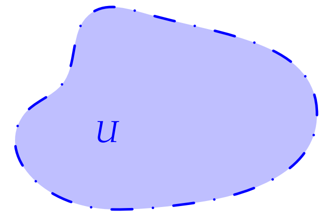
2 Vector Fields & Differential Forms

In preparation for our study of surfaces, we further develop the notion of a tangent vector. To permit easy differentiation, throughout this section all functions are assumed to be *smooth* (infinitely differentiable) and $U \subseteq \mathbb{R}^n$ will denote a *connected open set*: (informally) a region consisting of a single piece without edge points. As previously, n will always be 1, 2 or 3: when $n = 1$, $U = (a, b)$ is an open interval; the picture illustrates $n = 2$.



2.1 Directional Derivatives, Tangent Vectors & Vector Fields

First recall some basic objects and facts from elementary multivariable calculus.

Definition 2.1. The *gradient* of $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is the function $\nabla f : U \to \mathbb{R}^n$ defined by

$$\nabla f(x_1, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Given a point $p \in U$, a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, and a function $f : U \to \mathbb{R}$, the *directional derivative* of f at p in the direction $\mathbf{v}$ is the *scalar*

$$D_{\mathbf{v}} f(p) := \sum_{k=1}^{n} v_k \left. \frac{\partial f}{\partial x_k} \right|_p = \mathbf{v} \cdot (\nabla f(p))$$

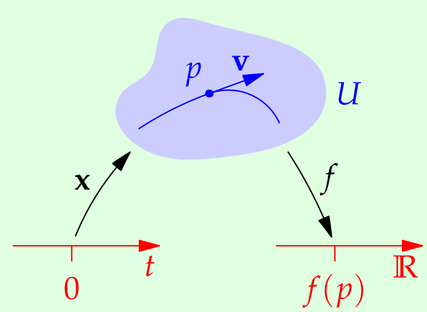
Example 2.2. Suppose $f(x, y, z) = x^2 - z \cos y$, $p = (1, \pi, 0)$, and $\mathbf{v} = (3, 5, 1)$. Then

$$\nabla f = \begin{pmatrix} 2x \\ z \sin y \\ -\cos z \end{pmatrix} \implies D_{\mathbf{v}} f(p) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 7$$

The directional derivative describes the rate of change of the value of f in a given direction.

Lemma 2.3. 1. *By the chain rule, if* $\mathbf{x}(t)$ *is a curve such that* $\mathbf{x}(0) = p$ *and* $\mathbf{x}'(0) = \mathbf{v}$*, then*

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} f(\mathbf{x}(t)) = \sum_{k=1}^{n} \left. \frac{\partial f}{\partial x_k} \right|_p x_k'(0) = D_{\mathbf{v}} f(p)$$



is the rate of change of f *at* p *as one travels along the curve.*

2. *If* t *is small, then* $f(p + t\mathbf{v}) \approx f(p) + D_{\mathbf{v}} f(p)\, t$.

3. *If* $\mathbf{v}$ *is a unit vector making angle* θ *with* $\nabla f(p)$*, then*

$$D_{\mathbf{v}} f(p) = \mathbf{v} \cdot \nabla f(p) = ||\nabla f(p)|| \cos \theta$$

is maximal when $\mathbf{v}$ *points in the same direction as* $\nabla f(p)$*. Otherwise said,* $\nabla f(p)$ *points in the direction of greatest increase of* f *at* p*; its magnitude measures the rate of change.*

By placing the function f at the end of the directional derivative, we are tempted to create an *operator*

$$D_{\mathbf{v}}\big|_p = \sum_{k=1}^{n} v_k \left.\frac{\partial}{\partial x_k}\right|_p$$

which takes a *function* $f : U \to \mathbb{R}$ and returns the *scalar* $D_{\mathbf{v}} f(p)$. This operator is a map (function) from the set of smooth functions $f : U \to \mathbb{R}$ to the real numbers. It is even more tempting to drop the point p and allow the components of $\mathbf{v}$ to be *smooth functions.* This yields a new definition of an old concept.

Definition 2.4. The set of directional derivative operators $D_{\mathbf{v}}\big|_p$ is the *tangent space* $T_p\mathbb{R}^n$ at $p \in \mathbb{R}^n$. A *vector field* v on $U \subseteq \mathbb{R}^n$ is a smooth choice for each $p \in U$ of an element of $T_p\mathbb{R}^n$: that is

$$v = \sum_{k=1}^{n} v_k \frac{\partial}{\partial x_k} \quad \text{where each } v_k : U \to \mathbb{R} \text{ is smooth}$$

Each operator $\frac{\partial}{\partial x_k}$ is termed a *co-ordinate vector field.*
If $f : U \to \mathbb{R}$ is smooth, we write $v[f] = \sum v_k \frac{\partial f}{\partial x_k}$ for the result of applying the vector field v to f; this is itself a smooth function $v[f] : U \to \mathbb{R}$.

Each tangent space $T_p\mathbb{R}^n$ is a vector space, with natural basis $\frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_n}\big|_p$. In this brave new world, a tangent vector $v_p = \sum v_k \frac{\partial}{\partial x_k}\big|_p$ corresponds to our previous notion $\mathbf{v}_p = (v_1, \ldots, v_n)$. While this might seem artificially complicated, the rational is simple: the purpose of tangent vectors is to measure how functions change in given directions (Lemma 2.3!).

Examples 2.5. 1. The vector field $v = 3x\frac{\partial}{\partial x} + 2xz\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ on $\mathbb{R}^3$ corresponds to the vector-valued function $\mathbf{v}(x, y, z) = (3x, 2xz, -x)$. Given $f(x, y, z) = xy^2 + z$, we have

$$v[f] = 3x\frac{\partial f}{\partial x} + 2xz\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial z} = 3xy^2 + 4x^2yz - x$$

which, as expected, is a smooth function $v[f] : \mathbb{R}^3 \to \mathbb{R}$.

2. Suppose, in $\mathbb{R}^2$, that we are given a *vector field* $v = y^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, a *function* $f(x, y) = x^2y$, and a *point* $p = (2, -1)$. These may be combined in various ways, for instance:

Vector field on $\mathbb{R}^2$	$fv = x^2y\left(y^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = x^2y^3\frac{\partial}{\partial x} - x^3y\frac{\partial}{\partial y}$
Tangent vector	$(fv)(p) = f(p)v_p = -4\left.\frac{\partial}{\partial x}\right\|_p + 8\left.\frac{\partial}{\partial y}\right\|_p \in T_p\mathbb{R}^2$
Function $\mathbb{R}^2 \to \mathbb{R}$	$v[f] = y^2\frac{\partial}{\partial x}(x^2y) - x\frac{\partial}{\partial y}(x^2y) = 2xy^3 - x^3$
Number	$\big(v[f]\big)(p) = -4 - 8 = -12$

Note the use of different brackets! Note also that fv denotes the vector field obtained by *multiplying* v by the *value* of f at each point. It does *not* mean *apply the function* f *to the vector field* v, which makes no sense!

Here are the basic rules of computation for vector fields. These are all essentially trivial if you take $v = \sum v_k \frac{\partial}{\partial x_k}$, etc., as in Definition 2.4. Just be careful with notation!

Lemma 2.6. *Let v, w be vector fields on U, let $f, g : U \to \mathbb{R}$ be smooth, and $a, b \in \mathbb{R}$ constant. Then,*

1. *$fv + gw$ is a vector field: at each $p \in U$, $(fv + gw)(p) := f(p)v_p + g(p)w_p$*
2. *Vector fields act linearly on smooth functions: $v[af + bg] = av[f] + bv[g]$*
3. *(Leibniz rule) Vector fields obey a product rule: $v[fg] = fv[g] + gv[f]$*

Examples 2.7. 1. We verify the Leibniz rule for the vector field $v = \frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}$ and functions $f(x, y) = x$ and $g(x, y) = ye^x$.

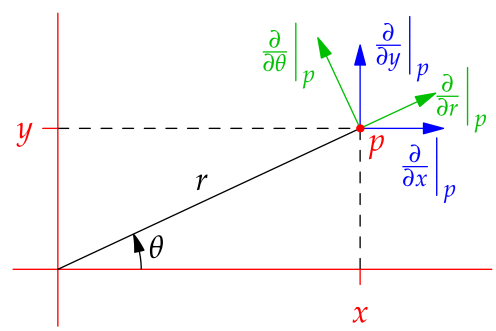
$$v[fg] = \left(\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\right)[xye^x] = ye^x + xye^x - x^2ye^x$$
$$fv[g] + gv[f] = x\left(\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\right)[ye^x] + ye^x\left(\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\right)[x] = x(ye^x - xye^x) + ye^x$$

2. (Polar co-ordinates) Let U be the plane without the non-positive x-axis. On U, the standard *rectangular co-ordinates* (x, y) are related to the *polar co-ordinates* (r, θ) via

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \leftrightsquigarrow \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\frac{y}{x} \quad (\text{or } \pm\frac{\pi}{2} \text{ if } x = 0) \end{cases}$$

The chain rule tells us that the co-ordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ are related via

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ &= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned}$$



At p, these point in the direction of maximal increase for the corresponding co-ordinate.

We could similarly compute $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by differentiating. For variety, we instead use linear algebra:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \implies \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$
$$\implies \begin{cases} \dfrac{\partial}{\partial x} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial y} = \sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta} \end{cases}$$

The first matrix is the familiar *Jacobian* $\frac{\partial(x,y)}{\partial(r,\theta)}$ from multivariable calculus. Strictly, we are viewing U as subsets of *two different versions of* $\mathbb{R}^2$:

- In rectangular co-ordinates, $U = \mathbb{R}^2 \setminus \{(x,0) : x \le 0\}$ is a cut plane.
- In polar co-ordinates, $U = (0,\infty) \times (-\pi,\pi)$ is an infinite open rectangle.

In practice, particularly since we are so familiar with polar co-ordinates, it is easier to stick to the first interpretation and draw all four co-ordinate tangent vectors on the same picture.

Exercises 2.1. 1. You are given the following vector fields and functions

$$u = 7\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} \qquad v = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \qquad w = \sin x\frac{\partial}{\partial x} - 2\cos x\frac{\partial}{\partial y}$$
$$f(x,y) = xy^2 \qquad g(x,y) = -y$$

Compute the *functions*:

(a) $u[f]$ (b) $v[f]$ (c) $w[f]$
(d) $v[fg]$ (e) $fu[g]$ (f) $v\big[w[g]\big]$

2. Revisit Example 2.7.2 on polar co-ordinates.

   (a) Use the chain rule to compute $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ directly in terms of r,θ, $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ and verify that you obtain the same expressions as the linear algebra approach.

   (b) Suppose $T_p\mathbb{R}^2$ is equipped with the standard dot product so that $\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$ are considered orthonormal.

      i. Show that $\frac{\partial}{\partial r}\Big|_p$ and $\frac{\partial}{\partial \theta}\Big|_p$ are perpendicular.

      ii. What are the lengths of $\frac{\partial}{\partial r}\Big|_p$ and $\frac{\partial}{\partial \theta}\Big|_p$?

3. Consider the spherical polar co-ordinate system

$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\sin\theta\cos\phi \\ z = r\sin\phi \end{cases} \qquad \text{where } r > 0,\ 0 < \theta < 2\pi \text{ and } -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

Show that

$$\frac{\partial}{\partial r} = \frac{1}{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

4. Prove the Leibniz rule (Lemma 2.6 part 3).

5. If f,g,h are smooth functions and v is a vector field, expand $v[fgh]$ using the Leibniz rule.

6. Let $s = x^2 - y^2$ and $t = 2xy$. Compute $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

   (*Hint: use the chain rule to find* $\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial y}$*, then invert the Jacobian*)

2.2 Differential 1-forms

Make sure you are comfortable with vector fields *before* you tackle this section and the next! There is a lot of new notation to get used to here, but with a little practice it is very easy to use.

Definition 2.8. Let $(x_1, \ldots, x_n)$ be co-ordinates on $U \subseteq \mathbb{R}^n$ and $p \in U$. The *(co-ordinate) 1-form* $\mathrm{d}x_k$ *at* p is the *linear map*[13] $\mathrm{d}x_k : T_p\mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathrm{d}x_k \left(\frac{\partial}{\partial x_j}\bigg|_p \right) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

A *1-form* $\alpha = \sum_{k=1}^{n} a_k \, \mathrm{d}x_k$ on U is a smooth assignment ($a_k : U \to \mathbb{R}$ *smooth*) of 1-forms.
If v is a vector field on U, we write $\alpha(v)$ for the *function* $U \to \mathbb{R}$ obtained by mapping $p \mapsto \alpha(v_p)$.

Examples 2.9. 1. Consider the vector field $v = xy\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. At each $p \in \mathbb{R}^2$, the components xy and -2 are *scalars* and thus ignored by the linear map $\mathrm{d}x : T_p\mathbb{R}^2 \to \mathbb{R}$. We therefore obtain a *function* $\mathrm{d}x(v) : \mathbb{R}^2 \to \mathbb{R}$

$$\mathrm{d}x(v) = \mathrm{d}x \left(xy\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} \right) = xy \, \mathrm{d}x \left(\frac{\partial}{\partial x} \right) - 2 \, \mathrm{d}x \left(\frac{\partial}{\partial y} \right) = xy$$

2. Again on $\mathbb{R}^2$, let $\alpha = 2x \, \mathrm{d}x + \mathrm{d}y$ and $v = x^2y\frac{\partial}{\partial x} - e^{xy}\frac{\partial}{\partial y}$. Then

$$\alpha(v) = (2x \, \mathrm{d}x + \mathrm{d}y) \left(x^2y\frac{\partial}{\partial x} - e^{xy}\frac{\partial}{\partial y} \right) = 2x^3y - e^{xy}$$

Remember that a 1-form α is linear *when restricted to each tangent space* $T_p\mathbb{R}^n$: if $v_p \in T_p\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$, we obtain a *real number*

$$\alpha_p(f(p)v_p) = f(p)\alpha_p(v_p) \in \mathbb{R}$$

by pointwise multiplication by the value of f. Taken over all points p, this means that scalar *functions* come straight through a 1-form: if v is a vector field on U, then

$$\alpha(fv) = f\alpha(v)$$

Definition 2.10. Let $f : U \to \mathbb{R}$ be *smooth*. The *exterior derivative* of f is the 1-form

$$\mathrm{d}f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \mathrm{d}x_k = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n$$

If a 1-form is the exterior derivative of a function, we say that it is *exact*.

[13] For those who've met dual vector spaces in linear algebra, the set of 1-forms at p is the *cotangent space* $T_p^*\mathbb{R}^n$, or the space of *covectors*. At each p, $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ is the *dual basis* to $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p \right\}$.

Our approach essentially splits a derivative into two pieces: for each k, we have $\mathrm{d}f\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial f}{\partial x_k}$. Moreover, since a linear map $(\mathrm{d}f_p : T_p\mathbb{R}^n \to \mathbb{R})$ is determined by what it does to a basis, the exterior derivative $\mathrm{d}f$ is the *unique* 1-form with the property that $\mathrm{d}f(v) = v[f]$ for all vector fields v on U. This says that the definition is *co-ordinate independent* (does not depend on $x_1, \ldots, x_n$).

Examples 2.11. 1. Let $f(x,y) = x^2y$, then $\mathrm{d}f = \alpha = 2xy\,\mathrm{d}x + x^2\mathrm{d}y$. As a sanity check, consider a general vector field $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ (remember that a, b are smooth functions!) and compute

$$\mathrm{d}f(v) = 2axy + bx^2 = v[x^2y]$$

2. If $\alpha = 4xy^2\,\mathrm{d}x + (4x^2y+1)\,\mathrm{d}y = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$ is exact, then 'partial integration' forces

$$f(x,y) = \int 4xy^2\,\mathrm{d}x = 2x^2y^2 + g(y) = \int 4x^2y + 1\,\mathrm{d}y = 2x^2y^2 + y + h(x)$$

for some functions g, h. Plainly g, h must be constant and $\alpha = \mathrm{d}(2x^2y^2 + y)$.

3. We could a similar game to see that $\alpha = 3x^2y\,\mathrm{d}x + 2\,\mathrm{d}y$ is *not* exact on $\mathbb{R}^2$. Alternatively, note that if $\alpha = \mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$, we obtain a contradiction by observing that the mixed partial derivative is simultaneously

$$3x^2 = \frac{\partial f_x}{\partial y} = f_{xy} = f_{yx} = \frac{\partial f_y}{\partial x} = 0$$

See Exercise 6 for the general result.

Lemma 2.12. *If f, g are smooth functions, then*

1. $\mathrm{d}(f+g) = \mathrm{d}f + \mathrm{d}g$
2. $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$
3. $\mathrm{d}f = 0 \iff f$ *is a constant function*

Proof. These follow straight from the definition of $\mathrm{d}f$. For instance

$$\mathrm{d}f = 0 \iff \frac{\partial f}{\partial x_j} = \mathrm{d}f\left(\frac{\partial}{\partial x_j}\right) = 0 \text{ for all } j = 1, \ldots, n \iff f \text{ is constant}$$

∎

Example (2.7.2 cont). The exterior derivative and part 2 of the Lemma make it easy to compute the relationship between the 1-forms $\mathrm{d}x, \mathrm{d}y, \mathrm{d}r, \mathrm{d}\theta$:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} \mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta \end{cases} \implies \begin{cases} \mathrm{d}r = \frac{1}{r}(x\,\mathrm{d}x + y\,\mathrm{d}y) \\ \mathrm{d}\theta = \frac{1}{r^2}(-y\,\mathrm{d}x + x\,\mathrm{d}y) \end{cases}$$

We may also verify directly that the dual basis relations hold; for instance,

$$\begin{aligned}\mathrm{d}r\left(\frac{\partial}{\partial r}\right) &= \frac{1}{r}(x\,\mathrm{d}x + y\,\mathrm{d}y)\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) = \frac{1}{r}(x\cos\theta + y\sin\theta) \\ &= \cos^2\theta + \sin^2\theta = 1\end{aligned}$$

Elementary Calculus & Line Integrals

It is worth reviewing some staples from basic calculus in our new language.

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then its exterior derivative $\mathrm{d}f = f'(x)\,\mathrm{d}x$ feels familiar.[14] To make sense of this as a relation between 1-forms we need *vector fields*: the derivative of f isn't the ratio of two 1-forms, rather it is the application of the 1-form $\mathrm{d}f$ to the vector field $\frac{\mathrm{d}}{\mathrm{d}x}$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[f] = \mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

Vector fields in $\mathbb{R}$ are written with a straight d rather than partial ∂ since there is only one direction in which to differentiate!

You've seen 1-forms before when integrating: we integrate 1-forms over oriented curves.

Definition 2.13. Let α be a 1-form on $U \subseteq \mathbb{R}^n$ and suppose $\mathbf{x} : [a,b] \to U$ parametrizes a smooth curve C. Our usual identification (Definition 2.4) produces the *tangent vector field*

$$\mathbf{x}'(t) = x_1'(t)\frac{\partial}{\partial x_1} + \cdots + x_n'(t)\frac{\partial}{\partial x_n}$$

along the curve. Now define the integral of α along C by

$$\int_C \alpha := \int_a^b \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t = \int_a^b \alpha\left(x_1'(t)\frac{\partial}{\partial x_1} + \cdots + x_n'(t)\frac{\partial}{\partial x_n}\right)\mathrm{d}t$$

Examples 2.14. 1. We integrate $\alpha = x\,\mathrm{d}y$ over the unit-circle $\mathbf{x}(t) = (\cos t, \sin t)$ counter-clockwise. Differentiate to obtain the tangent vector field $\mathbf{x}'(t) = -\sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y}$, then

$$\int_C \alpha = \int_0^{2\pi} \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t \int_0^{2\pi} \cos^2 t\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} 1 + \cos 2t\,\mathrm{d}t = \pi$$

2. Integrate $\alpha = y^2\,\mathrm{d}x - x^2\,\mathrm{d}y$ over the curve $\mathbf{x}(t) = (t, t^2)$ between $(0,0)$ and $(1,1)$:

$$\begin{aligned}\int_C \alpha &= \int_0^1 \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t = \int_0^1 \alpha\left(\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\right)\mathrm{d}t = \int_0^1 \Big(\big(y(t)\big)^2 - 2t\big(x(t)\big)^2\Big)\,\mathrm{d}t \\ &= \int_0^1 t^4 - 2t^3\,\mathrm{d}t = \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}\end{aligned}$$

Lemma 2.15. *The integral of a 1-form along a curve is independent of the choice of (orientation-preserving) parametrization.*

Otherwise said, if $\mathbf{x}(t) = \mathbf{y}\big(s(t)\big)$ parametrizes the same curve where $s'(t) > 0$, then

$$\int_a^b \alpha\big(\mathbf{x}'(t)\big)\,\mathrm{d}t = \int_{s(a)}^{s(b)} \alpha\big(\mathbf{y}'(s)\big)\,\mathrm{d}s$$

The proof is an easy exercise in interpreting old material (the chain rule/substitution).

[14] Consider the equivalence of notations $\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x)$, linear approximations (differentials) & integration by substitution.

Our final result from elementary calculus shows that integrals of exact forms are independent of path. This is essentially the fundamental theorem of calculus for curves.

Theorem 2.16 (Fundamental Theorem of Line Integrals). *If f is a function on $U \subseteq \mathbb{R}^2$ and C is a curve in U, then the integral of $\mathrm{d}f$ depends only on the values of f at the endpoints of C:*

$$\int_C \mathrm{d}f = f(\text{end of } C) - f(\text{start of } C)$$

The converse also holds: if $\int_C \alpha$ is independent of path, then α is exact.

Proof. Suppose $\mathbf{x} : [a,b] \to U$ parametrizes C, then

$$\begin{aligned}\int_C \mathrm{d}f &= \int_a^b \mathrm{d}f(\mathbf{x}')\,\mathrm{d}t = \int_a^b \mathbf{x}'[f]\,\mathrm{d}t = \int_a^b \left(x_1'(t)\frac{\partial f}{\partial x_1} + \cdots + x_n'(t)\frac{\partial f}{\partial x_n} \right) \mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\big(f(\mathbf{x}(t))\big)\,\mathrm{d}t = f\big(\mathbf{x}(b)\big) - f\big(\mathbf{x}(a)\big)\end{aligned}$$

The converse is sketched in an exercise. ∎

In elementary multivariable calculus this result was written $\int_C \nabla f \cdot \mathrm{d}\mathbf{x} = f\big(\mathbf{x}(b)\big) - f\big(\mathbf{x}(b)\big)$ which comports with our new notation when we view $\mathrm{d}\mathbf{x}$ as a vector of 1-forms:

$$\nabla f \cdot \mathrm{d}\mathbf{x} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix} = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n = \mathrm{d}f$$

The exterior derivative $\mathrm{d}f$ is just the gradient in disguise!

Example 2.17. If $\alpha = \cos(xy)(y\,\mathrm{d}x + x\,\mathrm{d}y)$, find the integral of α over any curve C joining the points $\left(\pi, \frac{1}{3}\right)$ and $\left(\frac{1}{2}, \pi\right)$. Since $\alpha = \mathrm{d}\sin(xy)$ is exact on $\mathbb{R}^2$, we see that

$$\int_C \alpha = \sin(xy)\Big|_{\left(\pi,\frac{1}{3}\right)}^{\left(\frac{1}{2},\pi\right)} = \sin\frac{\pi}{2} - \sin\frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2}$$

Summary

- Tangent vectors & vector fields encode *directional derivatives*, measuring how functions change in given directions.
- Vector fields and 1-forms break standard derivatives into two pieces: the result is a more flexible and extensible language for describing familiar results from multi-variable calculus.

The real pay-off comes once our new language is applied to surfaces and higher-dimensional objects. Here is a précis. A parametrized surface is a function $\mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{E}^3$; its exterior derivative $\mathrm{d}\mathbf{x}$ is a vector-valued 1-form which, at each point $p \in U$, describes a *linear map* between tangent spaces

$$\mathrm{d}\mathbf{x}_p : T_p\mathbb{R}^2 \to T_{\mathbf{x}(p)}\mathbb{E}^3$$

which maps the co-ordinate fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ on U to corresponding vector fields *tangent to the surface.*

Exercises 2.2. 1. In $\mathbb{R}^2$, let $\alpha = 2y\,\mathrm{d}x - 3\,\mathrm{d}y$ and $v = 3x^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Compute $\alpha(v)$, and $v\big[\alpha(v)\big]$.

2. On $\mathbb{R}^3$, suppose $f(x,y,z) = x^2\cos(yz)$ and $v = e^x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}$. Verify that $\mathrm{d}f(v) = v[f]$.

3. Find $\mathrm{d}r$ directly by taking the exterior derivative of the equation $r^2 = x^2 + y^2$.

4. Prove parts 1 and 2 of Lemma 2.12.

5. Continuing Example 2.7.2, verify that $\mathrm{d}\theta\left(\frac{\partial}{\partial\theta}\right) = 1$, and $\mathrm{d}r\left(\frac{\partial}{\partial\theta}\right) = 0 = \mathrm{d}\theta\left(\frac{\partial}{\partial r}\right)$.

6. Suppose that $\alpha = \sum a_k\,\mathrm{d}x_k$ is exact. Prove that $\frac{\partial a_k}{\partial x_j} = \frac{\partial a_j}{\partial x_k}$ for all j,k.

7. Decide whether the 1-forms α are exact on $\mathbb{R}^2$. If yes, find a function f such that $\alpha = \mathrm{d}f$.
   (a) $\alpha = 2x\,\mathrm{d}x + \mathrm{d}y$
   (b) $\alpha = \mathrm{d}x + 2x\,\mathrm{d}y$
   (c) $\alpha = \cos(x^2y)(2y\,\mathrm{d}x + x\,\mathrm{d}y)$
   (d) $\alpha = x\cos(x^2y)(2y\,\mathrm{d}x + x\,\mathrm{d}y)$

8. Let $\alpha = \frac{1}{x^2+y^2}(-y\,\mathrm{d}x + x\,\mathrm{d}y) = a\,\mathrm{d}x + b\,\mathrm{d}y$ be defined on the *punctured plane* $\mathbb{R}^2 \setminus \{(0,0)\}$.
   Show that $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$ but that α is *not exact*: the converse to Exercise 6 is *false!*[a]
   (*Hint: $\alpha = \mathrm{d}\theta$ except on the non-positive real axis; why is this a problem?*)

9. Evaluate the integral $\int_C \alpha$ given C and α.
   (a) $\alpha = \mathrm{d}x - x^{-1}\,\mathrm{d}y$, where C is parametrized by $\mathbf{x}(t) = (t^2, t^3)$, $0 \le t \le 1$.
   (b) $\alpha = 2x\tan^{-1}y\,\mathrm{d}x + \frac{x^2}{1+y^2}\,\mathrm{d}y$, where C is parametrized by $\mathbf{x}(t) = \left(\frac{1}{t+1}, 1\right)$, $0 \le t \le 2$.
   (c) $\alpha = \cos x\,\mathrm{d}x + \mathrm{d}y$, with C the graph of $y = \cos x$ over one period of the curve.

10. Which of the integrals in the previous question are path-independent?

11. Prove Lemma 2.15. Moreover, show that if we reverse the orientation of the curve ($s'(t) < 0$) then the order of the limits is reversed and $\int \alpha$ becomes $-\int \alpha$.

12. Suppose $s(x,y), t(x,y)$ are co-ordinates on $U \subseteq \mathbb{R}^2$. Find the relationship between the 2×2 matrix-valued functions J, K which satisfy
$$\begin{pmatrix}\mathrm{d}s\\ \mathrm{d}t\end{pmatrix} = J\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\frac{\partial}{\partial s}\\ \frac{\partial}{\partial t}\end{pmatrix} = K\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}$$

13. Let $p \in U \subseteq \mathbb{R}^2$ and let $\alpha = a\,\mathrm{d}x + b\,\mathrm{d}y$ be a 1-form on U. For each q define $f(q) := \int_C \alpha$ where we additionally assume this value is *independent of the path* C joining p to q.
   Let h be small and C_h the straight line from q to $q + h\mathbf{i}$. Integrate over C_h to show that
$$\left.\frac{\partial f}{\partial x}\right|_q = \lim_{h\to 0}\frac{f(q+h\mathbf{i}) - f(q)}{h} = a(q)$$
   Make a similar argument to conclude that $\alpha = \mathrm{d}f$ is exact.

14. (If you've done complex analysis) Let $f(x,y) = u(x,y) + iv(x,y)$ be a complex-valued function $f : \mathbb{R}^2 \to \mathbb{C}$ where u, v are real-valued. Viewing $z = x + iy$ and $\overline{z} = x - iy$ as co-ordinates on $\mathbb{R}^2$, prove that $\mathrm{d}f\left(\frac{\partial}{\partial\overline{z}}\right) = 0$ if and only if u, v satisfy the *Cauchy-Riemann equations*:
$$u_x = v_y, \quad v_x = -u_y$$

[a] Exercise 6 can be shown to be equivalent to the exactness of α provided the domain U is *simply-connected*: has no holes.

2.3 Higher-degree Forms

We introduce a new operation on forms which generalizes the cross product of vectors.

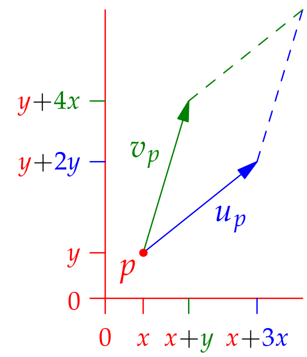
Definition 2.18. Given 1-forms α, β on U, their *wedge product* $\alpha \wedge \beta$ is the function which takes two vector fields and returns the *smooth function*

$$\alpha \wedge \beta(u, v) = \det \begin{pmatrix} \alpha(u) & \alpha(v) \\ \beta(u) & \beta(v) \end{pmatrix} : U \to \mathbb{R}$$

We call $\alpha \wedge \beta$ a *2-form.*

Example 2.19. Let x, y be the usual co-ordinates on $\mathbb{R}^2$. The *standard area form* is the object $\mathrm{d}x \wedge \mathrm{d}y$ which takes two vector fields $u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y}$ and $v = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$ and returns the determinant

$$\mathrm{d}x \wedge \mathrm{d}y(u, v) = \begin{vmatrix} \mathrm{d}x(u) & \mathrm{d}x(v) \\ \mathrm{d}y(u) & \mathrm{d}y(v) \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$



This gets its name since, at each point p, it returns the (signed) area of the parallelogram spanned by the tangent vectors u_p, v_p.

For instance, if $u = 3x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$ and $v = y \frac{\partial}{\partial x} + 4x \frac{\partial}{\partial y}$, then

$$\mathrm{d}x \wedge \mathrm{d}y(u, v) = \begin{vmatrix} 3x & y \\ 2y & 4x \end{vmatrix} = 12x^2 - 2y^2$$

Recall that determinants change sign if you switch its rows or columns, and that they are linear functions of both their rows and columns. This has two consequences for $\alpha \wedge \beta$.

Lemma 2.20. 1. *(Columns) At each $p \in U$, a wedge product of 1-forms is an* alternating, bilinear *function $\alpha \wedge \beta : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$: given vector fields u, v, w and functions $f, g : U \to \mathbb{R}$,*

$$\alpha \wedge \beta(v, u) = -\alpha \wedge \beta(u, v) \qquad \text{(alternating)}$$

$$\alpha \wedge \beta(fu + gv, w) = f\, \alpha \wedge \beta(u, w) + g\, \alpha \wedge \beta(v, w) \qquad \text{(linear in } 1^{\text{st}} \text{ slot)}$$

2. *(Rows) Wedge products are alternating and addition distributes over $\wedge$*

$$\beta \wedge \alpha = -\alpha \wedge \beta \quad \text{and} \quad \alpha \wedge \alpha = 0 \qquad \text{(alternating)}$$

$$(\alpha + \gamma) \wedge \beta = \alpha \wedge \beta + \gamma \wedge \beta \qquad \text{(distributivity in } 1^{\text{st}} \text{ slot)}$$

Linearity/distributivity in the second slot is similar in both cases.

The linearity and alternating properties tell us that every wedge product of 1-forms on $\mathbb{R}^2$ may be written

$$\alpha \wedge \beta = (a_1\, \mathrm{d}x + a_2\, \mathrm{d}y) \wedge (b_1\, \mathrm{d}x + b_2\, \mathrm{d}y) = (a_1 b_2 - a_2 b_1)\, \mathrm{d}x \wedge \mathrm{d}y$$

Notice the determinant again!

For higher order forms, we extend the same approach.

Definition 2.21. The *wedge product* of 1-forms $\alpha_1, \ldots, \alpha_k$ on $U \subseteq \mathbb{R}^n$ takes k *vector fields* and returns a *smooth function*:

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, \ldots, v_k) = \begin{vmatrix} \alpha_1(v_1) & \cdots & \alpha_1(v_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \cdots & \alpha_k(v_k) \end{vmatrix} : U \to \mathbb{R}$$

Let $x_1, \ldots, x_n$ be co-ordinates on U. A *k-form* on U (alternating form of degree k) is an expression

$$\alpha = \sum a_I \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}, \quad a_I : U \to \mathbb{R} \text{ smooth}$$

where we sum over all *increasing multi-indices* $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, 2, \ldots, n\}$ of length k. The *wedge product* of a k-form α and an l-form β is the $(k+l)$-form

$$\alpha \wedge \beta = \sum_{I,J} a_I b_J \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_l}$$

where the 1-forms $\mathrm{d}x$ may be rearranged/cancelled using the alternating property (Lemma 2.20.2).

By convention, a 0-form is a smooth function $f : U \to \mathbb{R}$, whose wedge product with anything is pointwise multiplication. At each point $p \in U$, the k-forms comprise the vector space of alternating multilinear maps with basis $\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : i_1 < \cdots < i_k\}$ and dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
In this course we'll never have reason to work in more then three dimensions!

The table describes all k-forms in 2 and 3 dimensions written in standard co-ordinates.

Analogous to Example 2.19, $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the *standard volume form* on $\mathbb{R}^3$.

k	$\mathbb{R}^2$	$\mathbb{R}^3$
0	function f	f
1	$f\,\mathrm{d}x + g\,\mathrm{d}y$	$f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$
2	$f\,\mathrm{d}x \wedge \mathrm{d}y$	$f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}x \wedge \mathrm{d}z + h\,\mathrm{d}y \wedge \mathrm{d}z$
3	None	$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
4+	None	None

Examples 2.22. 1. Given 1-forms $\alpha = 2\,\mathrm{d}x - 3x\,\mathrm{d}y$ and $\beta = y^2\,\mathrm{d}x + y\,\mathrm{d}y$ on $\mathbb{R}^2$,

$$\begin{aligned}\alpha \wedge \beta &= (2\,\mathrm{d}x - 3x\,\mathrm{d}y) \wedge \left(y^2\,\mathrm{d}x + y\,\mathrm{d}y\right) \\ &= 2y^2\,\mathrm{d}x \wedge \mathrm{d}x + 2y\,\mathrm{d}x \wedge \mathrm{d}y - 3xy^2\,\mathrm{d}y \wedge \mathrm{d}x - 3xy\,\mathrm{d}y \wedge \mathrm{d}y \\ &= \left(2y - 3xy^2\right) \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

2. Given the 1-forms $\alpha = \mathrm{d}x + 2\,\mathrm{d}y + x\,\mathrm{d}z$ and 2-form $\beta = 3z\,\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{R}^3$, the wedge product $\alpha \wedge \beta$ is the 3-form

$$\begin{aligned}\alpha \wedge \beta &= \mathrm{d}x \wedge (-\mathrm{d}y \wedge \mathrm{d}z) + 3xz\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= (3xz - 1)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\end{aligned}$$

Note how $\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ requires *two* swaps, so the sign is ultimately unchanged!

Lemma 2.23. *For any forms α, β,*

$$\beta \wedge \alpha = (-1)^{\deg \alpha \deg \beta} \alpha \wedge \beta$$

where $\deg \alpha = k$ means that α is a k-form.

This is true by definition when α, β are 1-forms, and trivially true when α is a 0-form. Check the previous examples to make sure they agree.

Example 2.24 (Polar co-ordinates). Changing to polar co-ordinates, the standard area form on $\mathbb{R}^2$ becomes

$$\mathrm{d}x \wedge \mathrm{d}y = (\cos\theta \,\mathrm{d}r - r\sin\theta \,\mathrm{d}\theta) \wedge (\sin\theta \,\mathrm{d}r + r\cos\theta \,\mathrm{d}\theta) = r \,\mathrm{d}r \wedge \mathrm{d}\theta$$

This should remind you of change of variables in integration: if $f(x,y) = g(r,\theta)$, then

$$\int f(x,y)\mathrm{d}x\mathrm{d}y = \int g(r,\theta) r \,\mathrm{d}r\mathrm{d}\theta$$

The example illustrates one of the advantages of forms: change of variables (Jacobians) are built in!

The Exterior Derivative Just as with functions, we can apply 'd' to forms.

Definition 2.25. The *exterior derivative* of a k-form $\alpha = \sum a_I \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ is the $(k+1)$-form

$$\mathrm{d}\alpha = \sum \mathrm{d}a_I \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

where $\mathrm{d}a_I = \sum_j \frac{\partial a}{\partial x_j} \,\mathrm{d}x_j$ is the usual exterior derivative of a function (Definition 2.10).

Example 2.26. In $\mathbb{R}^3$, let $\alpha = xy^2z \,\mathrm{d}x - xz \,\mathrm{d}z$. Then

$$\begin{aligned}
\mathrm{d}\alpha &= \mathrm{d}(xy^2z) \wedge \mathrm{d}x - \mathrm{d}(xy) \wedge \mathrm{d}z \\
&= ({\color{blue}y^2z \,\mathrm{d}x} + 2xyz \,\mathrm{d}y + xy^2 \,\mathrm{d}z) \wedge \mathrm{d}x - (z \,\mathrm{d}x + {\color{blue}x \,\mathrm{d}z}) \wedge \mathrm{d}z \\
&= -2xyz \,\mathrm{d}x \wedge \mathrm{d}y - (xy^2 + z) \,\mathrm{d}x \wedge \mathrm{d}z
\end{aligned}$$

Since $\mathrm{d}x \wedge \mathrm{d}x = 0 = \mathrm{d}z \wedge \mathrm{d}z$, there was no need to write the blue terms.

Theorem 2.27. *Let α, β be forms:*

1. $\mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$ *(α, β must have the same degree)*
2. $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$
3. $\mathrm{d}(\mathrm{d}\alpha) = 0$. *This is often written*[15] $\mathrm{d}^2\alpha = 0$*, or just* $\mathrm{d}^2 = 0$.

[15] A k-form α is *closed* if $\mathrm{d}\alpha = 0$, and *exact* if $\exists \beta$ such that $\alpha = \mathrm{d}\beta$. The result says that every exact form is closed.

Example (2.26 cont). We verify that $d^2\alpha = 0$:

$$\begin{aligned} d(d\alpha) &= d(-2xyz) \wedge dx \wedge dy - d(xy^2 + z) \wedge dx \wedge dz \\ &= -2xy\, dz \wedge dx \wedge dy - 2xy\, dy \wedge dx \wedge dz = 0 \end{aligned}$$

Proof. This is very easy to prove explicitly for the only forms we'll ever see (up to 3-forms in $\mathbb{R}^3$). Here are general arguments that work in any dimension.

For simplicity of notation, write $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, whenever $I = \{i_1 < \cdots < i_k\}$. Then

$$d(\alpha + \beta) = \sum_I da_I \wedge dx_I + db_I \wedge dx_I = \sum_I (da_I + db_I) \wedge dx_I = d\alpha + d\beta$$

Part 2 is an exercise. For part 3, we extend Exercise 2.2.6 which in fact shows that $d^2 f = 0$ for any function (0-form)

$$\begin{aligned} d(d\alpha) &= d \sum_I da_I \wedge dx_I = d \sum_{j \notin I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I = \sum_{i,j \notin I} \frac{\partial^2 a_I}{\partial x_i x_j} dx_i \wedge dx_j \wedge dx_I \\ &= \sum_{i<j \notin I} \left[\frac{\partial^2 a_I}{\partial x_i x_j} - \frac{\partial^2 a_I}{\partial x_j x_i} \right] dx_i \wedge dx_j \wedge dx_I = 0 \end{aligned}$$

since mixed partial derivatives commute. ∎

A New Take on Vector Calculus

The standard vector calculus operations of div, grad and curl in $\mathbb{E}^3$ are closely related to the exterior derivative. For instance, compare the curl of a vector field $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ with the exterior derivative of the 1-form $\alpha = a_1\, dx + a_2\, dy + a_3\, dz$:

$$\nabla \times \mathbf{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) \mathbf{k}$$

$$d\alpha = \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) dx \wedge dy$$

Comparing coefficients gives part of the dictionary for comparing forms and traditional vector fields.

$$\begin{tikzcd}
\text{function } f \arrow[d, "d"] \arrow[r, leftrightarrow] & \text{function } f \arrow[d, "\text{grad}"', "\nabla"] \\
a_1\, dx + a_2\, dy + a_3\, dz \arrow[d, "d"] \arrow[r, leftrightarrow] & a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \arrow[d, "\text{curl}"', "\nabla\times"] \\
b_1\, dy \wedge dz + b_2\, dz \wedge dx + b_3\, dx \wedge dy \arrow[d, "d"] \arrow[r, leftrightarrow] & b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \arrow[d, "\text{div}"', "\nabla\cdot"] \\
c\, dx \wedge dy \wedge dz \arrow[r, leftrightarrow] & \text{function } c
\end{tikzcd}$$

The exterior derivative d is div, grad and curl all in one tidy package! Moreover:

- The identity $\mathrm{d}^2 = 0$ translates to two familiar results from vector calculus:

$$\nabla \times (\nabla f) = \mathbf{0} \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{v}) = 0$$

- Under the above identification, the wedge product of 1-forms corresponds to the cross product, and the wedge product of a 1-form and a 2-form to the dot product. Various identities may be obtained this way: for instance, if α is a 1-form, then

$$\mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha \quad \longleftrightarrow \quad \nabla \times f\mathbf{v} = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v}$$

- Changes of co-ordinates are built into forms (e.g. Example 2.24).
- The exterior derivative and wedge product apply in any dimension, thus extending standard vector calculus and the cross product to arbitrary dimensions.

None of what we've done in this chapter is strictly necessary for the analysis of surfaces in $\mathbb{E}^3$. However, forms are the language of modern differential geometry (and other things besides) and it is easier to meet them first in a familiar setting. And if you want to do higher-dimensional geometry (e.g., general relativity), this new language becomes almost essential.

Exercises 2.3. 1. Compute $\alpha(u, v)$, given $\alpha = \mathrm{d}x \wedge \mathrm{d}y + z\,\mathrm{d}y \wedge \mathrm{d}z$, $u = \frac{\partial}{\partial x} - \frac{\partial}{\partial z}$ and $v = y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

2. Let $\alpha = y^2\,\mathrm{d}x \wedge \mathrm{d}z - \mathrm{d}y \wedge \mathrm{d}z$ and $u = x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$ and $v = -y\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y}$.
   (a) Compute $\alpha(u, v)$.
   (b) Find the 3-form $\mathrm{d}\alpha$.
3. Given $s = x^2 - y^2$ and $t = 2xy$, compute $\mathrm{d}s \wedge \mathrm{d}t$ in terms of $\mathrm{d}x \wedge \mathrm{d}y$
4. Revisit Lemma 2.20. State what it means for a wedge product of 1-forms $\alpha \wedge \beta$ to be linear in the second slot.
5. Let f, g be functions and consider the 1-form $\alpha = g\,\mathrm{d}f$. Show that $\alpha \wedge \mathrm{d}\alpha = 0$. Can the 1-form $\mathrm{d}x + y\,\mathrm{d}z$ be written in the form $g\,\mathrm{d}f$?
6. (a) Check the claim that the wedge product of 1-forms on $\mathbb{R}^3$ corresponds to the cross product.
   (b) Suppose α is a 2-form on $\mathbb{R}^3$. To what vector calculus identity does $\mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha$ correspond?
   (c) State an expression using forms, d and $\wedge$ which corresponds to the vector calculus identity

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

7. Let r, θ, ϕ be the spherical polar co-ordinate system in Exercise 2.1.3. Show that

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \cos\phi\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$$

8. A 2-form is *decomposable* if it can be written as a wedge product $\alpha \wedge \beta$ for some 1-forms α, β.

   (a) Show that every 2-form on $\mathbb{R}^3$ is decomposable.

   (b) If w, x, y, z are co-ordinates on $\mathbb{R}^4$, show that the 2-form $\mathrm{d}w \wedge \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}z$ is *not* decomposable.
   (*Hint: if a 2-form γ is decomposable, what is $\gamma \wedge \gamma$?*)

9. (Hard) Suppose α, β are forms, sketch an argument for why

$$\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha$$

   Now prove that

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$$

10. (Hard) Given vector fields u, v, their *Lie bracket* $[u, v]$ is the vector field such that

$$[u, v][f] := u\big[v[f]\big] - v\big[u[f]\big]$$

   for all functions f.

   (a) Compute $[u, v][f]$ where $u = 3x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $v = \frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $f(x, y) = x^2 y$.

   (b) If $u = \sum u_j \frac{\partial}{\partial x_j}$ and $v = \sum v_k \frac{\partial}{\partial x_k}$, show that $[u, v]$ really is a vector field by explicitly computing $[u, v][f]$ in the form $\sum c_j \frac{\partial f}{\partial x_j}$: how do the coefficients c_j of the vector field $[u, v]$ depend on those of u, v? Find the *field* $[u, v]$ when u, v are as in part (a).

   (c) If α is a 1-form and u, v are vector fields, prove that

$$\mathrm{d}\alpha(u, v) = u\big[\alpha(v)\big] - v\big[\alpha(u)\big] - \alpha\big([u, v]\big)$$

   This provides a co-ordinate-free definition of $\mathrm{d}\alpha$*; similar expressions exist for k-forms*
   (*Hint: Write everything out as sums over j, k so that all differentiations of scalars are with respect to the single variable x_k; now compare!*)