \log z, \beta \in \log w\} \\ &= \{|z| + i \operatorname{Arg} z + 2\pi ki + |w| + i \operatorname{Arg} w + 2\pi mi : k, m \in \mathbb{Z}\} \\ \log zw &= \{\ln |z|\,|w| + i \operatorname{Arg} zw + 2\pi ni : n \in \mathbb{Z}\}\end{aligned}$$

- There exist *particular choices* of the arguments of z, w and zw so that $\arg zw = \arg z + \arg w$.

Unless you are sure you won't make a mistake, it is therefore safer to write

$$\boxed{\log zw = \log z + \log w + 2\pi ni \quad \text{where} \quad n \in \mathbb{Z}}$$

Given its restricted range, we can be more precise for the principal logarithm:

$$\exists n \in \{0, \pm 1\} \text{ such that } \operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + 2\pi ni$$

Example 3.6. Let $z = -\sqrt{3} + i = 2e^{5\pi/6}$ and $w = \sqrt{2}(1 + i) = 2e^{\pi i/4}$. Then

$$\begin{aligned}&\log z = \ln 2 + \frac{5\pi}{6} i + 2\pi ki, \qquad \log w = \ln 2 + \frac{\pi}{4} i + 2\pi mi \\ &\log zw = \log(4e^{\frac{5\pi}{6} + \frac{\pi}{4}}) = \log(4e^{13\pi/12}) = \ln 4 + \frac{13\pi}{12} i + 2\pi ni\end{aligned}$$

For the above boxed formula to make sense for particular choices of logarithms, we'd need to select $k + m = n + 1$. More explicitly,

$$\begin{aligned}&\operatorname{Log} z = \ln 2 + \frac{5\pi}{6} i, \qquad \operatorname{Log} w = \ln 2 + \frac{\pi}{4} i \\ &\operatorname{Log} zw = \operatorname{Log}(4e^{-11\pi/12}) = \operatorname{Log}(4e^{-11\pi/12}) = \ln 4 - \frac{11\pi}{12} i = \operatorname{Log} z + \operatorname{Log} w - 2\pi i\end{aligned}$$

We can similarly demonstrate the other log law, with the same caveat:

$$\log \frac{z}{w} = \log z - \log w$$

You might assume that the final log law ($\log z^n = n \log z$ whenever $n \in \mathbb{N}$) also makes sense, but you'd be very wrong: an example should explain why you should ignore this law!

Example 3.7. Let $z = -\sqrt{3} + i = 2e^{5\pi/6}$ and compute what is generally meant by the *set* $2\log z$:

$$2\log z = 2\left(\ln 2 + \frac{5\pi}{6}i + 2\pi mi\right) = \ln 4 + \frac{5\pi}{3}i + 4\pi mi$$

that is, we double the value of everything in the set $\log z$. This is different from the set

$$\log z^2 = \log(4e^{10\pi/6}) = \ln 4 + \frac{5\pi}{3}i + 2\pi ki$$

However, in the language of the previous exercise, there would be no problem if each copy of $\log z$ gots its own multiples of $2\pi i$:

$$\begin{aligned}\log z + \log z &= \ln 4 + \frac{5\pi}{3}i + 2\pi(l+m)i \quad \text{where} \quad l, m \in \mathbb{Z} \\ &= \log z^2\end{aligned}$$

Since this is *not* how $2\log z$ is generally interpreted, it is safer to state that $\log z^2 \neq 2\log z$.

Since the principal logarithm is a function rather than a set, we can be more precise: for any $n \in \mathbb{N}$,

$$\operatorname{Log} z^n = n \operatorname{Log} z + 2\pi ki \quad \text{for some integer } k \text{ with } |k| \leq \frac{n}{2}$$

Example 3.8. Let $z = e^{-13\pi i/16}$ and consider z^{16}. We see that

$$\operatorname{Log} z^{16} = \operatorname{Log} e^{-13\pi i} = \operatorname{Log} e^{\pi i} = i\pi, \qquad 16 \operatorname{Log} z = -13\pi i \implies \operatorname{Log} z^{16} = 16 \operatorname{Log} z + 14\pi i$$

Exercises. 3.1.1. Compute the following:

(a) $\exp(3-\frac{\pi}{2}i)$ (b) $\operatorname{Log}(ie)$ (c) $\log(3-4i)$ (d) $\operatorname{Log}[(-1+i)^2]$

3.1.2. (a) If e^z is real, show that $\operatorname{Im} z = n\pi$ for some integer n.

(b) If e^z is imaginary, what restriction is placed on z?

3.1.3. Show in two ways that the function $f(z) = \exp(z^2)$ is entire, and find its derivative.

3.1.4. Prove, for any $z \in \mathbb{C}$, that $\left|\exp(z^2)\right| \le \exp|z|^2$. What must z satisfy if this is to be *equality*?

3.1.5. Find $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

3.1.6. Show that $\operatorname{Log} i^3 \neq 3 \operatorname{Log} i$.

3.1.7. The square roots of i are $\sqrt{i} = e^{\pi i/4}$ and $-\sqrt{i} = e^{-3\pi i/4}$.

(a) Compute $\operatorname{Log}\sqrt{i}$ and $\operatorname{Log}(-\sqrt{i})$ and check that $\operatorname{Log}\sqrt{i} = \frac{1}{2}\operatorname{Log} i$.

(b) Show that the set of all logarithms of all square roots of i is

$$\log i^{1/2} = \left(n+\frac{1}{4}\right)\pi i \quad \text{where} \quad n \in \mathbb{Z}$$

and therefore deduce that $\log i^{1/2} = \frac{1}{2}\log i$ *as sets.*

3.2 Multi-valued functions, Branch Cuts and the Power Function (§33, 35, 36)

The complex logarithm is often called a *multi-valued function*, since each $\log z$ represents a set of complex numbers. We have already seen several of these beasts:

- The argument of a complex number is any of the values $\arg z = \operatorname{Arg} z + 2\pi i n$ where $n \in \mathbb{Z}$. The logarithm is simply a modification of this: $\log z = \ln r + i \arg z$.
- The n^{th} root of $z \in \mathbb{C}$ is the set of values $z^{1/n} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$ where $\omega_n = e^{2\pi i/n}$.

It is something of an abuse of language to refer to a multi-valued *function* $f : D \to \mathbb{C}$, since a function must assign *exactly one* object to each element of a domain. While this problem can be fixed very simply using equivalence classes,[3] for the purposes of calculus another approach is more useful.

Definition 3.9. A *branch* of a multi-valued function f is a single-valued function F on a domain D which is *analytic* on D and such that each $F(z)$ is one of the values of $f(z)$.

Let $D = \mathbb{C} \setminus \ell$ where ℓ is a line or curve in $\mathbb{C}$. If $F : D \to \mathbb{C}$ is a branch of f, we call ℓ a *branch cut.* A *branch point* is any point common to all branch cuts.

[3]Define $z \sim w \iff z - w = 2\pi n$ for some $n \in \mathbb{Z}$, then we have a *single-valued* function whose codomain is a set of equivalence classes

$$\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}/\sim$$

The idea that $\log z$ sometimes means the set $\{\operatorname{Arg} z + 2\pi i n\}$ and sometimes a specific element from that set is just the common fudge of choosing a representative from an equivalence class.
The same approach can be used to make the n^{th} root single-valued: just define $z \sim w \iff z^n = w^n$.

The *principal logarithm* $\operatorname{Log} z = \ln r + i \operatorname{Arg} z$ is a branch of the logarithm function, sometimes called the *principal branch.* The branch cut in this case is the non-positive real axis.[4] More generally, for any angle α take the branch cut ℓ to be the line with argument α and define a branch of the logarithm by

$$\log z = \ln r + i\theta$$

There is still something to check here!

Lemma 3.10. *For any α, the branch $\log z$ defined above really is a branch: i.e. it is analytic on its domain. Moreover $\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}$.*

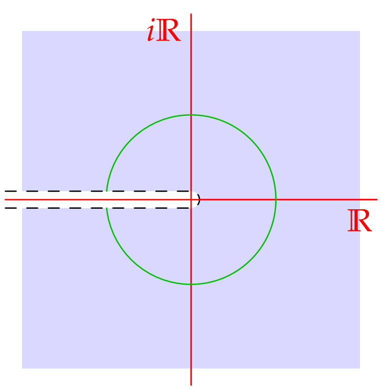
Proof. Simply check the Cauchy–Riemann equations in polar form:[a] since $\ln r + i\theta = u + iv$, we obtain

$$ru_r = 1 = v_\theta, \qquad u_\theta = 0 = -rv_r$$

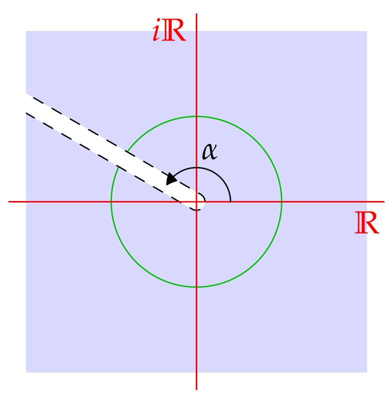
The partial derivatives are certainly continuous throughout D, whence $\log z$ is analytic and has derivative $\frac{\mathrm{d}}{\mathrm{d}z} \log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{r} e^{-i\theta} = \frac{1}{z}$. ∎

[a] If you struggle to remember these, quickly compute the partial derivatives for $z = re^{i\theta} = r\cos\theta + ir\sin\theta$!

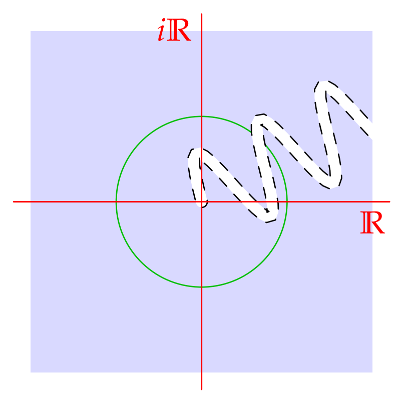
It is not too hard to describe all branches of the logarithm, since its multi-valued nature is precisely that of the argument. For $\arg z$ to be multi-valued on an open connected domain D, there must be a closed path lying in D which encircles the origin: traversing this path counter-clockwise increases the value of $\arg z$ by 2π. To make the logarithm single-valued, we must choose a branch cut which 'cuts' any such encircling path. Any (reasonable) curve from the origin to ∞ will do the trick. Three branch cuts for the logarithm are shown below. Note also that the origin is the only branch point for the logarithm.[5]



Principal Branch



α-Branch



An Esoteric Branch

It is very important to be clear when working with multi-valued functions: is $\log z$ a *set*, a particular element from that set, or a *branch*? Certain expressions may be true or false depending on the meaning.

[4] As an analytic function on an open domain, we need to restrict to $\operatorname{Arg} z \in (-\pi, \pi)$, so this is very slightly different to the earlier definition.

[5] If one considers the extended complex plane, a branch cut is then a curve joining the *two* branch points 0 and ∞.

Example 3.11. Consider $z = \frac{1}{\sqrt{2}}(1+i) = e^{\pi i/4}$. For the principal branch, we have

$$\operatorname{Log} z^2 = \operatorname{Log} e^{\pi i/2} = \frac{\pi}{2}i = 2\operatorname{Log} z$$

For the branch with $\alpha = \frac{\pi}{3}$, we have

$$z^2 = e^{\pi i/2} = e^{-3\pi i/2} \implies \log z^2 = -\frac{3\pi}{2} \neq 2\log z$$

Recall also (Example 3.7), that *as sets*, $\log z^2 \neq 2\log z$.

For particular branches of the logarithm, specific versions of the logarithm laws are available; for instance recall that the principal branch satisfies

$$\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + 2\pi n i \quad \text{for some } n \in \{0, \pm 1\}$$

In practice, it is not worth trying to remember these; just take care and think out the possibilities!

General Exponential Functions

The logarithm can be used to define exponential functions for any non-zero complex base c:

$$c^z := e^{z \log c}$$

Since $\log c$ is multi-valued, so also is c^z. Provided c is not a negative real number, then the standard interpretation of c^z uses the principal logarithm. Regardless, once a branch of logarithm is chosen, $\log c$ is a constant and the exponential function is analytic everywhere and satisfies the usual rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = c^z \log c$$

Example 3.12. Let $c = i = e^{\pi i/2}$ and use the principal logarithm to define

$$i^z := e^{z \operatorname{Log} i} = \exp\left(\frac{\pi z}{2} i\right) = \exp\left(-\frac{\pi}{2} y + i\frac{\pi}{2} x\right) = e^{-\pi y/2}\left[\cos\frac{\pi}{2}x + i\sin\frac{\pi}{2}x\right]$$

It is simple to check the Cauchy–Riemann equations for this function and see that it is analytic on $\mathbb{C}$. We could have chose another branch of the logarithm to define the function however; if we took the α-branch with $\alpha = \frac{\pi}{3}$, then $i = e^{-3\pi i/2}$ and so

$$i^z = \exp\left(\frac{-3\pi z}{2} i\right) = e^{3\pi y/2}\left[\cos\frac{3\pi}{2}x - i\sin\frac{3\pi}{2}x\right]$$

This is still entire, however it is a completely different function! Note that both definitions of i^z agree whenever z is an integer, but for $z = \frac{1}{2}$ these produce the two distinct square roots of i!

Power Functions

Similarly to the general exponential function, for any non-zero z and complex number c, we can define

$$z^c := e^{c \log z}$$

In this case, restricting to the principal branch of the logarithm gives an unambiguous function:

Definition 3.13. The *principal value* of z^c is the function

$$\mathrm{P.V.}\, z^c := e^{c \operatorname{Log} z}$$

with domain excluding the non-positive real axis.

The power function is usually multi-valued, although if $c = m$ is an integer it is not: indeed

$$z^m = e^{m(\operatorname{Log} z + 2\pi n i)} = e^{m \operatorname{Log} z} e^{2\pi m n i} = e^{m \operatorname{Log} z} = \mathrm{P.V.}\, z^m$$

is independent of the branch of logarithm chosen and recovers the usual notion of z^m. For exponents $c = \frac{1}{m}$, the principal value of $z^{1/m}$ is precisely the principal m^{th} root of z as defined earlier. Choosing another branch of the logarithm results in a different branch of $z^{1/m}$.

Example 3.14. Let $m = 3$ and define $z^{1/3}$ using the principal branch of the logarithm: writing $\Theta = \operatorname{Arg} z$, we recover

$$\mathrm{P.V.}\, z^{1/3} = e^{\frac{1}{3}(\ln r + i\Theta)} = e^{\ln r^{1/3} + \frac{i\Theta}{3}} = r^{1/3} e^{i\Theta/3}$$

precisely the principal cube-root of z.
If instead we choose an α-branch,[a] then we have $\theta = \arg z \in (\alpha, \alpha + 2\pi)$ from which $\theta = \Theta + \alpha + \pi$ and so

$$z^{1/3} = r^{1/3} e^{i\theta/3} = r^{1/3} e^{i\Theta/3} e^{i(\alpha+\pi)/3} = \sqrt[3]{z} e^{i(\alpha+\pi)/3}$$

In particular, if $\alpha = \pi$ we obtain $z^{1/3} = \sqrt[3]{z} e^{2\pi i/3} = \sqrt[3]{z}\omega_3$, one of the other cube-roots of z.

[a] The principal branch is when $\alpha = -\pi$!

Lemma 3.15. *Choose a branch of the logarithm so that $z^c = e^{c \log z}$ is defined. Then z^c is analytic on the same domain as the logarithm and moreover $\frac{\mathrm{d}}{\mathrm{d}z} z^c = z^{c-1}$, defined using the same branch of the logarithm as for z^c.*

Proof. Since $\log z$ is analytic, we simply use the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}z} z^c = \frac{\mathrm{d}}{\mathrm{d}z} e^{c \log z} = e^{c \log z} \frac{\mathrm{d}}{\mathrm{d}z}(c \log z) = e^{c \log z} \cdot \frac{c}{z} = c e^{c \log z} e^{-\log z} = c e^{(c-1)\log z} = c z^{c-1}$$

■

Example 3.16. f the principal branch of the logarithm is used, then

$$(zw)^c = \exp(c\,\mathrm{Log}(zw)) = \exp(c\,\mathrm{Log}\,z + c\,\mathrm{Log}\,w + 2\pi cni) = z^c w^c e^{2\pi cni}$$

for some $n \in \{0, \pm 1\}$. We do not therefore expect simple exponent rules such as $(ab)^c = a^c b^c$ to hold in complex analysis. Note, however, that this does work in the case where c is an integer.
As an example, again using the principal value, if $z = w = e^{3\pi i/4}$, then $zw = e^{3\pi i/2} = e^{-\pi i/2}$, whence

$$\text{P. V.}(zw)^{5i} = \exp\left(-5i\frac{\pi i}{2}\right) = e^{5\pi/2}$$
$$z^{5i} = \exp\left(5i\frac{3\pi i}{4}\right) \implies z^{5i}w^{5i} = e^{-15\pi/2} = e^{5\pi/2}e^{2\pi\cdot 5i\cdot ni}$$

with $n = 1$.

Exercises. 3.2.1. (a) Show that the function $f(z) = \mathrm{Log}(z-i)$ is analytic everywhere except on the portion $x \le 0$ of the line $y = 1$.

(b) Show that the function $f(z) = \frac{1}{z^2+i}\mathrm{Log}(z+4)$ is analytic everywhere except at the points $\pm\frac{1}{\sqrt{2}(1-i)}$ and on the portion $x \le -4$ of the real axis.

3.2.2. Show that the set $z^{1/4}$ as defined earlier in the course coincides with the set $z^{1/4} := \exp\left(\frac{1}{4}\log z\right)$ as defined in this section.

3.2.3. Show that $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)$ where $n \in \mathbb{Z}$.

3.2.4. Find the principal values of the following:

(a) i^{2i} (b) $(1-i)^{3i}$ (c) $(-\sqrt{3}+i)^{1+4\pi i}$

3.2.5. Suppose c, c_1, c_2 and z are complex numbers where $z \ne 0$. If all the powers involved are principal values, show that,

(a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$ (b) $(z^c)^n = z^{cn}$ for any $n \in \mathbb{N}$.

3.2.6. Check the claim at the bottom of Example 3.12: if $m \in \mathbb{Z}$, then i^m is the same value for the two definitions of i^z.

3.2.7. Continuing the previous question, suppose $c \ne 0$ and define $c^z = e^{z\log c}$ where any choice of the branch of the logarithm is made.

(a) Let $m \in \mathbb{Z}$. Prove that c^m produces the same value, regardless of the branch of logarithm used to define $\log c$.

(b) If $z = \frac{1}{m}$, show that c^z really is an m^{th} root of c. If the principal branch of the logarithm is used, show that c^z is the principal m^{th} root of c. For every m^{th} root of c, show that there exists a branch of the logarithm for which c^z equals the given m^{th} root.

3.3 Trigonometric and Inverse Trigonometric Functions (§37–40)

It is fairly easy to give a sensible definition of the basic trigonometric functions by modifying Euler's formula. For instance, if $y \in \mathbb{R}$, then

$$e^{iy} + e^{-iy} = \cos y + i \sin y + \cos y - i \sin y = 2 \cos y$$

This motivates the following.

Definition 3.17. For any $z \in \mathbb{C}$ we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

The upshot of what follows is that these functions do exactly what you expect them to do.

Theorem 3.18. *Sine and cosine are entire functions with derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \cos z = -\sin z$$

Sine and cosine satisfy the same identities, including double and multiple-angle formulæ, as their real counterparts: for instance

$$\sin^2 z + \cos^2 z = 1, \quad \cos(z+w) = \cos z \cos w - \sin z \sin w, \quad \cos 2z = 2\cos^2 z - 1, \quad \textit{etc.}$$

In particular, $\sin z = \cos(z - \frac{\pi}{2})$ *and* $\cos z = \sin(z + \frac{\pi}{2})$. *Sine and cosine are also both periodic with period* 2π *and have exactly the same zeros as their real versions:*

$$\sin z = 0 \iff z = n\pi, \quad \cos z = 0 \iff z = \frac{\pi}{2} + n\pi \quad \textit{where} \quad n \in \mathbb{Z}$$

The proofs are straightforward applications of properties of the exponential function. For instance;

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \frac{1}{2i} \frac{\mathrm{d}}{\mathrm{d}z} (e^{iz} - e^{-iz}) = \frac{1}{2i} (ie^{iz} + ie^{-iz}) = \cos z$$

Similarly,

$$\sin z = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 \iff e^{-2y}(\cos 2x + i \sin 2x) = 1 \iff z = \pi n$$

It is worth considering the real and imaginary parts: if $z = x + iy$, then

$$\begin{aligned} \sin z &= \frac{1}{2i}(e^{ix-y} - e^{-ix+y}) = \frac{1}{2i}(e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x) \\ &= \frac{1}{2}(e^{y} + e^{-y})\sin x + \frac{i}{2}(e^{y} - e^{-y})\cos x = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

where $\cosh y = \frac{1}{2}(e^y + e^{-y})$ and $\sinh y = \frac{1}{2}(e^y - e^{-y})$ are the usual (real) hyperbolic functions. Similarly,

$$\cos z = \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{i}{2}(e^{y} - e^{-y})\sin x = \cos x \cosh y - i \sin x \sinh y$$

For sanity's sake, we can easily check the Cauchy–Riemann equations for $\sin z$:

$$u_x = \frac{1}{2}(e^y + e^{-y})\cos x = v_y, \qquad u_y = \frac{1}{2}(e^y - e^{-y})\sin x = -v_x$$

The remaining trigonometric functions are defined in the expected way: for instance,

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \quad \text{etc.}$$

All are analytic except at their singularities and have the usual expressions for their derivatives: e.g.

$$\frac{\mathrm{d}}{\mathrm{d}z}\tan z = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z \quad \text{whenever} \quad z \neq \frac{\pi}{2} + n\pi$$

Hyperbolic Functions (for reference only)

Hyperbolic functions are often a convenient short-cut, but are never necessary. Feel free to use or ignore them as you like. All their properties can be derived from the definition and their relationship to trigonometric functions:

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cos iz, \qquad \sinh z = \frac{e^z - e^{-z}}{2} = -i \sin iz$$

For instance

$$\frac{\mathrm{d}}{\mathrm{d}y}\sinh y = \cosh y, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}y}\cosh y = \sinh y$$

Hyperbolic identities usually look almost the same as trig identities, you just have to be careful with $\pm$-signs: for instance

$$\cosh^2 z = \cos^2(iz) = 1 - \sin^2(iz) = 1 + \sinh^2 z \implies \cosh^2 z - \sinh^2 z = 1$$

Inverse Trigonometric Functions

The standard trig functions can be inverted very easily. As you might expect, however, this results in multi-valued functions. For example

$$z = \cos w \iff 2z = e^{iw} + e^{-iw} \iff (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

which is a quadratic equation in e^{iw}. Applying the quadratic formula, we see that

$$e^{iw} = \frac{2z + (4z^2 - 4)^{1/2}}{2} = z + (z^2 - 1)^{1/2} \iff w = -i \log\left[z + (z^2 - 1)^{1/2}\right]$$

Not only is the square-root two-valued, but the logarithm is infinite-valued!

Theorem 3.19. *The inverse sine, cosine and tangent functions are given by the expressions*

$$\sin^{-1} z = -i \log\left[iz + (1-z^2)^{1/2}\right] \qquad \cos^{-1} z = -i\log\left[z + i(1-z^2)^{1/2}\right]$$
$$\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$$

These are infinite-valued functions, however, once branches of the square-root and logarithm are chosen, they are analytic on their domains, satisfying the familiar expressions from real analysis:

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\tan^{-1} z = \frac{1}{1+z^2}$$

The branches of the square-root for the derivatives of inverse sine and cosine are identical to those used in the definitions of the original functions.

The analyticity of each function is clear since each is a composition of analytic functions. All we need is to apply the chain rule: for instance

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z &= \frac{-i}{z+i(1-z^2)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}z}\left[z+i(1-z^2)^{1/2}\right] = \frac{-i}{z+i(1-z^2)^{1/2}}\left[1 - \frac{iz}{(1-z^2)^{1/2}}\right]\\ &= \frac{-i}{z+i(1-z^2)^{1/2}}\cdot\frac{(1-z^2)^{1/2}-iz}{(1-z^2)^{1/2}}\\ &= \frac{-1}{(1-z^2)^{1/2}}\end{aligned}$$

Examples 3.20. 1. To evaluate $\sin^{-1}\frac{1}{\sqrt{2}}$ as a complex number, we first see that

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log\left[\frac{i}{\sqrt{2}} \pm \sqrt{1-\frac{1}{2}}\right] = -i\log\frac{1}{\sqrt{2}}(i\pm 1)$$

Now evaluate the logarithms separately:

$$\begin{aligned}-i\log\frac{1}{\sqrt{2}}(i+1) &= -i\log e^{\pi i/4} = -i\left[\frac{\pi i}{4}+2\pi n i\right] = \frac{\pi}{4}+2\pi n\\ -i\log\frac{1}{\sqrt{2}}(i-1) &= -i\log e^{3\pi i/4} = -i\left[\frac{3\pi i}{4}+2\pi n i\right] = \frac{3\pi}{4}+2\pi n\end{aligned}$$

The set of values $\sin^{-1}\frac{1}{\sqrt{2}}$ generated by all branches of the square-root and logarithm is precisely the set we'd have found by computing entirely within $\mathbb{R}$!

2. We can also evaluate inverse sines that would have no meaning in $\mathbb{R}$. For instance,

$$\begin{aligned}\sin^{-1} 7 &= -i\log[7i \pm \sqrt{-48}] = -i\log(7\pm 4\sqrt{3})i = -i\log(7\pm 4\sqrt{3})e^{\pi i/2}\\ &= -i\left[\ln(7\pm 4\sqrt{3}) + \frac{\pi i}{2} + 2\pi n i\right] = -i\ln(7\pm 4\sqrt{3}) + \frac{\pi}{2} + 2\pi n\end{aligned}$$

Note that $7 > 4\sqrt{3}$, so we are always taking natural log of a positive real number.

Finally we consider an example with inverse tangent:

Example 3.21. Compute $\tan^{-1}(i - 2\sqrt{3})$. First compute the fraction in polar form:

$$\frac{i + (i - 2\sqrt{3})}{i - (i - 2\sqrt{3})} = \frac{2i - 2\sqrt{3}}{2\sqrt{3}} = -1 + \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}} e^{5\pi/6}$$

It follows that

$$\tan^{-1}(i - 2\sqrt{3}) = \frac{i}{2}\left(\ln\frac{2}{\sqrt{3}} + \frac{5\pi}{6}i - 2\pi n i\right) = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}} + \pi n : \quad n \in \mathbb{Z}$$

Choosing the principal value of the logarithm yields $-\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.

Exercises. 3.3.1. Find the real and imaginary parts of $\sin i$, $\cos(1+i)$ and $\tan(2i\ln 5 + \pi/2)$

3.3.2. Prove the following double/multiple-angle formulæ using the definitions in this section:

(a) $\cos 2z = 2\cos^2 z - 1$

(b) $\sin(z - w) = \sin z \cos w - \cos z \sin w$

(c) $\tan(z + w) = \dfrac{\tan z + \tan z}{1 - \tan z \tan w}$

3.3.3. Find all the values of $\tan^{-1}(1+i)$.

3.3.4. Solve the equation $\cos z = \sqrt{2}$ for z.

3.3.5. Recall Exercise 3.21: check explicitly that $\tan w = i - 2\sqrt{3}$ when $w = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.
(*Hint: use* $\tan w = \frac{e^{2iw}-1}{i(e^{2iw}+1)}$. *Why…?*)

3.3.6. Suppose $z > 1$ is real. Prove that $\operatorname{Re}\sin^{-1} z = \frac{\pi}{2} + 2\pi n$ is independent of z. What is $\operatorname{Im}\sin^{-1} z$.

3.3.7. Derive the expressions for $\tan^{-1} z$ and its derivative in Theorem 3.19.

3.3.8. (a) Given that $\cosh z = \cos(-iz)$, find an expression in terms of the complex logarithm for $\cosh^{-1} z$.

(b) Using your answer to part (a), or otherwise, find all solutions to the equation $\cosh z = \sqrt{3}$.

(c) Find an expression for the derivative of $\cosh^{-1} z$.

4 Integrals

Differentiation of complex functions is already somewhat different from that of real functions. Integration in complex analysis is even more alien. At first glance it might appear straightforward: surely we must be able to write things like

$$\int z^2 \, \mathrm{d}z = \frac{1}{3}z^3 + c \tag{$*$}$$

just as in real calculus? This misses a major issue: the fundamental theorem of calculus works by allowing us to use anti-derivatives to evaluate *definite* integrals. When we write ($*$), but for a real function, we are actually evaluating infinitely many definite integrals of the form

$$\int_a^x t^2 \, \mathrm{d}t = \frac{1}{3}x^3 - \frac{1}{3}a^3$$

This instantly begs the question: what should a *definite* integral be in complex analysis? For instance, does it mean anything to write

$$\int_{3+i}^{4i} z^2 \, \mathrm{d}z \, ?$$

The major difficulty is one of *paths.* Any Riemann sum type definition of this integral would require us to first partition a curve in the plane: what curve? A straight line? But what if we wanted to compute $\int_{-i}^{i} \frac{1}{z} \, \mathrm{d}z$ where the integrand is undefined somewhere on the straight line joining the endpoints? As a result, we need to revisit the idea of a *contour* or *path integral* from multi-variable calculus.

4.1 Calculus with Functions of a Real Variable (§41,42)

The heart of the definition of a contour integral is the integration of a function of a real variable. Thankfully this is really simple.

Definition 4.1. *Derivatives* and *definite integrals* of complex functions of a single real variable $w(t) = u(t) + iv(t)$ are defined in terms of those of its real and imaginary parts u and v:

$$w'(t) = u'(t) + iv'(t), \qquad \int_a^b w(t) \, \mathrm{d}t = \int_a^b u(t) \, \mathrm{d}t + i \int_a^b v(t) \, \mathrm{d}t$$

Note that $[a, b]$ is a closed interval of *real numbers;* improper integrals are also permitted.
The function w is *differentiable/integrable* precisely when the same can be said about u and v.

Examples 4.2. 1. Let $w(t) = 5t^2 + it$. Then $w'(t) = 10t + i$. Moreover,

$$\int_1^2 w(t) \, \mathrm{d}t = \int_1^2 5t^2 \, \mathrm{d}t + i \int_1^2 t \, \mathrm{d}t = \frac{7}{3} + \frac{i}{2}$$

2. Let $w(t) = t^2 + e^{it} = t^2 + \cos t + i \sin t$. Then

$$w'(t) = 2t - \sin y + i \cos t, \qquad \int_0^{2\pi} w(t) \, \mathrm{d}t = \frac{1}{3}t^3 + \sin t - i \cos t \Big|_0^{2\pi} = \frac{8}{3}\pi^3$$

Most of the basic properties of these objects can be translated straight from real calculus. Indeed we are already implicitly making use of the fundamental theorem of calculus: if $U' = u$ and $V' = v$, then

$$\int_a^b w(t)\,\mathrm{d}t = U(t) + iV(t)\Big|_a^b$$

so that it makes sense to call $U(t) + iV(t)$ an *anti-dertivative* of $w(t)$. Properties such as linearity, the product rule, etc., also hold and can be proved in the obvious way: use the definition and multiply out. For instance, if $k = a + ib$ is constant, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}kw(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(a+ib)(u+iv) = \frac{\mathrm{d}}{\mathrm{d}t}\left[(au-bv)+i(av+bu)\right]\\ &= au' - bv' + i(av' + bu') = (a+ib)(u'+iv') = kw'(t)\end{aligned}$$

We also have the usual form of integration by substitution: if $s : [a, b] \to \mathbb{R}$ is continuously differentiable with non-zero derivative and F is integrable on the range of s, then,

$$\int_a^b F\big(s(t)\big)s'(t)\,\mathrm{d}t = \int_{s(a)}^{s(b)} F(s)\,\mathrm{d}s \tag{$*$}$$

This is immediate by considering the real and imaginary parts of F.

A particularly useful time-saving trick is the chain-rule.

Theorem 4.3. *Suppose $w(t) = f(z(t))$ where*

- *$z(t) = x(t) + iy(t)$ is differentiable on an interval containing t,*
- *$f(z) = u(x, y) + iv(x, y)$ is analytic at $z(t)$.*

Then w is differentiable at t, and $w'(t) = f'\big(z(t)\big)z'(t)$. In particular, in integral form we have

$$\int_a^b f'\big(z(t)\big)z'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}f\big(z(t)\big)\,\mathrm{d}t = f\big(z(b)\big) - f\big(z(a)\big)$$

While related to the substitution rule ($*$), this is worth a proof since we are now relying on the *complex* differentiability of f.

Proof. Apply the chain rule from real calculus:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = u_x x' + u_y y', \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = v_x x' + v_y y'$$

Thus

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} + i\frac{\mathrm{d}v}{\mathrm{d}t} &= u_x x' + u_y y' + i(v_x x' + v_y y') = (u_x + iv_x)x' + i(v_y - iu_y)y'\\ &= (u_x + iv_x)(x' + iy') &&\text{(Cauchy–Riemann equations)}\\ &= f'(z(t))z'(t)\end{aligned}$$

■

Examples 4.4. 1. Let $w(t) = e^{t-it^2}$, then $w'(t) = e^{t-it^2}\frac{\mathrm{d}}{\mathrm{d}t}(t-it^2) = e^{t-it^2}(1-2it)$. Compare this with the original method which, naturally, gives the same result,

$$w'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(e^t\cos t^2 - ie^t\sin t^2) = e^t(\cos t^2 - 2t\sin t^2) - ie^t(\sin t^2 + 2t\cos t^2)$$

2. Let $w(t) = (1-t+it)^{10}$. Then

$$w'(t) = 10(i-1)(1-t+it)^9,$$
$$\int_0^1 w(t)\,\mathrm{d}t = \frac{1}{11(i-1)}(1-t+it)^{11}\Big|_0^1 = \frac{-1-i}{22}(i^{11}-1) = \frac{(1+i)^2}{22} = \frac{i}{11}$$

Not everything goes through from real calculus however! A particular problem are existence results such as the mean value theorem which apply perfectly well to the real and imaginary parts, but not to the whole. For instance, if $w = u+iv$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists \xi, \eta \in (a,b)$ such that

$$\frac{w(b)-w(b)}{b-a} = \frac{u(b)-u(a)}{b-a} + i\frac{v(b)-v(a)}{b-a} = u'(\xi) + iv'(\eta)$$

There is no reason for ξ to equal η, so we do not expect the right hand side to be $w'(\xi)$.

Exercises. 4.1.1. Evaluate the derivatives and integrals:

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin t + i\sqrt{t}\right]$ (b) $\frac{\mathrm{d}}{\mathrm{d}t}(i+t^3)^2$ (c) $\int_0^1 e^{\pi it}\,\mathrm{d}t$ (d) $\int_0^{\pi/2} e^{it}(1+e^{it})^2\,\mathrm{d}t$

4.1.2. Show that if $m, n \in \mathbb{Z}$, then

$$\int_0^{2\pi} e^{imt}e^{-int}\,\mathrm{d}t = 2\pi\delta_{mn}$$

where $\delta_{mn} = 1$ when $m = n$ and 0 otherwise. Do this two ways:

(a) Using the exponential law and the chain rule/substitution.

(b) By multiplying out and working with real and imaginary parts.

4.1.3. Use the chain rule/substitution to evaluate the integral $\int_0^x e^{(1+i)t}\,\mathrm{d}t$.
Hence find both $\int_0^x e^t\cos t\,\mathrm{d}t$ and $\int_0^x e^t\sin t\,\mathrm{d}t$ *without* using integration by parts.

4.1.4. Prove the product rule for functions of a real variable:

$$\frac{\mathrm{d}}{\mathrm{d}t}(wz) = w'z + wz'$$

(*Hint: let* $w(t) = u(t) + iv(t)$, $z(t) = x(t) + iy(t)$ *and multiply out...*)

4.1.5. Check that the mean value theorem fails for the function $w(t) = \sqrt{t} + it^2$ on the interval $[0,1]$.

4.2 Contour Integrals (§43–46)

We now come to the question of computing integrals of complex functions along *curves*. The first question is, 'What sort of curve?'

Definition 4.5. An *smooth arc* is an oriented curve C in the complex plane for which there exists a *regular parametrization*. Otherwise said, there exists $z : [a, b] \to \mathbb{C}$, such that:

1. $z([a, b]) = C$ where $z(a)$ is the *start* of the curve and $z(b)$ is the *end*;
2. z is *continuously differentiable*[a] on $[a, b]$ with *non-zero* derivative.

In particular, the *unit tangent vector* $\mathbf{T}(t) = \frac{z'(t)}{|z'(t)|}$ is well-defined and continuous.

A *contour* is a oriented piecewise smooth curve C in the complex plane: there exists a continuous parametrization $z : [a, b] \to \mathbb{C}$ and a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that each $z : [t_j, t_{j+1}] \to \mathbb{C}$ parametrizes a smooth arc.

If we reverse the orientation of a contour C, the resulting contour is labelled $-C$.

Additionally, we say that a contour is:

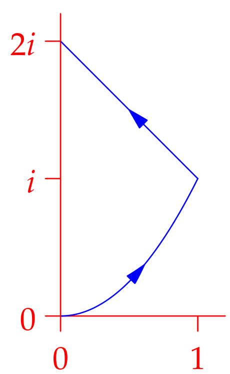
- *Closed* if it starts and ends at the same point, $z(a) = z(b)$;
- *Simple* if it does not cross itself (z is injective, $z(t) = z(s) \implies t = s$).

A simple closed curve is *positively oriented* if it is traversed counter-clockwise.

[a] We use the left- and right-derivatives for $z'(a)$ and $z'(b)$.

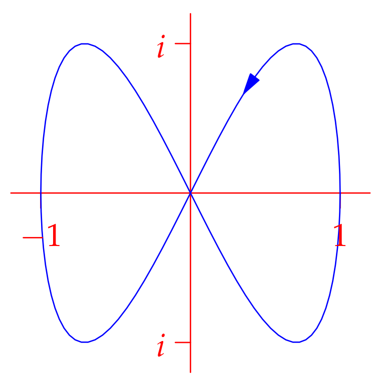
Integrals around positively oriented simple closed curves will be of most interest to us in this course.

A few examples of contours are sketched below with their orientations indicated:



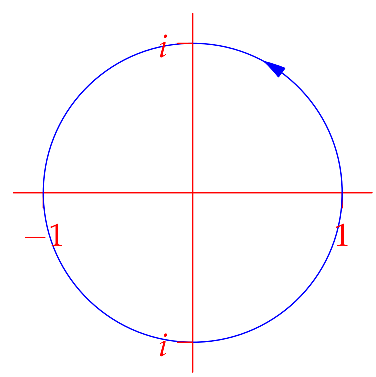
Simple, piecewise

$$z = \begin{cases} t + it^2 & t \in [0, 1] \\ 2 - t + it & t \in [1, 2] \end{cases}$$



Closed, non-simple

$$z = \cos t + i \sin 2t,\ t \in [0, 2\pi]$$



Positively oriented, simple, closed

$$z = e^{i\theta},\ \theta \in [0, 1]$$

When using circular paths, as above, it is common to parametrize using θ rather than t.
Note that a contour can have many different parametrizations! However, for a given smooth arc, any other parametrization must have the form $z(s(t))$ where s is continuously differentiable and $s' > 0$.

For any contour, we can define its *arc-length*: since $|z'(t)|$ is the speed of the curve, its length is simply the integral

$$\int_a^b |z'(t)| \, \mathrm{d}t$$

This is well-defined, since z' and thus the speed is assumed to be piecewise continuous. As you'll have seen in previous classes, the arc-length of a curve is very unlikely to be explicitly computable due to the square-root.

Examples 4.6.

1. Consider the simple piecewise curve above. Its arc-length is

$$\int_0^1 |1+2it| \, \mathrm{d}t + \int_1^2 |-1+i| \, \mathrm{d}t = \int_0^1 \sqrt{1+4t^2} \, \mathrm{d}t + \sqrt{2} = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}) + \sqrt{2}$$

   The integral can be done by substitution...

2. The second example, just for kicks, has arc-length

$$\int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t} \, \mathrm{d}t$$

   which must be estimated using power series or Riemann sums.

3. The third example has arc-length $\int_0^1 \left|2\pi i e^{2\pi i t}\right| \, \mathrm{d}t = 2\pi$, as of course it must, being the unit circle.

We will not be so interested in the arc-length along a curve. Instead, we mostly work with integrals that have more in common with the *line integrals* of multi-variable calculus. Happily, these are typically *much* easier to evaluate due to the lack of a square-root!

Definition 4.7. The *contour integral* of a function $f(z)$ along a contour C parametrized by $z : [a,b] \to \mathbb{C}$ is the integral

$$\int_C f(z) \, \mathrm{d}z := \int_a^b f\big(z(t)\big) z'(t) \, \mathrm{d}t$$

whenever the integral makes sense.

If C is a positively-oriented simple closed curve, we sometimes write $\oint_C f(z) \, \mathrm{d}z$.

If the value of $\int_C f(z) \, \mathrm{d}z$ depends only on the endpoints of C and is otherwise *independent of path*, it is legitimate to write $\int_{z_1}^{z_2} f(z) \, \mathrm{d}z$ where z_1, z_2 are the endpoints of C.

We shall be particularly interested in this last situation of path-indpendence. At the present, however, we cannot demonstrate that this holds for anything but the zero function $f(z) \equiv 0$ whose integral is automatically zero along any contour.

Examples 4.8. We evaluate several contour integrals.

1. For the contour C defined by $z(t) = t + it^2$, $t \in [0,1]$, we compute

$$\int_C z \, dz = \int_0^1 (t + it^2)(1 + 2it) \, dt = \int_0^1 t - 2t^3 + 3it^2 \, dt = \frac{1}{2} + i$$

2. For the contour $z(t) = e^{i\theta}$ with $\theta \in [0, \pi]$,

$$\int_C \frac{1}{z} \, dz = \int_0^\pi \frac{ie^{i\theta}}{e^{i\theta}} \, d\theta = \pi i, \quad \text{and}$$
$$\int_C z^2 + 1 \, dz = \int_0^\pi (e^{2i\theta} + 1) ie^{i\theta} \, d\theta = \frac{i}{3i} e^{3i\theta} + \frac{i}{i} e^{i\theta} \Big|_0^\pi = \frac{1}{3}(e^{3\pi i} - 1) + e^{\pi i} - 1 = -\frac{8}{3}$$

3. We compute the same integrals as the previous example, but over a different contour $\tilde{C}$ parametrized by $z(t) = e^{-i\theta}$ with $\theta \in [0, \pi]$. This time

$$\int_{\tilde{C}} \frac{1}{z} \, dz = \int_0^\pi \frac{-ie^{-i\theta}}{e^{-i\theta}} \, d\theta = -\pi i, \quad \text{and}$$
$$\int_{\tilde{C}} z^2 + 1 \, dz = \int_0^\pi (e^{-2i\theta} + 1)(-ie^{-i\theta}) \, d\theta = \frac{-i}{-3i} e^{-3i\theta} + \frac{-i}{-i} e^{-i\theta} \Big|_0^\pi = -\frac{8}{3}$$

Perhaps surprisingly, the sign of one integral changed while the other didn't: in fact the second integral will shortly be seen to be independent of path, so it doesn't matter which path you use to travel from $z = 1$ to $z = -1$, you'll still get the same result $\int_1^{-1} z^2 + 1 \, dz = -\frac{8}{3}$.

Basic rules for Contour Integrals

Some of the basic properties of contour integrals are immediate from real calculus and our discussion in the previous section: for instance linearity

$$\int_C (af(z) + bg(z)) \, dz = a \int_C f(z) \, dz + b \int_C g(z) \, dz$$

Of more importance are the following:

Theorem 4.9. *Suppose C is a contour parametrized by $z : [a, b] \to \mathbb{C}$ and $\int_C f(z) \, dz$ exists.*

1. *If $C = C_1 \cup C_2$ where C_1 and C_2 are contours such that the end of C_1 is the start of C_2, then*

$$\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz$$

2. *The contour integral is independent of (orientation-preserving) parametrization.*
3. *Reversing orientation changes the sign of the integral: $\int_{-C} f(z) \, dz = -\int_C f(z) \, dz$.*

Part 1 is a very easy exercise based on the well-known property $\int_a^b = \int_a^c + \int_c^b$. The other two parts benefit from a more formal proof.

Proof. By part 1, it is enough to check the remaining results on a single smooth arc C. Suppose $z : [a, b] \to \mathbb{C}$ and $w : [\alpha, \beta] \to \mathbb{C}$ are parametrizations of C: otherwise said, $w(s(t)) = z(t)$ for some continuously differentiable function s with *positive* derivative. Observe that

$$z'(t) = w'(s)s'(t), \qquad s(a) = \alpha, \quad s(b) = \beta$$

whence we can compare integrals by substitution:

$$\begin{aligned}\int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t &= \int_a^b f\big(w(s(t))\big)w'(s(t))\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_{s(a)}^{s(b)} f\big(w(s)\big)w'(s)\mathrm{d}s \\ &= \int_\alpha^\beta f\big(w(s)\big)w'(s)\mathrm{d}s\end{aligned}$$

This proves part 2. Part 3 is identical, except that to parametrize $-C$ we require $s' < 0$, $s(a) = \beta$ and $s(b) = \alpha$. The result is that the limits are flipped on the final integral. ∎

With a little of the basic properties out of the way, we return to our earlier examples.

Example 4.10. Suppose $z : [a, b] \to \mathbb{C}$ parametrizes a smooth arc such that $z(a) = 1$ and $z(b) = -1$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}[z(t)]^3 + z(t)\right) = \left([z(t)]^2 + 1\right) z'(t)$, we can apply Theorem 4.3 to see that

$$\begin{aligned}\int_C z^2 + 1\,\mathrm{d}z &= \int_a^b \left([z(t)]^2 + 1\right) z'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}[z(t)]^3 + z(t)\right)\mathrm{d}t \\ &= \frac{1}{3}[z(b)]^3 + z(b) - \frac{1}{3}[z(a)]^3 - z(a) \\ &= -\frac{1}{3} - 1 - \frac{1}{3} - 1 = -\frac{8}{3}\end{aligned}$$

The contour integral is independent of path!

The critical observation was that $f(z) = z^2 + 1$ is the derivative of a function $F(z) = \frac{1}{3}z^2 + z$. Just as in real calculus, we can use this *anti-derivative* to evaluate the integral.

Definition 4.11. A function $f : D \to \mathbb{C}$ has an *anti-derivative* $F : D \to \mathbb{C}$ if $F'(z) = f(z)$ on D.

Recall our discussion of analytic functions, where we saw that the only functions with zero derivative on an open connected domain are constants. This immediately forces an intuitive result.

Lemma 4.12. *Suppose F and G are anti-derivatives of f on an open connected domain D. Then $G(z) - F(z)$ is constant on D.*

In particular, this shows that we could have used *any* anti-derivative $\frac{1}{3}z^3 + z + k$ to evalute the previous example.

We can now state the first part of a powerful result relating anti-derivatives and the path-independence of contour integrals.

Theorem 4.13 (Fundamental Theorem of Line Integrals). *If F is an anti-derivative of f on D and C is any contour lying within D, then*

$$\int_C f(z)\,\mathrm{d}z = F(z_e) - F(z_s)$$

where z_s, z_e are the start and end points of the contour respectively: the contour integral is independent of path. Moreover, if C is a closed contour lying within D, then

$$\int_C f(z)\,\mathrm{d}z = 0$$

Proof. The result is already true for every smooth arc by Theorem 4.3. Now let $C = C_1 \cup \cdots \cup C_n$ be a contour where the start and end points of C_k are z_{k-1}, z_k respectively. By Theorem 4.9, part 1, we see that

$$\int_C f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \int_{C_k} f(z)\,\mathrm{d}z = \sum_{k=1}^{n} [F(z_k) - F(z_{k-1})] = F(z_n) - F(z_0)$$ ∎

This result will eventually have a converse: if *every* contour integral of f in D is independent of path then f has an anti-derivative. This will require some work however. For the present, we observe how crucial it is that an anti-derivative exist on a domain containing the full contour.

Example 4.14. If C is the unit circle, Example 4.8 says that $\oint_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$: explicitly

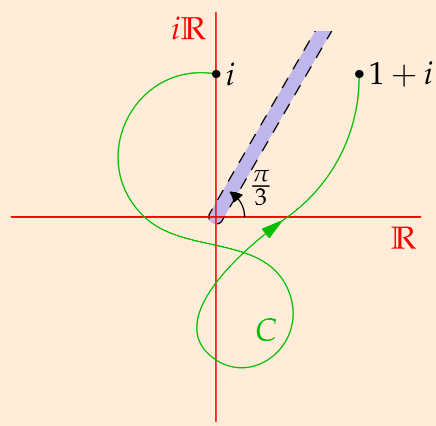
$$\oint_C f(z)\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,\mathrm{d}\theta = 2\pi i$$

By the fundamental theorem, $f(z) = \frac{1}{z}$ cannot have an anti-derivative on any domain containing the unit circle. But this is obvious: since

$$\frac{\mathrm{d}}{\mathrm{d}z} \log z = \frac{1}{z}$$

every anti-derivative has the form $F(z) = \log z + k$ which cannot be made single-valued on any path encircling the origin: no anti-derivative satisfies the hypotheses!

We can, however, use the fundamental theorem to evaluate $\int_C \frac{1}{z}\,\mathrm{d}z$ on any path staying within a single branch of the logarithm: in the picture, given the contour C, we choose the branch cut shown and evaluate



$$\begin{aligned}\int_C \frac{1}{z}\,\mathrm{d}z &= \log(1+i) - \log i = \log \sqrt{2}e^{\pi i/4} - \log e^{-3\pi i/2} \\ &= \ln\sqrt{2} + \frac{7\pi i}{4}\end{aligned}$$

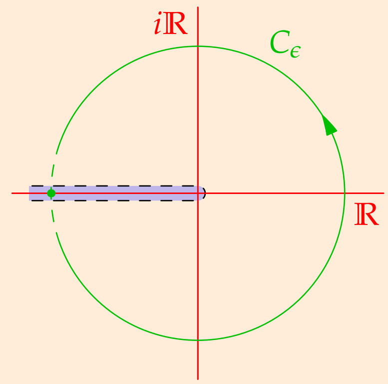
Note how the arguments were chosen so that $-\frac{5\pi}{3} < \theta < \frac{\pi}{3}$.

Integrals of functions with branch cuts

We sometimes wish to take integrals of functions which require a branch cut, but where the path cannot lie in a single branch. In such situations it is acceptable to take a branch cut so that the contour starts and/or finishes on the cut, one simply evaluates the integral by continuity. This is easiest to see in some examples.

Example 4.15. Compute $\oint_C z^{1/3}\,\mathrm{d}z$ where C is the unit circle, starting at the point $z=-1$.

$f(z) = z^{1/3}$ is multi-valued, but if we choose the principal branch, then $\arg z \in (-\pi, \pi)$ so the contour starts and finishes on the cut. For any small $\epsilon > 0$, let C_ϵ be the arc of the unit circle with argument in the interval $[-\pi+\epsilon, \pi-\epsilon]$, then



$$\int_{C_\epsilon} z^{1/3}\,\mathrm{d}z = \int_{-\pi+\epsilon}^{\pi-\epsilon} e^{i\theta/3} i e^{i\theta}\,\mathrm{d}\theta = \int_{-\pi+\epsilon}^{\pi-\epsilon} i e^{4i\theta/3}\,\mathrm{d}\theta$$
$$= \frac{3}{4} e^{4i\theta/3}\Big|_{-\pi+\epsilon}^{\pi-\epsilon} = \frac{3i}{2}\sin\frac{4(\pi-\epsilon)}{3}$$

Since this is continuous as $\epsilon \to 0^+$, we simply define

$$\oint_C z^{1/3}\,\mathrm{d}z = \lim_{\epsilon\to 0^+}\int_{C_\epsilon} z^{1/3}\,\mathrm{d}z = -\frac{3\sqrt{3}}{4} i$$

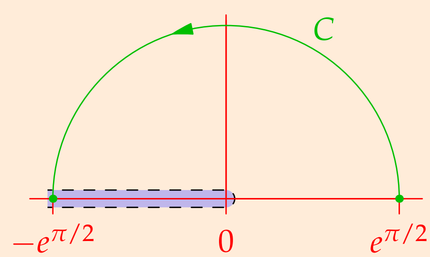
In practice, one need not mention ϵ at all. It is sufficient only to observe that the integrand $ie^{4it/3}$ is continuous on the interval $[-\pi, \pi]$. More generally, if we wish to compute $\int_C f(z)\,\mathrm{d}z$ where one or both ends $z(a), z(b)$ of C lie on a branch cut, then we can compute

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f\big(z(t)\big) z'(t)\,\mathrm{d}t$$

provided $f\big(z(t)\big)z'(t)$ is continuous on $[a, b]$. Here is one more example.

Example 4.16. Compute $\int_C z^i\,\mathrm{d}z$ where C is the counter-clockwise arc of the circle with radius $e^{\pi/2}$ centered at the origin from $z = e^{\pi/2}$ to $-e^{\pi/2}$ and we take the principal value of z^i.

We have $z(\theta) = e^{\pi/2}e^{i\theta}$ where $\theta \in [0, \pi]$. Ignoring the endpoint π, this just fits inside the principal branch of the logarithm $\arg z \in (-\pi, \pi)$. Indeed



$$z^i z' = e^{i\,\mathrm{Log}\,z} z' = \exp\left(i(\tfrac{\pi}{2} + i\theta)\right) ie^{i\theta} = -e^{(-1+i)\theta}$$

is continuous when $0 \le \theta \le \pi$, and so,

$$\int_C z^i\,\mathrm{d}z = \int_0^\pi -e^{(-1+i)\theta}\,\mathrm{d}\theta = \frac{1}{1-i}(e^{(-1+i)\pi} - 1)$$
$$= -\frac{1+i}{2}(1 + e^{-\pi})$$

Warning! It isn't quite as simple to allow C to *cross* a branch cut since one then must work with multiple branches simultaneously. This is more advanced and best avoided at the present…

Exercises. 4.2.1. Evaluate each contour integral $\int_C f(z)\,\mathrm{d}z$ by explicitly parametrizing C:

(a) $f(z) = z^2$; C is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = z$; C consists of the straight lines joining $z = 1$ to $1+i$ to $-1+i$ to -1.

(c) $f(z) = \operatorname{Log} z$; C is the circular arc of radius 3 centered at the origin, oriented counter-clockwise from $-3i$ to $3i$.

4.2.2. Explicitly check that $\int_C z\,\mathrm{d}z = \frac{1}{2}(B^2 - A^2)$ along the straight line joining A and B.
(*Hint: the line can be parametrized by* $z(t) = Bt + (1-t)A$ *where* $t \in [0,1]$)

4.2.3. Evaluate each contour integral $\int_C f(z)\,\mathrm{d}z$ using the fundamental theorem:

(a) $f(z) = z^5$; C is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = \frac{1}{z}$; C is the pair of straight lines from $z = 1$ to $-1-i$ to $-i$.

(c) $f(z) = \frac{1}{1+z^2}$; C is the straight line from $z = 1$ to $2+i$.

(d) $f(z) = \frac{1}{\sqrt{z}}$; C is any path $z(t)$ with $\operatorname{Re} z > 0$ joining $z = 1+i$ and $z = 4$.

4.2.4. Compute the integral of $z^{1/3}$ around the unit circle starting and finishing at $z = 1$.
(*Hint: take a branch cut along the positive real axis...*)

4.2.5. Compute the contour integral of the principal branch of z^{-1-2i} along the circular arc $e^{i\theta}$ from 1 to i.

4.2.6. Compute $\oint_{C_\alpha} z^{1/2}\,\mathrm{d}z$ where we take the α-branch cut of $z^{1/2}$ and the unit circle C_α traced from angle $\alpha - 2\pi$ to α.

4.2.7. Let $n \in \mathbb{Z}$ and let C_0 be the positively oriented circle centered at z_0 with radius $R > 0$. Explicitly parametrize this circle to show that

$$\oint_{C_0} (z - z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

4.2.8. Suppose $z : [a,b] \to \mathbb{C}$ is a regular parametrization of a smooth arc C. Define the function $s(t)$ to be the arc-length of the curve measured from $z(a)$: that is

$$s(t) := \int_a^t \left| z'(\tau) \right| \mathrm{d}\tau$$

Consider a new parametrization $w(s) = z(t(s))$, where $t(s)$ is the inverse function of $s(t)$. Prove that $\left|\frac{\mathrm{d}w}{\mathrm{d}s}\right| = 1$.
(*This proves that every smooth arc has a unit-speed parametrization*)

4.2.9. The multi-variable calculus version of the fundamental theorem of line integrals states that

$$\int_C \nabla\phi \cdot \mathrm{d}\mathbf{r} = \int_C \phi_x\,\mathrm{d}x + \phi_y\,\mathrm{d}y = \phi(B) - \phi(A) \qquad (*)$$

where $\nabla\phi$ is the gradient, $\mathrm{d}\mathbf{r} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$, and A, B are the start and end of C.

Let $F(z) = U(x,y) + iV(x,y)$ be an anti-derivative of $f(z)$ on a connected open domain D containing a contour C. Prove that $\phi = U$ and $\phi = V$ both satisfy $(*)$.

4.3 Estimates of contour integrals (§47)

Recall that for real integrals

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x \tag{$*$}$$

Our goal is to prove a similar result for contour integrals. Things are not quite as simple, since we have to take into account the length of the contour. The proof is very straightforward however.

Theorem 4.17. *Suppose $w : [a,b] \to \mathbb{C}$ is piecewise continuous. Then*

$$\left|\int_a^b w(t)\,\mathrm{d}t\right| \le \int_a^b |w(t)|\,\mathrm{d}t$$

Now suppose C is a contour with length L, and that f is piecewise continuous on C. Then $\exists M > 0$ such that $|f(z)| \le M$ on C, and moreover

$$\left|\int_C f(z)\,\mathrm{d}z\right| \le ML$$

Proof. Let $\int_a^b w(t)\,\mathrm{d}t = re^{i\theta}$. Since θ is constant, observe that $r = \int_a^b e^{-i\theta}w(t)\,\mathrm{d}t \in \mathbb{R}$. But then, appealing to $\operatorname{Re} z \le |z|$ and $(*)$, we see that

$$\left|\int_a^b w(t)\,\mathrm{d}t\right| = r = \int_a^b \operatorname{Re}(e^{-i\theta}w(t))\,\mathrm{d}t \le \int_a^b \left|e^{-i\theta}w(t)\right|\,\mathrm{d}t = \int_a^b |w(t)|\,\mathrm{d}t$$

We apply this to the contour integral after parametrizing by $z : [a,b] \to \mathbb{C}$. Since $f(z(t))$ is piecewise continuous on a closed bounded interval $[a,b]$, it is bounded and thus M exists. But now,

$$\left|\int_C f(z)\,\mathrm{d}z\right| = \left|\int_a^b f(z(t))z'(t))\,\mathrm{d}t\right| \le \int_a^b |f(z(t))z'(t)|\,\mathrm{d}t \le \int_a^b M\,|z'(t)|\,\mathrm{d}t = ML \qquad \blacksquare$$

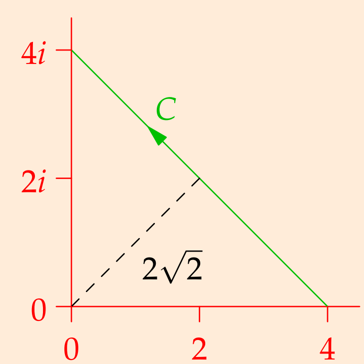
The theorem has limited utility in one sense: unless you work only with straight lines or circular arcs, computing the arc-length explicitly will likely be very hard. Its use also tends to rely heavily on the triangle inequalities.

Example 4.18. On the straight line C joining $z = 4$ to $4i$, we see that

$$\begin{aligned} &2\sqrt{2} = |2(1+i)| \le |z| \le 4 \\ &\implies |z+1| \le |z| + 1 \le 5, \qquad \left|z^4 + 4\right| \ge \left||z|^4 - 4\right| \ge 60 \end{aligned}$$

Since C has length $4\sqrt{2}$, it follows that

$$\left|\int_C \frac{z+1}{z^4+4}\,\mathrm{d}z\right| \le \frac{20\sqrt{2}}{60} = \frac{\sqrt{2}}{3}$$



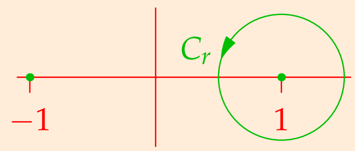
One of the major applications of integral-estimations lies in considering integrals round very large circles. This is particularly appropriate for polynomials and rational functions.

Examples 4.19. 1. For the function in the previous example, consider the circle C_R with radius R centered at the origin. Provided $R > \sqrt[4]{2}$ we have $|z|^4 > 4$ and so all singularities of $f(z) = \frac{z+1}{z^4+4}$ lie inside the circle. Moreover, on C,

$$\left|\frac{z+1}{z^4+4}\right| \le \frac{|z|+1}{|z^4|-4} = \frac{R+1}{R^4-4} \implies \left|\oint_{C_R} \frac{z+1}{z^4+4}\,\mathrm{d}z\right| \le \frac{2\pi R(R+1)}{R^4-4}$$

In particular, this approaches zero as $R \to \infty$.

2. Let $r < 2$ and let C_r be the circle radius r centered at $z = 1$. Then



$$|z-1| = r, \qquad |z+1| = |2-(1-z)| \ge 2-r$$

from which

$$\left|\oint_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z\right| \le \frac{2\pi r}{(2-r)r} = \frac{2\pi}{2-r} \xrightarrow[r\to 0]{} \pi$$

In fact it will be seen later that $\lim_{r\to 0} \oint_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z = -\pi i$

Exercises. 4.3.1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $2i$. Without evaluating the integral, show that

(a) $\left|\int_C \frac{z+4}{z^3-1}\,\mathrm{d}z\right| \le \frac{6\pi}{7}$ (b) $\left|\int_C \frac{\mathrm{d}z}{z^2-1}\right| \le \frac{\pi}{3}$

4.3.2. If C is the straight line joining the origin to $1+i$, show that

$$\left|\int_C z^3 e^{2iz}\,\mathrm{d}z\right| \le 4$$

4.3.3. If C is the boundary of the triangle with vertices $0, 3i$ and -4, prove that

$$\left|\oint_C (e^z - \overline{z})\,\mathrm{d}z\right| \le 60$$

(Hint: show that $|e^z - \overline{z}| \le e^x + \sqrt{x^2+y^2}\ldots$*)*

4.3.4. Let C_R be the circle of radius $R > 1$ centered at the origin. Prove that

$$\left|\oint_{C_R} \frac{\operatorname{Log} z}{z^2}\,\mathrm{d}z\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

and thus prove that $\lim_{R\to\infty} \oint_{C_R} \frac{\operatorname{Log} z}{z^2}\,\mathrm{d}z = 0$

4.4 The Fundamental Theorem

As we saw in the fundamental theorem of line integrals, the existence of an anti-derivative allows one to easily compute contour integrals and we see that such integrals are independent of path. We now state the full version of this result, which essentially says that path-independence is equivalent to the existence of an anti-derivative.

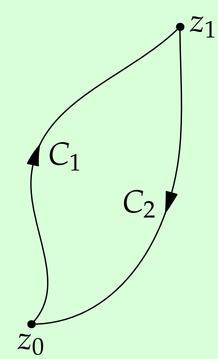
Theorem 4.20 (Fundamental Theorem, mark II). *Suppose $f(z)$ is continuous on an open domain D. The following are equivalent:*

1. *$f(z)$ has an anti-derivative $F(z)$ on D;*
2. *For any contour in D, $\int_C f(z)\,\mathrm{d}z$ is independent of path, depending only on the endpoints.*
3. *The integral of $f(z)$ round any closed contour in D is zero.*

Moreover, $\int_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0)$ where $F(z)$ is any anti-derivative of $f(z)$.

Proof. $(1 \Rightarrow 2)$ This is Theorem 4.13, as is the conclusion that $\int_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0)$.

$(2 \Leftrightarrow 3)$ Let C be any closed contour. Choose any two points $z_0, z_1 \in C$ and decompose $C = C_1 \cup C_2$ into two contours from z_0 to z_1 and back again. But then,



$$\begin{aligned}
\int_C f(z)\,\mathrm{d}z &= \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z && \text{(Theorem 4.9, part 1)}\\
&= \int_{C_1} f(z)\,\mathrm{d}z - \int_{-C_2} f(z)\,\mathrm{d}z && \text{(Theorem 4.9, part 3)}\\
&= 0
\end{aligned}$$

since C_1 and $-C_2$ share the same endpoints.

Conversely, if $C_1, -C_2$ are any contours with the same endpoints z_0, z_1, then $C = C_1 \cup C_2$ is a closed contour and the above shows that $\int_{C_1} f(z)\,\mathrm{d}z = \int_{-C_2} f(z)\,\mathrm{d}z$.

Before continuing, you may find it helpful to review the proof of the fundamental theorem of calculus part I: f continuous $\implies \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$. The argument we are about to give is very similar, though there are a couple of subtleties in the complex case:

- One must choose how to travel from one endpoint of the integral to the other: path independence says it doesn't matter how.
- We need to use Theorem 4.17 to bound the modulus of a contour integral rather than the more familiar real version $\left|\int_a^b g(x)\,\mathrm{d}x\right| \le \int_a^b |g(x)|\,\mathrm{d}x$.

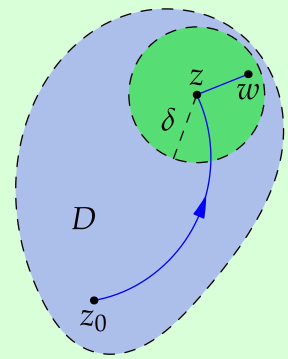
Proof (continued). (2 ⇒ 1) Fix $z_0 \in D$ and *define*

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta \tag{$*$}$$

where the integral is taken along *any* curve joining z_0 to z. This is well-defined by the assumption of path-independence. We need only check that $F(z)$ is analytic with derivative $f(z)$.

Let $z \in D$ be fixed and $\epsilon > 0$ be given. Since f is continuous and D is open,

$$\exists \delta > 0 \text{ such that } |\zeta - z| < \delta \implies \zeta \in D \text{ and } |f(\zeta) - f(z)| < \epsilon$$



Let $w \in D$ such that $|w - z| < \delta$. Then

$$F(w) - F(z) = \int_{z}^{w} f(\zeta)\, d\zeta$$

along any curve joining z and w. We may therefore *choose* this to be the straight line segment. It follows that

$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| = \left| \frac{1}{w - z} \int_{z}^{w} f(\zeta) - f(z)\, d\zeta \right| < \frac{1}{|w - z|} |w - z|\, \epsilon = \epsilon$$

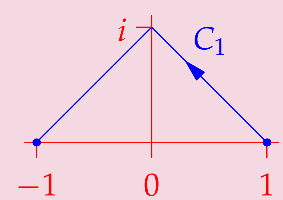
where we made use of Theorem 4.17 and the fact the path is a straight line with length $|w - z|$. We conclude that $\lim_{w \to z} \frac{F(w)-F(z)}{w-z} = f(z)$ exists for all $z \in D$: thus F is differentiable with $F'(z) = f(z)$. ∎

Exercises. 4.4.1. Let $n \in \mathbb{N}_0$. Prove that for every contour C from a point z_0 to $z_1 \in \mathbb{C}$

$$\int_C z^n\, dz = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1} \right)$$

4.4.2. If C is a closed curve not containing z_0, and $n \in \mathbb{Z} \setminus \{0\}$, prove that $\int_C (z - z_0)^{n-1}\, dz = 0$.

4.4.3. Let $f(z) = z^{1/3}$ be the branch where $\arg z \in [0, 2\pi]$. Evaluate the integral $\int_{C_1} f(z)\, dz$ where C_1 is the curve shown in the picture.



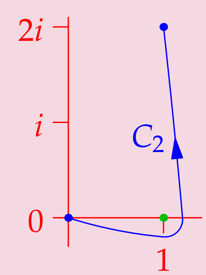
4.4.4. Evaluate each integral where the function is defined on the domain D.

(a) $\int_0^{1+2i} \frac{1}{1-z^2}\, dz$ where $D = \mathbb{C}$ except for the real axis where $|z| \geq 1$

(b) $\int_{2i}^{1+i} \operatorname{Log} z\, dz$ D where D is the upper half-plane.

4.4.5. *Use an anti-derivative* to evaluate $\int_{-1}^{1} z^{2-i}\, dz$ where z^{2-i} is the principal branch, and the integral is over any contour which, apart from its endpoints, lies above the real axis.

4.4.6. (Hard!) Evaluate $\int_0^{1+2i} \frac{1}{1-z^2}\, dz$ along a curve C_2 looping to the *right* of $z = 1$ as shown in the picture. Where do you need to make branch cuts?



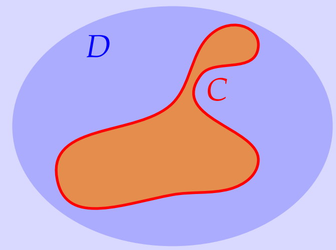
4.5 The Cauchy–Goursat Theorem (§50–53)

In this section we begin to extend the fundamental theorem. Our goal is to see that having an anti-derivative is essentially identical to being analytic. To properly discuss this, we first need to consider the notion of *simple-connectedness.*

Definition 4.21. Suppose D is a connected region of the plane.

We say that D is *simply-connected* if every closed contour in D can by shrunk smoothly to a point without any part leaving D.

Otherwise said, if C is a simple closed contour in D then everything *inside* C also lies in D.



Theorem 4.22 (Cauchy–Goursat, version 1). *If f is analytic on a simply-connected domain D and C is a simple closed contour in D, then $\int_C f(z)\,\mathrm{d}z = 0$.*

We prove a simpler version, which relies on *Green's Theorem*: you should have seen this in multi-variable calculus:

Lemma 4.23 (Green's Theorem). *If $P, Q : D \to \mathbb{R}$ are functions on a closed bounded simply-connected domain D with continuous partial derivatives and C is the boundary of D, then*

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

Sketch Proof of Cauchy–Goursat. Suppose $f(z) = u + iv$ is written in real and imaginary parts and parametrize the boundary curve by $z(t) = x(t) + iy(t)$ where $a \le t \le b$: assume WLOG that this is positively oriented. Suppose additionally that the partial derivatives of u, v are continuous. Then

$$\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,\mathrm{d}z \\
&= \int_a^b \big(u(z(t)) + iv(z(t))\big)\big(x'(t) + iy'(t)\big)\,\mathrm{d}t \\
&= \int_a^b ux' - vy'\,\mathrm{d}t + i\int_a^b vx' + uy'\,\mathrm{d}t \\
&= \oint_C u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_C v\,\mathrm{d}x + u\,\mathrm{d}y \\
&= \iint_D -v_x - u_y\,\mathrm{d}A + i\iint_D u_y - v_x\,\mathrm{d}A = 0
\end{aligned}$$

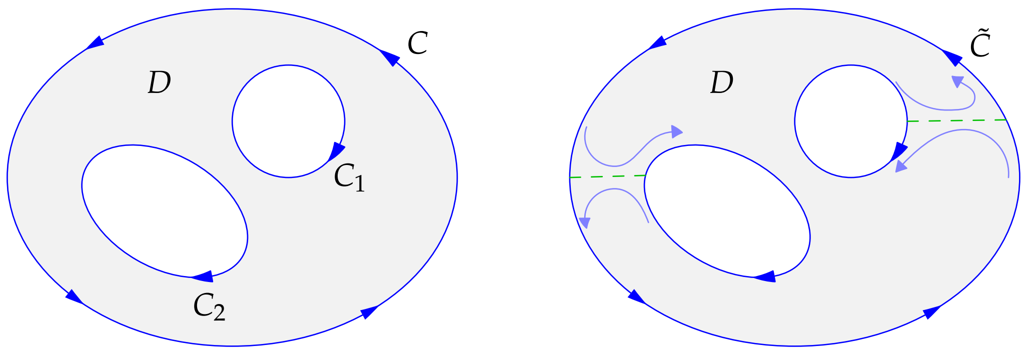
by Green's Theorem and the Cauchy–Riemann equations. ∎

This is a very straight-forward calculation, though it relies on Green's Theorem, which is not so straight-forward to prove. More importantly, the proof requires the continuity of the partial derivatives of u, v. A proof without this restriction is much longer and challenging, so we omit it. Indeed it can be shown that every analytic function must have continuous partial derivatives and so its derivative is also analytic.

We can easily extend this to multiply-connected domains using a, hopefully, familiar extension of Green's Theorem.

Definition 4.24. A *multiply-connected domain* is a region D which results from subtracting finitely many simply-connected sets from a simply-connected set.

We can remove individual *points* so that, for instance, the punctured plane $\mathbb{C} \setminus \{0\}$ is multiply-connected. Suppose we remove only regions within simple closed contours $C_1, \ldots, C_k$. As shown in the picture, the convention is to orient the interior boundary contours clockwise and the outer curve counter-clockwise so that the region D always lies to the *left* of the curves.



By cutting (dashed green lines in the above picture) one can easily join the boundary curves into a single curve $\tilde{C}$, at the cost of traversing each cut twice in opposite directions. With a little stretching of the definition, $\tilde{C}$ is a simple closed curve, and so Cauchy's Theorem applies $\int_{\tilde{C}} f(z)\,\mathrm{d}z = 0$. Since contour integrals count negatively when traversed backwards, we thus have a proof of Cauchy's Theorem for multply-connected domains.

Theorem 4.25 (Cauchy–Goursat, version 2). *Suppose C is a simple closed contour, oriented counter-clockwise, lying in a multiply-connected domain D. Suppose that the part of D interior to C is simply-connected except for the removal of the interiors of finitely many simple, closed, non-intersecting contours $C_1, \ldots, C_k$, oriented clockwise. If $f(z)$ is analytic on D, then*

$$\int_C f(z)\,\mathrm{d}z + \sum_{j=1}^{k} \int_{C_j} f(z)\,\mathrm{d}z = 0$$

This version allows us to *compare* integrals around difficult to parametrize contours using much simpler contours.

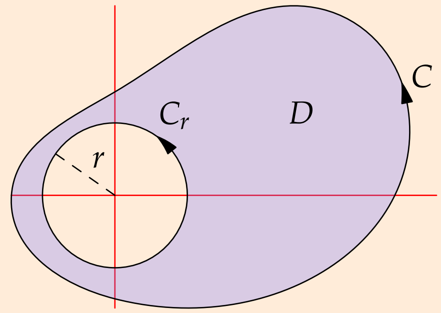
Corollary 4.26. *Suppose C_1, C_2 are positively oriented simple closed contours where C_1 is interior to C_2. If f is analytic on the region between and including the curves, then*

$$\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z$$

Here is the classic example.

Example 4.27. Suppose C is a simple closed contour staying away from the origin. If the origin is *exterior* to C, then $f(z) = \frac{1}{z}$ is analytic inside and on C, whence $\oint_C \frac{1}{z}\,\mathrm{d}z = 0$.
If the origin is *interior* to C, choose any circle C_r with radius r lying interior to C. Since $f(z) = \frac{1}{z}$ is analytic on the region D between and including C and C_r, we conclude that

$$\oint_C \frac{1}{z}\,\mathrm{d}z = \oint_{C_r} \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{re^{i\theta}} rie^{i\theta}\,\mathrm{d}\theta = 2\pi i$$



Exercises. 4.5.1. Apply the Cauchy–Goursat Theorem to show that $\oint_C f(z)\,\mathrm{d}z = 0$ when the contour C is the unit circle $|z| = 1$.

(a) $f(z) = \dfrac{z^2}{z+3}$ (b) $f(z) = ze^{-z}$ (c) $f(z) = \mathrm{Log}(z+2)$

4.5.2. Let C_1 denote the square with sides along the lines $x = \pm 1, y = \pm 1$, and C_2 be the circle $|z| = 4$: explain why

$$\oint_{C_1} \frac{1}{3z^2+1}\,\mathrm{d}z = \oint_{C_2} \frac{1}{3z^2+1}\,\mathrm{d}z$$

4.5.3. Let C be the square with sides $x = 0, 1$ and $y = 0, 1$. Evaluate the integral

$$\oint_C \frac{1}{z-a}\,\mathrm{d}z$$

when:

(a) a is *exterior* to the square;
(b) a is *interior* to the square.

4.5.4. Let C be the positively-oriented boundary of the half-disk $0 \le r \le 1$, $0 \le \theta \le \pi$ and define $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$ and $f(0) = 0$ using the branch of $z^{1/2}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Prove that

$$\oint_C f(z)\,\mathrm{d}z = 0$$

by evaluating three contour integrals: over the semicircle, and over two segments of the real axis joining 0 to ± 1. Why does Cauchy–Goursat not apply here?

4.5.5. If C is a positively-oriented simple closed contour, prove that the area enclosed by C can be written

$$\frac{1}{2i}\oint_C \overline{z}\,\mathrm{d}z$$

(*Hint: Mirror the sketch proof of Cauchy–Goursat, even though $\overline{z}$ isn't analytic…*)

4.6 Cauchy's Integral Formula (§54–58)

We have already seen that $\int_C \frac{1}{z-z_0}\,dz = 2\pi i$ whenever C is a positively-oriented simple closed contour encircling z_0. This generalizes to one of the absolutely central results of complex analysis.

Theorem 4.28 (Cauchy's Integral Formula). *Suppose $f(z)$ is analytic everywhere on and inside a simple closed contour C. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

Moreover, f is infinitely differentiable at z_0 (every derivative is analytic) and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Example 4.29. If $f(z)$ is a polynomial, we can check the integral formula explicitly by appealing to the Cauchy–Goursat Theorem and Exercise 4.2.7.: if C is a simple closed contour encircling z_0, then

$$\oint_C (z-z_0)^{n-1}\,dz = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

For instance, if $f(z) = 3z^2+2$ and $z_0 = 0$, then

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{3z^2+2}{z}\,dz &= \frac{1}{2\pi i}\oint_C 3z\,dz + \frac{1}{2\pi i}\oint_C \frac{2}{z}\,dz = 0+2 = f(0) \\
\frac{1!}{2\pi i}\oint_C \frac{3z^2+2}{z^2}\,dz &= \frac{1}{2\pi i}\oint_C 3\,dz + \frac{1}{2\pi i}\oint_C \frac{2}{z^2}\,dz = 0+0 = f'(0) \\
\frac{2!}{2\pi i}\oint_C \frac{3z^2+2}{z^3}\,dz &= \frac{2}{2\pi i}\oint_C \frac{3}{z}\,dz + \frac{2}{2\pi i}\oint_C \frac{2}{z^3}\,dz = 6+0 = f''(0)
\end{aligned}$$

More generally, if $f(z) = \sum_{j=1}^m a_j z^j$, then, as expected,

$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz = \frac{n!}{2\pi i}\sum_{j=1}^m a_j \oint_C z^{j-n-1}\,dz = \begin{cases} n!a_n & \text{if } n \le m \\ 0 & \text{otherwise} \end{cases}$$

A quick trick Here is a nice pattern for remembering the integral formula: observe that

$$\frac{d}{dz_0}\frac{n!}{(z-z_0)^{n+1}} = \frac{(n+1)!}{(z-z_0)^{n+2}}$$

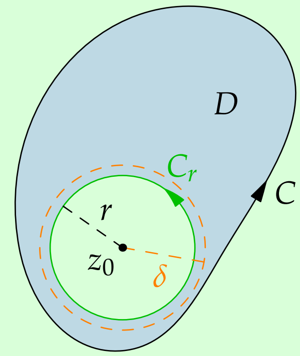
It appears therefore as if the integral formula simply follows from repeated differentiation:

$$f^{(n+1)}(z_0) = \frac{d}{dz_0} f^{(n)}(z_0) \stackrel{?}{=} \frac{n!}{2\pi i}\oint_C \frac{d}{dz_0}\frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Of course one cannot blindly take a derivative inside a contour integral! Thus…

Sketch Proof. Denote by D the open region interior to C. Let $\epsilon > 0$ be given. Since f is analytic, it is also continuous. Thus

$$\exists \delta > 0 \text{ such that } |z - z_0| < \delta \implies z \in D \text{ and } |f(z) - f(z_0)| < \epsilon$$



Choose a positive $r < \delta$ and draw the circle C_r of radius r centered at z_0. This lies entirely in D. Since $\frac{f(z)}{z-z_0}$ is analytic between and on C and C_r, Corollary 4.26 tells us that

$$\oint_C \frac{f(z)}{z - z_0}\,\mathrm{d}z = \oint_{C_r} \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

We need only bound an integral:

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,\mathrm{d}z - f(z_0)\right| = \left|\frac{1}{2\pi i}\oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z\right| < \frac{1}{2\pi}2\pi r\frac{\epsilon}{r} = \epsilon$$

Now we extend the formula. As above, let $\delta > 0$ be such that $|w - z_0| < \delta \implies w \in D$, and let $|\Delta z| < \delta$. For any $z \in C$, we have

$$|z - z_0| \geq \delta, \quad |z - z_0 - \Delta z| \geq \big||z - z_0| - |\Delta z|\big| \geq \delta - |\Delta z| > 0$$

Then

$$\begin{aligned}
&\left|\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,\mathrm{d}z\right| \\
&\qquad = \left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{\Delta z}\left(\frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0}\right) - \frac{f(z)}{(z - z_0)^2}\,\mathrm{d}z\right| \\
&\qquad = \frac{1}{2\pi}\left|\oint_C \frac{f(z)\big((z - z_0)^2 - (z - z_0)(z - z_0 - \Delta z) - \Delta z(z - z_0 - \Delta z)\big)}{\Delta z(z - z_0 - \Delta z)(z - z_0)^2}\,\mathrm{d}z\right| \\
&\qquad = \frac{1}{2\pi}\left|\oint_C \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\,\mathrm{d}z\right| \leq \frac{ML}{2\pi(\delta - \Delta z)\delta^2}\,|\Delta z|
\end{aligned}$$

where M is an upper bound for $|f(z)|$ on C, and L is the length of C. Clearly this goes to zero as $\Delta z \to 0$: we conclude that $f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$.
One can repeat this exercise for higher derivatives: for instance,

$$\begin{aligned}
&\left|\frac{f'(z_0 + \Delta z) - f'(z_0)}{\Delta z} - \frac{2}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^3}\,\mathrm{d}z\right| \\
&\qquad = \left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{\Delta z}\left(\frac{1}{(z - z_0 - \Delta z)^2} - \frac{1}{(z - z_0)^2}\right) - \frac{2f(z)}{(z - z_0)^3}\,\mathrm{d}z\right| \\
&\qquad = \frac{1}{2\pi}\left|\oint_C \frac{(3(z - z_0) - 2\Delta z)f(z)\Delta z}{(z - z_0 - \Delta z)^2(z - z_0)^3}\,\mathrm{d}z\right| \leq \frac{(3K + 2\delta)ML}{2\pi(\delta - \Delta z)^2\delta^3}\,|\Delta z|
\end{aligned}$$

where $|z - z_0| \leq K$ is the largest distance from z_0 to a point on C. Again this tends to zero as $\Delta z \to 0$.

The nastier calculations for the higher derivatives are omitted for sanity's sake... ∎

Examples 4.30. We can use the integral formula to evaluate certain integrals that would be difficult if not impossible to evaluate by parametrization.

1. If C is the circle centered at $z = i$ with radius 1, then $f(z) = \frac{3z\sin z}{z+i}$ is analytic on an inside C, whence

$$\oint_C \frac{3z\sin z}{z^2+1}\,\mathrm{d}z = \oint_C \frac{3z\sin z}{(z+i)(z-i)}\,\mathrm{d}z = 2\pi i f(i) = 2\pi i\frac{3i\sin i}{2i} = 3\pi i\sin i = \frac{3}{2}(e^{-1}-e^{1})$$

   Contrast this with parametrizing $z(\theta) = i + e^{i\theta}$ and attempting to evaluate directly!

$$\int_0^{2\pi} \frac{3(i+e^{i\theta})\sin(i+e^{i\theta})ie^{i\theta}}{(i+e^{i\theta})^2+1}\,\mathrm{d}\theta\ldots$$

2. Let C be a simple closed contour staying away from $z_0 = 4$. Since $f(z) = \frac{3z^2+7}{e^z}$ is entire, we see that

$$\oint_C \frac{3z^2+7}{e^z(z-4)}\,\mathrm{d}z = \begin{cases} 0 & \text{if } z_0 = 4 \text{ is outside } C \\ 2\pi i f(4) = 110\pi i e^{-4} & \text{if } z_0 = 4 \text{ is inside } C \end{cases}$$

3. Let C be the circle of radius 2 centered at $z = 1+i$. Then $g(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$ is analytic on and inside C, except at $z = i$. We conclude that

$$\oint_C g(z)\,\mathrm{d}z = \oint_C \frac{1}{(z+i)^3(z-i)^3}\,\mathrm{d}z = \frac{2\pi i}{2!}\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right|_i (z+i)^{-3} = 12\pi i(2i)^{-5} = \frac{3\pi}{8}$$

We finish this section with an easy corollary of the integral formula.

Corollary 4.31. *If f is analytic at z_0 then it is infinitely differentiable and all derivatives are analytic. In particular, the real and imaginary parts of f have continuous partial derivatives of all orders.*

Proof. If $f(z)$ is analytic at z_0, then it is analytic on an open set D containing z_0. Draw a circle C_r centered at z_0 lying inside D. By the Cauchy integral formula, $f''(z)$ exists at every point inside C_r and so $f'(z)$ is analytic at z_0. Now induct. ∎

Finally, note that if $f(z)$ has an anti-derivative $F(z)$, then $F(z)$ is necessarily analytic and so, by the Corollary, is $f(z)$ itself. We can combine this with the Fundamental Theorem and Cauchy–Goursat to obtain the following summary:

Suppose $f(z)$ is continuous on an open domain D.

$$\begin{tikzcd}
\text{all } \oint_C f(z)\,\mathrm{d}z = 0 \arrow[d, Leftrightarrow] & & f \text{ analytic on } D \arrow[ll, Rightarrow, "\text{If } D \text{ simply-connected}"'] \\
\text{all } \int_C f(z)\,\mathrm{d}z \text{ path independent} \arrow[rr, Leftrightarrow] & & f \text{ has an anti-derivative on } D \arrow[u, Rightarrow]
\end{tikzcd}$$

Exercises. 4.6.1. Let C denote the boundary of the square with sides $x = \pm 2, y = \pm 2$. Evaluate the following:

(a) $\displaystyle\oint_C \frac{e^{-\frac{\pi z}{2}}\,\mathrm{d}z}{z-i}$ (b) $\displaystyle\oint_C \frac{e^z + e^{-z}}{z(z^2+10)}\,\mathrm{d}z$ (c) $\displaystyle\oint_C \frac{z\,\mathrm{d}z}{3z+i}$ (d) $\displaystyle\oint_C \frac{\sec(z/2)}{(z-1-i)^2}\,\mathrm{d}z$

4.6.2. Evaluate the integral $\oint_C g(z)\,\mathrm{d}z$ around the circle of radius 3 centered at $z = i$ when:

(a) $g(z) = \dfrac{1}{z^2+9}$ (b) $g(z) = \dfrac{1}{(z^2+9)^2}$

4.6.3. Prove that if f is analytic on and inside a simple closed contour C and z_0 is not on C, then

$$\oint_C \frac{f'(z)}{z-z_0}\,\mathrm{d}z = \oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$$

4.6.4. Let C be the unit circle $z = e^{i\theta}$ where $-\pi < \theta \le \pi$ and suppose $a \in \mathbb{R}$ is constant. By first evaluating $\oint_C z^{-1}e^{az}\,\mathrm{d}z$, prove that

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,\mathrm{d}\theta = \pi$$

4.6.5. (a) Suppose that $f(z)$ is *continuous* on and inside a simple closed contour C. Prove that the function $g(z)$ defined by

$$g(z_0) := \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

is analytic at every point z_0 inside C and that

$$g'(z_0) := \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$$

(b) If $f(z) = x(1-x)(1-y)$ and C is the square with vertices $0, 1, 1+i, i$ compute $g(z_0)$.

4.6.6. Suppose we have a polynomial centered at z_0:

$$p(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$$

Prove that $a_k = \frac{p^{(k)}(z_0)}{k!}$ is the usual Taylor coefficient.

4.6.7. Suppose one attempted to prove Cauchy's integral formula $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$ purely for a *circular path* C. Explain why we still need the full Cauchy–Goursat Theorem to complete the proof.

If we didn't, then the full version would be redundant: Corollary 4.31 would show that every analytic function $f(z) = u + iv$ had continuous partial derivatives u_x, u_y, v_x, v_y, whence our sketch proof of Cauchy–Goursat (Theorem 4.22) relying on Green's Theorem would be enough to establish the complete integral formula…

4.7 Liouville's Theorem and The Maximum Modulus Principle

In this section we derive some powerful corollaries of the Cauchy integral formula. The first is easy.

Lemma 4.32 (Cauchy's Inequality). *Let C_R be the circle of radius R centered at z_0. If $f(z)$ is analytic on and inside C_R and $|f(z)| \le M_R$ on C_R, then*

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z\right| \le \frac{n!M_R}{R^n}$$

Here is a surprising application. Suppose $f(z)$ is entire and bounded: $|f(z)| \le M$ for all z. Since $f(z)$ is analytic on and inside *every* circle centered at z_0, Cauchy's inequality says that

$$\forall z_0, R,\ \left|f'(z_0)\right| \le \frac{M}{R}$$

Plainly $f'(z_0) = 0$ for all z_0, and we've proved the following.

Theorem 4.33 (Liouville). *The only bounded entire functions are constants.*

We now come to one of the most famous results in mathematics. It has many proofs, though this is one of the easiest, provided you are comfortable with all the machinery we've developed thusfar!

Theorem 4.34 (Fundamental Theorem of Algebra). *Every polynomial $p(z)$ of degree $n \ge 1$ has a zero. Moreover, every such $p(z)$ factors uniquely up to ordering as a product of linear factors:*

$$p(z) = a(z-z_1)\cdots(z-z_n)$$

Proof. Suppose WLOG that $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ has no zeros. Then $\frac{1}{p(z)}$ is entire: we prove that it is also *bounded* on $\mathbb{C}$. Since $p(z)$ is not constant, this contradicts Liouville's Theorem.

For each $k \le n-1$, define $R_k := (2n\,|a_k|)^{\frac{1}{n-k}} \ge 0$ and set $R := \max\{R_0, \ldots, R_{n-1}\}$. Then, if $a_k \ne 0$,

$$|z| > R \implies \frac{|a_k|}{|z|^{n-k}} < \frac{|a_k|}{R^{n-k}} \le \frac{|a_k|}{R_k^{n-k}} = \frac{1}{2n}$$

The inequality is trivially true if $|a_k| = 0$. But then

$$|z| > R \implies \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| < \frac{1}{2} \implies |p(z)| = |z|^n \left|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \ge \frac{1}{2}|z|^n > \frac{1}{2}R^n$$

whence $\frac{1}{|p(z)|} < \frac{2}{R^n}$ is bounded on the domain $|z| > R$. Since $\frac{1}{p(z)}$ is continuous, it is also bounded on the closed disk $|z| \le R$, and thus on $\mathbb{C}$. We have our contradiction.

The second result follows by applying the *factor theorem* n times (exercise): if $p(z)$ has a zero z_1, then there exists a polynomial $q(z)$ with

$$p(z) = (z-z_1)q(z) \quad \text{where} \quad \deg q = n-1\ldots$$

∎

In the previous proof we used the fact that a continuous function $f(z)$ on a closed bounded domain K is itself bounded. As you should recall from real analysis, the least upper bound is in fact achieved:

$$\exists z_0 \in K \text{ such that } |f(z_0)| = \sup\{|f(z)| : z \in K\}$$

For *analytic* functions, we can say something more restrictive and surprising. The maximum modulus of an analytic function on a closed bounded domain is always and only achieved at an *edge point.*

Example 4.35. Let $f(z) = e^z$ on the unit disk $|z| \le 1$. Then $|f(z)| = e^x$ which has its maximum at $z = 1$, on the edge of the disk.

We state the general result a little differently.

Theorem 4.36 (Maximum Modulus Principle). *If $f(z)$ is analytic and non-constant on an* bounded, open, connected *domain D, then $|f(z)|$ has no maximum value on D.*

Proof. Suppose $f(z)$ is analytic on a bounded, open, connected D and that it attains a maximum value: $|f(z)| \le |f(z_0)|$ for all $z \in D$. We prove that $f(z)$ is constant.

Let $w \in D$ and join z_0 to w by a simple contour C. Recall that there exist finitely many[a] open disks $B_0, B_1, \ldots, B_n \subseteq D$ of some radius $\epsilon > 0$, centered on C and points $z_1, \ldots, z_n \in C$ such that $z_k \in B_{k-1} \cap B_k$. We may also assume that B_0 is centered at z_0.

Let $r < \epsilon$ and apply the Cauchy integral formula around the circle C_r of radius r centered at z_0:

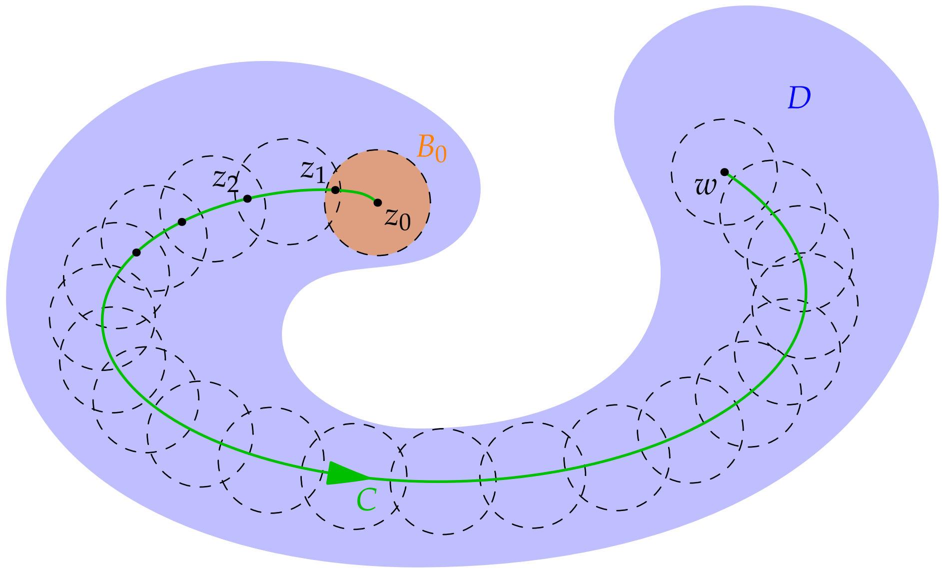
$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} \, \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, \mathrm{d}\theta \\
\implies |f(z_0)| &\le \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right| \mathrm{d}\theta \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \, \mathrm{d}\theta = |f(z_0)| \\
\implies |f(z_0)| &= \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right| \mathrm{d}\theta \\
\implies &\int_0^{2\pi} |f(z_0)| - \left| f(z_0 + re^{i\theta}) \right| \mathrm{d}\theta = 0
\end{aligned}$$

Since the integrand is non-negative and continuous, we conclude that it is zero. Since $|f(z)| = |f(z_0)|$ on all circles of radius $< \epsilon$ centered at z_0, we see that it is constant on B_0. Since $f(z)$ is analytic, it is also constant[b] on B_0.

To finish the proof, note that $z_1 \in B_0 \cap B_1 \implies f(z_1) = f(z_0)$ and that $|f(z)| \le |f(z_1)|$ on B_1. Thus $f(z) = f(z_0)$ on B_1 also. Repeating this process n times, we see that $f(z) = f(z_0)$ on every disk B_k: in particular, $f(w) = f(z_0)$. ∎

[a] Let $\epsilon > 0$ be at most the minimum distance from C to the boundary of D. The curve C is then a subset (is covered by) the family of open disks of radius ϵ centered at each $z \in C$. Since C is a compact set, finitely many of these cover C. A picture is shown below.

[b] Recall: $|f(z)| = k \ne 0 \implies \overline{f(z)} = \frac{k^2}{f(z)}$ is analytic, so $u - iv$ satisfy the Cauchy–Riemann equations…



Exercises. 4.7.1. (a) Suppose $f(z)$ is entire and that $|f(z)| \le c\,|z|$ for some constant $c \in \mathbb{R}^+$. Prove that $f(z) = kz$ where $k \in \mathbb{C}$ satisfies $|k| \le c$.

(b) What can you say about $f(z)$ if it is entire and there exists some linear polynomial $cz + d$ with $c \ne 0$ such that $|f(z)| \le |cz + d|$ for all $z \in \mathbb{C}$?

4.7.2. If $f(z) = u + iv$ is entire and $u(x, y)$ is bounded above, apply Liouville's Theorem to $\exp(f(z))$ to prove that $u(x, y)$ is constant.

4.7.3. If $f(z)$ is a non-zero analytic function on a closed bounded domain, show that the minimum value of $|f(z)|$ occurs on the boundary of the domain.
(*Hint: consider the function* $g(z) = \frac{1}{f(z)}$)

4.7.4. Find the maximum and minimum values of $\left|z^2 + 4i\right|$ on the unit disk $|z| \le 1$.
(*Hint: maximize and minimize* $\left|z^2 + 4i\right|^2$)

4.7.5. On the rectangle $0 \le x \le \pi$, $0 \le y \le 1$, show that $|\sin z|$ attains its maximum value at the point $z = \frac{\pi}{2} + i$.
(*Hint: first show that* $|\sin z|^2 = \sin^2 x + \sinh^2 y$)

4.7.6. In the proof of the fundamental theorem of algebra, why must R be positive?

4.7.7. (a) Prove the factor theorem: if $p(z_1) = 0$, then $p(z) = (z - z_1)q(z)$ for some polynomial $q(z)$. If you want more of a challenge, prove the full division algorithm first: if $f(z), g(z)$ are polynomials with $\deg f \ge \deg g$, then there exist unique polynomials $q(z), r(z)$ for which

$$f(z) = g(z)q(z) + r(z) \quad \text{and} \quad \deg r < \deg g$$

(b) Complete the proof of the fundamental theorem of algebra.

5 Series

The discussion of series in complex analysis differs significantly from the real situation. In particular:

- Taylor's Theorem: Analytic functions equal their Taylor series. This is false in real analysis where differentiable functions need not have, nor equal, a Taylor series.
- Laurent Series: By series, we also include negative powers such as $z^{-1} + 3z^{-2} + \cdots$

Most of this chapter is devoted to these topics. First we quickly review the basics of sequences and infinite series (of non-negative powers) which should largely have been covered in a previous course.

5.1 A Brief Review of Sequences and Infinite Series

Post real analysis, there is little specific to say regarding sequences of complex numbers. The notions of limit, convergence and sequential continuity are essentially identical in $\mathbb{C}$ and $\mathbb{R}^2$. For instance:

Definition 5.1. A sequence (z_n) has limit $z \in \mathbb{C}$, and we write $\lim z_n = z$, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |z_n - z| < \epsilon$$

Writing $z_n = x_n + iy_n$ and $z = x + iy$ in real and imaginary parts, we see that

$$|z_n - z| \le |x_n - x| + |y_n - y| \le 2\,|z_n - z|$$

In particular, it should be obvious that

Lemma 5.2. $\lim z_n = z \iff \lim x_n = x$ *and* $\lim y_n = y$*, in which case* $\lim z_n = \lim x_n + i \lim y_n$.

Warning! While this mostly translates to the polar representation $z_n = r_n e^{i\theta_n}$, there is a caveat: the discontinuity of $\operatorname{Arg} z = \Theta$ when z is a non-positive real number means that (Θ_n) need not converge even if (z_n) does.

Example 5.3. The sequence with $z_n = 2i - \frac{1+i}{n}$ has limit $z = 2i$: given $\epsilon > 0$, let $N = \sqrt{2}\epsilon$, then

$$n > N \implies |z_n - z| = \frac{\sqrt{2}}{n} < \frac{\sqrt{2}}{N} = \epsilon$$

The real and imaginary parts are $x_n = -\frac{1}{n}$ and $y_n = 2 - \frac{1}{n}$ which clearly converge to $x = 0$ and $y = 2$ respectively. In polar co-ordinates things are also as expected[a]

$$\lim r_n = \lim \sqrt{\frac{1 + (2n-1)^2}{n^2}} = \lim \frac{2}{n}\sqrt{n^2 - n} = 2$$
$$\lim \Theta_n = \lim \left(\pi \tan^{-1} \frac{(2n-1)/n}{-1/n} \right) = \pi + \tan^{-1}(1 - 2n) = \frac{\pi}{2}$$

[a] Note that z_n lies in the second quadrant!

Definition 5.4. Given a sequence[a] (z_n), define the sequence of partial sums (s_n) by

$$s_n = \sum_{k=0}^{n} z_k = z_0 + \cdots + z_n$$

Define the *infinite series* $\sum z_n = \lim s_n$. This is said to converge (diverge) if the sequence of partial sums converges (diverges).
The sequence of *remainders* is defined by $\rho_n = \sum z_n - s_n$. Clearly $\sum z_n$ converges $\iff \lim \rho_n = 0$.
The series *converges absolutely* if $\sum |z_n|$ converges.

[a]For brevity, we assume the initial term is z_0: nothing prevents the initial term being z_{n_0} for any natural number n_0.

Here is a quick summary of some basic facts about series.

Theorem 5.5. *Let $\sum z_n$ and $\sum w_n$ be series of complex numbers.*

1. *If $z_n = x_n + iy_n$, then $\sum z_n$ converges if and only if $\sum x_n$ and $\sum y_n$ both converge, in which case*

$$\sum z_n = \sum x_n + i \sum y_n$$

2. *If $a \neq 0$ and $\sum z_n$ and $\sum w_n$ converge, then $\sum az_n + w_n$ converges, in which case*

$$\sum az_n + w_n = a \sum z_n + \sum w_n$$

3. *(n^{th} term/divergence test)* *If $\sum z_n$ converges, then $\lim z_n = 0$.*
4. *Absolute convergence implies convergence.*[a]

[a]A series *converges conditionally* if it converges but not absolutely.

Proof. 1. This is immediate from 5.2.

2, 3, 4. These follow from 1 and the corresponding results for the real series $\sum x_n, \sum y_n$.

5. Since $\sum |z_n|$ is a convergent series of non-negative terms and $|x_n| \leq |z_n|$, the (real!) comparison test proves that $\sum x_n$ is absolutely convergent and thus convergent.
Since $\sum y_n$ converges similarly, part 1 shows that $\sum z_n$ converges. ∎

Exercises. 5.1.1. Prove, using the ϵ-N definition, that $\lim \frac{2+in}{n} = i$.

5.1.2. Provide a rigorous proof of 5.2. Sketch a proof of the corresponding statement for the polar representation whenever $\lim z_n$ is non-zero and not a negative real number.

5.1.3. Explicitly prove part 2 of Theorem 5.5.

5.1.4. Fix $\theta \in (-\pi, \pi)$. Prove that the sequence defined by $z_n = e^{in\theta}$ converges if and only if $\theta = 0$.

5.1.5. Use the ϵ-N defineion to prove that $\lim \sqrt{i + \frac{1}{n}} = \frac{1+i}{\sqrt{2}}$ where we use the principal value.

5.2 Power Series, Taylor Series and Taylor's Theorem

We first make the identical definition to that in real analysis.

Definition 5.6. A power series about $z_0 \in \mathbb{C}$ is a function $f : D \to \mathbb{C}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where D is the region on which the series converges.

For the purposes of review, we consider the canonical example:

Example 5.7. (Geometric series) Consider the power series $\sum_{n=0}^{\infty} z^n$. By the n^{th} term test, this certainly diverges if $|z| \geq 1$. Otherwise, its partial sums satisfy

$$s_n - zs_n = 1 - z^{n+1} \implies s_n = \frac{1 - z^{n+1}}{1 - z}$$

Since (z^{n+1}) converges (to zero) when $|z| < 1$, we conclude that the geometric series converges inside the unit circle, and that

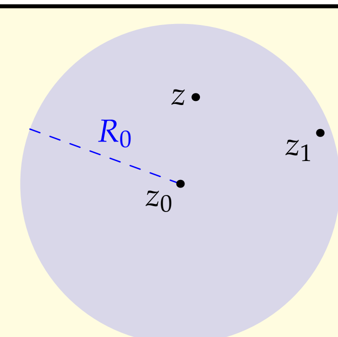
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ whenever } |z| < 1$$

Note how the geometric series converges on a disk. As the next result shows, this is the case for *every* power series, analogous to the concept of the interval/radius of convergence in real analysis.

Theorem 5.8. *If a power series $f(z) = \sum a_n (z - z_0)^n$ converges at a point $z_1 \neq z_0$, then it is absolutely convergent at every point z satisfying $|z - z_0| < |z_1 - z_0|$. Moreover, if we define*

$$R_0 = \sup \{|z - z_0| : f(z) \text{ converges}\}$$

then $f(z)$ converges absolutely whenever $|z - z_0| < R_0$ and diverges whenever $|z - z_0| > R_0$.



Proof. The sequence $(a_n (z_1 - z_0)^n)$ converges (to 0) and is therefore bounded by some $M \in \mathbb{R}^+$. Thus

$$|a_n| \, |z - z_0|^n = |a_n| \, |z_1 - z_0|^n \left(\frac{|z - z_0|}{|z_1 - z_0|} \right)^n \leq Mr^n \quad \text{where} \quad r = \frac{|z - z_0|}{|z_1 - z_0|} < 1$$

Since $\sum Mr^n$ converges, we conclude (real comparison test) that $\sum |a_n| \, |z - z_0|^n$ converges.
The second part is clear since R_0 is the least upper bound. ∎

Definition 5.9. The R_0 in the theorem is the *radius of convergence* of the power series.

- If $R_0 = \infty$, the series is (absolutely) convergent on $\mathbb{C}$;
- If $R_0 = 0$, the series converges only when $z = z_0$;
- Otherwise, the circle radius R_0 centered at z_0 is the *circle of convergence* for the series. As in real analysis, convergence on the circle itself must be checked separately.

Rather than computing any examples, we first revisit a familiar definition and observe a startling difference between the real and complex case that again marks the special nature of analytic functions.

Definition 5.10. Suppose $f(z)$ is analytic at $z = z_0$. Its *Taylor series* about z_0 is the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

The *Taylor coefficients* are $a_n = \frac{f^{(n)}(z_0)}{n!}$. The *Maclaurin series* is the Taylor series with $z_0 = 0$.

Theorem 5.11 (Taylor's Theorem). *If $f(z)$ is analytic on a disk $|z - z_0| < R_0$, then it equals its Taylor series on that disk.*

This is *very* special compared to real analysis where there exists many functions which do not equal their Taylor series (see Exercise 5.2.4.).

Clearly R_0 must be no larger than the radius of convergence of the Taylor series. If $f(z)$ is entire, then the result holds for all positive R_0, whence the disk may be taken to be $\mathbb{C}$. For finite R_0, we shall later see that R_0 is the distance from z_0 to the nearest point at which $f(z)$ is non-analytic (Exercise 5.3.7.).

Examples 5.12. The familiar examples of Maclaurin series translate over from real analysis.

1. $f(z) = \frac{1}{1-z}$ has $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ whence $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ whenever $|z| < 1$.
2. Since $f(z) = e^z$ is entire and $f^{(n)}(0) = 1$, we see that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for all } z \in \mathbb{C}$$

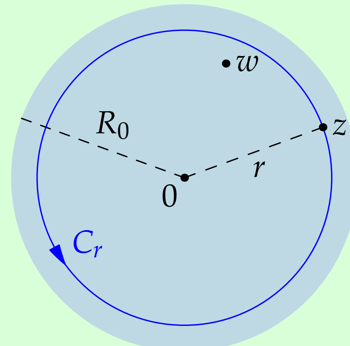
3. $f(z) = \sin z$ is entire with $f^{(2n)}(z) = (-1)^n \sin z$ and $f^{(2n+1)}(z) = (-1)^n \cos z$. Its Maclaurin series is therefore

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \text{for all } z \in \mathbb{C}$$

Why does is Taylor's Theorem so much more specific than in the real case? The answer is largely due to Cauchy's integral formula...

Proof. By relabelling $g(z) = f(z - z_0)$, it is enough to prove Taylor's Theorem about the origin.

Suppose w is given where $|w| < R_0$. Then $\exists r < R_0$ such that $|w| < r$. Consider the circle C_r centered at the origin with radius r. By Example 5.7, for any $z \in C_r$, we have



$$\sum_{k=0}^{n-1} \left(\frac{w}{z}\right)^k = \frac{1 - \left(\frac{w}{z}\right)^n}{1 - \frac{w}{z}}$$

$$\implies \frac{1}{z-w} = \frac{1}{z\left(1 - \frac{w}{z}\right)} = \frac{1}{z}\sum_{k=0}^{n-1} \left(\frac{w}{z}\right)^k + \frac{1}{z-w}\left(\frac{w}{z}\right)^n$$

Apply Cauchy's integral formula, both to evaluate $f(w)$ and to convert contour integrals back to derivatives of f:

$$f(w) = \sum_{k=0}^{n-1} \frac{w^k}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{k+1}}\,\mathrm{d}z + \frac{w^n}{2\pi i} \oint_{C_r} \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} w^k + \frac{w^n}{2\pi i} \oint_{C_r} \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z$$

All that remains is to control the last integral. Since $f(z)$ is analytic on C_r, it is bounded by some $M \in \mathbb{R}^+$. Moreover, for $z \in C_r$ we have $|z - w| \geq r - |w|$. Thus

$$\begin{aligned}\left| f(w) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} w^k \right| &= \frac{|w|^n}{2\pi} \left| \oint_{C_r} \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z \right| \leq \frac{|w|^n}{2\pi} 2\pi r \frac{M}{r^n(r - |w|)} \\ &= \frac{Mr}{(r - |w|)} \left(\frac{|w|}{r}\right)^n \xrightarrow[n\to\infty]{} 0\end{aligned}$$

∎

Exercises. 5.2.1. Compute the Maclaurin series of $\cos z$ directly from the definition.

5.2.2. Evaluate the Taylor series of $\sin z$ about $z_0 = \frac{\pi}{2}$ and confirm that it equals your answer to question 1 where z is replaced with $z - \frac{\pi}{2}$.

5.2.3. Consider $f(z) = \frac{1}{z}$. For any $z_0 \neq 0$, find the Taylor series of $f(z)$ about z_0. What is its circle of convergence?

5.2.4. Consider the function

$$f(z) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

When $z \in \mathbb{R}$ this provides the classic example of an infinitely differentiable function whose Maclaurin series (being identically zero) does not equal the original function except at the origin. When $z \in \mathbb{C}$, why $f(z)$ does not contradict Taylor's Theorem.

5.3 Uniform Convergence: Continuity, Integrability and Differentiability

In this section we perform a little housekeeping. Analogously to real analysis, we want to establish the following facts about power series:

1. Representations are unique: if two power series are equal, their coefficients are equal;
2. Power series are continuous, indeed analytic;
3. Power series may be differentiated and integrated term-by-term.

The arguments are intertwined, so we do not establish them in order. Since the arguments are often similar to the real case, in some cases identical, we will be brief and postpone all examples until the end. The critical ingredient is uniform convergence.

Definition 5.13. Suppose $f(z) = \sum a_n(z-z_0)^n$ is a power series with partial sum $s_n(z)$ and remainder $\rho_n(z) = f(z) - s_n(z)$. We say that the series *converges uniformly* on a domain D, if

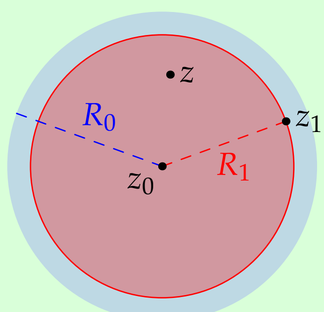
$$\forall \epsilon > 0,\ \exists N \text{such that } n > N, z \in D \implies |\rho_n(z)| < \epsilon$$

Uniformity refers to the fact that the same N works for all $z \in D$. If the value of N had to depend on z, we'd refer to the convergence as *pointwise.*

Theorem 5.14. *Suppose R_0 is the radius of convergence of a power series about z_0. If $R_1 < R_0$, then the series converges uniformly on the closed disk $|z - z_0| \le R_1$.*

Proof. Let $\epsilon > 0$ be given. Suppose z_1 satisfies $|z_1 - z_0| = R_1$. Since $R_1 < R_0$, the series converges absolutely at z_1. Denote the n^{th} remainder at z_1 of this absolutely convergent series by

$$\sigma_n = \sum_{k=n+1}^{\infty} |a_k|\,|z_1 - z_0|^k$$



By the n^{th} term test, $\lim\limits_{n\to\infty} \sigma_n = 0$:

$$\exists N \text{ such that } n > N \implies \sigma_n < \epsilon$$

Now suppose $|z - z_0| \le R_1$: for any $m \ge n$, we have

$$\left|\sum_{k=n+1}^{\infty} a_k(z-z_0)^k\right| \le \sum_{k=n+1}^{\infty} |a_k|\,|z-z_0|^k \le \sum_{k=n+1}^{\infty} |a_k|\,|z_1-z_0|^k$$

Both sides converge as $m \to \infty$, yielding $|\rho_n(z)| \le \sigma_n$, whence

$$n > N \implies |\rho_n(z)| \le \sigma_n < \epsilon$$

■

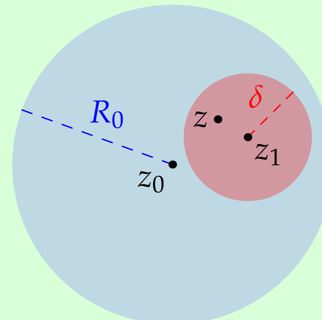
Note that the convergence need not be uniform all the way to the edge of the circle of convergence: see, for instance, Exercise 5.3.8..

Theorem 5.15 (Continuity). *Suppose $f(z) = \sum a_n(z-z_0)^n$ has radius of convergence R_0. Then $f(z)$ is continuous whenever z lies inside the circle of convergence $|z-z_0| < R_0$.*

Proof. We present the famous $\frac{\epsilon}{3}$-proof. Fix z_1 inside the circle and let $\epsilon > 0$ be given. Observe:

- The circle of convergence is open and the partial sums are continuous (polynomials!):

$$\exists \delta > 0 \text{ such that } |z-z_1| < \delta \implies \begin{cases} |z-z_0| < R_0 \\ |s_n(z) - s_n(z_1)| < \frac{\epsilon}{3} \end{cases}$$



- Uniform convergence at z and z_1:

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{3} \text{ and } |\rho_n(z_1)| < \frac{\epsilon}{3}$$

Now put it together to see that $f(z)$ is continuous at z_1:

$$\begin{aligned} |z-z_1| < \delta \implies |f(z) - f(z_1)| &= |f(z) - s_n(z) + s_n(z) - s_n(z_1) + s_n(z_1) - f(z_1)| \\ &\le |\rho_n(z)| + |s_n(z) - s_n(z_1)| + |\rho_n(z_1)| < \epsilon \end{aligned}$$

■

Since power series are continuous, we may now define contour integrals along any contour staying within the circle of convergence. The powerful tools of contour integration, particularly Cauchy's integral formula, will allow us to prove the remaining results, all of which follow from a general version of term-by-term integration.

Theorem 5.16. *Let $f(z) = \sum a_n(z-z_0)^n$ be a power series, C a contour lying inside the circle of convergence, and let $g(z)$ be continuous on C. Then*

$$\int_C g(z)f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\,\mathrm{d}z$$

Proof. Since f, g are continuous on C, the integral $\int_C g(z)f(z)\,\mathrm{d}z$ certainly exists. Moreover $g(z)$ is bounded on C by some $M \in \mathbb{R}^+$. Let C have length L and let $\epsilon > 0$ be given. Since $f(z)$ converges uniformly,

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{ML}$$

Now take integrals and moduli of

$$g(z)f(z) - \sum_{k=0}^{n} a_k g(z)(z-z_0)^k\,\mathrm{d}z = g(z)\rho_n(z)$$

to see that

$$n > N \implies \left| \int_C g(z)f(z)\,\mathrm{d}z - \sum_{k=0}^{n} a_k \int_C g(z)(z-z_0)^k\,\mathrm{d}z \right| = \left| \int_C g(z)\rho_n(z)\,\mathrm{d}z \right| < \epsilon$$

■

Everything we want now follows by choosing specific functions $g(z)$.

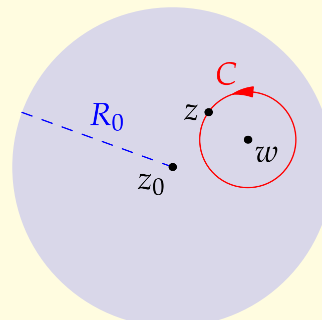
Corollary 5.17. *Suppose $f(z) = \sum a_n(z-z_0)^n$ is a power series with radius of convergence $R_0 > 0$.*

1. *(Term-by-term integration) Let $g(z) = 1$ to see that*

$$\int_C f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^n \Big|_{C(start)}^{C(end)}$$

2. *(Analyticity) Let $g(z) = 1$, then $\oint_C f(z)\,\mathrm{d}z = 0$ round every simple closed contour, whence $f(z)$ is analytic inside the circle of convergence.*

3. *(Term-by-term differentiation) Given w inside the circle of convergence, let $g(z) = \dfrac{1}{2\pi i(z-w)^2}$ and apply the Cauchy integral formula on a small circle around w:*



$$\begin{aligned} f'(w) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-w)^2}\,\mathrm{d}z = \sum \frac{a_n}{2\pi i}\oint \frac{(z-z_0)^n}{(z-w)^2}\,\mathrm{d}z \\ &= \sum a_n \left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=w} (z-z_0)^n = \sum a_n n(z-z_0)^{n-1} \end{aligned}$$

4. *(Uniqueness of representation) The only power series representation of a function about z_0 is its Taylor series. In particular, if two power series are equal on a domain, then their coefficients are equal.*

Part 4 is the converse to Taylor's Theorem: an argument is in the Exercises.

Examples 5.18. By uniqueness of representation, we can compute Taylor/Maclaurin series *algebraically*: if a function equals a series, then that series is the Taylor/Maclaurin series of the function.

1. $f(z) = z^3 e^{z^2} = z^3 \sum_{n=0}^{\infty} \dfrac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{z^{2n+3}}{n!}$. Moreover, this is the Maclaurin series of $f(z)$. Since $f(z)$ is entire, the function equals its Maclaurin series everywhere on $\mathbb{C}$.
2. If $f(z) = \dfrac{1}{z^4+16i}$, then we may find its Maclaurin series algebraically:

$$f(z) = \frac{1}{16i\left(1 - \frac{z^4}{-16i}\right)} = \frac{1}{16i}\sum_{n=0}^{\infty}\left(\frac{z^4}{-16i}\right)^n = \sum_{n=0}^{\infty} \frac{i^{n-1}}{16^{n+1}} z^{4n}$$

This converges whenever $\left|\frac{z^4}{-16i}\right| < 1 \iff |z| < 2$. If C is the straight line joining $z = 0$ to $z = i$, then

$$\int_C f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{i^{n-1}}{16^{n+1}} \int_C z^{4n}\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{i^{n-1}}{16^{n+1}(4n+1)} i^{4n+1} = \sum_{n=0}^{\infty} \frac{i^n}{16^{n+1}(4n+1)}$$

Exercises. 5.3.1. Find a power series representation and the radius of convergence:

(a) $f(z) = \dfrac{z}{4-z}$ about $z_0 = 0$;

(b) $f(z) = z \sin z^2$ about $z_0 = 0$;

(c) $f(z) = \cosh 3z$ about $z_0 = \dfrac{i\pi}{9}$

5.3.2. *Without computing derivatives*, find the Taylor series for $f(z) = \frac{1}{z}$ about $z_0 \neq 0$. By differentiating term-by-term, find the Taylor series of $\frac{1}{z^2}$ about z_0.

5.3.3. By expressing it as a Taylor series, show that the following function is entire:

$$f(z) = \begin{cases} \frac{1}{z^2}(1-\cos z) & \text{if } z \neq 0 \\ \frac{1}{2} & \text{if } z = 0 \end{cases}$$

5.3.4. (a) By integrating the Taylor series for z^{-1} about $z_0 = 1$, prove that

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \quad \text{whenever } |z-1| < 1$$

(b) Prove that the following function is analytic on the domain $0 < |z|$, $\operatorname{Arg} z \in (-\pi, \pi)$:

$$f(z) = \begin{cases} \frac{\operatorname{Log} z}{z-1} & \text{if } z \neq 1 \\ 1 & \text{if } z = 1 \end{cases}$$

5.3.5. Consider the Maclaurin series $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ on the disk $|z| < 1$. Show that $h(z) = \frac{1}{z^2+1}$ is the analytic continuation of $f(z)$ to $\mathbb{C} \setminus \{i, -i\}$.

5.3.6. Prove part 4 of Corollary 5.17: if $f(z) = \sum a_n(z-z_0)^n$, prove that $f^{(m)}(z_0) = m! a_m$ so that the series really is the Taylor series of $f(z)$.
(*Hint: let* $g(z) = \frac{m!}{2\pi i(z-z_0)^{m+1}}$ *in Theorem 5.16*)

5.3.7. Suppose $f(z)$ is analytic at z_0 and define

$$R_0 = \inf\{|\hat{z} - z_0| : f(z) \text{ non-analytic or undefined at } \hat{z}\,\}$$

Essentially R_0 is the distance from z_0 to the nearest point $\hat{z}$ at which $f(z)$ is non-analytic or undefined. Prove that R_0 is the radius of convergence of the Taylor series of $f(z)$ about z_0.

5.3.8. Consider $f(z) = \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ on $|z| < 1$.

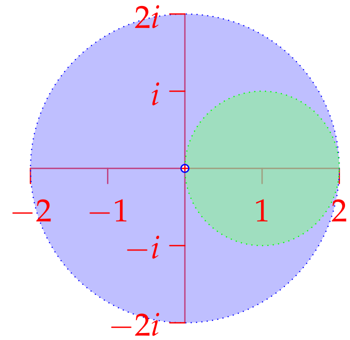
(a) Let $R_1 < 1$. Explicitly check uniform convergence when $|z| \leq R_1$. That is, given $\epsilon > 0$, find an explicit N such that

$$n > N \implies |\rho_n(z)| = \left| f(z) - \sum_{k=0}^{n} z^k \right| < \epsilon \text{ whenever } |z| \leq R_1$$

(b) Prove that $f(z)$ is *not* uniformly convergent on $|z| < 1$.
(*Hint: Let $\epsilon = 1$ and try to get a contradiction...*)

5.4 Laurent Series

While Taylor series are undeniably useful, they have some weaknesses. In particular, they are only valid on a *disk*. For instance $f(z) = \frac{1}{z(2-z)}$ can be written as a Taylor series centered at $z = 1$:



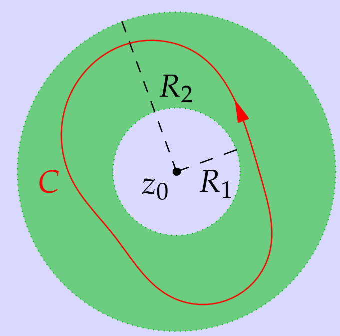
$$f(z) = \frac{1}{1-(z-1)^2} = \sum_{n=0}^{\infty} (z-1)^{2n} \quad \text{whenever } |z-1| < 1$$

Unfortunately, the most interesting aspects of $f(z)$ involve its behavior near the poles $z = 0, 2$. Because of its limited domain, we can't use the Taylor series to loop around these points. As an alternative, we expand $\frac{1}{2-z}$ in a power series centered at 0 to obtain

$$f(z) = \frac{1}{2z(1-\frac{z}{2})} = \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}} = \frac{1}{2z} + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16} + \cdots$$

This second series, containing a negative power, is known as a *Laurent series*. By construction, it is valid on the punctured disk $0 < |z| < 2$. The picture compares the domains of the two series. The larger domain, particularly encircling the origin, is an obvious advantage of the Laurent series.

Definition 5.19. Suppose a function $f(z)$ is analytic on an *annulus* $R_1 < |z - z_0| < R_2$. Its *Laurent series* about z_0 is the expression[a]



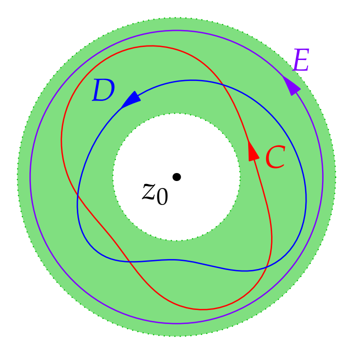
$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where each} \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

and C is a simple closed contour encircling z_0 within the annulus.

[a] If you prefer, write $\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ where $b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,\mathrm{d}z$.

A few remarks:

1. As above, the inner radius can be $R_1 = 0$ so that the domain is a punctured disk. As with Taylor series, the outer radius can also be infinite.
2. The choice of curve C is irrelevant. If D is another simple closed curve encircling z_0, then choose a circle E outside both C and D. Since $\frac{f(z)}{(z-z_0)^{n+1}}$ is analytic on the annulus, two applications of the Cauchy–Goursat theorem yield



$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \oint_E \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \oint_D \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

   In particular, the coefficients a_n are independent of the curve C.
3. If $f(z)$ is analytic on the *disk* $|z - z_0| < R_2$, then the Laurent series is the Taylor series of $f(z)$:
   - $n \geq 0 \implies a_n = \frac{f^{(n)}(z_0)}{n!}$ by the Cauchy integral formula;
   - $n < 0 \implies a_n = 0$ by the Cauchy–Goursat theorem.

It is usually difficult to compute a Laurent series directly using the definition, since it requires the computation of infinitely many contour integrals. Thankfully, as we shall see in the following results, if a function equals a series with negative powers, then the series is the Laurent series of the function. This makes computing examples much easier!

Examples 5.20. 1. We revisit our example $f(z) = \frac{1}{z(2-z)}$ about $z_0 = 0$ and compute the Laurent series explicitly. First observe that $f(z)$ is analytic on the punctured disk $0 < |z| < 2$. Let C be the unit circle centered at $z = 0$ and compute:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, \mathrm{d}z = \frac{1}{2\pi i} \oint_C \frac{1}{z^{n+2}(2-z)} \, \mathrm{d}z$$

There are two cases:

- If $n \leq -2$, then $a_n = \dfrac{1}{2\pi i} \displaystyle\oint_C \frac{z^{-2-n}}{2-z} \, \mathrm{d}z = 0$ since the integrand is analytic on and within C;
- If $n \geq -1$, the Cauchy integral formula tells us that

$$a_n = \frac{1}{(n+1)!} \left.\frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\right|_{z=0} (2-z)^{-1} = \frac{(n+1)!}{(n+1)!} (2-z)^{-2-n}\big|_{z=0} = \frac{1}{2^{n+2}}$$

The Laurent series of $f(z)$ is therefore as claimed above: $\displaystyle\sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}}$

2. Since e^z has Maclaurin series $\sum \frac{z^n}{n!}$ valid on the entire complex plane, we obtain the Laurent series expansion

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

on the punctured plane $z \neq 0$. It would be extremely irritating to have to calculate the integrals $a_n = \frac{1}{2\pi i} \oint_C \frac{e^{1/z}}{z^{n+1}} \, \mathrm{d}z$ explicitly.

3. We obtain another Laurent series valid on $|z| > 0$:

$$\frac{1}{z^7} \sin z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n-5} = z^{-5} - \frac{1}{6} z^{-1} + \frac{1}{120} z^3 - \frac{1}{5040} z^7 + \cdots$$

4. Multiplying term-by-term, and since we need *both* Maclaurin series to be valid, we obtain a Laurent series valid on the punctured disk $0 < |z| < 1$:

$$\begin{aligned}
\frac{1}{z(z-1)(z-2i)} &= \frac{1}{z} \left(\sum_{n=0}^{\infty} (-1)^n z^n \right) \left(\sum_{m=0}^{\infty} \left(\frac{i}{2}\right)^m z^m \right) \\
&= \frac{1}{z} \left(1 - z + z^2 - z^3 + \cdots \right) \left(1 + \frac{i}{2} z - \frac{1}{4} z^2 - \frac{i}{8} z^3 + \cdots \right) \\
&= \frac{1}{z} + \left(-1 + \frac{i}{2} \right) + \left(\frac{3}{4} - \frac{i}{2} \right) z + \left(-\frac{3}{4} + \frac{3i}{8} \right) z^2 + \cdots
\end{aligned}$$

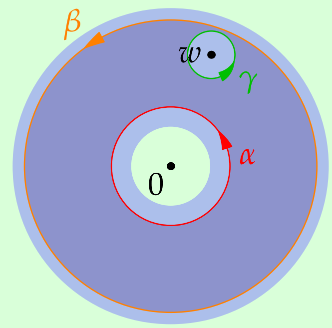
Theory time

Having seen a few examples, we should properly state and prove the main properties of Laurent series. These are very similar to the corresponding results for Taylor series, as are the proofs: mostly it is just an issue of keeping track of two series at once.

Theorem 5.21 (Laurent's Theorem). *An analytic function on an open annulus equals its Laurent series.*

Proof. By simple translation, it is enough to prove when $z_0 = 0$. Let w in the annulus be given.

Since the annulus is open, we may choose three non-overlapping circles α, β, γ with radii $R_\alpha, R_\beta, R_\gamma$ as in the picture:



- γ a small circle centered at w inside the annulus;
- α, β centered at 0, α inside and β outside w.

Since $\frac{f(z)}{z-w}$ is analytic on the region inside β with interior boundaries α and γ, Cauchy–Goursat says that

$$\left(\oint_\beta - \oint_\alpha - \oint_\gamma\right) \frac{f(z)}{z-w}\,\mathrm{d}z = 0 \implies f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = \frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,\mathrm{d}z$$

As in the proof of Taylor's theorem, we expand

$$\frac{1}{z-w} = \frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k + \frac{1}{z-w}\left(\frac{w}{z}\right)^n = -\frac{1}{w}\sum_{k=1}^{n}\left(\frac{z}{w}\right)^{k-1} + \frac{1}{z-w}\left(\frac{z}{w}\right)^n$$

and use this to attack the two integrals:

$$\frac{1}{2\pi i}\oint_\beta \frac{f(z)}{z-w}\,\mathrm{d}z = \sum_{k=0}^{n-1} \underbrace{\frac{w^k}{2\pi i}\oint_\beta \frac{f(z)}{z^{k+1}}\,\mathrm{d}z}_{a_k w^k} + \frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z$$

$$\frac{-1}{2\pi i}\oint_\alpha \frac{f(z)}{z-w}\,\mathrm{d}z = \sum_{k=1}^{n} \underbrace{\frac{1}{2\pi i w^k}\oint_\alpha z^{k-1} f(z)\,\mathrm{d}z}_{a_{-k} w^{-k}} - \frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,\mathrm{d}z$$

Since $f(z)$ is continuous on the closed bounded annulus between α, β, it has an upper bound M. Moreover, whenever $z \in \alpha \cup \beta$, we have $|z-w| > R_\gamma$. The triangle inequality finishes things off:

$$\begin{aligned}\left|f(w) - \sum_{k=-n}^{n-1} a_k w^k\right| &= \left|\frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,\mathrm{d}z - \sum_{k=-n}^{n-1} a_k w^k\right| \\ &\le \left|\frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z\right| + \left|\frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,\mathrm{d}z\right| \\ &\le \frac{MR_\beta}{R_\gamma}\left(\frac{|w|}{R_\beta}\right)^n + \frac{MR_\alpha}{R_\gamma}\left(\frac{R_\alpha}{|w|}\right)^n \xrightarrow[n\to\infty]{} 0\end{aligned}$$

∎

By substituting $w = (z - z_0)^{-1}$ in a series of negative powers

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} a_{-n} w^n$$

and applying Theorems 5.8, 5.14 and 5.15 to the power series in w, we may rapidly conclude the following:

Corollary 5.22. *Given a series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, define*

$$R_1 = \inf\{|z - z_0| : \text{series converges}\}, \qquad R_2 = \sup\{|z - z_0| : \text{series converges}\}$$

Then:

1. *The series converges absolutely on the annulus $R_1 < |z - z_0| < R_2$ to a continuous function.*
2. *The convergence is uniform on any closed sub-annulus.*

We also obtain the analogues of Theorem 5.16 and Corollary 5.17: some details are in the exercises.

Corollary 5.23. *Suppose $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z - z_0)^n$ has annulus of convergence $R_1 < |z - z_0| < R_2$.*

1. *If C is any contour in the annulus and $g(z)$ is continuous on C, then*

$$\int_C g(z) f(z)\, dz = \sum_{n=-\infty}^{\infty} a_n \int_C g(z)(z - z_0)^n\, dz$$

   In particular $f(z)$ may be integrated term-by-term along C.

2. *$f(z)$ is analytic on the annulus and $f'(z) = \sum\limits_{n=-\infty}^{\infty} a_n n (z - z_0)^{n-1}$.*
3. *Uniqueness: $\sum\limits_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is the Laurent series of $f(z)$.*

Now all the abstraction is out of the way, we can more easily compute Laurent series. In particular all the above examples are now valid. Indeed we have more, for instance:

Examples 5.24. 1. In accordance with part 2 of the Corollary, observe that

$$\frac{d}{dz} e^{1/z} = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=1}^{\infty} \frac{-z^{-1-n}}{(n-1)!} = -\frac{1}{z^2} \sum_{n=1}^{\infty} \frac{z^{-(n-1)}}{(n-1)!} = -\frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = -\frac{1}{z^2} e^{1/z}$$

2. To compute the integral $\oint_C \frac{1}{z^5} \sin z^2\, dz$ on a simple closed contour encircling the origin, we use the Laurent series and observe that all but one of the integrals evaluates to zero:

$$\oint_C \frac{1}{z^7} \sin z^2\, dz = \sum_{n=0}^{\infty} \oint_C \frac{(-1)^n}{(2n+1)!} z^{4n-5}\, dz = \oint_C \frac{(-1)^1}{(2+1)!} z^{4-5} = -\frac{1}{3}\pi i$$

Exercises. 5.4.1. Find a Laurent series representation for each function. Also find $\oint_C f(z)\,dz$ where C is a simple closed curve in the given domain encircling the origin.

(a) $f(z) = \frac{3}{z^2}e^{2z}$ whenever $|z| > 0$;

(b) $f(z) = \cos\frac{i}{z}$ whenever $|z| > 0$;

(c) $f(z) = \frac{1}{1+z^3}$ when $1 < |z|$ (*Hint: let* $w = z^{-1}$).

5.4.2. On each domian, find a Laurent series about $z_0 = 0$ for the function

$$f(z) = \frac{1}{z(z-2i)} = \frac{i}{2}\left(\frac{1}{z} - \frac{1}{z-2i}\right)$$

(a) $D_1 = \{z : 0 < |z| < 2\}$;

(b) $D_2 = \{z : |z| > 2\}$ (*again let* $w = z^{-1}$).

5.4.3. Repeat the previous question for

$$f(z) = \frac{1-2i}{(z-1)(z-2i)} = \frac{1}{z-1} - \frac{1}{z-2i}$$

Also find $\oint_C f(z)\,dz$ where C is a simple closed curve in the given domain encircling the origin.

(a) $D_1 = \{z : 0 < |z| < 1\}$ (*this is a Taylor series*);

(b) $D_2 = \{z : 1 < |z| < 2\}$;

(c) $D_3 = \{z : |z| > 2\}$.

5.4.4. Show that when $0 < |z-1| < 2$, we have

$$\frac{z}{(z-1)(z-3)} = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

5.4.5. Let a be a real number in the interval $(-1, 1)$. Show that

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad \text{whenever } |a| < |z|$$

5.4.6. Suppose $f(z)$ is a series satisfying the hypotheses of Corollary 5.23.

(a) Suppose part 1 has been proved. Explain why the function $f(z) - a_{-1}(z-z_0)^{-1}$ is analytic on the annulus. Hence conclude that $f(z)$ is analytic on the annulus.
(*This is different to Corollary 5.17 since* $a_{-1}(z-z_0)^{-1}$ *has no anti-derivative on the annulus!*)

(b) In order to mimic the proof of Corollary 5.17 to show that $f(z)$ is differentiable term-by-term, what properties must the curve C have?

(c) Prove part 3 (*recall Exercise 5.3.6. - the same hint works!*).

6 Residues and Poles

6.1 Residues and Cauchy's Residue Theorem

The goal of this section is the efficient computation of contour integrals of analytic functions. There are two crucial facts relating to contour integrals around simple closed curves:

- (Cauchy–Goursat) $\oint_C f(z)\,\mathrm{d}z = 0$ if $f(z)$ is analytic on and inside C;
- If C encircles z_0, then $\oint_C (z-z_0)^n\,\mathrm{d}z = 2\pi i$ if $n=-1$ and is zero otherwise.

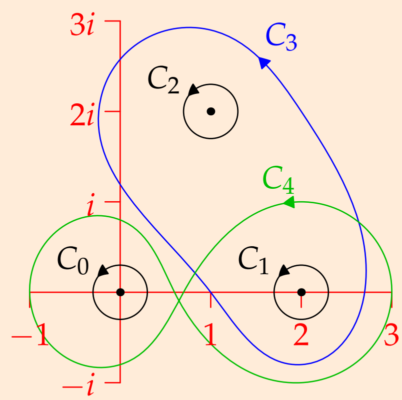
It turns out that these facts can be combined to allow us to evaluate many integrals round closed curves very efficiently.We start with an example.

Example 6.1. Consider the function

$$f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$$

which is analytic except at the points $z_0 = 0$, $z_1 = 2$, $z_2 = 1+2i$.

Several curves are drawn and the following values should be obvious using the above facts.



$$\oint_{C_0} f(z)\,\mathrm{d}z = 6\pi i, \qquad \oint_{C_1} f(z)\,\mathrm{d}z = -10\pi$$

$$\oint_{C_2} f(z)\,\mathrm{d}z = 2\pi i$$

More interesting are the curves C_3 and C_4. Cauchy–Goursat deals with the first, since $f(z)$ is analytic on and between C_3 and C_1/C_2:

$$\oint_{C_3} f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \oint_{C_2} f(z)\,\mathrm{d}z = 2\pi(i-5)$$

C_4 is a little trickier, though it can be thought of as *two* contours, the first encircling z_1 *counter-clockwise* and the second passing *clockwise* around z_0. We conclude that

$$\int_{C_4} f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z - \oint_{C_0} f(z)\,\mathrm{d}z = -2\pi(5+3i)$$

The upshot of the example is that the only properties we really needed to consider about the curves were how many times and in what direction they encircled each of the points z_0, z_1, z_2. Once we know these numbers (essentially the integrals round C_0, C_1, C_2), we can compute the integral round any closed curve by taking linear combinations of the numbers.

To develop this idea further, we first need a little formality.

Isolated Singularities and their Types

Definition 6.2. Suppose $f(z)$ is analytic on an punctured neighbourhood $0 < |z - z_0| < \epsilon$ of a point z_0, but not at z_0 itself. We call z_0 an *isolated singularity* of $f(z)$.

Note that $f(z)$ can have a singularity either because:

- It is undefined; for instance $f(z) = \frac{1}{z}$ has asingularity at zero;
- Because it is defined but not analytic; for instance $f(z) = \sqrt{z}$ at zero.

In Example 6.1, the points z_0, z_1, z_2 are isolated singularities: indeed choosing $\epsilon = 0.3$ creates the punctured neighborhoods inside C_0, C_1, C_2 in the picture.

It is worth spending a little time discussing the three types of isolated singularity, and to consider a couple of ways in which a singularity can fail to be isolated. For the classification, the critical observation comes from Laurent's Theorem: if z_0 is an isolated singularity of $f(z)$, then the function equals its Laurent series centered at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

and C is any simple closed curve in the disk encircling z_0. The structure of this series allows us to classify isolated singular points.

Removable Singularity The Laurent series is a Taylor series: there are no negative powers and so the series may be extended to z_0 analytically.

Pole of order m The highest negative power in the Laurent series is $(z - z_0)^{-m}$. A pole of order 1 is called a *simple pole.*

Essential Singularity There are infinitely many non-zero negative terms in the Laurent series.

Examples 6.3. 1. $f(z) = \sum_{n=0}^{\infty} 3^{-n}(z - 2i)^n$ defined on $0 < |z - 2i| < 3$ has a removable singularity at $z_0 = 2i$. Indeed the function is a geometric series and thus equals

$$f(z) = \frac{1}{1 - \frac{z-2i}{3}} = \frac{3}{3 + 2i - z}$$

on the punctured disk. Certainly this extends analytically to $f(2i) = 1$.

2. Here is a function with a pole of order 2 at zero, of order 13 at $3i$ and a simple pole at 7.

$$f(z) = \frac{1}{z^2} + \frac{1}{(z - 3i)^{13}} + \frac{1}{z - 7}$$

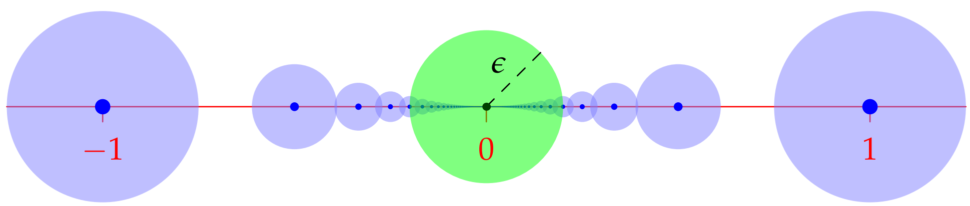
3. $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ has an essential singularity at zero.

It is also important to be on the lookout for the two common ways in which a singularity *fails* to be isolated.

Limit Points A function can have infinitely many singularities in a finite region and thus a limit point within the set. For example, the function

$$f(z) = \left(e^{\frac{2\pi i}{z}} - 1\right)^{-1}$$

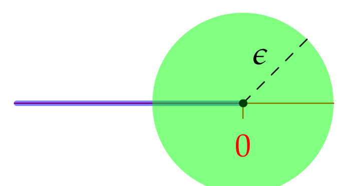
has singularities at $z_0 = 0$ and whenever $z_n = \frac{1}{n}$ for each non-zero integer n. Each of the non-zero singularities is isolated (choose $\epsilon = \frac{1}{(|n|+1)^2}$ for instance), but the limit point singularity at $z - 0$ is non-isolated: for any $\epsilon > 0$, the domain $0 < |z| < \epsilon$ contains non-zero singularities and thus $f(z)$ fails to be analytic on the domain.



Branch Cuts A function is undefined on a branch cut, so these points are singularities. Since branch cuts are curves, these singularities are never isolated. For example, the function

$$f(z) = \sqrt{z}$$

requires a branch cut (e.g. the non-positive real axis) to be analytic. Clearly $z_0 = 0$ is a singularity, but any domain $0 < |z| < \epsilon$ contains other points on the branch cut and so $z_0 = 0$ is non-isolated.



Residues

Since a function always has a Laurent series about any isolated singularity, we make an easy definition.

Definition 6.4. The *residue* of an analytic function $f(z)$ at an isolated singularity z_0 is

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

where C is any circle centered at z_0 such that $f(z)$ is analytic on and inside C, except at z_0.

For the function $f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$ in Example 6.1, we therefore have

$$\operatorname*{Res}_{z=0} f(z) = 3, \quad \operatorname*{Res}_{z=2} f(z) = 5i, \quad \operatorname*{Res}_{z=1+2i} f(z) = 1$$

We also saw in the example that residues can be summed. More generally we have the following key result.

Theorem 6.5 (Cauchy's Residue Theorem). *Suppose $f(z)$ is analytic on and inside a simple closed curve C, except at finitely many singular points*[a] $z_1, \ldots, z_n$. *Then*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

[a]These are *necessarily* isolated since there are only finitely many of them...

To prove this, center a small circle C_k at each z_k such that no other singularities lie on or inside C_k, before applying Cauchy–Goursat on the multiply-connected domain between C and the C_k: the picture in Example 6.1 should help.

Examples 6.6.

1. Let $f(z) = \frac{3(1+iz)}{z(z-3i)}$. If we use the definition to compute its residues at the isolated singular points $z_1 = 0$ and $z_2 = 3i$, then we need a lot of work:

$$\frac{3(1+iz)}{z(z-3i)} = \frac{i-z}{z(1-\frac{iz}{3})} = \left(iz^{-1}-1\right)\sum_{n=0}^{\infty}\left(\frac{i}{3}\right)^n z^n = \frac{i}{z} + \text{power series}$$
$$\frac{3(1+iz)}{z(z-3i)} = \frac{z-3i+2i}{(1+\frac{z-3i}{3i})(z-3i)} = \left(\frac{2i}{z-3i}+1\right)\sum_{n=0}^{\infty}\left(\frac{3i-z}{3i}\right)^n = \frac{2i}{z-3i} + \text{power series}$$

   Clearly both singularities are simple poles. If C is the circle of radius 4 centered at the origin, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=3i} f(z)\right) = 2\pi i(i+2i) = -6\pi$$

   We could also do this more simply using partial fractions!

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \frac{i}{z} + \frac{2i}{z-3i}\,\mathrm{d}z = 2\pi i(i+2i) = -6\pi$$

   One can also compute the residues directly via the Cauchy integral theorem.

2. Again let C be the circle centered at 0 with radius 4. Since $e^{1/z} = \sum\limits_{n=0}^{\infty} \frac{1}{n!} z^{-n}$ has $\operatorname{Res}_{z=0} e^{1/z} = 1$, we see that

$$\oint_C 3e^{1/z} + \frac{2}{z-7} + \frac{2i}{z-i}\,\mathrm{d}z = 2\pi i\left(\operatorname*{Res}_{z=0} 3e^{1/z} + \operatorname*{Res}_{z=i}\frac{2i}{z-i}\right) = 2\pi i(3-2i)$$

   Note that while $z = 7$ is an isolated singular point, it lies *outside* the curve C and therefore does not contribute to the residue.

As the first example above shows, unless a function is given to you as a Laurent series, or as a sum of simple terms, the computation of residues is a lot of work. In the next section we'll see another method which works very efficiently for a large class of functions.

Exercises. 6.1.1. Find the residue at $z = 0$ of each function:

(a) $\dfrac{1}{z+3z^2}$ (b) $z\cos\dfrac{1}{z}$ (c) $\dfrac{z-\sin z}{z}$

6.1.2. Evaluate the integral of each function round the positively oriented circle $|z| = 3$ using Cauchy's residue theorem:

(a) $\dfrac{e^{-z}}{z^2}$ (b) $\dfrac{e^{-z}}{(z-1)^2}$ (c) $z^2 e^{1/z}$ (d) $\dfrac{z+1}{z^2-2z}$

6.1.3. Suppose a closed contour C loops twice counter-clockwise around $z = i$ and three times clockwise around $z = 2$. Use residues to compute the integral

$$\int_C \frac{z+3}{(z-2)^2(z-i)}\,\mathrm{d}z$$

6.1.4. Identify the type of singular point of each of the following functions and determine the residue:

(a) $\dfrac{1-\cosh z}{z^3}$ (b) $\dfrac{1-e^{2z}}{z^4}$ (c) $\dfrac{e^{2z}}{(z-1)^2}$

6.1.5. Suppose $f(z)$ is analytic at z_0 and define $g(z) = (z-z_0)^{-1}f(z)$. Prove:

(a) If $f(z_0) \neq 0$, then z_0 is a simple pole of $g(z)$ with $\operatorname{Res}_{z=z_0} g(z) = f(z_0)$;

(b) If $f(z_0) = 0$, then z_0 is a removable singularity of $g(z)$.

6.1.6. Let $P(z)$ and $Q(z)$ be polynomials whose degrees satisfy $2 + \deg P \leq \deg Q$ and assume C is a simple closed contour such that all zeros of $Q(z)$ lie interior to C.

(a) Prove that $\displaystyle\oint_C \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0$

(*Hint: Try the substitution* $w = \frac{1}{z}\ldots$)

(b) What can you conclude if $\deg Q = \deg P + 1$?

6.2 Poles & Zeros

If we have a pole of known order, we can be a little more specific in how we compute residues. Recall that $f(z)$ has a pole of order m at z_0 if and only if its Laurent series has the form

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n = (z-z_0)^{-m} \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n \text{where } a_{-m} \neq 0$$

The following is then immediate:

Lemma 6.7. *$f(z)$ has a pole of order m at z_0 if and only if $f(z) = (z-z_0)^{-m}\phi(z)$ where $\phi(z)$ is* analytic *at z_0 and $\phi(z_0) \neq 0$.*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\phi^{(m-1)}(z_0)$$

This specializes to $\operatorname{Res}_{z=z_0} f(z) = \phi(z_0)$ *for a simple pole.*

The residue formula comes from applying Cauchy's integral formula to a small circle round z_0, and observing that

$$a_{-1} = \frac{1}{2\pi i}\oint_C \frac{\phi(z)}{(z-z_0)^m}\,\mathrm{d}z$$

While the lemma can make computing residues much easier, it is only effective when you know you have a pole of a given order! In particular, this makes factorized rational functions particularly easy, as long as the order of the pole is not very large.

Examples 6.8. 1. The rational function $f(z) = \frac{1-2iz}{(z-1)(z-2i)^3}$ has two poles:

- A simple pole at $z_0 = 1$. We can write $f(z) = (z-1)^{-1}\phi_0(z)$ where $\phi_0(z) = \frac{1-2iz}{(z-2i)^3}$, which is plainly analytic at $z_0 = 1$. It follows that

$$\operatorname*{Res}_{z=1} f(z) = \phi_0(1) = \frac{1-2i}{(1-2i)^3} = \frac{1}{(1-2i)^2} = \frac{4i-3}{25}$$

- A pole of order three at $z_1 = 2i$. This time $f(z) = \frac{\phi_1(z)}{(z-2i)^3}$ where $\phi_1(z) = \frac{1-2iz}{z-1} = -2i + \frac{1-2i}{z-1}$ is analytic at $2i$, and

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{(3-1)!}\phi_1''(2i) = \left.\frac{1-2i}{(z-1)^3}\right|_{z=2i} = \frac{-1}{(2i-1)^2} = \frac{3-4i}{25}$$

2. We find the residue of $f(z) = \frac{e^z}{z^3-z^2-z+1}$ at $z_0 = 1$. This is a little trickier, since the function isn't quite in the right form. However a quick factorization tells us that that

$$f(z) = \frac{e^z}{(z-1)^2(z+1)} = \frac{\phi(z)}{(z-1)^2} \quad \text{with} \quad \phi(z) = \frac{e^z}{z+1}$$

so that $f(z)$ has a pole of order two at $z_0 = 1$. we therefore have

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\phi'(1) = \left.\frac{(z+1)e^z - e^z}{(z+1)^2}\right|_{z=1} = \frac{1}{4}e$$

3. Be careful not to simply stare at the denominator! For instance,

$$f(z) = \frac{6\sin z - 6z + z^3}{z^6}$$

might appear to have a pole of order *six* at $z_0 = 0$. However, if we write $f(z) = z^{-6}\phi(z)$, then $\phi(0) = 0$, so our criteria are not met. In this case one should first use the Maclaurin series for sine to obtain

$$\begin{aligned} f(z) &= \frac{1}{z^6}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} - z + \frac{1}{6}z^3\right) = z^{-6}\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= z^{-1}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+5)!} z^{2m} = \frac{1}{120z} + \frac{1}{5040} + \cdots \\ \implies & \operatorname*{Res}_{z=0} f(z) = \frac{1}{120} \end{aligned}$$

Zeros of Analytic Functions It turns out that poles and zeros of analytic functions are intimately related. We essentially mirror the observation made at the start of this section.

Definition 6.9. Suppose $f(z)$ is analytic at z_0 and $f(z_0) = 0$. We say that z_0 is a *zero of order m* if $f^{(m)}(z_0)$ is the first *non-zero* derivative.

As with poles, we refer to a *simple zero* when $m = 1$.

Example 6.10. It should be obvious, though you can check it directly if you like that the function $f(z) = z^2(z-2i)^3$ has zero of order two at $z_0 = 0$ and a zero of order three at $z_1 = 2i$.

The example motivates the following: in common with poles, we have the analogue of Lemma 6.7.

Lemma 6.11. *$f(z)$ has a zero z_0 of order m if and only if the Taylor series of $f(z)$ about z_0 has the form*

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m \psi(z)$$

where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$.

We omit the proof since it is straightforward.

Example 6.12. $f(z) = \frac{(z^2+1)^2}{z-3} = \frac{(z-i)^2(z+i)^2}{z-3}$ has zeros of order two at the points $\pm i$. It also has a simple pole at $z = 3$. The reciprocal exactly the reverse arrangement: poles of order two at $\pm i$ and a simple zero at 3:

$$\frac{1}{f(z)} = \frac{z-3}{(z-i)^2(z+i)^2}$$

The example is typical in that the zeros of a non-zero analytic function must be *isolated* just as we have for poles, and that taking reciprocals switches zeros and poles.

Theorem 6.13. *If $f(z)$ is analytic and has a zero of order m at z_0, then the function $g(z) = \frac{1}{f(z)}$ has a pole of order m at z_0.*

The result for poles is identical: if z_1 is a pole of $f(z)$ order m, then it is a zero of order m for $g(z)$.

Proof. Write $f(z) = (z-z_0)^m\psi(z)$ where $\psi(z)$ is analytic and $\psi(z_0) \neq 0$. Then $g(z) = (z-z_0)^{-m}\frac{1}{\psi(z)}$. Since $\psi(z_0)$ is non-zero, $\frac{1}{\psi(z)}$ is analytic on some disk centered at z_0. ∎

We'll discuss the notions of isolated and 'non-isolated' zeros shortly. Before this, it's time for a few examples, and the following useful short cut:

Lemma 6.14. *Suppose $p(z), q(z)$ are analytic at z_0 with $p(z_0) \neq 0$ and where $q(z)$ has a simple zero at z_0. Then, writing $q(z) = (z-z_0)\psi(z)$ where $\psi(z_0) = q'(z_0) \neq 0$, we see that*

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \operatorname*{Res}_{z=z_0} (z-z_0)^{-1}\frac{p(z)}{\psi(z)} = \frac{p(z_0)}{q'(z_0)}$$

Examples 6.15.
1. The function in the previous example has a simple pole at $z = 3$. We check that

$$\operatorname*{Res}_{z=3} f(z) = \operatorname*{Res}_{z=3} \frac{(z^2+1)^2}{z-3} = \frac{(3^2+1)^2}{1} = 100$$

2. $f(z) = \frac{\sin z}{z^2+4} = \frac{\sin z}{(z-2i)(z+2i)}$ has two simple poles at $\pm 2i$ with

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Res}_{z=-2i} f(z) = \frac{\sin 2i}{4i} = \frac{1}{8}(e^2 - e^{-2}) = \frac{1}{4}\sinh 2$$

The function also has simple zeros at $z = n\pi$ for every $n \in \mathbb{Z}$: this is clear since

$$f'(n\pi) = \cos n\pi = (-1)^n \neq 0$$

The reciprocal therefore has simple poles at $z = n\pi$, and moreover

$$\operatorname*{Res}_{z=n\pi} \frac{1}{f(z)} = \frac{1}{f'(n\pi)} = (-1)^n$$

3. Be careful to choose a sensible $q(z)$ if you use the Lemma! For instance, observe that

$$f(z) = \frac{z^3+7}{(z+i)^2 \operatorname{Log}(z+2)(e^z-1)}$$

has a simple pole at $z=0$, since

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

Choosing $p(z) = \frac{z^3+7}{(z+i)^2 \operatorname{Log}(z+2)}$ and $q(z) = e^z - 1$ is the way forward, since it the derivative is trivial to compute,

$$\operatorname*{Res}_{z=0} f(z) = \frac{p(0)}{q'(0)} = \frac{-7}{\ln 2}$$

If we'd instead chosen $q(z) = (z+i)^2 \operatorname{Log}(z+2)(e^z-1)$, the differentiation would have been much worse!

Isolated Zeros

We finish by considering the notion of an isolated zero z_0 of $f(z)$. By this we mean that there exists some punctured neighborhood $0 < |z - z_0| < \epsilon$ on which $f(z) \neq 0$. It is barely worth defining this however, for the next result shows that every 'interesting' zero of an analytic function is isolated.

Theorem 6.16. *Let z_0 be a zero of an analytic function $f(z)$. The following are equivalent:*

1. *$f(z)$ is not identically zero on any neighborhood $|z - z_0| < \epsilon$ of z_0.*
2. *z_0 is a zero of some positive order m.*
3. *z_0 is an isolated zero.*

The distinction between condition 1 and 'isolated zero' is important: condition 1 is a weaker assumption and the equivalence is *false* for non-analytic functions. For example, $f(z) = z + \overline{z} = 2x$ satisfies condition 1 around $z_0 = 0$, but the zero is not isolated. There is something to prove here!

Proof. $(1 \Rightarrow 2)$ The Taylor series of $f(z)$ cannot be zero, else $f(z)$ would be zero on the disk. There must therefore be some minimum $m \in \mathbb{N}$ such that $f^{(m)}(z_0) \neq 0$: thus z_0 is a zero of order m.

$(2 \Rightarrow 3)$ $f(z) = (z - z_0)^m \psi(z)$ where $\psi(z)$ is analytic and $\psi(z_0) \neq 0$. Since $\psi(z)$ is continuous, it must be non-zero on some disk $|z - z_0| < \epsilon$, whence so also is $f(z)$. We conclude that z_0 is an isolated zero.

$(3 \Rightarrow 1)$ This is trivial. ∎

We finish with the famous fact about analytic functions with *non-isolated* zeros (they're boring...).

Corollary 6.17. *If $f(z)$ is analytic on a connected open domain D containing z_0, and $f(z) = 0$ at each point of some contour C containing z_0, then $f(z) \equiv 0$ on D.*

Proof. This is just the negation of the situation in the Theorem: plainly z_0 is not isolated and so $f(z) \equiv 0$ on some disk centered on z_0. The usual patching argument extends this to D. ■

Exercises. 6.2.1. Show that any singular point of the given function is a pole. Determine the order of each pole and its residue.

(a) $f(z) = \dfrac{z+1}{z^2+9}$ (b) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$

6.2.2. Show that:

(a) $\operatorname{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ when $|z| > 0$ and $\arg z \in (0, 2\pi)$

(b) $\operatorname{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$

(c) $\operatorname{Res}_{z=z_n} z \sec z = (-1)^{n+1} z_n$, where $z_n = \frac{\pi}{2} + n\pi$ and $n \in \mathbb{Z}$

6.2.3. Find the value of the integral

$$\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\,\mathrm{d}z$$

when C is each of the circles (a) $|z-2| = 2$, (b) $|z| = 4$.

6.2.4. Let C denote the circle $|z| = 2$. Evaluate $\oint_C \tan z\,\mathrm{d}z$.

6.2.5. Suppose $f(z)$ is an analytic function at z_0 such that $g(z) = \frac{1}{f(z)}$ has a pole of order m at z_0. Prove that $f(z)$ has a zero of order m at z_0.

6.2.6. Show that

$$\oint_C \frac{\mathrm{d}z}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}$$

where C is the rectangle whose sides are the lines $x = \pm 2, y = 0$ and $y = 1$.
(*Hint: show that the integrand has simple poles at $\pm w$ and $\pm\overline{w}$, where $w = \sqrt{1+\sqrt{3}i}$. Only two of these lie inside the curve: now use Lemma 6.14...*)

7 Applications of Residues

7.1 Improper Integrals

A natural application of residues involves the easy evaluation of certain *real* improper integrals of the form $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$. We first need to properly define what we mean by this:

Definition 7.1. The *Cauchy principal value* of the improper integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is the limit

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

provided it exists.

This is merely one interpretation of the improper integral, and it can be a misleading one. In single-variable calculus, the definition is usually different, requiring *two* limits:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x := \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,\mathrm{d}x + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,\mathrm{d}x$$

both of which must exist in order for the integral to converge. It should be clear that if $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ converges, then it equals its Cauchy principal value. Unfortunately there exists many functions for which the Cauchy principal value exists and yet the full improper integral diverges.

Example 7.2. If $f(x)$ is *any* odd function ($f(-x) = -f(x)$) then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x = \lim_{R\to\infty} 0 = 0$$

If either of the 1-sided improper integrals diverges, then the the full integral also diverges. For instance

$$\text{P.V.} \int_{-\infty}^{\infty} x^3\,\mathrm{d}x = 0 \quad \text{but} \int_{0}^{\infty} x^3\,\mathrm{d}x \text{ diverges} \implies \int_{-\infty}^{\infty} x^3\,\mathrm{d}x \text{ diverges}$$

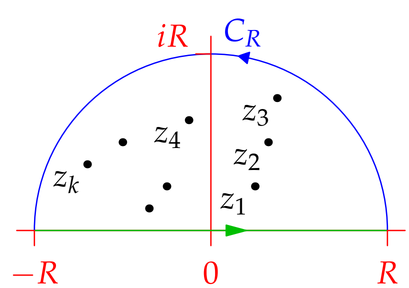
Complex analysis offers a neat trick for computing Cauchy principal values:

1. Suppose $f(x)$ is the restriction to the real line of a *complex rational function* $f(z) = \frac{p(z)}{q(z)}$ with poles $z_1, \ldots, z_n$ in the upper half-plane $\operatorname{Im} z > 0$;

2. Choose $R > 0$ so that $R > |z_k|$ for each k and let C_R be the semi-circle counter-clockwise from $z = R$ to $-R$, then

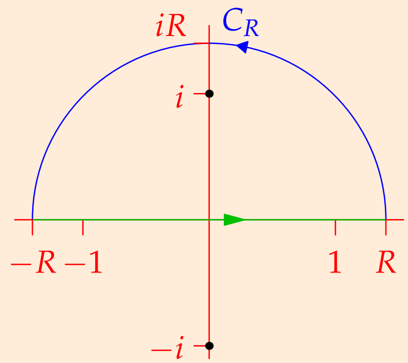
$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

3. If $\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0$, then $\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$



Examples 7.3. 1. $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$ has simple poles at $\pm i$. Provided $|z| = R > 1$, we see that

$$\left|z^2+1\right| \geq \left||z|^2 - 1\right| = R^2 - 1 \implies \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$$



It follows that

$$\left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0$$

whence

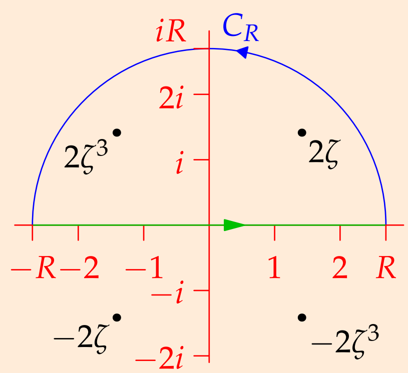
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{2\pi i}{2i} = \pi$$

This fits with the value obtained using the standard method from single-variable calculus:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = -\lim_{R_1\to\infty} \tan^{-1} R_1 + \lim_{R_2\to\infty} \tan^{-1} R_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

2. $f(z) = \frac{4(z^2-1)}{z^4+16}$ has simple poles at $\pm 2\zeta, \pm 2\zeta^3$ where

$$\zeta = \sqrt{i} = e^{\pi i/4} = \frac{1+i}{\sqrt{2}}$$



When $|z| = R > 2$, we see that

$$\left|z^4+16\right| \geq \left||z|^4 - 16\right| = R^4 - 16$$
$$\implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{4\pi R(R^2+1)}{R^4-16} \xrightarrow[R\to\infty]{} 0$$

To compute the residues, let $p(z) = 16(z^2-1)$ and $q(z) = z^4 + 16$, so that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0^2 - 1}{z_0^3}$$

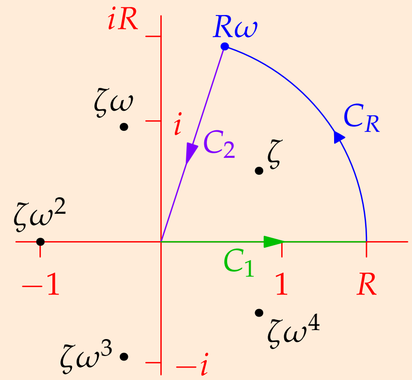
from which

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{4(x^2-1)}{x^4+16}\,\mathrm{d}x &= 2\pi i \left(\operatorname*{Res}_{z=2\zeta} f(z) + \operatorname*{Res}_{z=2\zeta^3} f(z)\right) \\
&= 2\pi i \left(\frac{4\zeta^2-1}{8\zeta^3} + \frac{4\zeta^6-1}{8\zeta^9}\right) = \frac{2\pi i}{\zeta}\left(\frac{4}{8} - \frac{1}{8i} + \frac{4}{8i} - \frac{1}{8}\right) \\
&= \frac{2\pi i\sqrt{2}}{1+i} \cdot \frac{3}{8}(1-i) = \frac{3\pi}{2\sqrt{2}}
\end{aligned}$$

Variations on this construction are possible, for instance by taking only part of a semi-circular arc.

Example 7.4. The function $f(z)=\frac{1}{z^5+1}$ has five simple poles: the fifth-roots of -1,

$$(-1)^{\frac{1}{5}}=\left\{\zeta,\zeta\omega,\zeta\omega^2,\zeta\omega^3,\zeta\omega^4:\zeta=e^{\frac{\pi i}{5}},\omega=\zeta^2=e^{\frac{2\pi i}{5}}\right\}$$



Since the pole $\zeta\omega^2=-1$ lies on the negative real axis, the integral $\int_{-\infty}^{\infty}f(x)\,\mathrm{d}x$ diverges. Instead we can consider the arcs in the picture when $R>1$. Parametrize arc C_2 via $z(t)=t\omega$ where $t:R\to 0$ to see that

$$\int_{C_2}\frac{1}{z^5+1}\,\mathrm{d}z=\int_R^0\frac{\omega}{t^5+1}\,\mathrm{d}t=-\omega\int_0^R\frac{1}{t^5+1}\,\mathrm{d}t$$
$$=-\omega\int_{C_1}\frac{1}{z^5+1}\,\mathrm{d}z$$

$$\implies (1-\omega)\int_0^R\frac{1}{x^5+1}\,\mathrm{d}x+\int_{C_R}\frac{1}{z^5+1}\,\mathrm{d}z=2\pi i\operatorname*{Res}_{z=\zeta}\frac{1}{z^5+1}=\frac{2\pi i}{5\zeta^4}=\frac{2\pi i}{5\omega^2}$$

When $|z|=R>1$, we see that $\left|z^5+1\right|\geq R^5-1 \implies \left|\int_{C_R}\frac{1}{z^5+1}\,\mathrm{d}z\right|\leq\frac{2\pi R}{5(R^5-1)}\xrightarrow[R\to\infty]{}0$. We conclude

$$\int_0^\infty\frac{1}{x^5+1}\,\mathrm{d}x=\frac{2\pi i}{5(\omega^2-\omega^3)}=\frac{2\pi i}{5\zeta\omega^2(\zeta^{-1}-\zeta)}=\frac{2\pi i}{5(2i\sin\frac{\pi}{5})}=\frac{\pi}{5}\csc\frac{\pi}{5}$$

Exercises. 7.1.1. Use residues to establish the value of the improper integral:

(a) $\displaystyle\int_0^\infty\frac{\mathrm{d}x}{(x^2+1)^2}=\frac{\pi}{4}$ (b) $\displaystyle\int_0^\infty\frac{x^2\,\mathrm{d}x}{x^6+1}=\frac{\pi}{6}$

(c) $\displaystyle\int_0^\infty\frac{x^2\,\mathrm{d}x}{(x^2+1)(x^2+4)}=\frac{\pi}{6}$ (d) $\displaystyle\int_0^\infty\frac{x^2\,\mathrm{d}x}{(x^2+9)(x^2+4)^2}=\frac{\pi}{200}$

7.1.2. Find the Cauchy principal value of the integrals:

(a) $\displaystyle\int_{-\infty}^\infty\frac{\mathrm{d}x}{x^2+2x+2}$ (b) $\displaystyle\int_{-\infty}^\infty\frac{x\mathrm{d}x}{(x^2+1)(x^2+2x+2)}$

7.1.3. Let m,n be integers where $0\leq m\leq n-2$. By mimicking Example 7.4, prove that

$$\int_0^\infty\frac{x^m}{x^n+1}\,\mathrm{d}x=\frac{\pi}{n}\csc\frac{(m+1)\pi}{n}$$

7.1.4. Prove the following:

(a) If the improper integral $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ converges, then it equals its Cauchy principal value.

(b) If $f(x)=\frac{p(x)}{q(x)}$ is a rational function where $q(x)$ has no zeros and where $2+\deg p\leq\deg q$, prove that $\int_0^\infty f(x)\,\mathrm{d}x$ converges.
(*Hint: assume p,q are monic, show that $f(x)>0$ for all large x, and recall the comparison test for improper integrals...*)

7.2 Further Improper Integrals and Jordan's Lemma - sec 87,88,89

In this section we consider several modifications of the method considered previously.

It is often useful, particularly when computing Fourier transforms,[6] to evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\cos ax\,\mathrm{d}x + i\int_{-\infty}^{\infty} f(x)\sin ax\,\mathrm{d}x$$

where $a > 0$ is a real constant and $f : \mathbb{R} \to \mathbb{C}$ is a given function. If $f(x)$ is real-valued, then the above breaks the integral into real and imaginary parts. Given reasonable conditions on $f(x)$, the same method as in the previous section can typically be employed for these.

Example 7.5. The function $f(z) = \frac{e^{3iz}}{z^2+4}$ is analytic on the upper half-plane except at the simple pole $z = 2i$. With $R > 2$ and C_R the usual semi-circle, we see that

$$\left|e^{3iz}\right| = e^{-3y} \le 1 \implies \left|\int_{C_R} f(z)e^{3iz}\,\mathrm{d}z\right| \le \frac{\pi R}{R^2-4} \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{3ix}}{x^2+4}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{3iz}}{z^2+4} = 2\pi i\frac{e^{-6}}{4i} = \frac{1}{2}\pi e^{-6}$$

Since this is real, we see that it is in fact just the integral $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,\mathrm{d}x$. We don't need the Cauchy principal value of the integral here since the full improper integral converges. The corresponding integral with sine is trivially zero since $\frac{\sin x}{x^2+1}$ is an odd function.

It should be clear that the method will work for any integral $\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x$ provided $\frac{1}{f(x)}$ is a polynomial of degree ≥ 2 with no real zeros. More generally, we state without proof the following existence result, where C_R is the usual semi-circle in the upper half-plane.

Theorem 7.6 (Jordan's Lemma). *Let $R_0 > 0$ be given and suppose $f(z)$ is analytic at all points exterior to C_{R_0} in the upper half-plane. Suppose also that*

$$\forall R > R_0,\ \exists M_R \text{ such that } |f(z)| \le M_R \text{ and } \lim_{R\to\infty} M_R = 0$$

Then $\lim\limits_{R\to\infty} \int_{C_R} f(z)e^{iaz}\,\mathrm{d}z = 0$

In particular, if $f(z)$ is also analytic on the real axis with finitely many poles $z_1, \ldots, z_k$ in the upper half-plane, then

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_k} f(z)e^{iaz}$$

[6] The Fourier transform of $f(x)$ is the function $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}\,\mathrm{d}x$.

Example 7.7. Easy examples relying on Jordan's Lemma include when $f(x)$ goes to zero as $\frac{1}{x}$ as $x \to \pm\infty$. For instance, if $f(x) = \frac{x+1}{x^2+9}$, and if $R > 3$, then

$$|f(z)| = \frac{|z+2|}{|z^2+9|} \le \frac{R+2}{R^2-9} = M_R \xrightarrow[R\to\infty]{} 0$$

We'd conclude that, say

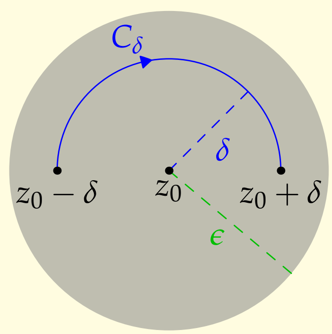
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{(x+2)e^{iax}}{x^2+9}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=3i} \frac{(z+2)e^{iaz}}{z^2+9} = \frac{2\pi i(2+3i)e^{-3a}}{6i} = \frac{\pi(2+3i)}{3}e^{-3a}$$

By considering even and odd functions, etc., we can rewrite this as

$$\int_0^\infty \frac{\cos ax}{x^2+9}\,\mathrm{d}x = \frac{\pi}{6}e^{-3a} \qquad \int_0^\infty \frac{x\sin ax}{x^2+9}\,\mathrm{d}x = \frac{\pi}{2}e^{-3a}$$

Indented paths Another modification allows us to deal with the situation where an analytic function $f(z)$ has a simple pole at z_0 on the real axis. First a preparatory result:

Lemma 7.8. *Let D be the disk $|z - z_0| \le \epsilon$, let $\delta < \epsilon$ and let C_δ be the clockwise half-circle drawn.*



1. *If $\phi(z)$ is analytic on D, then $\lim\limits_{\delta\to 0} \int_{C_\delta} \phi(z)\,\mathrm{d}z = 0$.*
2. *If $f(z)$ is analytic on $D \setminus \{z_0\}$ and has a simple pole at z_0, then*

$$\lim_{\delta\to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -\pi i \operatorname*{Res}_{z=z_0} f(z)$$

More generally, if C_δ is a circular arc θ radians clockwise round z_0, then $\lim\limits_{\delta\to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -i\theta \operatorname*{Res}\limits_{z=z_0} f(z)$

Proof. 1. ϕ is continuous on D and so bounded by some M. Now simply bound the integral

$$\left| \int_{C_\delta} \phi(z)\,\mathrm{d}z \right| \le M\pi\delta$$

2. The Laurent series expansion of $f(z)$ on $D \setminus \{z_0\}$ is

$$f(z) = \frac{a_{-1}}{z - z_0} + \phi(z)$$

where $a_{-1} = \operatorname*{Res}\limits_{z=z_0} f(z)$ and $\phi(z)$ is analytic on D. Now evaluate

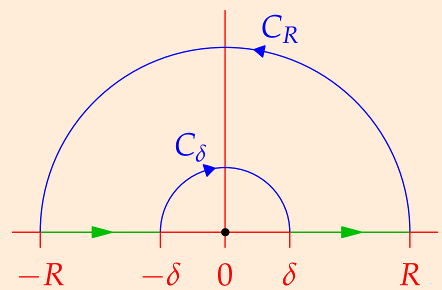
$$\int_{C_\delta} \frac{a_{-1}}{z-z_0}\,\mathrm{d}z = a_{-1}\int_\pi^0 \frac{1}{\delta e^{i\theta}} i\delta e^{i\theta}\,\mathrm{d}\theta = -ia_{-1}\int_0^\pi \mathrm{d}\theta = -\pi i a_{-1}$$

■

Example 7.9. Consider $f(z) = \frac{e^{iz}}{z}$. If $0 < \delta < R$, then

$$\left(\int_{-R}^{-\delta} + \int_{\delta}^{R}\right) f(x)\,\mathrm{d}x = \left(\int_{-R}^{-\delta} + \int_{\delta}^{R}\right) \frac{\cos x + i\sin x}{x}\,\mathrm{d}x = 2i\int_{\delta}^{R} \frac{\sin x}{x}\,\mathrm{d}x$$

by even/oddness. Moreover, by Lemma 7.8,



$$\lim_{\delta\to 0}\int_{C_\delta} \frac{e^{iz}}{z}\,\mathrm{d}z = -i\pi \operatorname*{Res}_{z=0} f(z) = -i\pi$$

and, since $|f(z)| = \frac{e^{-y}}{R} \leq \frac{1}{R}$ on C_R, Jordan's lemma tells us that

$$0 = 2i\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - i\pi \implies \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

The example relied on the evenness of $\frac{\sin x}{x}$ and on the fact that the region of the half-plane between C_R and C_δ contains no poles of $f(z)$. We essentially evaluated $\int_0^R \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2}\int_{-R}^{R} \frac{\sin x}{x}\,\mathrm{d}x$ using an *indented path* mostly lying on the x-axis, but just dodging round the simple pole at zero. Many more versions of this trick are possible!

Exercises. 7.2.1. Suppose $f(x)$ is an even function on the real line and that P.V. $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists. Prove that the full improper integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ also exists and has the same value.

7.2.2. Prove the integration formulae:

(a) $$\int_{-\infty}^{\infty} \frac{\cos x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \text{ if } a > b > 0$$

(b) $$\int_0^\infty \frac{\cos ax\,\mathrm{d}x}{(x^2+b^2)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab} \text{ if } a, b > 0$$

7.2.3. Evaluate the integrals:

(a) $$\int_{-\infty}^{\infty} \frac{x\sin x\,\mathrm{d}x}{(x^2+1)(x^2+4)}$$ (b) $$\int_0^\infty \frac{x^3\sin x\,\mathrm{d}x}{(x^2+1)(x^2+9)}$$

7.2.4. If a is any real number and $b > 0$, find the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\cos x\,\mathrm{d}x}{(x+a)^2+b^2}$

7.2.5. Use the function $f(z) = z^{-2}(e^{iaz} - e^{ibz})$ and an indented contour around $z_0 = 0$ to prove that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} = \frac{\pi}{2}(b-a) \quad a, b \geq 0$$

7.2.6. By integrating the function $f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp(-\frac{1}{2}\log z)}{z^2+1}$ where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ along an indented contour, prove that

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

7.2.7. What happens to part 2 of Lemma 7.8 if $f(z)$ is analytic on $D \setminus \{z_0\}$ but has a pole of order $m \geq 2$ at z_0.