Math 147 — Complex Analysis

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1 Complex Numbers

1.1 Definition and Basic Algebraic Properties

In the 1500's, Italian mathematician Rafael Bombelli posited a solution to the seemingly absurd equation $x^2 = -1$. By supposing that it behaved according to the 'usual' rules of algebra, Bombelli and others were able to describe the solutions to any quadratic equation. To some extent, this was math for its own sake; Bombelli always considered his solutions to be entirely 'fictitious.'

For a modern definition, we start with the Cartesian plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

Definition 1.1. Given real numbers x, y, the *complex number* $z = x + iy$ is the point $(x, y) \in \mathbb{R}^2$. Its *real* and *imaginary parts* are the co-ordinates

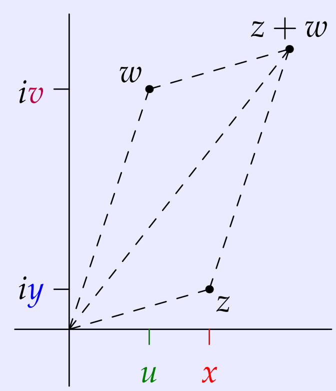
$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y$$

The *complex numbers* $\mathbb{C}$ comprise the *real* vector space $\mathbb{R}^2$ with the extra operation of *complex multiplication*: If $z = x + iy$ and $w = u + iv$, define

(Vector) Addition: $z + w := (x + u) + i(y + v)$

Complex multiplication: $zw := (xu - yv) + i(xv + yu)$

When drawn with axes, the complex plane is known as the *Argand diagram* and we refer, respectively, to the *real* and *imaginary axes*.



Since $\mathbb{C} = \mathbb{R}^2$ is a real vector space under addition, we have several immediate properties:

Lemma 1.2 (Basic properties of complex addition).

Associativity: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

Commutativity: $z + w = w + z$ *(this is the parallelogram law as illustrated in the picture)*

(Real) scalar multiplication: $\forall \lambda \in \mathbb{R}$, $\lambda(x + iy) = \lambda x + i\lambda y$

Additive inverse: $-z = -(x + iy) = (-x) + i(-y) = -x - iy$

Example 1.3. If $z = 3 + 4i$ and $w = 2 - 7i$, then

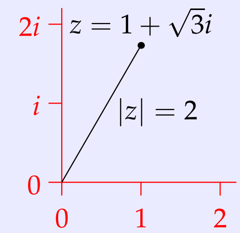
$$z - w = z + (-w) = (3 - 2) + (4 - (-7))i = 1 + 11i$$

The natural distance measure from $\mathbb{R}^2$ transfers to $\mathbb{C}$ (the complex numbers are a real metric space).

Definition 1.4. The *modulus* of a complex number $z = x + iy$ is the Euclidean distance of the point (x, y) from the origin:

$$|z| := \sqrt{x^2 + y^2}$$

In the picture, $z = 1 + \sqrt{3}i$ has modulus $|z| = \sqrt{1+3} = 2$.



Some natural inequalities following straight from the picture in Definition 1.1.

Lemma 1.5 (Triangle inequalities). *For all $z, w \in \mathbb{C}$,*

$$|z + w| \le |z| + |w| \quad \text{and} \quad |z + w| \ge \big||z| - |w|\big|$$

In the second inequality, we take the *absolute value* of the difference of the moduli. Unlike in $\mathbb{R}$, these inequalities follow from an honest triangle! We can easily extend the first by induction,

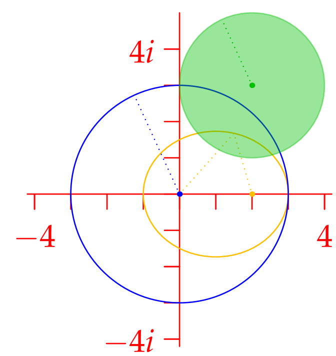
$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + \cdots + |z_n|$$

The modulus may be used to describe various curves and regions in the plane.

Examples 1.6.

1. $|z| = 3$ describes the circle radius 3 centered at the origin. In Cartesian co-ordinates, this becomes $x^2 + y^2 = 9$.
2. $|z - 2 - 3i| \le 2$ describes the *disk* of radius 2 centered at $2 + 3i$.
3. $|z| + |z - 2| = 4$ describes an *ellipse* with foci 0 and 2. This is more familiar after multiplying out (try it!):

$$\sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 4 \implies (x-2)^2 + y^2 = \cdots$$
$$\implies \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$$



Complex multiplication, division and the complex conjugate

Multiplication is the source of all the distinct structure of the complex numbers. For starters, we instantly see that i is a solution to Bombelli's absurd equation:

$$\boxed{i^2 = (0 + 1i)(0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1}$$

The upshot is that we can treat complex addition, subtraction and multiplication as if we are working with *linear polynomials*[1] in the abstract variable i; simply replace i^2 with -1 when needed.

[1] This is precisely the definition you'll see if you take a course in Rings & Fields, where $\mathbb{C}$ is the *factor ring* of real polynomials modulo the *ideal* $\langle x^2 + 1 \rangle$.

Example 1.7. If $z = 3 + 4i$ and $w = 2 - 7i$,

$$zw = (3 + 4i)(2 - 7i) = 3 \cdot 2 + 4i \cdot 2 - 3 \cdot 7i - 4i \cdot 7i = 6 + 8i - 21i - 28i^2$$
$$= 6 + 8i - 21i + 28 = 34 - 13i$$

The basic algebraic properties of complex multiplication are straightforward, if tedious, to verify:

Lemma 1.8 (Basic properties of multiplication). *For any complex numbers z_1, z_2, z_3,*

Associativity: $z_1(z_2z_3) = (z_1z_2)z_3$

Commutativity: $z_1z_2 = z_2z_1$

Distributivity: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

To develop division, it is is helpful to introduce a new concept.

Definition 1.9. The *(complex) conjugate* of $z = x + iy$ is $\overline{z} := x - iy$ (read z-bar). Geometrically, $\overline{z}$ is obtained by reflection in the real axis.

It is immediate that $z\overline{z} = x^2 + y^2 = |z|^2$, which helps us conclude:

Lemma 1.10. *Every non-zero complex number $z = x + iy$ has a unique multiplicative inverse*

$$z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

Proof. That $zz^{-1} = 1$ is trivial. For uniqueness, suppose we also have $zw = 1$ and use associativity and commutativity to conclude that

$$w = (z^{-1}z)w = z^{-1}(zw) = z^{-1}$$

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Division is simply multiplication by an inverse.

Example 1.11. Given $z = 3 + 4i$ and $w = 2 - 7i$, we compute

$$\frac{w}{z} = \frac{2 - 7i}{3 + 4i} = wz^{-1} = \frac{w\overline{z}}{|z|^2} = \frac{(2 - 7i)(3 - 4i)}{|3 + 4i|^2} = \frac{2 \cdot 3 - 2 \cdot 4i - 7i \cdot 3 + 7i \cdot 4i}{3^2 + 4^2}$$
$$= \frac{6 - 8i - 21i + 28i^2}{25} = \frac{-22 - 29i}{25}$$

If you prefer, you can think about this as multiplying the numerator and denominator by the conjugate[2] of the denominator:

$$\frac{2 - 7i}{3 + 4i} = \frac{2 - 7i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \cdots$$

[2]Compare this approach with elementary algebra where, for example, $5 + \sqrt{3}$ is the *conjugate* of $5 - \sqrt{3}$, and we use it to compute $\frac{1}{5-\sqrt{3}} = \frac{5+\sqrt{3}}{(5-\sqrt{3})(5+\sqrt{3})} = \frac{5+\sqrt{3}}{22}$.

Exercises 1.1 1. For any $z \in \mathbb{C}$, prove that $\operatorname{Re}(iz) = -\operatorname{Im} z$ and that $\operatorname{Im}(iz) = \operatorname{Re} z$.

2. (a) Check explicitly that $z = 2 + 3i$ and its conjugate $\overline{z} = 2 - 3i$ solve the quadratic equation $z^2 - 4z + 13 = 0$.

   (b) Suppose $a, b, c \in \mathbb{R}$ where $\omega := 4ac - b^2 > 0$. Check that $z = \frac{-b+i\sqrt{\omega}}{2a}$ and its conjugate $\overline{z}$ both solve the quadratic equation $az^2 + bz + c = 0$.
   (*Since $i^2 = -1$, we write $\sqrt{-\omega} = i\sqrt{\omega}$: the quadratic formula now applies to all real quadratics*)

3. Explicitly prove the commutativity of complex multiplication (Lemma 1.8) using the vector definition of $\mathbb{C}$ (Definition 1.1).

4. Evaluate the following in the form $x + iy$:

   (a) $\dfrac{2-i}{3-5i}$ (b) $(1+i)^4$ (c) $(2+3i)^{-2} - (2-3i)^{-2}$

5. Prove the following: you should write $z = x + iy$ rather than using the vector definition.

   (a) $\overline{\overline{z}} = z$ (b) $(z^{-1})^{-1} = z$ (c) $\overline{zw} = \overline{z} \cdot \overline{w}$

6. (a) For any z, w, prove that $|z + w| \geq \big||z| - |w|\big|$.

   (b) What relationship between z, w corresponds to *equality* here? Draw a picture!

7. Suppose that $|z| \geq 2$ and consider the polynomial $P(z) = z^3 + 3z - 1$.

   (a) Prove that $\left|\frac{3z-1}{z^3}\right| \leq \frac{7}{8}$

   (b) Write $|P(z)| = \left|z^3 + 3z - 1\right| = \left|z^3\right| \left|1 + \frac{3z-1}{z^3}\right|$. Use the extended triangle inequality (Exercise 6(a)) to prove that $|P(z)| \geq 1$.
   (*This shows that all zeros of $P(z)$ lie inside the circle $|z| < 2$.*)

8. By considering the inequality $(|x| - |y|)^2 \geq 0$, prove that

$$\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$$

9. Prove that the hyperbola $x^2 - y^2 = 1$ can be written in the form $z^2 + \overline{z}^2 = 2$.

10. Draw a picture of the ellipse satisfying the equation $|z| + |z - 4i| = 6$. Find the equation of the curve in Cartesian coordinates: $\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1$ where (c, d) is the center of the ellipse and a, b are the semi-axes.
    (*Hint: write $|z - 4i| = 6 - |z|$, square both sides, cancel x^2, y^2 terms and repeat...*)

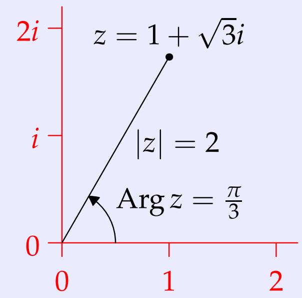
1.2 The Exponential or Polar Form of a Complex Number

Recall Definition 1.4 of the *modulus* of a complex number. We extend this to also consider the *angle*.

Definition 1.12. A complex number can be written in polar co-ordinates:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Plainly $r = |z|$ is the modulus. The angle $\arg z = \theta$ is the *argument* of z.
The argument is multi-valued in that we also have $\arg z = \theta + 2\pi n$ for any integer n. We therefore distinguish the *principal argument* $\operatorname{Arg} z$ by insisting that $-\pi < \operatorname{Arg} z \leq \pi$.



Note that 0 has no argument: it is the only complex number without an argument!

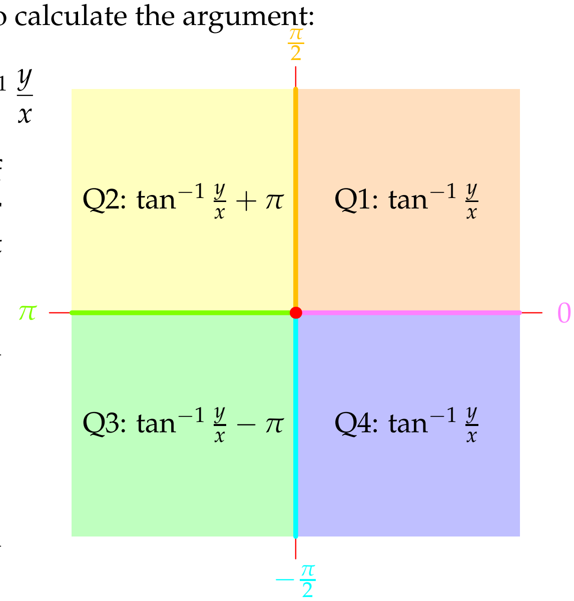
Example 1.13. In the above picture, $z = 1 + \sqrt{3}i$ has principal argument $\operatorname{Arg} z = \frac{\pi}{3}$. You can write the argument either as many different values, or as a set:[3]all the following are legitimate

$$\arg z = \{\tfrac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\}, \quad \text{or} \quad \arg z = \tfrac{\pi}{3}, \quad \text{or} \quad \arg z = \tfrac{7\pi}{3}$$

Provided $x \neq 0$, the following is (almost) all we need to calculate the argument:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x} \overset{???}{\implies} \arg z = \tan^{-1}\frac{y}{x}$$

This isn't quite right. Since arctan has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, if z lies in the second or third quadrants, an addition or subtraction of π might be required to find the correct value of the principal argument $\operatorname{Arg} z$.



Example 1.14. Since $z = -3 - 3i$ lies in the third quadrant, we have

$$\operatorname{Arg} z = \tan^{-1}\frac{-3}{-3} - \pi = -\frac{3\pi}{4}$$

If we preferred the argument to be positive, we could instead choose a non-principal argument $\arg z = \frac{5\pi}{4}$.

The polar form allows us to define a crucial new function.

Definition 1.15. Given $\theta \in \mathbb{R}$, the *exponential* $e^{i\theta}$ is defined by *Euler's formula*

$$e^{i\theta} := \cos\theta + i\sin\theta$$

The *polar form* of a complex number can now be written $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$.
If $w = u + iv$ is complex, then its exponential is defined by $e^w := e^u e^{iv} = e^u(\cos v + i\sin v)$.

[3]This is merely the common mathematical fudge of denoting an equivalence class $\{\frac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\}$ by any of its representatives, e.g. $\frac{\pi}{3}$ and $\frac{7\pi}{3}$.

There are several reasons why Euler's formula provides a sensible definition of $e^{i\theta}$: in particular it fits with two common definitions of the exponential in real analysis:

1. If $k \in \mathbb{R}$, then $e^{k\theta}$ is the solution to the initial value problem $y' = ky$ with $y(0) = 1$. Assuming that differentiation works when $k = i$, Euler's formula satisfies this criterion

$$\frac{\mathrm{d}}{\mathrm{d}\theta}e^{i\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}$$

2. The real and imaginary parts of the Maclaurin series $\exp z = \sum \frac{z^n}{n!}$ evaluated at $z = i\theta$ are, respectively, the Maclaurin series of $\cos\theta$ and $\sin\theta$.

Another reason is that the definition satisfies the usual exponential laws.

Lemma 1.16 (Exponential laws). *Let $z = re^{i\theta}$ and $w = se^{i\psi}$ be written in polar form. Then*

1. $zw = rse^{i(\theta+\psi)}$, *in particular* $|zw| = |z|\,|w|$ *and* $\arg zw = \arg z + \arg w$
2. $\frac{z}{w} = \frac{r}{s}e^{i(\theta-\psi)}$
3. $z^n = r^n e^{in\theta}$, $n \in \mathbb{Z}$

Note that the *principal argument* might not behave so nicely for products; the best we can say is that

$$\boxed{\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w + 2\pi n \text{ for some } n = 0, \pm 1}$$

Proof. Part 1 follows from the multiple-angle formulæ for sine and cosine:

$$\begin{aligned} e^{i(\theta+\psi)} &= \cos(\theta+\psi) + i\sin(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi + i(\sin\theta\cos\psi + \cos\theta\sin\psi) \\ &= (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) = e^{i\theta}e^{i\psi} \end{aligned}$$

Parts 2 and 3 are now straightforward. ∎

Examples 1.17. 1. Given $z = -7 + i$ and $w = 3 + 4i$, we find the modulus and argument of zw in two ways:

(a) First find the polar forms of z, w, then apply the Lemma:

$$\begin{aligned} z = |z|\,e^{i\arg z} = 5\sqrt{2}\exp\left(i(\pi - \tan^{-1}\tfrac{1}{7})\right), \quad w = |w|\,e^{i\arg w} = 5\exp\left(i\tan^{-1}\tfrac{4}{3}\right) \\ \implies |zw| = |z|\,|w| = 25\sqrt{2}, \quad \arg zw = \arg z + \arg w = \pi - \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{4}{3} \end{aligned}$$

(b) First find $zw = (-7+i)(3+4i) = -25 - 25i$ then compute its polar form:

$$zw = 25\sqrt{2}e^{-\frac{3\pi i}{4}} \implies |zw| = 25\sqrt{2}, \qquad \operatorname{Arg} zw = -\frac{3\pi}{4}$$

The first answer is plainly uglier. It is usually better to use the second approach unless the arguments of z, w are exactly computable. In Exercise 7, we check that these values correspond.

2. We compute z^{10} when $z = \sqrt{3} - i$. First observe that $z = 2e^{-\frac{\pi i}{6}}$, from which

$$z^{10} = 2^{10}e^{-\frac{5\pi i}{3}} = 1024e^{\frac{\pi i}{3}} = 512(1 + \sqrt{3}i)$$

3. The identity $(e^{i\theta})^n = e^{in\theta}$ is known as *de Moivre's formula*; it is usually written

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Many trigonometric identities follow from this by taking real or imaginary parts. For instance, when $n = 3$,

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$
$$\implies \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$$

Exercises 1.2 1. Use induction to prove that for any $n \in \mathbb{N}_{\geq 2}$ we have

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)}$$

2. Find the principal argument of $(1+i)^{2022}$.
3. Prove that $\left|e^{i\theta}\right| = 1$ and that $\overline{e^{i\theta}} = e^{-i\theta}$.
4. (a) Show that if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, then $\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w$.
   (b) If z and w both lie in quadrant 2, explain why $\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w - 2\pi$.
5. Prove that non-zero $z, w \in \mathbb{C}$ have the same modulus if and only if $\exists p, q \in \mathbb{C}$ such that $z = pq$ and $w = p\overline{q}$.
6. Use de Moivre's formula to establish the identity

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

7. (a) Let $\alpha = \tan^{-1}\frac{4}{3}$ and $\beta = \tan^{-1}\frac{1}{7}$. Use right-triangles to show that

$$\cos\alpha = \frac{3}{5}, \qquad \sin\alpha = \frac{4}{5}, \qquad \cos\beta = \frac{7}{\sqrt{50}}, \qquad \sin\beta = \frac{1}{\sqrt{50}}$$

   Now use the cosine multiple-angle formula to check that $\alpha - \beta = \frac{\pi}{4}$.
   (*This shows that* $\arg zw = \frac{5\pi}{4}$ *in Example 1.17(a)*)
   (b) Generalize the approach in part (a) to prove the multiple-angle formula for tangent

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

   when α, β are acute angles.
8. The polar form of a complex number is well-suited to describing *circles.* For instance the circle centered at i with radius 3 may be parametrized by $z = i + 3e^{i\theta}$ where $\theta \in (-\pi, \pi]$.
   (a) Describe the circle centered at $z_0 = 3 + 4i$ with radius 2.
   (b) Show that the points $z = re^{i\theta}$ for which $r = 2a\cos\theta$ describe a circle.
   (*Hint: Multiply by r*)

1.3 Roots of Complex Numbers

A naïve approach to taking roots in $\mathbb{C}$ is very messy.

Example 1.18. To find c such that $c^2 = -5 + 12i$, we need to solve an equation:

$$-5 + 12i = c^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \iff \begin{cases} x^2 - y^2 = -5 \\ xy = 6 \end{cases}$$

Substituting $y = 6x^{-1}$ into the first equation yields a quadratic in x^2:

$$x^4 + 5x^2 - 36 = (x^2 - 4)(x^2 + 9)$$

from which we conclude that $x = \pm 2$ and obtain the square roots $\pm c = \pm(2 + 3i)$.

The example is reassuring in that we obtain precisely two square roots. However, attempting to extend the method to cube, or higher, roots is utterly doomed! Instead we use the polar form. Suppose $n \in \mathbb{N}$ and that c, z satisfy $z = c^n$. In polar form

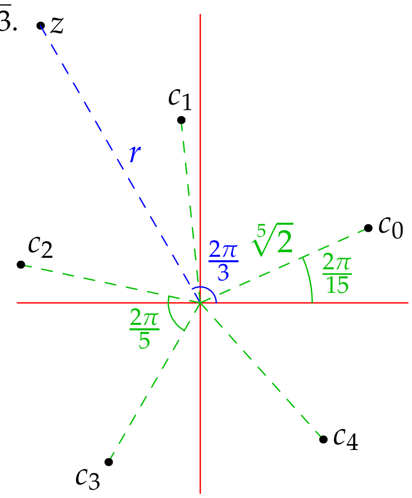
$$z = re^{i\theta}, \quad c = se^{i\psi} \implies re^{i\theta} = s^n e^{in\psi}$$

By equating moduli and arguments, we conclude that

$$r = s^n, \qquad n\psi = \theta + 2\pi k \tag{$*$}$$

where k is some integer. We'll shortly put this together to obtain a proper definition, but we already have enough for a calculation.

Example 1.19. We compute the fifth roots of $z = 2e^{\frac{2\pi i}{3}} = -1 + i\sqrt{3}$.



In the above language,

$$s = \sqrt[5]{2}, \quad \psi = \frac{1}{5}\left(\frac{2\pi}{3} + 2\pi k\right) = \frac{2\pi}{15}(1 + 3k)$$

which results in the fifth roots

$$c_0 = \sqrt[5]{2}e^{\frac{2\pi i}{15}} \qquad c_1 = \sqrt[5]{2}e^{\frac{8\pi i}{15}} \qquad c_2 = \sqrt[5]{2}e^{\frac{14\pi i}{15}}$$

$$c_3 = \sqrt[5]{2}e^{\frac{20\pi i}{15}} = \sqrt[5]{2}e^{-\frac{10\pi i}{15}} \qquad c_4 = \sqrt[5]{2}e^{\frac{26\pi i}{15}} = \sqrt[5]{2}e^{-\frac{4\pi i}{15}}$$

Note how there are precisely *five* fifth roots: once $k \geq n(= 5)$ in $(*)$, the roots start repeating. The last two roots were written both with positive and principal arguments: both have advantages!

Observe also how the fifth roots form the vertices of a regular pentagon, equally spaced around the circle of radius $\sqrt[5]{2}$.

Finally, note how essential the polar form was to this calculation. We could convert back to rectangular form, but since $\cos\frac{2\pi}{15}$ and $\sin\frac{2\pi}{15}$ do not have friendly values, this is of limited utility.

Definition 1.20. Given a non zero complex number $z = re^{i\theta}$ and a positive integer n, the n^{th} *roots* of z are the n complex numbers

$$c_k = \sqrt[n]{r}\exp\frac{(\theta+2k\pi)i}{n} \qquad k = 0, 1, \ldots, n-1$$

where $\sqrt[n]{r}$ is the usual real (positive!) n^{th} root.

There are some conventions to follow. Let $\theta = \operatorname{Arg} z$ be the *principal argument* of z:

(a) The *principal n^{th} root* is $\sqrt[n]{z} := \sqrt[n]{r}e^{\frac{i\theta}{n}}$.

(b) The *set of n^{th} roots* is $z^{\frac{1}{n}} := \{c_0, \ldots, c_{n-1}\}$.

Denote by $\omega_n = e^{\frac{2\pi i}{n}}$ a *primitive n^{th} root of unity*, then the full set of n^{th} *roots of unity* is

$$1^{\frac{1}{n}} = \{\omega_n^k : k = 0, \ldots, n-1\} = \{e^{\frac{2\pi ki}{n}} : k = 0, \ldots, n-1\}$$

The n^{th} roots of z may be written in terms of the principal root and the n^{th} roots of unity

$$z^{\frac{1}{n}} = \sqrt[n]{z}\,1^{\frac{1}{n}} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$$

The geometric effect of multiplying by $\omega_n^k = e^{\frac{2\pi ki}{n}}$ is to rotate counter-clockwise by $\frac{2\pi k}{n}$ radians:

$$\arg \sqrt[n]{z}\omega_n^k = \arg \sqrt[n]{z} + \arg \omega_n^k = \arg \sqrt[n]{z} + \frac{2\pi k}{n}$$

It follows that the n^{th} roots of $z = re^{i\theta}$ form the vertices of a regular n-gon spaced equally round the circle of radius $\sqrt[n]{r}$. Compare this with the previous example.

Examples 1.21. 1. First compare what happens when $z = r = 16$ and $n = 4$.

- $\sqrt[4]{16} = 2$ is the principal sixth root.
- In the real numbers, we have *two* fourth roots: $16^{\frac{1}{4}} = \pm 2$.
- In complex analysis, there are *four* fourth roots: $16^{\frac{1}{4}} = \{2, 2i, -2, -2i\}$ where $i = \omega_4 = e^{\frac{i\pi}{2}}$ is a primitive fourth root of unity.

2. We compute the fourth roots of $z = 8\sqrt{2}(1+i)$.

   First we write in polar form: $z = 16e^{\frac{\pi i}{4}}$. Since $\operatorname{Arg} z = \frac{\pi}{4}$, the principal fourth root is

$$\sqrt[4]{8\sqrt{2}(1+i)} = 2e^{\frac{\pi i}{16}}$$

   To find all fourth roots, simply multiply by the fourth roots of unity $1^{\frac{1}{4}} = \{1, i, -1, -i\}$:

$$\left(8\sqrt{2}(1+i)\right)^{\frac{1}{4}} = \{\pm 2e^{\frac{\pi i}{16}}, \pm 2ie^{\frac{\pi i}{16}}\} = \{2e^{\frac{\pi i}{16}}, 2e^{\frac{9\pi i}{16}}, 2e^{-\frac{15\pi i}{16}}, 2e^{-\frac{7\pi i}{16}}\}$$

   Evaluating these in rectangular form is messy but possible (see Exercise 7). In practice it is better to leave such expressions in polar form.

Exercises 1.3 1. Find the square roots of $-\sqrt{3}+i$ and express them in rectangular co-ordinates. (*Hint: you may find it useful that* $(\sqrt{3}-1)^2 = 4-2\sqrt{3}$)

2. Find the sixth roots of i in polar co-ordinates. Which is the principal root?

3. Use the fact that the cube roots of unity are $1, \omega_3 = \frac{-1+\sqrt{3}i}{2}$ and $\omega_3^2 = \frac{-1-\sqrt{3}i}{2}$ to evaluate the cube roots of -27 in rectangular co-ordinates.

4. We previously found the fourth roots of 16. Use these to find the fourth roots of -16. Hence factorize the equation $z^4+16=0$ as a product of two quadratic equations with real coefficients.

5. If ω is an n^{th} root of unity *other than 1*, prove that $\sum\limits_{k=0}^{n-1} \omega^k = 0$.

   (*Hint: recall geometric series*)

6. (a) Suppose that $a, b, c \in \mathbb{C}$ with $a \neq 0$ and suppose that z satisfies the quadratic equation $az^2+bz+c=0$. Prove the quadratic formula:

   $$z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$$

   Note that $(b^2-4ac)^{1/2}$ is the *set* of square roots of b^2-4ac, so that this provides *two* solutions whenever $b^2-4ac \neq 0$.

   (b) Find the roots of the equation $iz^2+(1+i)z+3=0$ in rectangular form.

7. Use the half-angle formula $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$ to explicitly evaluate $\cos\frac{\pi}{8}$ and then $\cos\frac{\pi}{16}$. Hence find an expression for the rectangular form of $\sqrt[4]{8\sqrt{2}(1+i)} = 2e^{\frac{\pi i}{16}}$ using only square roots.

8. Recall Example 1.18. Verify that the method in Definition 1.20 gives the same value for the principal square root $\sqrt{-5+12i}$.

   (*You'll need some trig identities...*)

2 Holomorphic Functions

In this chapter we discuss functions of a complex variable and what it means for such to be *differentiable.* This turns out to be much more subtle and restrictive than in real analysis.

2.1 Functions of a Complex Variable

A function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ may be defined in the obvious manner, using a *rule.*

Example 2.1. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^3 - z$. This evaluates as expected; e.g.

$$f(2+i) = (2+i)^3 - (2+i) = 2^3 + 3 \cdot 2^2 i + 3 \cdot 2i^2 + i^3 - 2 - i = 10i$$

As in real analysis, it is common to express a function simply using a rule: its *implied domain* is the largest possible set $D \subseteq \mathbb{C}$ on which the rule is defined.

Example 2.2. $f(z) = \frac{1}{z^2+9}$ has implied domain $D = \mathbb{C} \setminus \{\pm 3i\}$.
The function may also be expressed in two other common ways.
Real and imaginary parts: Write $f(z) = u(x,y) + iv(x,y)$ where $u, v : D \to \mathbb{R}$. In this case,

$$f(z) = \frac{1}{(x+iy)^2+9} = \frac{1}{x^2-y^2+9+2ixy} = \frac{x^2-y^2+9}{(x^2-y^2+9)^2+4x^2y^2} + i\frac{-2xy}{(x^2-y^2+9)^2+4x^2y^2}$$

Polar form: Write $z = re^{i\theta}$ to obtain

$$f(z) = \frac{1}{r^2e^{2i\theta}+9}$$

Depending on the function, each of these approaches might have certain advantages or disadvantages. You might also wish to combine the approaches: for instance writing u, v as functions of r, θ.

Examples 2.3. We consider the function $f(z) = z^2$ and some relations in more detail.

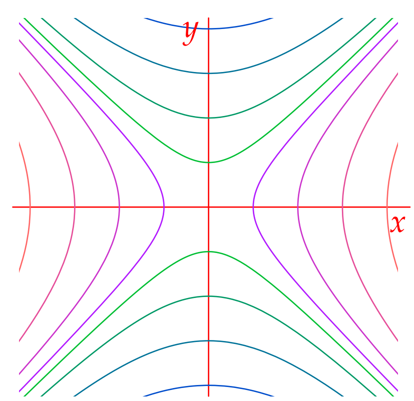
First we write f in the three standard forms of Example 2.2,

$$f(z) = z^2 = x^2 - y^2 + 2ixy = r^2e^{2i\theta}$$

We cannot straightforwardly *graph* $f(z) = z^2$ (don't think about a parabola!). However, we can visualize the real and imaginary parts

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy$$

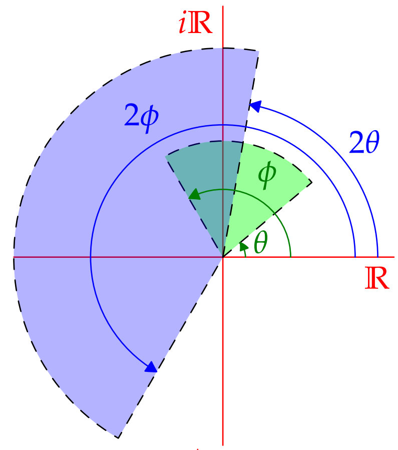
as graphs of functions $\mathbb{R}^2 \to \mathbb{R}$. Both are *saddle surfaces* and can be analyzed using the standard tools of multivariable calculus. For instance, the *level curves* $u =$ constant and $v =$ constant are *hyperbolæ.*



Level curves of $u = x^2 - y^2$

Now consider the polar form and what it means for the argument:

$$f(z) = r^2 e^{2i\theta} \implies \arg(z^2) = 2\arg z$$

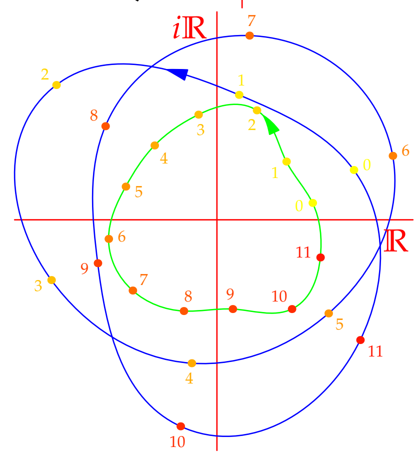


This can be visualized by considering sectors of the plane: the sector between arguments θ and ϕ is *doubled* in angle to the sector between 2θ and 2ϕ.

Away from the origin, $f(\pm z) = z^2$ shows that f is a *two-to-one* function: f maps the sector with $\arg z \in [0, \pi)$ onto the entire plane.

We can also visualize this pathwise. If z traces a path around the origin, z^2 traces a new path *twice* round the origin! The colored dots on the two paths correspond under $z \mapsto z^2$.

Similar behavior occurs with higher powers. For instance, $z \mapsto z^3$ maps a single loop round the origin to a *triple* loop; away from the origin we have a *three-to-one* function. The function $z \mapsto z^n$ is an *n-to-one* map.



By contrast, the principal square root function $\sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ *halves* the principal argument $\theta = \operatorname{Arg} z$, from which $\arg\sqrt{z} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Written instead using fractional exponent notation, we see that

$$z \mapsto z^{1/2} = \{\sqrt{r}e^{\frac{i\theta}{2}}, -\sqrt{r}e^{\frac{i\theta}{2}}\}$$

is a *two-valued* function.

Basic 2D geometry with complex numbers

Complex functions can be used to describe the primary geometric transformations of the plane.

Translation If w is constant, the function $f(z) = z + w$ translates the plane, shifting the origin to w.

Scaling If $R \in \mathbb{R}$ is constant, the function $f(z) = Rz$ scales the complex plane, inflating the distance from the origin.

Rotation Given ϕ, the function $f(z) = e^{i\phi}z$ rotates the complex plane by ϕ radians counter-clockwise around the origin. The easy way to see this is to write the function in polar form itself:

$$f(z) = f(re^{i\theta}) = e^{i\phi}re^{i\theta} = re^{i(\theta+\phi)}$$

Reflection Complex conjugation $f(z) = \overline{z} = x - iy$ reflects the complex plane in the horizontal axis. Combining with rotation, we can produce the reflection in any line through the origin. To reflect in the line through 0 and w with $\operatorname{Arg} w = \phi$;

1. Rotate the plane by $-\phi$: $z \mapsto e^{-i\phi}z$
2. Reflect in the real axis: $z \mapsto \overline{e^{-i\phi}z} = e^{i\phi}\overline{z}$
3. Rotate the plane back by ϕ: the result is the function $f(z) = e^{2i\phi}\overline{z}$

Combining these functions allows one to rotate around any point and reflect across any line.

Examples 2.4. 1. The function $f(z) = e^{\frac{i\pi}{3}}(z-2i) + 2i$ rotates the complex plane by $\frac{\pi}{3}$ radians around the point $2i$.

2. We compute the function that reflects across the line joining $\alpha = 2+i$ and $\beta = 4+3i$.

   Since $\beta - \alpha = 2+2i$ has argument $\phi = \frac{\pi}{4}$, we combine reflection across the line making ϕ through the origin with translation by α (translate by $-\alpha$, reflect, translate back by α):

$$\begin{aligned} f(z) &= e^{2i\phi}(\overline{z-\alpha}) + \alpha = e^{\frac{i\pi}{2}}(\overline{z-2-i}) + 2 + i = i(\overline{z} - 2 + i) + 2 + i \\ &= i\overline{z} + 1 - i \end{aligned}$$

   As a sanity check, you should verify that $f(\alpha) = \alpha$ and $f(\beta) = \beta$: why?

Exercises 2.1 1. For each function, describe its implied domain (page 11).

   (a) $f(z) = \dfrac{1}{4+z^2}$ (b) $f(z) = \dfrac{z-1}{e^z - 1}$ (c) $f(z) = \dfrac{z^2+z+1}{z^4-1}$

2. Write the function in terms of its real and imaginary parts: $f(z) = u(x,y) + iv(x,y)$.

   (a) $f(z) = z^3 - 4z^2 + 2$ (b) $f(z) = \dfrac{z^2}{1-\overline{z}}$ (c) $f(z) = e^{\overline{z}}$

3. Write the function $f(z) = \dfrac{1}{|z|^2}\overline{z}$ in polar form.

4. Find an expression for the function which reflects across the vertical line through the point $\alpha = -1$.

5. For Example 2.42 evaluate the function $g(z) = e^{2i\phi}(\overline{z-\beta}) + \beta$. Why are you not surprised by the result?

6. Let $\phi = \tan^{-1}\frac{3}{4}$. Find the result (in rectangular co-ordinates) of rotating $z = -2+i$ counter-clockwise by ϕ radians around the origin.
   (*Hint: consider a 3:4:5 triangle!*)

7. Prove, using the expressions on page 12, that the composition of two reflections is a rotation, and that the composition of a rotation and a reflection is a reflection.

2.2 Open sets, Limits and Continuity

Before we can differentiate, we need to understand limits and continuity. Much of this section should be treated as a reference, where we review without proof several preliminaries which are identical to (multivariable) real analysis.

Definition 2.5 (Sequences and Limits). Let $(z_n) = (z_1, z_2, \ldots)$ be a *sequence* of complex numbers.

1. (z_n) *converges* to z_0, and we write $\lim_{n\to\infty} z_n = z_0$ or simply $z_n \to z_0$, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |z_n - z_0| < \epsilon$$

2. (z_n) is *Cauchy* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |z_m - z_n| < \epsilon$$

Theorem 2.6. 1. *Cauchy completeness: (z_n) converges if and only if it is Cauchy.*

2. *Bolzano–Weierstraß: If (z_n) is bounded, then it has a convergent subsequence.*

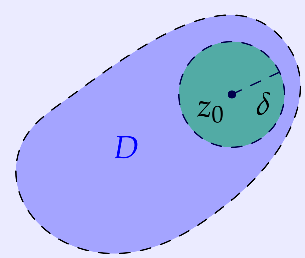
Definition 2.7 (Disks, Sets and Neighborhoods). The *open disk* (or *δ-ball*) centered at $z_0 \in \mathbb{C}$ with radius δ is the set

$$B_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

The *punctured open disk* centered at z_0 with radius δ is the set

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$$

Let $D \subseteq \mathbb{C}$ be a subset.



An open neighborhood of z_0

- D is *open* if every point is *interior*: at every point we can center an open disk $B_\delta(z_0) \subseteq D$:

$$\forall z_0 \in D,\ \exists \delta > 0 \text{ such that } |z - z_0| < \delta \implies z \in D$$

- D is a *neighborhood* of z_0 if it contains some $B_\delta(z_0)$. A neighborhood can, but need not be, open. A *punctured neighborhood* instead contains a punctured disk.
- D is *closed* if its complement $\mathbb{C} \setminus K$ is open.

Theorem 2.8. *K is closed if and only if every Cauchy sequence $(z_n) \subseteq K$ has its limit in K.*

We define continuity of functions in the usual way.

Definition 2.9 (Continuity). Let $f : D \to \mathbb{C}$ and $z_0 \in D$. We say that f is *continuous at z_0* if

$$\forall \text{ sequences } (z_n) \subseteq D \text{ with } z_n \to z_0 \text{ we have } f(z_n) \to f(z_0)$$

f is *continuous* (on D) if it is continuous at all points $z_0 \in D$.

As in real analysis, it is helpful to relate this to limits of functions.

Definition 2.10 (Limits of Functions). Let $f : D \to \mathbb{C}$, where D contains an open punctured neighborhood of z_0. We say that w_0 is the *limit of f as z approaches z_0* and write $\lim_{z \to z_0} f(z) = w_0$, if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

Theorem 2.11. *If z_0 is an* interior *point to D, then $f : D \to \mathbb{C}$ is continuous at z_0 if and only if*

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Otherwise said, for any ϵ-ball $B_\epsilon\big(f(z_0)\big)$, there exists a δ-ball $B_\delta(z_0)$ such that $f\big(B_\delta(z_0)\big) \subseteq B_\epsilon\big(f(z_0)\big)$. We won't spend much time on calculations since these tend to proceed similarly to real analysis.

Example 2.12. We show that $f(z) = z^2$ is continuous by proving that $\lim_{z \to z_0} z^2 = z_0^2$ for all $z_0 \in \mathbb{C}$.

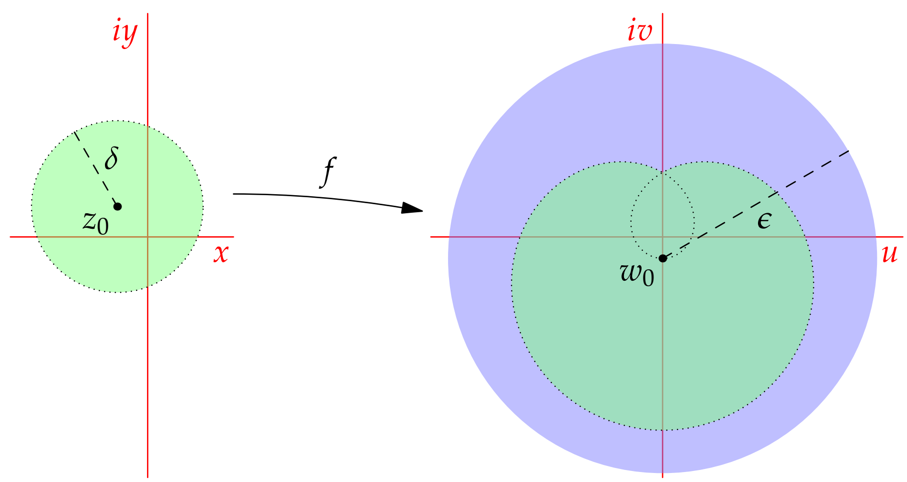
Let $z_0 \in \mathbb{C}$ and $\epsilon > 0$ be given. Define $\delta = \min\{1, \frac{\epsilon}{1+2|z_0|}\}$. By the triangle-inequality,

$$|z - z_0| < \delta \implies |z + z_0| = |z - z_0 + 2z_0| \le |z - z_0| + 2|z_0| < \delta + 2|z_0| \le 1 + 2|z_0|$$

from which

$$|z - z_0| < \delta \implies \left|z^2 - z_0^2\right| = |z - z_0|\,|z + z_0| < \delta(1 + 2|z_0|) \le \epsilon$$

The picture below should help you visualize this. Given the ϵ-ball centered at $w_0 = z_0^2$, we've described how to choose $\delta > 0$ so that the punctured δ-ball centered at z_0 is mapped to a region *inside* the original ϵ-ball.



The picture illustrates the case when

$$z_0 = \frac{1}{2}e^{\frac{3\pi i}{4}} = \frac{1}{2\sqrt{2}}(-1+i), \quad w_0 = -\frac{i}{4}, \quad \epsilon = \frac{5}{2} \quad \text{and} \quad \delta = \min\left\{1, \frac{5/2}{1+2\cdot\frac{1}{2}}\right\} = 1$$

Theorem 2.13 (Basic Limit Results). *Throughout, let $f, g : D \to \mathbb{C}$ be functions and $z_0 = x_0 + iy_0$ be a point satisfying the assumptions of Definition 2.10.*

1. *Limits are unique: If w_0 and $\widetilde{w_0}$ satisfy Definition 2.10, then $\widetilde{w_0} = w_0$.*
2. *If $f(z) = u(x,y) + iv(x,y)$ and $w_0 = u_0 + iv_0$, then*

$$\lim_{z\to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

   In particular $\lim_{z\to z_0} \overline{z} = \overline{z_0}$

3. *Suppose $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$:*
   (a) *For any $a, b \in \mathbb{C}$, $\lim_{z\to z_0} (af(z) + bg(z)) = aw_0 + bw_1$*
   (b) *$\lim_{z\to z_0} (f(z)g(z)) = w_0w_1$*
   (c) *$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$, provided $w_1 \neq 0$ and $g(z) \neq 0$ on a punctured neighbourhood of z_0.*
   (d) *If h is a function such that $\lim_{w\to w_0} h(w) = w_2$, then $\lim_{z\to z_0} h(f(z)) = w_2$*

Most of the above should feel familiar, and all should be intuitive. Parts 2 & 3 have obvious corollaries for continuous functions. For instance:

f is continuous if and only if its real and imaginary parts $u, v : D \to \mathbb{R}$ are also.

Several of the properties in part 3 follow from part 2 by considering real and imaginary parts and the limit laws for functions $\mathbb{R}^2 \to \mathbb{R}$. For instance write $f(z) = u_1 + iv_1$ and $g(z) = u_2 + iv_2$, then

$$\lim_{z\to z_0} (f(z)g(z)) = \lim_{z\to z_0} (u_1u_2 - v_1v_2 + i(u_1v_2 + v_1u_2)) = w_0w_1$$

Examples 2.14. 1. $\lim_{z\to 1+3i} z^2 - i\overline{z} = (1+3i)^2 - i\overline{1+3i} = 1 + 6i - 9 - i(1-3i) = -11 + 5i.$

2. Every polynomial function is continuous on $\mathbb{C}$.
3. Every rational function $f(z) = \frac{p(z)}{q(z)}$ where p, q are polynomials, is continuous on its implied domain $\{z : q(z) \neq 0\}$.
4. Since the exponential, cosine and sine are continuous on $\mathbb{R}$, we see that the exponential function

$$\exp(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

   is continuous on $\mathbb{C}$.

Limits and the Point at Infinity

The treatment of infinity in complex analysis is very different to that in (single-variable) real analysis, where there are *two* infinities ($\pm\infty$), one for each *direction*. The convention in complex analysis is to have only one: that is, the sequences $z_n = n$ and $w_n = in$ diverge to the *same* infinity.

Definition 2.15. The *extended complex plane* or *Riemann sphere* is the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where the symbol ∞ denotes the *point at infinity*.
A *neighborhood* of ∞ is any set containing an *open disk* at ∞; a subset of the form

$$\{\infty\} \cup \{z \in \mathbb{C} : |z| > M\}$$

We extend the notion of limit to the point at infinity.

1. $\lim\limits_{z\to z_0} f(z) = \infty$ means: $\forall M > 0, \exists \delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z)| > M$
2. $\lim\limits_{z\to\infty} f(z) = w_0$ means: $\forall \epsilon > 0, \exists N > 0$ such that $|z| > N \implies |f(z) - w_0| < \epsilon$
3. $\lim\limits_{z\to\infty} f(z) = \infty$ means: $\forall M > 0, \exists N > 0$ such that $|z| > N \implies |f(z)| > M$

Example 2.16. We verify that $\lim\limits_{z\to -3i} \frac{z^2}{z+3i} = \infty$.

Let $M > 0$ be given and define $\delta = \min\{1, \frac{4}{M}\}$. Then

$$\begin{aligned} 0 < |z + 3i| < \delta &\implies |z| \geq |3i| - |z + 3i| > 3 - \delta \geq 2 \\ &\implies \left|\frac{z^2}{z+3i}\right| > \frac{4}{\delta} \geq M \end{aligned}$$

The intuitive relationships between functions, limits, zero and infinity feel like the dubious claims $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$!

Theorem 2.17. *Provided all limits make sense, Theorem 2.13 also applies to limits involving infinity. Moreover, we have th additional relationships:*

1. $\lim\limits_{z\to z_0} f(z) = \infty \iff \lim\limits_{z\to z_0} \frac{1}{f(z)} = 0$
2. $\lim\limits_{z\to\infty} f(z) = w_0 \iff \lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = w_0$
3. $\lim\limits_{z\to\infty} f(z) = \infty \iff \lim\limits_{z\to 0} \frac{1}{f(1/z)} = 0$

Sketch Proof. Suppose $\lim\limits_{z\to\infty} f(z) = w_0$. Simply let $\delta = \frac{1}{N}$ to see that

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < \tfrac{1}{z} < \delta \implies \left|f(\tfrac{1}{z}) - w_0\right| < \epsilon$$

The other five results are similar. ∎

Example 2.18. Consider $f(z)=\frac{5iz+1}{3z-2i}$. Plainly

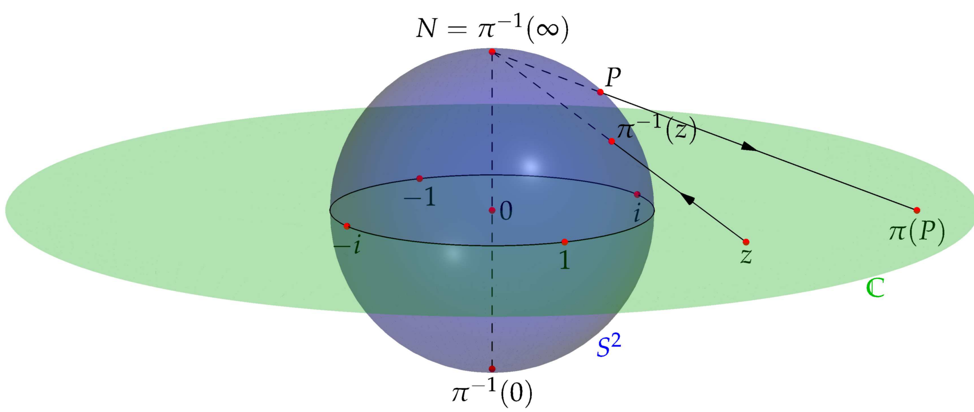
$$\lim_{z\to\frac{2}{3}i}\frac{1}{f(z)}=\lim_{z\to\frac{2}{3}i}\frac{3z-2i}{5iz+1}=0\implies\lim_{z\to\frac{2}{3}i}f(z)=\infty$$
$$\lim_{z\to\infty}f(z)=\lim_{z\to\infty}\frac{5i+1/z}{3-2i/z}=\frac{5i+\lim\frac{1}{z}}{3-2i\lim\frac{1}{z}}=\frac{5}{3}i$$

In view of these calculations, the function is really a map $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ of the Riemann sphere to itself:

$$f(z):=\begin{cases}\frac{5iz+1}{3z-2i} & \text{if } z\neq\frac{2}{3}i,\infty\\ \infty & \text{if } z=\frac{2}{3}i\\ \frac{5}{3}i & \text{if } z=\infty\end{cases}$$

In fact f is a continuous bijection (exercise)!

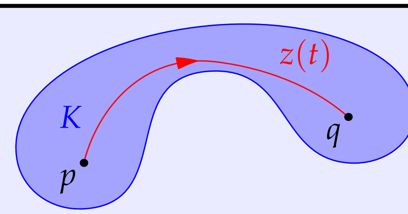
For us, the Riemann sphere is merely a fun diversion. Unless indicated otherwise, all sets should be assumed to be subsets of the (*finite*) complex plane.[4] The Riemann sphere gets its name because $\overline{\mathbb{C}}$ can be visualized as a sphere S^2 where ∞ plays the role of the north pole. The rest of the sphere is identified bijectively with the equatorial plane $\mathbb{C}$ via *stereographic projection* $\pi:S^2\to\overline{\mathbb{C}}$: in the picture, the image of $P\in S^2$ is the intersection $\pi(P)$ of $\mathbb{C}$ with the line through P and $N=\pi^{-1}(\infty)$.



Compactness, Path-Connectedness & Continuity

We finish with two further properties we might wish our domains to have.

Definition 2.19. We say that a subset $K\subseteq\mathbb{C}$ is:



- *Compact* if it is closed and bounded.
- *Path-connected* if any two points can be joined by a path lying entirely within K: more precisely,

$$\forall p,q\in K,\ \exists\text{ continuous } z:[0,1]\to K \text{ such that } z(0)=p \text{ and } z(1)=q$$

[4]Computations of limits involving infinity are examinable, but the Riemann sphere and its interpretation are not.

What you should be familiar with are the analogues of these ideas in real analysis. Compactness generalizes the idea of a closed bounded interval: a compact subset of $\mathbb{R}$ is a union of finitely many closed bounded intervals. Path-connectedness generalizes the notion of an interval; that a set consists of only one piece.[5] In view of this, we translate over two familiar results from real analysis:

Extreme Value Theorem A continuous function on a closed bounded interval is bounded and attains its bounds.

Interval preservation A continuous function maps intervals to intervals. Essentially this is the intermediate value theorem.

Theorem 2.20. *Suppose $f : K \to \mathbb{C}$ is continuous.*

1. *If K is compact, so is the image $f(K)$.*
2. *If K is path-connected, so is $f(K)$.*

Proof.

1. The trick is to consider the *real-valued* function $|f|$.
   - Let $M = \sup\{|f(z)| : z \in K\}$. Since $|f|$ is real-valued, $\exists (z_n) \subseteq K$ such that $|f(z_n)| \to M$.
   - K is bounded; the Bolzano–Weierstraß theorem says (z_n) has a convergent subsequence (z_{n_k}) with limit z_0.
   - K is closed; thus $z_0 \in K$ and $f(z_0)$ is defined.
   - f is continuous; thus $f(z_{n_k}) \to f(z_0)$, necessarily $M = f(z_0)$ is finite and so $f(K)$ is bounded.
   - The closure of $f(K)$ is a short exercise.

   The same argument works even if K is a subset of the real numbers.
2. This is also an exercise.

■

We'll also have reason to use one final fact about compactness, which we again state without proof.

Theorem 2.21. *Let K be compact and suppose that we have a collection of open sets U_j for which $K \subseteq \bigcup_{j \in I} U_j$. Then there exists a* finite subset *$J \subseteq I$ such that $K \subseteq \bigcup_{j \in J} U_j$.*

This is usually said as "every open cover has a finite subcover." In topology, this statement is typically taken as the *definition* of compactness. That this is equivalent to being closed and bounded in $\mathbb{C}$ (or indeed any Euclidean space) is the famous Heine–Borel Theorem. The full details of this, and a thorough discussion of (path-)connectedness, are properly subjects for a course in topology.

[5] Path-connectedness is more useful to us than the related concept of *connectedness*: both convey the idea that a set consists of only one piece (or *component*). For open sets, and for typical domains of functions in this course, connectedness and path-connectedness mean the same thing.

Exercises 2.2 1. Use the ϵ–δ definition (2.10) to prove the following.

(a) $\lim_{z\to z_0} \overline{z} = \overline{z_0}$ (b) $\lim_{z\to 0} \frac{\overline{z}^2}{z} = 0$ (c) $\lim_{z\to 2} \frac{1}{z-i} = \frac{1}{2-i}$ (d) $\lim_{z\to z_0} z^3 = z_0^3$

2. Show that the function $f(z) = \left(\frac{z}{\overline{z}}\right)^2$ has value 1 at all non-zero points on the real and imaginary axes, but that it has the value -1 at all non-zero points on the line $y = x$. Hence explain why $\lim_{z\to 0} f(z)$ does not exist.

3. Prove part 3(c) of Theorem 2.13.

4. Suppose $\lim_{z\to z_0} f(z) = w_0$. Prove that $\lim_{z\to z_0} |f(z)| = |w_0|$.

5. Use Definition 2.15 to prove part of Theorem 2.17: $\lim_{z\to z_0} f(z) = \infty \implies \lim_{z\to z_0} \frac{1}{f(z)} = 0$.

6. Use Definition 2.15 to prove that

(a) $\lim_{z\to 2i} \frac{iz-1}{z-2i} = \infty$ (b) $\lim_{z\to \infty} \frac{iz-1}{z-2i} = i$.

7. (a) Show that $f(z) = \frac{5iz+1}{3z-2i}$ defines a bijection of the Riemann sphere. (*Hint: let $w = f(z)$ and solve for z . . .*)

   (b) More generally, for any complex numbers $\alpha, \beta, \gamma, \delta$, consider the function $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Prove that this defines a bijection of the Riemann sphere if and only if $\alpha\delta - \beta\gamma \neq 0$. How does this discussion relate to the 2×2 matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$?

8. We complete the proof of Theorem 2.20. Suppose $f : K \to \mathbb{C}$ is continuous.

   (a) Let $K \subseteq \mathbb{C}$ be compact. Suppose $(w_n) \subseteq K$ is a sequence such that $(f(w_n))$ is convergent in $\mathbb{C}$. Show that there exists a convergent subsequence (w_{n_k}) and use it to show that $\lim f(w_n) \in f(K)$. Hence conclude that $f(K)$ is closed.

   (b) Suppose K is path-connected. If $f(p), f(q) \in f(K)$, show that $\exists w : [0,1] \to f(K)$ such that $w(0) = f(p)$ and $w(1) = f(q)$. Hence conclude that $f(K)$ is path-connected.

2.3 Derivatives & the Cauchy–Riemann Equations

Differentiation is where complex analysis shows significant differences from real analysis, though it won't appears so at first, since the definition of derivative is exactly what you are used to.

Definition 2.22. Let $f : D \to \mathbb{C}$ be a complex function and z_0 an *interior* point of D. We say that f is *differentiable at* z_0 if the following limit exists.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

We call this limit the *derivative* of f at z_0 and denote it by $f'(z_0)$. If f is differentiable everywhere[a] on D then the derivative is a function; $f'(z)$ or $\frac{\mathrm{d}f}{\mathrm{d}z}$.

[a]Such *holomorphic* functions are the main topic of the course: we'll consider them more properly in the next section. Necessarily, D must be an open set if f is to be holomorphic (think about the definition of the limit!).

As with limits, we quickly consider an easy example and record the basic facts.

Example 2.23. The function $f(z) = z^2$ is everywhere differentiable,[a]

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = \lim_{z \to z_0} z + z_0 = 2z_0$$

[a]Remember that when taking limits, we only compute on a punctured disk $0 < |z - z_0| < \delta$, hence the red equality.

The basic rules of differentiation are identical those in real analysis and can be proved similarly. We state them for reference.

Theorem 2.24. *Suppose f and g are differentiable (either at a point z_0 or as functions).*

1. *(Linearity)* *For any constants $a, b \in \mathbb{C}$,* $\frac{\mathrm{d}}{\mathrm{d}z}(af(z) + bg(z)) = af'(z) + bg'(z)$
2. *(Power Law)* *For any $n \in \mathbb{N}_0$,* $\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1}$
3. *(Product rule)* $\frac{\mathrm{d}}{\mathrm{d}z}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z)$
4. *(Quotient Rule)* *If $g(z) \neq 0$, then* $\frac{\mathrm{d}}{\mathrm{d}z} \frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$
5. *(Chain Rule)* *If h is differentiable at $g(z_0)$, then $h \circ g$ is differentiable at z_0 and*

$$(h \circ g)'(z_0) = h'\big(g(z_0)\big)g'(z_0)$$

We immediately see that all polynomials and rational function are differentiable. Indeed we can easily compute familiar examples without caring whether we are in $\mathbb{R}$ or $\mathbb{C}$!

Example 2.25. $$\frac{\mathrm{d}}{\mathrm{d}z} \frac{3(z^2 - 2)^5 + z^2}{z^3 + 1} = \frac{[30z(z^2 - 2)^4 + 2z)](z^3 + 1) - 3z^2[3(z^2 - 2)^5 + z^2]}{(z^3 + 1)^2}$$

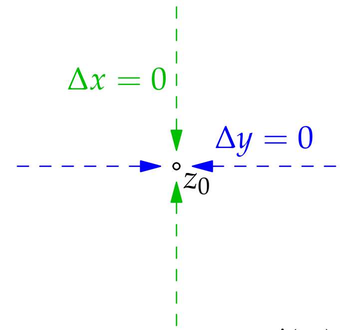
The Cauchy–Riemann Equations

Thus far, differentiation seems uncontroversial. Here is example that might change your mind...

Example 2.26. Consider the complex conjugate function $f(z) = \overline{z}$. We attempt to compute the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{(x,y)\to(x_0,y_0)} \frac{x-iy-x_0+iy_0}{x+iy-x_0-iy_0} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

For $f'(z)$ to exist, we must obtain the same value *regardless of how* $(\Delta x, \Delta y) \to (0,0)$. There are two obvious ways to take the limit:



Horizontally $\quad \Delta y = 0 \implies \dfrac{f(z)-f(z_0)}{z-z_0} = 1$

Vertically $\quad \Delta x = 0 \implies \dfrac{f(z)-f(z_0)}{z-z_0} = -1$

The quotient takes different values depending on how $(\Delta x, \Delta y) \to (0,0)$. We conclude that $f'(z_0)$ *does not exist*, and that the function $f(z) = \overline{z}$ is not differentiable anywhere!

If the example doesn't surprise you, read it again. There is barely a simpler complex-valued function than the complex conjugate, yet this is not differentiable! We now extend the approach to consider (non-)differentiability more generally.

Let $f(z) = u(x,y) + iv(x,y)$ be written in terms of its real and imaginary parts. As before, we write $z = x + iy$, $z_0 = x_0 + iy_0$ and denote the differences by

$$\Delta z = z - z_0 = \Delta x + i\Delta y = (x - x_0) + i(y - y_0)$$

We attempt to evaluate the limit of the difference quotient

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x,y)-u(x_0,y_0)+i(v(x,y)-v(x_0,y_0))}{\Delta x + i\Delta y}$$

If this exists, then we *must* have the same result when evaluating along straight lines approaching z_0 both horizontally and vertically:

Horizontally We have $\Delta y = 0$ and so the limit becomes

$$\lim_{\Delta x\to 0} \frac{u(x,y_0)-u(x_0,y_0)+i(v(x,y_0)-v(x_0,y_0))}{\Delta x} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x_0,y_0)$$

where we see the *partial derivatives* of the functions u and v.

Vertically We have $\Delta x = 0$, from which

$$\lim_{\Delta y\to 0} \frac{u(x_0,y)-u(x_0,y_0)+i(v(x_0,y)-v(x_0,y_0))}{i\Delta y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)(x_0,y_0)$$

If $f'(z)$ exists, then these limits are equal (to $f'(z)$ itself). We have therefore proved:

Theorem 2.27 (Cauchy–Riemann equations). *If $f(z)$ is complex-differentiable, then the real and imaginary parts of f satisfy the* Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \textit{equivalently} \quad u_x = v_y, \quad u_y = -v_x$$

In such a situation, we can write

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Examples 2.28. 1. $f(z) = \overline{z} = x - iy$ has $u(x,y) = x$ and $v(x,y) = -y$. We quickly see that

$$u_x = 1 \neq -1 = v_y, \qquad u_y = 0 = v_x$$

Since u, v do not satisfy the Cauchy–Riemann equations anywhere (not both of them simultaneously!), f fails to be differentiable anywhere.

2. $f(z) = |z| = \sqrt{x^2+y^2}$ has $u = \sqrt{x^2+y^2}$ and $v = 0$. The Cauchy–Riemann equations are

$$u_x = \frac{x}{\sqrt{x^2+y^2}} = 0 = v_y, \quad u_y = \frac{y}{\sqrt{x^2+y^2}} = 0 = -v_x$$

These equations are satisfied nowhere, and so $f(z)$ is nowhere differentiable.[a]

3. $f(z) = z^2 = x^2 - y^2 + 2ixy$ has $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We check

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x$$

As expected, u, v satisfy the Cauchy–Riemann equations. Moreover,

$$f'(z) = 2z = 2x + 2iy = u_x + iv_x = v_y - iu_y$$

4. $f(z) = \dfrac{z}{2+|z|^2} = \dfrac{x}{2+x^2+y^2} + \dfrac{iy}{2+x^2+y^2}$. We compute

$$\begin{aligned} u_x &= \frac{2-x^2+y^2}{(2+x^2+y^2)^2} & v_y &= \frac{2+x^2-y^2}{(2+x^2+y^2)^2} \\ u_y &= \frac{-2xy}{(2+x^2+y^2)^2} & -v_x &= \frac{2xy}{(2+x^2+y^2)^2} \end{aligned}$$

The Cauchy–Riemann equations are satisfied if and only if $xy = 0 = x^2 - y^2$, which is if and only if $x = y = 0$. We conclude that f is not differentiable at any non-zero $z \in \mathbb{C}$.

Note that the Cauchy–Riemann equations only provide a *necessary* condition for differentiability. We do not (yet!) have a sufficient condition. However, we can easily check that this function is differentiable at $z = 0$, straight from the definition of derivative and the continuity of $|z|$:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{1}{2+|z|^2} = \frac{1}{2}$$

The function f is therefore differentiable at precisely one point.

[a]Note that $u = \sqrt{x^2+y^2}$ isn't even differentiable at $(x,y) = (0,0)$.

A Sufficient Condition for Differentiability?

Since Theorem 2.27 only provides a necessary condition for differentiability, is is most useful in the contrapositive form:

Cauchy–Riemann not satisfied $\implies$ f not differentiable

We'd like this to be bidirectional, so that the Cauchy–Riemann equations become a sufficient condition for differentiability. This is indeed possible, with one small caveat...

Suppose a complex function $f(z) = u(x,y) + iv(x,y)$ has partial derivatives on an open neighborhood D of a point $z_0 = x_0 + iy_0$. Also suppose f satisfies the Cauchy–Riemann equations and that the partial derivatives $u_x = v_y$ and $u_y = -v_x$ are *continuous* at z_0. The rough idea is to use the linear approximation: for $z \in D$,

$$\begin{aligned} f(z) - f(z_0) &\approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y \\ &= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y \\ &= (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y \\ &= (u_x + iv_x)(\Delta x + i\Delta y) = (u_x + iv_x)\Delta z \\ \implies \frac{f(z) - f(z_0)}{z - z_0} &\approx u_x(x_0,y_0) + iv_x(x_0,y_0) \end{aligned}$$

The continuity of the partial derivatives at (x_0, y_0) means that the approximation approaches equality as $\Delta z \to 0$. We omit the complete proof since there are too many details. The upshot is a near converse to Theorem 2.27.

Theorem 2.29. *Let u, v have partial derivatives on a neighbourhood of $z_0 = x_0 + iy_0$. Assume $f = u + iv$ satisfies the Cauchy–Riemann equations and has continuous partial derivatives at z_0. Then f is differentiable at z_0 and*

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,v_0)$$

Examples 2.30. 1. Revisiting Example 2.28.4, we see that $f(z) = \frac{z}{2+|z|^2}$ has partial derivatives everywhere; these are continuous and satisfy Cauchy–Riemann at $z = 0$, whence f is differentiable there. Moreover

$$f'(0) = u_x(0,0) + iv_x(0,0) = \frac{1}{2}$$

2. The exponential function is differentiable everywhere: indeed

$$f(z) = e^z = e^x \cos y + ie^x \sin y$$

satisfies

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin x = -v_x$$

where these are certainly continuous on $\mathbb{C}$. Moreover, as expected,

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$$

Exercises 2.3 1. Use Theorem 2.24 to find the derivatives of the following functions:

(a) $f(z) = \dfrac{1}{z^2 + 2z}$ (b) $f(z) = (z^3 + 2iz + 1)^7$ (c) $f(z) = \dfrac{(3z^2 - i)^3}{(iz^3 + 4)^2}$

2. Use the limit definition of the derivative to compute the derivative of the functions:

(a) $f(z) = 3z^3 - iz^2$ (b) $f(z) = \dfrac{1}{z^2}$

3. Give a proof of the quotient rule, directly using the definition of the derivative.

4. Use the quotient rule to prove the power law for negative integer exponents: that is

$$\forall n \in \mathbb{N}, \quad \frac{\mathrm{d}}{\mathrm{d}z} z^{-n} = -nz^{-n-1}$$

5. Suppose $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exists, where $g'(z_0) \neq 0$. Use the definition of the derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

6. Prove that the functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not differentiable anywhere.

7. Exactly as in Example 2.30.2, prove that $\dfrac{\mathrm{d}}{\mathrm{d}z} e^{kz} = ke^{kz}$ for any complex constant k.

8. Consider the Cauchy–Riemann equations for the following functions: what can you conclude, if anything?

(a) $f(z) = \dfrac{1}{\overline{z} - i}$ (b) $f(z) = z^3 - \dfrac{2}{z}$ (c) $f(z) = (|z|^2 + z)^2$

9. Write a complex function $f(z) = f(z, \overline{z})$ as a function of z and $\overline{z}$. For example,

$$f(z) = |z|^2 = z\overline{z}$$

Noting that $x = \dfrac{1}{2}(z + \overline{z})$ and $y = \dfrac{1}{2i}(z - \overline{z})$, use the chain rule

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}}$$

to prove that f satisfies the Cauchy–Riemann equations if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$.

Hence give a quick proof that $f(z) = z\overline{z}^2$ is not differentiable when $z \neq 0$.

2.4 Holomorphic and Harmonic Functions

We tend to be most interested in functions which are differentiable on their whole domain.

Definition 2.31. Let $f : D \to \mathbb{C}$ be a function where D is open.

- If f is differentiable at every point of D we say that it is *holomorphic* (or *analytic*[a]) on D.
- We say that f is *holomorphic at* $z_0 \in D$ if it differentiable (holomorphic) on some open set containing z_0.
- An *entire function* is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ (domain $= \mathbb{C}$).

[a] We'll use these terms interchangeably though strictly they have different definitions: analyticity being related to power series representations. A major part of the course involves showing that these definitions are equivalent.

Examples 2.32. 1. The function $f(z) = e^{4z}$ is entire, as is every polynomial.

2. The function $f(z) = \dfrac{iz}{z^2+4}$ is holomorphic on its implied domain $\operatorname{dom} f = \mathbb{C} \setminus \{\pm 2i\}$; indeed, by the quotient rule

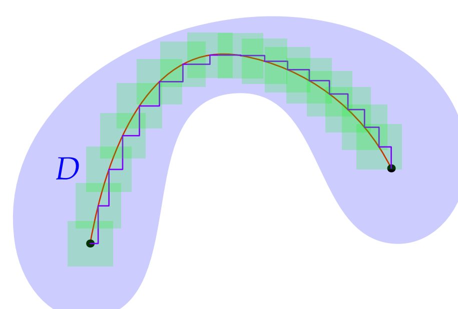
$$f'(z) = \frac{i(z^2+4) - 2iz^2}{(z^2+4)^2} = \frac{i(4-z^2)}{(z^2+4)^2}$$

Our first major result should seem very familiar.

Theorem 2.33. *If $f'(z) = 0$ on a (path-)connected open domain D, then $f(z)$ is constant.*

The set-up relies on Theorems 2.20 and 2.21, though the calculation should be compared to that in real analysis, which also uses the mean value theorem.

Proof. Choose any $p, q \in D$ and join these with a path.
Since D is open, at each point of the path we can choose an open square lying within D centered on the path.
The path is compact, whence it may be covered by finitely many boxes. A zigzag path consisting of finitely many horizontal and vertical segments may now be described within these boxes.
Since $f'(z) = u_x + iv_x = v_y - iu_y = 0$, we see that all four partial derivatives are zero.



For each horizontal segment ($x_1 \le x \le x_2$, y fixed), we apply the mean value theorem to the functions $x \mapsto u(x,y)$ and $x \mapsto v(x,y)$. We therefore have $\widehat{x}, \widetilde{x} \in (x_1, x_2)$ with

$$\frac{u(x_2,y) - u(x_1,y)}{x_2 - x_1} = u_x(\widehat{x}, y) = 0 \quad \text{and} \quad \frac{v(x_2,y) - v(x_1,y)}{x_2 - x_1} = v_x(\widetilde{x}, y) = 0$$

whence u and v, and thus $f(z)$, are constant along any horizontal segment.
The same holds along any vertical segment, this time after utilizing the fact that $u_y = v_y = 0$.
We conclude that $f(q) = f(p)$: since p, q were arbitrary points of D, f is constant. ∎

Corollary 2.34. *If $f(z)$ is holomorphic and $|f(z)|$ is constant, then $f(z)$ is constant.*

This is essentially trivial in the real case: think about why! In the complex case, we need a proof.

Proof. Clearly $|f(z)|^2 = f(z)\overline{f(z)} = k$ is constant. If $k = 0$, we are done. Otherwise, $\overline{f(z)} = \frac{k}{f(z)}$ is holomorphic (quotient rule!). Write $f(z) = u + iv$, whence $\overline{f(z)} = u - iv$, and consider the Cauchy–Riemann equations for *both*:

$$u_x = v_y, \qquad u_y = -v_x, \qquad u_x = -v_y, \qquad u_y = v_x$$

We conclude that all partial derivatives are zero: $f'(z) = 0$ and so f is constant. ∎

We now come to a significant contrast with the real case.

Theorem 2.35. *If $f(z) = u + iv$ is holomorphic, then f is* infinitely differentiable. *Otherwise said:*

- *$f^{(n)}(z)$ exists and is continuous for all $n \in \mathbb{N}$.*
- *u and v have continuous partial derivatives of all orders.*

In real analysis, even functions which are differentiable everywhere need not be *twice* differentiable, let alone infinitely so (Exercise 4). We'll prove this important result later in the course once we've developed some integration theory.

Harmonic Functions We may now consider what happens with the *second* partial derivatives of a holomorphic function:

$$u_{xx} = \frac{\partial}{\partial x}u_x \overset{CR1}{=} \frac{\partial}{\partial x}v_y = v_{yx} \overset{(*)}{=} v_{xy} = \frac{\partial}{\partial y}v_x \overset{CR2}{=} -\frac{\partial}{\partial y}u_y = -u_{yy}$$

Equality of the mixed partial derivatives $(*)$ follows because all derivatives are continuous (Theorem 2.35 and Clairaut's Theorem). The same equation holds for v. We conclude:

Corollary 2.36. *If $f = u + iv$ is holomorphic, then u and v are* harmonic functions*; solutions to* Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Laplace's equation is one of the most important partial differential equations, and is widely used throughout mathematics and physics.

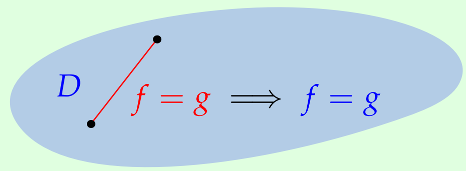
Example 2.37. $f(z) = \frac{1}{z} = \frac{x-iy}{x^2+y^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$; its real and imaginary parts are therefore harmonic away from the origin. Indeed,

$$\begin{aligned} u_{xx} + u_{yy} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{x}{x^2+y^2} = \frac{\partial}{\partial x}\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{\partial}{\partial y}\frac{2xy}{(x^2+y^2)^2} \\ &= \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3} - \frac{2x(x^2+y^2) - 8xy^2}{(x^2+y^2)^3} = 0 \end{aligned}$$

Analytic Continuations Though we cannot yet prove it, now is a good time to introduce another surprising property of holomorphic/analytic functions. Since this result will be seen to depend on power series, we'll stick to using the term *analytic* here.

Theorem 2.38. *Suppose that f and g are analytic functions on an open connected domain D and assume that $f(z) = g(z)$ on some path contained in D.*

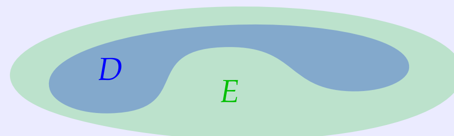
Then $f(z) = g(z)$ throughout D.



This is highly counter-intuitive; you need only know the values of an analytic function on a tiny path to know the full function on its whole (connected) domain! This leads to a new concept.

Definition 2.39. Let $D \subseteq E$ be open connected domains and $g : E \to \mathbb{C}$ an analytic function. Let $f : D \to \mathbb{C}$ be the restriction of g to D; that is $f(z) = g(z)$ on D.

We call g *the analytic continuation* of f to E.

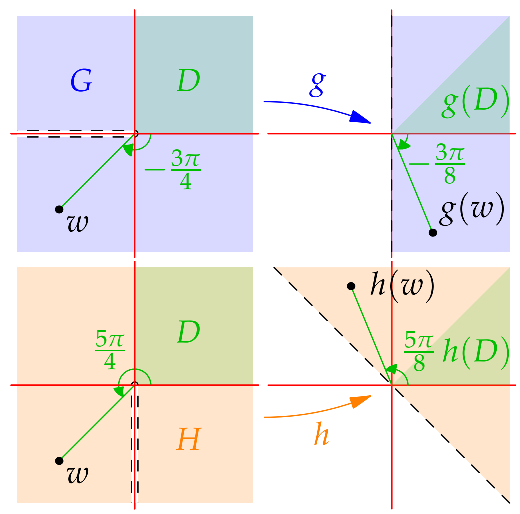


By Theorem 2.38, the analytic continuation of f to E must be unique. There are, however, some subtleties, which we explore a little in the next example:

- Given f analytic on D and $D \subseteq E$, an analytic continuation is not guaranteed to exist.
- The choice of extended domain E *really* matters.

Example 2.40. Consider the principal square root $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ with domain D the first quadrant. In Exercise 10, we'll see that f is differentiable on D.

Consider two analytic continuations of f: in both cases we use the same formula $z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}$ and we distinguish the point $w = e^{-\frac{3\pi i}{4}} = e^{\frac{5\pi i}{4}}$ for comparison.



Let $G = \mathbb{C} \setminus \{-x : x \in \mathbb{R}_0^+\}$ be the plane omitting the *non-positive real axis* and let

$$g : G \to \mathbb{C} : z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}, \quad \theta = \operatorname{Arg} z \in (-\pi, \pi)$$

The codomain of g is the right half-plane and $g(w) = e^{-\frac{3\pi i}{8}}$ lies in the *fourth quadrant*.

Let $H = \mathbb{C} \setminus \{-iy : y \in \mathbb{R}_0^+\}$ omit the *non-positive imaginary axis* and let

$$h : H \to \mathbb{C} : z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}, \quad \theta = \arg z \in (-\tfrac{\pi}{2}, \tfrac{3\pi}{2})$$

This time the codomain of h is the upper-right half-plane; moreover $h(w) = e^{\frac{5\pi i}{8}} = -g(w)$ lies in the *second quadrant*.

We have two analytic continuations of f which disagree on the intersection of their domains!

It can moreover be seen that the omissions chosen for G, H are necessary: there is no analytic continuation of f to the punctured plane $\mathbb{C} \setminus \{0\}$, or indeed to any domain in which it is possible to loop completely around the origin. We shall return to this topic later…

Exercises 2.4 1. Suppose $g'(z) = h'(z)$ on an open connected domain D. Prove that $h(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

(*Equivalently: if $g(z)$ and $h(z)$ are anti-derivatives of $f(z)$ on an open connected domain D then $g(z) - h(z)$ is constant on D.*)

2. Check explicitly that $u = e^{nx} \cos ny$ and $v = e^{nx} \sin ny$ are harmonic functions for any $n \in \mathbb{Z}$.

3. Prove or disprove: if $u, v : D \to \mathbb{R}$ are harmonic functions on an open set $D \subseteq \mathbb{R}^2$, then $f(z) := u(x, y) + iv(x, y)$ is holomorphic.

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Show that f is differentiable, but not *twice* differentiable.

(*Such properties are impossible for complex functions*)

5. Suppose $f(z) = u + iv$ and write $z = x + iy = re^{i\theta}$ in polar form.

   (a) Use the chain rule applied to the polar co-ordinate relations

   $$x = r\cos\theta, \qquad y = r\sin\theta$$

   to compute the partial derivatives u_r, u_θ, v_r and v_θ.

   (b) Deduce the polar form of the Cauchy–Riemann equations:

   $$ru_r = v_\theta \qquad u_\theta = -rv_r, \qquad f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{-i}{z}(u_\theta + iv_\theta)$$

6. Prove the polar form of Laplace's equation:

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0$$

7. Show that $u = r^n \cos n\theta$ is a harmonic function for any $n \in \mathbb{N}$: find *two* ways to show that this is true!

8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(iy) = -iy^3$ whenever $z = iy$ lies on the imaginary axis. What is the value $f(2)$? Explain your answer.

9. The function $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$. Explain why there is no analytic continuation of f that is analytic at $z = 0$.

10. (a) Use Exercise 5 to prove that $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ is analytic on the first quadrant, and find $f'(z)$. Moreover, explain why the functions g, h in Example 2.40 are analytic and therefore analytic continuations of f.

    (b) Prove that there exists no analytic continuation of g to any set larger than G.
    (*Hint: suppose the extended domain contains $-r \in \mathbb{R}^-$. Now use the fact that an analytic continuation must be continuous at $-r \ldots$*)

3 Elementary Functions

We've already considered polynomials, rational functions and, to some extent, n^{th} roots and the exponential. We now develop the logarithmic and trigonometric functions.

3.1 Exponential and Logarithmic Functions

The exponential function $\exp : \mathbb{C} \to \mathbb{C} : z \mapsto e^z$ was defined earlier using Euler's formula

$$\exp(z) = e^z := e^x \cos y + ie^x \sin y \tag{$*$}$$

For reference, we collect some basic properties: at least parts 1–4 should be familiar.

Lemma 3.1. *Throughout let $z, w \in \mathbb{C}$.*

1. *The exponential function is entire and has derivative $\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$*
2. $e^z \neq 0$
3. $e^{z+w} = e^z e^w$ and $e^{z-w} = \dfrac{e^z}{e^w}$
4. *For all $n \in \mathbb{Z}$, $(e^z)^n = e^{nz}$*
5. *e^z is* periodic *with period $2\pi i$. Moreover,*

$$e^z = e^w \iff z - w = 2\pi in \text{ for some } n \in \mathbb{Z}$$

Sketch Proof. 1. We saw this earlier: check Cauchy–Riemann and compute $\frac{\mathrm{d}}{\mathrm{d}z}e^z = u_x + iv_x$.

2. This follows trivially from ($*$): $e^x > 0$, while $\cos y$ and $\sin y$ are never both zero.
3. Recall the multiple-angle formulæ for cosine and sine.
4. This requires an induction using part 3 with $z = w$.
5. Certainly $e^{w+2\pi in} = e^w$ by the periodicity of sine and cosine. Now suppose $e^z = e^w$ where $z = x + iy$ and $w = u + iv$. By considering the modulus and argument, we see that

$$e^x e^{iy} = e^u e^{iv} \implies \begin{cases} e^x = e^u \\ y = v + 2\pi in \text{ for some } n \in \mathbb{Z} \end{cases}$$

   We conclude that $x = u$ and so $z - w = i(y - v) = 2\pi in$.

■

Example 3.2. Find all $z \in \mathbb{C}$ such that $e^z = 5(-1+i)$.
Following the Lemma, write $z = x + iy$ and take the polar form of $5(-1+i)$ to see that

$$e^z = 5(-1+i) \iff e^x e^{iy} = 5\sqrt{2}e^{\frac{3\pi i}{4}} \iff \begin{cases} x = \ln(5\sqrt{2}) \\ y = \frac{3\pi}{4} + 2\pi n \quad \text{for some } n \in \mathbb{Z} \end{cases}$$

$$\iff z = \ln(5\sqrt{2}) + \left(\frac{3\pi}{4} + 2\pi n\right) i \quad \text{for some} \quad n \in \mathbb{Z}$$

We see that there are *infinitely many* suitable z!

Duplicate Notation Warning! When $n \in \mathbb{N}$, the expression $e^{\frac{1}{n}}$ can now mean two things. For instance $e^{\frac{1}{3}}$ can mean:

1. The *set* of cube roots of e, namely $\{\sqrt[3]{e}, \sqrt[3]{e}e^{\frac{2\pi i}{3}}, \sqrt[3]{e}e^{-\frac{2\pi i}{3}}\}$;
2. The *real value* $\sqrt[3]{e} \in \mathbb{R}^+$.

Given that e^z is such a common function, in both real and complex analysis, we default to the second meaning: if you mean the set of n^{th} roots, say so! Remember that you can always write $\exp(z)$ for the function if you want to be unambiguous.

The periodicity of the exponential leads to the far more interesting notion of the complex *logarithm.*

Definition 3.3. Let $z = re^{i\theta}$ be a non-zero complex number with principal argument $\theta = \operatorname{Arg} z$. The *principal logarithm* of z is the value

$$\operatorname{Log} z := \ln r + i\theta = \ln|z| + i \operatorname{Arg} z$$

where ln is the usual natural logarithm. The *logarithm* of z is any (and all!) of the values[6]

$$\log z = \ln|z| + i \arg z = \ln r + i(\theta + 2\pi n) : n \in \mathbb{Z}$$

Examples 3.4. 1. Since $-4 = 4e^{\pi i}$, we see that

$$\operatorname{Log}(-4) = \ln 4 + \pi i \quad \text{and} \quad \log(-4) = \ln 4 + (1 + 2n)\pi i$$

2. Again write in polar form to compute:

$$\operatorname{Log}(\sqrt{3} - i) = \operatorname{Log}(2e^{-\frac{\pi i}{6}}) = \ln 2 - \frac{\pi i}{6} \quad \text{and} \quad \log(\sqrt{3} - i) = \ln 2 - \frac{\pi i}{6} + 2\pi n i$$

These examples involve solving equations of the form $e^w = z$: writing $z = re^{i\theta} = e^{\ln r + i\theta}$ as above, and appealing to part 5 of Lemma 3.1, we instantly see that

$$e^w = z \iff w = \log z$$

Read this carefully, remembering that the logarithm is multi-valued and the exponential periodic:

$$e^{\log z} = z \qquad \text{and} \qquad \log e^w = w + 2\pi n i \quad \text{where } n \in \mathbb{Z}$$

Before moving on, we clear up some of the basic properties of the principal logarithm function. All parts of this should be clear from Definition 3.3.

Lemma 3.5. *Throughout, z and w are complex numbers with $z \neq 0$, and $n \in \mathbb{Z}$.*

- $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \{w \in \mathbb{C} : \operatorname{Im} w \in (-\pi, \pi]\}$ *is a bijection with inverse* exp.
- $\operatorname{Log} e^w = w + 2\pi n i$ *where $n \in \mathbb{Z}$ is chosen such that* $\operatorname{Im}(\operatorname{Log} e^w) = \operatorname{Im} w + 2\pi n \in (-\pi, \pi]$.
- *If $z \in \mathbb{R}^+$, then* $\operatorname{Log} z = \ln z$ *is the usual* natural *logarithm.*

[6]This is identical to how we use $\arg z$, which, depending on context, means either the *set* $\{\operatorname{Arg} z + 2\pi n i\}$ or some particular value from this set. We'll more formally discuss such *multi-valued* functions in Section 3.2.

The Logarithm Laws Just as the standard rules for exponentiation (Lemma 3.1 parts 3 and 4) apply to the complex exponential, something similar works for the log laws. However, the multi-valued nature of the logarithm makes this a little more subtle.
Suppose non-zero z, w are given: since $|zw| = |z|\,|w|$ and $\arg zw = \arg z + \arg w$, we conclude that

$$\begin{aligned}\log zw &= \ln(|z|\,|w|) + i(\arg z + \arg w) = \ln|z| + i\arg z + \ln|w| + i\arg w \\ &= \log z + \log w\end{aligned}$$

Be very careful with this expression; it is *not* an identity of *functions.* What it really means is the following:

1. We have *set equality*: in particular, the following sets are identical:
$$\begin{aligned}\log z + \log w &= \{\alpha + \beta : \alpha \in \log z, \beta \in \log w\} \\ &= \{|z| + i\operatorname{Arg} z + 2\pi ki + |w| + i\operatorname{Arg} w + 2\pi mi : k, m \in \mathbb{Z}\} \\ \log zw &= \{\ln|zw| + i\operatorname{Arg}(zw) + 2\pi ni : n \in \mathbb{Z}\}\end{aligned}$$
2. There exist *particular choices* of the arguments of z, w and zw so that $\arg zw = \arg z + \arg w$.

Unless you are sure you won't make a mistake, it is therefore safer to write

$$\boxed{\log zw = \log z + \log w + 2\pi ni \quad \text{for some} \quad n \in \mathbb{Z}}$$

Given its restricted range, we can be more precise for the principal logarithm:

$$\boxed{\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + 2\pi ni \quad \text{for some} \quad n = 0, -1, 1}$$

Example 3.6. Let $z = -\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}$ and $w = \sqrt{2}(1+i) = 2e^{\frac{\pi i}{4}}$. Then

$$\begin{aligned}&\log z = \ln 2 + \frac{5\pi i}{6} + 2\pi ki, \qquad \log w = \ln 2 + \frac{\pi i}{4} + 2\pi mi \\ &\log zw = \log(4e^{\frac{5\pi i}{6} + \frac{\pi i}{4}}) = \log(4e^{\frac{13\pi i}{12}}) = \ln 4 + \frac{13\pi i}{12} + 2\pi ni = \log z + \log w \iff k + m = n\end{aligned}$$

For principal logarithms, however, we need to make a different, explicit, choice:

$$\begin{aligned}&\operatorname{Log} z = \ln 2 + \frac{5\pi i}{6}, \qquad \operatorname{Log} w = \ln 2 + \frac{\pi i}{4} \\ &\operatorname{Log} zw = \operatorname{Log}(4e^{-\frac{11\pi}{12}}) = \operatorname{Log}(4e^{-\frac{11\pi}{12}}) = \ln 4 - \frac{11\pi i}{12} = \operatorname{Log} z + \operatorname{Log} w - 2\pi i\end{aligned}$$

We can similarly demonstrate the other log law, with the same caveat:

$$\boxed{\log \frac{z}{w} = \log z - \log w}$$

You might assume that the final log law ($\log z^n = n \log z$ whenever $n \in \mathbb{N}$) also makes sense, but you'd be very wrong: an example should explain why you should ignore this law!

Example 3.7. Let $z = -\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}$ and compute what would be meant by the *set* $2\log z$:

$$2\log z = 2\left(\ln 2 + \frac{5\pi i}{6} + 2\pi m i\right) = \ln 4 + \frac{5\pi i}{3} + 4\pi m i$$

This is different from the set

$$\log z^2 = \log(4e^{\frac{10\pi i}{6}}) = \ln 4 + \frac{5\pi i}{3} + 2\pi k i$$

However, in the language of the previous exercise, there would be no problem if each copy of $\log z$ got its own multiples of $2\pi i$: It is therefore safer to state that $\log z^2 \neq 2\log z$.

Since the principal logarithm is a function rather than a set, we can be more precise: for any $n \in \mathbb{N}$,

$$\boxed{\operatorname{Log} z^n = n \operatorname{Log} z + 2\pi k i \quad \text{for some integer } k \text{ with } |k| \leq \frac{n}{2}}$$

Example 3.8. Let $z = e^{-\frac{13\pi i}{16}}$ and consider z^{16}. We see that

$$\operatorname{Log} z^{16} = \operatorname{Log} e^{-13\pi i} = \operatorname{Log} e^{\pi i} = i\pi, \qquad 16\operatorname{Log} z = -13\pi i \implies \operatorname{Log} z^{16} = 16\operatorname{Log} z + 14\pi i$$

Exercises 3.1

1. Compute the following:

   (a) $\exp(3 - \frac{\pi}{2}i)$ (b) $\operatorname{Log}(ie)$ (c) $\log(3-4i)$ (d) $\operatorname{Log}[(-1+i)^2]$

2. (a) If e^z is real, show that $\operatorname{Im} z = n\pi$ for some integer n.

   (b) If e^z is imaginary, what restriction is placed on z?

3. Show in two ways that the function $f(z) = \exp(z^2)$ is entire, and find its derivative.

4. Prove, for any $z \in \mathbb{C}$, that $\left|\exp(z^2)\right| \leq \exp|z|^2$. What must z satisfy if this is to be *equality*?

5. Find $\operatorname{Re} e^{\frac{1}{z}}$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

6. Show that $\operatorname{Log} i^3 \neq 3\operatorname{Log} i$.

7. Show that $\operatorname{Re}(\log(z-1)) = \frac{1}{2}\ln[(x-1)^2 + y^2]$ whenever $z \neq 1$.

8. Prove the above boxed formula for $\operatorname{Log} z^n$.

9. The square roots of i are $\sqrt{i} = e^{\frac{\pi i}{4}}$ and $-\sqrt{i} = e^{-\frac{3\pi i}{4}}$.

   (a) Compute $\operatorname{Log}\sqrt{i}$ and $\operatorname{Log}(-\sqrt{i})$ and check that $\operatorname{Log}\sqrt{i} = \frac{1}{2}\operatorname{Log} i$.

   (b) Show that the set of all logarithms of all square roots of i is

   $$\log i^{\frac{1}{2}} = \left(n + \frac{1}{4}\right)\pi i \quad \text{where} \quad n \in \mathbb{Z}$$

   and therefore deduce that $\log i^{\frac{1}{2}} = \frac{1}{2}\log i$ *as sets*.

3.2 Multi-valued Functions, Branch Cuts and the Power Function

Since each $\log z$ represents a *set* of complex numbers, the complex logarithm is often called a *multi-valued function*. We have already seen several of these beasts:

- The argument of a complex number is any of the values $\arg z = \operatorname{Arg} z + 2\pi n$ where $n \in \mathbb{Z}$. The logarithm is simply a modification of this: $\log z = \ln r + i \arg z$.
- The n^{th} root of z is the set of values $z^{\frac{1}{n}} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$ where $\omega_n = e^{\frac{2\pi i}{n}}$ is an n^{th} root of unity and $\sqrt[n]{z}$ the principal root.

It is an abuse of language to refer to a multi-valued *function*, since a function should assign *exactly one* object to each element of a domain. While this problem can be fixed using equivalence classes, another approach is simpler to visualize.

Definition 3.9. A *branch* of a multi-valued function f is a single-valued function F on a domain D which is *holomorphic* on D and such that each $F(z)$ is one of the values of $f(z)$.

Let $D = \mathbb{C} \setminus \ell$ where ℓ is a line or curve in $\mathbb{C}$. If $F : D \to \mathbb{C}$ is a branch of f, we call ℓ a *branch cut*. A *branch point* is any point common to all branch cuts.

Branches of the Logarithm The *principal branch* of the logarithm is a slightly restricted version of the principal logarithm

$$\operatorname{Log} z = \ln r + i\theta \quad \text{where} \quad \theta = \operatorname{Arg} z \in (-\pi, \pi)$$

The branch cut in this case is the non-positive real axis. To check this, verify the Cauchy–Riemann equations (use polar form[7]):

$$r u_r = r\frac{\partial}{\partial r}\ln r = 1 = \frac{\partial}{\partial \theta}\theta = v_\theta, \qquad u_\theta = 0 = -r v_r$$

The partial derivatives are certainly continuous, whence $\log z$ is holomorphic with derivative

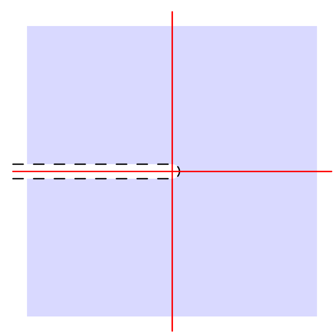
$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = e^{-i\theta}(u_r + i v_r) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

More generally, for any angle α we could take the branch cut ℓ to be the line with argument α and define a branch of the logarithm by

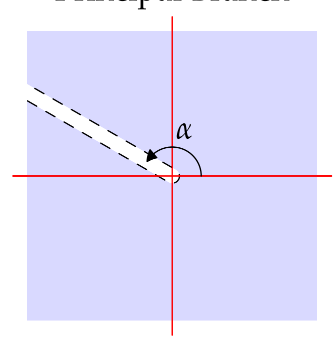
$$\log z = \ln r + i\theta \quad \text{where } \theta \in (\alpha, \alpha + 2\pi)$$

The principal branch corresponds to $\alpha = -\pi$. Note that choosing $\alpha = \pi$ produces the *same branch cut* as for $\operatorname{Log} z$, but a *different branch*!

$$\log z = \ln r + i\theta \text{ where } \theta \in (\pi, 3\pi)$$



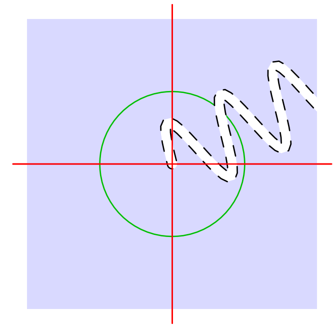
Principal Branch



α-Branch

[7] If you struggle to remember these, compute the partial derivatives of $z = re^{i\theta} = r\cos\theta + ir\sin\theta$!

More esoteric branch cuts are possible. The problem with the logarithm is that if we travel counter-clockwise around the origin, its value increases by $2\pi i$. It is therefore impossible for a branch to be *continuous* (let along *holomorphic*) on any domain D containing such an encircling path; to make $\log z$ single-valued, a branch cut must 'cut' any such path, and must therefore connect the two *branch points* 0 and ∞.



Clarity is crucial here: when you write $\log z$, do you mean a *set*, a particular *element* of that set, or a *branch*? Certain expressions may be true or false depending on the meaning.

Example 3.10. Consider $z = \frac{1}{\sqrt{2}}(1+i) = e^{\frac{\pi i}{4}}$. For the principal branch, we have

$$\operatorname{Log} z^2 = \operatorname{Log} e^{\frac{\pi i}{2}} = \frac{\pi i}{2} = 2 \operatorname{Log} z$$

For the branch with $\alpha = \frac{\pi}{3}$, we have

$$z^2 = e^{\frac{\pi i}{2}} = e^{-\frac{3\pi i}{2}} \implies \log z^2 = -\frac{3\pi i}{2} \neq 2 \log z$$

Recall also (Example 3.7), that *as sets*, $\log z^2 \neq 2 \log z$.

For particular branches of the logarithm, specific versions of the logarithm laws are available. It is not worth trying to remember these; just take care and think out the possibilities!

General Exponential Functions The logarithm can be used to define exponential functions for any non-zero complex base c: choose a value $\log c$ and define

$$\boxed{c^z := e^{z \log c}}$$

Provided c is not a negative real number, the standard is to use the principal logarithm. Regardless of the choice, $\log c$ is constant and the exponential function is holomorphic everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = c^z \log c$$

Example 3.11. Let $c = i = e^{\frac{\pi i}{2}}$ and use the principal logarithm to define

$$i^z := e^{z \operatorname{Log} i} = \exp\left(\frac{\pi i z}{2}\right) = \exp\left(-\frac{\pi}{2} y + i \frac{\pi}{2} x\right) = e^{-\frac{\pi y}{2}} \left[\cos \frac{\pi}{2} x + i \sin \frac{\pi}{2} x\right]$$

It is simple to check the Cauchy–Riemann equations for this function and see that it is holomorphic on $\mathbb{C}$.

If we instead took the α-branch of the logarithm with $\alpha = \frac{\pi}{3}$, then $i = e^{-\frac{3\pi i}{2}}$ and so

$$i^z = \exp\left(\frac{-3\pi i z}{2}\right) = e^{\frac{3\pi y}{2}} \left[\cos \frac{3\pi}{2} x - i \sin \frac{3\pi}{2} x\right]$$

This is still entire, though it is a completely different function! Note that both definitions of i^z agree whenever z is an integer, but for $z = \frac{1}{2}$ these produce the two distinct square roots of i!

Power Functions Similarly to the general exponential function, for any non-zero z and complex number c, we define the (typically multi-valued) function

$$\boxed{z^c := e^{c \log z}}$$

In this case, restricting to the principal branch of the logarithm gives an unambiguous function.

Definition 3.12. The *principal value* of z^c is the function

$$\text{P. V. } z^c := e^{c \operatorname{Log} z}$$

whose domain excludes the non-positive real axis.

Example 3.13. Using the principal branch of the logarithm ($\Theta = \operatorname{Arg} z$), we obtain

$$\text{P. V. } z^{\frac{1}{3}} = \exp\left(\frac{1}{3}(\ln r + i\Theta)\right) = \exp\left(\ln \sqrt[3]{r} + \frac{i\Theta}{3}\right) = \sqrt[3]{r}e^{\frac{i\Theta}{3}} = \sqrt[3]{z}$$

precisely the principal cube-root of z as defined previously.
If we instead choose an α-branch, then $\theta = \arg z \in (\alpha, \alpha + 2\pi)$, from which

$$e^{\frac{1}{3}\log z} = \exp\left(\frac{1}{3}(\ln r + i\theta)\right) = \sqrt[3]{r}\exp\left(\frac{i}{3}(\Theta + (\theta - \Theta))\right) = \sqrt[3]{z}e^{\frac{i(\theta-\Theta)}{3}}$$

Since, for any z, the difference in the arguments $\theta - \Theta = 2\pi n$ is a multiple of 2π, this expression really does return a cube-root of z.

Lemma 3.14. *Choose a branch of the logarithm so that $z^c = e^{c \log z}$ is single-valued. Then z^c is holomorphic on the same domain as the logarithm; moreover $\frac{\mathrm{d}}{\mathrm{d}z} z^c = z^{c-1}$.*

Proof. Since $\log z$ is holomorphic, simply use the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}z} z^c = \frac{\mathrm{d}}{\mathrm{d}z} e^{c \log z} = e^{c \log z}\frac{\mathrm{d}}{\mathrm{d}z}(c \log z) = e^{c \log z} \cdot \frac{c}{z} = ce^{c \log z}e^{-\log z} = ce^{(c-1)\log z} = cz^{c-1}$$

∎

Example 3.15. If the principal branch of the logarithm is used, then

$$(zw)^c = \exp(c \operatorname{Log}(zw)) = \exp(c \operatorname{Log} z + c \operatorname{Log} w + 2\pi cni) = z^c w^c e^{2\pi cni}$$

for some $n \in \{0, \pm 1\}$. We do not therefore expect simple exponent rules such as $(ab)^c = a^c b^c$ to hold in complex analysis. Note, however, that this does work in the case where c is an integer.

As an example, again using the principal value, if $z = w = e^{\frac{3\pi i}{4}}$, then $zw = e^{\frac{3\pi i}{2}} = e^{-\frac{\pi i}{2}}$, whence

$$\text{P. V.}(zw)^{5i} = \exp\left(-5i\frac{\pi i}{2}\right) = e^{\frac{5\pi}{2}}$$

$$z^{5i} = \exp\left(5i\frac{3\pi i}{4}\right) = e^{-\frac{15\pi}{4}} \implies z^{5i}w^{5i} = e^{-\frac{15\pi}{2}} = e^{\frac{5\pi}{2}}e^{2\pi \cdot 5i \cdot ni} \quad \text{with } n = 1$$

Exercises 3.2

1. (a) Show that the function $f(z) = \operatorname{Log}(z - i)$ is holomorphic everywhere except on the portion $x \le 0$ of the line $y = 1$.

   (b) Show that the function $f(z) = \frac{1}{z^2+i}\operatorname{Log}(z+4)$ is holomorphic everywhere except at the points $\pm\frac{1}{\sqrt{2}(1-i)}$ and on the portion $x \le -4$ of the real axis.

2. Show that the set $z^{\frac{1}{4}}$ as defined earlier in the course coincides with the set $z^{\frac{1}{4}} := \exp\left(\frac{1}{4}\log z\right)$ as defined in this section.

3. Show that $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)$ where $n \in \mathbb{Z}$.

4. Find the principal values of the following:

   (a) i^{2i} (b) $(1-i)^{3i}$ (c) $(-\sqrt{3}+i)^{1+4\pi i}$

5. Suppose c, c_1, c_2 and z are complex numbers where $z \neq 0$. If all the powers involved are principal values, show that,

   (a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$ (b) $(z^c)^n = z^{cn}$ for any $n \in \mathbb{N}$.

6. The power function z^c is *usually* multi-valued. However, if $c = m$ is an integer, prove that z^m is single-valued: i.e. it is independent of the branch of logarithm used in its definition.

7. Check the claim at the bottom of Example 3.11: if $m \in \mathbb{Z}$, then i^m is the same value for the two definitions of i^z.

8. Continuing the previous question, suppose $c \neq 0$ and define $c^z = e^{z\log c}$ where any choice of the branch of the logarithm is made.

   (a) Let $m \in \mathbb{Z}$. Prove that c^m produces the same value, regardless of the branch of logarithm used to define $\log c$.

   (b) If $z = \frac{1}{m}$, show that c^z really is an m^{th} root of c. If the principal branch of the logarithm is used, show that c^z is the principal m^{th} root of c. For every m^{th} root of c, show that there exists a branch of the logarithm for which c^z equals the given m^{th} root.

3.3 Trigonometric and Inverse Trigonometric Functions

It is straightforward to give a sensible definition of the basic trigonometric functions simply by modifying Euler's formula. For instance, if $y \in \mathbb{R}$, then

$$e^{iy} + e^{-iy} = \cos y + i \sin y + \cos y - i \sin y = 2\cos y$$

This motivates the following.

Definition 3.16. For any $z \in \mathbb{C}$ we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Example 3.17. $\cos(\frac{\pi}{4} + i) = \frac{1}{2}(e^{\frac{i\pi}{4}-1} + e^{\frac{-i\pi}{4}+1}) = \frac{1}{2}\left(\frac{e^{-1}}{\sqrt{2}}(1+i) + \frac{e}{\sqrt{2}}(1-i)\right) = \frac{e+e^{-1}}{2\sqrt{2}} - i\frac{e-e^{-1}}{2\sqrt{2}}$

Theorem 3.18. *Sine and cosine are entire functions with derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z$$

Sine and cosine satisfy the same identities, including double and multiple-angle formulæ, as their real counterparts: for instance

$$\sin^2 z + \cos^2 z = 1, \quad \cos(z+w) = \cos z \cos w - \sin z \sin w, \quad \cos 2z = 2\cos^2 z - 1, \quad \textit{etc.}$$

In particular, $\sin z = \cos(z - \frac{\pi}{2})$ *and* $\cos z = \sin(z + \frac{\pi}{2})$*. Sine and cosine are also* 2π*-periodic and have exactly the same zeros as their real versions:*

$$\sin z = 0 \iff z = n\pi, \quad \cos z = 0 \iff z = \frac{\pi}{2} + n\pi \quad \text{where} \quad n \in \mathbb{Z}$$

The upshot of the Theorem is that sine and cosine behave exactly as you'd expect.
The proofs are straightforward applications of properties of the exponential function. For instance;

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \frac{1}{2i}\frac{\mathrm{d}}{\mathrm{d}z}(e^{iz} - e^{-iz}) = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \cos z$$

and,

$$\sin z = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 \iff e^{-2y}(\cos 2x + i \sin 2x) = 1 \iff z = \pi n$$

The remaining trigonometric functions are defined in the expected way: for instance,

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \quad \text{whenever} \quad z \neq \frac{\pi}{2} + n\pi$$

All trigonometric functions are holomorphic wherever defined and have the usual expressions for their derivatives: e.g.

$$\frac{\mathrm{d}}{\mathrm{d}z}\tan z = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z$$

Aside: Hyperbolic Functions By considering real and imaginary parts,

$$\begin{aligned}\sin z &= \frac{1}{2i}(e^{ix-y} - e^{-ix+y}) = \frac{1}{2i}(e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x)\\ &= \frac{1}{2}(e^{y}+e^{-y})\sin x + \frac{i}{2}(e^{y}-e^{-y})\cos x = \sin x\cosh y + i\cos x\sinh y\\ \cos z &= \cos x\cosh y - i\sin x\sinh y\end{aligned}$$

where $\cosh y = \frac{1}{2}(e^y + e^{-y})$ and $\sinh y = \frac{1}{2}(e^y - e^{-y})$ are the usual (real) hyperbolic functions. We could have written Example 3.17 this way

$$\cos\left(\frac{\pi}{4}+i\right) = \frac{e+e^{-1}}{2\sqrt{2}} - i\frac{e-e^{-1}}{2\sqrt{2}} = \cos\frac{\pi}{4}\cosh 1 - i\sin\frac{\pi}{4}\sinh 1$$

Hyperbolic functions are a convenient short-cut, but never necessary; use or ignore as you like. All their properties can be derived from their relationship to exponential and trigonometric functions:

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cos iz, \qquad \sinh z = \frac{e^z - e^{-z}}{2} = -i\sin iz$$

For instance

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\sinh z = -i\frac{\mathrm{d}}{\mathrm{d}z}\sin iz = -i^2\cos iz = \cosh z, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z\\ &\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1\end{aligned}$$

Inverse Trigonometric Functions The standard trigonometric functions can also be inverted. As you might expect, this results in multi-valued functions.

Example 3.19. We find an expression for $\cos^{-1} z$ and compute its derivative.

$$w = \cos^{-1} z \iff z = \cos w \iff 2z = e^{iw} + e^{-iw} \iff (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

which is a quadratic equation in e^{iw}. Applying the quadratic formula, we see that

$$e^{iw} = \frac{2z + (4z^2-4)^{\frac{1}{2}}}{2} = z + i(1-z^2)^{\frac{1}{2}} \iff \cos^{-1} z = w = -i\log\left[z + i(1-z^2)^{\frac{1}{2}}\right]$$

A branch of $\cos^{-1}$ requires a choice of branches of both the square-root *and* the logarithm.
Inverse cosine is holomorphic since it is a composition of holomorphic functions. By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z &= \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}z}\left[z + i(1-z^2)^{\frac{1}{2}}\right] = \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\left[1 - \frac{iz}{(1-z^2)^{\frac{1}{2}}}\right]\\ &= \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\cdot\frac{(1-z^2)^{\frac{1}{2}} - iz}{(1-z^2)^{\frac{1}{2}}} = \frac{-1}{(1-z^2)^{\frac{1}{2}}}\end{aligned}$$

If we fix a branch of the square-root (and logarithm) so that $\cos^{-1} z$ is single-valued, this is necessarily the same branch that appears in the expression for the derivative.

Expressions such as these are not worth memorizing; instead you should become comfortable *deriving* them when needed. Here is the complete list.

Theorem 3.20. *The inverse sine, cosine and tangent functions are given by the expressions*

$$\sin^{-1} z = -i\log\left[iz + (1-z^2)^{1/2}\right] \qquad \cos^{-1} z = -i\log\left[z + i(1-z^2)^{1/2}\right]$$

$$\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$$

Once branches of the square-root and logarithm are chosen, these are holomorphic on their domains and have familiar derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin^{-1} z = \frac{1}{(1-z^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z = \frac{-1}{(1-z^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\tan^{-1} z = \frac{1}{1+z^2}$$

The branches of the square-root for the derivatives of inverse sine and cosine are identical to those used in the definitions of the original functions.

Examples 3.21. 1. To evaluate $\sin^{-1}\frac{1}{\sqrt{2}}$ as a complex number, first observe that

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log\left[\frac{i}{\sqrt{2}} \pm \sqrt{1-\frac{1}{2}}\right] = -i\log\frac{1}{\sqrt{2}}(i\pm 1)$$

Now evaluate the logarithms separately:

$$-i\log\frac{1}{\sqrt{2}}(i+1) = -i\log e^{\frac{\pi i}{4}} = -i\left[\frac{\pi i}{4} + 2\pi n i\right] = \frac{\pi}{4} + 2\pi n$$

$$-i\log\frac{1}{\sqrt{2}}(i-1) = -i\log e^{\frac{3\pi i}{4}} = -i\left[\frac{3\pi i}{4} + 2\pi n i\right] = \frac{3\pi}{4} + 2\pi n$$

The set of values $\sin^{-1}\frac{1}{\sqrt{2}}$ generated by all branches of the square-root and logarithm is precisely the set we'd have found by computing entirely within $\mathbb{R}$!

2. Of course, we can also evaluate inverse sines that would have no meaning in $\mathbb{R}$. For instance,

$$\begin{aligned}\sin^{-1} 7 &= -i\log[7i \pm \sqrt{-48}] = -i\log(7\pm 4\sqrt{3})i = -i\log(7\pm 4\sqrt{3})e^{\frac{\pi i}{2}}\\ &= -i\left[\ln(7\pm 4\sqrt{3}) + \frac{\pi i}{2} + 2\pi n i\right] = -i\ln(7\pm 4\sqrt{3}) + \frac{\pi}{2} + 2\pi n\end{aligned}$$

Note that $7 > 4\sqrt{3}$, so we are always taking natural log of a positive real number.

3. Compute $\tan^{-1}(i - 2\sqrt{3})$. First compute the required fraction in polar form:

$$\frac{i + (i-2\sqrt{3})}{i - (i-2\sqrt{3})} = \frac{2i - 2\sqrt{3}}{2\sqrt{3}} = -1 + \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}}e^{\frac{5\pi i}{6}}$$

It follows that

$$\tan^{-1}(i-2\sqrt{3}) = \frac{i}{2}\left(\ln\frac{2}{\sqrt{3}} + \frac{5\pi}{6}i - 2\pi n i\right) = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}} + \pi n : \quad n \in \mathbb{Z}$$

Choosing the principal value of the logarithm yields $-\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.

Exercises 3.3 1. Find the real and imaginary parts of $\sin i$, $\cos(1+i)$ and $\tan(2i\ln 5 + \frac{\pi}{2})$.

2. As a sanity check, if $w = -i\log\left[z + (z^2-1)^{\frac{1}{2}}\right]$, compute $\cos w = \frac{1}{2}(e^{iw} + e^{-iw})$ directly and verify that you obtain z, *irrespective* of which branches are chosen.

3. Using the real and imaginary parts of $\sin z$, directly verify that the Cauchy–Riemann equations are satisfied.

4. Prove the following double/multiple-angle formulæ using the definitions in this section:

   (a) $\cos 2z = 2\cos^2 z - 1$

   (b) $\sin(z-w) = \sin z\cos w - \cos z\sin w$

   (c) $\tan(z+w) = \dfrac{\tan z + \tan z}{1 - \tan z\tan w}$

5. Find all the values of $\tan^{-1}(1+i)$.

6. Solve the equation $\cos z = \sqrt{2}$ for z.

7. Recall Exercise 3: check explicitly that $\tan w = i - 2\sqrt{3}$ when $w = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.
   (*Hint: use* $\tan w = \frac{e^{2iw}-1}{i(e^{2iw}+1)}$. *Why…?*)

8. Suppose $z > 1$ is real. Prove that $\operatorname{Re}\sin^{-1} z = \frac{\pi}{2} + 2\pi n$ is independent of z. What is $\operatorname{Im}\sin^{-1} z$.

9. If the same branch of square-root is chosen in each case, prove that $\sin^{-1} z + \cos^{-1} z$ is constant.

10. Derive the expressions for $\tan^{-1} z$ and its derivative in Theorem 3.20.

11. (a) Given that $\cosh z = \cos(-iz)$, find an expression in terms of the complex logarithm for $\cosh^{-1} z$.

    (b) Using your answer to part (a), or otherwise, find all solutions to the equation $\cosh z = \sqrt{3}$.

    (c) Find an expression for the derivative of $\cosh^{-1} z$.

4 Integration

At first glance, integration might appear straightforward; surely we can write the following?

$$\int z^2 \, \mathrm{d}z = \frac{1}{3}z^3 + c$$

Recall that such a statement *in real analysis* would reflect the equivalence of two distinct concepts:

Anti-derivatives $\frac{1}{3}z^3$ is an anti-derivative of z^2

Definite Integrals The 'area' under the curve $y = z^2$ on the interval $[0, z]$ equals $\frac{1}{3}z^3$

The equivalence of these concepts is so amazing that we call it the *fundamental* theorem of calculus. While anti-derivatives make immediate sense in complex analysis, definite integrals are more delicate. For instance, what should we mean by

$$\int_{3+i}^{4i} z^2 \, \mathrm{d}z \; ?$$

A Riemann sum construction requires partitioning some *curve* joining the points $3+i$ and $4i$; but which curve? Does it matter? This question leads us to revisit the idea of a *contour* or *path integral.*

4.1 Functions of a Real Variable and Contour Integrals

We start by considering complex-valued functions of a *real* variable. Derivatives and definite integrals of such functions are built from those of their real and imaginary parts:

$$w'(t) = u'(t) + iv'(t), \qquad \int_a^b w(t) \, \mathrm{d}t = \int_a^b u(t) \, \mathrm{d}t + i \int_a^b v(t) \, \mathrm{d}t \qquad (a, b \text{ are } \textit{real} \text{ or } \pm\infty)$$

Examples 4.1. 1. If $w(t) = 5t^2 + it$, then

$$w'(t) = 10t + i, \qquad \int_1^2 w(t) \, \mathrm{d}t = \int_1^2 5t^2 \, \mathrm{d}t + i \int_1^2 t \, \mathrm{d}t = \frac{7}{3} + \frac{i}{2}$$

2. If $w(t) = t^2 + e^{it} = t^2 + \cos t + i \sin t$, then

$$w'(t) = 2t - \sin t + i \cos t, \qquad \int_0^{2\pi} w(t) \, \mathrm{d}t = \frac{1}{3}t^3 + \sin t - i \cos t \Big|_0^{2\pi} = \frac{8}{3}\pi^3$$

The natural extension of the (real) fundamental theorem is on show here; $\frac{1}{3}t^3 + \sin t - i \cos t$ is plainly an *anti-derivative* of $w(t)$.

The obvious calculus laws are easily verified by considering real and imaginary parts separately.

Lemma 4.2.
- *Linearity: if* $k \in \mathbb{C}$, *then* $\frac{\mathrm{d}}{\mathrm{d}t} kw(t) = kw'(t)$ *and* $\int_a^b kw(t) \, \mathrm{d}t = k \int_a^b w(t) \, \mathrm{d}t$
- *Product rule:* $\frac{\mathrm{d}}{\mathrm{d}t} w(t)z(t) = w'(t)z(t) + w(t)z'(t)$
- *Chain rule: If* $s(t)$ *is a* real *function then* $\frac{\mathrm{d}}{\mathrm{d}t} w\big(s(t)\big) = w'\big(s(t)\big)s'(t)$

Complex substitutions are a little more subtle, and benefit from a proof.

Lemma 4.3 (Complex Chain Rule). *Suppose $w(t) = F\big(z(t)\big)$ where*

- $z(t) = x(t) + iy(t)$ *is differentiable at t, and,*
- $F(z) = u(x,y) + iv(x,y)$ *is holomorphic at $z(t)$*

Then w is differentiable at t, and $w'(t) = F'\big(z(t)\big)z'(t)$.
If $z'(t)$ is integrable on $[a,b]$ and F is holomorphic on $z([a,b])$, we can put this in integral form

$$\int_a^b F'\big(z(t)\big)z'(t)\,\mathrm{d}t = F\big(z(b)\big) - F\big(z(a)\big)$$

Proof. Apply the multi-variable chain rule from real calculus and the Cauchy–Riemann equations:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} + i\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + i\left(\frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = (u_x + iv_x)\frac{\mathrm{d}x}{\mathrm{d}t} + i(v_y - iu_y)\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (u_x + iv_x)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right) \qquad \text{(Cauchy–Riemann)} \\ &= F'\big(z(t)\big)z'(t)\end{aligned}$$

■

Examples 4.4.
1. Let $w(t) = e^{t-it^2} = F\big(z(t)\big)$ where $F(z) = e^z$ and $z(t) = t - it^2$. Then

   $$w'(t) = e^{t-it^2}\frac{\mathrm{d}}{\mathrm{d}t}(t - it^2) = e^{t-it^2}(1 - 2it)$$

   Compare this with the method of Example 4.1 which gives the same result, if more slowly

   $$w'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(e^t\cos t^2 - ie^t\sin t^2) = e^t(\cos t^2 - 2t\sin t^2) - ie^t(\sin t^2 + 2t\cos t^2)$$

2. Since $w(t) = F\big(z(t)\big) = \frac{1}{10}(1 - t^2 + it)^{10}$ has $w'(t) = (i - 2t)(1 - t^2 + it)^9$, we see that

   $$\int_0^1 (i - 2t)(1 - t^2 + it)^9\,\mathrm{d}t = \frac{1}{10}(1 - t^2 + it)^{10}\Big|_0^1 = \frac{i^{10} - 1}{10} = -\frac{1}{5}$$

   This would be horrific if we had to multiply out to work with real and imaginary parts!

3. Sometimes the real and imaginary part approach is simply not tenable;

   $$\int_0^1 3it\sqrt{1 + it^2}\,\mathrm{d}t = (1 + it^2)^{3/2}\Big|_0^1 = (1 + i)^{3/2} - 1 = 2\sqrt{2}e^{\frac{3\pi i}{8}} - 1$$

   Everything is evaluated using the principal square root since $1 + it^2$ lies in the first quadrant.

While most of the basic rules of real calculus translate to complex-valued functions of a real variable, not everything goes through. Be particularly careful of existence results such as the mean value theorem which apply perfectly well to real and imaginary parts, but not to the whole…

Contours and Contour Integrals

We now turn our attention to integrating complex functions along curves. But what sort of curves?

Definition 4.5. A *smooth arc* is an oriented curve C in the complex plane for which there exists a *regular parametrization*; a differentiable function $z : [a,b] \to \mathbb{C}$ such that,

1. $z([a,b]) = C$ where $z(a)$ is the *start* of the curve and $z(b)$ is the *end*;
2. $z'(t)$ is *continuous* on $[a,b]$ and *non-zero* on (a,b).

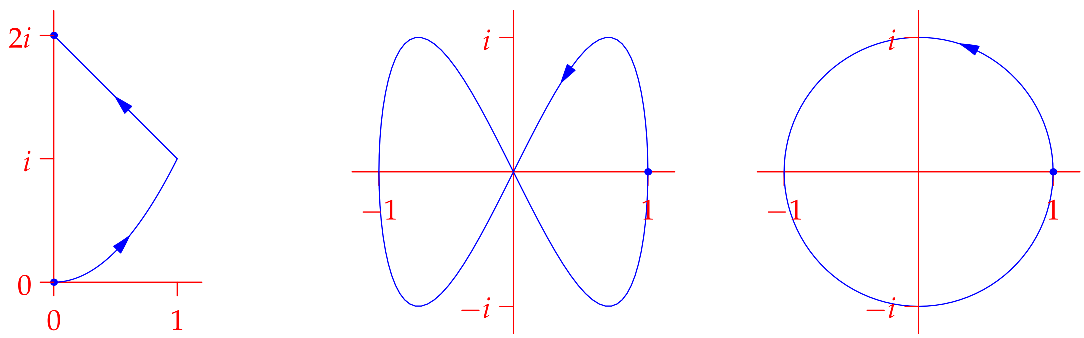
A *contour* is a piecewise smooth arc C consisting of finitely many smooth arcs joined end-to-end. A parametrization $z(t)$ of C is therefore continuous with piecewise continuous derivative.

If we *reverse the orientation* of a contour C, the resulting contour is labelled $-C$.

Additionally, we say that a contour is:

- *Closed* if it starts and ends at the same point, $z(a) = z(b)$;
- *Simple* if it does not cross itself (z is injective, $z(t) = z(s) \implies t = s$).
- *Positively oriented* if it is simple, closed and traversed counter-clockwise.

Examples 4.6. Here are three contours with explicit parametrizations:



Simple, piecewise

$$z = \begin{cases} t + it^2 & t \in [0,1] \\ 2 - t + it & t \in [1,2] \end{cases}$$

Closed, non-simple

$$z = \cos t + i \sin 2t,\ t \in [0, 2\pi]$$

Positively oriented

$$z = e^{it},\ t \in [0, 2\pi]$$

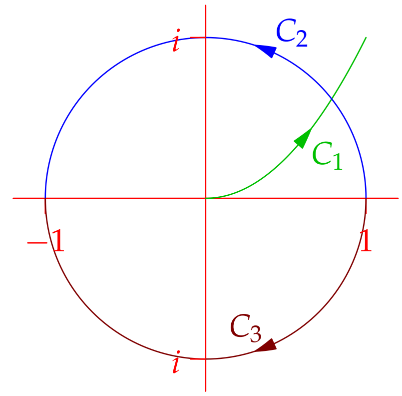
Definition 4.7. Let C be a contour parametrized by $z : [a,b] \to \mathbb{C}$ and suppose that $f(z)$ is a complex function defined on the range of z. The *contour integral* of $f(z)$ along C is

$$\int_C f = \int_C f(z)\,\mathrm{d}z := \int_a^b f(z(t))z'(t)\,\mathrm{d}t$$

This is often written $\oint_C f$ if C is positively oriented (simple and closed).

We plainly require the integrability of the function $f(z(t))z'(t)$ on the *real interval* $[a,b]$; typically we assume that this expression is piecewise continuous. As we'll see shortly, the choice of parametrization is irrelevant.

Examples 4.8. We evaluate several contour integrals.



1. For the contour C_1 parametrized by $z(t) = t + it^2$, $t \in [0, 1]$, we compute

$$\int_{C_1} z\,\mathrm{d}z = \int_0^1 z(t)z'(t)\,\mathrm{d}t = \int_0^1 (t + it^2)(1 + 2it)\,\mathrm{d}t$$
$$= \int_0^1 t - 2t^3 + 3it^2\,\mathrm{d}t = i$$

2. For the contour C_2 with $z(t) = e^{it}$ with $t \in [0, \pi]$,

$$\int_{C_2} \frac{1}{z}\,\mathrm{d}z = \int_0^\pi \frac{ie^{it}}{e^{it}}\,\mathrm{d}t = \pi i$$

$$\int_{C_2} z^2 + 1\,\mathrm{d}z = \int_0^\pi (e^{2it} + 1)ie^{it}\,\mathrm{d}t = \frac{i}{3i}e^{3it} + \frac{i}{i}e^{it}\Big|_0^\pi = \frac{1}{3}(e^{3\pi i} - 1) + e^{\pi i} - 1 = -\frac{8}{3}$$

3. We compute the same integrals as the previous example, but over the lower semi-circle C_3 parametrized by $z(t) = e^{-it}$ with $t \in [0, \pi]$. This time

$$\int_{C_3} \frac{1}{z}\,\mathrm{d}z = \int_0^\pi \frac{-ie^{-it}}{e^{-it}}\,\mathrm{d}t = -\pi i$$
$$\int_{C_3} z^2 + 1\,\mathrm{d}z = \int_0^\pi (e^{-2it} + 1)(-ie^{-it})\,\mathrm{d}t = \frac{-i}{-3i}e^{-3it} + \frac{-i}{-i}e^{-it}\Big|_0^\pi = -\frac{8}{3}$$

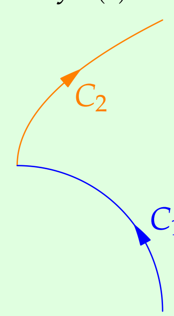
The *sign* of one integral changed but the other did not! We'll return to this problem shortly...

Before considering more examples we develop some of the basic properties of contour integrals. Several are immediate from our earlier discussion, for instance linearity:

$$\int_C \big(af(z) + bg(z)\big)\,\mathrm{d}z = a\int_C f(z)\,\mathrm{d}z + b\int_C g(z)\,\mathrm{d}z$$

Of more importance are the following:

Theorem 4.9 (Basic rules for contour integrals). *Suppose C is a contour parametrized by $z(t)$.*



1. *If $C = C_1 \cup C_2$, where C_1 and C_2 are contours such that the end of the first is the start of the second, then*

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z$$

2. *$\int_C f$ is independent of (orientation-preserving) parametrization.*

3. *Reversing orientation changes the sign of the integral: $\int_{-C} f = -\int_C f$.*

Example 4.10. By the previous example and part 3 of the Theorem, if C is the unit circle centered at the origin, then $\oint_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$. This can also be verified directly.

Proof. Part 1 follows from the well-known property $\int_a^b = \int_a^c + \int_c^b$ of real integrals. Armed with this, it is enough to check the other parts for a single smooth arc C.

2. Suppose $z : [a, b] \to \mathbb{C}$ and $w : [\alpha, \beta] \to \mathbb{C}$ are parametrizations of C. Then $w\big(s(t)\big) = z(t)$ for some continuously differentiable s with *positive* derivative. Now compute,

$$\begin{aligned} &z'(t) = w\big(s(t)\big)s'(t), \qquad s(a) = \alpha, \quad s(b) = \beta, \quad \text{from which,} \\ &\int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t = \int_a^b f\big(w(s(t))\big)w'(s(t))\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_{s(a)}^{s(b)} f\big(w(s)\big)w'(s)\,\mathrm{d}s \\ &\qquad\qquad\qquad\qquad = \int_\alpha^\beta f\big(w(s)\big)w'(s)\,\mathrm{d}s \end{aligned}$$

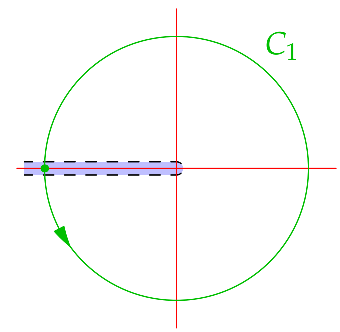
3. This is almost identical, except that $-C$ requires a reparametrization $w(s)$ with $s' < 0$, $s(a) = \beta$ and $s(b) = \alpha$. The upshot is that the limits flip on the final integral:

$$\int_C f = \int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t = \int_\beta^\alpha f\big(w(s)\big)w'(s)\,\mathrm{d}s = -\int_\alpha^\beta f\big(w(s)\big)w'(s)\,\mathrm{d}s = -\int_{-C} f \qquad \blacksquare$$

Contour Integrals of multi-valued functions If a function is multi-valued, we must specify a *branch* before integrating. It is acceptable to have the contour start and/or finish on the branch cut.[8]

Examples 4.11. 1. Compute $\oint_{C_1} z^{1/2}\,\mathrm{d}z$ over the unit circle using the principal square root.

Since the question specifies the principal branch, we must must start and finish the contour at $z = -1$. Parametrize via $z(t) = e^{it}$ where $t \in (-\pi, \pi)$. Since



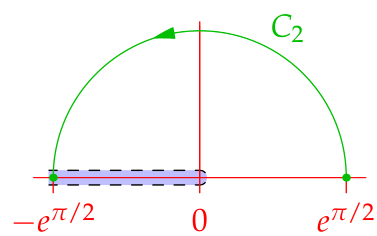
$$\sqrt{z(t)}\,z'(t) = e^{\frac{it}{2}}ie^{it} = ie^{\frac{3it}{2}}$$

is continuous on $[-\pi, \pi]$, we compute

$$\int_{C_1} z^{1/2}\,\mathrm{d}z = \int_{-\pi}^{\pi} ie^{\frac{3it}{2}}\,\mathrm{d}t = \frac{2}{3}e^{\frac{3it}{2}}\Big|_{-\pi}^{\pi} = \frac{2}{3}(e^{\frac{3\pi i}{2}} - e^{-\frac{3\pi i}{2}}) = -\frac{4i}{3}$$

2. Find $\int_{C_2} z^i\,\mathrm{d}z$ over the semi-circle shown using P. V. z^i.

Let $z(t) = e^{\frac{\pi}{2}}e^{it}$ where $t \in [0, \pi]$. Since



$$z^iz' = e^{i\,\mathrm{Log}\,z}z' = \exp\big(i(\tfrac{\pi}{2} + it)\big)\,e^{\frac{\pi}{2}}ie^{it} = -e^{\frac{\pi}{2}}e^{(-1+i)t}$$

is continuous when $0 \le t \le \pi$, we have

$$\int_{C_2} z^i\,\mathrm{d}z = \int_0^\pi -e^{\frac{\pi}{2}}e^{(-1+i)t}\,\mathrm{d}t = \frac{e^{\frac{\pi}{2}}}{1-i}\Big(e^{(-1+i)\pi} - 1\Big) = -\frac{1+i}{2}(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}})$$

Warning! Things are more difficult if we permit a contour to *cross* a branch cut since we must then work with multiple branches simultaneously. We won't do this.

[8]We extend the integrand continuously: recall that if $g(t)$ is *uniformly continuous* on a bounded open interval (a, b), then it has a continuous extension to the closed interval $[a, b]$ and we can therefore define $\int_a^b g(t)\,\mathrm{d}t$.

Exercises 4.1 1. Evaluate the derivatives and integrals:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\sin t + i\sqrt{t}\right]$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(i+t^3)^2$ (c) $\displaystyle\int_0^1 e^{\pi i t}\,\mathrm{d}t$

(d) $\displaystyle\int_0^{\pi/2} e^{it}(1+e^{it})^2\,\mathrm{d}t$ (e) $\displaystyle\int_0^1 (i-t)^6\,\mathrm{d}t$ (f) $\displaystyle\int_0^{\pi} (i-1)\cos\big((1+i)t\big)\,\mathrm{d}t$

2. Show that if $m, n \in \mathbb{Z}$, then
$$\int_0^{2\pi} e^{imt}e^{-int}\,\mathrm{d}t = 2\pi\delta_{mn}$$
where $\delta_{mn} = 1$ when $m = n$ and 0 otherwise. Do this two ways:
   (a) Using the exponential law and the chain rule/substitution.
   (b) By multiplying out and working with real and imaginary parts.

3. Use the chain rule to evaluate the integral $\int_0^x e^{(1+i)t}\,\mathrm{d}t$.
Hence find both $\int_0^x e^t \cos t\,\mathrm{d}t$ and $\int_0^x e^t \sin t\,\mathrm{d}t$ *without* using integration by parts.

4. Prove the product rule for functions of a real variable:
$$\frac{\mathrm{d}}{\mathrm{d}t}(wz) = w'z + wz'$$
(*Hint: let* $w(t) = u(t) + iv(t),\ z(t) = x(t) + iy(t)$ *and multiply out...*)

5. Check that the mean value theorem fails for the function $w(t) = \sqrt{t} + it^2$ on the interval $[0, 1]$.
That is, there is no $\xi \in (0, 1)$ for which $w'(\xi) = \frac{w(1)-w(0)}{1-0}$.

6. Justify the integral form of the complex chain rule by considering the real and imaginary parts of F. What facts from real calculus are you using?

7. Evaluate each contour integral $\int_C f(z)\,\mathrm{d}z$ by explicitly parametrizing C:
   (a) $f(z) = z^2$; C is the straight line from $z = 1$ to $z = i$.
   (b) $f(z) = z$; C consists of the straight lines joining $z = 1$ to $1+i$ to $-1+i$ to -1.
   (c) $f(z) = \operatorname{Log} z$; C is the circular arc of radius 3 centered at the origin, oriented counter-clockwise from $-3i$ to $3i$.

8. Explicitly check that $\int_C z\,\mathrm{d}z = \frac{1}{2}(B^2 - A^2)$ along the straight line joining A and B.
(*Hint: the line can be parametrized by* $z(t) = (1-t)A + tB$ *where* $t \in [0, 1]$)

9. Let $n \in \mathbb{Z}$ and let C_0 be the positively oriented circle centered at z_0 with radius $R > 0$. Explicitly parametrize this circle to show that
$$\oint_{C_0} (z - z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

10. Suppose $z : [a,b] \to \mathbb{C}$ is a regular parametrization of a smooth arc C. Then the *arc-length* of the curve is the integral of the *speed* of the parametrization:

$$L = \int_a^b \left|z'(t)\right| \mathrm{d}t$$

   (a) Compute the arc-length of the circle of radius R centered at the origin.

   (b) Compute the arc-length of the simple piecewise curve in Example 4.6.
   (*This requires a tough substitution; perhaps look up a table of integrals…*)

   (c) By commenting on the proof of Theorem 4.9, explain why a reparametrization of C does not change the arc-length.

   (d) Let $s(t) = \int_a^t |z'(\tau)| \,\mathrm{d}\tau$ be the arc-length as a *function* of $t \in [a,b]$. Consider a new parametrization $w(s) = z(t(s))$, where $t(s)$ is the inverse function of $s(t)$.
   Prove that $\left|\frac{\mathrm{d}w}{\mathrm{d}s}\right| = 1$.
   (*This proves that every smooth arc has a unit-speed parametrization*)

The last three questions elaborate a little on the approach in Examples 4.11.

11. (a) Compute the integral of the principal value of $z^{1/3}$ around the positively oriented unit circle starting and finishing at $z = -1$.

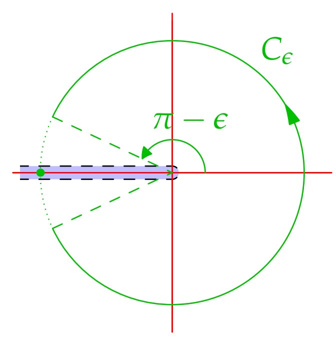
    (b) Now consider the branch $z^{1/3} = \exp(\frac{1}{3}\log z)$ where $\arg z \in (0, 2\pi)$. Integrate this around the positively oriented unit circle starting and finishing at $z = 1$. What do you observe?

12. Compute $\oint_C z^{1/2}\,\mathrm{d}z$ where we take the α-branch $z^{1/2} = \exp(\frac{1}{2}\log z)$ with $\arg z \in (\alpha, \alpha + 2\pi)$ and the unit circle C traced from angle α to $\alpha + 2\pi$.

13. Let $\epsilon > 0$ be small and suppose that C_ϵ is the circular arc of radius 1 centered at the origin, traversed counter-clockwise from angle $-\pi + \epsilon$ to $\pi - \epsilon$.
By parametrizing C_ϵ, explicitly evaluate $\int_{C_\epsilon} \sqrt{z}\,\mathrm{d}z$ and verify that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \sqrt{z}\,\mathrm{d}z = -\frac{4i}{3}$$

is the value obtained previously for $\oint_C \sqrt{z}\,\mathrm{d}z$.



4.2 Path-independence, the Fundamental Theorem & Integral Estimation

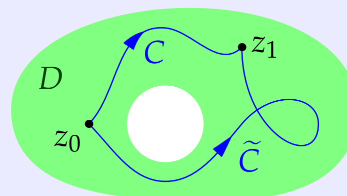
We start by revisiting an earlier example.

Example (4.8, parts 2 & 3). Let $F(z) = \frac{1}{3}z^3 + z$ and observe that $F'(z) = z^2 + 1$. If $z : [a, b] \to \mathbb{C}$ parametrizes a smooth arc C such that $z(a) = 1$ and $z(b) = -1$, then Lemma 4.3 shows that

$$\begin{aligned} \int_C z^2 + 1 \, \mathrm{d}z &= \int_a^b \big(z(t)^2 + 1\big) z'(t) \, \mathrm{d}t = \int_a^b F'\big(z(t)\big) z'(t) \, \mathrm{d}z = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F\big(z(t)\big) \, \mathrm{d}t \\ &= F\big(z(b)\big) - F\big(z(b)\big) = -\frac{4}{3} - \frac{4}{3} = -\frac{8}{3} \end{aligned}$$

The contour integral is independent of the choice of arc C!

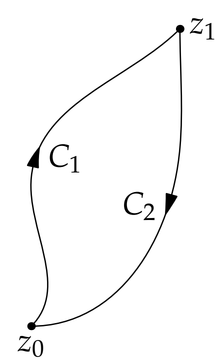
Definition 4.12. Suppose $f : D \to \mathbb{C}$ where D is path-connected. We say that a contour integral $\int_C f$ is *path-independent* if its value depends only on the *endpoints* of C and not on the how the contour travels between these points.
Otherwise said, if $\widetilde{C}$ is any other contour with the same endpoints, then $\int_C f = \int_{\widetilde{C}} f$.
In such a situation it is permissible to write $\int_C f = \int_{z_0}^{z_1} f$.



Before tackling the main result, we tidy up a connection between path independence and closed curves.

Lemma 4.13. *Every contour integral $\int_C f(z) \, \mathrm{d}z$ is path-independent if and only if $\int_C f(z) \, \mathrm{d}z = 0$ around every closed contour.*

Proof. ($\Rightarrow$) Assume all integrals are path independent and let C be a closed contour. Choose any points $z_0, z_1 \in C$ and decompose $C = C_1 \cup C_2$ into two contours from z_0 to z_1 and back again. Then,



$$\begin{aligned} \int_C f(z) \, \mathrm{d}z &= \int_{C_1} f(z) \, \mathrm{d}z + \int_{C_2} f(z) \, \mathrm{d}z && \text{(Theorem 4.9, part 1)} \\ &= \int_{C_1} f(z) \, \mathrm{d}z - \int_{-C_2} f(z) \, \mathrm{d}z && \text{(Theorem 4.9, part 3)} \\ &= 0 \end{aligned}$$

since C_1 and $-C_2$ share the same endpoints.
($\Leftarrow$) Conversely, suppose $\int_C f(z) \, \mathrm{d}z = 0$ round any closed contour. Suppose $C_1, -C_2$ are any contours with the same endpoints z_0, z_1, then $C = C_1 \cup C_2$ is a closed contour and the above calculation shows that $\int_{C_1} f(z) \, \mathrm{d}z = \int_{-C_2} f(z) \, \mathrm{d}z$. ∎

The critical observation in the example was that the integrand $f(z) = z^2 + 1$ had an *anti-derivative* $F(z) = \frac{1}{3}z^2 + z$; this demonstrated path-independence and facilitated the easy computation of the integral. This should seem very familiar. . .

Theorem 4.14 (Fundamental Theorem). *Let f be continuous on an open domain D. Then (on D),*

$$f \text{ has an anti-derivative } \iff \text{ all contour integrals of } f \text{ are path-independent}$$

In such situations, $\int_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0)$ where $F(z)$ is any anti-derivative of $f(z)$.

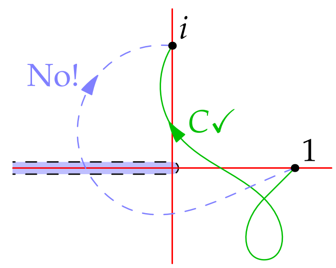
The assumptions of openness and continuity are necessary both because f has an anti-derivative, and because they'll be used explicitly in the proof. The ($\Rightarrow$) direction is sometimes known as the Fundamental Theorem of Line Integrals, especially in multi-variable calculus.
Since the proof requires a little work, we postpone it until after some examples.

Examples 4.15. 1. $f(z) = (z+i)^3$ has anti-derivative $F(z) = \frac{1}{4}(z+i)^4$. It follows that

$$\int_0^{1-2i} (z+i)^4 = \frac{1}{4}(z+i)^4\Big|_0^{1-2i} = \frac{1}{4}\left[(1-i)^4 - i^4\right] = \frac{1}{4}(-4-1) = -\frac{5}{4}$$

regardless of the contour used to travel from $z = 0$ to $1-2i$.

2. Let $f(z) = z^{1/2}$ where we take the principal value. This has anti-derivative $F(z) = \frac{2}{3}z^{3/2}$. If C is any contour joining $z_0 = 1$ and $z_1 = i$, which does not cross the branch cut, then



$$\int_C z^{1/2}\,\mathrm{d}z = \frac{2}{3}z^{3/2}\Big|_1^i = \frac{2}{3}(i^{3/2} - 1^{3/2}) = \frac{2}{3}\left(e^{\frac{3\pi i}{4}} - 1\right)$$

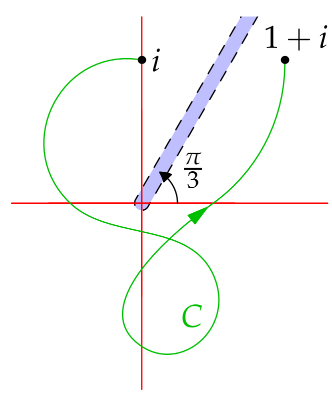
It is important to use the *same (principal) branch* to evaluate the anti-derivative: since $z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z}$, we also have $z^{3/2} = e^{\frac{3}{2}\operatorname{Log} z}$.

3. Since $f(z) = \sin z$ has an anti-derivative $F(z) = -\cos z$ valid at all $z \in \mathbb{C}$, we see that $\int_C \sin z\,\mathrm{d}z = 0$ round any closed curve C.

4. By Example 4.10, $\oint_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$ round the unit circle. By the fundamental theorem, $f(z) = \frac{1}{z}$ cannot have an anti-derivative on any domain containing the circle. But this is obvious from our earlier discussion of the logarithm, any branch of which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z}$$

Since the logarithm cannot be made single-valued on any path encircling the origin, no anti-derivative of $f(z) = \frac{1}{z}$ exists on any domain containing the circle C.
We can, however, use the fundamental theorem to evaluate $\int_C \frac{1}{z}\,\mathrm{d}z$ over any contour staying within a single branch of the logarithm. In the picture, given the contour C, we choose the branch cut shown and evaluate



$$\begin{aligned}\int_C \frac{1}{z}\,\mathrm{d}z &= \log(1+i) - \log i = \log\sqrt{2}e^{\frac{\pi i}{4}} - \log e^{-\frac{3\pi i}{2}} \\ &= \ln\sqrt{2} + \frac{\pi i}{4} - \frac{3\pi i}{2} = \frac{1}{4}(\ln 4 + 7\pi i)\end{aligned}$$

Note how the arguments were chosen so that $-\frac{5\pi}{3} < \theta < \frac{\pi}{3}$.

Proving the Fundamental Theorem

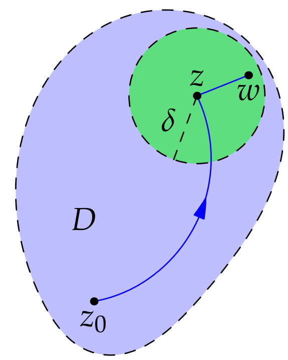
The forward direction is relatively straightforward. Before reading the converse however, you may find it helpful to review part I of the fundamental theorem from real analysis, namely

$$f \text{ continuous} \implies \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

We proceed by mimicking the original argument though there are some extra subtleties.

Proof. ($\Rightarrow$) The result is already true for every smooth arc by Lemma 4.3. Now let $C = C_1 \cup \cdots \cup C_n$ be a contour where the smooth arcs C_k are arranged in order with start/end points z_{k-1}, z_k. By Theorem 4.9, part 1, we see that

$$\int_C f = \sum_{k=1}^{n} \int_{C_k} f = \sum_{k=1}^{n} [F(z_k) - F(z_{k-1})] = F(z_n) - F(z_0)$$



($\Leftarrow$) Fix $z_0 \in D$ and *define*

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta$$

where the integral is taken along *any* curve joining z_0 to z. This is well-defined by the assumption of path-independence. To complete the proof, we need only show that $\lim_{w\to z} \frac{F(w)-F(z)}{w-z} = f(z)$ on D.
Fix $z \in D$ and let $\epsilon > 0$ be given. Since f is continuous and D open,

$$\exists \delta > 0 \text{ such that } |\zeta - z| < \delta \implies \zeta \in D \text{ and } |f(\zeta) - f(z)| < \frac{\epsilon}{2}$$

Let $w \in D$ be such that $|w - z| < \delta$. Evaluating along any curve joining z, w we obtain

$$F(w) - F(z) = \int_z^w f(\zeta)\,\mathrm{d}\zeta$$

We may therefore *choose* the straight line segment. The proof is almost complete:

$$\begin{aligned}\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| &= \left|\frac{1}{w-z}\int_z^w f(\zeta) - f(z)\,\mathrm{d}\zeta\right| = \frac{1}{|w-z|}\left|\int_z^w f(\zeta) - f(z)\,\mathrm{d}\zeta\right| \\ &\leq \frac{1}{|w-z|}|w-z|\frac{\epsilon}{2} < \epsilon\end{aligned}$$

∎

The second last inequality should give you pause. We are tempted to argue that

$$\left|\int_z^w f(\zeta) - f(z)\,\mathrm{d}\zeta\right| \leq \int_z^w |f(\zeta) - f(z)|\,\mathrm{d}\zeta \leq \int_z^w \frac{\epsilon}{2}\,\mathrm{d}\zeta = |w-z|\frac{\epsilon}{2}$$

While this makes perfect sense in *real* analysis, it is utter nonsense in *complex* analysis, since

$$\int_z^w |f(\zeta) - f(z)|\,\mathrm{d}\zeta = \int_a^b \big|f\big(\zeta(t)\big) - f(z)\big|\,\zeta'(t)\,\mathrm{d}t$$

need not be a real number! We had better tidy this up.

Theorem 4.16 (Integral Estimation). 1. *Suppose $w : [a,b] \to \mathbb{C}$ is piecewise continuous. Then*

$$\left| \int_a^b w(t)\,\mathrm{d}t \right| \le \int_a^b |w(t)|\,\mathrm{d}t$$

2. *Suppose C is a contour with length L, and let f be piecewise continuous on C. Then $|f(z)|$ is bounded by some $M \in \mathbb{R}_0^+$ on C, and*

$$\left| \int_C f(z)\,\mathrm{d}z \right| \le ML$$

Part 2 justifies the suspect inequality in the proof of the Fundamental Theorem: $f(\zeta) - f(z)$ is bounded by $M = \frac{\epsilon}{2}$ and the path of integration is the straight line of length $L = |w - z|$.

Proof. 1. Let $\int_a^b w(t)\,\mathrm{d}t = re^{i\theta}$. Since θ is constant, observe that $r = \int_a^b e^{-i\theta} w(t)\,\mathrm{d}t$ is *real*. In particular,

$$r = \int_a^b \mathrm{Re}\big(e^{-i\theta} w(t)\big) + i\,\mathrm{Im}\big(e^{-i\theta} w(t)\big)\,\mathrm{d}t = \int_a^b \mathrm{Re}\big(e^{-i\theta} w(t)\big)\,\mathrm{d}t$$

Appealing to $\mathrm{Re}\, z \le |z|$, we see that

$$\left| \int_a^b w(t)\,\mathrm{d}t \right| = r = \int_a^b \mathrm{Re}(e^{-i\theta} w(t))\,\mathrm{d}t \le \int_a^b \left| e^{-i\theta} w(t) \right|\,\mathrm{d}t = \int_a^b |w(t)|\,\mathrm{d}t$$

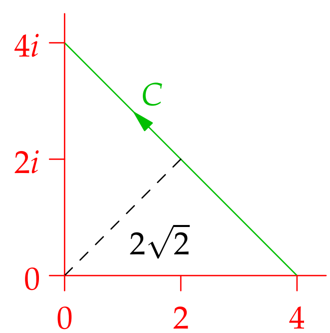
2. Parametrize the contour integral by $z : [a,b] \to \mathbb{C}$. Since $f(z(t))$ is piecewise continuous on a closed bounded interval $[a,b]$, it is bounded and thus M exists. But now,

$$\left| \int_C f(z)\,\mathrm{d}z \right| = \left| \int_a^b f\big(z(t)\big) z'(t)\,\mathrm{d}t \right| \le \int_a^b \left| f\big(z(t)\big) z'(t) \right| \mathrm{d}t \le \int_a^b M \left| z'(t) \right| \mathrm{d}t = ML$$

■

While the computation of arc-length is usually impractical, for circular and straight arcs it is straightforward. Indeed the estimation of integrals around both large and small circles will prove crucial for the rest of the course.

Examples 4.17. 1. On the straight line C joining $z = 4$ to $4i$, we see that



$$2\sqrt{2} = |2(1+i)| \le |z| \le 4 \implies |z+1| \le |z| + 1 \le 5$$

By the extended triangle inequality,

$$\left| z^4 + 4 \right| \ge \left| |z|^4 - 4 \right| \ge 60$$

Since C has length $4\sqrt{2}$, it follows that

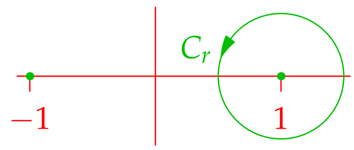
$$\left| \int_C \frac{z+1}{z^4+4}\,\mathrm{d}z \right| \le \frac{20\sqrt{2}}{60} = \frac{\sqrt{2}}{3}$$

2. For the function in the previous example, consider the circle C_R with radius $R > \sqrt[4]{4}$ centered at the origin. On C_R, we have $|z|^4 > 4$ whence the triangle inequality tells us that

$$\left|\frac{z+1}{z^4+4}\right| \le \frac{|z|+1}{|z^4|-4} = \frac{R+1}{R^4-4} \implies \left|\oint_{C_R} \frac{z+1}{z^4+4}\,\mathrm{d}z\right| \le \frac{2\pi R(R+1)}{R^4-4}$$

In particular, this approaches zero as $R \to \infty$.

3. Let C_r be the circle with radius $r < 2$ centered at $z = 1$. Then



$$|z-1| = r \text{ and } |z+1| = |2-(1-z)| \ge 2-r$$

from which

$$\left|\oint_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z\right| \le \frac{2\pi r}{(2-r)r} = \frac{2\pi}{2-r} \xrightarrow[r\to 0]{} \pi$$

It will shortly be verified that $\oint_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z = -\pi i$ for *any* $r < 2$.

Exercises 4.2 1. Evaluate each contour integral $\int_C f(z)\,\mathrm{d}z$ using the fundamental theorem:

(a) $f(z) = z^5$ where C is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = \dfrac{1}{z}$ where C is the pair of straight lines from $z = 1$ to $-1-i$ to $-i$.

(c) $f(z) = iz\sin z^2$, where C is the straight line from the origin to $z = i\sqrt{\pi}$.

(d) $f(z) = \dfrac{1}{1+z^2}$ where C is the straight line from $z = 1$ to $2+i$.

(e) $f(z) = \dfrac{1}{\sqrt{z}}$ where C is any path $z(t)$ with $\operatorname{Re} z > 0$ joining $z = 1+i$ and $z = 4$.

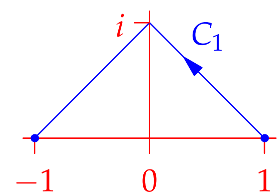
(f) $f(z) = \text{P.V.}\, z^{-1-2i}$ along the quarter circle $z(t) = e^{it}$ where $0 \le t \le \frac{\pi}{2}$.

2. Let $n \in \mathbb{N}_0$. Prove that for every contour C from z_0 to z_1

$$\int_C z^n\,\mathrm{d}z = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$

3. If C is a closed curve not containing z_0, and $n \in \mathbb{Z} \setminus \{0\}$, prove that $\int_C (z-z_0)^{n-1}\,\mathrm{d}z = 0$.

4. Let $f(z) = z^{1/3}$ be the branch where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Evaluate the integral $\int_{C_1} f(z)\,\mathrm{d}z$ where C_1 is the curve shown in the picture.



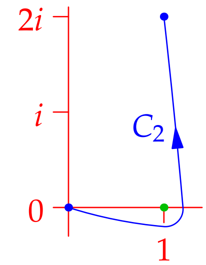
5. Evaluate $\int_{2i}^{1+i} \operatorname{Log} z\,\mathrm{d}z$ where the curve C lies in the upper half-plane. (*Hint: use integration by parts*)

6. Evaluate $\int_{-1}^{1} z^{2-i}\,\mathrm{d}z$ where z^{2-i} is the principal branch, and the integral is over any contour which, apart from its endpoints, lies above the real axis.

7. (a) Find an anti-derivative of $\frac{1}{1-z^2}$ on the domain $D = \mathbb{C}$ except for the real axis where $|z| \geq 1$. Evaluate $\int_0^{1+2i} \frac{1}{1-z^2}\,\mathrm{d}z$ along any curve in D.

   (b) Prove directly that $\oint_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z = -i\pi$ where C_r is the circle of radius $r < 2$ centered at $z = 1$.
   (*Hint: use the anti-derivative and the circle starting and finishing on the branch cut at $z = 1 + r$*)

   (c) Evaluate $\int_0^{1+2i} \frac{1}{1-z^2}\,\mathrm{d}z$ along a curve C_2 looping to the *right* of $z = 1$ as shown in the picture. Compare your answer with parts (a) and (b).



8. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $2i$. Without evaluating the integral, show that

   (a) $\left|\int_C \frac{z+4}{z^3-1}\,\mathrm{d}z\right| \leq \frac{6\pi}{7}$ (b) $\left|\int_C \frac{\mathrm{d}z}{z^2-1}\right| \leq \frac{\pi}{3}$

9. If C is the straight line joining the origin to $1 + i$, show that $\left|\int_C z^3 e^{2iz}\,\mathrm{d}z\right| \leq 4$

10. If C is the boundary of the triangle with vertices $0, 3i$ and -4, prove that $\left|\oint_C (e^z - \overline{z})\,\mathrm{d}z\right| \leq 60$
(*Hint: show that* $|e^z - \overline{z}| \leq e^x + \sqrt{x^2 + y^2} \ldots$)

11. Let C_R be the circle of radius $R > 1$ centered at the origin. Prove that

$$\left|\oint_{C_R} \frac{\operatorname{Log} z}{z^2}\,\mathrm{d}z\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

and thus prove that $\lim_{R\to\infty} \oint_{C_R} \frac{\operatorname{Log} z}{z^2}\,\mathrm{d}z = 0$.

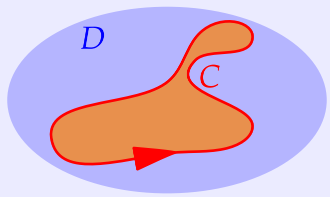
4.3 The Cauchy–Goursat Theorem and Cauchy's Integral Formula

We begin to extend the fundamental theorem with the goal of more fully understanding and evaluating holomorphic functions. We first require another piece of topology.

Definition 4.18. Suppose D is a connected region of the plane.

We say that D is *simply-connected* if every closed contour in D can by shrunk smoothly to a point without any part leaving D.

Otherwise said, if C is a simple closed contour in D then everything *inside* C also lies in D.



Theorem 4.19 (Cauchy–Goursat, version 1). *Suppose C is a closed curve in a simply-connected region D. If f is holomorphic on D, then $\int_C f(z)\,\mathrm{d}z = 0$.*

We prove a basic version where the real and imaginary parts of f are assumed to have continuous partial derivatives; we then invoke *Green's Theorem*, which you should have seen in a multi-variable calculus course. A proof without the restriction is significantly longer and more challenging.

Lemma 4.20 (Green's Theorem). *Suppose D is a closed bounded simply-connected domain with boundary C and that $P, Q : D \to \mathbb{R}$ have continuous partial derivatives. Then*

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

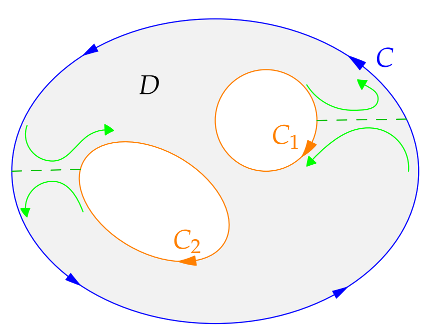
Sketch Proof of Cauchy–Goursat. Suppose $f(z) = u + iv$ has continuous partial derivatives and parametrize C: assume WLOG that this is positively oriented. Then

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t = \int_a^b \big(u(z(t)) + iv(z(t))\big)\big(x'(t) + iy'(t)\big)\,\mathrm{d}t \\ &= \int_a^b ux' - vy'\,\mathrm{d}t + i\int_a^b vx' + uy'\,\mathrm{d}t \\ &= \oint_C u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_C v\,\mathrm{d}x + u\,\mathrm{d}y \qquad \text{(definition of line integral in } \mathbb{R}^2\text{)} \\ &= \iint_D -v_x - u_y\,\mathrm{d}A + i\iint_D u_y - v_x\,\mathrm{d}A = 0\end{aligned}$$

by Green's Theorem and the Cauchy–Riemann equations. ∎

We now perform a sneaky trick. Starting with a simple closed curve C, remove from its interior the regions within simple closed non-intersecting contours $C_1, \ldots, C_k$. We orient these clockwise so that the interior region D lies on the curves' *left*.

By cutting, we can join the boundary curves into a single curve at the cost of traversing each cut twice in opposite directions. Stretching credulity slightly, we have a new simple closed curve to which we can apply Cauchy–Goursat…



Corollary 4.21 (Cauchy–Goursat, version 2). *Suppose C is a simple closed contour, oriented counter-clockwise. Let $C_1, \ldots, C_k$ be non-intersecting simple closed curves in the interior of C, oriented clockwise. If $f(z)$ is holomorphic on the region between and including C and the interior boundaries $C_1, \ldots, C_k$, then*

$$\int_C f(z)\,\mathrm{d}z + \sum_{j=1}^{k} \int_{C_j} f(z)\,\mathrm{d}z = 0$$

The power of this result lies in how it allows us to *compare* integrals around different contours.

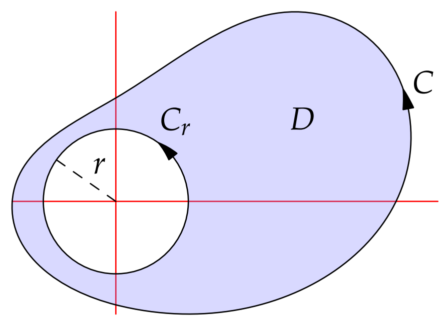
Corollary 4.22. *Suppose C_1, C_2 are non-intersecting positively oriented simple closed contours. If f is holomorphic on the region between and including the curves, then*

$$\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z$$

Examples 4.23. 1. Recall Example 4.17.3. Since $f(z) = \frac{1}{1-z^2}$ is holomorphic on and between any two circles C_r centered at $z = 1$ and with radius $r < 2$, we see that the value of $\int_{C_r} \frac{1}{1-z^2}\,\mathrm{d}z$ is independent of the radius r. At the moment, we still need to evaluate on at least one such circle to obtain the explicit value $-i\pi$.

2. We compute the integral of $f(z) = \frac{1}{z}$ round *any* simple closed contour C staying away from the origin.

   - If the origin is *exterior* to C, then $f(z)$ is holomorphic on and inside C, whence $\oint_C \frac{1}{z}\,\mathrm{d}z = 0$.
   - If the origin is *interior* to C, then there is some minimum distance d of C to the origin. Choose any circle C_r with radius $r < d$ centered at the origin. Since $f(z)$ is holomorphic on the region D between and including C and C_r, we conclude that



$$\oint_C \frac{1}{z}\,\mathrm{d}z = \oint_{C_r} \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{re^{i\theta}} rie^{i\theta}\,\mathrm{d}\theta = 2\pi i$$

   More generally, if C is a positively-oriented simple closed contour not containing z_0, then

$$\oint_C \frac{1}{z-z_0}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } z_0 \text{ lies inside } C \\ 0 & \text{if } z_0 \text{ lies outside } C \end{cases}$$

   This example generalizes to one of the central results of complex analysis…

Theorem 4.24 (Cauchy's Integral Formula). *Suppose $f(z)$ is holomorphic everywhere on and inside a simple closed contour C. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

Moreover, f is infinitely differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

Example 4.25. If $f(z)$ is a polynomial, we can check the integral formula explicitly by appealing to the Cauchy–Goursat Theorem and Exercise 4.1.9: if C is a simple closed contour encircling z_0, then

$$\oint_C (z-z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

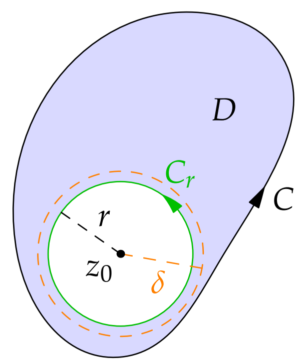
For instance, if $f(z) = 3z^2+2$ and $z_0 = 0$, then

$$\frac{1}{2\pi i}\oint_C \frac{3z^2+2}{z}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_C 3z\,\mathrm{d}z + \frac{1}{2\pi i}\oint_C \frac{2}{z}\,\mathrm{d}z = 0+2 = f(0)$$

$$\frac{2!}{2\pi i}\oint_C \frac{3z^2+2}{z^3}\,\mathrm{d}z = \frac{2}{2\pi i}\oint_C \frac{3}{z}\,\mathrm{d}z + \frac{2}{2\pi i}\oint_C \frac{2}{z^3}\,\mathrm{d}z = 6+0 = f''(0)$$

Proof of the basic Integral formula. Denote by D the open region interior to C. Let $\epsilon > 0$ be given. Since f is holomorphic, it is also continuous. Thus

$$\exists\delta > 0 \text{ such that } |z-z_0| < \delta \implies z \in D \text{ and } |f(z)-f(z_0)| < \frac{\epsilon}{2}$$



Choose a positive $r < \delta$ and draw the circle C_r of radius r centered at z_0. This lies entirely in D. Since $\frac{f(z)}{z-z_0}$ is holomorphic between and on C and C_r, Corollary 4.22 tells us that

$$\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{C_r} \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

We need only bound an integral to obtain the basic formula:

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z - f(z_0)\right| = \left|\frac{1}{2\pi i}\oint_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,\mathrm{d}z\right| \le \frac{1}{2\pi}2\pi r\frac{\epsilon}{2r} < \epsilon$$ ■

We postpone a sketch proof of the derivative formula[9] to the exercises.

[9] Informally, it appears as if the formula follows from repeated differentiation:

$$f^{(n+1)}(z_0) = \frac{\mathrm{d}}{\mathrm{d}z_0}f^{(n)}(z_0) \overset{???}{=} \frac{n!}{2\pi i}\oint_C \frac{\partial}{\partial z_0}\frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z = \frac{(n+1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+2}}\mathrm{d}z$$

Of course, one cannot blindly bring a derivative inside an integral like this, so a formal proof is required.

Examples 4.26. We can use the integral formula to evaluate certain integrals that would be difficult if not impossible to evaluate by parametrization.

1. If C is the circle centered at $z = i$ with radius 1, then $f(z) = \frac{3z\sin z}{z+i}$ is holomorphic on an inside C, whence
$$\oint_C \frac{3z\sin z}{z^2+1}\,\mathrm{d}z = \oint_C \frac{3z\sin z}{(z+i)(z-i)}\,\mathrm{d}z = 2\pi i f(i) = 2\pi i\frac{3i\sin i}{2i} = 3\pi i\sin i = \frac{3}{2}(e^{-1}-e^{1})$$
Contrast this with parametrizing $z(t) = i + e^{it}$ and attempting to evaluate directly!
$$\int_0^{2\pi} \frac{3(i+e^{it})\sin(i+e^{it})ie^{it}}{(i+e^{it})^2+1}\,\mathrm{d}t\ldots$$

2. Let C be a simple closed contour staying away from $z_0 = 4$. Since $f(z) = \frac{3z^2+7}{e^z}$ is entire,
$$\oint_C \frac{3z^2+7}{e^z(z-4)}\,\mathrm{d}z = \begin{cases} 0 & \text{if } z_0 = 4 \text{ is outside } C \\ 2\pi i f(4) = 110\pi i e^{-4} & \text{if } z_0 = 4 \text{ is inside } C \end{cases}$$

3. Let C be the circle of radius 2 centered at $z = 1+i$. Then $g(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$ is holomorphic on and inside C, except at $z = i$. We conclude that
$$\oint_C g(z)\,\mathrm{d}z = \oint_C \frac{1}{(z+i)^3(z-i)^3}\,\mathrm{d}z = \frac{2\pi i}{2!}\left.\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right|_i (z+i)^{-3} = 12\pi i(2i)^{-5} = \frac{3\pi}{8}$$

We finish this section with an easy yet powerful corollary of the integral formula that we mentioned in Chapter 2.

Corollary 4.27. *If f is holomorphic, then it is infinitely differentiable and all derivatives are holomorphic. In particular, the real and imaginary parts of f have continuous partial derivatives of all orders.*

Proof. If f is holomorphic at w_0, then it is holomorphic on an open set D containing w_0. Draw a circle C_r centered at w_0 lying inside D. By the Cauchy integral formula, $f''(z_0)$ exists at every z_0 inside C_r. Otherwise said, f' is holomorphic at w_0. Now induct... ∎

Finally, note that if f has an anti-derivative F, then F is necessarily holomorphic and so, by the Corollary, is f itself. Combining our results yields the following.

Theorem (Summary). *Suppose f is continuous on an open domain D and that curves C lie in D.*

$$\begin{tikzcd}
\text{all } \oint_C f(z)\,\mathrm{d}z = 0 \arrow[d, Leftrightarrow, "\text{Lemma 4.13}"'] & & f \text{ holomorphic on } D \arrow[ll, Rightarrow, "\text{If } D \text{ simply-connected}"', "\text{Cauchy–Goursat}"] \\
\text{all } \int_C f(z)\,\mathrm{d}z \text{ path independent} \arrow[rr, Leftrightarrow, "\text{Fundamental Thm}"'] & & f \text{ has an anti-derivative on } D \arrow[u, Rightarrow, "\text{Cauchy Integral Formula}"']
\end{tikzcd}$$

Exercises 4.3 1. Apply the Cauchy–Goursat Theorem to show that $\oint_C f(z)\,\mathrm{d}z = 0$ when the contour C is the unit circle $|z| = 1$.

(a) $f(z) = \dfrac{z^2}{z+3}$ (b) $f(z) = ze^{-z}$ (c) $f(z) = \operatorname{Log}(z+2)$

2. Let C_1 denote the square with sides along the lines $x = \pm 1, y = \pm 1$, and C_2 be the circle $|z| = 4$: explain why

$$\oint_{C_1} \frac{1}{3z^2+1}\,\mathrm{d}z = \oint_{C_2} \frac{1}{3z^2+1}\,\mathrm{d}z$$

3. Let C be the square with sides $x = 0, 1$ and $y = 0, 1$. Evaluate the integral $\oint_C \frac{1}{z-a}\,\mathrm{d}z$ when:

(a) a is *exterior* to the square;

(b) a is *interior* to the square.

4. Let C be the positively-oriented boundary of the half-disk $0 \le r \le 1$, $0 \le \theta \le \pi$ and define $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$ and $f(0) = 0$ using the branch of $z^{1/2}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Prove that

$$\oint_C f(z)\,\mathrm{d}z = 0$$

by evaluating three contour integrals: over the semicircle, and over two segments of the real axis joining 0 to ± 1. Why does Cauchy–Goursat not apply here?

5. If C is a positively-oriented simple closed contour, prove that the area enclosed by C is

$$\frac{1}{2i}\oint_C \overline{z}\,\mathrm{d}z$$

(*Hint: Mirror the sketch proof of Cauchy–Goursat, even though $\overline{z}$ isn't holomorphic. . .*)

6. Let C denote the boundary of the square with sides $x = \pm 2, y = \pm 2$. Evaluate the following:

(a) $\displaystyle\oint_C \frac{e^{-\frac{\pi z}{2}}\,\mathrm{d}z}{z-i}$ (b) $\displaystyle\oint_C \frac{e^z+e^{-z}}{z(z^2+10)}\,\mathrm{d}z$ (c) $\displaystyle\oint_C \frac{z\,\mathrm{d}z}{3z+i}$ (d) $\displaystyle\oint_C \frac{\sec(z/2)}{(z-1-i)^2}\,\mathrm{d}z$

7. Evaluate the integral $\oint_C g(z)\,\mathrm{d}z$ around the circle of radius 3 centered at $z = i$ when:

(a) $g(z) = \dfrac{1}{z^2+9}$ (b) $g(z) = \dfrac{1}{(z^2+9)^2}$

8. Prove that if f is holomorphic on and inside a simple closed contour C and z_0 is not on C, then

$$\oint_C \frac{f'(z)}{z-z_0}\,\mathrm{d}z = \oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$$

9. For the final time, recall Example 4.17.3. Use Cauchy's integral formula on a circle C_r of radius $r < 2$ centered at $z = 1$ to prove that

$$\oint_{C_r} \frac{1}{1-z^2}\mathrm{d}z = -i\pi$$

10. Suppose we have a polynomial $p(z) = \sum_{k=0}^{n} a_k (z - z_0)^k$ centered at z_0. Use Cauchy's integral formula to prove that $a_k = \frac{p^{(k)}(z_0)}{k!}$ is the usual Taylor coefficient.

11. Let C be the unit circle $z = e^{i\theta}$ where $-\pi < \theta \le \pi$ and suppose $a \in \mathbb{R}$ is constant. By first evaluating $\oint_C z^{-1} e^{az} \, \mathrm{d}z$, prove that

$$\int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta) \, \mathrm{d}\theta = \pi$$

12. (a) Suppose that $f(z)$ is *continuous* on and inside a simple closed contour C. Prove that the function $g(z)$ defined by

$$g(z_0) := \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

is holomorphic at every point z_0 inside C and that

$$g'(z_0) := \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, \mathrm{d}z$$

(*Hint: consider the formula in part (b) of the next question, after replacing the first two f's by g*)

(b) If $f(z) = x(1-x)(1-y)$ and C is the square with vertices $0, 1, 1+i, i$ compute $g(z_0)$.

13. We prove the 1st derivative version of Cauchy's integral formula. As previously, let $\delta > 0$ be such that $|w - z_0| < \delta \implies w \in D$.

(a) If $|\Delta z| < \delta$ and $z \in C$ explain why

i. $|z - z_0| \ge \delta$

ii. $|z - z_0 - \Delta z| > 0$

(b) Use the basic integral formula on C to evaluate $f(z_0 + \Delta z) - f(z_0)$ and thus prove that

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, \mathrm{d}z \right| \le \frac{ML}{2\pi(\delta - \Delta z)\delta^2} |\Delta z|$$

where M is an upper bound for $|f(z)|$ on C, and L is the length of C. Hence conclude that $f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} \, \mathrm{d}z$.

If you want a real challenge, do the same thing for higher derivatives!

4.4 Liouville's Theorem and The Maximum Modulus Principle

In this section we derive some powerful corollaries of the Cauchy integral formula. The first is easy.

Lemma 4.28 (Cauchy's Inequality). *If f is holomorphic on and inside the circle C of radius R centered at z_0, and $|f(z)| \leq M$ on C, then*

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z\right| \leq \frac{n!M}{R^n}$$

This seems innocuous, but it has surprising applications. If f is entire and bounded ($|f(z)| \leq M$ for all z) then Cauchy's inequality applies for *every* circle centered at *every* z_0:

$$\forall z_0, R,\ \left|f'(z_0)\right| \leq \frac{M}{R} \implies \forall z_0,\ f'(z_0) = 0$$

We've therefore proved:

Theorem 4.29 (Liouville). *The only bounded entire functions are constants.*

We now come to perhaps the simplest proof of one of the most famous results in mathematics.

Theorem 4.30 (Fundamental Thm. of Algebra). *Every non-constant polynomial has a root in $\mathbb{C}$.*

Proof. It costs nothing to assume that the polynomial is monic. Suppose $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ has no roots, then $\frac{1}{p(z)}$ is entire. Our goal is to prove that $\frac{1}{p(z)}$ is also *bounded* on $\mathbb{C}$: since $p(z)$ is non-constant, this contradicts Liouville's Theorem.

In fact it is enough to bound $\frac{1}{p(z)}$ for all *large* z: that is we want to find $R > 0$ such that

$$|z| > R \implies \frac{1}{p(z)} \text{ bounded}$$

This is since $\frac{1}{p(z)}$, being continuous, is already bounded on the compact disk $|z| \leq R$. So how do we find R? The idea is to use the triangle inequality to write

$$|z| > R \implies |p(z)| = |z|^n \left|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \geq |z|^n \left|1 - \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right|\right|$$

We need only choose R such that $\left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \leq \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \leq \frac{1}{2}$ to complete the proof:

$$|z| > R \implies |p(z)| \geq \frac{1}{2}R^n \implies \frac{1}{|p(z)|} \leq \frac{2}{R^n}$$

Forcing each term in the sum to be $\leq \frac{1}{2n}$ is enough: it is easy to check that this is accomplished by

$$R := \max\left\{(2n\,|a_k|)^{\frac{1}{n-k}} : k = 0, \ldots, n-1\right\}$$

∎

In the proof, we used the fact that a continuous function $f(z)$ on a compact (closed bounded) domain K is bounded. As you should recall from real analysis, the least upper bound is achieved

$$\exists z_0 \in K \text{ such that } |f(z_0)| = \sup\{|f(z)| : z \in K\}$$

For *holomorphic* functions, we can say something more restrictive and surprising. The maximum modulus of an holomorphic function on a compact domain is always and only achieved at an *edge point*.

Theorem 4.31 (Maximum Modulus Principle). *Suppose $f(z)$ is holomorphic and non-constant on an bounded, open, connected domain D. Then $|f(z)|$ has no maximum value on D.*

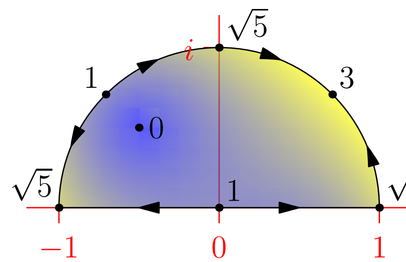
Examples 4.32. 1. Let $f(z) = e^z$ on the unit disk $|z| \le 1$. Then $|f(z)| = e^x$ which has its maximum at $z = 1$, on the edge of the disk.

2. Consider $f(z) = 2z^2 + i$ on the upper semi-disk with radius 1. There are two boundaries:
   $y = 0$: plainly $|f(z)| = \sqrt{4x^4 + 1}$ is maximal at $z = \pm 1$.
   $r = 1$: write $f(z) = 2e^{2i\theta} + i$ from which

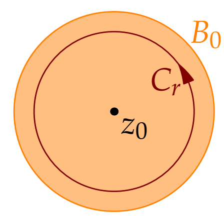
$$|f(z)| = \sqrt{(2\cos 2\theta)^2 + (2\sin 2\theta + 1)^2} = \sqrt{5 + 4\sin 2\theta}$$

   which is maximal when $\theta = \frac{\pi}{4}$ with $\left|f(e^{\frac{i\pi}{4}})\right| = 3$.

   The color indicates that value of $|f(z)|$, and the arrows its direction of increase on the boundary.



Proof. First we prove a special case.
Fix $\delta > 0$, assume $f(z)$ is holomorphic on and inside $B_0 = \{z : |z - z_0| \le \delta\}$ and suppose $|f(z)|$ attains its maximum at z_0.
Let $r \le \delta$ and apply the Cauchy integral formula around the circle C_r of radius r centered at z_0:



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_r} \frac{f(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t$$

$$\implies |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{it})\right| \mathrm{d}t \qquad \text{(Theorem 4.16)}$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,\mathrm{d}t = |f(z_0)| \qquad (|f(z_0)| = \max\{|f(z)| : z \in B_0\})$$

The inequalities are therefore *equalities*, the middle of which now becomes

$$\int_0^{2\pi} |f(z_0)| - \left|f(z_0 + re^{it})\right| \mathrm{d}t = 0$$

Since the integrand is non-negative and continuous, it must be zero. But then $|f(z)| = |f(z_0)|$ on *all circles* C_r, whence $|f(z)|$ is *constant* on B_0. Since $f(z)$ is holomorphic, it is also constant[10] on B_0.

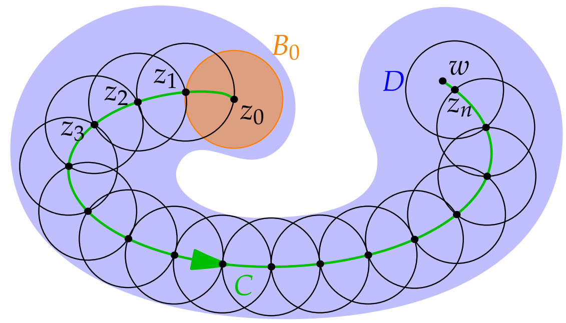
[10] Recall from chapter 2: $|f(z)| = k \neq 0 \implies \overline{f(z)} = \frac{k^2}{f(z)}$ is holomorphic, whence $u - iv$ satisfies the Cauchy–Riemann equations...

Now suppose that f is holomorphic on some open connected domain D and that it attains its maximum at $z_0 \in D$.

Let $w \in D$ be any other point and join z_0 to w by a simple contour C.

Since D is open, $\exists \delta > 0$ such that all points $\leq \delta$ distance of C lie within D.

Take points $z_1, \ldots, z_n$ separated by δ along C and disks $B_k = \{z : |z - z_k| \leq \delta\}$ centered at each z_k such that $w \in B_n$: this is possible since C has *finite length*.



By the simple case, $f(z)$ is constant on B_0. Since $z_1 \in B_0$, this shows that $|f(z_1)| = |f(z_0)|$ is also the maximum modulus on D. The special case now says that $f(z)$ is constant in $B_0 \cup B_1$. By induction, f is constant on all $\bigcup B_k$ and so $f(w) = f(z_0)$.

■

Exercises 4.4 1. (a) Suppose f is entire and that $|f(z)| \leq c\,|z|$ for some constant $c \in \mathbb{R}^+$. Prove that $f(z) = kz$ where $k \in \mathbb{C}$ satisfies $|k| \leq c$.

(b) What can you say about f if it is entire and there exists some linear polynomial $cz + d$ with $c \neq 0$ such that $|f(z)| \leq |cz + d|$ for all $z \in \mathbb{C}$?

2. If $f(z) = u + iv$ is entire and $u(x, y)$ is bounded above, apply Liouville's Theorem to $\exp(f(z))$ to prove that u is constant.

3. Let $f(z)$ be a non-zero holomorphic function on a closed bounded domain. By considering $g(z) = \frac{1}{f(z)}$, show that the minimum value of $|f(z)|$ also occurs on the boundary.

4. Find the maximum and minimum values of $\left|z^2 + 4i\right|$ on the unit disk $|z| \leq 1$.

5. On the rectangle $0 \leq x \leq \pi, 0 \leq y \leq 1$, show that $|\sin z|$ is maximal at the point $z = \frac{\pi}{2} + i$.

   (*Hint: first show that* $|\sin z|^2 = \sin^2 x + \sinh^2 y$)

6. Revisit the standard method from multivariable calculus (compute $\nabla\,|f(z)|^2$) to check that the maximum value of $\left|2z^2 + i\right|$ in Exercise 4.32 really does occur on the boundary.

7. Complete the proof of the fundamental theorem of algebra:

   (a) Verify that $R := \max\{(2n\,|a_k|)^{\frac{1}{n-k}}\}$ is *positive.*

   (b) Prove that $|z| > R \implies \forall k,\ \frac{|a_k|}{|z|^{n-k}} < \frac{1}{2n}$.

8. (a) (Hard and long) Prove the factor theorem: if $p(z_1) = 0$, then $p(z) = (z - z_1)q(z)$ for some polynomial $q(z)$.

   For a challenge, prove the full division algorithm: if $\deg f \geq \deg g$, *then there exist unique polynomials* $q(z), r(z)$ *for which*

   $$f(z) = g(z)q(z) + r(z) \quad \textit{and} \quad \deg r < \deg g$$

   (b) Prove that a degree $n \geq 1$ polynomial $p(z)$ factors uniquely over $\mathbb{C}$: up to order, exist unique $z_1, \ldots, z_n \in \mathbb{C}$ such that

   $$p(z) = a(z - z_1) \cdots (z - z_n)$$

5 Series

The discussion of series in complex analysis differs significantly from the real situation, particularly with regard to two concepts.

- Taylor's Theorem: Holomorphic functions *equal* their Taylor series. This is false in real analysis where differentiable functions need not have, nor equal, a Taylor series.
- Laurent Series: *series* also include negative powers such as $z^{-1} + 3z^{-2} + \cdots$

First we review the basics of sequences and infinite series (of non-negative powers). Hopefully you are familiar with all of this.

5.1 A Brief Review of Sequences and Infinite Series

Post real analysis, there is little specific to say regarding sequences of complex numbers. The notions of limit, convergence and sequential continuity are essentially identical in $\mathbb{C}$ and $\mathbb{R}^2$. For instance:

Definition 5.1. A sequence (z_n) has limit $z \in \mathbb{C}$, written $\lim z_n = z$, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |z_n - z| < \epsilon$$

Writing $z_n = x_n + iy_n$ and $z = x + iy$ in real and imaginary parts, we see that

$$|z_n - z| \le |x_n - x| + |y_n - y| \le 2\,|z_n - z|$$

from which:

Lemma 5.2. *If $z_n = x_n + iy_n$, then (z_n) converges if and only if both (x_n) and (y_n) converge, in which case*

$$\lim z_n = \lim x_n + i \lim y_n$$

Warning! While this mostly translates to the polar representation $z_n = r_n e^{i\theta_n}$, there is a caveat: the discontinuity of $\operatorname{Arg} z = \Theta$ when z is a non-positive real number means that (Θ_n) need not converge even if (z_n) does.

Example 5.3. The sequence with $z_n = 2i - \frac{1+i}{n}$ has limit $z = 2i$: given $\epsilon > 0$, let $N = \sqrt{2}\epsilon$, then

$$n > N \implies |z_n - z| = \frac{\sqrt{2}}{n} < \frac{\sqrt{2}}{N} = \epsilon$$

The real and imaginary parts are $x_n = -\frac{1}{n}$ and $y_n = 2 - \frac{1}{n}$ which clearly converge to $x = 0$ and $y = 2$ respectively. In polar co-ordinates things are also as expected

$$\lim r_n = \lim \sqrt{\frac{1 + (2n-1)^2}{n^2}} = \lim \frac{2}{n}\sqrt{n^2 - n} = 2$$

$$\lim \Theta_n = \lim \left(\pi \tan^{-1} \frac{(2n-1)/n}{-1/n} \right) = \pi + \tan^{-1}(1 - 2n) = \frac{\pi}{2}$$

Since z_n lies in the second quadrant and $z = 2i$, we never get near the non-negative real axis where Θ_n is discontinuous.

Definition 5.4 (Infinite Series). Given a sequence[a] (z_n), its sequence of *partial sums* (s_n) is

$$s_n = \sum_{k=0}^{n} z_k = z_0 + \cdots + z_n$$

The *infinite series* $\sum z_n := \lim s_n$ is said to converge (diverge) if the sequence (s_n) converges (diverges).
The series *converges absolutely* if $\sum |z_n|$ converges.
The series *converges conditionally* if it converges but not absolutely.

[a]For brevity, assume the initial term is z_0: nothing prevents the initial term being z_{n_0} for any natural number n_0.

Theorem 5.5 (Basic Series Facts). *Let $\sum z_n$ and $\sum w_n$ be series of complex numbers.*

1. *If $z_n = x_n + iy_n$, then $\sum z_n$ converges if and only if $\sum x_n$ and $\sum y_n$ both converge, in which case*
$$\sum z_n = \sum x_n + i \sum y_n$$
2. *If $a \in \mathbb{C}$, and $\sum z_n$ and $\sum w_n$ converge, then $\sum az_n + w_n$ converges, in which case*
$$\sum az_n + w_n = a \sum z_n + \sum w_n$$
3. *(n^{th} term/divergence test)* *If $\sum z_n$ converges, then $\lim z_n = 0$.*
4. *The (real!) comparison, ratio and root tests apply to the series $\sum |z_n|$.*
5. *Absolute convergence implies convergence: moreover $|\sum z_n| \le \sum |z_n|$.*

Proof. 1. This is immediate from Lemma 5.2.

2, 3. These follow from 1 and the corresponding results for the real series $\sum x_n, \sum y_n$.

4. This requires no proof: $\sum |z_n|$ is a series of non-negative real numbers.

5. Since $\sum |z_n|$ is a convergent series of non-negative terms and $|x_n| \le |z_n|$, the comparison test proves that $\sum x_n$ is absolutely convergent and thus convergent. Since $\sum y_n$ converges similarly, part 1 shows that $\sum z_n$ converges.
Finally, apply the triangle inequality $\left| \sum_{k=0}^{m} z_k \right| \le \sum_{k=0}^{m} |z_k| \le \sum_{n=0}^{\infty} |z_n|$ and take limits as $m \to \infty$. ∎

Exercises 5.1 1. Use the ϵ-N definition to prove that $\lim \frac{2+in}{n} = i$.

2. Give a rigorous proof of 5.2. Sketch a proof of the corresponding statement for the polar representation whenever $\lim z_n$ is non-zero and not a negative real number.

3. Explicitly prove part 2 of Theorem 5.5.

4. Fix $\theta \in (-\pi, \pi]$. Prove that the sequence defined by $z_n = e^{in\theta}$ converges if and only if $\theta = 0$.

5. Use the ϵ-N definition to prove that $\lim \sqrt{i + \frac{1}{n}} = \frac{1+i}{\sqrt{2}} = \sqrt{i}$ where we use the principal value.

5.2 Power Series, Taylor Series and Taylor's Theorem

We first make the identical definition to that in real analysis.

Definition 5.6. A *power series* centered at $z_0 \in \mathbb{C}$ is a function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

A function $f : D \to \mathbb{C}$ is *analytic* if every $z_0 \in D$ has an open neighborhood on which $f(z)$ equals a power series centered at z_0. That is,

$$\forall z_0 \in D, \exists \delta > 0, (a_n) \text{ such that } |z - z_0| < \delta \implies f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

To be analytic at a point z_0 is to be analytic on some open neighborhood of z_0.

The goal of the next two sections is the establishment of a key observation:

A function is analytic if and only if it is holomorphic (differentiable)

Here is the canonical example of a power series.

Example 5.7. (Geometric series) By the n^{th} term test, the power series $\sum z^n$ diverges if $|z| \geq 1$. Otherwise, inside the unit circle $|z| < 1$ we have $z^{n+1} \to 0$, and so

$$s_n - z s_n = 1 - z^{n+1} \implies s_n = \frac{1 - z^{n+1}}{1 - z} \implies \sum_{n=0}^{\infty} z^n = \lim \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}$$

Now let $z_0 \neq 1$; by substituting $z \mapsto \frac{z - z_0}{1 - z_0}$ in the above, observe that

$$\frac{1}{1-z} = \frac{1}{1 - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{1 - z_0}} = \frac{1}{1 - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{1 - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{1}{(1 - z_0)^{n+1}} (z - z_0)^n$$

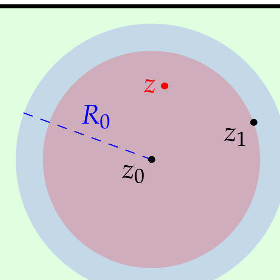
provided $|z - z_0| < |1 - z_0|$. It follows that $f(z) = \frac{1}{1-z}$ is analytic on its domain $\mathbb{C} \setminus \{1\}$.

Note how the geometric series converges on a *disk*. As the next result shows, this is the case for *every* power series, analogous to the concept of the interval/radius of convergence in real analysis.

Theorem 5.8. *Consider a power series $f(z) = \sum a_n (z - z_0)^n$.*

1. *If f converges at a point $z_1 \neq z_0$, then it is absolutely convergent at every point z satisfying $|z - z_0| < |z_1 - z_0|$.*
2. *Define $R_0 := \sup \{|z - z_0| : f(z) \text{ converges}\}$.*

   Then $f(z)$ converges absolutely whenever $|z - z_0| < R_0$ and diverges whenever $|z - z_0| > R_0$.



Proof. 1. By the n^{th} term test, the sequence $(a_n(z_1 - z_0)^n)$ converges (to 0) and is therefore bounded by some $M \in \mathbb{R}^+$. Thus

$$|a_n|\,|z - z_0|^n = |a_n|\,|z_1 - z_0|^n \left(\frac{|z - z_0|}{|z_1 - z_0|}\right)^n \leq Mr^n \quad \text{where} \quad r = \frac{|z - z_0|}{|z_1 - z_0|} < 1$$

Since $\sum Mr^n$ converges, we conclude (comparison test) that $\sum |a_n|\,|z - z_0|^n$ converges.

2. By standard properties of the supremum, if $|z - z_0| < R_0$, then $\exists z_1$ such that $f(z_1)$ converges and $|z - z_0| < |z_1 - z_0|$: now apply part (a). The remaining part is an exercise. ∎

Definition 5.9. The value R_0 in the theorem is the *radius of convergence* of the power series.

- If $R_0 = \infty$, the series is (absolutely) convergent on $\mathbb{C}$;
- If $R_0 = 0$, the series converges only when $z = z_0$;
- Otherwise, we have a *disk of convergence* with radius $R_0 \in \mathbb{R}^+$ centered at z_0. The *circle of convergence* is the boundary circle of this disk.

As in real analysis:

1. If $R_0 \in \mathbb{R}^+$, we have to test separately for convergence/divergence on the circle of convergence. A key technique for doing this is *Abel's Test* (see Exercise 4).
2. We could use the ratio/root tests to explicitly compute: $R_0 = \liminf |a_n|^{-1/n}$, etc. This is mostly redundant since, as we'll see later, the radius of convergence is usually easier to spot as the distance from z_0 to the nearest point at which f fails to be differentiable.

Before computing further examples, we first revisit a familiar definition and observe a startling difference between the real and complex case.

Definition 5.10. If a function $f(z)$ is infinitely differentiable at z_0, then its *Taylor series* is the power series

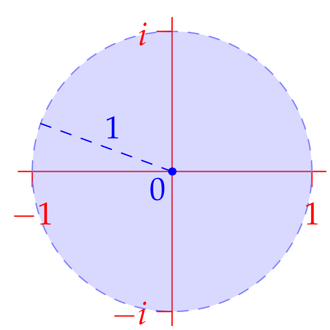
$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

The *Taylor coefficients* are $a_n = \frac{f^{(n)}(z_0)}{n!}$. A *Maclaurin series* is a Taylor series with $z_0 = 0$.

We continue Example 5.7. On the disk $|z| < 1$, the function $f(z) = \frac{1}{1-z}$ has

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \implies \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n \overset{!!}{=} f(z)$$

The Maclaurin series is therefore the geometric power series representation of $f(z)$ on the open disk $|z| < 1$! In fact this situation is completely general...



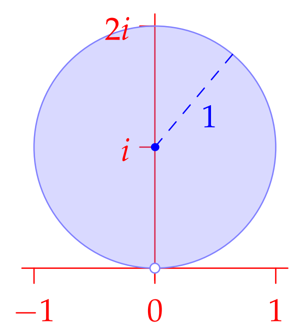
Theorem 5.11 (Taylor's Theorem). *If $f(z)$ is holomorphic on a disk $|z-z_0|<R$, then it equals its Taylor series on that disk.*

This is a *very* strong statement in comparison to real analysis, where there exist infinitely differentiable functions which do not equal their Taylor series (see Exercise 5).

Clearly R cannot be larger than the radius of convergence R_0 of the Taylor series. If f is entire, then the result holds for all positive R and the series has radius of convergence is infinity.

Examples 5.12. Further familiar examples translate over from real analysis.

1. Since $f(z)=e^z$ is entire and $f^{(n)}(0)=1$ for all n, we see that
$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}\quad\text{for all } z\in\mathbb{C}$$
2. $f(z)=\sin z$ is entire with $f^{(2n)}(z)=(-1)^n\sin z$ and $f^{(2n+1)}(z)=(-1)^n\cos z$. Its Maclaurin series is therefore
$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\quad\text{for all } z\in\mathbb{C}$$
3. $f(z)=\operatorname{Log} z$ is holomorphic on the open disk $|z-i|<1$. Whenever $n\geq 1$, we have
$$f^{(n)}(i)=\left.\frac{(-1)^{n-1}(n-1)!}{z^n}\right|_{z=i}=-i^n(n-1)!$$
and we obtain the Taylor series
$$\operatorname{Log} z=\operatorname{Log} i-\sum_{n=1}^{\infty}\frac{i^n}{n}(z-i)^n=\frac{\pi i}{2}-\sum_{n=1}^{\infty}\frac{(iz+1)^n}{n}$$



Convergence when $|z-i|<1$ can be directly verified using the comparison test:
$$|z-i|=r<1\implies\frac{|z-i|^n}{n}\leq r^n\implies\sum\frac{i^n(z-i)^n}{n}\text{ converges absolutely}$$
At $z=0$, we recognize the divergent harmonic series $\sum\frac{1}{n}$: the radius of convergence is $R_0=1$. Exercise 4 shows that the series converges everywhere else on the boundary circle $|z-i|=1$.

We'll see the proof in a moment, but first observe that if $f:D\to\mathbb{C}$ is holomorphic, then, for every $z_0\in D$ it is holomorphic on some disk $|z-z_0|<R$; by Taylor's theorem $f(z)$ equals its Taylor series on this disk and we've therefore proved half of our key observation.

Corollary 5.13. *Every holomorphic function is analytic.*

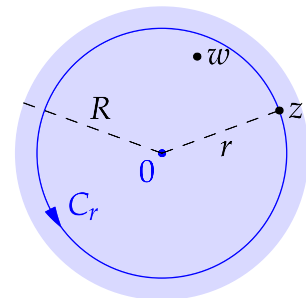
We'll obtain the converse in the next section.

Why is Taylor's Theorem so much more specific in complex analysis? The answer is that we have available a very powerful tool, namely Cauchy's integral formula.

Proof of Taylor's Theorem. By relabelling $\tilde{f}(z) = f(z - z_0)$, it is enough to prove when $z_0 = 0$, that is for Maclaurin series.

Let w be given where $|w| < R$. By Example 5.7, if $z \neq 0$,

$$\frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k = \frac{1-\left(\frac{w}{z}\right)^n}{z(1-\frac{w}{z})} = \frac{1}{z-w} - \frac{1}{z-w}\left(\frac{w}{z}\right)^n \qquad (*)$$



Choose any circle C_r centered at the origin with radius $r \in (|w|, R)$. Since both 0 and w lie inside C_r, we may apply Cauchy's integral formula *twice*:

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}\mathrm{d}z && \text{(Cauchy for } C_r \text{ around } w\text{)}\\
&= \sum_{k=0}^{n-1}\frac{w^k}{2\pi i}\oint_{C_r}\frac{f(z)}{z^{k+1}}\,\mathrm{d}z + \frac{w^n}{2\pi i}\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z && \text{(subsititute for } \tfrac{1}{z-w} \text{ using } (*)\text{)}\\
&= \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k + \frac{w^n}{2\pi i}\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z && \text{(Cauchy for } C_r \text{ around } 0\text{)}
\end{aligned}$$

All that remains is to control the final integral. Since f is holomorphic on C_r, it is bounded by some $M \in \mathbb{R}^+$. Moreover, for $z \in C_r$ we have $|z - w| \geq= ||z| - |w|| = r - |w|$. Thus

$$\begin{aligned}
\left|f(w) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k\right| &= \frac{|w|^n}{2\pi}\left|\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z\right|\\
&\leq \frac{|w|^n M\cdot 2\pi r}{2\pi r^n(r-|w|)} = \frac{Mr}{(r-|w|)}\left(\frac{|w|}{r}\right)^n
\end{aligned}$$

This last plainly converges to zero since $|w| < r$. Otherwise said

$$f(w) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n$$

so that $f(z)$ equals its Maclaurin series whenever $|z| < R$. ∎

Exercises 5.2 1. (a) Compute the Maclaurin series of $\cos z$ directly from the definition.

(b) Evaluate the Taylor series of $\sin z$ about $z_0 = \frac{\pi}{2}$ and confirm that it equals your answer to part (a) when z is replaced with $z - \frac{\pi}{2}$.

2. Consider $f(z) = \frac{1}{z}$. For any $z_0 \neq 0$, find the Taylor series of $f(z)$ about z_0. What is its disk of convergence?

3. Finish the proof of Theorem 5.8. Suppose $|z - z_0| > R_0$. Prove that $f(z)$ diverges.

4. The alternating series test was often useful in real analysis to decide convergence at the end-points of an interval of convergence. Here is a generalization to the complex situation.

   Consider the power series $\sum a_n z^n$ where (a_n) is a *real* sequence such that

   $$a_n \geq 0, \quad a_{n+1} \leq a_n, \quad \lim_{n\to\infty} a_n = 0$$

   (a) Write $s_n(z) = \sum_{k=0}^{n} a_k z^k$ for the partial sum and prove that

   $$(1-z)s_n(z) = a_0 - a_n z^{n+1} + \sum_{k=1}^{n}(a_k - a_{k-1})z^k$$

   (b) Prove *Abel's Test*: $\sum a_n z^n$ converges everywhere on the *closed* unit disk $|z| \leq 1$, *except perhaps* when $z = 1$.
   (*Hint: show that* $\sum(a_k - a_{k-1})z^k$ *converges absolutely by comparison with a telescoping series*)

   (c) Verify that $\operatorname{Log} z = \frac{\pi i}{2} - \sum_{n=1}^{\infty} \frac{(iz+1)^n}{n}$ converges whenever $|z - i| = 1$, except when $z = 0$.

   (d) Prove that the real series $\sum \frac{\cos n\theta}{n}$ converges except when θ is divisible by 2π. For what values of θ does the series $\sum \frac{\sin n\theta}{n}$ converge?

   (e) Find all values of z for which the series $\sum \frac{1+i}{(n+i)(4+3i)^n}(z-1+2i)^n$ converges and sketch the disk of convergence.
   (*Hint: let* $w = \frac{z-1+2i}{4+3i}$)

5. Consider the function

   $$f(z) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

   When $z \in \mathbb{R}$ this provides the classic example of an infinitely differentiable function whose Maclaurin series (being identically zero) does not equal the original function except at the origin. When $z \in \mathbb{C}$, why $f(z)$ does not contradict Taylor's Theorem.

6. Why do we need to introduce r in part 1 of the proof of Theorem 5.8? That is, explain why can't we use the comparison test to say

   $$|a_n|\,|z - z_0|^n < |a_n|\,|z_1 - z_0|^n \implies \sum |a_n|\,|z - z_0|^n \text{ converges}$$

5.3 Uniform Convergence: Continuity, Integrability and Differentiability

As in real analysis, we want to establish the following useful facts:

1. Representations are unique: if two power series are equal, their coefficients are equal.
2. Power series are continuous, indeed differentiable, inside their disk of convergence.
3. Power series may be differentiated and integrated term-by-term.

The arguments are intertwined. Since these are often similar, even identical, to the real case, we will be brief and postpone all examples until the end. The critical ingredient is uniform convergence.

Definition 5.14. Suppose $f(z) = \sum a_n(z-z_0)^n$ is a power series with n^{th} partial sum $s_n(z)$ and remainder $\rho_n(z) = f(z) - s_n(z)$. We say that the series *converges uniformly* on a domain D if

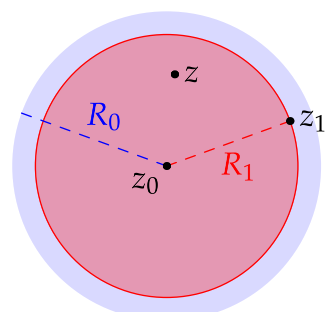
$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N,\ z \in D \implies |\rho_n(z)| < \epsilon$$

Uniformity means that $N = N(\epsilon)$ is independent of the location $z \in D$. If $N = N(\epsilon, z)$ is permitted to depend on z, we'd refer to the convergence as *pointwise.*

Theorem 5.15. *Suppose R_0 is the radius of convergence of a power series about z_0. If $R_1 < R_0$, then the series converges uniformly on the closed disk $|z - z_0| \le R_1$.*

Proof. As preparation, suppose z_1 satisfies $|z_1 - z_0| = R_1$. Since $R_1 < R_0$, the series converges absolutely at z_1 (Theorem 5.8). Denote the n^{th} remainder of this absolutely convergent series by

$$\sigma_n = \sum_{k=n+1}^{\infty} |a_k| \, |z_1 - z_0|^k = \sum_{k=n+1}^{\infty} |a_k| \, R_1^k$$



Now let $\epsilon > 0$ be given. Since the above series converges, we have $\lim_{n\to\infty} \sigma_n = 0$:

$$\exists N \text{ such that } n > N \implies \sigma_n < \epsilon \qquad (*)$$

By the comparison test, if z satisfies $|z - z_0| \le R_1$, then

$$\begin{aligned} |\rho_n(z)| &= \left| \sum_{k=n+1}^{\infty} a_k (z-z_0)^k \right| \le \sum_{k=n+1}^{\infty} |a_k| \, |z-z_0|^k \\ &\le \sum_{k=n+1}^{\infty} |a_k| \, |z_1 - z_0|^k = \sigma_n < \epsilon \end{aligned}$$

∎

Note where the uniformity comes from: we were able to choose N depending only[11] on ϵ, not z.

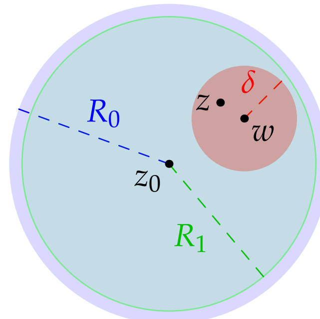
That R_1 is *strictly less* than the radius of convergence R_0 is important. In Exercise 8, we'll see that the convergence of a power series need not be uniform on the full disk of convergence.

[11] It looks as if N might also depend on our choice of z_1 in the first line. However, any suitable z_1 has the same value for $|z_1 - z_0| = R_1$ and thus produces the same sequence (σ_n): it is from the convergence of this sequence that we get N.

Theorem 5.16 (Continuity). *Suppose $f(z) = \sum a_n(z-z_0)^n$ has radius of convergence R_0. Then $f(z)$ is continuous whenever z is interior to the disk of convergence: $|z-z_0| < R_0$.*

This is identical to the famous $\frac{\epsilon}{3}$-proof seen in real analysis.

Proof. Fix w and R_1 such that $|w-z_0| < R_1 < R_0$. Let $\epsilon > 0$ be given. Observe:



- Uniform convergence whenever $|z-z_0| \le R_1$:

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{3} \text{ and } |\rho_n(w)| < \frac{\epsilon}{3}$$

- Openness and continuity (s_n is a polynomial!): for any $n > N$,

$$\exists \delta > 0 \text{ such that } |z-w| < \delta \implies \begin{cases} |z-z_0| < R_1 \\ |s_n(z) - s_n(w)| < \frac{\epsilon}{3} \end{cases}$$

Now put it together to see that $f(z)$ is continuous at w: for any $n > N$,

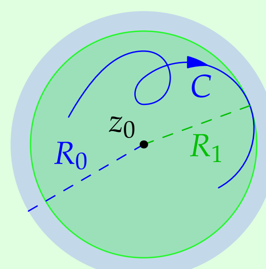
$$\begin{aligned} |z-w| < \delta \implies |f(z) - f(w)| &= |f(z) - s_n(z) + s_n(z) - s_n(w) + s_n(w) - f(w)| \\ &\le |\rho_n(z)| + |s_n(z) - s_n(w)| + |\rho_n(w)| < \epsilon \end{aligned}$$

■

Our treatment now splits from that in real analysis. Since power series are continuous, we may define *contour integrals*. The remaining results follow from a general version of term-by-term integration.

Theorem 5.17. *Let $f(z) = \sum a_n(z-z_0)^n$ have radius of convergence R_0, and C be a contour interior to the disk of convergence: $z \in C \implies |z-z_0| < R_0$. If $g(z)$ is continuous on C, then*

$$\int_C g(z) f(z) \, \mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n \, \mathrm{d}z$$



Proof. The integral $\int_C g(z) f(z) \, \mathrm{d}z$ exists since f, g are continuous on C. Since C is a compact set:

- C lies inside some closed disk $|z-z_0| \le R_1 < R_0$ on which the series $f(z)$ converges uniformly.
- $g(z)$ is bounded on C by some $M \in \mathbb{R}^+$.

Let C have length L and let $\epsilon > 0$ be given. Since $f(z)$ converges uniformly when $|z-z_0| \le R_1$,

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{ML}$$

Now take integrals and moduli to see that

$$n > N \implies \left| \int_C g(z) f(z) \, \mathrm{d}z - \sum_{k=0}^{n} a_k \int_C g(z)(z-z_0)^k \, \mathrm{d}z \right| = \left| \int_C g(z) \rho_n(z) \, \mathrm{d}z \right| < \epsilon$$

■

Everything we want now follows by choosing specific functions $g(z)$.

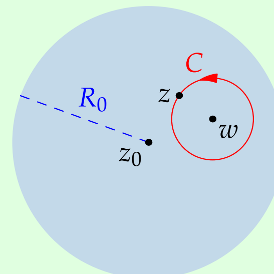
Corollary 5.18. *Suppose $f(z) = \sum a_n(z-z_0)^n$ is a power series with radius of convergence $R_0 > 0$.*

1. *(Term-by-term integration) Let $g(z) = 1$ to see that*

$$\int_C f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}\Big|_{C(\text{start})}^{C(\text{end})}$$

2. *(Holomorphicity) $\oint_C f(z)\,\mathrm{d}z = 0$ for every simple closed contour, whence $f(z)$ is holomorphic inside the circle of convergence. In particular, every analytic function is holomorphic.*

3. *(Term-by-term differentiation) Given $|w-z_0| < R_0$, let $g(z) = \frac{1}{2\pi i(z-w)^2}$ and apply Cauchy's integral formula on a small circle around w:*



$$f'(w) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-w)^2}\,\mathrm{d}z = \sum \frac{a_n}{2\pi i}\oint \frac{(z-z_0)^n}{(z-w)^2}\,\mathrm{d}z$$
$$= \sum a_n \frac{\mathrm{d}}{\mathrm{d}z}\Big|_{z=w}(z-z_0)^n = \sum a_n n(z-z_0)^{n-1}$$

4. *(Unique representation) The power series is the Taylor series of $f(z)$: that is, $a_n = \frac{f^{(n)}(z_0)}{n!}$.*

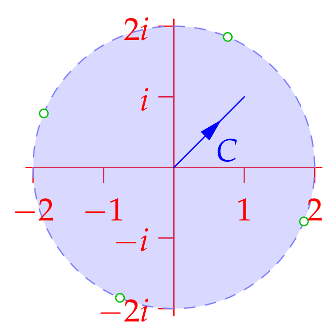
Exercise 6 considers part 4 and its implications. Since analytic and holomorphic are now equivalent, we'll retire the latter for the rest of the course.

Examples 5.19. By uniqueness of representation, we can compute Taylor/Maclaurin series *algebraically*: if a function equals a series, that's the one we want regardless of how we found it!

1. $f(z) = z^3e^{z^2} = z^3\sum_{n=0}^{\infty}\frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{2n+3}}{n!}$ is the Maclaurin series of $f(z)$. Since the radius of convergence is infinite, the function equals its Maclaurin series everywhere on $\mathbb{C}$.

2. The function $f(z) = \begin{cases}\frac{\sin z}{z} & \text{if } z \neq 0\\ 1 & \text{if } z = 0\end{cases}$ is entire since it equals the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$.

3. We find the Maclaurin series of $f(z) = \frac{1}{z^4+16i}$ algebraically:



$$f(z) = \frac{1}{16i\left(1-\frac{z^4}{-16i}\right)} = \frac{1}{16i}\sum_{n=0}^{\infty}\left(\frac{z^4}{-16i}\right)^n = \sum_{n=0}^{\infty}\frac{i^{n-1}}{16^{n+1}}z^{4n}$$

This converges whenever $\left|\frac{z^4}{-16i}\right| < 1 \iff |z| < 2$, equalling the distance from the center to the nearest point(s) that $f(z)$ fails to be analytic. If C is the straight line from $z=0$ to $z=1+i$, then

$$\int_C f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty}\frac{i^{n-1}}{16^{n+1}}\int_C z^{4n}\,\mathrm{d}z = \sum_{n=0}^{\infty}\frac{i^{n-1}(1+i)^{4n+1}}{16^{n+1}(4n+1)} = \sum_{n=0}^{\infty}\frac{1-i}{16(4n+1)}\left(\frac{-i}{4}\right)^n$$

Exercises 5.3 1. Find a power series representation and the radius of convergence:

(a) $f(z) = \dfrac{z}{4-z}$ about $z_0 = 0$;

(b) $f(z) = z \sin z^2$ about $z_0 = 0$;

(c) $f(z) = \cosh 3z$ about $z_0 = \dfrac{i\pi}{9}$

2. *Without computing derivatives*, find the Taylor series for $f(z) = \frac{1}{z}$ about $z_0 \neq 0$. By differentiating term-by-term, find the Taylor series of $\frac{1}{z^2}$ about z_0.

3. By expressing it as a Maclaurin series, show that the following function is entire:

$$f(z) = \begin{cases} \frac{1}{z^2}(1-\cos z) & \text{if } z \neq 0 \\ \frac{1}{2} & \text{if } z = 0 \end{cases}$$

4. (a) By integrating the Taylor series for z^{-1} about $z_0 = 1$, prove that

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \quad \text{whenever } |z-1| < 1$$

(b) Prove that the following function is analytic on the domain $0 < |z|,\ \operatorname{Arg} z \in (-\pi, \pi)$:

$$f(z) = \begin{cases} \frac{\operatorname{Log} z}{z-1} & \text{if } z \neq 1 \\ 1 & \text{if } z = 1 \end{cases}$$

5. Consider the Maclaurin series $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ on the disk $|z| < 1$. Show that $h(z) = \frac{1}{z^2+1}$ is the analytic continuation of $f(z)$ to $\mathbb{C} \setminus \{i, -i\}$.

6. (a) Prove part 4 of Corollary 5.18: if $f(z) = \sum a_n(z-z_0)^n$, prove that $f^{(m)}(z_0) = m! a_m$ so that the series really is the Taylor series of $f(z)$.
(*Hint: let* $g(z) = \frac{m!}{2\pi i(z-z_0)^{m+1}}$ *in Theorem 5.17*)

(b) Explain carefully why every power series defines an analytic function.
(*Think carefully about the definitions and what we've proved in the last two sections!*)

7. Suppose that the series $\sum a_n(z-z_0)^n$ has radius of convergence R_0 and that $f(z) = \sum a_n(z-z_0)^n$ whenever $|z-z_0| < R_0$. Prove that

$$R_0 = \inf\{|\hat{z} - z_0| : f(z) \text{ non-analytic or undefined at } \hat{z}\}$$

(*R_0 is essentially the distance from z_0 to the nearest point at which $f(z)$ is non-analytic*)

8. (Hard) Consider $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ on $|z| < 1$.

(a) Let $R_1 < 1$. Explicitly check uniform convergence when $|z| \le R_1$. That is, given $\epsilon > 0$, find an explicit N such that

$$n > N \implies |\rho_n(z)| = \left| f(z) - \sum_{k=0}^{n} z^k \right| < \epsilon \text{ whenever } |z| \le R_1$$

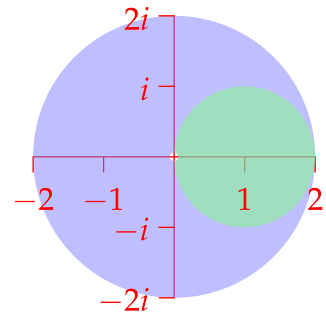
(b) Prove that $f(z)$ is *not* uniformly convergent on $|z| < 1$.
(*Hint: Let $\epsilon = 1$ and try to get a contradiction...*)

5.4 Laurent Series

While Taylor series are undeniably useful, they also have key weaknesses, particularly with regard to their domains being *disks*. We motivate a more general construction with an example.

Example 5.20. $f(z) = \frac{1}{z(2-z)}$ can be written as a Taylor series centered at $z = 1$:

$$f(z) = \frac{1}{1-(z-1)^2} = \sum_{n=0}^{\infty} (z-1)^{2n} \quad \text{whenever } |z-1| < 1$$



However, the most interesting aspects of $f(z)$ involve its behavior near the points $z = 0, 2$. Because of their disk-domains, we can't use Taylor series to loop around these points.

As an alternative, expand $\frac{1}{2-z}$ in a power series centered at 0:

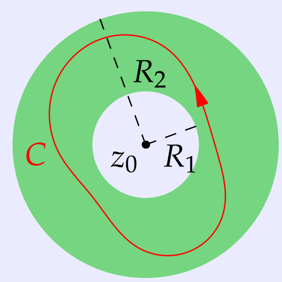
$$f(z) = \frac{1}{2z(1-\frac{z}{2})} = \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}} = \frac{1}{2z} + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16} + \cdots$$

By construction, this second series is valid on the punctured disk $0 < |z| < 2$. The larger domain, particularly the fact that it encircles the origin, provides an obvious advantage over the Taylor series.

Definition 5.21. Let $R_1 < R_2$ and suppose $f(z)$ is analytic on the *annulus* $R_1 < |z - z_0| < R_2$. Its *Laurent series* about z_0 is the expression[a]

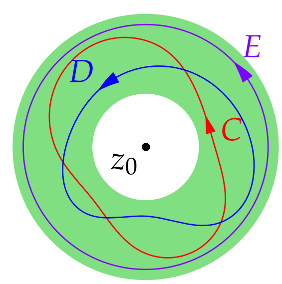
$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

and C is a simple closed contour encircling z_0 within the annulus.



[a]If you prefer, write $\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ where $b_n = \frac{1}{2\pi i} \oint_C (z-z_0)^{n-1} f(z) \, dz$.

- As in Example 5.20, the inner radius can be $R_1 = 0$ and the domain a punctured disk. As with Taylor series, the outer radius can be infinite.
- The coefficients a_n are independent of the choice of contour C.
  To see this, suppose D is another simple closed curve encircling z_0, and choose a circle E outside both C and D. Since $\frac{f(z)}{(z-z_0)^{n+1}}$ is analytic on the annulus, two applications of Cauchy–Goursat yield



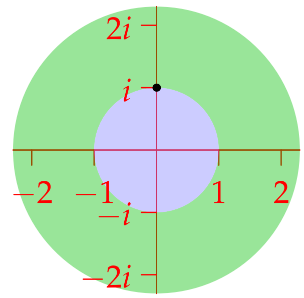
$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \oint_E \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \oint_D \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

- If $f(z)$ is analytic on the *disk* $|z - z_0| < R_2$, then the Laurent series equals the Taylor series:
  - $n \geq 0 \implies a_n = \frac{f^{(n)}(z_0)}{n!}$ by Cauchy's integral formula;
  - $n < 0 \implies a_n = 0$ by Cauchy–Goursat.

It is usually difficult to compute a Laurent series directly using the definition, since it requires infinitely many contour integrals! Thankfully, as we'll see shortly, all the standard facts regarding Taylor series translate to this new situation. In particular, if $f(z) = \sum a_n(z-z_0)^n$, then the series is the Laurent series of $f(z)$ (Corollary 5.26). This makes computing examples much easier!

Examples 5.22. 1. Whenever $|z| < 1$ we have the Taylor series

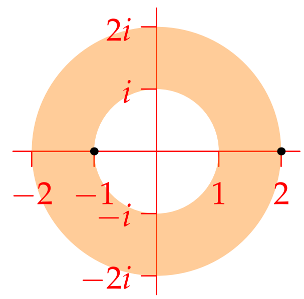
$$\frac{1}{z-i} = \frac{1}{-i(1-\frac{z}{i})} = i\sum_{n=0}^{\infty}(-iz)^n = i + z - iz^2 - z^3 + iz^4 + \cdots$$



When $|z| > 1$, we have the Laurent series

$$\frac{1}{z-i} = \frac{z}{(1-\frac{i}{z})} = \sum_{n=0}^{\infty} i^n z^{-n-1} = \frac{i}{z} - \frac{1}{z^2} - \frac{i}{z^3} + \frac{1}{z^4} + \cdots$$

2. Whenever $1 < |z| < 2$ we have a Laurent series



$$\begin{aligned}\frac{3}{(2-z)(1+z)} &= \frac{1}{2-z} + \frac{1}{1+z} = \frac{1}{2(1-\frac{z}{2})} + \frac{1}{z(1+\frac{1}{z})}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{z}\sum_{m=0}^{\infty}(-z)^{-m}\\ &= \cdots + z^{-3} - z^{-2} + z^{-1} + \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots\end{aligned}$$

3. Since e^z has Maclaurin series $\sum \frac{z^n}{n!}$ valid on the entire complex plane, we obtain the Laurent series expansion

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

on the punctured plane $z \neq 0$. Explicitly evaluating the integrals $a_n = \frac{1}{2\pi i}\oint_C \frac{e^{1/z}}{z^{n+1}}\,dz$ would be extremely irritating!

4. Again using Maclaurin series, we obtain another Laurent series valid on the punctured plane $z \neq 0$:

$$\frac{1}{z^7}\sin z^2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{4n-5} = z^{-5} - \frac{1}{6}z^{-1} + \frac{1}{120}z^3 - \frac{1}{5040}z^7 + \cdots$$

5. Multiplying term-by-term, and since we need *both* Maclaurin series to be valid, we obtain a Laurent series valid on the punctured disk $0 < |z| < 1$:

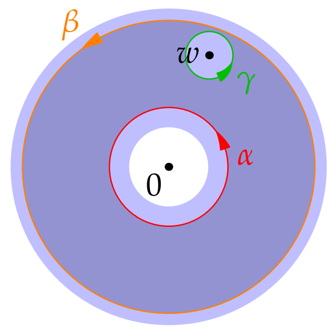
$$\begin{aligned}\frac{1}{z(z-1)(z-2i)} &= \frac{1}{z}\left(\sum_{n=0}^{\infty}(-1)^n z^n\right)\left(\sum_{m=0}^{\infty}\left(\frac{i}{2}\right)^m z^m\right)\\ &= \frac{1}{z}\left(1 - z + z^2 - z^3 + \cdots\right)\left(1 + \frac{i}{2}z - \frac{1}{4}z^2 - \frac{i}{8}z^3 + \cdots\right)\\ &= \frac{1}{z} + \left(-1 + \frac{i}{2}\right) + \left(\frac{3}{4} - \frac{i}{2}\right)z + \left(-\frac{3}{4} + \frac{3i}{8}\right)z^2 + \cdots\end{aligned}$$

Theory time!

Having seen a few examples, we should properly state and prove the main properties of Laurent series. These are very similar to the corresponding arguments for Taylor series; mostly it is an issue of keeping track of two series at once.

Theorem 5.23 (Laurent's Theorem). *An analytic function on an open annulus equals its Laurent series.*

Proof. By a simple translation, it is enough to prove when $z_0 = 0$. Let w in the annulus be given.



Since the annulus is open, we may choose three non-overlapping circles α, β, γ with radii $R_\alpha, R_\beta, R_\gamma$ as in the picture:

- γ a small circle centered at w inside the annulus;
- α, β centered at 0, α inside and β outside w.

Since $\frac{f(z)}{z-w}$ is analytic on the region inside β with interior boundaries α and γ, Cauchy–Goursat says that

$$\left(\oint_\beta - \oint_\alpha - \oint_\gamma\right)\frac{f(z)}{z-w}\,dz = 0 \implies f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,dz$$

As in the proof of Taylor's theorem, we expand

$$\frac{1}{z-w} = \frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k + \frac{1}{z-w}\left(\frac{w}{z}\right)^n = -\frac{1}{w}\sum_{k=1}^{n}\left(\frac{z}{w}\right)^{k-1} + \frac{1}{z-w}\left(\frac{z}{w}\right)^n$$

and use this to attack the two integrals:

$$\frac{1}{2\pi i}\oint_\beta \frac{f(z)}{z-w}\,dz = \sum_{k=0}^{n-1}\underbrace{\frac{w^k}{2\pi i}\oint_\beta \frac{f(z)}{z^{k+1}}\,dz}_{a_k w^k} + \frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,dz$$

$$\frac{-1}{2\pi i}\oint_\alpha \frac{f(z)}{z-w}\,dz = \sum_{k=1}^{n}\underbrace{\frac{1}{2\pi i w^k}\oint_\alpha z^{k-1}f(z)\,dz}_{a_{-k}w^{-k}} - \frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,dz$$

Since $f(z)$ is continuous on the closed bounded annulus between α, β, it has an upper bound M. Moreover, whenever $z \in \alpha \cup \beta$, we have $|z-w| > R_\gamma$. The triangle inequality finishes things off:

$$\begin{aligned}\left|f(w) - \sum_{k=-n}^{n-1} a_k w^k\right| &= \left|\frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,dz - \sum_{k=-n}^{n-1} a_k w^k\right| \\ &\le \left|\frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,dz\right| + \left|\frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,dz\right| \\ &\le \frac{MR_\beta}{R_\gamma}\left(\frac{|w|}{R_\beta}\right)^n + \frac{MR_\alpha}{R_\gamma}\left(\frac{R_\alpha}{|w|}\right)^n \xrightarrow[n\to\infty]{} 0\end{aligned}$$

■

By substituting $w = (z - z_0)^{-1}$ in a series of negative powers

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} a_{-n} w^n$$

and applying Theorems 5.8, 5.15 and 5.16 to the power series in w, we may conclude:

Corollary 5.24. *Given a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, define*

$$R_1 = \inf\{|z - z_0| : f(z) \text{ converges}\}, \qquad R_2 = \sup\{|z - z_0| : f(z) \text{ converges}\}$$

Then:

1. *The series converges absolutely on the annulus $R_1 < |z - z_0| < R_2$ to a continuous function.*
2. *The convergence is uniform on any closed sub-annulus.*

Definition 5.25. The annulus $R_1 < |z - z_0| < R_2$ is the (open) *annulus of convergence* of the Laurent series. As with power series, convergence on the boundary circles must be checked separately.

We also obtain the analogues of Theorem 5.17 and Corollary 5.18: some details are in the exercises.

Corollary 5.26. *Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ has annulus of convergence $R_1 < |z - z_0| < R_2$.*

1. *(Term-by-term Integration) If $g(z)$ is continuous on a contour C lying inside the annulus, then*

$$\int_C g(z) f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \int_C g(z)(z - z_0)^n\,\mathrm{d}z$$

   In particular, $f(z)$ may be integrated term-by-term along C.

2. *(Analyticity/Derivatives) $f(z)$ is analytic on the annulus and $f'(z) = \sum_{n=-\infty}^{\infty} a_n n (z - z_0)^{n-1}$*
3. *(Uniqueness) $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is the Laurent series of $f(z)$.*

Now all the abstraction is out of the way, we can more easily compute Laurent series and Examples 5.22 are all valid. Here are a couple more.

Examples 5.27. 1. In accordance with part 2 of Corollary 5.26,

$$\frac{\mathrm{d}}{\mathrm{d}z} e^{1/z} = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=1}^{\infty} \frac{-z^{-1-n}}{(n-1)!} = -\frac{1}{z^2} \sum_{n=1}^{\infty} \frac{z^{-(n-1)}}{(n-1)!} = -\frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = -\frac{1}{z^2} e^{1/z}$$

2. To compute the integral $\oint_C \frac{1}{z^5} \sin z^2\,\mathrm{d}z$ on a simple closed contour encircling the origin, we use the Laurent series and observe that all but one of the integrals evaluates to zero:

$$\oint_C \frac{1}{z^7} \sin z^2\,\mathrm{d}z = \sum_{n=0}^{\infty} \oint_C \frac{(-1)^n}{(2n+1)!} z^{4n-5}\,\mathrm{d}z = \oint_C \frac{(-1)^1}{(2+1)!} z^{4-5} = -\frac{1}{3}\pi i$$

Exercises 5.4 1. Using Definition 5.21, directly compute the Laurent series of $f(z) = \frac{1}{z(2-z)}$ on the punctured disk $0 < |z| < 2$ and verify that you obtain the series in Example 5.20.

2. Find a Laurent series representation for each function. Also find $\oint_C f(z)\,dz$ where C is a simple closed curve in the given domain encircling the origin.

   (a) $f(z) = \frac{3}{z^2}e^{2z}$ whenever $|z| > 0$;

   (b) $f(z) = \cos\frac{i}{z}$ whenever $|z| > 0$;

   (c) $f(z) = \frac{1}{1+z^3}$ when $1 < |z|$ (*Hint: let* $w = z^{-1}$).

3. On each domain, find a Laurent series about $z_0 = 0$ for the function

$$f(z) = \frac{1}{z(z-2i)} = \frac{i}{2}\left(\frac{1}{z} - \frac{1}{z-2i}\right)$$

   (a) $D_1 = \{z : 0 < |z| < 2\}$;

   (b) $D_2 = \{z : |z| > 2\}$ (*again let* $w = z^{-1}$).

4. Repeat the previous question for

$$f(z) = \frac{1-2i}{(z-1)(z-2i)} = \frac{1}{z-1} - \frac{1}{z-2i}$$

   Also find $\oint_C f(z)\,dz$ where C is a simple closed curve in the given domain encircling the origin.

   (a) $D_1 = \{z : 0 < |z| < 1\}$ (*this is a Taylor series*);

   (b) $D_2 = \{z : 1 < |z| < 2\}$;

   (c) $D_3 = \{z : |z| > 2\}$.

5. Show that when $0 < |z-1| < 2$, we have

$$\frac{z}{(z-1)(z-3)} = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

6. Let a be complex number. Show that

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad \text{whenever } |a| < |z|$$

7. Suppose $f(z) = \sum a_n(z-z_0)^n$ is a series satisfying the hypotheses of Corollary 5.26.

   (a) Suppose part 1 has been proved. Explain why the function $f(z) - a_{-1}(z-z_0)^{-1}$ is analytic on the annulus. Hence conclude that $f(z)$ is analytic on the annulus.
   (*This is different to Corollary 5.18 since* $a_{-1}(z-z_0)^{-1}$ *has no anti-derivative on the annulus!*)

   (b) In order to mimic the proof of Corollary 5.18 to show that $f(z)$ is differentiable term-by-term, what properties must the curve C have?

   (c) Prove part 3 (*recall Exercise 5.3.6 - the same hint works!*).

6 Residues and Poles

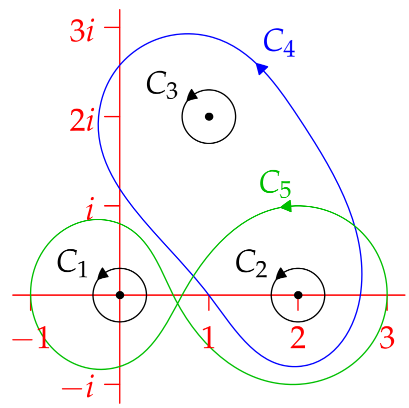
6.1 Residues and Cauchy's Residue Theorem

The goal of this section is the efficient computation of contour integrals of analytic functions. Essentially everything will depend on two crucial facts:

- The Cauchy–Goursat Theorem (f analytic on and inside $C \implies \oint_C f = 0$), and its extension to a region with finitely many interior boundary curves.
- If C encircles z_0, then $\oint_C (z-z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise} \end{cases}$

Make sure you are familiar with these statements before proceeding!

Example 6.1. Consider the function



$$f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$$

which is analytic except at the points $z_1 = 0$, $z_2 = 2$, $z_3 = 1+2i$. Several curves are drawn. The integral round the small circle C_1 should be clear from the 'crucial' facts:

$$\begin{aligned}\oint_{C_1} f(z)\,dz &= 3\oint_{C_1}\frac{dz}{z} + \oint_{C_1}\frac{dz}{z^2} + \oint_{C_1}\frac{5i\,dz}{z-2} + \oint_{C_1}\frac{dz}{z-1-2i} \\ &= 3\cdot 2\pi i + 0 + 0 + 0 = 6\pi i\end{aligned}$$

since the latter three integrands are analytic on and inside C_1. Similarly,

$$\oint_{C_2} f(z)\,dz = 5i\oint_{C_2}\frac{dz}{z-2} = -10\pi, \qquad \oint_{C_3} f(z)\,dz = \oint_{C_3}\frac{dz}{z-1-2i} = 2\pi i$$

More interesting are the curves C_4 and C_5. Since $f(z)$ is analytic on and between C_4 and C_2/C_3, Cauchy–Goursat tells us that

$$\oint_{C_4} f(z)\,dz = \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz = 2\pi(i-5)$$

C_5 appears a little trickier, though it becomes easy once you visualize it as *two* contours: the first encircles z_2 *counter-clockwise* while the second passes *clockwise* around z_1. We conclude that

$$\int_{C_5} f(z)\,dz = \oint_{C_2} f(z)\,dz - \oint_{C_1} f(z)\,dz = -2\pi(5+3i)$$

The example suggests that the value of any integral round a simple closed contour can be evaluated as a linear combination

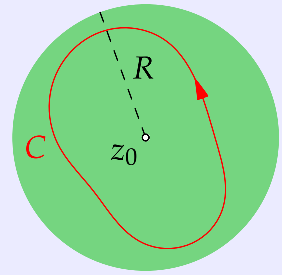
$$\int_C f(z)\,dz = \lambda_1\oint_{C_1} f + \lambda_2\oint_{C_2} f + \lambda_3\oint_{C_3} f$$

where λ_k denotes the number of times C orbits z_k in a counter-clockwise direction.

To properly develop this idea, we need a little formality.

Isolated Singularities and their Types

Definition 6.2. Suppose $f(z)$ is analytic on an punctured disk $0 < |z - z_0| < R$ of a point z_0, but not at z_0 itself. We call z_0 an *isolated singularity* of $f(z)$.



By Laurent's Theorem, $f(z)$ equals its Laurent series on this domain:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, \mathrm{d}z$$

and C is any simple closed contour encircling z_0. The *residue* of $f(z)$ at z_0 is the coefficient

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, \mathrm{d}z$$

The type of isolated singularity is determined by the structure of the Laurent series:

Removable Singularity The Laurent series is a Taylor series. There are no negative powers; the series, and $f(z)$, may be extended analytically to z_0.

Pole of order m The highest negative power in the Laurent series is $(z - z_0)^{-m}$. A pole of order 1 is typically called a *simple pole.*

Essential Singularity The Laurent series has infinitely many many non-zero negative terms.

Examples 6.3. 1. The series $f(z) = \sum_{n=0}^{\infty} 3^{-n}(z - 2i)^n$ defined on the punctured disk $0 < |z - 2i| < 3$ has a removable singularity at $z_0 = 2i$ with residue $\operatorname*{Res}_{z=2i} f(z) = 0$. Indeed the function is a geometric series and thus equals

$$f(z) = \frac{1}{1 - \frac{z-2i}{3}} = \frac{3}{3 + 2i - z}$$

on the punctured disk. Certainly this extends analytically to $f(2i) = 1$.

2. (Example 6.1) The function $f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$ is analytic on the punctured disk $0 < |z| < 0.3$ (inside the circle C_0). Since $\frac{5i}{z-2} + \frac{1}{z-1-2i}$ is also analytic at zero, the Laurent series of $f(z)$ has the form

$$f(z) = \frac{3}{z} + \frac{1}{z^2} + \sum_{n=0}^{\infty} a_n z^n$$

We conclude that $f(z)$ has a pole of order 2 at $z_0 = 0$ and residue $\operatorname*{Res}_{z=0} f(z) = 3$. Similarly, $f(z)$ has simple poles (order 1) at $z_1 = 2$ and $z_2 = 1 + 2i$ with

$$\operatorname*{Res}_{z=2} f(z) = 5i, \qquad \operatorname*{Res}_{z=1+2i} f(z) = 1$$

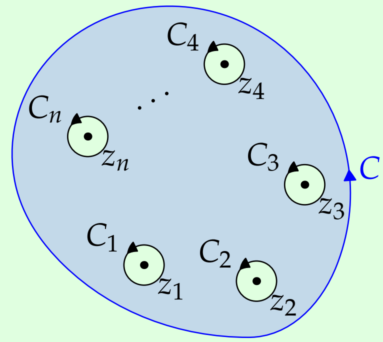
3. $e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$ has an essential singularity at zero with $\operatorname*{Res}_{z=0} e^{1/z} = 1$.

Theorem 6.4 (Cauchy's Residue Theorem). *If $f(z)$ is analytic on and inside a simple closed C, except at finitely many singular points $z_1, \ldots, z_n$, then*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

More generally, if C is closed and orbits the point z_k counter-clockwise w_k times (the winding number), then

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} w_k \operatorname*{Res}_{z=z_k} f(z)$$



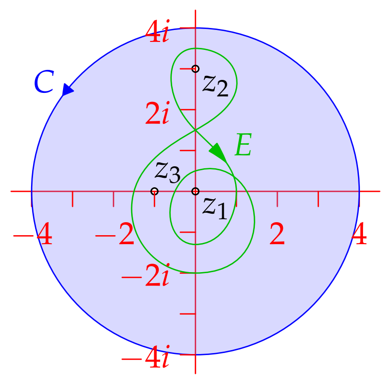
Proof. (Simple case). Center a small circle C_k at each z_k such that no other singularities lie on or inside C_k. Now apply Cauchy–Goursat to the domain on and between C and $C_1 \cup \cdots \cup C_n$. ∎

Examples 6.5. Let C be the circle with radius 4 centered at the origin and E the green curve drawn.

1. The function $f(z) = \frac{3(1+iz)}{z(z-3i)}$ has simple poles at $z_1 = 0$ and $z_2 = 3i$. There are several ways to compute the residues and thus the integrals $\oint_C f(z)\mathrm{d}z$ and $\oint_E f(z)\mathrm{d}z$.

   Partial Fractions For *this* example, this is very easy.

$$f(z) = \frac{i}{z} + \frac{2i}{z-3i} \implies \operatorname*{Res}_{z=0} f(z) = i, \quad \operatorname*{Res}_{z=3i} f(z) = 2i$$
$$\implies \oint_C f(z)\,\mathrm{d}z = 2\pi i(i+2i) = -6\pi$$



   The curve E is the union of three closed curves; twice *clockwise* around z_1 and once *counter-clockwise* around z_2. Therefore

$$\int_E f(z)\,\mathrm{d}z = 2\pi i\left[-2\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=3i} f(z)\right] = 0$$

   Laurent series Remember that we only need the z^{-1} terms for the residues!

$$\frac{3(1+iz)}{z(z-3i)} = \frac{i-z}{z(1-\frac{iz}{3})} = \left(iz^{-1}-1\right)\sum_{n=0}^{\infty}\left(\frac{iz}{3}\right)^n = \frac{i}{z} + \text{power series}$$
$$\frac{3(1+iz)}{z(z-3i)} = \frac{z-3i+2i}{(1+\frac{z-3i}{3i})(z-3i)} = \left(\frac{2i}{z-3i}+1\right)\sum_{n=0}^{\infty}\left(\frac{3i-z}{3i}\right)^n = \frac{2i}{z-3i} + \text{power series}$$

   Cauchy's formula Let C_k be a small circle around z_k, then

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2\pi i}\oint_{C_1} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C_1}\frac{3(1+iz)}{z(z-3i)}\,\mathrm{d}z = \left.\frac{3(1+iz)}{z-3i}\right|_{z=0} = i$$
$$\operatorname*{Res}_{z=3i} f(z) = \frac{1}{2\pi i}\oint_{C_2} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C_2}\frac{3(1+iz)}{z(z-3i)}\,\mathrm{d}z = \left.\frac{3(1+iz)}{z}\right|_{z=3i} = 2i$$

   We'll revisit this last approach in the next section.

2. Plainly $f(z) = z^2 \sin \frac{1}{z}$ has one isolated singularity at the origin. Using the Maclaurin series for $\sin z$, we see that this is an essential singularity, and can easily evaluate the required integrals:

$$z^2 \sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{1-2n} \implies \oint_C z^2 \sin \frac{1}{z} \, dz = 2\pi i \operatorname*{Res}_{z=0} \left(z^2 \sin \frac{1}{z} \right) = -\frac{\pi i}{3}$$

Since E loops twice *clockwise* around the origin, we obtain

$$\int_E z^2 \sin \frac{1}{z} \, dz = 2\pi i \cdot (-2) \operatorname*{Res}_{z=0} \left(z^2 \sin \frac{1}{z} \right) = \frac{2\pi i}{3}$$

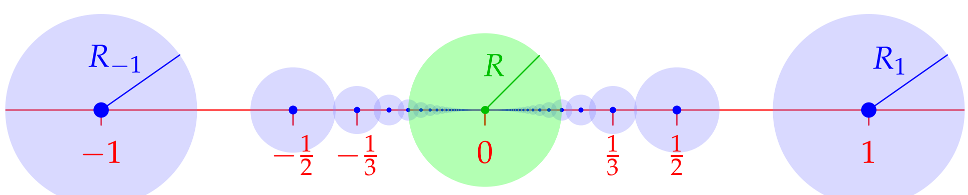
3. The function $f(z) = 3e^{1/z} + \frac{4}{z-7i} + \frac{2i}{z+1}$ has an essential singularity at the origin and simple poles at -1 and $7i$. Since the last of these lies *outside* the curves C, E, it does not contribute to either integral. Moreover, note that E loops *twice* clockwise around the origin and *once* clockwise around $z_3 = -1$. We therefore have

$$\oint_C f(z) \, dz = 2\pi i \left(\operatorname*{Res}_{z=0} 3e^{1/z} + \operatorname*{Res}_{z=-1} \frac{2i}{z+1} \right) = 2\pi i (3 + 2i)$$
$$\oint_E f(z) \, dz = 2\pi i \left(-2 \operatorname*{Res}_{z=0} 3e^{1/z} - \operatorname*{Res}_{z=-1} \frac{2i}{z+1} \right) = 2\pi i (-6 - 2i) = 4\pi (1 - 3i)$$

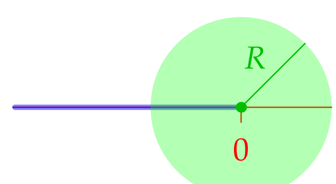
Non-isolated Singularities

Consider the converse of Definition 6.2: a singularity z_0 is non-isolated if *every* punctured disk $0 < |z - z_0| < R$ contains at least one point where $f(z)$ is non-analytic. Necessarily this requires $f(z)$ to be non-analytic at infinitely many points. Non-isolated singularities typically appear in two flavors.

Examples 6.6. 1. The function $f(z) = \left(e^{\frac{2\pi i}{z}} - 1\right)^{-1}$ has singularities at $z_0 = 0$ and whenever $z_n = \frac{1}{n}$ for each non-zero integer n. Each of the non-zero singularities is isolated (e.g. choose $R_n = \frac{1}{(|n|+1)^2}$). The remaining *limit point* $z_0 = 0$ is a non-isolated singularity: for any $R > 0$, the punctured disk $0 < |z| < R$ contains non-zero singularities.



2. The function $f(z) = \sqrt{z}$ has *branch point* $z_0 = 0$. For f to be analytic, we need to make a branch cut: for instance, the principal value of $\sqrt{z}$ has the non-positive real axis as a branch cut.
Since any domain $0 < |z| < R$ contains other points of the branch cut (where f is non-analytic), it follows that a branch point is a non-isolated singularity *for any branch* of f.



Exercises 6.1 1. For each of the following types of singularity, what, if anything, can you say about the value of the residue $\operatorname*{Res}_{z=z_0} f(z)$? Choose from 'Equals zero,' 'Non-zero,' 'No restriction'.

(a) Removable singularity.

(b) Simple pole.

(c) Pole of order $m \geq 2$.

(d) Essential singularity.

2. Find the residue at $z = 0$ of each function:

(a) $\dfrac{1}{z + 3z^2}$ (b) $z \cos \dfrac{1}{z}$ (c) $\dfrac{z - \sin z}{z}$

3. Let C be the circle $|z| = 3$. Evaluate the integrals using Cauchy's residue theorem:

(a) $\oint_C \dfrac{e^{-z}}{z^2}\, dz$ (b) $\oint_C \dfrac{e^{-z}}{(z-1)^2}\, dz$ (c) $\oint_C z^2 e^{1/z}\, dz$ (d) $\oint_C \dfrac{z+1}{z^2 - 2z}\, dz$

4. Suppose a closed contour C loops twice counter-clockwise around $z = i$ and three times clockwise around $z = 2$. Use residues to compute the integral

$$\int_C \frac{z+3}{(z-2)^2(z-i)} dz$$

5. Identify the type of singular point of each of the following functions and determine the residue:

(a) $\dfrac{1 - \cosh z}{z^3}$ (b) $\dfrac{1 - e^{2z}}{z^4}$ (c) $\dfrac{e^{2z}}{(z-1)^2}$

6. Suppose $f(z)$ is analytic at z_0 and define $g(z) = (z - z_0)^{-1} f(z)$. Prove:

(a) If $f(z_0) \neq 0$, then z_0 is a simple pole of $g(z)$ with $\operatorname*{Res}_{z=z_0} g(z) = f(z_0)$;

(b) If $f(z_0) = 0$, then z_0 is a removable singularity of $g(z)$.

7. Let $P(z)$ and $Q(z)$ be polynomials whose degrees satisfy $2 + \deg P \leq \deg Q$ and assume C is a simple closed contour such that all zeros of $Q(z)$ lie interior to C.

(a) Prove that $\oint_C \dfrac{P(z)}{Q(z)}\, dz = 0$

(*Hint: Try the substitution* $w = \frac{1}{z}$)

(b) What can you conclude if $\deg Q = \deg P + 1$?

6.2 Poles & Zeros

Recall Example 6.5 where we used Cauchy's integral formula as one of the methods for computing a residue. If the order of a pole is known, this approach is often fairly efficient.

Theorem 6.7. *A function $f(z)$ has a pole of order m at z_0 if and only if $f(z) = (z-z_0)^{-m}\phi(z)$ where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. In such a case,* [a]

$$f(z) = (z-z_0)^{-m}\sum_{n=0}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \implies \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\phi^{(m-1)}(z_0)$$

This specializes to $\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)$ for a simple pole.

[a] This formula works even if $\phi(z_0) = 0$; you only need $\phi(z)$ analytic and *non-constant* at z_0: a naïve application means you won't know the order of the pole and you'll have to differentiate more times than necessary!

Examples 6.8. 1. (Example 6.5) Write $f(z) = \dfrac{3(1+iz)}{z(z-3i)} = \dfrac{\phi_1(z)}{z} = \dfrac{\phi_2(z)}{z-3i}$ and verify:

Simple pole at $z_1 = 0$: $\phi_1(z) = \frac{3(1+iz)}{z-3i}$ is analytic, and $\phi_1(0) = \frac{3}{-3i} = i = \operatorname*{Res}_{z=0} f(z)$

Simple pole at $z_2 = 3i$: $\phi_2(z) = \frac{3(1+iz)}{z}$ is analytic, and $\phi_2(3i) = \frac{3(1-3)}{3i} = 2i = \operatorname*{Res}_{z=3i} f(z)$

2. Write $f(z) = \dfrac{1-2iz}{(z-1)(z-2i)^3} = \dfrac{\phi_1(z)}{z-1} = \dfrac{\phi_2(z)}{(z-2i)^3}$ and compute:

Simple pole at $z_1 = 1$: $\phi_1(z) = \frac{1-2iz}{(z-2i)^3}$ is analytic and non-zero at $z_1 = 1$. It follows that

$$\operatorname*{Res}_{z=1} f(z) = \phi_1(1) = \frac{1-2i}{(1-2i)^3} = \frac{1}{(1-2i)^2} = \frac{4i-3}{25}$$

Pole of order three at $z_2 = 2i$: $\phi_2(z) = \frac{1-2iz}{z-1} = -2i + \frac{1-2i}{z-1}$ is analytic and non-zero at $2i$, and

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{(3-1)!}\phi_2''(2i) = \frac{1-2i}{(z-1)^3}\bigg|_{z=2i} = \frac{-1}{(2i-1)^2} = \frac{3-4i}{25}$$

Proof. ($\Rightarrow$) By Laurent's Theorem, $f(z)$ equals its Laurent series. It moreover has a pole of order m at z_0 if and only if

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n = (z-z_0)^{-m}\sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n \quad \text{where } a_{-m} \neq 0 \tag{$*$}$$

Plainly $\phi(z) := \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$ is analytic at z_0 and satisfies $\phi(z_0) = a_{m-n} \neq 0$.

($\Leftarrow$) Taylor's Theorem says that $\phi(z)$ equals its Taylor series $\sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$, whence (uniqueness of representation) $(*)$ is *the* Laurent series of $f(z)$ and $f(z)$ has a pole of order m. ∎

As the examples show, the method is very effective when $f(z)$ is rational with low-order poles; as a bonus, it saves us from partial fractions! Its utility is more variable for other functions...

Examples 6.9. 1. Taking $\phi(z) = \frac{e^z}{z+1}$ shows that the non-rational function

$$f(z) = \frac{e^z}{(z-1)^2(z+1)} = \frac{\phi(z)}{(z-1)^2}$$

has a pole of order two at $z_0 = 1$ and moreover that

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\phi'(1) = \left.\frac{(z+1)e^z - e^z}{(z+1)^2}\right|_{z=1} = \frac{1}{4}e$$

2. Don't let the denominator fool you! At first glance we appear to have a pole of order *six*:

$$f(z) = \frac{6\sin z - 6z + z^3}{z^6} = \frac{\widetilde{\phi}(z)}{z^6} \overset{??}{\Longrightarrow} \operatorname*{Res}_{z=0} f(z) = \frac{1}{5!}\widetilde{\phi}^{(5)}(0) = \frac{6}{120} = \frac{1}{20}$$

However, if we apply the Maclaurin series for sine, we instead find a *simple pole*:

$$\begin{aligned} f(z) &= \frac{1}{z^6}\left(6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} - 6z + z^3\right) = z^{-6}\sum_{n=2}^{\infty}\frac{6(-1)^n}{(2n+1)!}z^{2n+1} \\ &= z^{-1}\sum_{m=0}^{\infty}\frac{6(-1)^m}{(2m+5)!}z^{2m} = \frac{1}{20z} + \frac{1}{840} + \cdots \implies \operatorname*{Res}_{z=0} f(z) = \frac{1}{20} \end{aligned}$$

Even though the residue was correct, our original $\widetilde{\phi}$ was wrong ($\widetilde{\phi}(0) \neq 0$). The correct function is the series $\phi(z) = 6\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+5)!}z^{2m}$.

Zeros of Analytic Functions

It turns out that poles and zeros of analytic functions are intimately related. We start by mirroring Definition 6.2 and Theorem 6.7.

Definition 6.10. Suppose $f(z)$ is analytic at z_0 and $f(z_0) = 0$. We say that z_0 is a *zero of order m* if $f^{(m)}(z_0)$ is the first *non-zero* derivative. We refer to a *simple zero* when $m = 1$.

A zero z_0 is *isolated* if it has some neighborhood with no other zeros:

$$\exists R > 0 \text{ such that } 0 < |z - z_0| < R \implies f(z) \neq 0$$

We are used to the idea of polynomial having a zero z_0 if and only if we can factorize out $z - z_0$. The tight link-up with Taylor series makes essentially this observation hold for *any* analytic function!

Lemma 6.11. *A function $f(z)$ has a zero z_0 of order m if and only if $f(z) = (z-z_0)^m\psi(z)$ where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$. Indeed, on some disk $|z - z_0| < R_0$,*

$$f(z) = \sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m\psi(z)$$

Examples 6.12. 1. $f(z) = z^4(z-2i)^{10} = z^4\psi_1(z) = (z-2i)^{10}\psi_2(z)$ has two zeros:

Order four at $z_1 = 0$: $\psi_1(z) = (z-2i)^{10} \implies \psi_1(0) = -1024 \neq 0$.

Order ten at $z_2 = 2i$: $\psi_2(z) = z^4 \implies \psi_2(2i) = 16 \neq 0$.

2. $g(z) = 17(z-4i)^3 \cos z$ has a zero of order three at $4i$, and simple zeros at each half-integer multiple $\left(\frac{1}{2} \pm k\right)\pi$ of π. For instance

$$\begin{aligned} g(z) &= 17(z-4i)^3 \cos\left(z - \frac{\pi}{2} + \frac{\pi}{2}\right) = -17(z-4i)^3 \sin\left(z - \frac{\pi}{2}\right) \\ &= -17(z-4i)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(z - \frac{\pi}{2}\right)^{2n+1} \\ &= \left(z - \frac{\pi}{2}\right)\left[-17(z-4i)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(z - \frac{\pi}{2}\right)^{2n}\right] = \left(z - \frac{\pi}{2}\right)\psi(z) \end{aligned}$$

The examples are typical: just as with singularities, the typical arrangement is for a zero of an analytic function to be *isolated.* Indeed, analytic functions with non-isolated zeros are very boring…

Theorem 6.13. *Let z_0 be a zero of an analytic function $f(z)$. The following are equivalent:*

1. *$f(z)$ is non-zero at some point of every neighborhood $|z - z_0| < \epsilon$ of z_0.*
2. *z_0 is a zero of some positive order m.*
3. *z_0 is an isolated zero.*

The distinction between conditions 1 and 3 is important: the first is weaker and the the equivalence is *false* for non-analytic functions. For example, at $z_0 = 0$ the non-analytic function $f(z) = z + \bar{z} = 2x$ satisfies condition 1 but not 3 (e.g. $f(\frac{\epsilon}{2}) \neq 0$). There is something to prove here!

Proof. ($1 \Rightarrow 2$) The Taylor series of $f(z)$ is non-zero, else $f(z)$ would be zero on some disk $|z - z_0| < \epsilon$. There must therefore be some minimum $m \in \mathbb{N}$ such that $f^{(m)}(z_0) \neq 0$, whence z_0 is a zero of order m.

($2 \Rightarrow 3$) $f(z) = (z - z_0)^m \psi(z)$ where $\psi(z)$ is analytic and $\psi(z_0) \neq 0$. Since $\psi(z)$ is continuous, it is non-zero on some disk $|z - z_0| < \epsilon$, and so also is $f(z)$. We conclude that z_0 is an isolated zero.

($3 \Rightarrow 1$) This is trivial.

■

Corollary 6.14. *If $f(z)$ is analytic on a connected open domain D containing z_0, and $f(z) = 0$ at each point of some contour C containing z_0, then $f(z) \equiv 0$ on D.*

Proof. This is the negation of the situation in the Theorem: plainly z_0 is not isolated and so $f(z) \equiv 0$ on some disk centered on z_0. The usual patching argument extends this to D.

■

This essentially proves the result regarding unique analytic continuations from earlier in the course.

Reciprocals switch poles and zeros

It seems intuitive that we can turn poles into zeros just by flipping a function upside down.

Theorem 6.15. *Let $f(z)$ be analytic at z_0 and $g(z) = \frac{1}{f(z)}$. Then, at z_0,*

$$f(z) \text{ has a zero of order } m \iff g(z) \text{ has a pole of order } m$$

The proof is an easy exercise in combining Theorem 6.7 and Lemma 6.11.

This approach yields a very quick method for computing residues of functions with *simple poles*, and is particularly useful which a function is multi-valued.

Corollary 6.16. *Suppose p, q are analytic at z_0 such that $f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)\psi(z)}$ has a simple pole at z_0. Plainly $q'(z_0) = \psi(z_0)$, from which*

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Examples 6.17. 1. The function $f(z) = \frac{p(z)}{q(z)} = \frac{(z^2+1)^2}{z-3} = \frac{(z-i)^2(z+i)^2}{z-3}$ has zeros of order two at $\pm i$ and a simple pole at $z = 3$. The reciprocal has the reverse arrangement: poles of order two at $\pm i$ and a simple zero at 3. Moreover,

$$\operatorname*{Res}_{z=3} f(z) = \frac{p(3)}{q'(3)} = \frac{(3^2+1)^2}{1} = 100$$

2. The function $g(z) = \frac{p(z)}{q(z)} = \frac{\sin z}{z^2+4} = \frac{\sin z}{(z-2i)(z+2i)}$ has simple poles at $\pm 2i$ with

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Res}_{z=-2i} f(z) = \frac{p(\pm 2)}{q'(\pm 2)} = \frac{\sin 2i}{4i} = \frac{1}{8}(e^2 - e^{-2}) = \frac{1}{4}\sinh 2$$

The reciprocal has simple poles at $z = n\pi$ for every $n \in \mathbb{Z}$: moreover

$$\operatorname*{Res}_{z=n\pi} \frac{1}{f(z)} = \frac{q(n\pi)}{p'(n\pi)} = \frac{n^2\pi^2+4}{\cos n\pi} = (-1)^n(n^2\pi^2+4)$$

3. Since $q(z) = e^{2z} - 1 = \sum_{n=1}^{\infty} \frac{2^n z^n}{n!} = z\sum_{n=0}^{\infty} \frac{2^n z^n}{(n+1)!}$ has a simple zero at $z = 0$, we see that

$$f(z) = \frac{\sqrt{z+4i}}{(z+i)^2\operatorname{Log}(z+2)(e^{2z}-1)} = \frac{p(z)}{q(z)}$$

has a simple pole at $z = 0$ (we use the principal value of $\sqrt{z+4i}$). Moreover

$$\operatorname*{Res}_{z=0} f(z) = \frac{p(0)}{q'(0)} = \frac{e^{\frac{i\pi}{4}}}{\ln 2} = \frac{1+i}{\sqrt{2}\ln 2}$$

We could instead have chosen $q(z) = (z+i)^2\operatorname{Log}(z+2)(e^{2z}-1)$, but the differentiation would have been much worse!

Counting Poles and Zeros

Definition 6.18. A function f is *meromorphic* on a domain D if it is analytic except at isolated *poles*. That is, f cannot have essential (or removable) singularities.

Theorem 6.19. *Suppose C is a positively oriented closed contour. If f is meromorphic on and inside C, then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = Z - P$$

where Z and P are the number of zeros and poles of f inside C, counted up to multiplicity.

Example 6.20. Consider, $f(z) = \frac{(z-i)^2\sin z}{(z-5)^4}$ where C is a large circle surrounding the points 0, i and 5. Plainly $Z = 2+1 = 3$ and $P = 4$. By an unpleasant application of the quotient rule (or better using logarithms, just be careful!), we obtain

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_C \frac{2}{z-i} + \frac{\cos z}{\sin z} - \frac{4}{z-5}\,\mathrm{d}z = 2 + \cos 0 - 4 = -1 = Z - P$$

Proof. If f has a zero of order m at $z = z_k$, then $f(z) = (z-z_k)^m\psi(z)$ on and inside a some small circle C_k centered at z_k, where $\psi(z)$ is analytic and *non-zero*. But then

$$\begin{aligned}\oint_{C_k} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \oint_{C_k} \frac{m(z-z_0)^{m-1}\psi(z) + (z-z_0)^m\psi'(z)}{(z-z_0)^m\psi(z)}\,\mathrm{d}z = \oint_{C_k} \frac{m}{z-z_0} + \frac{\psi'(z)}{\psi(z)}\,\mathrm{d}z \\ &= 2\pi i m\end{aligned}$$

Similarly, if f has a pole of order m, then we repeat with $f(z) = (z-z_k)^{-m}\phi(z)$ to obtain

$$\begin{aligned}\oint_{C_k} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \oint_{C_k} \frac{-m(z-z_0)^{-m-1}\phi(z) + (z-z_0)^{-m}\phi'(z)}{(z-z_0)^{-m}\phi(z)}\,\mathrm{d}z = \oint_{C_k} -\frac{m}{z-z_0} + \frac{\phi'(z)}{\phi(z)}\,\mathrm{d}z \\ &= -2\pi i m\end{aligned}$$

Cauchy's residue theorem completes the proof. ∎

Properties of Singularities

We finish by considering some equivalent conditions for the various types of singularities.

Theorem 6.21 (Removable Singularities). *Suppose $f(z)$ has an isolated singularity at z_0 (and is therefore analytic on a punctured disk $0 < |z| < R$). The following are equivalent:*

1. *The singularity is removable.*
2. *$\lim_{z\to z_0} f(z)$ exists and is finite.*
3. *$f(z)$ is bounded on some $0 < |z - z_0| < \delta$.*

Proof. For simplicity, suppose that $z_0 = 0$.

(1 ⇒ 2) $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $0 < |z| < R$, whence $\lim_{z \to 0} f(z) = a_0$.

(2 ⇒ 3) This is almost tautological but it bears repeating: If $\lim_{z \to 0} f(z) = a_0$ exists and is finite then choose $\epsilon = |a_0|$ in the definition;

$$\begin{aligned} \exists \delta > 0 \text{ such that } 0 < |z| < \delta &\implies |f(z) - a_0| < |a_0| \\ &\implies |f(z)| = |f(z) - a_0 + a_0| \le 2\,|a_0| \end{aligned}$$

(3 ⇒ 1) Consider

$$g(z) = \begin{cases} z^2 f(z) & \text{if } 0 < z < \delta \\ 0 & \text{if } z = 0 \end{cases}$$

Since f is bounded, we may compute the limit

$$\lim_{z \to 0} \frac{g(z) - g(0)}{z} = \lim_{z \to 0} z f(z) = 0$$

whence $g(z)$ is differentiable at zero! Since it is already differentiable on the punctured neighborhood $0 < |z| < \min\{\delta, R\}$, it is therefore analytic on the *disk* $|z| < \min\{\delta, R\}$ and equals its Maclaurin series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

However $g(0) = 0 = g'(0) \implies b_0 = b_1 = 0$ and so

$$g(z) = z^2 \sum_{n=0}^{\infty} b_{n-2} z^n \implies f(z) = \sum_{n=0}^{\infty} b_{n-2} z^n \text{ on } 0 < |z| < \min\{\delta, R\}$$

whence f has a removable singularity. ∎

We leave the remaining results as exercises.

Theorem 6.22. *Suppose f has an isolated singularity at $z = z_0$.*

1. *z_0 is a pole if and only if $\lim_{z \to z_0} f(z) = \infty$.*
2. *If z_0 is essential and $w \in \mathbb{C} \cup \{\infty\}$, then $\exists z_n \to z_0$ such that $f(z_n) \to w$.*

This second result is the *Casorati–Weierstrass Theorem*; the range of $f(z)$ is *dense* in a neighborhood of an essential singularity. A stronger result is available, through its proof is beyond us.

Theorem 6.23 (Picard). *If z_0 is an essential singularity of $f(z)$, then $f(z)$ takes every complex value* except at most one *in any neighborhood of z_0.*

Example 6.24. Let $f(z) = e^{1/z}$ at $z_0 = 0$. If $w = e^{1/z}$, write $w = re^{i\theta}$ with $0 \le \theta < 2\pi$, from which

$$e^{\ln r + i\theta} = e^{1/z} \implies \frac{1}{z} = \ln r + i\theta + 2\pi i n$$

for any integer n. If $n > 0$, observe that

$$\left|\frac{1}{z}\right| = \sqrt{(\ln r)^2 + (\theta + 2\pi n)^2} > 2\pi n$$

whence $|z|$ can be chosen arbitrarily small. A suitable z thus exists in any punctured disk $0 < |z| < \delta$.

Exercises 6.2 1. Determine the order of each pole and its residue.

(a) $f(z) = \dfrac{z+1}{z^2+9}$ (b) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$

2. Show that:

(a) $\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ when $|z| > 0$ and $\arg z \in (0, 2\pi)$

(b) $\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$

(c) $\operatorname*{Res}_{z=z_n} (z \sec z) = (-1)^{n+1} z_n$, where $z_n = \frac{\pi}{2} + n\pi$ and $n \in \mathbb{Z}$

3. Find the value of the integral

$$\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

when C is each of the circles: (a) $|z-2| = 2$ and (b) $|z| = 4$.

4. Let C be the circle $|z| = 2$. Evaluate $\oint_C \tan z\,dz$.

5. Prove Theorem 6.15.

6. Let $f(z) = \left(z \sin \frac{\pi}{z}\right)^{-1}$

(a) Evaluate $\operatorname*{Res}_{z=\frac{1}{n}} f(z)$ for each $n \in \mathbb{Z}$.

(b) Why doesn't $\operatorname*{Res}_{z=0} f(z)$ make sense?

7. Suppose $f(z)$ is analytic and non-constant at z_0. Prove that

$$\exists \epsilon > 0 \text{ such that } 0 < |z - z_0| < \epsilon \implies f(z) \neq f(z_0)$$

8. Suppose that C is the rectangle whose sides are the lines $x = \pm 2, y = 0$ and $y = 1$. Prove that

$$\oint_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}$$

(*Hint: the integrand has four simple poles, only two of which lie inside* C)

The last two questions are more of a challenge

9. Prove Theorem 6.22. For simplicity, assume $z_0 = 0$.

   Hint 1: $f(z)$ has a pole if and only if $\frac{1}{f(z)}$ has a zero.

   Hint 2: If no such sequence exists, show that $g(z) := \frac{1}{f(z)-w}$ is analytic and bounded.

10. Suppose $f(z)$ is analytic on and inside a simple closed curve C, and that it has no zeros on C. We consider the integral

$$I = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, \mathrm{d}z$$

   (a) Explain why I counts the number of zeros (including multiplicity) of $f(z)$ inside C.

   (b) (Winding number) Explain why I also counts the number times the curve $\gamma = f(C)$ orbits the origin counter-clockwise.
   (*Hint: let $w = f(z)$*)

   (c) (Rouche's Theorem) Suppose that $|g(z)| < |f(z)|$ for all $z \in C$. Prove that the number of zeros of $f + g$ inside C equals that of f.
   (*Hint: Apply part (a) to the product $f + g = f \cdot (1 + \frac{g}{f})$ and consider why the function $1 + \frac{g}{f}$ has winding number zero*)

   (d) Since $\left|4z^3\right| > \left|z^{22} + 2i\right|$ on the circle $|z| = 1$, how many solutions (up to multiplicity) are there to the equation $z^{22} + 4z^3 + 2i = 0$ on the domain $|z| < 1$?

6.3 Improper Integrals

We describe a natural application of residues to the evaluation of certain *real* improper integrals. We start with an alternative definition of improper integral more suited to our purposes.

Definition 6.25. Provided the limit exists, the *Cauchy principal value* of $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is the limit

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

This is a potentially misleading interpretation of the improper integral. In standard calculus the definition requires *two* limits, *both* of which must exist for the integral to converge:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x := \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,\mathrm{d}x + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,\mathrm{d}x$$

If $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ converges, then it certainly equals its Cauchy principal value. However, the converse isn't true.

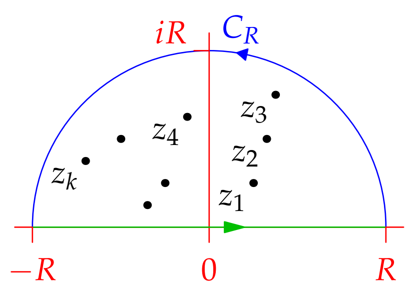
Example 6.26. If $f(x)$ is *any* odd function $\big(f(-x) = -f(x)\big)$, then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x = \lim_{R\to\infty} 0 = 0$$

If either of the 1-sided improper integrals diverges, then the the full integral also diverges: e.g.

$$\text{P.V.} \int_{-\infty}^{\infty} x^3\,\mathrm{d}x = 0 \quad \text{but} \quad \int_{0}^{\infty} x^3\,\mathrm{d}x \text{ diverges} \quad \Longrightarrow \quad \int_{-\infty}^{\infty} x^3\,\mathrm{d}x \text{ diverges}$$

Residue theory supplies a neat trick for computing Cauchy principal values:



1. Suppose $f(x)$ is the restriction to the real line of a *complex function* $f(z)$ which is analytic except at finitely many poles $z_1, \ldots, z_n$ in the upper half-plane $\operatorname{Im} z > 0$;

2. Choose $R > 0$ so that $R > |z_k|$ for each k and let C_R be the counter-clockwise upper semi-circle centered at the origin with radius R. By Cauchy's Residue Theorem,

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

3. If $\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0$, then $\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$

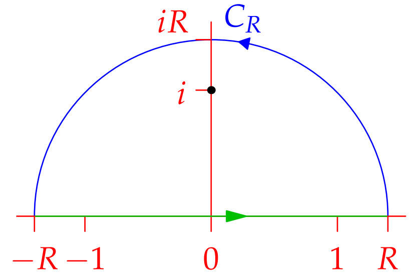
Mostly we will apply the method to rational functions $f(z) = \frac{p(z)}{q(z)}$, though it works more generally. Beyond ease of residue calculation, the reason is that $\deg q \geq \deg p + 2$ is enough to guarantee that step 3 applies (Exercise 5).

Examples 6.27. 1. $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$ has simple poles at $\pm i$. Provided $|z| = R > 1$,

$$\left|z^2+1\right| \geq \left||z|^2 - 1\right| = R^2 - 1 \implies \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$$

$$\implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{2\pi i}{2i} = \pi$$



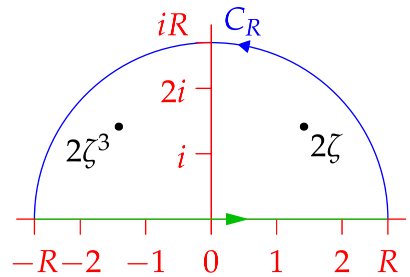
Compare with the usual calculus method:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1} x\Big|_{-R_1\to-\infty}^{R_2\to\infty} = \pi$$

2. $f(z) = \frac{4(z^2-1)}{z^4+16}$ has simple poles at $\pm 2\zeta, \pm 2\zeta^3$ where $\zeta = e^{\frac{\pi i}{4}}$. Let $p(z) = 16(z^2-1)$ and $q(z) = z^4+16$, so that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0^2-1}{z_0^3}$$

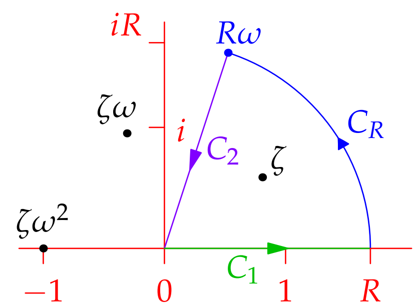
When $|z| = R > 2$, we see that



$$\left|z^4+16\right| \geq R^4 - 16 \implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{4\pi R(R^2+1)}{R^4-16} \to 0$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \left(\operatorname*{Res}_{z=2\zeta} f(z) + \operatorname*{Res}_{z=2\zeta^3} f(z)\right) = 2\pi i \left(\frac{4\zeta^2-1}{8\zeta^3} + \frac{4\zeta^6-1}{8\zeta^9}\right) = \frac{3\pi}{2\sqrt{2}}$$

3. Variations are possible, for instance by taking only part of a semi-circular arc. The function $f(z) = \frac{1}{z^5+1}$ has five simple poles: the fifth-roots of -1.
Since the pole $\zeta\omega^2 = -1$ lies on the negative real axis, the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ diverges. Instead consider the arcs in the picture when $R > 1$. Parametrizing C_2 via $z(t) = t\omega$,



$$\int_{C_2} \frac{1}{z^5+1}\,\mathrm{d}z = \int_R^0 \frac{\omega}{t^5+1}\,\mathrm{d}t = -\omega \int_0^R \frac{1}{t^5+1}\,\mathrm{d}t$$

$$= -\omega \int_{C_1} \frac{1}{z^5+1}\,\mathrm{d}z$$

$$\implies (1-\omega) \int_0^R \frac{1}{x^5+1}\,\mathrm{d}x + \int_{C_R} \frac{1}{z^5+1}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=\zeta} \frac{1}{z^5+1} = \frac{2\pi i}{5\zeta^4} = \frac{2\pi i}{5\omega^2}$$

When $|z| = R > 1$, we see that $\left|z^5+1\right| \geq R^5 - 1 \implies \left|\int_{C_R} \frac{1}{z^5+1}\,\mathrm{d}z\right| \leq \frac{2\pi R}{5(R^5-1)} \xrightarrow[R\to\infty]{} 0$, and we conclude

$$\int_0^{\infty} \frac{1}{x^5+1}\,\mathrm{d}x = \frac{2\pi i}{5(\omega^2-\omega^3)} = \frac{2\pi i}{5\zeta\omega^2(\zeta^{-1}-\zeta)} = \frac{2\pi i}{5(2i\sin\frac{\pi}{5})} = \frac{\pi}{5}\csc\frac{\pi}{5}$$

Jordan's Lemma

It is often useful, particularly when computing Fourier transforms,[12] to evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\cos ax\,\mathrm{d}x + i\int_{-\infty}^{\infty} f(x)\sin ax\,\mathrm{d}x$$

where $a > 0$ is a real constant and $f : \mathbb{R} \to \mathbb{C}$ is a given function. If $f(x)$ is real-valued, then the above breaks the integral into real and imaginary parts. Given reasonable conditions on $f(x)$, the above method can often be employed.

Example 6.28. The function $f(z) = \frac{e^{3iz}}{z^2+4}$ is analytic on the upper half-plane except at the simple pole $z = 2i$. With $R > 2$ and C_R the usual semi-circle, we see that

$$\left|e^{3iz}\right| = e^{-3y} \le 1 \implies \left|\int_{C_R} f(z)e^{3iz}\,\mathrm{d}z\right| \le \frac{\pi R}{R^2-4} \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{3ix}}{x^2+4}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{3iz}}{z^2+4} = 2\pi i\frac{e^{-6}}{4i} = \frac{1}{2}\pi e^{-6}$$

Since this is real, we see that the result is in fact the integral $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,\mathrm{d}x$. We don't need the Cauchy principal value of the integral here since the full improper integral converges. The corresponding imaginary integral is trivially zero since $\frac{\sin x}{x^2+1}$ is an odd function.

To assist with these computations, we state the following result without proof.

Theorem 6.29 (Jordan's Lemma). *Let $a, R_0 > 0$ be given and suppose $f(z)$ is analytic at all points exterior to C_{R_0} in the upper half-plane. Suppose also that*

$$\forall R > R_0,\ \exists M_R \text{ such that } |f(z)| \le M_R \text{ and } \lim_{R\to\infty} M_R = 0$$

Then $\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,\mathrm{d}z = 0$. If $f(z)$ also satisfies the hypotheses of our method, then

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x = 2\pi i\sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)e^{iaz}$$

Example 6.30. If $f(x) = \frac{x+1}{x^2+9}$ and $R > 3$, then

$$|f(z)| = \frac{|z+2|}{|z^2+9|} \le \frac{R+2}{R^2-9} = M_R \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{(x+2)e^{iax}}{x^2+9}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=3i} \frac{(z+2)e^{iaz}}{z^2+9} = \frac{2\pi i(2+3i)e^{-3a}}{6i} = \frac{\pi(2+3i)}{3}e^{-3a}$$

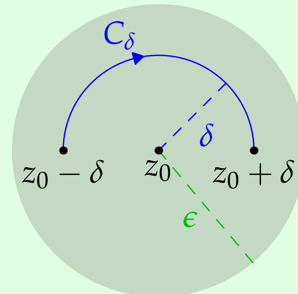
By considering even and odd functions, etc., we can rewrite this as

$$\int_0^{\infty} \frac{\cos ax}{x^2+9}\,\mathrm{d}x = \frac{\pi}{6}e^{-3a} \qquad \int_0^{\infty} \frac{x\sin ax}{x^2+9}\,\mathrm{d}x = \frac{\pi}{2}e^{-3a}$$

[12] The Fourier transform of $f(x)$ is the function $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x$.

Indented paths Another modification allows $f(z)$ to have a simple pole on the real axis.

Lemma 6.31. *Let D be the disk $|z - z_0| \le \epsilon$, let $\delta < \epsilon$, and let C_δ be the clockwise semi-circle in the picture.*



1. *If $\phi(z)$ is analytic on D, then $\lim\limits_{\delta \to 0} \int_{C_\delta} \phi(z)\,\mathrm{d}z = 0$.*

2. *If $f(z)$ is analytic on $D \setminus \{z_0\}$ with a simple pole at z_0, then*

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -\pi i \operatorname*{Res}_{z=z_0} f(z)$$

More generally, if C_δ spans θ radians clockwise round z_0, then $\lim\limits_{\delta\to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -i\theta \operatorname*{Res}\limits_{z=z_0} f(z)$.

Proof. 1. ϕ is continuous on D and thus bounded by some M: but then

$$\left| \int_{C_\delta} \phi(z)\,\mathrm{d}z \right| \le M\pi\delta$$

2. The Laurent series expansion of $f(z)$ on $D \setminus \{z_0\}$ is

$$f(z) = \frac{a_{-1}}{z - z_0} + \phi(z)$$

where $a_{-1} = \operatorname*{Res}\limits_{z=z_0} f(z)$ and $\phi(z)$ is analytic on D. Now evaluate

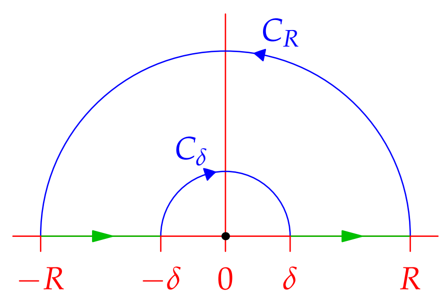
$$\int_{C_\delta} \frac{a_{-1}}{z - z_0}\,\mathrm{d}z = a_{-1} \int_\pi^0 \frac{1}{\delta e^{i\theta}} i\delta e^{i\theta}\,\mathrm{d}\theta = -ia_{-1} \int_0^\pi \mathrm{d}\theta = -\pi i a_{-1}$$

■

Example 6.32. Consider $f(z) = \frac{e^{iz}}{z}$. If $0 < \delta < R$, then

$$\left(\int_{-R}^{-\delta} + \int_\delta^R \right) f(x)\,\mathrm{d}x = \left(\int_{-R}^{-\delta} + \int_\delta^R \right) \frac{\cos x + i \sin x}{x}\,\mathrm{d}x = 2i \int_\delta^R \frac{\sin x}{x}\,\mathrm{d}x$$

by even/oddness. Moreover, by Lemma 6.31,



$$\lim_{\delta \to 0} \int_{C_\delta} \frac{e^{iz}}{z}\,\mathrm{d}z = -i\pi \operatorname*{Res}_{z=0} f(z) = -i\pi$$

Since $|f(z)| = \frac{e^{-y}}{R} \le \frac{1}{R}$ on C_R, Jordan's lemma tells us that

$$0 = 2i \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - i\pi \implies \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

The example relied on the evenness of $\frac{\sin x}{x}$ and on the fact that the region of the half-plane between C_R and C_δ contains no poles of $f(z)$. We essentially evaluated $\int_0^R \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2}\int_{-R}^R \frac{\sin x}{x}\,\mathrm{d}x$ using an *indented path* lying on the x-axis but dodging round the simple pole at zero. Many other versions of this trick are possible!

Exercises 6.3 Many of these problems require extensive calculation to evaluate using residues: take your time and use it as an excuse to practice the previous section.

1. Use residues to verify the values of the improper integrals:

   (a) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4}$ (b) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{x^6+1} = \frac{\pi}{6}$

   (c) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$ (d) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$

2. Find the Cauchy principal value of the integrals:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^2+2x+2}$ (b) $\displaystyle\int_{-\infty}^\infty \frac{x\mathrm{d}x}{(x^2+1)(x^2+2x+2)}$

3. Let m, n be integers where $0 \le m \le n-2$. By mimicking Example 6.27.3, prove that

$$\int_0^\infty \frac{x^m}{x^n+1}\,\mathrm{d}x = \frac{\pi}{n}\csc\frac{(m+1)\pi}{n}$$

4. (a) If $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ converges, prove that it equals its Cauchy principal value.

   (b) Suppose $f(x)$ is an even function and that P.V. $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists. Prove that $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists and has the same value.

5. If $f(x) = \frac{p(x)}{q(x)}$ is a rational function where $q(x)$ has no zeros and where $2 + \deg p \le \deg q$, prove that $\int_0^\infty f(x)\,\mathrm{d}x$ converges.

   (*Hint: let p, q be monic and recall the comparison test for improper integrals*)

6. Prove the integration formulae:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$ if $a > b > 0$

   (b) $\displaystyle\int_0^\infty \frac{\cos ax\,\mathrm{d}x}{(x^2+b^2)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab}$ if $a, b > 0$

7. Evaluate the integrals:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{x\sin x\,\mathrm{d}x}{(x^2+1)(x^2+4)}$ (b) $\displaystyle\int_0^\infty \frac{x^3\sin x\,\mathrm{d}x}{(x^2+1)(x^2+9)}$

8. If a is any real number and $b > 0$, find the Cauchy principal value of $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x+a)^2+b^2}$

9. Use the function $f(z) = z^{-2}(e^{iaz} - e^{ibz})$ and an indented contour around $z_0 = 0$ to prove that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} = \frac{\pi}{2}(b-a) \quad a, b \ge 0$$

10. By integrating the function $f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp(-\frac{1}{2}\log z)}{z^2+1}$ where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ along an indented contour, prove that

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

11. What happens to part 2 of Lemma 6.31 if $f(z)$ is analytic on $D \setminus \{z_0\}$ but has a pole of order $m \geq 2$ at z_0.

12. (Hard) A similar trick can be applied with sequences of boundary curves C_N. For instance, for each $N \in \mathbb{N}$, let C_N denote the positively oriented boundary of the square whose edges lie along the lines $x, y = \pm\left(N + \frac{1}{2}\right)\pi$. Prove that

$$\oint_{C_N} \frac{\mathrm{d}z}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

Hence conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$