3 Surfaces

3.1 Regular Parametrized Surfaces

We approach surfaces in $\mathbb{E}^3$ similarly to how we considered curves; a parametrized surface will be a *function* $\mathbf{x} : U \to \mathbb{E}^3$ where U is some open subset of the plane $\mathbb{R}^2$. Our main purpose is to develop and measure the *curvature* of a surface in terms of the parametrizing function $\mathbf{x}$.

Our primary definition should mostly be familiar from elementary multivariable calculus.

Definition 3.1. A *(smooth local) surface*[16] is the range $S = \mathbf{x}(U)$ of a smooth function $\mathbf{x} : U \to \mathbb{E}^3$, where U is a connected open subset of $\mathbb{R}^2$.

Given co-ordinates u, v on U, the *co-ordinate tangent vector fields* are the partial derivatives $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$.

The *exterior derivative* or *differential* of the surface is the vector-valued 1-form $\mathrm{d}\mathbf{x} = \mathbf{x}_u \,\mathrm{d}u + \mathbf{x}_v \,\mathrm{d}v$.

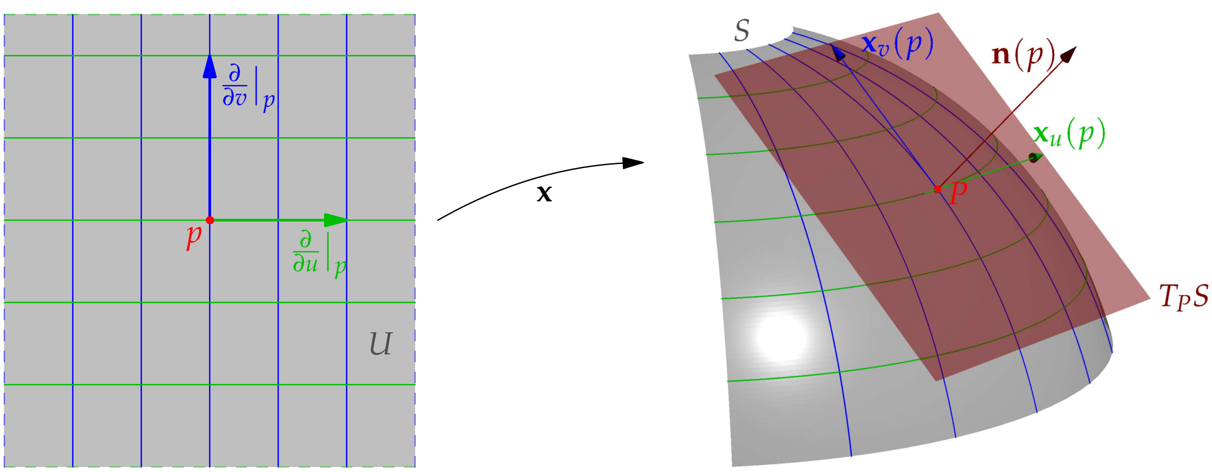
A surface is *regular* at $P = \mathbf{x}(p)$ if the tangent vectors $\mathbf{x}_u(p)$ and $\mathbf{x}_v(p)$ are *linearly independent*: otherwise said, at P, the surface has a well-defined

$$\textit{Tangent plane } \; T_P S = \operatorname{Span}\{\mathbf{x}_u(p), \mathbf{x}_v(p)\} \text{ (a 2-dim subspace of } T_P\mathbb{E}^3\text{), and}$$

$$\textit{Unit normal vector } \; \mathbf{n}(p) = \frac{\mathbf{x}_u(p) \times \mathbf{x}_v(p)}{||\mathbf{x}_u(p) \times \mathbf{x}_v(p)||} \in T_P\mathbb{E}^3$$

A surface is *regular* if it is regular everywhere. An *orientation* is a smooth choice of *unit normal vector field* $\mathbf{n}$ (this is not always possible!).

In what follows, we will often refer to the function $\mathbf{x}$ as the surface.



The partial derivatives $\mathbf{x}_u(p), \mathbf{x}_v(p)$ really are *tangent to the surface* at $\mathbf{x}(p)$: if $p = (u_0, v_0)$ then the *curve* $\mathbf{y}(t) := \mathbf{x}(t, v_0)$ lies in the surface and passes through $P = \mathbf{x}(p)$; its tangent vector at P is then

$$\mathbf{y}'(u_0) = \lim_{h \to 0} \frac{\mathbf{y}(u_0 + h) - \mathbf{y}(u_0)}{h} = \lim_{h \to 0} \frac{\mathbf{x}(u_0 + h, v_0) - \mathbf{x}(p)}{h} = \mathbf{x}_u(p)$$

[16] A surface is typically parametrized by several overlapping functions $\mathbf{x}$. Our definition is *local* since there is only one $\mathbf{x}$.

To help distinguish between the domain and codomain, we standardize notation.

Domain $U \subseteq \mathbb{R}^2$: *Points* are written *lower case* or as *row vectors*: e.g. $p = (u_0, v_0) \in U$. Typically we'll use u, v as *co-ordinates* unless it is more natural to use angles such as ϕ, θ.

Tangent vectors/fields are written with an arrow in our *new* notation: e.g. $\vec{w}_p = \frac{\partial}{\partial u}\Big|_p \in T_p\mathbb{R}^2$.

Codomain $\mathbb{E}^3$: *Points* are written *upper case* or as *row vectors*, e.g. $P = (3, 4, 8) \in \mathbb{E}^3$. Co-ordinates on $\mathbb{E}^3$ will typically be x, y, z.

Vectors are written *bold-face* as either row or column vectors: e.g. $\mathbf{x}(u, v) = \big(u, v, u^2 + v^2\big)$.

Tangent vectors/fields use the *old* notation:[17] e.g. if $P = \mathbf{x}(p)$, then $\mathbf{x}_u(p) = \frac{\partial \mathbf{x}}{\partial u}\Big|_p \in T_P\mathbb{E}^3$.

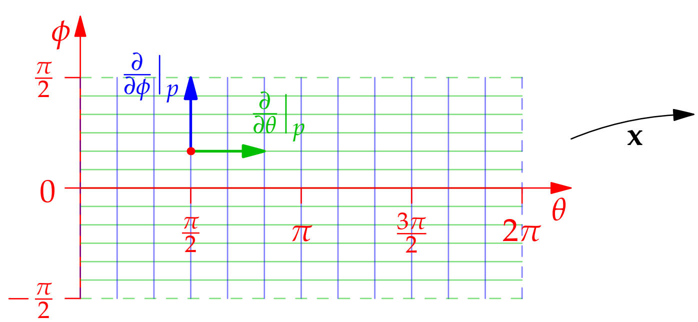
Example 3.2. Consider the sphere of radius a parametrized using spherical polar co-ordinates:

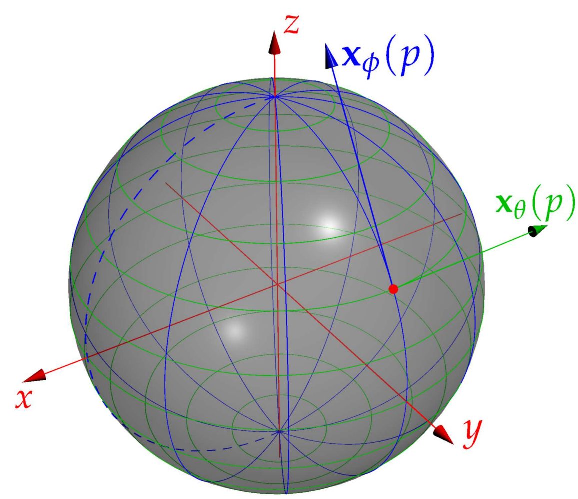
$$\mathbf{x}(\theta, \phi) = a\begin{pmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ \sin\phi\end{pmatrix}, \qquad \mathrm{d}\mathbf{x} = \mathbf{x}_\theta\,\mathrm{d}\theta + \mathbf{x}_\phi\,\mathrm{d}\phi = a\cos\phi\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}\mathrm{d}\theta + a\begin{pmatrix}-\cos\theta\sin\phi\\ -\sin\theta\sin\phi\\ \cos\phi\end{pmatrix}\mathrm{d}\phi$$

Here $\mathbf{x} : U \to \mathbb{E}^3$, where $U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is an open rectangle. The image $S = \mathbf{x}(U)$ is the sphere minus the (dashed) semicircle $\mathbf{x}(0, \phi)$. While we could extend θ to wrap round the sphere, note that the co-ordinates cannot be extended to the north or south poles without sacrificing *regularity*:

$$\mathbf{x}_\theta = a\cos\phi\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix} = \mathbf{0} \text{ when } \phi = \pm\frac{\pi}{2}$$

The unit normal field is simply $\mathbf{n} = \frac{1}{a}\mathbf{x}$.





Also observe how the tangent vectors $\frac{\partial}{\partial \phi}\Big|_p, \frac{\partial}{\partial \theta}\Big|_p \in T_p\mathbb{R}^2$ are mapped by the differential $\mathrm{d}\mathbf{x}$ to the tangent vectors

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\phi}\bigg|_p = \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial \phi}\bigg|_p\right), \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\theta}\bigg|_p = \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial \theta}\bigg|_p\right) \in T_{\mathbf{x}(p)}S$$

17 To also use our new notation in $\mathbb{E}^3$ would require a subtle redefinition of $\mathrm{d}\mathbf{x}$: if $\vec{w}$ is a vector field on U, then $\mathrm{d}\mathbf{x}(\vec{w})$ is the vector field on S such that $\big(\mathrm{d}\mathbf{x}(\vec{w})\big)[f] = \vec{w}[f \circ \mathbf{x}]$ for all $f : S \to \mathbb{R}$. In co-ordinates this benefits from tensor notation:

$$\mathbf{x}(u_1, u_2) = \Big(x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2)\Big) \implies \mathrm{d}\mathbf{x} = \sum_{i,j} \frac{\partial x_j}{\partial u_i}\frac{\partial}{\partial x_j} \otimes \mathrm{d}u_i$$

This approach is necessary in more general situations, but is overkill for our purposes.

Theorem 3.3. *Let $S = \mathbf{x}(U)$ be a smooth surface containing the point $P = \mathbf{x}(p)$:*

1. *The differential at p is a linear map $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P\mathbb{E}^3$ mapping tangent vectors in $\mathbb{R}^2$ to vectors tangent to S.*

2. *S is regular at P if and only if $\mathrm{d}\mathbf{x}$ is* injective (1–1) *at p. In such a case we can view it as an invertible linear map $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P S$.*

Proof. 1. The differential at p is linear since the co-ordinate 1-forms $\mathrm{d}u, \mathrm{d}v$ are linear: indeed

$$\mathrm{d}\mathbf{x}\left(a\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p\right) = \mathbf{x}_u(p)\,\mathrm{d}u\left(a\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p\right) + \mathbf{x}_v(p)\,\mathrm{d}v\left(a\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p\right)$$
$$= a\mathbf{x}_u(p) + b\mathbf{x}_v(p) = a\,\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial u}\bigg|_p\right) + b\,\mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial v}\bigg|_p\right)$$

This expression is moreover tangent to S at $\mathbf{x}(p)$: if this last assertion is unconvincing, see Exercise 7.

2. The range of $\mathrm{d}\mathbf{x}$ at p is plainly $\operatorname{Span}\{\mathbf{x}_u(p), \mathbf{x}_v(p)\}$. This is 2-dimensional (and thus defines the tangent plane) if and only if $\operatorname{rank} \mathrm{d}\mathbf{x} = 2 \iff \mathrm{d}\mathbf{x}$ is 1–1. ■

It is worth reiterating the two crucially important properties of $\mathrm{d}\mathbf{x}$:

- At a regular point, $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_P S$ is an invertible linear map. We shall shortly use this to *pull-back* calculations from S to U.

- It is co-ordinate independent and thus does not depend on the parametrization of S. Recall that this follows since $\mathrm{d}\mathbf{x}$ is the unique 1-form which satisfying $\mathrm{d}\mathbf{x}(\vec{w}) = \vec{w}[\mathbf{x}]$ for all vector fields $\vec{w}$ on U; a categorization that does not depend on co-ordinates.

Aside: change of co-ordinates To really spell this out, suppose $\mathbf{y}(s,t) = \mathbf{x}\big(F(s,t)\big)$ where $F(s,t) = (u,v)$ is a change of co-ordinates on U. By the chain rule,

$$\begin{pmatrix}\mathbf{y}_s\\ \mathbf{y}_t\end{pmatrix} = \begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} \quad\text{and}\quad (\mathrm{d}u\ \mathrm{d}v) = (\mathrm{d}s\ \mathrm{d}t)\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}$$

from which

$$\mathrm{d}\mathbf{y} = (\mathrm{d}s\ \mathrm{d}t)\begin{pmatrix}\mathbf{y}_s\\ \mathbf{y}_t\end{pmatrix} = (\mathrm{d}u\ \mathrm{d}v)\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} = (\mathrm{d}u\ \mathrm{d}v)\begin{pmatrix}\mathbf{x}_u\\ \mathbf{x}_v\end{pmatrix} = \mathrm{d}\mathbf{x}$$

The matrix of partial derivatives is the *Jacobian* of the co-ordinate change.

To be completely strict, $\mathrm{d}\mathbf{x}$ and $\mathrm{d}\mathbf{y}$ are not identical since they feed on tangent vectors with respect to different co-ordinates. Formally

$$\mathbf{y} = \mathbf{x} \circ F \implies \mathrm{d}\mathbf{y} = \mathrm{d}\mathbf{x} \circ \mathrm{d}F$$

where $\mathrm{d}F$ maps tangent vectors in $\operatorname{Span}\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\}$ to those in $\operatorname{Span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$: in matrix language, $\mathrm{d}F$ is precisely the above Jacobian!

Common Surfaces

You should have met many of these families/examples in multi-variable calculus.

Graphs If $f(x,y)$ is a smooth function, its graph may be parametrized by $\mathbf{x}(u,v) = \big(u,v,f(u,v)\big)$. Its differential and unit normal field are

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}1\\0\\f_u\end{pmatrix}\mathrm{d}u + \begin{pmatrix}0\\1\\f_v\end{pmatrix}\mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1+f_u^2+f_v^2}}\begin{pmatrix}-f_u\\-f_v\\1\end{pmatrix}$$

This is regular at all points, regardless of f.

Examples 3.4. 1. The standard circular paraboloid may be parametrized $\mathbf{x}(u,v) = \big(u,v,u^2+v^2\big)$.

2. The upper half of the unit sphere is the graph of $z = f(x,y) = \sqrt{1-x^2-y^2}$ where $x^2+y^2 < 1$.

3. A plane has equation $ax+by+cz = d$ where a,b,c,d are constant. Since at least one of a,b,c must be non-zero, this may be written as a function and graphed. For instance, if $b \neq 0$ we have $y = f(x,z) = \frac{1}{b}(d-ax-cz)$ and $\mathbf{n} = \frac{1}{\sqrt{a^2+b^2+c^2}}\big(a,b,c\big)$.

Surfaces of Revolution If a smooth positive function $x = f(z)$ is rotated around the z-axis, we obtain a parametrization

$$\mathbf{x}(\theta,v) = \big(f(v)\cos\theta, f(v)\sin\theta, v\big), \quad (\theta,v) \in (0,2\pi)\times\mathrm{dom}(f)$$

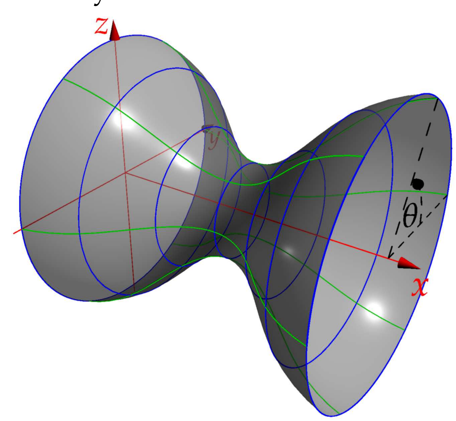
with differential and unit normal field

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}-f(v)\sin\theta\\f(v)\cos\theta\\0\end{pmatrix}\mathrm{d}\theta + \begin{pmatrix}f'(v)\cos\theta\\f'(v)\sin\theta\\1\end{pmatrix}\mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1+f'(v)^2}}\begin{pmatrix}\cos\theta\\\sin\theta\\-f'(v)\end{pmatrix}$$

Examples 3.5. 1. The simplest example ($f(z) \equiv 1$) is the right circular cylinder of radius 1.

2. We may rotate around any axis! For instance, if we rotate the curve $z = 2+\cos x$ around the x-axis, the resulting surface may be parametrized

$$\mathbf{x}(\theta,v) = (2+\cos v)\begin{pmatrix}0\\\cos\theta\\\sin\theta\end{pmatrix} + \begin{pmatrix}v\\0\\0\end{pmatrix}$$



This time v measures distance along the x-axis and θ the angle of rotation around it.

The differential and unit normal field are

$$\mathrm{d}\mathbf{x} = (2+\cos v)\begin{pmatrix}0\\-\sin\theta\\\cos\theta\end{pmatrix}\mathrm{d}\theta + \begin{pmatrix}1\\-\sin v\cos\theta\\-\sin v\sin\theta\end{pmatrix}\mathrm{d}v \qquad \mathbf{n} = \frac{1}{\sqrt{1+\sin^2 v}}\begin{pmatrix}\sin v\\\cos\theta\\\sin\theta\end{pmatrix}$$

Note the *orientation* of the surface: the unit normal field points *outward*, away from the x-axis.

Ruled Surfaces Given functions $\mathbf{y}(u), \mathbf{z}(u)$, define

$$\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{z}(u)$$

Through each point $P = \mathbf{x}(u_0, v_0)$ passes a line $t \mapsto \mathbf{x}(u_0, t) = \mathbf{y}(u_0) + t\mathbf{x}(u_0)$ lying in the surface. The surface can be visualized as moving a *ruler* through space. Ruled surfaces are common in engineering applications since they may be constructed using straight beams.

Definition 3.6. The *tangent developable* of a smooth curve $\mathbf{y}$ is the special case when $\mathbf{z} = \mathbf{y}'$.

Examples 3.7. 1. Every plane is a ruled surface! Let $\mathbf{y}$ be a line in the plane and $\mathbf{z}$ any other tangent direction. For instance, the plane passing through $(1,0,9)$ and spanned by $(2,-3,-5)$ ad $(1,2,3)$ may be parametrized

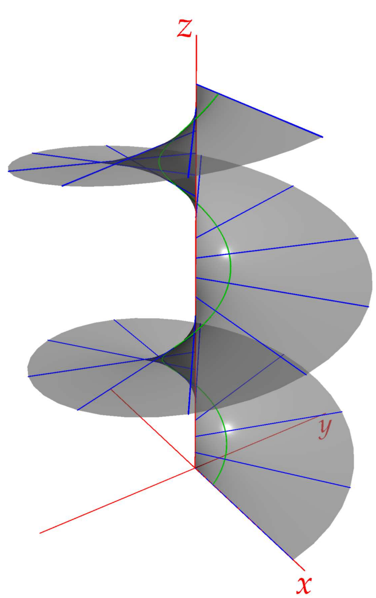
$$\mathbf{x}(u,v) = \underbrace{(1,0,9) + (2,-3,-5)u}_{\mathbf{y}(u)} + \underbrace{(1,2,3)}_{\mathbf{z}(u)}v$$

2. A *helicoid* is built by joining each point of a helix to its axis of rotation. From the standard helix, we obtain the helicoid $\mathbf{x}(u,v) = \big(v\cos u, v\sin u, u\big)$ for $v > 0$.

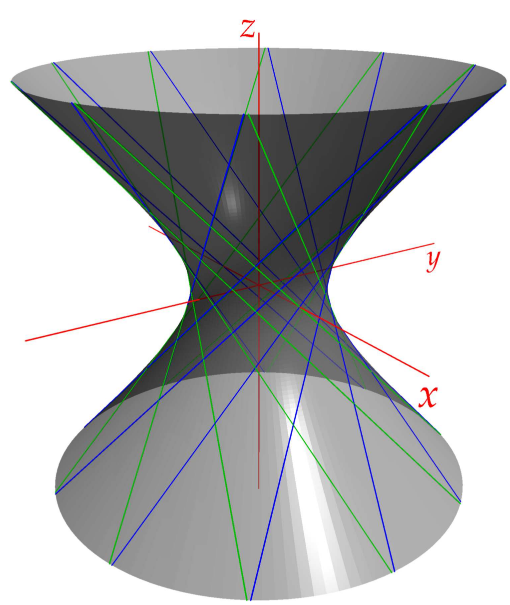
3. The *hyperboloid of one sheet* is a *doubly ruled surface*: through each point there are *two* lines lying on the surface. It may be parametrized as a ruled surface by

$$\mathbf{x}(u,v) = \begin{pmatrix}1\\0\\0\end{pmatrix} + u\begin{pmatrix}0\\1\\1\end{pmatrix} + v\begin{pmatrix}2u\\u^2-1\\u^2+1\end{pmatrix}$$

though convincing yourself there are *two* lines through each point might take a little more work…



Helicoid



Hyperboloid

Implicitly Defined Surfaces

Definition 3.8. A *regular implicitly defined surface* is the zero set of a smooth function $f : \mathbb{E}^3 \to \mathbb{R}$ for which $\mathrm{d}f \neq 0$ (equivalently $\nabla f \neq \mathbf{0}$).

Recall that the directional derivative of f in the direction $\mathbf{v}$ is $D_{\mathbf{v}} f(P) = \mathbf{v} \cdot \nabla f(P)$. This is zero if and only if $\mathbf{v}$ is orthogonal to $\nabla f(P)$. In particular, this says that ∇f provides a *normal field* to an implicitly defined surface.

Examples 3.9. 1. Let a, b, c, d be constant. The function $f(x, y, z) = ax + by + cz - d$ has

$$\mathrm{d}f = a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z$$

which is non-zero provided at least one of a, b, c are non-zero. This defines a plane with unit normal field $\mathbf{n} = \frac{1}{||\nabla f||} \nabla f = \frac{1}{\sqrt{a^2+b^2+c^2}}(a, b, c)$.

2. The sphere of radius a is the zero set of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$. It has unit normal field

$$\mathbf{n} = \frac{1}{||\nabla f||} \nabla f = \frac{1}{a}(x, y, z)$$

The sphere is everywhere regular since at least one of x, y, z is non-zero at all points of the sphere. Contrast this with our earlier example of the *parametrized* sphere which could not be made regular at the north and south poles. The lack of regularity in this case is an aspect of the parametrization, not the surface itself.

3. The function $f(x, y, z) = x^2 + y^2 - z^2 - c$ has

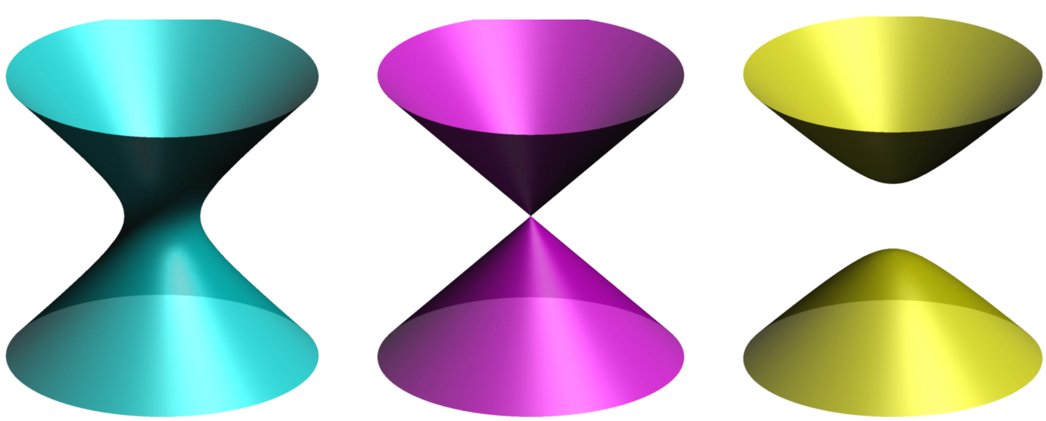
$$\mathrm{d}f = 2(x\,\mathrm{d}x + y\,\mathrm{d}y - z\,\mathrm{d}z),$$

which is non-zero away from $(x, y, z) = (0, 0, 0)$. Depending on the sign of c, the zero set is a hyperboloid or a cone; visualize the horizontal cross-sectional circles to determine which.

$c > 0$ Hyperboloid of 1-sheet: $x^2 + y^2 = z^2 + c > 0$ for all z

$c = 0$ Cone: $x^2 + y^2 = z^2$ contains a non-regular point $(0, 0, 0)$

$c < 0$ Hyperboloid of 2-sheets: $x^2 + y^2 = z^2 - |c| \geq 0$ only when $|z| \geq \sqrt{|c|}$



Our next result, a version of the famous *implicit function theorem*, ties together the notions of *regularity*. In particular, it says that we can always assume the existence of *local* co-ordinates.

Theorem 3.10. *A regular implicitly defined surface $f(x,y,z) = 0$ is (locally) the image of a regular local surface.*

Proof. Suppose $P = (x_0, y_0, z_0)$ lies on the surface and $\nabla f(P) \neq \mathbf{0}$. At least one of the partial derivatives of f is non-zero; suppose WLOG that $f_z(P) \neq 0$. By the implicit function theorem, there exists $U \subseteq \mathbb{R}^2$ and a function $g : U \to \mathbb{R}$ for which $g(x_0, y_0) = z_0$ and $f\big(x, y, g(x,y)\big) = 0$. The surface is then (locally) the graph of $z = g(x,y)$. ∎

Example 3.11. The zero set of $f(x,y,z) = x^2 + y^2 - z^2 - 6$ is a hyperboloid of one sheet. It has unit normal vector field

$$\mathbf{n}(x,y,z) = \frac{1}{||\nabla f||}\nabla f = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ -z \end{pmatrix} = \frac{1}{\sqrt{6+2z^2}}\begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

whenever (x,y,z) is a point on the hyperboloid. For instance, at $P = (3,1,2)$ the unit normal is $\mathbf{n}(P) = \frac{1}{\sqrt{14}}(3,1,2)$, and the tangent plane has equation

$$3x + y - 2z = 6$$

Alternatively, the hyperboloid could have been parametrized in several ways.

(a) In the language of the proof, near $P = (3,1,2)$ it is the graph of $z = g(x,y) = \sqrt{x^2+y^2-6}$. This results in a (local) regular parametrization

$$\mathbf{x}(u,v) = \big(u, v, \sqrt{u^2+v^2-6}\big)$$

(b) It is a surface of revolution around the z-axis:

$$\mathbf{x}(\theta, v) = \begin{pmatrix} \sqrt{6+v^2}\cos\theta \\ \sqrt{6+v^2}\sin\theta \\ v \end{pmatrix}$$

The differential and normal field are then

$$\mathrm{d}\mathbf{x} = \sqrt{6+v^2}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\mathrm{d}\theta + \frac{1}{\sqrt{6+v^2}}\begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \sqrt{6+v^2} \end{pmatrix}\mathrm{d}v$$

$$\mathbf{n} = \frac{\mathbf{x}_\theta \times \mathbf{x}_v}{||\mathbf{x}_\theta \times \mathbf{x}_v||} = \frac{1}{\sqrt{6+2v^2}}\begin{pmatrix} \sqrt{6+v^2}\cos\theta \\ \sqrt{6+v^2}\sin\theta \\ -v \end{pmatrix}$$

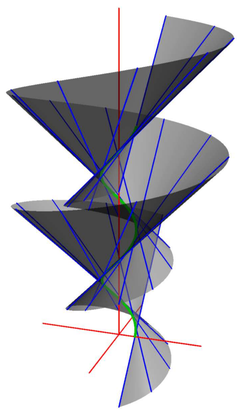
which is precisely what we obtained above.

Yet another expression could be obtained using a parametrization as a ruled surface (e.g. page 51).

Exercises 3.1. 1. Show that parametrization $\mathbf{x}(r,\theta) = \left(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\right)$ of the upper hemisphere is non-regular at $r = 0$.

2. (a) Compute $\mathrm{d}\mathbf{x}$ and $\mathbf{n}$ for the paraboloid $\mathbf{x}(u,v) = \left(u, v, u^2 + v^2\right)$.
   (b) Repeat for the polar co-ordinate parametrization $\mathbf{y}(r,\theta) = \left(r\cos\theta, r\sin\theta, r^2\right)$. Is this parametrization everywhere regular?
   (c) Using $x = r\cos\theta$, etc., write $\mathrm{d}\mathbf{x}$ in terms of $r, \theta, \mathrm{d}r, \mathrm{d}\theta$. What do you observe?
   (d) By viewing the paraboloid as the zero set of $f(x,y,z) = z - x^2 - y^2$, find another expression for the unit normal field.

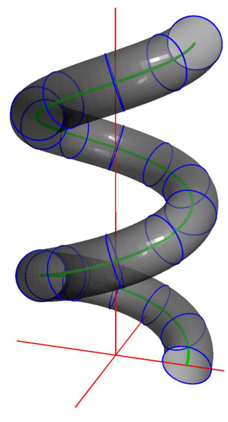
3. (a) Find a parametrization for the tangent developable of the helix $\mathbf{y}(u) = \left(\cos u, \sin u, u\right)$. Compute its differential and unit normal field.
   (*The picture covers $v \in (-3,6)$ with the original curve $\mathbf{y}(u)$ in green*)
   (b) If $\mathbf{y}$ is a unit speed biregular curve, prove that its tangent developable $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ is a regular surface except when $v = 0$. Express the differential and unit normal field in terms of the Frenet frame of $\mathbf{y}$.



4. Let $f(x,y,z) = z^2$. Show that the zero set of f has a regular parametrization despite the gradient of f vanishing at $z = 0$.

5. Let a, b, c be positive constants and define $\mathbf{x}(\theta,\phi) = \begin{pmatrix} a\cos\theta\cos\phi \\ b\sin\theta\cos\phi \\ c\sin\phi \end{pmatrix}$, $(\theta,\phi) \in (0,2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$
   (a) Show that $\mathbf{x}$ parametrizes the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. What part(s) of the ellipsoid are 'missing' from the parametrization?
   (b) Describe geometrically the curves $\theta =$ constant and $\phi =$ constant on the ellipsoid.
   (c) Calculate the differential of $\mathbf{x}$ and show that $\mathrm{d}\mathbf{x}$ is 1–1 for each $p \in U$.

6. The tube of radius $a > 0$ centered on a curve $\mathbf{y}(t)$ may be parametrized in terms of the Frenet frame of $\mathbf{y}$:
$$\mathbf{x}(\phi, t) = \mathbf{y}(t) + a\cos\phi\,\mathbf{N}(t) + a\sin\phi\,\mathbf{B}(t)$$
   (a) *Briefly* explain why the unit normal field is $\mathbf{n} = \cos\phi\,\mathbf{N}(t) + \sin\phi\,\mathbf{B}(t)$.
   (b) Suppose $\mathbf{y}$ is unit speed. Prove that $\mathbf{x}$ is everywhere regular if and only if $\kappa(t) < \frac{1}{a}$ at all points of the generating curve.



7. Let $c(t) = \left(u(t), v(t)\right)$ be a curve in U and define $\mathbf{y}(t) = \mathbf{x}\left(c(t)\right)$ to be the corresponding curve in the surface $\mathbf{x} : U \to \mathbb{E}^3$. Prove that $\mathrm{d}\mathbf{x}\left(c'(0)\right) = \mathbf{y}'(0)$.
   (*Hint: Recall how to write $c'(t)$ as a* vector field)

3.2 The Fundamental Forms

Our immediate goal is to use differentials to describe the shape of a surface. Before making the main definition, we need another product of 1-forms.

Definition 3.12. Given 1-forms α, β on U, define the *symmetric 2-form* $\alpha\beta$ by

$$\alpha\beta(\vec{v}, \vec{w}) = \frac{1}{2}\big(\alpha(\vec{v})\beta(\vec{w}) + \alpha(\vec{w})\beta(\vec{v})\big)$$

where $\vec{v}, \vec{w}$ are vector fields on U. Note that $\alpha^2(\vec{v}, \vec{w}) := \alpha\alpha(\vec{v}, \vec{w}) = \alpha(\vec{v})\alpha(\vec{w})$.

Symmetric 2-forms behave the way you expect they should.

Lemma 3.13. *On each tangent space, $\alpha\beta : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ is a symmetric and bilinear. Moreover $\alpha\beta = \beta\alpha$, and the product is linear in each slot:*

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad \text{and} \quad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$$

Take care when using co-ordinate 1-forms; convention is that $\mathrm{d}x^2 = (\mathrm{d}x)^2$ is a symmetric 2-form.[18]

Example 3.14. Let $\vec{v} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $\vec{w} = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$. Then

$$\mathrm{d}x^2(\vec{v}, \vec{w}) = ac, \quad \mathrm{d}y^2(\vec{v}, \vec{w}) = bd, \quad \mathrm{d}x\mathrm{d}y(\vec{v}, \vec{w}) = \frac{1}{2}(ad + bc)$$

In particular, $\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)(\vec{v}, \vec{w}) = ac + bd$ is essentially the dot product in disguise.
It is typical to evaluate symmetric 2-forms with respect to co-ordinates; linearity and the above are then all you need in $\mathbb{R}^2$. For instance, if $\alpha = x\,\mathrm{d}x - \mathrm{d}y$ and $\beta = xy\,\mathrm{d}y$, then $\alpha\beta = x^2y\,\mathrm{d}x\mathrm{d}y - xy\,\mathrm{d}y^2$.

If α, β take values in $\mathbb{E}^n$, we use the dot product for multiplication of the resulting *vectors* $\alpha(\vec{v})$, etc.

$$(\alpha \cdot \beta)(\vec{v}, \vec{w}) := \frac{1}{2}\big(\alpha(\vec{v}) \cdot \beta(\vec{w}) + \alpha(\vec{w}) \cdot \beta(\vec{v})\big)$$

Definition 3.15. The *first and second fundamental forms* of a regular local surface $\mathbf{x} : U \to \mathbb{E}^3$ are

$$\mathrm{I} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x}, \qquad \mathbb{I} = -\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{n}$$

where $\mathrm{d}\mathbf{n}$ is the differential of the unit normal field ($\mathbb{I}$ requires that the surface be oriented).

Example 3.16. 1. If $\mathbf{x}(u, v) = \big(u, uv, 1 + u\big)$, then

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 1 \\ v \\ 1 \end{pmatrix} \mathrm{d}u + \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} \mathrm{d}v, \qquad \mathbf{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathrm{d}\mathbf{n} = \mathbf{0}$$

from which $\mathrm{I} = (2 + v^2)\,\mathrm{d}u^2 + 2uv\,\mathrm{d}u\mathrm{d}v + u^2\,\mathrm{d}v^2$ and $\mathbb{I} = 0$.

[18] This is *not* the same thing as the exterior derivative (1-form) $\mathrm{d}(x^2) = 2x\,\mathrm{d}x$.

Basic meaning $\mathrm{I}(\vec{v},\vec{w}) = \mathrm{d}\mathbf{x}(\vec{v})\cdot\mathrm{d}\mathbf{x}(\vec{w})$ pulls back the dot product from $T_P S$ to $T_p\mathbb{R}^2$. The length of and angle between tangent vectors to S may now be computed in $T_p\mathbb{R}^2$.

$\mathbb{I}(\vec{v},\vec{w}) = -\frac{1}{2}\big(\mathrm{d}\mathbf{x}(\vec{v})\cdot\mathrm{d}\mathbf{n}(\vec{w}) + \mathrm{d}\mathbf{x}(\vec{w})\cdot\mathrm{d}\mathbf{n}(\vec{v})\big)$ describes how the normal field $\mathbf{n}$ changes over the surface. In the example, $\mathbb{I}\equiv 0$ encapsulates the fact that the normal field is *constant*: the surface is (part of) a plane $\mathbf{x}\cdot(1,0,-1) = -1$.

Co-ordinate invariance Since $\mathrm{d}\mathbf{x}$ is independent of co-ordinates, so also is I. The unit normal field is independent of *oriented* co-ordinate changes. More formally, if $\mathbf{x}(u,v) = \mathbf{y}(s,t)$ are parametrizations of the same surface, then[19]

$$\mathrm{I}_{\mathbf{y}} = \mathrm{I}_{\mathbf{x}} \quad \text{and} \quad \mathbb{I}_{\mathbf{y}} = \begin{cases} \mathbb{I}_{\mathbf{x}} & \text{if the orientations are identical} \\ -\mathbb{I}_{\mathbf{x}} & \text{if the orientations are reversed} \end{cases}$$

The upshot is that the fundamental forms provide a co-ordinate independent way to compute information about a surface from within the *parametrization space* U. We'll think more about this later.

Example 3.17. For the sphere of radius a in spherical polar co-ordinates, recall Exercise 3.2:

$$\mathbf{x}(\theta,\phi) = a\begin{pmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ \sin\phi\end{pmatrix} \implies \mathrm{d}\mathbf{x} = a\cos\phi\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}\mathrm{d}\theta + a\begin{pmatrix}-\cos\theta\sin\phi\\ -\sin\theta\sin\phi\\ \cos\phi\end{pmatrix}\mathrm{d}\phi$$
$$\implies \mathrm{I} = a^2\big(\cos^2\!\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$$

If you revisit the pictures in Example 3.2, the effect of I is easy to visualize:

- $\mathrm{I}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) = \lvert\lvert\mathbf{x}_\theta\rvert\rvert^2 = a^2\cos^2\!\phi$: the tangent vector $\mathbf{x}_\theta$ is *shorter* near the poles, where $\cos\phi\to 0$.
- $\mathrm{I}(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}) = \lvert\lvert\mathbf{x}_\phi\rvert\rvert^2 = a^2$: the tangent vector $\mathbf{x}_\phi$ always has the *same length.*
- $\mathrm{I}(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}) = \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0$: the co-ordinate tangent vectors are always *orthogonal.*

At a point $\mathbf{x}(p)$ on the sphere, if we increase the co-ordinates by tiny quantities $\Delta p = (\Delta\theta,\Delta\phi)$, then the distance Δs travelled along the surface satisfies[20]

$$(\Delta s)^2 \approx \big\lvert\big\lvert\mathbf{x}_\theta\,\Delta\theta + \mathbf{x}_\phi\,\Delta\phi\big\rvert\big\rvert^2 = a^2\cos^2\!\phi\,(\Delta\theta)^2 + a^2(\Delta\phi)^2$$

Near the poles, a change in the angle θ corresponds to a smaller distance on the sphere. This is analogous to how a standard *map* of the Earth works, with distances appearing distorted near the poles. We'll return to this idea shortly…

Computing $\mathbb{I}$ is very easy for the sphere, since $\mathbf{n} = \frac{1}{a}\mathbf{x}$ is merely the scaled position vector:

$$\mathbb{I} = -\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{n} = -\frac{1}{a}\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = -\frac{1}{a}\mathrm{I} = -a\big(\cos^2\!\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$$

[19] In the language of page 49, the $\pm$ in the expressions for $\mathbb{I}$ is the sign of the *determinant* of the Jacobian $\mathrm{d}F$ of the change of co-ordinates $(u,v) = F(s,t)$.

[20] For this reason the first fundamental form is also commonly denoted $\mathrm{d}s^2$.

It is typical to compute $\mathrm{I}, \mathbb{I}$ directly in terms of the co-ordinates u, v without explicitly finding $\mathrm{d}\mathbf{n}$.

Theorem 3.18. *Given $\mathbf{x} : U \to \mathbb{E}^3$ with unit normal field $\mathbf{n}$, define functions E, F, G and l, m, n via*

$$\begin{aligned} &E = \mathbf{x}_u \cdot \mathbf{x}_u && F = \mathbf{x}_u \cdot \mathbf{x}_v && G = \mathbf{x}_v \cdot \mathbf{x}_v \\ &l = \mathbf{x}_{uu} \cdot \mathbf{n} = -\mathbf{x}_u \cdot \mathbf{n}_u && m = \mathbf{x}_{uv} \cdot \mathbf{n} = -\mathbf{x}_u \cdot \mathbf{n}_v = -\mathbf{x}_v \cdot \mathbf{n}_u && n = \mathbf{x}_{vv} \cdot \mathbf{n} = -\mathbf{x}_v \cdot \mathbf{n}_v \end{aligned}$$

Then

$$\mathrm{I} = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2 \quad \text{and} \quad \mathbb{I} = l\,\mathrm{d}u^2 + 2m\,\mathrm{d}u\mathrm{d}v + n\,\mathrm{d}v^2$$

The expressions for $\mathbb{I}$ come from differentiating $\mathbf{x}_u \cdot \mathbf{n} = 0 = \mathbf{x}_v \cdot \mathbf{n}$.

Example 3.19. If we parametrize the graph of $z = f(x, y)$ by $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$, we obtain,

$$\begin{aligned} \mathbf{x}_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}, \quad \mathbf{x}_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix} &\implies E = 1 + f_u^2, \quad F = f_u f_v, \quad G = 1 + f_v^2 \\ &\implies \mathrm{I} = (1 + f_u^2)\,\mathrm{d}u^2 + 2f_u f_v\,\mathrm{d}u\mathrm{d}v + (1 + f_v^2)\,\mathrm{d}v^2 \end{aligned}$$

$$\mathbf{x}_{uu} = \begin{pmatrix} 0 \\ 0 \\ f_{uu} \end{pmatrix}, \quad \mathbf{x}_{uv} = \begin{pmatrix} 0 \\ 0 \\ f_{uv} \end{pmatrix}, \quad \mathbf{x}_{vv} = \begin{pmatrix} 0 \\ 0 \\ f_{vv} \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$\begin{aligned} &\implies l = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad m = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}}, \quad n = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}} \\ &\implies \mathbb{I} = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \Big(f_{uu}\,\mathrm{d}u^2 + 2f_{uv}\,\mathrm{d}u\mathrm{d}v + f_{vv}\,\mathrm{d}v^2 \Big) \end{aligned}$$

For instance, the circular paraboloid $z = x^2 + y^2$ has fundamental forms

$$\begin{aligned} \mathrm{I} &= (1 + 4u^2)\,\mathrm{d}u^2 + 8uv\,\mathrm{d}u\mathrm{d}v + (1 + 4v^2)\,\mathrm{d}v^2 = \mathrm{d}u^2 + \mathrm{d}v^2 + 4\big(u\,\mathrm{d}u + v\,\mathrm{d}v\big)^2 \\ \mathbb{I} &= \frac{2}{\sqrt{1 + 4u^2 + 4v^2}} (\mathrm{d}u^2 + \mathrm{d}v^2) \end{aligned}$$

As a sanity check, compare this with the parametrization of the same paraboloid in polar co-ordinates $\mathbf{y}(r, \theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$ (recall Exercise 3.1.2). By computing the partial derivatives $\mathbf{y}_r, \mathbf{y}_\theta, \mathbf{y}_{rr}, \mathbf{y}_{r\theta}, \mathbf{y}_{\theta\theta}$ directly, it is easy to verify that

$$\mathrm{I} = (1 + 4r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \qquad \mathbb{I} = \frac{2}{\sqrt{1 + 4r^2}} (\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$$

These expressions are identical to the originals (same orientation!) via

$$\begin{cases} \mathrm{d}u = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}v = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta \end{cases} \implies \begin{cases} \mathrm{d}u^2 + \mathrm{d}v^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \\ \big(u\,\mathrm{d}u + v\,\mathrm{d}v\big)^2 = r^2\,\mathrm{d}r^2 \end{cases}$$

Curves in Surfaces: interpreting I and $\mathrm{I\!I}$

Given a surface $\mathbf{x} : U \to \mathbb{E}^3$ and a curve $c(t)$ in U, we may transfer this curve to the surface $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$. Its tangent vector (Exercise 3.1.7) and speed are

$$\mathbf{y}'(t) = \mathrm{d}\mathbf{x}\big(c'(t)\big) \implies ||\mathbf{y}'(t)|| = \sqrt{\mathrm{d}\mathbf{x}\big(c'(t)\big)\cdot \mathrm{d}\mathbf{x}\big(c'(t)\big)} = \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}$$

We can do something similar for the second fundamental form.

Theorem 3.20. *Let $\mathbf{y}(t) = \mathbf{x}\big(c(t)\big)$ parametrize a curve in a surface $\mathbf{x}$ with unit normal $\mathbf{n}$.*

1. *If $a < b$, then the* arc-length *of $\mathbf{y}$ between $\mathbf{y}(a)$ and $\mathbf{y}(b)$ is $\int_a^b \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}\,\mathrm{d}t$.*
2. *The* normal acceleration *of the curve is $\mathbf{y}''(t)\cdot\mathbf{n} = \mathrm{I\!I}(c', c')$.*

This puts a little flesh on our earlier observations (page 56): the first fundamental form measures *infinitesimal distance* on the surface, while the second measures how the surface bends away from the normal field (recall how the curvature of a curve was motivated in terms of force/acceleration).

Proof. 1. Arc-length is the integral of the speed $||\mathbf{y}'(t)|| = \sqrt{\mathrm{I}\big(c'(t), c'(t)\big)}$.

2. Since $\mathbf{y}'$ lies in the tangent plane, we have $\mathbf{y}'\cdot\mathbf{n} \equiv 0$. Differentiate this to obtain

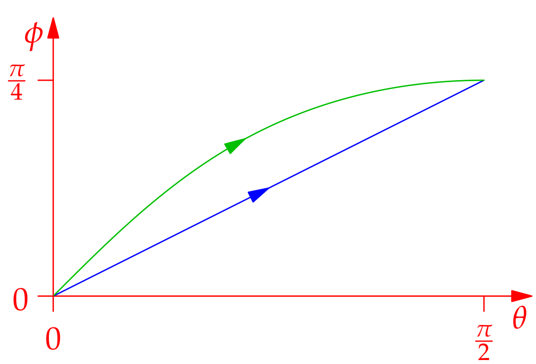
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{y}'\cdot\mathbf{n}) = \mathbf{y}''\cdot\mathbf{n} + \mathbf{y}'\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}\big(c(t)\big) = \mathbf{y}''\cdot\mathbf{n} + \mathrm{d}\mathbf{x}(c')\cdot\mathrm{d}\mathbf{n}(c') = \mathbf{y}''\cdot\mathbf{n} - \mathrm{I\!I}(c', c')$$

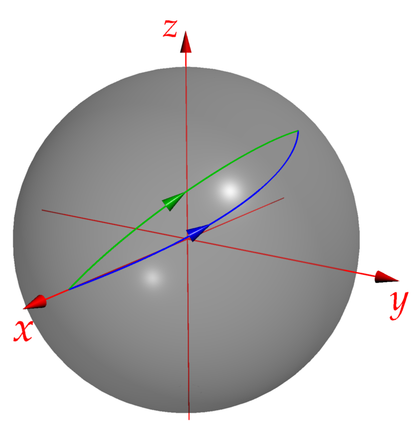
■

Example (3.17, cont). Consider the curve $c(t) = (\theta(t), \phi(t)) = \big(2t, t\big)$ where $0 \le t \le \frac{\pi}{4}$. This has tangent field $c'(t) = 2\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}$. Translated to the unit sphere, the resulting curve has arc-length

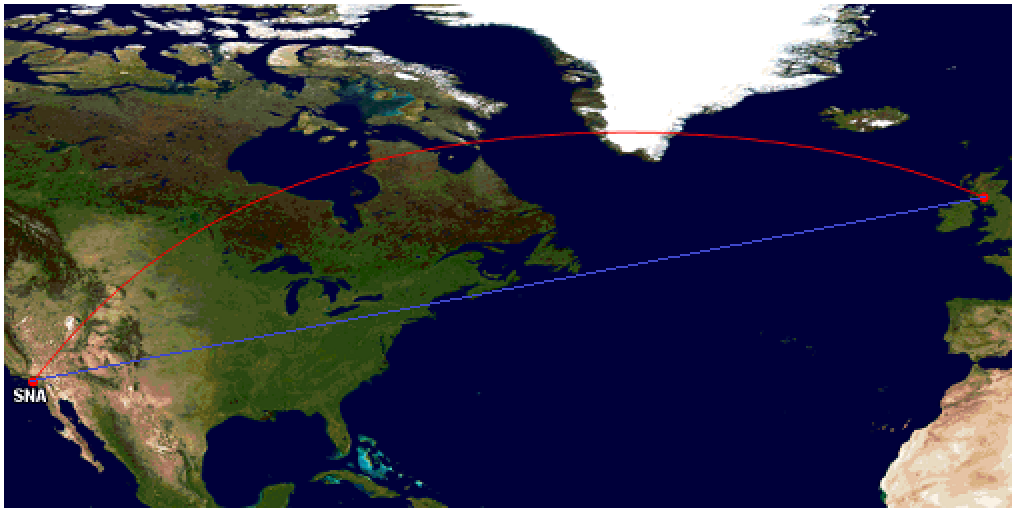
$$\int_0^{\frac{\pi}{4}} \sqrt{\mathrm{I}(c', c')}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} \sqrt{4\cos^2 t + 1}\,\mathrm{d}t \approx 1.619$$

In the parametrization space U, $c(t)$ is a straight line. The shortest path between the endpoints of the curve *on the sphere* is the great circle arc with length $\frac{2\pi}{4} = \frac{\pi}{2} \approx 1.571$; its pre-image in U *appears* longer but isn't due to the $\cos^2\phi$ factor in the first fundamental form. By spending more time at northerly latitudes, I is smaller for more of the great circle arc and the resulting arc-length is smaller.





Provided a map of the Earth covers only a small range of latitudes (almost constant $\phi \approx \phi_0$), the first fundamental form looks almost identical to a standard dot product $\mathrm{I} \approx (a\cos\phi_0 \, d\theta)^2 + (a \, d\phi)^2$. If not, say when we travel by plane, the distortion becomes apparent.



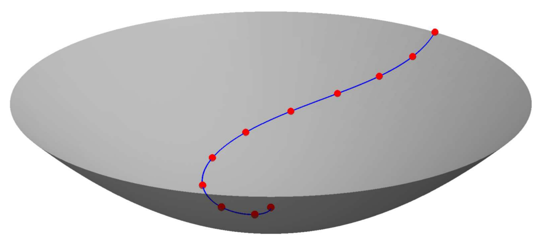
The above map shows the shortest path from Irvine (California) to Irvine (Scotland); the path flown by an aircraft in the absence of wind. The straight line on the map corresponds to a *longer path.*

If we travel at constant speed, it can be checked that great circles are precisely the curves whose acceleration is entirely in the normal direction; this observation, and its relation to *geodesics* (paths of shortest distance), is a matter for another course.

Example 3.21. A skater descends into a paraboloidal bowl $z = \frac{1}{2}r^2$ following the a path described by $c(t) = (r(t), \theta(t)) = (1-t, 4t^2)$ in polar co-ordinates. If we parametrize the surface in polar co-ordinates $\mathbf{x}(r,\theta) = \left(r\cos\theta, r\sin\theta, \frac{1}{2}r^2\right)$, the fundamental forms are easily seen to be

$$\mathrm{I} = (1+r^2)\,dr^2 + r^2\,d\theta^2$$

$$\mathbb{I} = \frac{1}{\sqrt{1+r^2}}(dr^2 + r^2\,d\theta^2)$$

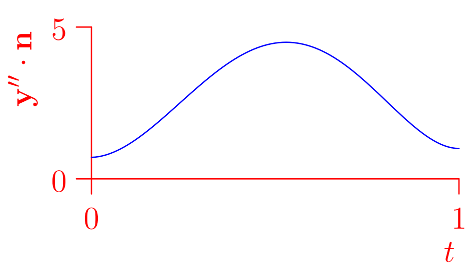


For the skater's path, $c'(t) = -\frac{\partial}{\partial r} + 8t\frac{\partial}{\partial\theta}$, whence

$$\mathrm{I}(c',c') = (1+(1-t)^2) + 64t^2(1-t)^2$$

The path therefore has arc-length

$$\int_0^1 \sqrt{\mathrm{I}(z',z')}\,dt = \int_0^1 \sqrt{1+(64t^2+1)(1-t)^2}\,dt \approx 1.82$$



and normal acceleration

$$\mathbb{I}(c',c') = \frac{1}{\sqrt{1+(1-t)^2}}\left(1+64t^2(1-t)^2\right)$$

By Newton's second law, this is proportional to the component of the force experienced by the skater pushing out from the surface.

Exercises 3.2. 1. Verify the final details of Example 3.19: that is, compute $\mathrm{I}, \mathbb{I}$ using the polar coordinate parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$.

2. Compute the fundamental forms for the surface of revolution $\mathbf{x}(\theta, v) = \big(f(v)\cos\theta, f(v)\sin\theta, v\big)$

3. Compute the first fundamental forms of the following parametrized surfaces wherever they are regular (a, b, c are constants). Where does each parametrization fail to be regular?
   (a) Ellipsoid $\mathbf{x}(\theta,\phi) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi, c\sin\phi)$
   (b) Elliptic paraboloid $\mathbf{x}(r,\theta) = (ar\cos\theta, br\sin\theta, r^2)$

4. Calculate the fundamental forms of Enneper's surface
$$\mathbf{x}(u,v) = \big(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + vu^2, u^2 - v^2\big)$$

5. Compute $\mathrm{d}\mathbf{y}$ for the parametrization $\mathbf{y}(r,\theta) = \big(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\big)$ of the upper unit hemisphere. Verify that the first fundamental form is the same as in Example 3.17.

6. Recall Exercise 3.1.3 where $\mathbf{x}$ is the tangent developable of a unit speed biregular curve $\mathbf{y}$.
   (a) Compute the fundamental forms of $\mathbf{x}$ in terms of the curvature and torsion of $\mathbf{y}$.
   (b) If $\mathbf{y}(u) = \big(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\big)$ is the unit speed helix, show that
$$\mathrm{I} = \big(1 + \tfrac{v^2}{4}\big)\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2, \qquad \mathbb{I} = -\tfrac{v}{4}\,\mathrm{d}u^2$$

7. Prove that $\mathbb{I} \equiv 0$ if and only if $\mathbf{x}$ is (part of) a plane.

8. Paramatrize the great circle arc in Example 3.17 (and cont) by $\mathbf{z}(t) = \big(\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\sin t\big)$, $0 \le t \le \frac{\pi}{2}$. Use this to verify:
   (a) The arc has length $\frac{\pi}{2}$.
   (b) The acceleration of $\mathbf{z}$ is entirely *normal*; $\mathbf{z}'' = (\mathbf{z}'' \cdot \mathbf{n})\mathbf{n}$.

9. Equip the upper half plane $y > 0$ with the abstract first fundamental form $\mathrm{I} = \frac{1}{y^2}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)$. Compute the arc-length between the points $(1,1)$ and $(-1,1)$ in two ways:
   (a) Over the circular arc $c(t) = \sqrt{2}\big(\cos t, \sin t\big)$ centered at the origin.
   (b) Over the 'straight' line $y = 1$.

   Compare your answers!

   (*This is the Poincaré half-plane model of hyperbolic space. There is no surface* $\mathbf{x} : U \to \mathbb{E}^3$ *and no second fundamental form!*)

10. (Hard) The *torus* obtained by rotating the unit circle in the x,z-plane centered at $(2,0,0)$ around the z-axis may be parametrized
$$\mathbf{x}(u,v) = \big((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi\big), \quad (\theta,\phi) \in \mathbb{R}^2$$
    Let $k \neq 0$ be constant and consider the curve $\mathbf{y}(t) = \mathbf{x}(kt, t)$ on the torus.
   (a) Prove that $\mathbf{y}(t)$ has a self-intersection ($\exists s \neq t$ such that $\mathbf{y}(t) = \mathbf{y}(s)$) if and only if $k \in \mathbb{Q}$.
   (b) If $k \in \mathbb{Q}$, show that the curve is *periodic* in that there exists a minimum positive T for which $\mathbf{y}(t+T) = \mathbf{y}(t)$ for all t. Find T in terms of k and write down (don't evaluate!) the integral for the arc-length of the curve over one period.

3.3 Principal, Gauss & Mean Curvatures

Since I and II are symmetric bilinear forms on each tangent space $T_p\mathbb{R}^2$, they may be expressed in matrix form: their matrices with respect to vector fields $\vec{s}, \vec{t}$ are

$$[\mathrm{I}] = \begin{pmatrix} \mathrm{I}(\vec{s},\vec{s}) & \mathrm{I}(\vec{s},\vec{t}) \\ \mathrm{I}(\vec{s},\vec{t}) & \mathrm{I}(\vec{t},\vec{t}) \end{pmatrix} \quad \text{and} \quad [\mathrm{II}] = \begin{pmatrix} \mathrm{II}(\vec{s},\vec{s}) & \mathrm{II}(\vec{s},\vec{t}) \\ \mathrm{II}(\vec{s},\vec{t}) & \mathrm{II}(\vec{t},\vec{t}) \end{pmatrix}$$

Otherwise said

$$\mathrm{I}\big(f\vec{s}+g\vec{t}, h\vec{s}+k\vec{t}\big) = \begin{pmatrix} f & g \end{pmatrix} [\mathrm{I}] \begin{pmatrix} h \\ k \end{pmatrix}$$

and similarly for II. Matters are simplest when these matrices are *diagonal*...

Definition 3.22. Vector fields $\vec{s}, \vec{t}$ are said to be *orthogonal* if $\mathrm{I}(\vec{s},\vec{t}) = 0$. They additionally describe *curvature directions* if $\mathrm{II}(\vec{s},\vec{t}) = 0$.
Co-ordinates u, v are *orthogonal*/*curvature-line* if the above apply to the co-ordinate fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$.

In the language of Theorem 3.18, the matrices of the fundamental forms with respect to $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ are

$$A := \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

Co-ordinates are orthogonal if $F = \mathbf{x}_u \cdot \mathbf{x}_v \equiv 0$ (I has no $\mathrm{d}u\mathrm{d}v$ term). We have curvature line co-ordinates if II is also diagonal:

$$\mathrm{I} = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2 \quad \text{and} \quad \mathrm{II} = l\,\mathrm{d}u^2 + n\,\mathrm{d}v^2$$

While the meaning of *orthogonal* is clear, the reason for the term *curvature-line* will take a little work.

Examples 3.23. 1. Since the sphere of radius a has $\mathrm{II} = -\frac{1}{a}\mathrm{I}$, any orthogonal co-ordinates on the sphere are curvature-line! E.g., spherical polar co-ordinates: $\mathrm{I} = a^2\big(\cos^2\phi\,\mathrm{d}\theta^2 + \mathrm{d}\phi^2\big)$.

2. (Example 3.2.3.19) For the paraboloid $z = r^2$, standard polar co-ordinates are curvature line:

$$\mathrm{I} = (1+4r^2)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \qquad \mathrm{II} = \frac{2}{\sqrt{1+4r^2}}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\big)$$

A Little Linear Algebra The existence of curvature directions is equivalent to the simultaneous diagonalization of both forms. Doing this requires us to extend the concepts of eigenvalues and eigenvectors.

Definition 3.24. Let A, B be square matrices. A vector $\vec{v} \neq \vec{0}$ is an *eigenvector* of B with respect to A with *eigenvalue* λ if

$$(B - \lambda A)\vec{v} = \vec{0}$$

If $A = I$, these are standard eigenvalues/eigenvectors. We compute in the usual way: start by solving the characteristic polynomial $\det(B - \lambda I) = 0$...

Theorem 3.25. *Let A and B be symmetric and A* positive-definite.[21]

1. *There exists a basis of eigenvectors of B with respect to A, and all eigenvalues are* real.
2. *If $\vec{s}, \vec{t}$ are eigenvectors corresponding to distinct eigenvalues, then $\vec{s}^T A\vec{t} = 0 = \vec{s}^T B\vec{t}$.*

Example 3.26. Let $A = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 5 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 3 \end{smallmatrix}\right)$. Note that A has eigenvalues $\frac{1}{2}(7 \pm \sqrt{45}) > 0$.

$$\det(B - \lambda A) = \begin{vmatrix} -2\lambda & 1-3\lambda \\ 1-3\lambda & 3-5\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1$$

$$\lambda_1 = 1 \implies (B - A)\vec{v}_1 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \vec{v}_1, \qquad \lambda_2 = -1 \implies (B + A)\vec{v}_2 = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \vec{v}_2$$

Thus $\{\vec{v}_1, \vec{v}_2\} = \left\{ \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right) \right\}$ is an eigenbasis for B with respect to A.

Proof. 1. This follows from the famous spectral theorem in linear algebra. If you've not seen this, the proof[22] may safely be ignored.

2. Assume $B\vec{s} = k_1 A\vec{s}$ and $B\vec{t} = k_2 A\vec{t}$ where $k_1 \neq k_2$, and apply the symmetry of A and B,

$$\left.\begin{array}{l} \vec{s}^T B\vec{t} = \vec{s}^T (k_2 A\vec{t}) = k_2 \vec{s}^T A\vec{t} \\ \quad \| \\ \vec{t}^T B\vec{s} = \vec{t}^T (k_1 A\vec{s}) = k_1 \vec{t}^T A\vec{s} \end{array}\right\} \implies (k_2 - k_1)\vec{s}^T A\vec{t} = 0 \implies \vec{s}^T A\vec{t} = 0$$

■

Application to Surfaces

With respect to any basis fields, the matrices A, B of I, II are symmetric and A is positive-definite:

$$\forall \vec{w} \neq \vec{0} \implies \vec{w}^T A\vec{w} = \mathrm{I}(\vec{w}, \vec{w}) = d\mathbf{x}(\vec{w}) \cdot d\mathbf{x}(\vec{w}) = ||d\mathbf{x}(\vec{w})||^2 > 0$$

We may therefore apply Theorem 3.25.

Definition 3.27. The *principal curvatures* $k_1, k_2 : U \to \mathbb{R}$ of an oriented surface $\mathbf{x} : U \to \mathbb{E}^3$ are the eigenvalues of II with respect to I. Corresponding eigenvectors are *curvature directions.*
The *Gauss* and *mean curvatures* are, respectively, $K := k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$.
A point $\mathbf{x}(p)$ is *umbilic* if $k_1(p) = k_2(p)$.

[21] For all non-zero vectors, $\vec{v}^T A\vec{v} > 0$. Equivalently, all eigenvalues of A are positive. This means that $\langle \vec{v}, \vec{w} \rangle := \vec{v}^T A\vec{w}$ defines an *inner product* on $\mathbb{R}^n$.

[22] Since A is symmetric, it has an orthogonal eigenbasis $\{\vec{x}_1, \ldots, \vec{x}_n\}$. Since each eigenvalue μ_i is positive, we may scale the eigenvectors such that $||\vec{x}_i||^2 = \frac{1}{\mu_i}$. Let $X = (\vec{x}_1 \ \cdots \ \vec{x}_n)$ so that $X^T AX = I$ is the identity matrix. But then,

$$\det(B - \lambda A) = \det(X^T)^{-1} \det(X^T BX - \lambda I) \det(X^{-1}) = 0 \iff \det(X^T BX - \lambda I) = 0$$

Since $X^T BX$ is symmetric, it also has an orthogonal eigenbasis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ and real eigenvalues λ_k. Each $\vec{v}_k := X\vec{w}_k$ is then an eigenvector of B with respect to A with eigenvalue λ_k; since X is invertible, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis.

These definitions are independent of *oriented* co-ordinate changes: if we reverse the orientation, then k_1, k_2 and H change sign, while $K = k_1 k_2$ is unchanged.

At non-umbilic points, Theorem 3.25 says that curvature directions diagonalize both fundamental forms, in line with Definition 3.22.

At umbilic points, $\mathrm{I\!I} = k\mathrm{I}$ and all directions are curvature directions; any orthogonal directions necessarily diagonalize both fundamental forms.

Examples 3.28. Here are two *totally umbilic* surfaces where the curvatures are constant.

1. A plane: $\mathrm{I\!I} \equiv 0 \implies$ all curvatures are zero.
2. A sphere of radius a: $\mathrm{I\!I} = -\frac{1}{a}\mathrm{I} \implies k_1 = k_2 = -\frac{1}{a}$, $K = \frac{1}{a^2}$ and $H = -\frac{1}{a}$.

In fact these comprise all totally umbilic surfaces.

Theorem 3.29. 1. *In co-ordinates, the Gauss and mean curvatures are given by*

$$K = \frac{ln - m^2}{EG - F^2} = \frac{\det B}{\det A} \quad \text{and} \quad H = \frac{lG + nE - 2mF}{2(EG - F^2)} = \frac{1}{2}\operatorname{tr} A^{-1}B$$

2. *At non-umbilic points, the curvatures k_1, k_2, K, H are smooth functions and the curvature directions may be described locally by (smooth) vector fields.*

Proof. 1. The principal curvatures are the solutions to the quadratic equation

$$\det\left(\begin{pmatrix} l & m \\ m & n \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = (EG - F^2)\lambda^2 - (lG + nE - 2mF)\lambda + (ln - m^2)$$

of which K and H are the product and half the sum of the roots.

2. The roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of a quadratic are smooth functions of the coefficients unless $b^2 - 4ac = 0$, in which case we have a repeated root ($k_1 = k_2$). Moreover, each eigenspace is one-dimensional so there is no difficulty choosing smooth eigenvectors.[23]

■

Examples 3.30. 1. (Example 3.19) For the paraboloid $\mathbf{x}(r, \theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$, standard polar co-ordinates are curvature line:

$$A = [\mathrm{I}] = \begin{pmatrix} 1 + 4r^2 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad B = [\mathrm{I\!I}] = \begin{pmatrix} \frac{2}{\sqrt{1+4r^2}} & 0 \\ 0 & \frac{2r^2}{\sqrt{1+4r^2}} \end{pmatrix}$$

The curvatures are therefore

$$k_1 = \frac{2}{(1+4r^2)^{3/2}}, \quad k_2 = \frac{2}{\sqrt{1+4r^2}}, \quad K = \frac{4}{(1+4r^2)^2}, \quad H = \frac{2+4r^2}{(1+4r^2)^{3/2}}$$

The curvatures make sense at the single umbilic point ($r = 0$), but the co-ordinates are not curvature line since the parametrization fails to be regular ($\mathbf{x}_\theta(0, \theta) = \mathbf{0}$).

[23] If $\mathbf{x}(p)$ is umbilic, then the eigenspace at p is 2-dimensional and $\lim_{q \to p} \vec{v}(q)$ might not exist.

2. Parametrize the graph of a function $z = f(x, y)$ in the usual way $\mathbf{x}(u, v) = \big(u, v, f(u, v)\big)$, then

$$A = [\mathrm{I}] = \begin{pmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{pmatrix} \qquad B = [\mathrm{II}] = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

Theorem 3.29 tells us that

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2} \qquad H = \frac{f_{vv}(1 + f_u^2) + f_{uu}(1 + f_v^2) - 2 f_u f_v f_{uv}}{2(1 + f_u^2 + f_v^2)^{3/2}}$$

In the abstract, solving for the curvatures and curvature directions is disgusting. As a sanity check, you should verify that $f(u, v) = u^2 + v^2$ recovers exactly the curvatures in the previous example!

3. (Exercise 3.2.6) The tangent developable of the unit speed helix has

$$A = [\mathrm{I}] = \begin{pmatrix} 1 + \frac{v^2}{4} & 1 \\ 1 & 1 \end{pmatrix} \qquad B = [\mathrm{II}] = \begin{pmatrix} -\frac{v}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

The principal curvatures solve

$$\det \begin{pmatrix} -\frac{v}{4} - \lambda \left(1 + \frac{v^2}{4}\right) & -\lambda \\ -\lambda & -\lambda \end{pmatrix} = \frac{v^2}{4} \lambda^2 + \frac{v}{4} \lambda = 0$$

from which

$$k_1 = 0, \quad k_2 = -\frac{1}{v}, \quad K = 0, \quad H = -\frac{1}{2v}$$

In this case explicitly computing the curvature directions is not so difficult:

$$\begin{aligned} k_1 = 0 &\implies \mathcal{N}(B - k_1 A) = \mathcal{N} \begin{pmatrix} -\frac{v}{4} & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{Span} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \vec{s} = \frac{\partial}{\partial v} \\ k_2 = -\frac{1}{v} &\implies \mathcal{N}(B - k_2 A) = \mathcal{N} \begin{pmatrix} \frac{1}{v} & \frac{1}{v} \\ \frac{1}{v} & \frac{1}{v} \end{pmatrix} = \operatorname{Span} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies \vec{t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \end{aligned}$$

where we made the natural choice of vector fields $\vec{s}, \vec{t}$. As a sanity check, here are the matrices of the fundamental forms with respect to $\vec{s}, \vec{t}$:

$$\begin{aligned} \mathrm{I}(\vec{s}, \vec{s}) = (0\ 1) A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1, \ldots \implies [\mathrm{I}] &= \begin{pmatrix} \mathrm{I}(\vec{s}, \vec{s}) & \mathrm{I}(\vec{s}, \vec{t}) \\ \mathrm{I}(\vec{s}, \vec{t}) & \mathrm{I}(\vec{t}, \vec{t}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{v^2}{4} \end{pmatrix} \\ [\mathrm{II}] &= \begin{pmatrix} \mathrm{II}(\vec{s}, \vec{s}) & \mathrm{II}(\vec{s}, \vec{t}) \\ \mathrm{II}(\vec{s}, \vec{t}) & \mathrm{II}(\vec{t}, \vec{t}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{v}{4} \end{pmatrix} \end{aligned}$$

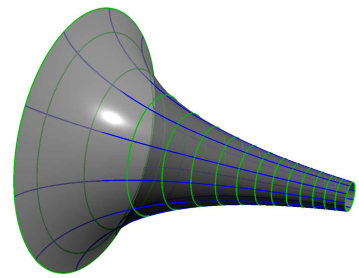
in which the principal curvatures are clearly visible:

$$0 = k_1 1, \qquad -\frac{v}{4} = k_2 \frac{v^2}{4}$$

The Gauss and mean curvatures are extremely important quantities. Here are a couple of ideas built on these concepts.

Minimal Surfaces $H \equiv 0$ These are so-called because they minimize *surface area.* Consider a closed curve in $\mathbb{E}^3$; among all surfaces with this boundary, the surface with minimal surface area has $H \equiv 0$. This is the shape a soap film takes if you dip the curve in soapy water: a minimal surface minimizes the 'total tension' of the soap film.

More generally, *constant mean curvature* (CMC) surfaces may be used to model soap *bubbles.*



Constant Gauss Curvature Surfaces Spheres are examples of surfaces of constant positive Gauss curvature. Planes, cones and cylinders are examples of surfaces with $K = 0$. A *pseudosphere* with constant $K = -1$ is drawn.

Curvature-Line Co-ordinates

At non-umbilic points, Theorems 3.25 and 3.29 tells us how to find curvature *directions* as vector fields $\vec{s}, \vec{t}$. But what about *co-ordinates*? Otherwise said, we want functions $s, t : U \to \mathbb{R}$ whose resulting vector fields are *parallel* to $\vec{s}, \vec{t}$: there exist functions $f, g : U \to \mathbb{R}$ for which

$$\frac{\partial}{\partial s} = f\vec{s} \quad \text{and} \quad \frac{\partial}{\partial t} = g\vec{t} \tag{$*$}$$

This is equivalent to $\vec{s}[t] = 0 = \vec{t}[s]$. These are in fact solvable in general, if only *locally.*

Theorem 3.31. *Let $p \in U$ and let $\vec{s}, \vec{t}$ be linearly independent vector fields on U. There exists a neighborhood V of p and co-ordinates $s, t : V \to \mathbb{R}$ such that*

$$\vec{s}[t] = 0 = \vec{t}[s]$$

In particular, if $\mathbf{x}(p)$ is a non-umbilic point on a surface $\mathbf{x} : U \to \mathbb{E}^3$, then there exists a neighborhood V of p and curvature-line co-ordinates s, t on V.

We state this without proof. Since it is merely an *existence* result, it is very unlikely you'll be able to find *explicit* curvature-line co-ordinates for a given surface. Of course, you *might* get lucky…

Example (3.30.3 cont). Recall that we chose curvature direction fields $\vec{s} = \frac{\partial}{\partial v}$ and $\vec{t} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$. By inspection, the functions $s = u + v$ and $t = u$ solve the required equations:

$$\vec{s}[u] = \frac{\partial}{\partial v}[u] = 0, \qquad \vec{t}[u+v] = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)[u+v] = 1 - 1 = 0$$

Indeed we see that

$$\mathrm{I} = \frac{v^2}{4}\,\mathrm{d}u^2 + \mathrm{d}(u+v)^2 = \mathrm{d}s^2 + \frac{v^2}{4}\,\mathrm{d}t^2, \qquad \mathrm{II} = 0\,\mathrm{d}s^2 - \frac{v}{4}\,\mathrm{d}t^2$$

In general this is very unlikely to work, though co-ordinate functions can always be approximated numerically.

Exercises 3.3. 1. Find the eigenvalues of $B = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ with respect to $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. If $\vec{s}, \vec{t}$ are corresponding eigenvectors, verify that $\vec{s}^T A\vec{t} = 0 = \vec{s}^T B\vec{t}$.

2. Parametrize the graph of $x = z^2$; compute its fundamental forms and its principal, Gauss and mean curvatures.

3. Use Theorem 3.29 to find the Gauss and mean curvatures of the graph of $y = x^2 - z^2$.

4. Show that Enneper's surface (Exercise 3.2.4) is minimal.

5. Let $\mathbf{x}(u, v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ be the tangent developable of a unit speed biregular curve $\mathbf{y}$.

   (a) Find the principal curvatures, Gauss and mean curvatures of $\mathbf{x}$.

   (b) Compute the curvature directions and find curvature line co-ordinates.

   (*This is very similar to Example 3.30.3 - keep track of the changes!*)

6. Rotate $y = f(x)$ around the x-axis and parametrize the surface via

   $$\mathbf{x}(\phi, v) = \big(v, f(v)\cos\phi, f(v)\sin\phi\big)$$

   (a) Verify that the co-ordinates ϕ, v are curvature-line, compute the principal curvatures, and show that the Gauss and mean curvatures are

   $$K = -\frac{f''(v)}{f(v)\big(1 + f'(v)^2\big)^2}, \qquad H = \frac{f(v)f''(v) - 1 - f'(v)^2}{2f(v)\big(1 + f'(v)^2\big)^{3/2}}$$

   (b) Demonstrate the following (choose suitable $f(v)$ if necessary):

      i. A cylinder has $K = 0$;

      ii. A cone has $K = 0$;

      iii. A sphere of radius a has $K = \frac{1}{a^2}$.

      iv. A *catenoid* $f(v) = a^{-1}\cosh(av - c)$ is a minimal surface.

7. We reverse some of the analysis in the previous question for surfaces of revolution around the x-axis.

   (a) Suppose the surface is minimal $H \equiv 0$. Write $g(v) = 1 + \big(f'(v)\big)^2$, use the differential equation above to show that

   $$1 + f'^2 = g^2 = a^2 f^2$$

   for some constant a. By making the substitution $f(v) = a^{-1}\cosh\big(ah(v)\big)$ for some unknown function $h(v)$, show that the surface is a *catenoid* (see previous question).

   (b) Plainly $K \equiv 0$ if and only if $f''(v) \equiv 0$. What are these surfaces? More generally, if the surface has constant non-zero Gauss curvature K, show that f satisfies a non-linear ODE

   $$Kf^2 = (1 + f'^2)^{-1} + c$$

   for some constant c.
   (*If $c \neq 0$, this requires evaluating an elliptic integral, so don't try! There are thus a great many constant Gauss curvature surfaces of revolution*)

8. The *tractrix* is parametrized by

$$\mathbf{y}(t) = \begin{pmatrix} \sinh^{-1} t - t(1+t^2)^{-1/2} \\ (1+t^2)^{-1/2} \end{pmatrix}$$

By revolving this curve around the x-axis, show that the resulting surface is a pseudosphere with $K \equiv -1$.

9. We know that the Gauss and mean curvature are defined in terms of the principal curvatures. By writing down a suitable quadratic polynomial, prove that knowing of H, K is sufficient to recover the principal curvatures.

10. The graph of a function $z = f(x,y)$ is parametrized by $\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$. What can you say about the surface if (u,v) are curvature-line co-ordinates?

    (*Hint: recall Example 3.19*)

11. Suppose that a surface $\mathbf{x} : U \to \mathbb{E}^3$ is totally umbilic $\mathbb{I} = k\mathrm{I}$ for some function $k : U \to \mathbb{R}$.

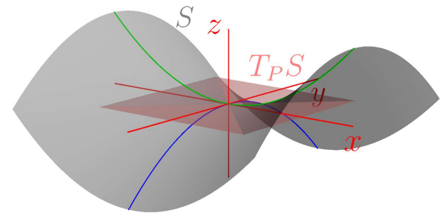
    (a) Explain why we have $\mathbf{n}_u = -k\mathbf{x}_u$ and $\mathbf{n}_v = -k\mathbf{x}_v$.
    (*Hint:* $\mathbf{x}_u \cdot (\mathbf{n}_u + k\mathbf{x}_u) = (-\mathbb{I} + k\mathrm{I})(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})$*, etc.*)

    (b) Compute the mixed partial derivative $\mathbf{n}_{uv} = \mathbf{n}_{vu}$ to prove that k is constant.

    (c) Prove that $\mathbf{x}$ is (part of) a plane or a sphere.
    (*Hint: If* $k \neq 0$ *consider* $\mathbf{c} := \mathbf{x} + \frac{1}{k}\mathbf{n} \ldots$)

3.4 Power Series Expansions and Euler's Theorem

In this section we intersect a surface with certain planes and consider the resulting curves. The curvatures provide data about these curves and thus tell us something about the local shape of the surface. The key is to see how curvatures describe a quadratic approximation to a surface.



At a regular point P on a surface S, choose axes such that P is the origin and the (x,y)-plane is tangent[24] to S. By Theorem 3.10, S is locally the graph of a function $z = f(x,y)$, which we may parametrize in the usual manner

$$\mathbf{x}(u,v) = \big(u, v, f(u,v)\big)$$

The unit normal vector $\mathbf{n}_P = \mathbf{k}$ is therefore the standard vertical basis vector. Since the tangent plane at P is the (x,y)-plane, we see that $f_u(0,0) = 0 = f_v(0,0)$; substituting into Example 3.19 yields the fundamental forms *at* P:

$$\begin{aligned} \mathrm{I}_P &= \mathrm{d}u^2 + \mathrm{d}v^2 \\ \mathbb{I}_P &= f_{uu}\,\mathrm{d}u^2 + 2f_{uv}\,\mathrm{d}u\mathrm{d}v + f_{uv}\,\mathrm{d}v^2 \end{aligned} \qquad [\mathrm{I}]_P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\mathbb{I}]_P = \operatorname{Hess} f = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

The last matrix is the *Hessian* of f, and the Gauss and mean curvatures at P are

$$K(P) = \det \operatorname{Hess} f(0,0) \quad \text{and} \quad H(P) = \frac{1}{2} \operatorname{tr} \operatorname{Hess} f(0,0)$$

It bears repeating that these expressions are only valid *at the origin* $O \in U$ (equivalently $P \in S$). Although the co-ordinates u, v will extend nearby on the surface, the first fundamental form need not be diagonal anywhere except at the origin.

Now suppose we rotate the (x,y)-plane so that the axes point in the principal directions. Then the Hessian is also diagonal ($f_{uv}(0,0) = 0$) and the principal curvatures at P are

$$k_1 = f_{uu}(0,0) \quad \text{and} \quad k_2 = f_{vv}(0,0)$$

Theorem 3.32. *If the graph of $z = f(x,y)$ is tangent to the (x,y)-plane at the origin O so that the axes are the curvature directions, then the Maclaurin approximation of the function $f(x,y)$ is*

$$\begin{aligned} f(x,y) &\approx f(O) + (x\ y)\ \nabla f|_O + \frac{1}{2}(x\ y) \operatorname{Hess} f(O) \begin{pmatrix} x \\ y \end{pmatrix} + \textit{ higher order terms} \\ &= \frac{1}{2}k_1(O)x^2 + \frac{1}{2}k_2(O)y^2 + \textit{ higher order terms} \end{aligned}$$

Example 3.33. Let $f(x,y) = x^2 - y^2$ (above picture). At the origin, $\mathbf{x}(u,v) = \big(u, v, u^2 - v^2\big)$ has

$$\mathrm{I} = \mathrm{d}u^2 + \mathrm{d}v^2, \quad \mathbb{I} = 2(\mathrm{d}u^2 - \mathrm{d}v^2), \quad k_1 = 2, \quad k_2 = -2, \quad K = -4, \quad H = 0$$

In this case the Maclaurin approximation is exact!

$$\frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2 = x^2 - y^2 = f(x,y)$$

[24] This amounts to applying a rigid motion (direct isometry) to the surface, which does nothing to the fundamental forms.

Level Curves: intersections with planes parallel to the tangent plane

If c is small, then the intersection of S with a plane $c\mathbf{n}_P + T_PS$ parallel to the tangent plane is a *level curve*; in our analysis, they correspond to level curves $f(x,y) = \text{constant}$. Theorem 3.32 tells us how level curves depend on the curvatures. For instance, if k_1, k_2 have opposite signs, then for small c,

$$k_1x^2 + k_2y^2 \approx 2c$$

is approximately a *hyperbola.*

Definition 3.34. Suppose k_1, k_2, K, H are the curvatures of a surface S at a point P. We say that P is:

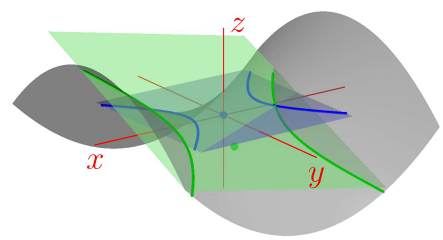
Elliptic $\iff K > 0 \iff k_1, k_2 \neq 0$ and have the same sign.
Level curves near P are approximately *ellipses.*

Hyperbolic $\iff K < 0 \iff k_1, k_2 \neq 0$ and have opposite signs.
Level curves near P are approximately *hyperbolæ.*

Parabolic $\iff K = 0$ and $H \neq 0 \iff$ exactly one of k_1, k_2 is zero.
Level curves near P are approximately a pair of *parallel lines*, e.g. $x = \pm c$.

Planar $\iff K = H = 0 \iff k_1 = k_2 = 0$.
The curvatures provide no data as to the level curves near P.

Example (3.33, mk. II). For the graph of $z = x^2 - y^2$, the level curve $x^2 - y^2 = c \neq 0$ is a hyperbola. In fact this is true everywhere on this surface: under the usual parametrization $\mathbf{x}(u,v) = \big(u, v, u^2 - v^2\big)$, we have



$$K = -\frac{4}{(1+4u^2+4v^2)^2} \quad \text{and} \quad H = \frac{4(v^2-u^2)}{(1+4u^2+4v^2)^{3/2}}$$

Since $K < 0$ everywhere, all points are hyperbolic.

In the picture, shifted tangent planes $c\mathbf{n}_P + T_PS$ and their intersections with the surface are drawn for two points. In both cases the level curves are genuine hyperbolæ.

Normal Curvature: intersections with planes containing the normal vector

Theorem 3.32 is the surface analogy of Exercise 1.6.5: a regular curve in $\mathbb{E}^2$ passing through the origin horizontally at $t = 0$ has its graph given locally by

$$y = \frac{1}{2}\kappa(0)x^2 + \text{higher order terms} \qquad (*)$$

We put this to work by considering the curvature of curves passing through a point on a surface.

Definition 3.35. Let S be a surface and $\mathbf{v}_P \in T_PS$ a non-zero tangent vector.
The *normal curvature* $\nu(\mathbf{v}_P)$ is the curvature at P of the curve[25] $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$.
We say that $\mathbf{v}_P$ is *asymptotic* if $\nu(\mathbf{v}_P) = 0$.

[25]The curve is the *connected component* of $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$ containing P.

The normal curvature is surprisingly easy to compute.

Theorem 3.36 (Euler). *Suppose $\mathbf{v}_P$ makes angle ψ with the first principal curvature direction. Then*

$$\nu(\mathbf{v}_P) = k_1 \cos^2 \psi + k_2 \sin^2 \psi$$

Moreover, the principal curvatures are the extremes of $\nu(\mathbf{v}_P)$: e.g. if $k_1 \le k_2$, then

$$k_1 \le \nu(\mathbf{v}_P) \le k_2$$

Proof. Choose axes such that the principal curvature directions at P are $\mathbf{i}, \mathbf{j}$, and the unit normal is $\mathbf{n}_P = \mathbf{k}$. The surface is locally the graph of a function $z = f(x, y)$ satisfying Theorem 3.32. If (r, ψ) are standard polar co-ordinates on the (x, y)-plane, then

$$z = \tfrac{1}{2}k_1(r\cos\psi)^2 + \tfrac{1}{2}k_2(r\sin\psi)^2 + \cdots = \frac{1}{2}(k_1\cos^2\psi + k_2\sin^2\psi)r^2 + \cdots$$

Now fix ψ. Without loss of generality, we may assume $\mathbf{v}_P$ has unit length since only its direction matters. Our curve of interest $\mathbf{y}_P \subset \Pi(\mathbf{v}_P)$ may be parametrized

$$\mathbf{y}(r) = r\mathbf{v}_P + f(r\cos\psi, r\sin\psi)\,\mathbf{n}_P = \begin{pmatrix} r\cos\psi \\ r\sin\psi \\ f(r\cos\psi, r\sin\psi) \end{pmatrix} = \begin{pmatrix} r\cos\psi \\ r\sin\psi \\ \frac{1}{2}\nu r^2 + \cdots \end{pmatrix}$$

The last equality used observation $(*)$, where ν is the normal curvature. Compare the z-expressions for the main result. For the final observation, note that

$$\nu = k_1 + (k_2 - k_1)\sin^2\phi \in [k_1, k_2]$$

■

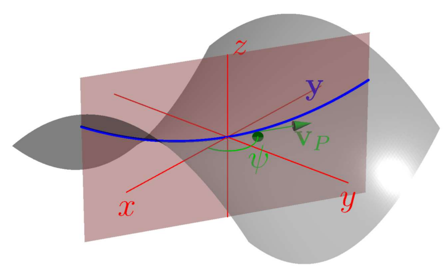
Examples 3.37. 1. If P is a planar point, then all normal curvatures are zero and all directions are asymptotic.

2. For the graph of the hyperbolic paraboloid $z = x^2 - y^2$ (Example 3.33) at the origin, if $\psi = \frac{5\pi}{6}$, then

$$\nu(\mathbf{v}_O) = 2\cos^2\frac{5\pi}{6} - 2\sin^2\frac{5\pi}{6} = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$$

Indeed since every point is hyperbolic, there are always two asymptotic directions at each point: these correspond to the angles $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for which

$$k_1\cos^2\psi + k_2\sin^2\psi = 0 \iff \tan\psi = \pm\sqrt{-\frac{k_1}{k_2}}$$



3. A elliptic paraboloid $z = x^2 + y^2$ has no asymptotic directions at any point. In its usual parametrization, the principal curvatures are both positive everywhere and the normal curvature at an point is therefore $\mathbf{n}(\mathbf{v}_P) = k_1\cos^2\theta + k_2\sin^2\theta > 0$.

The Second Fundamental Form and the Local Shape of a Surface

Part of our standard approach is to transfer calculations and analysis on surfaces to the parametrization space. By appealing to the ideal of *normal acceleration* (Theorem 3.20), we may easily transfer the notion of an asymptotic vector to a property of the second fundamental form.

Theorem 3.38. *Suppose* $\mathbf{x} : U \to \mathbb{E}^3$ *is an oriented surface and let* $\vec{w}_p \in T_p\mathbb{R}^2$ *be a non-zero tangent vector. Then* $\mathrm{d}\mathbf{x}(\vec{w}_p)$ *is asymptotic iff* $\mathbb{I}(\vec{w}_p, \vec{w}_p) = 0$*.*

We tend to call such a tangent vector $\vec{w}_p$ asymptotic in its own right.

We can moreover restate the point-types introduced in Definition 3.34 by introducing a related object.

Definition 3.39. The *Dupin indicatrix* at $p \in U$ is the set of tangent vectors $\vec{w}_p \in T_p\mathbb{R}^2$ such that $\mathbb{I}(\vec{w}_p, \vec{w}_p) = \pm 1$.

If $\vec{s}_p, \vec{t}_p$ are orthonormal[26] curvature directions and $\vec{w}_p = a\vec{s}_p + b\vec{t}_p$, then the Dupin indicatrix has equation

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = a^2\mathbb{I}(\vec{s}_p, \vec{s}_p) + 2ab\mathbb{I}(\vec{s}_p, \vec{t}_p) + b^2\mathbb{I}(\vec{t}_p, \vec{t}_p) = k_1a^2 + k_2b^2 = \pm 1$$

This defines a curve in the tangent space $T_p\mathbb{R}^2$ whose type depends on the signs of the principal curvatures. In essence, the Dupin indicatrix is a depiction of the level curves obtained by taking the intersection $S \cap (c\mathbf{n}_P + T_P S)$ for *infinitesimal* c; unlike real level curves, the indicatrix is an genuine conic! We summarize all possibilities in a table.

type of point	# of asymptotic directions	Dupin indicatrix
elliptic	0	ellipse
hyperbolic	2	pair of hyperbolæ
parabolic	1	pair of parallel lines
planar	∞	empty

The advantage of this approach is that it is co-ordinate independent: $\mathbb{I}(\vec{w}_p, \vec{w}_p) = \pm 1$ describes the same *type* of curve regardless of which basis vectors/co-ordinates one uses to evaluate $\mathbb{I}$.

Examples 3.40. For a parametrized surface $\mathbf{x}$, at a given point $p = (u_0, v_0)$, write $\vec{w}_p = a\,\frac{\partial}{\partial u}\Big|_p + b\,\frac{\partial}{\partial v}\Big|_p$.

1. The tangent developable of the unit speed helix has

$$\mathbb{I}(\vec{w}_p, \vec{w}_p) = -\frac{v_0}{4}\,\mathrm{d}u^2(\vec{w}_p, \vec{w}_p) = -\frac{v_0}{4}a^2$$

The single asymptotic direction is $\vec{w}_p = \frac{\partial}{\partial v}\Big|_p$, the direction of zero curvature. The Dupin indicatrix is a pair of parallel lines

$$-\frac{v_0}{4}a^2 = \pm 1 \implies \vec{w}_p = \pm\frac{2}{\sqrt{|v_0|}}\,\frac{\partial}{\partial u}\bigg|_p + b\,\frac{\partial}{\partial v}\bigg|_p$$

[26] I.e. $\mathrm{I}(\vec{s}_p, \vec{s}_p) = 1 = \mathrm{I}(\vec{t}_p, \vec{t}_p)$ and $\mathrm{I}(\vec{s}_p, \vec{t}_p) = 0$.

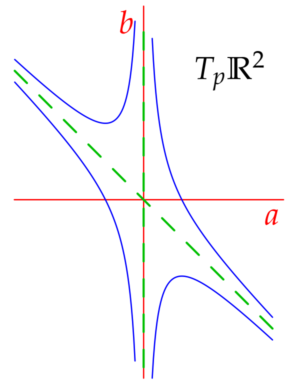
2. The graph of the surface $z = x^2 y$ has second fundamental form

$$\mathrm{I\!I} = \frac{2}{\sqrt{1+4u^2v^2+u^4}}\left(v\,\mathrm{d}u^2 + 2u\,\mathrm{d}u\mathrm{d}v\right)$$

At the point $p = (1, 2)$ (i.e. $\mathbf{x}(p) = (1, 2, 2)$), we see that

$$\mathrm{I\!I}(\vec{w}_p, \vec{w}_p) = \frac{4}{\sqrt{10}}\left(a^2 + ab\right) = \frac{4}{\sqrt{10}} a(a+b)$$

The point is hyperbolic with asymptotic directions $\frac{\partial}{\partial v}\Big|_p$ and $\frac{\partial}{\partial u}\Big|_p - \frac{\partial}{\partial v}\Big|_p$ (corresponding to $a = 0$ and $a + b = 0$). The indicatrix comprises the two hyperbolæ $a(a+b) = \pm\frac{\sqrt{10}}{4}$.



Exercises 3.4. 1. Consider the graph of the function $z = x^2 - 3y^2 + 7xy^3 + 9y^4$.

(a) Find the Gauss and mean curvatures at the origin.

(b) Find the normal curvature at the origin for the curve in the surface described by $x = y$.

2. For the elliptic paraboloid $z = x^2 + y^2$, let $P = (1, 2, 5)$ be a fixed point.

(a) Find the maximum and minimum values for the normal curvature at P.

(b) Find the Dupin indicatrix at P.

3. For the hyperbolic paraboloid $z = x^2 - y^2$, let $p = (u_0, v_0)$ and $P = \left(u_0, v_0, u_0^2 - v_0^2\right)$. If $c \neq 0$, prove that the intersection of the parallel plane $c\mathbf{n}_P + T_P S$ with the paraboloid may be expressed

$$(x-u_0)^2 - (y-v_0)^2 = \text{constant}, \quad z = x^2 - y^2$$

That is, the level curves really are hyperbolæ.

4. Consider the graph of the surface $z = x^2 + y^4$.

(a) Compute the Gauss curvature and classify all points according to Definition 3.34.

(b) Sketch the level curves $z = c$ where $c = 1$, $\frac{1}{100}$ and $\frac{1}{10000}$ and compare these to the Dupin indicatrix at the origin.
(*This should convince you of the importance that c be small!*)

5. Prove Theorem 3.38 by considering the normal acceleration of the curve $S \cap \operatorname{Span}\{\mathbf{v}_P, \mathbf{n}_P\}$.

3.5 Adaptive Frames & Gauss' Remarkable Theorem

We repurpose an idea first encountered when studying curves.

Definition 3.41. Let $\mathbf{x} : U \to \mathbb{E}^3$ parametrize a surface $S = \mathbf{x}(U)$. A *moving frame* for S is a triple of smooth functions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ on U such that, for each $p \in U$,

$$\{\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p)\} \text{ is a positively oriented orthonormal basis of } T_{\mathbf{x}(p)}\mathbb{E}^3$$

If S is oriented, we say that a frame is *adaptive* if $\mathbf{e}_3 = \mathbf{n}$ is the unit normal field.

For an adaptive frame, the tangent plane at each point is $T_{\mathbf{x}(p)}S = \operatorname{Span}\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$.
We will often refer to the matrix-valued function $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) : U \to SO_3(\mathbb{R})$ as the frame.

Examples 3.42. We'll repeatedly analyze three examples through this section.

1. The parabolic cylinder $\mathbf{x}(u, v) = (u, v, \frac{1}{2}u^2)$ has an adaptive frame

$$\mathbf{e}_1 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}1\\0\\u\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}-u\\0\\1\end{pmatrix}$$

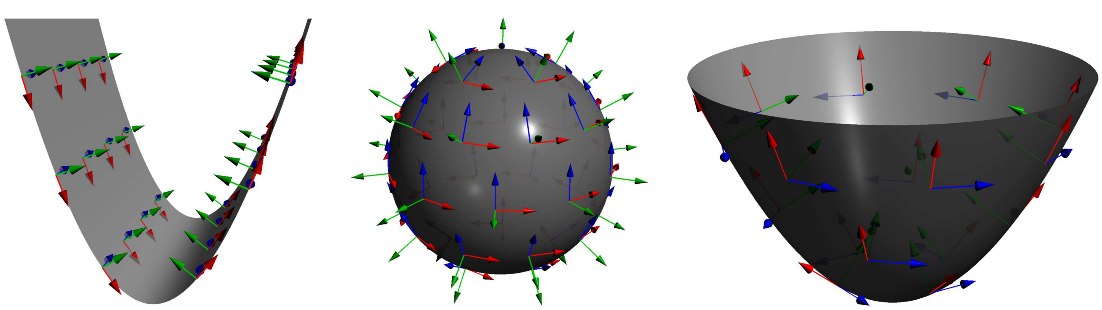
2. The sphere of radius R in spherical polar co-ordinates $\mathbf{x}(\psi, \phi)$ has an adaptive frame[27]

$$\mathbf{e}_1 = \begin{pmatrix}-\sin\psi\\ \cos\psi\\ 0\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}-\cos\psi\sin\phi\\ -\sin\psi\sin\phi\\ \cos\phi\end{pmatrix} \qquad \mathbf{e}_3 = \mathbf{x} = \begin{pmatrix}\cos\psi\cos\phi\\ \sin\psi\cos\phi\\ \sin\phi\end{pmatrix}$$

3. The paraboloid $\mathbf{x}(r, \psi) = (r\cos\psi, r\sin\psi, \frac{1}{2}r^2)$ has an adaptive frame

$$\mathbf{e}_1 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}\cos\psi\\ \sin\psi\\ r\end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix}-\sin\psi\\ \cos\psi\\ 0\end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix}-r\cos\psi\\ -r\sin\psi\\ 1\end{pmatrix}$$

In the pictures we've reduced the lengths of the frame vectors for clarity.



In each case the first two vectors were obtained by differentiating with respect to the co-ordinates (and normalizing if necessary). This only works because all these co-ordinate systems are *orthogonal*!

[27] We use ψ instead of θ since we'll need the latter for something else in a moment...

As with the Frenet frame, our strategy is to treat the analysis of a surface $\mathbf{x}: U \to \mathbb{E}^3$ two stages:

1. Describe how $\mathbf{x}$ moves with respect to the frame.
2. Describe how the frame itself moves.

As we're now used to, we describe infinitesimal change using 1-forms, following an approach pioneered by Élie Cartan around 1899.

Definition 3.43. Let $\mathbf{x}: U \to \mathbb{E}^3$ be a smooth map and $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ a moving frame. The *metric forms* θ_j and the *connection forms* ω_{jk} are the 1-forms on U defined by

$$\theta_j := \mathbf{e}_j \cdot \mathrm{d}\mathbf{x}, \quad \omega_{jk} = \mathbf{e}_j \cdot \mathrm{d}\mathbf{e}_k$$

where $j, k \in \{1, 2, 3\}$.

Since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthonormal, these forms are simply the co-ordinates of $\mathrm{d}\mathbf{x}$, $\mathrm{d}\mathbf{e}_1$, $\mathrm{d}\mathbf{e}_2$, $\mathrm{d}\mathbf{e}_3$ with respect to the moving frame:

$$\mathrm{d}\mathbf{x} = \sum_{j=1}^{3} (\mathbf{e}_j \cdot \mathrm{d}\mathbf{x})\mathbf{e}_j = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2 + \mathbf{e}_3\theta_3, \qquad \mathrm{d}\mathbf{e}_k = \sum_{j=1}^{3} \mathbf{e}_j \omega_{jk} \tag{$*$}$$

If the frame is adaptive, then $\theta_3 = 0$. Moreover, for any frame, there are only three independent connection forms.

Lemma 3.44. *For all j, k, we have $\omega_{jk} = -\omega_{kj}$. In particular $\omega_{jj} = 0$.*

Proof. Apply d to the identity $\mathbf{e}_j \cdot \mathbf{e}_k = 0$ or 1 to see that

$$0 = \mathrm{d}\mathbf{e}_j \cdot \mathbf{e}_k + \mathbf{e}_j \cdot \mathrm{d}\mathbf{e}_k = \omega_{kj} + \omega_{jk}$$

■

If ($*$) are arranged in matrix format, the subscripts follow the usual row/column convention:

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} \begin{pmatrix}\theta_1 \\ \theta_2 \\ \theta_3\end{pmatrix} = \mathcal{E}\Theta, \qquad \mathrm{d}\mathcal{E} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} = \mathcal{E}\omega$$

The second expression should remind you of the Frenet-Serret equations for a curve! The metric forms get their name because they *measure* small changes on the surface. The connection forms tell us how nearby frames are related (*connected*): if $\vec{s}_p \in T_p\mathbb{R}^2$, then

$$\mathcal{E}(p + \vec{s}_p) - \mathcal{E}(p) \approx \mathrm{d}\mathcal{E}(\vec{s}_p) = \mathcal{E}\omega(\vec{s}_p)$$

The fundamental forms of $\mathbf{x}$ can be written in terms of Θ and ω; in an adaptive frame this is particularly simple.

Lemma 3.45. *In an adaptive frame*

$$\mathrm{I} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \theta_1^2 + \theta_2^2 \quad \textit{and} \quad \mathrm{II} = -\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{e}_3 = -\theta_1\omega_{13} - \theta_2\omega_{23}$$

Examples (3.42, mk. II). You needn't compute all exterior derivatives $\mathrm{d}\mathbf{e}_k$; use the skew-symmetry of ω to help and look for which frame fields are easier to differentiate! The expressions for the fundamental forms should be a sanity check, since we know how to compute them already.

1. The parabolic cylinder has

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}1\\0\\u\end{pmatrix}\mathrm{d}u + \begin{pmatrix}0\\1\\0\end{pmatrix}\mathrm{d}v = \sqrt{1+u^2}\,\mathbf{e}_1\,\mathrm{d}u + \mathbf{e}_2\,\mathrm{d}v \implies \theta_1 = \sqrt{1+u^2}\,\mathrm{d}u,\ \ \theta_2 = \mathrm{d}v$$
$$\implies \mathrm{I} = (1+u^2)\,\mathrm{d}u^2 + \mathrm{d}v^2$$

Since $\mathbf{e}_2$ is constant, we have $\mathrm{d}\mathbf{e}_2 = \mathbf{0}$ from which

$$\omega_{12} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_2 = 0, \quad \omega_{23} = -\omega_{32} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_2 = 0$$

The final connection form requires a derivative:

$$\omega_{13} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_3 = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}1\\0\\u\end{pmatrix}\cdot\left[\frac{-u}{(1+u^2)^{3/2}}\begin{pmatrix}-u\\0\\1\end{pmatrix} + \frac{1}{\sqrt{1+u^2}}\begin{pmatrix}-1\\0\\0\end{pmatrix}\right] = \frac{-1}{1+u^2}\mathrm{d}u$$

Putting it together, we have

$$\omega = \frac{1}{1+u^2}\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\mathrm{d}u \quad\text{and}\quad \mathbb{I} = -\theta_1\omega_{13} - \theta_2\omega_{23} = \mathrm{d}u^2$$

2. For the sphere of radius R, $\mathrm{d}\mathbf{x} = R\cos\phi\,\mathbf{e}_1\,\mathrm{d}\psi + R\mathbf{e}_2\,\mathrm{d}\phi$, whence

$$\theta_1 = R\cos\phi\,\mathrm{d}\psi,\ \ \theta_2 = R\mathrm{d}\phi \implies \mathrm{I} = R^2\big(\cos^2\!\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2\big)$$
$$\mathrm{d}\mathbf{e}_1 = \begin{pmatrix}-\cos\psi\\-\sin\psi\\0\end{pmatrix}\mathrm{d}\psi \implies \begin{cases}\omega_{12} = -\mathbf{e}_2\cdot\mathrm{d}\mathbf{e}_1 = -\sin\phi\,\mathrm{d}\psi\\ \omega_{13} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_1 = \cos\phi\,\mathrm{d}\psi\end{cases}$$
$$\omega_{23} = \mathbf{e}_2\cdot\mathrm{d}\mathbf{e}_3 = \begin{pmatrix}-\cos\psi\sin\phi\\-\sin\psi\sin\phi\\\cos\phi\end{pmatrix}\cdot\left[\begin{pmatrix}-\sin\psi\cos\phi\\\cos\psi\cos\phi\\0\end{pmatrix}\mathrm{d}\psi + \begin{pmatrix}-\cos\psi\sin\phi\\-\sin\psi\sin\phi\\\cos\phi\end{pmatrix}\mathrm{d}\phi\right] = \mathrm{d}\phi$$
$$\implies \mathbb{I} = -\theta_1\omega_{13} - \theta_2\omega_{23} = -R\big(\cos^2\!\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2\big)$$

3. For the paraboloid,

$$\mathrm{d}\mathbf{x} = \begin{pmatrix}\cos\psi\\\sin\psi\\r\end{pmatrix}\mathrm{d}r + r\begin{pmatrix}-\sin\psi\\\cos\phi\\0\end{pmatrix}\mathrm{d}\psi = \sqrt{1+r^2}\,\mathbf{e}_1\,\mathrm{d}r + r\mathbf{e}_2\,\mathrm{d}\psi$$
$$\implies \theta_1 = \sqrt{1+r^2}\,\mathrm{d}r, \quad \theta_2 = r\,\mathrm{d}\psi \implies \mathrm{I} = (1+r^2)\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2$$

The connection forms are comparatively ugly. The low-hanging fruit is $\mathrm{d}\mathbf{e}_2 = \left(\begin{smallmatrix}-\cos\psi\\-\sin\psi\\0\end{smallmatrix}\right)\mathrm{d}\psi$, which quickly yields two of them:

$$\omega_{12} = \mathbf{e}_1\cdot\mathrm{d}\mathbf{e}_2 = -\frac{\mathrm{d}\psi}{\sqrt{1+r^2}}, \quad \omega_{23} = -\omega_{32} = -\mathbf{e}_3\cdot\mathrm{d}\mathbf{e}_2 = \frac{-r\,\mathrm{d}\psi}{\sqrt{1+r^2}}$$

The last connection form requires a nastier differentiation, though only one of the three terms in $d\mathbf{e}_3$ provides a non-zero result when dotted with $\mathbf{e}_1$:

$$\omega_{13} = \mathbf{e}_1 \cdot d\mathbf{e}_3 = \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} \cos\psi \\ \sin\psi \\ r \end{pmatrix} \cdot \left[\cdots + \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} -\cos\psi \\ -\sin\psi \\ 0 \end{pmatrix} dr \right] = \frac{-dr}{1+r^2}$$

We therefore obtain the connection form matrix

$$\omega = \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} 0 & -d\psi & \frac{-1}{\sqrt{1+r^2}}\,dr \\ d\psi & 0 & -r\,d\psi \\ \frac{1}{\sqrt{1+r^2}}\,dr & r\,d\psi & 0 \end{pmatrix}$$

and second fundamental form

$$\mathbb{I} = -\sqrt{1+r^2}\,dr\frac{-dr}{1+r^2} - r\,d\psi\frac{-r\,d\psi}{\sqrt{1+r^2}} = \frac{1}{\sqrt{1+r^2}}(dr^2 + r^2 d\psi^2)$$

The Structure Equations for a Moving Frame

The metric and connection forms satisfy matrix equations $d\mathbf{x} = \mathcal{E}\Theta$ and $d\mathcal{E} = \mathcal{E}\omega$. Since $d^2 = 0$, something nice happens when we take the exterior derivatives of these expressions:

$$\mathbf{0} = d^2\mathbf{x} = d(d\mathbf{x}) = d(\mathcal{E}\,\Theta) = d\mathcal{E}\wedge\Theta + \mathcal{E}\,d\Theta = \mathcal{E}(\omega\wedge\Theta + d\Theta)$$
$$0 = d^2\mathcal{E} = d(d\mathcal{E}) = d(\mathcal{E}\omega) = d\mathcal{E}\wedge\omega + \mathcal{E}d\omega = \mathcal{E}\big(\omega\wedge\omega + d\omega\big)$$

The notation $\omega\wedge\Theta$ means matrix multiplication using the wedge product of forms to evaluate each entry.[28] Since $\mathcal{E}$ is always an invertible matrix, we may conclude two identities.

Theorem 3.46. *The metric and connection forms satisfy the* structure equations*: each amounts to three separate equations after multiplying out the matrix expressions.*

1. $d\Theta + \omega\wedge\Theta = \mathbf{0} \iff d\theta_j + \sum_{k\neq j}\omega_{jk}\wedge\theta_k = 0$ *for each* $j = 1,2,3$
2. $d\omega + \omega\wedge\omega = 0 \iff d\omega_{jk} + \omega_{ji}\wedge\omega_{ik} = 0$ *where* i,j,k *are distinct.*

These aren't hard to remember if you pay attention to the indices. In an adaptive frame ($\theta_3 = 0$), things are a little simpler and some of the equations get special names:

First structure equations	$\begin{cases} d\theta_1 + \omega_{12}\wedge\theta_2 = 0 \\ d\theta_2 + \omega_{21}\wedge\theta_1 = 0 \end{cases}$
Symmetry equation	$\omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = 0$
Gauss equation	$d\omega_{12} + \omega_{13}\wedge\omega_{32} = 0$
Codazzi equations	$\begin{cases} d\omega_{13} + \omega_{12}\wedge\omega_{23} = 0 \\ d\omega_{23} + \omega_{21}\wedge\omega_{13} = 0 \end{cases}$

[28]Be careful not to reverse the order: $\Theta\wedge\omega$ makes no sense since the dimensions of the matrices are incompatible! Similarly, $\omega\wedge\omega$ is unlikely to be zero.

Examples (3.42, mk. III). 1. Since $\Theta = \begin{pmatrix} \sqrt{1+u^2}\,\mathrm{d}u \\ \mathrm{d}v \\ 0 \end{pmatrix}$ and $\omega = \frac{1}{1+u^2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\mathrm{d}u$, all the structure equations are trivial:

$$\mathrm{d}\Theta = \mathbf{0} = -\omega \wedge \Theta, \quad \mathrm{d}\omega = 0 = -\omega \wedge \omega$$

2. For the sphere $\Theta = R\begin{pmatrix} \cos\phi\,\mathrm{d}\psi \\ \mathrm{d}\phi \\ 0 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & \sin\phi\,\mathrm{d}\psi & -\cos\phi\,\mathrm{d}\psi \\ -\sin\phi\,\mathrm{d}\psi & 0 & \mathrm{d}\phi \\ \cos\phi\,\mathrm{d}\psi & -\mathrm{d}\phi & 0 \end{pmatrix}$, from which

$$\mathrm{d}\Theta = R\begin{pmatrix} -\sin\phi \\ 0 \\ 0 \end{pmatrix}\mathrm{d}\phi \wedge \mathrm{d}\psi = -\omega \wedge \Theta$$

$$\mathrm{d}\omega = \begin{pmatrix} 0 & \cos\phi & \sin\phi \\ -\cos\phi & 0 & 0 \\ -\sin\phi & 0 & 0 \end{pmatrix}\mathrm{d}\phi \wedge \mathrm{d}\psi = -\omega \wedge \omega$$

3. This time we have $\Theta = \begin{pmatrix} \sqrt{1+r^2}\mathrm{d}r \\ r\,\mathrm{d}\psi \\ 0 \end{pmatrix}$ and $\omega = \frac{1}{\sqrt{1+r^2}}\begin{pmatrix} 0 & -\mathrm{d}\psi & -\frac{\mathrm{d}r}{\sqrt{1+r^2}} \\ \mathrm{d}\psi & 0 & -r\,\mathrm{d}\psi \\ \frac{\mathrm{d}r}{\sqrt{1+r^2}} & r\,\mathrm{d}\psi & 0 \end{pmatrix}$.

The first equations aren't too bad to check:

$$\mathrm{d}\Theta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}r \wedge \mathrm{d}\psi = -\omega \wedge \Theta$$

The second is a little nastier: you should check that

$$\mathrm{d}\omega = \frac{1}{(1+r^2)^{3/2}}\begin{pmatrix} 0 & r & 0 \\ -r & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\mathrm{d}r \wedge \mathrm{d}\psi = -\omega \wedge \omega$$

Gauss' Remarkable Theorem

Suppose we have an adaptive frame for an oriented local surface $\mathbf{x}$. If θ_1, θ_2 were linearly dependent at p, then the differential $\mathrm{d}\mathbf{x} : T_p\mathbb{R}^2 \to T_{\mathbf{x}(p)}S = \operatorname{Span}\{\mathbf{e}_1(p), \mathbf{e}_2(p)\}$ would have rank ≤ 1 and thus not be a bijection. We conclude that $\{\theta_1, \theta_2\}$ forms a basis of the space of 1-forms at p and that any other 1-form may be written as a linear combination thereof. . .

Lemma 3.47. *There exist unique functions a, b, c such that*

$$\omega_{13} = a\theta_1 + b\theta_2, \quad \omega_{23} = b\theta_1 + c\theta_2$$

With respect to these functions, the second fundamental form, Gauss and mean curvatures are

$$\mathbb{I} = -a\theta_1^2 - 2b\theta_1\theta_2 - c\theta_1^2, \qquad K = ac - b^2, \qquad H = -\frac{1}{2}(a+c)$$

Proof. That $\omega_{13} = a\theta_1 + b\theta_2$ and $\omega_{23} = \hat{b}\theta_1 + c\theta_2$ are linear combinations of θ_1, θ_2 is the above discussion. The symmetry equation says that $\hat{b} = b$:

$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2 = (-b + \hat{b})\,\theta_1 \wedge \theta_2$$

and the formula for $\mathbb{I}$ follows from Lemma 3.45.

Moreover, if $\vec{w}_1$ and $\vec{w}_2$ are the dual basis fields to θ_1, θ_2 at each point, then the matrices of $\mathrm{I}, \mathbb{I}$ with respect to these fields are the identity matrix[29] and $B = \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}$. The Gauss and mean curvatures are therefore the determinant and half the trace of B. ■

Now consider the final connection form. Since θ_1, θ_2 form a basis at each point, we may write

$$\omega_{12} = f\theta_1 + g\theta_2$$

for some functions $f, g : U \to \mathbb{R}$. Applying the 1st structure equations, we see that

$$\begin{aligned} d\theta_1 &= -\omega_{12} \wedge \theta_2 = -f\theta_1 \wedge \theta_2 \\ d\theta_2 &= -\omega_{21} \wedge \theta_1 = -\theta_1 \wedge \omega_{12} = -g\theta_1 \wedge \theta_2 \end{aligned}$$

whence f, g, and thus ω_{12} are determined by θ_1, θ_2. This brings us to the capstone result of the course.

Theorem 3.48 (Gauss' Theorem Egregium). *The Gauss curvature depends only on the first fundamental form* I.

Proof. By the above discussion, ω_{12} (and thus $d\omega_{12}$) depends only on θ_1, θ_2, which may be recovered from I by writing it as a sum of squares. Now observe that the Gauss equation reads

$$d\omega_{12} = \omega_{13} \wedge \omega_{23} = (a\theta_1 + b\theta_2) \wedge (b\theta_1 + c\theta_2) = (ac - b^2)\theta_1 \wedge \theta_2 = K\theta_1 \wedge \theta_2$$

■

An explicit formula for K as a function of the coefficients E, F, G of I can be found; see Exercise 9.

In Latin, *egregium* means *remarkable* or *outstanding*; this is the (modest!) term Gauss used upon proving his result in 1827. Why did he consider it so remarkable? The original definition of K relied on the normal field; an object *outside* the surface which helps describe its position/orientation in $\mathbb{E}^3$. However, Gauss' Theorem says that K is *intrinsic* to the surface: it depends only on the *metric* (first fundamental form) which may be understood by an occupant of the surface with no ability to escape (travel out of the surface) in order to view its shape. By contrast, the second fundamental form and the mean curvature depend on how a surface is embedded; these are *extrinsic* quantities.

The result provides what is often a faster method for calculating the Gauss curvature.

1. Compute $\mathrm{I} = d\mathbf{x} \cdot d\mathbf{x}$ and express it as a sum of squares $\mathrm{I} = \theta_1^2 + \theta_2^2$.
2. Write $\omega_{12} = f\theta_1 + g\theta_2$ and compute f, g using the 1st structure equations.
3. Use the Gauss equation to find K.

We need only calculate 1-forms $\theta_1, \theta_2, \omega_{12}$ that are related to the *tangent* part of the moving frame. The unit normal $\mathbf{e}_3$ doesn't need to be considered or calculated.

[29] $\theta_j(\vec{w}_k) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ implies that $d\mathbf{x}(\vec{w}_1) = \mathbf{e}_1$ and $d\mathbf{x}(\vec{w}_2) = \mathbf{e}_2$ are orthonormal.

Examples (3.42, mk. IV). We return to our examples one last time. We'll pretend we have *only* the 1st fundamental form to work with; even though we've calculated the connection forms already, the goal is to see that $\omega_{12} = f\theta_1 + g\theta_2$ and thus K may be found directly from I.

1. The parabolic cylinder has $\mathrm{I} = (1+u^2)\,\mathrm{d}u^2 + \mathrm{d}v^2$ so the natural choice is
$$\theta_1 = \sqrt{1+u^2}\,\mathrm{d}u \quad \text{and} \quad \theta_2 = \mathrm{d}v$$
Since $\mathrm{d}\theta_1 = 0 = \mathrm{d}\theta_2$ we see that $f = g = 0$. We conclude that
$$\omega_{12} = 0 \implies \mathrm{d}\omega_{12} = 0 \implies K = 0$$

2. For the sphere $\mathrm{I} = R^2(\cos^2\phi\,\mathrm{d}\psi^2 + \mathrm{d}\phi^2)$ so we choose $\theta_1 = R\cos\phi\,\mathrm{d}\psi$ and $\theta_2 = R\,\mathrm{d}\phi$. Certainly $\mathrm{d}\theta_2 = 0 \implies g = 0$. Moreover,
$$\mathrm{d}\theta_1 = -f\theta_1 \wedge \theta_2 \implies R\sin\phi\,\mathrm{d}\psi \wedge \mathrm{d}\phi = -fR^2\cos\phi\,\mathrm{d}\psi \wedge \mathrm{d}\phi \implies f = -R^{-1}\tan\phi$$
We conclude that $\omega_{12} = -R^{-1}\tan\phi\,\theta_1 = -\sin\phi\,\mathrm{d}\psi$, from which
$$\mathrm{d}\omega_{12} = \cos\phi\,\mathrm{d}\psi \wedge \mathrm{d}\phi = \frac{1}{R^2}\theta_1 \wedge \theta_2 \implies K = \frac{1}{R^2}$$

3. For the paraboloid, $\mathrm{I} = (1+r^2)\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2$ so we choose $\theta_1 = \sqrt{1+r^2}\,\mathrm{d}r$ and $\theta_2 = r\,\mathrm{d}\psi$. This time $\mathrm{d}\theta_1 = 0 \implies f = 0$ and
$$\mathrm{d}\theta_2 = -g\theta_1 \wedge \theta_2 \implies \mathrm{d}r \wedge \mathrm{d}\psi = -gr\sqrt{1+r^2}\,\mathrm{d}r \wedge \mathrm{d}\psi \implies g = -\frac{1}{r\sqrt{1+r^2}}$$
We conclude that $\omega_{12} = -\frac{1}{r\sqrt{1+r^2}}\,\theta_2 = -\frac{1}{\sqrt{1+r^2}}\,\mathrm{d}\psi$, from which
$$\mathrm{d}\omega_{12} = \frac{r}{(1+r^2)^{3/2}}\,\mathrm{d}r \wedge \mathrm{d}\psi = \frac{1}{(1+r^2)^2}\theta_1 \wedge \theta_2 \implies K = \frac{1}{(1+r^2)^2}$$

Since K depends only on the metric, it is invariant under *isometric* transformations of the surface. This helps explain why the Gauss curvature of a cylinder and a cone are both zero: both may constructed by rolling up a flat plane without otherwise distorting it.

The contrapositive is also important: if two surfaces have different Gauss curvatures, then they cannot be isometric. Since the metric I determines how we measure angle and length, this explains why we can never get a perfect flat map ($K = 0$) of any part of the Earth ($K = \frac{1}{R^2}$). The holy grail of map-making would be a map that is free of length, angle and shape distortion: for instance,

1. Straight lines on the map correspond to paths of shortest distance on the Earth.
2. Angles on the map equal the corresponding angles on the surface of the Earth.
3. Areas on the map and the Earth are in constant ratio.

Gauss' Theorem ultimately implies that you cannot have all these properties in one map; you can have one, but only one, at a time!

Riemannian Geometry

We can even employ the method when there is no surface! The idea is to equip a domain with an abstract first fundamental form and use it to compute lengths, angles, area, geodesics, curvature, etc.

Example 3.49. The Poincare disk model of hyperbolic space is the unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ equipped with the metric (first fundamental form)

$$\mathrm{I} = \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - x^2 - y^2)^2} = \frac{4(\mathrm{d}r^2 + r^2\,\mathrm{d}\psi^2)}{(1 - r^2)^2}$$

The idea is that distances get larger as one approaches the boundary of the disk. The natural choice is $\theta_1 = \frac{2\mathrm{d}r}{1-r^2}$ and $\theta_2 = \frac{2r\,\mathrm{d}\psi}{1-r^2}$, from which $\mathrm{d}\theta_1 = 0 \implies f = 0$ and

$$\mathrm{d}\theta_2 = -g\theta_1 \wedge \theta_2 \implies \frac{2(1 + r^2)}{(1 - r^2)^2}\mathrm{d}r \wedge \mathrm{d}\psi = -\frac{4gr}{(1 - r^2)^2}\,\mathrm{d}r \wedge \mathrm{d}\psi \implies g = -\frac{1 + r^2}{2r}$$

from which

$$\mathrm{d}\omega_{12} = \mathrm{d}(g\theta_2) = -\mathrm{d}\left(\frac{1 + r^2}{1 - r^2}\right) \wedge \mathrm{d}\psi = \frac{-4r}{(1 - r^2)^2}\,\mathrm{d}r \wedge \mathrm{d}\psi = -\theta_1 \wedge \theta_2 \implies K = -1$$

Hyperbolic space is the canonical example of a negatively curved geometry. There is no surface here, no second fundamental form and the mean curvature is meaningless!

The Gauss curvature of a surface is merely the simplest avatar of a more general object called the *Riemann curvature tensor.* Take general relativity, for instance.[30] Mass is construed as changing the metric of spacetime (i.e. I); by a similar analysis it can be seen that this metric is compatible with unique connection (essentially ω) from which the curvature ($\mathrm{d}\omega + \omega \wedge \omega$) may be computed. When a physicist claims that *spacetime is curved*, this is what they mean: there is no *exterior* to spacetime from which we can measure curvature, so everything is computed intrinsically. Since there is no surface, it is harder to imagine what K means in this context (e.g. Section 3.4); a proper treatment of its consequences will have to be postponed to another course.[31]

The Fundamental Theorem of Surfaces

Recall our discussion of the equivalence of spacecurves up to rigid motions (Theorem 1.38) and the Fundamental Theorem of Biregular Spacecurves (Corollary 1.42). A similar discussion is available for surfaces if we replace curvature and torsion with the fundamental forms $\mathrm{I}, \mathbb{I}$.

The equivalence problem is almost identical. Suppose $\mathbf{x} : U \to \mathbb{E}^3$ is an oriented surface, and that $A \in O_3(\mathbb{R})$ and $\mathbf{b} = \mathbb{E}^3$ are constants. Then $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ is the result of an isometry applied to $\mathbf{x}$. Any moving frame for $\mathbf{x}$ may be transformed into such for $\mathbf{y}$ via

$$\mathcal{E}_{\mathbf{y}} = \begin{pmatrix} A\mathbf{e}_1 & A\mathbf{e}_2 & \pm A\mathbf{e}_3 \end{pmatrix}$$

[30]Really this is *pseudo*-Riemannian geometry, since I is not positive-definite.

[31]For instance, if you join three nearby points with paths of shortest length (geodesics), then $K < 0$ means the angle sum of the resulting 'triangle' is *less than* 180°. When $K > 0$ (e.g. a sphere), the angle sum is greater than 180°.

where $\pm 1 = \det A$. The upshot is that $\mathbf{n_y} = (\det A)A\mathbf{n_x}$, whence $\mathrm{I}, \mathbb{I}$ transform exactly as κ, τ:

$$\mathrm{I}_{\mathbf{y}} = \mathrm{d}\mathbf{y} \cdot \mathrm{d}\mathbf{y} = (A\mathrm{d}\mathbf{x}) \cdot (A\mathrm{d}\mathbf{x}) = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \mathrm{I}_{\mathbf{x}}$$
$$\mathbb{I}_{\mathbf{y}} = -\mathrm{d}\mathbf{y} \cdot \mathrm{d}\mathbf{n_y} = -(\det A)(A\mathrm{d}\mathbf{x}) \cdot (A\mathrm{d}\mathbf{n_x}) = (\det A)\mathbb{I}_{\mathbf{x}}$$

As with curves, we may ask the question in reverse. If we know the fundamental forms, can we also recover the surface up to a rigid motion? The answer is yes, though with a caveat: unlike κ, τ for spacecurves, the fundamental forms cannot be chosen independently.

Theorem 3.50 (Bonnet). *Suppose* I *and* $\mathbb{I}$ *are symmetric bilinear forms where* I *is positive-definite. Provided the Gauss–Codazzi equations are satisfied, there exists a local parametrized surface with these fundamental forms, which is moreover unique up to rigid motions.*

Everything ultimately depends on a generalization of the existence/uniqueness theorem for ODE known as the *Frobenius Theorem.* Here is a rough sketch of how the process works.

1. Suppose we are given $\mathrm{I}, \mathbb{I}$ on U, and initial conditions at some $p \in U$ for the surface $\mathbf{x}(p) = \mathbf{x}_0 \in \mathbb{E}^3$ and frame $\mathcal{E}(p) = \mathcal{E}_0 \in \mathrm{SO}_3(\mathbb{R})$.

2. Since I is positive-definite, it may be written $\mathrm{I} = \theta_1^2 + \theta_2^2$.

3. The first structure equations determine ω_{12} and $\mathbb{I}$ determines ω_{13} and ω_{23} (Lemma 3.47).

4. The adaptive frame $\mathcal{E}$ satisfies an initial value problem

   $$\mathrm{d}\mathcal{E} = \mathcal{E}\omega \qquad \mathcal{E}(p) = \mathcal{E}_0 \tag{$*$}$$

   The Frobenius Theorem shows that this has a unique local solution provided ω satisfies the Gauss–Codazzi equations $\mathrm{d}\omega + \omega \wedge \omega = 0$. The analogue of Corollary 1.41 shows that the solution $\mathcal{E}$ is $\mathrm{SO}_3(\mathbb{R})$-valued.

5. To find the surface, we need to solve a second initial value problem

   $$\mathrm{d}\mathbf{x} = \mathcal{E}\Theta \qquad \mathbf{x}(p) = \mathbf{x}_0$$

   Frobenius says this has a unique solution provided $\mathrm{d}\Theta + \omega \wedge \Theta = 0$. Since this is precisely what we used to determine ω in step 2, we don't need to check this condition.

6. Any other choice of metric forms in step 2 merely results in rotating $\mathcal{E}$ around $\mathbf{n} = \mathbf{e}_3$ and does not affect the resulting surface.

It is a little easier to understand the integrability condition in co-ordinates: $(*)$ is a linear system of eighteen PDE in nine unknowns

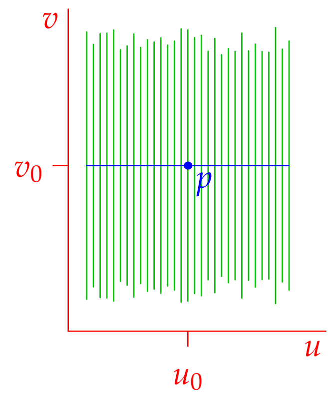
$$\begin{cases} \frac{\partial \mathcal{E}}{\partial u} = \mathcal{E}P \\ \frac{\partial \mathcal{E}}{\partial v} = \mathcal{E}Q \end{cases} \quad \text{where } P = \omega\big(\tfrac{\partial}{\partial u}\big),\ Q = \omega\big(\tfrac{\partial}{\partial v}\big) \text{ are skew-symmetric matrix functions}$$

The Gauss–Codazzi equations are essentially the fact that mixed partial derivatives commute:[32]

$$0 = \mathcal{E}_{uv} - \mathcal{E}_{vu} = \mathcal{E}_v P + \mathcal{E}P_v - \mathcal{E}_u Q + \mathcal{E}Q_u = \mathcal{E}\big(P_v - Q_u - [P,Q]\big)$$
$$P_v - Q_u - [P,Q] = \tfrac{\partial}{\partial v}\omega\big(\tfrac{\partial}{\partial u}\big) - \tfrac{\partial}{\partial u}\omega\big(\tfrac{\partial}{\partial v}\big) - \big[\omega\big(\tfrac{\partial}{\partial u}\big), \omega\big(\tfrac{\partial}{\partial v}\big)\big] = (\mathrm{d}\omega + \omega \wedge \omega)\left(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial u}\right)$$

[32] $[P,Q] = PQ - QP$ and $\mathrm{d}\omega$ is evaluated as in Exercise 2.3.10.

The part that requires some proof is that compatibility condition is *sufficient* for a solution. This is not as hard as it sounds; here is another sketch:



1. If $p = (u_0, v_0)$, use the ODE existence/uniqueness theorem to solve an initial value problem on the horizontal line $v = v_0$:

$$\frac{\mathrm{d}\widetilde{\mathcal{E}}}{\mathrm{d}u} = \widetilde{\mathcal{E}}P(u, v_0), \qquad \widetilde{\mathcal{E}}(u_0, v_0) = \mathcal{E}_O$$

2. For *each* u_1, apply the ODE theorem to solve another IVP on the vertical line $u = u_1$:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}v} = \mathcal{E}Q(u_1, v), \qquad \mathcal{E}(u_1, v_0) = \widetilde{\mathcal{E}}(u_1, v_0)$$

3. Finally, one shows that the resulting $\mathcal{E}$ is differentiable with respect to u, and uses the compatibility condition to check that $\mathcal{E}_u = \mathcal{E}P$ as required.

The first two steps may be accomplished approximately using a numerical method to desired accuracy, so this amounts to an algorithm for the approximation of $\mathcal{E}$. The same approach can then be followed to approximate $\mathbf{x}$.

The Gauss–Codazzi equations in curvature-line co-ordinates Suppose we wish to define a surface with curvature-line co-ordinates u, v; we choose fundamental forms

$$\mathrm{I} = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2, \qquad \mathrm{II} = k_1 E\,\mathrm{d}u^2 + k_2 G\mathrm{d}v^2 \tag{†}$$

where E, G are positive functions and k_1, k_2 the principal curvatures. It is sensible to choose metric forms $\theta_1 = \sqrt{E}\,\mathrm{d}u$ and $\theta_2 = \sqrt{G}\,\mathrm{d}v$. In the language of Lemma 3.47,

$$a = -k_1, \quad b = 0, \quad c = -k_2, \quad \omega_{13} = -k_1\sqrt{E}\,\mathrm{d}u, \quad \omega_{23} = -k_2\sqrt{G}\,\mathrm{d}v$$

The first structure equations determine (Exercise 9)

$$\omega_{12} = \frac{1}{2\sqrt{EG}}\left(E_v\,\mathrm{d}u - G_u\,\mathrm{d}v\right)$$

The Gauss–Codazzi equations turn out to be equivalent to

$$\mathrm{d}\omega_{12} + \omega_{21}\wedge\omega_{13} = 0 \iff \left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v = -2k_1k_2\sqrt{EG}$$
$$\mathrm{d}\omega_{13} + \omega_{12}\wedge\omega_{23} = 0 \iff 2(k_1)_v E = (k_2 - k_1)E_v$$
$$\mathrm{d}\omega_{23} + \omega_{21}\wedge\omega_{13} = 0 \iff 2(k_2)_u G = (k_1 - k_2)G_u$$

This makes it clear that there is an interdependence between I and II: we cannot independently choose the metric and the curvatures. However, if E, G, k_1, k_2 satisfy these equations, Bonnet's theorem guarantees that there indeed exists a surface with fundamental forms †, unique up to rigid motions.

Even though I, II cannot be chosen independently, Bonnet's result is still considered the best description of the *minimal data* for a surface. You might suspect/hope that knowledge of K, H would be enough to determine a surface up to rigid motions, but Exercise 10 shows such hope to be vain!

Exercises 3.5. 1. The unit cylinder $\mathbf{x}(\phi, v) = \big(\cos\phi, \sin\phi, v\big)$ has adaptive frame

$$\mathbf{e}_1 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

(a) Directly compute the metric forms θ_j and connection forms ω_{jk}.

(b) That the six structure equations are satisfied should be obvious from your answers to (a): why?

(c) Why is it completely obvious from your answer to (a) that $K \equiv 0$?

2. For a general regular surface, explain why we cannot, in general, find co-ordinates u, v for which $\mathrm{I} = \mathrm{d}u^2 + \mathrm{d}v^2$.

3. For the paraboloid example (3.42.3) verify the Gauss–Codazzi equations $\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0$.
(*Hint: this is easier if you treat the three equations separately!*)

4. Verify that the metric $\mathrm{I} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ on the upper half-plane $y > 0$ has curvature $K = -1$.
(*Hint: Recall Example 3.49 and Exercise 3.2.9*)

5. Consider the catenoid $\mathbf{x}(u, v) = \big(\cos u \cosh v, \sin u \cosh v, v\big)$ obtained by revolving the catenary $x = \cosh z$ around the z-axis.

(a) Show that there exists a moving frame for which the metric forms are

$$\theta_1 = \cosh v\, \mathrm{d}u, \quad \theta_2 = \cosh v\, \mathrm{d}v$$

(b) Show that $\omega_{12} = \tanh v\, \mathrm{d}u = \frac{\sinh v}{\cosh v}\, \mathrm{d}u$ and use it to prove that the Gaussian curvature of the catenoid is

$$K = -\frac{1}{\cosh^4 v}$$

6. We re-prove Exercise 3.3.11 using our new language.

(a) Suppose a surface $\mathbf{x}$ is totally umbilic: $\mathbb{I} = \lambda \mathrm{I}$, where λ is some function. Explain why $\omega_{13} = -\lambda\theta_1$ and $\omega_{23} = -\lambda\theta_2$.

(b) Use the 1$^{\text{st}}$ structure equations and the Codazzi equations to prove that $\mathrm{d}\lambda = 0$.

(c) If $a = 0$, what is $\mathbf{x}$?

(d) If $a \neq 0$, define $\mathbf{c} := \mathbf{x} - \frac{1}{a}\mathbf{e}_3$. Prove that $\mathrm{d}\mathbf{c} = \mathbf{0}$ and hence conclude that the surface is (part of a) round sphere.

7. Suppose $\mathcal{E} = \big(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\big)$ is an adaptive frame for a surface. Any other adaptive frame (with the same orientation) is obtained by *rotating* around $\mathbf{e}_3$: that is $\hat{\mathcal{E}} = \big(\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \mathbf{e}_3\big)$ where

$$\hat{\mathbf{e}}_1 = \cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2, \quad \hat{\mathbf{e}}_2 = -\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2$$

for some smooth function $\varphi : U \to \mathbb{R}$.

(a) Compute θ_1, θ_2 in terms of $\hat{\theta}_1, \hat{\theta}_2$ and conclude that $\hat{\theta}_1 \wedge \hat{\theta}_2 = \theta_1 \wedge \theta_2$.

(b) Use Definition 3.43 to compute $\hat{\omega}_{12}$ in terms of ω_{12} and φ. Verify that $\mathrm{d}\hat{\omega}_{12} = \mathrm{d}\omega_{12}$ so that the Gauss equation is identical for the new moving frame.

8. Suppose I is the 1^{st} fundamental form of a surface. Suppose $\mathrm{I} = \theta_1^2 + \theta_2^2$ for some 1-forms θ_1, θ_2. Prove that there exists a moving frame $\mathcal{E} = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ for which $\mathrm{d}\mathbf{x} = \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2$.

   (*Hint: consider the dual vector fields to* θ_1, θ_2)

9. Suppose u, v are orthogonal co-ordinates so that $\theta_1 = \sqrt{E}\,\mathrm{d}u$ and $\theta_2 = \sqrt{G}\,\mathrm{d}v$.

   (a) Use the structure equations to prove that

   $$\omega_{12} = \frac{1}{2\sqrt{EG}}\left(E_v\,\mathrm{d}u - G_u\,\mathrm{d}v\right)$$

   (b) Hence deduce an explicit formula for the Gauss curvature in terms of the coefficients of the 1^{st} fundamental form:

   $$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right)$$

   This can be multiplied out to remove the square roots, though you'll get more terms. A nastier expression (the Brioshi formula) may be found for general co-ordinates with $F \neq 0$.

10. In Exercise 3.3.5 we saw that the tangent developable $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$ of a unit-speed curve has curvatures $K = 0$, $H = -\frac{\tau}{2v\kappa}$. Use this to describe *two* surfaces with the same curvature functions which are not related by a direct isometry.

11. Show that the surfaces parametrized by

    $$\mathbf{x}(u,v) = \left(u\cos\phi, u\sin\phi, \ln u\right), \quad \mathbf{y}(u,v) = \left(u\cos\phi, u\sin\phi, \phi\right)$$

    have the same Gauss curvature but distinct first fundamental forms $\mathrm{I}_{\mathbf{x}} \neq \mathrm{I}_{\mathbf{y}}$. To do this properly, you should argue that there is no reparametrization of $\mathbf{y}$ so that $K_{\mathbf{x}} = K_{\mathbf{y}}$ and $\mathrm{I}_{\mathbf{x}} = \mathrm{I}_{\mathbf{y}}$.

    (*Gauss' Theorem isn't bidirectional: surfaces can have the same K without being locally isometric*)

12. Consider the family of surfaces

    $$\mathbf{x}^t(u,v) = \cos t \begin{pmatrix} \sin u \sinh v \\ -\cos u \sinh v \\ u \end{pmatrix} + \sin t \begin{pmatrix} \cos u \cosh v \\ \sin u \cosh v \\ v \end{pmatrix}, \qquad t \in [0, \tfrac{\pi}{2}]$$

    When $t = 0$ this is a *helicoid*. When $t = \frac{\pi}{2}$ this is the *catenoid* from Exercise 5.

    (a) Compute the first fundamental form of $\mathbf{x}^t$ and show that it is independent of t (the family $\mathbf{x}^t$ is therefore *isometric*).

    (b) Show that the unit normal of $\mathbf{x}^t$ is also independent of t:

    $$\mathbf{n}^t = \frac{1}{\cosh v}\begin{pmatrix} \cos u \\ \sin u \\ -\sinh v \end{pmatrix}$$

    Hence compute the second fundamental form of $\mathbf{x}^t$ for each t.

    (c) Find the Gauss and mean curvature of all surfaces $\mathbf{x}^t$. What is special about this family? Relate this to Gauss' Theorem.