Math 162A - Introduction to Differential Geometry

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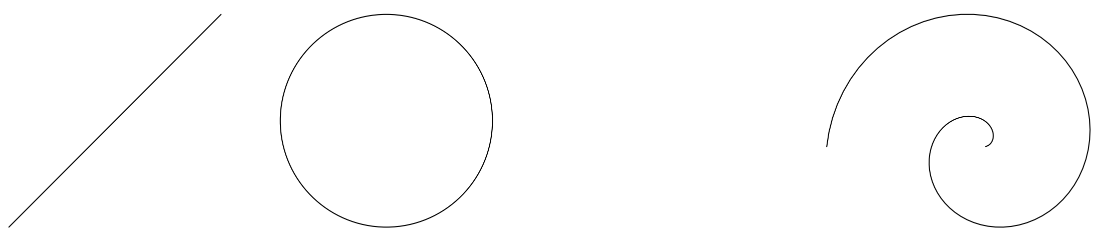
Introduction

Classical Differential Geometry is the study of curves and surfaces in the plane and three-dimensional space using *multi-variable calculus, linear algebra* & *differential equations.* At a more advanced level, topology, analysis and abstract algebra become more important, but none of this is required for our treatment.

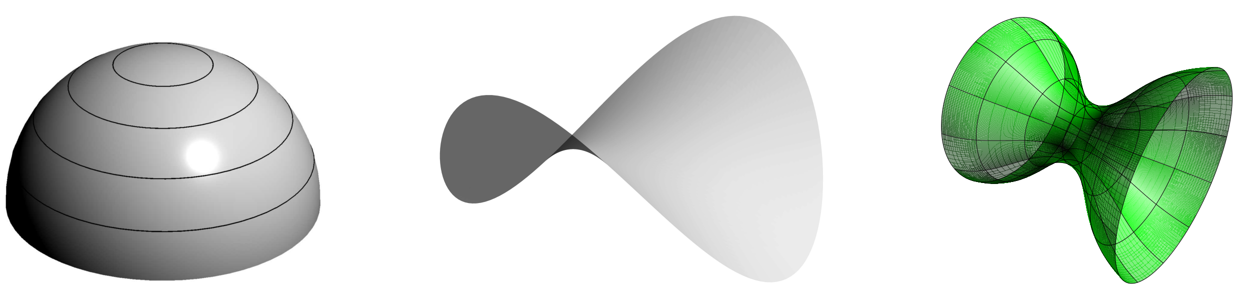
Of particular interest is the notion of *curvature*: a measure of the 'bendiness' of a curve or surface. Intuitively, a straight line should have zero curvature, while the curvature of a circle should vary inversely as the radius: a very large circle should have very small curvature.



Zero curvature Small curvature Larger curvature Variable curvature

Understanding and quantifying this concept for more complicated curves is our first important goal. The rough idea is to imagine a curve as a roller-coaster along which you travel at a constant speed; the curvature is then the *force* necessary to keep you travelling along the curve.

Curvature is a more difficult concept for surfaces. In particular, we will hunt for quantities which measure how much a surface appears to be dome- or saddle-shaped.



Dome-shaped Saddle-shaped More complicated

The third surface is saddle-shaped near the narrow neck and dome-shaped away from it.

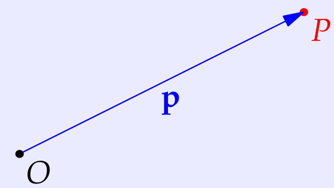
1 Curves in Euclidean Space

1.1 Euclidean Space, Tangent Vectors & Regular Curves

We begin by refreshing and developing a little notation.

Definition 1.1. The set of *n-tuples* of real numbers is denoted $\mathbb{R}^n$.
An element can be thought of either as a *point* P or as its *position vector* $\mathbf{p} = \overrightarrow{OP}$ connecting the origin $O = (0, \ldots, 0)$ to P.
In co-ordinates, points are typically written as row vectors

$$P = (p_1, \ldots, p_n) \text{ where each } p_i \in \mathbb{R}$$



P has position vector $\mathbf{p}$

For vectors, either row or column vector notation is acceptable.
For each i, the *co-ordinate function* $x_i : \mathbb{R}^n \to \mathbb{R}$ returns the i^{th} co-ordinate of a point: $x_i(P) = p_i$.

Since the focus of the course is curves and surfaces in 2- and 3-dimensions, we'll mostly restrict to $n \leq 3$ and quote theorems in this context.[1] We typically use x, y, z for the standard (rectangular) co-ordinate functions

$$x(P) = p_1, \quad y(P) = p_2, \quad z(P) = p_3$$

You should be comfortable with this notation from previous classes and, in particular, with *partial derivatives* of functions defined in terms of the co-ordinate functions x, y, z.

Examples 1.2. 1. If $P = (3, 1, 5) \in \mathbb{R}^3$, then $y(P) = 1$.

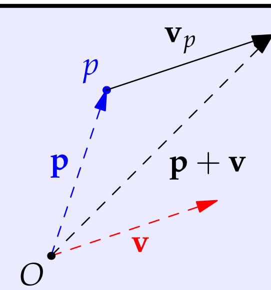
2. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f = x^3 \sin(yz)$ has partial derivatives

$$\frac{\partial f}{\partial x} = 3x^2 \sin(yz) \qquad \frac{\partial f}{\partial y} = x^3 z \cos(yz) \qquad \frac{\partial f}{\partial z} = x^3 y \cos(yz)$$

A *vector* is a directed line segment joining two *points*. We've already seen the *position vector* of a point P, namely $\overrightarrow{OP}$. In differential geometry it is crucial to distinguish the vectors *based* at a given point.

Definition 1.3. A *tangent vector* $\mathbf{v}_p$ is a *pair* of elements of $\mathbb{R}^3$: a *base point* p and a *direction* $\mathbf{v}$. It is the directed line segment from the point with position vector $\mathbf{p}$ to the point with position vector $\mathbf{p} + \mathbf{v}$.

The *tangent space at* p is the set $T_p\mathbb{R}^3$ of all tangent vectors based at p. At each point, $\mathbb{R}^3$ has a *different tangent space*!

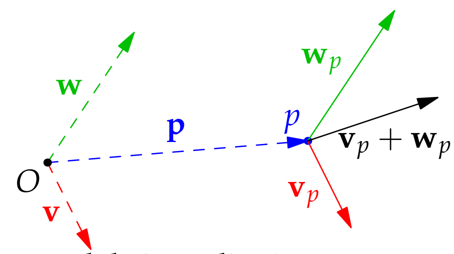


Be aware that $\mathbf{v}_p = \mathbf{w}_q \iff p = q$ *and* $\mathbf{v} = \mathbf{w}$: the same *direction* at different *base points* means a different tangent vector!

[1] For simplicity's sake, we'll almost always state theorems in $\mathbb{R}^3$. The majority are valid in $\mathbb{R}^n$ with a simple notational modification $\{x, y, z\} \rightsquigarrow \{x_1, \ldots, x_n\}$. For $\mathbb{R}^2$ just delete $z = x_3$; many results even make sense in $\mathbb{R} = \mathbb{R}^1$!

The tangent space at p is suitably named, for it is indeed a *vector space*: to add tangent vectors $\mathbf{v}_p, \mathbf{w}_p \in T_p\mathbb{R}^3$, simply sum the *direction vectors*

$$\mathbf{v}_p + \mathbf{w}_p := (\mathbf{v} + \mathbf{w})_p \qquad (*)$$



Scalar multiplication is similar: $\lambda \mathbf{v}_p := (\lambda \mathbf{v})_p$.
We will return later to a more abstract discussion of tangent vectors and their application.

Euclidean Space: $\mathbb{E}^n$ versus $\mathbb{R}^n$

To describe curves and surfaces in differential geometry, we *parametrize* using functions.

Example 1.4. There are multiple ways to do this for a given curve: for instance

$$\mathbf{x} : (-\pi, \pi] \to \mathbb{R}^2 : t \mapsto \big(\cos t, \sin t\big) \quad \text{and} \quad \mathbf{y} : \mathbb{R} \to \mathbb{R}^2 : s \mapsto \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}\right)$$

both parametrize (most of) the unit circle in the plane ($\mathbf{y}$ ignores the point $(-1, 0)$).

Plainly the *codomain* $\mathbb{R}^2$ is where the geometric action is: in the above we have the same circle, and concepts such as *length* and *angle* can be measured. This extra structure motivates us to distinguish the codomain with new notation.

Definition 1.5. *Euclidean space* $\mathbb{E}^n$ is $\mathbb{R}^n$ equipped with the usual *dot product*. Specifically in $\mathbb{E}^3$:

The *dot product* of $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{p} \cdot \mathbf{q} = \mathbf{p}^T\mathbf{q} = (p_1\ p_2\ p_3)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = p_1q_1 + p_2q_2 + p_3q_3$

The *length* of $\mathbf{p}$ is $||\mathbf{p}|| = \sqrt{\mathbf{p} \cdot \mathbf{p}} = \sqrt{p_1^2 + p_2^2 + p_3^2}$

The *angle* θ between $\mathbf{p}$ and $\mathbf{q}$ satisfies $\cos\theta = \dfrac{\mathbf{p} \cdot \mathbf{q}}{||\mathbf{p}||\,||\mathbf{q}||}$

Vectors are *orthogonal*/*perpendicular* if $\mathbf{p} \cdot \mathbf{q} = 0$, equivalently $\theta = \frac{\pi}{2}$; we write $\mathbf{p} \perp \mathbf{q}$.

Curves in $\mathbb{E}^2$ and $\mathbb{E}^3$

This course is primarily concerned with *functions* $\mathbf{x} : U \subseteq \mathbb{R}^m \to \mathbb{E}^n$. In particular:

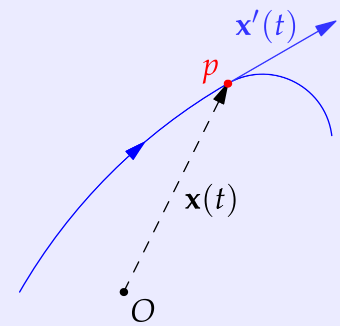
Plane curves: $m = 1$ and $n = 2$; for example the above circle.

Spacecurves: $m = 1$ and $n = 3$; we'll see several momentarily.

Surfaces: $m = 2$ and $n = 3$. For instance, the parametrization $\mathbf{x} : \mathbb{R}^2 \mapsto \mathbb{E}^3 : (u, v) \mapsto (u, v, u^2 + v^2)$ of a *paraboloid* should be familiar.

Surfaces are the focus of the second half of the course. It is now time for the formal definition of a curve.

Definition 1.6. A *(smooth parametrized) curve* is a function, $\mathbf{x} : I \to \mathbb{E}^3$, $\mathbf{x}(t) = \big(x(t), y(t), z(t)\big)$, defined on an interval I and whose components x, y, z are infinitely differentiable[2] everywhere on I. Its *derivative* is denoted



$$\mathbf{x}'(t) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \big(x'(t), y'(t), z'(t)\big)$$

The curve's *speed* is the continuous scalar function

$$v(t) = ||\mathbf{x}'(t)|| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

A curve is *regular* if its *tangent vector* $\mathbf{x}'(t)$ is everywhere non-zero.

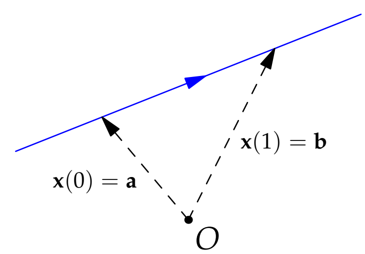
In the context of Definitions 1.1 and 1.3, note that for each $t \in I$:

$\mathbf{x}(t)$ is a *position vector* whose nose describes the location of a point on the curve.

$\mathbf{x}'(t) \in T_p\mathbb{E}^3$ is a *tangent vector* based at the point p with position vector $\mathbf{x}(t)$.

A parametrized curve has an *orientation* (indicated by the blue arrow): as t increases along the interval I, the point $\mathbf{x}(t)$ moves in a particular direction along the curve.

Examples 1.7. *Straight line*: The line through points with position vectors $\mathbf{a}, \mathbf{b}$ may be parametrized by



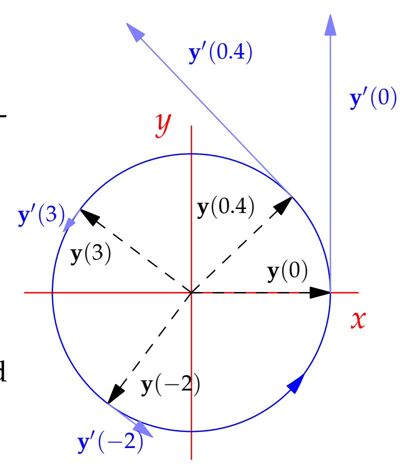
$$\mathbf{x}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1 - t)\mathbf{a} + t\mathbf{b}$$

The tangent vector at $\mathbf{x}(t)$ is the constant $\mathbf{x}'(t) = \mathbf{b} - \mathbf{a}$ and the parametrization has constant speed $||\mathbf{b} - \mathbf{a}||$. For instance,

$$\mathbf{x}(t) = \big(2 + t, 3 - 2t\big)$$

has constant velocity $\mathbf{x}'(t) = (1, -2)$ and speed $v(t) = \sqrt{5}$.

Circle (Example 1.4) The parametrization $\mathbf{x}(t) = \big(\cos t, \sin t\big)$ has velocity $\mathbf{x}'(t) = \big(-\sin t, \cos t\big)$ and constant speed $v(t) = 1$.



By contrast, $\mathbf{y}(s) = \frac{1}{1+s^2}\big(1 - s^2, 2s\big)$ has non-constant speed

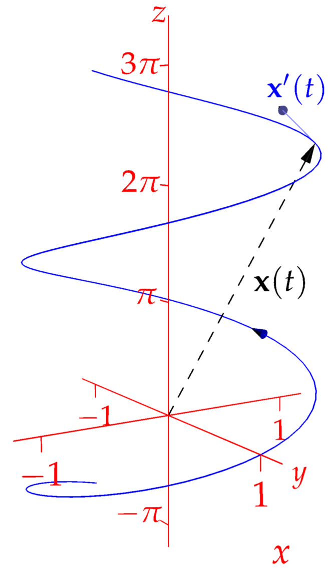
$$\mathbf{y}'(s) = \frac{2}{(1+s^2)^2}\big(-2s, 1 - s^2\big) \qquad v(s) = \frac{2}{1+s^2}$$

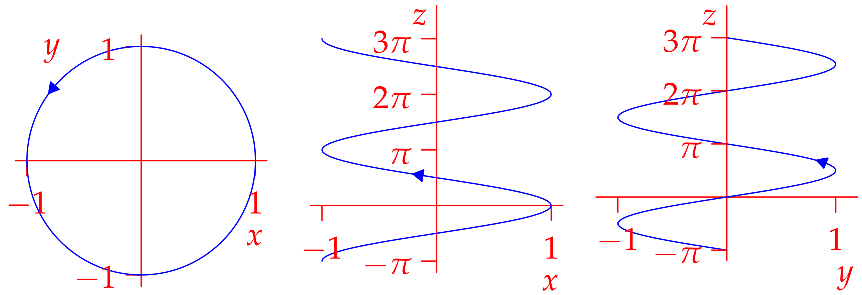
A particle moves quickest at $s = 0$ when $v(0) = 2$ and the speed tends to zero as $s \to \pm\infty$ (see the linked animation).

[2]The meaning of *smooth* depends on the author: at a minimum it means that x, y, z must be *differentiable with continuous derivative*. We take the maximal approach for simplicity.

Helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ parametrizes a *helix* (ascending spiral).

To help visualize this, imagine sitting on top of the z-axis and looking down; you'd see its horizontal projection $t \mapsto (\cos t, \sin t)$ (a counter-clockwise circle). Since $z(t) = t$, the curve moves upwards at constant speed. One can similarly project onto the xz- and yz-planes.





The tangent vector at $\mathbf{x}(t)$ is $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$ and the speed is constant $v(t) = \sqrt{2}$.

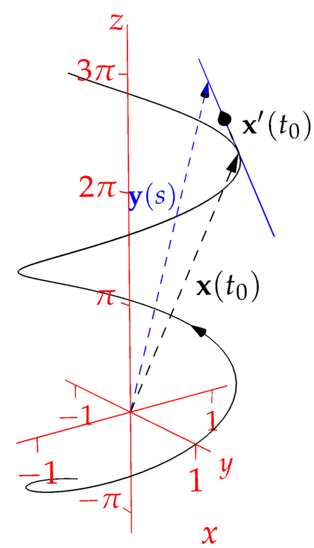
Tangent Line Let $\mathbf{x} : I \to \mathbb{E}^3$ be regular and $t_0 \in I$ be fixed. The *tangent line at* $\mathbf{x}(t_0)$ is simply the straight line through the point with position vector $\mathbf{x}(t_0)$ oriented in the direction of the tangent vector $\mathbf{x}'(t_0)$. It is itself a parametrized curve, $\mathbf{y} : \mathbb{R} \to \mathbb{E}^3$:

$$\mathbf{y}(s) = \mathbf{x}(t_0) + s\mathbf{x}'(t_0)$$

For example, the tangent line to the above helix at $t_0 = \frac{7\pi}{3}$ is

$$\mathbf{y}(s) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{7\pi}{3}\right) + \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right) s$$

The tangent line has the same speed as the helix $\sqrt{2}$.



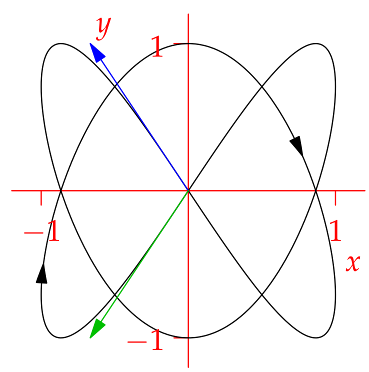
Self-intersections These are no problem for our formulation! The curve

$$\mathbf{x}(t) = \left(\sin \tfrac{2t}{3}, \cos t\right), \quad t \in [0, 6\pi)$$

passes through the origin at both $t_1 = \frac{3\pi}{2}$ and $t_2 = \frac{9\pi}{2}$, with corresponding tangent vectors

$$\mathbf{x}'\left(\tfrac{3\pi}{2}\right) = \left(-\tfrac{2}{3}, 1\right), \qquad \mathbf{x}'\left(\tfrac{9\pi}{2}\right) = \left(-\tfrac{2}{3}, -1\right)$$

In this example, we shouldn't talk about *the* tangent vector to the curve *at the origin*, since it is non-unique. Rather we should refer to the *co-ordinates* $\frac{3\pi}{2}$ or $\frac{9\pi}{2}$.



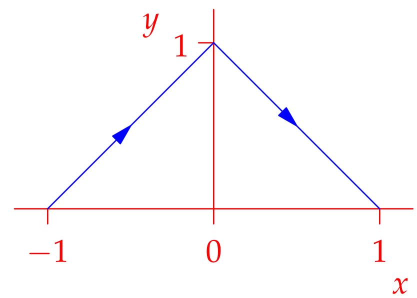
The linked animation shows the variable speed $v(t) = \sqrt{\frac{4}{9}\cos^2\frac{2t}{3} + \sin^2 t}$ of this curve.

Corners and Cusps To ensure that a tangent direction exists, a regular curve has everywhere non-zero derivative. Here are a couple of examples of curves with non-regular points.

Examples 1.8. *Corner* A curve might enter and leave a point in different directions. For example, $\mathbf{x}(t) = \big(t, 1-|t|\big)$ has derivative

$$\mathbf{x}'(t) = \begin{cases} (1,1) & \text{if } t<0 \\ (1,-1) & \text{if } t>0 \end{cases}$$

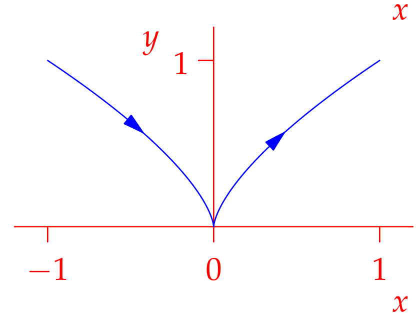
At $\mathbf{x}(0) = (0,1)$ the curve is non-differentiable and thus non-smooth and non-regular.



Cusp The curve $\mathbf{x}(t) = (t^3, t^2)$ has derivative

$$\mathbf{x}'(t) = \big(3t^2, 2t\big)$$

The origin is a *cusp*, a special type of corner where the curve leaves the point in the opposite direction to how it entered. In this case the curve is differentiable at the origin, but is non-regular since its speed $v(0)$ is zero.



Exercises 1.1. 1. A twice-differentiable curve $\mathbf{x}(t)$ has the property that its second derivative $\mathbf{x}''(t)$ is identically zero. What can be said about $\mathbf{x}$?

2. Find the unique curve such that $\mathbf{x}(0) = (1,0,5)$ and $\mathbf{x}'(t) = (t^2, t, e^t)$.

3. An ellipse in the plane has equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. By modifying the standard parametrization of the circle, find a regular parametrization of this ellipse. What is its speed?

4. Show that $\mathbf{x}(t) = \big(\frac{e^t+e^{-t}}{2}, \frac{e^t-e^{-t}}{2}\big)$ parametrizes half of the hyperbola $x^2 - y^2 = 1$. How would you parametrize the other half?

5. (a) Find the speed of the re-parametrized standard helix $\mathbf{y}(s) = \mathbf{x}(s^3) = \big(\cos s^3, \sin s^3, s^3\big)$.
   (b) More generally, if $\mathbf{x}(t)$ is a regular curve, show that $\mathbf{y}(s) := \mathbf{x}(s^3)$ is non-regular.

6. Verify that our cusp example (above) may instead be parametrized $\mathbf{y}(u) = \big(u, u^{2/3}\big)$. Is the new parametrization still non-regular at the origin? Explain.

7. Show that the tangent vectors to the regular curve $\mathbf{x}(t) = \big(3t, 3t^2, 2t^3\big)$ make a constant angle with the vector $(1,0,1)$.

8. Consider the plane curve $\mathbf{x}(t) = \big(t-1+e^{-t}, e^{-t}\big)$. Find the equation of its tangent line at $t = t_0$ and find where the tangent line intersects the x-axis.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Find a parametrization for the graph of $y = f(x)$ and find its tangent line when $x = x_0$.

10. Find a parametrization of the straight line through the points $(1,-3,-1)$ and $(6,2,1)$. Does this line meet the line through the points $(-1,1,0)$ and $(-5,-1,-1)$?

1.2 The Arc-length Parametrization and Curvature

As we've already seen, the same 'curve' (subset of $\mathbb{E}^3$) may be parametrized in different ways. For instance, in Exercise 1.1.5 we saw that the standard helix parametrized by $\mathbf{x}(t) = (\cos t, \sin t, t)$ may be re-parametrized to obtain

$$\mathbf{y}(s) = (\cos s^3, \sin s^3, s^3) \tag{$*$}$$

We also saw that this new parametrization is non-regular at $s = 0$; it slows down and pauses before resuming its journey up the helix! This shows that regularity is not an intrinsic property of a curve viewed as a *set* (range $\mathbf{x}$), rather it is a property of the parametrization.

Thankfully it is easy to create new parametrizations that remain regular.

Lemma 1.9. *If $\mathbf{x} : I \to \mathbb{E}^3$ is regular and $\alpha : J \to I$ is smooth with nowhere-zero derivative, then*

$$\mathbf{y} : J \to \mathbb{E}^3, \quad \mathbf{y}(s) := \mathbf{x}\big(\alpha(s)\big)$$

is also regular.

Proof. By the chain rule, $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}s} = \alpha'(s)\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$, which is non-zero by assumption. ∎

Since $\alpha'(s)$ is continuous and non-zero, there are two distinct cases:[3]

$\alpha(s)$ *increasing* We call this an *orientation-preserving* re-parametrization, since a 'particle' travels along the curve in the same direction.

$\alpha(s)$ *decreasing* The re-parametrization is *orientation-reversing.*

In the language of the Lemma, $(*)$ turned a regular parametrization into a non-regular one because $\alpha(s) = s^3$ has $\alpha'(s) = 3s^2$ which is zero at $s = 0$.

Our next goal is to develop a special parametrization for regular curves. First we recall a concept from multi-variable calculus.

Definition 1.10. The (signed) *arc-length* of a curve $\mathbf{x} : I \to \mathbb{E}^3$ measured from $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$ is the integral of the speed

$$s(t) = \int_{t_0}^{t} ||\mathbf{x}'(T)||\,\mathrm{d}T = \int_{t_0}^{t} v(T)\,\mathrm{d}T$$

The arc-length is *signed* because it is negative if $t < t_0$: we are measuring length against the orientation of the curve. Of course if $\mathbf{x} : [a, b] \to \mathbb{E}^3$ has domain a closed bounded interval, then it is most sensible to measure arc-length from $t_0 = a$ so that $s(t) \geq 0$ everywhere on the curve.

Example 1.11. The standard helix $\mathbf{x}(t) = \big(\cos t, \sin t, t\big)$ has constant speed $\sqrt{2}$, whence the arc-length measured from $\mathbf{x}(0)$ is simply $s(t) = \sqrt{2}t$.

[3] The observation here is that $\alpha'(s)$ is either *always positive* or *always negative.* In particular, $\alpha(s)$ is 1–1. If, in addition, α is *onto*, then $\mathbf{x}, \mathbf{y}$ parametrize precisely the same subset of $\mathbb{E}^3$.

Recall the Fundamental Theorem of Calculus: is $s(t)$ is the arc-length of a regular curve, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} ||\mathbf{x}'(T)||\, \mathrm{d}T = ||\mathbf{x}'(t)|| = v(t)$$

is the curve's speed, which is *positive* and *continuous.* The same is therefore true for its *inverse function*

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{s'(t)} = \frac{1}{v(t)} > 0$$

Definition 1.12. An *arc-length parameter* for a regular curve $\mathbf{x}(t)$ is the inverse $\alpha(s) = t(s)$ of an arc-length function $s(t)$.

Lemma 1.9 tells us that $\mathbf{y}(s) = \mathbf{x}(\alpha(s))$ is a regular re-parametrization of our original curve. Indeed it is a re-parametrization with a very special property:

$$||\mathbf{y}'(s)|| = \alpha'(s)\, ||\mathbf{x}'(\alpha(s))|| = \frac{1}{v(t)}v(t) = 1$$

The curve $\mathbf{y}(s)$ has *unit-speed.* We have therefore proved a key result.

Theorem 1.13. *Every regular curve has a unit-speed parametrization, namely by an arc-length parameter (measured from wherever you like).*

The usefulness of the Theorem is abstract; by *assuming* that we have a unit-speed parametrization, certain analyses become much simpler. As a practical matter, explicitly computing an arc-length parametrization might be essentially impossible since it requires evaluating an integral and inverting a function.

Examples 1.14.
1. Since the standard helix has arc-length parameter $s(t) = \sqrt{2}t$, it is trivial to observe that the re-parametrization

   $$\mathbf{y}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

   has unit speed.

2. More generally, if $\mathbf{x}(t)$ has constant speed v, then $s(t) = vt$ is an arc-length parameter and $\mathbf{y}(s) = \mathbf{x}(\frac{s}{v})$ a unit-speed re-parametrization.

3. The graph of $y = \frac{2}{3}x^{3/2}$ $(t \geq 0)$ may be parametrized by $\mathbf{x}(t) = (t, \frac{2}{3}t^{3/2})$. The arc-length measured from the origin is then

   $$s(t) = \int_0^t \sqrt{1+T}\,\mathrm{d}T = \tfrac{2}{3}\left[(1+t)^{3/2} - 1\right] \implies \alpha(s) = t(s) = \left(1 + \tfrac{3}{2}s\right)^{2/3} - 1$$

   We've obtained an explicit unit-speed parametrization

   $$\mathbf{y}(s) = \mathbf{x}(\alpha(s)) = \left(\left(1+\tfrac{3}{2}s\right)^{2/3} - 1, \tfrac{2}{3}\left[\left(1+\tfrac{3}{2}s\right)^{2/3} - 1\right]^{3/2}\right)$$

   though is it really something you ever want to compute with?!

Armed with unit-speed curves, we can now define our principal notion of bendiness.

Definition 1.15. The *curvature* of a unit-speed curve $\mathbf{x} : I \to \mathbb{E}^3$ is

$$\kappa(s) = ||\mathbf{x}''(s)||$$

We modify this slightly for curves in the plane: $\kappa(s)$ is positive/negative if the tangent vector rotates *counter-clockwise/clockwise* as we traverse the curve. This corresponds to the usual *right hand rule.*

By Newton's second law, a unit mass travelling along the curve at unit speed experiences a *transverse force* of magnitude $\kappa(s)$.

Examples 1.16.
1. A straight line has curvature zero. For example, the line joining $(1,4)$ and $(-3,1)$ has unit-speed parametrization $\mathbf{x}(s) = \left(-3 + \frac{4}{5}s, 1 + \frac{3}{5}s\right)$, whence $\mathbf{x}''(s) = \mathbf{0} \implies \kappa(s) = 0$.

2. The circle of radius r has unit-speed parametrization $\mathbf{x}(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$, whence

$$\mathbf{x}''(s) = -\frac{1}{r}\begin{pmatrix}\cos\frac{s}{r}\\ \sin\frac{s}{r}\end{pmatrix} \implies \kappa(s) = \frac{1}{r}$$

This is positive since the tangent vector rotates counter-clockwise. Observe that $\kappa = \frac{1}{r}$ is inversely proportional to the radius: smaller circles have larger curvature.

3. The standard helix with unit-speed parametrization $\mathbf{x}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ has

$$\mathbf{x}''(s) = -\frac{1}{2}\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right) \implies \kappa(s) = \frac{1}{2}$$

Since finding a unit-speed parametrization is difficult, there few curves for which this approach is sensible. What we want is a method that works for *arbitrary parametrization.* This is indeed possible, though for spacecurves it will take a while. For curves in the *plane* however, things are fairly easy.

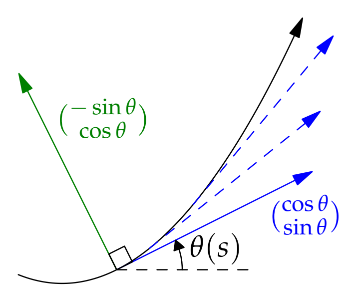
Curvature of Plane Curves If $\mathbf{y} : I \to \mathbb{E}^2$ has unit-speed, we can write

$$\mathbf{y}'(s) = \begin{pmatrix}\cos\theta(s)\\ \sin\theta(s)\end{pmatrix}$$

where $\theta(s)$ is the *angle* between the tangent line and the positive x-axis. Now observe that

$$\mathbf{y}''(s) = \theta'(s)\begin{pmatrix}-\sin\theta(s)\\ \cos\theta(s)\end{pmatrix}$$

Since $(-\sin\theta, \cos\theta)$ points to the *left* of $\mathbf{y}'(s)$, we conclude:



$\kappa(s) = \theta'(s) > 0$

Theorem 1.17. *The curvature of a unit-speed plane curve is the rate of change $\kappa(s) = \theta'(s)$ of the angle of its tangent line.*

This should be intuitive for constant curvature examples such as the straight line and the circle.

Now suppose $\mathbf{x}(t) = \big(x(t), y(t)\big)$ is any regular parametrization of the same curve; its speed satisfies

$$v(t) = \sqrt{x'(t)^2 + y'(t)^2} = s'(t)$$

where $s(t)$ is an arc-length function for $\mathbf{x}(t)$. Moreover, the angle $\theta(s)$ plainly satisfies

$$\theta(s) = \tan^{-1}\frac{y'(t)}{x'(t)}$$

Now differentiate and applying the chain rule:

$$\kappa(s) = \frac{\mathrm{d}}{\mathrm{d}s}\tan^{-1}\frac{y'(t)}{x'(t)} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}\frac{y'(t)}{x'(t)} = \cdots$$

The result is a formula for the curvature as a function of an arbitrary regular parametrization.

Corollary 1.18. *A regular curve* $\mathbf{x}(t) = \big(x(t), y(t)\big)$ *has curvature*

$$\kappa(t) = \frac{y''x' - x''y'}{[x'^2 + y'^2]^{3/2}} = \frac{y''x' - x''y'}{v^3} = \frac{\mathbf{x}'' \cdot J\mathbf{x}'}{v^3}$$

where $J\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -y' \\ x' \end{pmatrix}$. *In particular, the graph of a smooth function* $y = f(x)$ *has curvature*

$$\kappa(x) = \frac{f''(x)}{\Big[1 + \big(f'(x)\big)^2\Big]^{3/2}}$$

Examples 1.19. 1. The graph of $y = \frac{2}{3}x^{3/2}$ has curvature

$$\kappa(x) = \frac{\frac{1}{2}x^{-1/2}}{(1+x)^{3/2}} = \frac{1}{2\sqrt{x(1+x)^3}}$$

2. If $f(x) = \sin x$, then $\kappa(x) = \dfrac{-\sin x}{(1+\sin^2 x)^{3/2}}$

3. The spiral $\mathbf{x}(t) = \big(t\cos t, t\sin t\big)$ has

$$\mathbf{x}'(t) = \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{pmatrix}, \quad \mathbf{x}''(t) = \begin{pmatrix} -2\sin t - t\cos t \\ 2\cos t - t\sin t \end{pmatrix}$$

$$\implies \kappa(t) = \frac{(2\cos t - t\sin t)(\cos t - t\sin t) - (-2\sin t - t\cos t)(\sin t + t\cos t)}{[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2]^{3/2}}$$

$$= \frac{2+t^2}{[1+t^2]^{3/2}}$$

Exercises 1.2. 1. Compute the arc-length of the following curves by parametrizing and evaluating an integral:

(a) The straight line between points $(3,1,2)$ and $(1,1,0)$.

(b) The circle centered at $(1,-2)$ with radius 5 measured *clockwise* from $(6,-2)$ to $(1,3)$.

(c) The graph of the function $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ for $1 \le x \le 9$.

2. Find the curvature of the following plane curves (use Corollary 1.18).

(a) The graph of $y = x^2$.

(b) The catenary: the graph of $y = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

(c) The figure-eight curve $\mathbf{x}(t) = \big(\cos t, \sin 2t\big)$

(d) The exponential spiral $\mathbf{x}(t) = \big(e^t \cos t, e^t \sin t\big)$.

3. Find a unit-speed parametrization of the straight line between points with position vectors $\mathbf{a} \neq \mathbf{b}$ in $\mathbb{E}^3$ and hence verify that its curvature is zero.

4. Suppose $\mathbf{x} : \mathbb{R} \to \mathbb{E}^3$ has unit speed. Verify that $\mathbf{x}$ is parametrized by an arc-length parameter.

5. Find the curvature of the spacecurve $\mathbf{x}(s) = \big(\frac{5}{13}\cos s, \sin s, \frac{12}{13}\cos s\big)$. What is this curve?

6. (a) Find the arc-length of the standard helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ between $t = -\pi$ and $t = 2\pi$.

(b) Suppose that a particle travels *down* the helix starting at $(1,0,2\pi)$ at time $T = 0$ such that its speed is $v(T) = 2\sqrt{2}T$. Find a parametrization of the helix which describes this motion.

(c) Let r, h be positive constants. Find the curvature of the general circular helix

$$\mathbf{x}(t) = \big(r\cos t, r\sin t, ht\big)$$

and interpret how it depends on r and h.

7. Check the evaluation of $\kappa(t)$ and $\kappa(x)$ in the proof of Corollary 1.18.

8. We find the curvature of the exponential spiral $\mathbf{x}(t) = \big(e^t \cos t, e^t \sin t\big)$ the hard way.

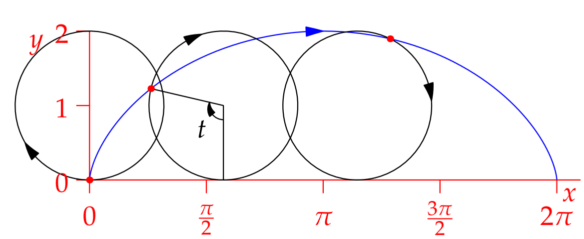
(a) Calculate the arc-length $s(t)$ measured from $\mathbf{x}(0)$.

(b) Find a unit-speed parametrization $\mathbf{y}(s)$ where $\mathbf{y}(0) = (1,0)$.

(c) Hence compute $\kappa(s)$ and show that it equals your answer from Exercise 2d.

9. A circle of radius 1 rolls at constant speed without slipping along the x-axis so that the angle indicated in the picture is t at time t.

The curve described by a point on the circumference of the rolling circle is a *cycloid*.



(a) Find a parametrization of the cycloid $\mathbf{x} : [0, 2\pi] \to \mathbb{E}^2$.

(b) Find the curvature of the cycloid as a function of t.

(c) Compute the arc-length of the cycloid over a complete rotation of the circle.

1.3 Orthogonality, Moving Frames & The Structure Equations

Our plan is to analyze a curve with respect to a family of *moving* orthonormal bases. Before embarking on this, we summarize the relevant ideas from linear algebra. The proofs are not critical so we omit most of them, what matters is that the concepts are mostly familiar. As usual, definitions and results are stated in 3-dimensions, but are valid in others, particularly 2-dimensions.

In $\mathbb{E}^3$, points are typically denoted with reference to the *standard basis* $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. For instance,

$$\mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

The numbers $3, 4, 6$ are the *co-ordinates* of $\mathbf{v}$ with respect to the standard basis. Of course other bases are available...

Definition 1.20. A set $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subseteq \mathbb{E}^3$ is a *basis* if every vector $\mathbf{v} \in \mathbb{E}^3$ can be expressed uniquely[4] as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$: that is

$$\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \tag{$*$}$$

for unique $c_1, c_2, c_3 \in \mathbb{R}$, the *co-ordinates* of $\mathbf{v}$ with respect to β.

A basis is *orthonormal* if $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

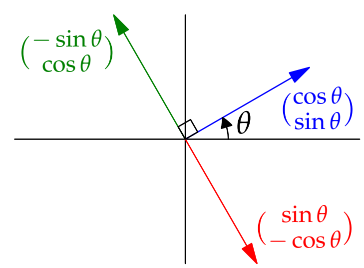
Consider the (invertible) matrix $E = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$ whose columns are the elements of β viewed as column vectors (with respect to the standard basis). A basis is *positively oriented* if $\det E > 0$.

Examples 1.21.
1. $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a *negatively* oriented orthonormal basis of $\mathbb{E}^3$ ($\det E = -1 < 0$).

2. Every orthonormal basis of $\mathbb{E}^2$ has the form

$$\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \right\}$$

for some angle θ. The first is positively oriented ($\det = 1 > 0$) and the second negatively ($\det = -1 < 0$).



A positively oriented orthonormal basis in $\mathbb{E}^3$ satisfies the *right-hand rule*: $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. In $\mathbb{E}^2$, positive orientation means that $\mathbf{e}_2$ is obtained by rotating $\mathbf{e}_1$ counter-clockwise by 90°: we can write this as

$$\mathbf{e}_2 = J\mathbf{e}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{e}_1$$

[4] In a linear algebra class this is usually broken into two definitions which imply, respectively, the existence and uniqueness of the linear combination $(*)$.

Spanning Set Every $\mathbf{v} \in \mathbb{E}^3$ can be expressed as a linear combination $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Linear Independence The only linear combination summing to $\mathbf{0}$ is trivial: $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0} \implies c_1 = c_2 = c_3 = 0$.

Finding the co-ordinates of a vector with respect to a basis $(*)$ is really a matrix problem[5]

$$\mathbf{v} = E\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = E^{-1}\mathbf{v}$$

Inverting a 3×3 matrix is tedious. Thankfully the co-ordinates can be found more easily if the basis is *orthonormal* just by taking dot products!

$$\mathbf{v} \cdot \mathbf{e}_i = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3) \cdot \mathbf{e}_i = c_i$$

Lemma 1.22. *If $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis, then for any vector $\mathbf{v} \in \mathbb{E}^3$,*

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}_3)\mathbf{e}_3$$

Example 1.23. $\beta = \{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\frac{1}{5}\left(\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right), \frac{1}{5}\left(\begin{smallmatrix} -3 \\ 4 \end{smallmatrix}\right)\right\}$ is a positively oriented orthonormal basis of $\mathbb{E}^2$. With respect to β, the vector $\mathbf{v} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ can be written

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = \frac{7}{5}\mathbf{e}_1 + \frac{1}{5}\mathbf{e}_2$$

Orthogonal Matrices

Recall Definition 1.5. Given $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and its associated matrix $E = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3)$, observe that

$$E^T E = \begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{pmatrix} (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) = \begin{pmatrix} ||\mathbf{e}_1||^2 & \mathbf{e}_1 \cdot \mathbf{e}_2 & \mathbf{e}_1 \cdot \mathbf{e}_3 \\ \mathbf{e}_2 \cdot \mathbf{e}_1 & ||\mathbf{e}_2||^2 & \mathbf{e}_2 \cdot \mathbf{e}_3 \\ \mathbf{e}_3 \cdot \mathbf{e}_1 & \mathbf{e}_3 \cdot \mathbf{e}_2 & ||\mathbf{e}_3||^2 \end{pmatrix}$$

When β is an orthonormal basis, this matrix is very simple.

Definition 1.24. A 3×3 matrix A is *orthogonal* if $A^T A = I$ (equivalently $AA^T = I$). The set of all such is denoted $\mathrm{O}_3(\mathbb{R})$. In addition, if $\det A = 1$, we write $A \in \mathrm{SO}_3(\mathbb{R})$ (*special orthogonal matrices*).

Lemma 1.25. 1. *If $A \in \mathrm{O}_3(\mathbb{R})$, then it is invertible with inverse A^T (also orthogonal).*

2. *The product of two orthogonal matrices is orthogonal.*
3. *A is orthogonal if and only if $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{E}^3$.*
4. *Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $E = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) \in M_3(\mathbb{R})$:*
   - *(a) $E \in \mathrm{O}_3(\mathbb{R}) \iff \beta$ is an orthonormal basis.*
   - *(b) $E \in \mathrm{SO}_3(\mathbb{R}) \iff \beta$ is a positively oriented orthonormal basis.*

Parts 1 and 2 together say that $\mathrm{O}_3(\mathbb{R})$ forms a *group* under matrix multiplication; it is therefore often known as the *orthogonal group*.

[5] For obvious reasons, this is known as the *change of co-ordinate matrix* from β to the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

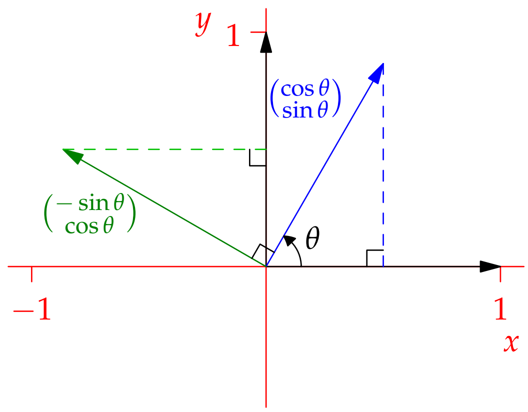
Examples (1.21 cont). 1. It is no fun to check $E^T E = I$ directly, but since we know we have an orthonormal basis, the Lemma tells us that

$$E = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \in O_3(\mathbb{R})$$

2. Every 2×2 orthogonal matrix has one of two forms:

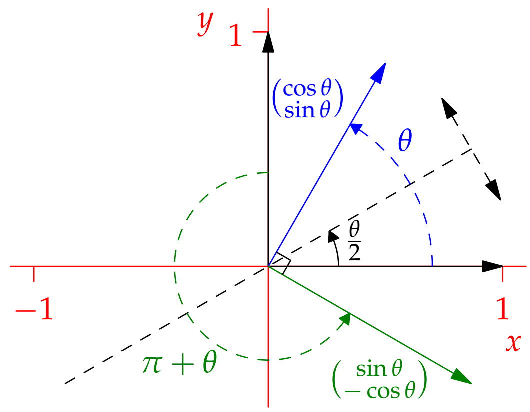
Rotations $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO_2(\mathbb{R})$

The effect of the map $\mathbf{x} \mapsto A_\theta \mathbf{x}$ is to *rotate* $\mathbf{x}$ counter-clockwise by θ radians.[6]



Reflections $B_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ $(\det B_\theta = -1)$

The effect of $\mathbf{x} \mapsto B_\theta \mathbf{x}$ is to *reflect* $\mathbf{x}$ across the line making angle $\frac{\theta}{2}$ with the positive x-axis.



Motivated by the 2×2 case, it is common to refer to every orthogonal matrix in $O_3(\mathbb{R})$ as either a rotation $(\det = 1)$ or a reflection $(\det = -1)$.[7]

Part 3 of Lemma 1.25 says that multiplication by an orthogonal matrix preserve the dot product and thus (Definition 1.5) the *lengths* of vectors and the *angles* between them. We use this to define a useful family of transformations of $\mathbb{E}^3$.

Definition 1.26. An *isometry*[8] is a function $S : \mathbb{E}^3 \to \mathbb{E}^3$ acting on points/position vectors by

$$S(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

where $\mathbf{b}$ is a constant vector and $A \in O_3(\mathbb{R})$. We call S a *direct isometry* or *rigid motion* if $\det A = 1$ $(A \in SO_3(\mathbb{R}))$, and an *indirect isometry* otherwise.

Congruent geometric objects are precisely those which are related by an isometry. Rigid motions are precisely the *orientation-preserving* isometries.

[6] Recall that the matrix of a linear map is found by evaluating the map on the standard basis: thus the 1st columns of A_θ is the column vector $A_\theta \mathbf{i} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. The pictures should help you verify the remaining columns; for B_θ you might find it helpful to consider how the required reflections of the standard basis vectors $\mathbf{i}, \mathbf{j}$ may be computed using *rotations*.

[7] A full analysis is more complicated; for instance the map $\mathbf{x} \mapsto E\mathbf{x}$ in the first example is the composition of a reflection across a plane in $\mathbb{E}^3$ followed by a rotation in that in that plane.

[8] Literally *equal length*; it can be seen that every function $S : \mathbb{E}^3 \to \mathbb{E}^3$ which preserves distances between all pairs of points has this form.

Moving Frames

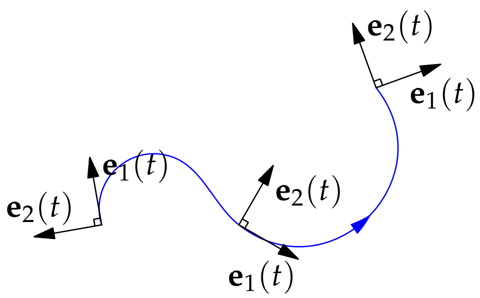
Thus far we have analysed curves with reference to the standard orthonormal basis $\epsilon = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. We replace this *static* frame of reference with one that *moves*.

Definition 1.27. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a smooth curve. Suppose that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are smooth functions on I such that, for each $t \in I$,

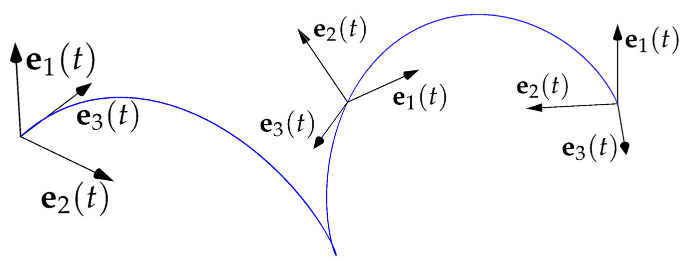
$$\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\} \text{ is a positively oriented orthonormal basis of the tangent space } T_{\mathbf{x}(t)}\mathbb{E}^3$$

We call this family of functions a *moving frame* along $\mathbf{x}$.

Equivalently, $E(t) = \begin{pmatrix}\mathbf{e}_1(t) & \mathbf{e}_2(t) & \mathbf{e}_3(t)\end{pmatrix}$ is a smooth function $E : I \to SO_3(\mathbb{R})$. We will often refer to $E(t)$ as a moving frame.



A moving frame in $\mathbb{E}^2$



A moving frame in $\mathbb{E}^3$

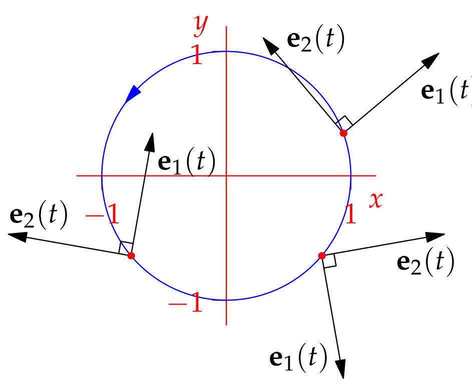
To be a little more precise; at each point on the curve, the tangent space $T_{\mathbf{x}(t)}\mathbb{E}^3$ has a standard basis of tangent vectors $\{\mathbf{i}_{\mathbf{x}(t)}, \mathbf{j}_{\mathbf{x}(t)}, \mathbf{k}_{\mathbf{x}(t)}\}$, and we can write

$$\mathbf{e}_j(t) = a_j(t)\mathbf{i}_{\mathbf{x}(t)} + b_j(t)\mathbf{j}_{\mathbf{x}(t)} + c_j(t)\mathbf{k}_{\mathbf{x}(t)} = \begin{pmatrix} a_j(t) \\ b_j(t) \\ c_j(t) \end{pmatrix}$$

We require that the functions $a_j, b_j, c_j : I \to \mathbb{R}$ be smooth. Strictly speaking, $\mathbf{e}_j(t)$ is a *smooth vector field* along the curve.

Example 1.28. The following functions define a moving frame along the unit circle $\mathbf{x}(t) = (\cos t, \sin t)$:

$$\mathbf{e}_1(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \qquad \mathbf{e}_2(t) = \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$



Click on the picture to see how the frame rotates twice as one travels once round the circle!

In accordance with the definition, for each t,

$$E(t) = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \in SO_3(\mathbb{R})$$

The goal is to find natural choices of frame with respect to which fundamental properties of a curve become clear. This advantage comes at a price: we have to understand how a moving frame *moves.*

Theorem 1.29 (Structure equations). *Suppose $\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\}$ is a moving frame. Then there exist unique functions $w_{ij}(t) = \mathbf{e}_i \cdot \mathbf{e}_j' = -\mathbf{e}_i' \cdot \mathbf{e}_j$ $(i < j)$ such that*

$$\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3'\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}0 & w_{12} & w_{13}\\ -w_{12} & 0 & w_{23}\\ -w_{13} & -w_{23} & 0\end{pmatrix}$$

In matrix form the structure equations can be written $E' = EW$, where each $W(t)$ is *skew-symmetric.* In $\mathbb{E}^2$ there is only a single function w_{12}. As we've done already, we often drop the (t) to make things more readable; just remember that everything is still a *function*!

Proof. Since $\mathbf{e}_i \cdot \mathbf{e}_j$ is constant (equals 0 or 1), the product rule says that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_i \cdot \mathbf{e}_j) = \mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j'$$

Now use Lemma 1.22 to compute the co-ordinates of $\mathbf{e}_i'$ with respect to the basis $\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)\}$. For instance, the first column of $E'(t)$ follows from

$$\mathbf{e}_1'(t) = (\mathbf{e}_1' \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{e}_1' \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{e}_1' \cdot \mathbf{e}_3)\mathbf{e}_3 = -(\mathbf{e}_1 \cdot \mathbf{e}_2')\mathbf{e}_2 - (\mathbf{e}_1 \cdot \mathbf{e}_3')\mathbf{e}_3$$

∎

Examples 1.30. 1. Example 1.28 described a moving frame in $\mathbb{E}^2$:

$$w_{12}(t) = \mathbf{e}_1(t) \cdot \mathbf{e}_2'(t) = \begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix} \cdot \begin{pmatrix}-2\cos 2t\\ -2\sin 2t\end{pmatrix} = -2$$

whence the structure equations are

$$\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2'\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{pmatrix}\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}$$

2. Without mentioning a curve $\mathbf{x}$, here is a moving frame in $\mathbb{E}^3$ with non-constant functions w_{ij}

$$\mathbf{e}_1(t) = \begin{pmatrix}\cos^2 t\\ \cos t \sin t\\ \sin t\end{pmatrix} \qquad \mathbf{e}_2(t) = \begin{pmatrix}\sin t\\ -\cos t\\ 0\end{pmatrix} \qquad \mathbf{e}_3(t) = \begin{pmatrix}\sin t \cos t\\ \sin^2 t\\ -\cos t\end{pmatrix}$$

We compute

$$\begin{aligned}
w_{12}(t) &= \mathbf{e}_1 \cdot \mathbf{e}_2' = \cos^3 t + \cos t \sin^2 t = \cos t,\\
w_{13}(t) &= \mathbf{e}_1 \cdot \mathbf{e}_3' = \cos^2 t(\cos^2 t - \sin^2 t) + 2(\cos t \sin t)^2 + \sin^2 t = 1\\
w_{23}(t) &= \mathbf{e}_2 \cdot \mathbf{e}_3' = \sin t(\cos^2 t - \sin^2 t) - \cos t(2\cos t \sin t) = -\sin t
\end{aligned}$$

The structure equations are therefore

$$\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3'\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}0 & \cos t & 1\\ -\cos t & 0 & -\sin t\\ -1 & \sin t & 0\end{pmatrix}$$

Exercises 1.3. 1. Express $\mathbf{v} = \left(\begin{smallmatrix}5\\1\end{smallmatrix}\right) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ as a linear combination with respect to the orthonormal basis $\beta = \{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\frac{1}{13}\left(\begin{smallmatrix}5\\12\end{smallmatrix}\right), \frac{1}{13}\left(\begin{smallmatrix}12\\-5\end{smallmatrix}\right)\right\}$ of $\mathbb{E}^2$.

2. (a) Show that $\beta = \left\{\frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix}-1\\-1\\4\end{pmatrix}\right\}$ is an orthonormal basis of $\mathbb{E}^3$. Is it positively oriented?

   (b) Find the co-ordinates of $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ with respect to β.

3. (a) Explain why the product rule $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{x}'\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{y}'$ holds for differentiable curves $\mathbf{x}, \mathbf{y}$.

   (b) Let $\mathbf{x}, \mathbf{y}$ be differentiable on an interval and use the product rule to answer the following:

      i. If $\mathbf{x}(t_0)$ and $\mathbf{x}'(t)$ are orthogonal to a fixed vector $\mathbf{v}$ for all t, show that $\mathbf{x}(t)$ is always orthogonal to $\mathbf{v}$.

      ii. If $\mathbf{y}(t_0)$ is a point on $\mathbf{y}$ which is closest to the origin, show that $\mathbf{y}(t_0) \perp \mathbf{y}'(t_0)$.

4. Find the function w_{12} for the moving frame $\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\frac{1}{1+t^2}\begin{pmatrix}2t\\1-t^2\end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix}t^2-1\\2t\end{pmatrix}\right\}$

5. Find the structure equations for the moving frame

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \left\{\begin{pmatrix}\cos t\\0\\\sin t\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-\sin t\\0\\\cos t\end{pmatrix}\right\}$$

6. (a) Explain why every moving frame in $\mathbb{E}^2$ has the form $\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{\left(\begin{smallmatrix}\cos\theta(t)\\\sin\theta(t)\end{smallmatrix}\right), \left(\begin{smallmatrix}-\sin\theta(t)\\\cos\theta(t)\end{smallmatrix}\right)\right\}$ for some function θ.

   (b) Find the structure equations for this frame: how does w_{12} relate to θ?

   (c) If $\mathbf{x}(t)$ is parametrized at unit speed such that $\mathbf{e}_1(t) = \mathbf{x}'(t)$, what is $w_{12}(t)$?

7. (a) Let $E(t)$ be a matrix-valued function. Show that $\frac{\mathrm{d}}{\mathrm{d}t}(E(t))^{-1} = -E^{-1}E'E^{-1}$.

   (b) Suppose $E : I \to O_3(\mathbb{R})$ is differentiable and define $W(t) := E^{-1}(t)E'(t)$. Show that $W(t)$ is skew-symmetric ($W^T = -W$).

8. (a) Verify parts 2 and 3 of Lemma 1.25.

   (b) Suppose f, g are rigid motions. Show that $f \circ g$ and f^{-1} are also rigid motions.

9. Let $\mathbf{i} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Suppose $\mathbf{p} \in \mathbb{E}^2$ and a unit vector $\mathbf{v}$ are given. Prove that there is a unique rigid motion $S : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ such that

$$S(\mathbf{0}) = \mathbf{p} \quad \text{and} \quad S(\mathbf{i}) = \mathbf{p} + \mathbf{v}$$

   Viewing $\mathbf{i_0} = (\mathbf{0}, \mathbf{i}) \in T_\mathbf{0}\mathbb{E}^2$ and $\mathbf{v_p} = (\mathbf{p}, \mathbf{v}) \in T_\mathbf{p}\mathbb{E}^2$ as *tangent vectors*, explain why it is reasonable to write $\mathbf{v_p} = S(\mathbf{i_0}) = (A\mathbf{i})_\mathbf{p}$: that is, only the matrix A affects the *directional part* of a tangent vector.

10. (Hard) Suppose that a moving frame has structure equations

$$\mathbf{e}_1' = -\frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3), \qquad \mathbf{e}_2' = \frac{1}{\sqrt{2}}\mathbf{e}_1, \qquad \mathbf{e}_3' = \frac{1}{\sqrt{2}}\mathbf{e}_1$$

   (a) By considering $\mathbf{e}_1''$, show that the vector $\mathbf{e}_1 \times \mathbf{e}_1'$ is constant.

   (b) Show that $||\mathbf{e}_1'||$ is constant.

   (c) Prove that there exists a constant positively oriented orthonormal basis $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{e}_1(t) = \cos t\,\mathbf{a} + \sin t\,\mathbf{b}$ and compute $\mathbf{e}_2, \mathbf{e}_3$ in terms of this basis.

1.4 The Frenet Frame for a Spacecurve

In this section we analyze spacecurves with respect to a moving frame *adapted* to the curve. To do this, we need to restrict our class of curves slightly. For this section, we work exclusively in $\mathbb{E}^3$.

Definition 1.31. A regular curve $\mathbf{x} : I \to \mathbb{E}^3$ is *biregular* if it has *non-zero curvature* κ.

Every biregular curve is necessarily regular, but the converse is false. For instance, a straight line is regular but *not biregular.* Indeed for a biregular curve, the vectors $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$ must be linearly independent.

Definition 1.32. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a biregular unit-speed curve. The *Frenet frame* $E(t) = (\mathbf{T}\ \mathbf{N}\ \mathbf{B})$ is the moving frame defined as follows:

$\mathbf{T} := \mathbf{x}'$ is the *unit tangent* vector field

$\mathbf{N} := \frac{1}{\kappa}\mathbf{T}'$ is the *principal normal* vector field

$\mathbf{B} := \mathbf{T} \times \mathbf{N}$ is the *binormal* vector field

Theorem 1.33. *The Frenet frame is indeed a moving frame. Moreover, its structure equations are*

$$\begin{pmatrix}\mathbf{T}' & \mathbf{N}' & \mathbf{B}'\end{pmatrix} = \begin{pmatrix}\mathbf{T} & \mathbf{N} & \mathbf{B}\end{pmatrix}\begin{pmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix} \qquad \begin{cases}\mathbf{T}' = \kappa\mathbf{N}\\ \mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}\\ \mathbf{B}' = -\tau\mathbf{N}\end{cases}$$

where $\kappa > 0$ is the curvature and a new function $\tau = \mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot \mathbf{B}'$ called the torsion.

Proof. Certainly $\mathbf{T} = \mathbf{x}'$ has unit length.

Since $\mathbf{x}$ is *biregular* we have $\kappa = ||\mathbf{x}''|| = ||\mathbf{T}'|| \neq 0$. The principal normal vector $\mathbf{N} = \frac{1}{\kappa}\mathbf{T}'$ is therefore the unit vector pointing in the same direction as $\mathbf{T}'$.

By definition, the binormal vector has unit length[9] and results in a positively oriented basis $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

The smoothness of the frame follows from the smoothness of $\mathbf{x}$ and the fact the κ is never zero (think about how the product/quotient rule could be used to differentiate infinitely many times...)

To establish the structure equations, it remains only to verify that $w_{13} = 0$. This is straightforward since

$$w_{13} = -\mathbf{T}' \cdot \mathbf{B} = -\frac{1}{\kappa}\mathbf{N} \cdot \mathbf{B} = 0$$

■

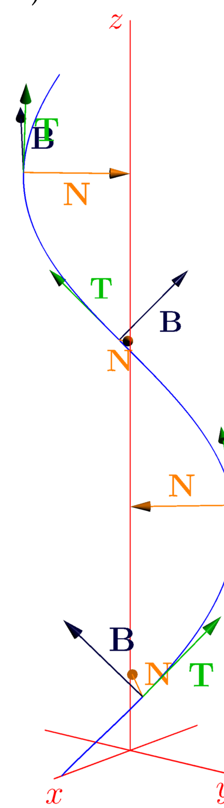
The structure equations for the Frenet frame are known as the *Frenet–Serret equations.* The moving planes spanned by pairs of these vectors have special names:

Span$\{\mathbf{T}, \mathbf{N}\}$, Span$\{\mathbf{T}, \mathbf{B}\}$ and Span$\{\mathbf{N}, \mathbf{B}\}$ are the *osculating*, *rectifying* and *normal* planes.

The tangent line at any point lies in the osculating plane.

[9]Recall that $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta$ where θ is the angle between the vectors.

Examples 1.34. 1. We compute the Frenet frame and its structure equations for the standard helix $\mathbf{x}(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ parametrized by arc-length (3D pic)(animation)



$$\mathbf{T}(s) = \mathbf{x}'(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\frac{s}{\sqrt{2}} \\ \cos\frac{s}{\sqrt{2}} \\ 1 \end{pmatrix} \implies \mathbf{T}'(s) = -\frac{1}{2}\begin{pmatrix} \cos\frac{t}{\sqrt{2}} \\ \sin\frac{t}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\implies \mathbf{N}(s) = -\begin{pmatrix} \cos\frac{t}{\sqrt{2}} \\ \sin\frac{t}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \kappa(s) = \frac{1}{2}$$

$$\implies \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sin\frac{s}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} \\ 1 \end{pmatrix}$$

$$\tau(s) = \mathbf{N}'(s)\cdot\mathbf{B}(s) = \frac{1}{2}\begin{pmatrix} \sin\frac{s}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} \\ 0 \end{pmatrix}\cdot\begin{pmatrix} \sin\frac{s}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} \\ 1 \end{pmatrix} = \frac{1}{2}$$

The Frenet–Serret equations for the helix are therefore

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' & \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

2. Let $\mathbf{x}(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{\sqrt{2}}s, \frac{1}{3}(1-s)^{3/2}\right)$ for $s \in (-1,1)$. First we verify this is unit-speed

$$\mathbf{x}'(s) = \frac{1}{2}\begin{pmatrix} \sqrt{1+s} \\ \sqrt{2} \\ -\sqrt{1-s} \end{pmatrix} \implies v(s) = \left|\left|\mathbf{x}'(s)\right|\right| = \frac{1}{2}\sqrt{1+s+2+1-s} = 1$$

It follows that $\mathbf{T} = \mathbf{x}'$. Now compute the rest of the Frenet apparatus:

$$\mathbf{T}' = \frac{1}{4}\begin{pmatrix} (1+s)^{-1/2} \\ 0 \\ (1-s)^{-1/2} \end{pmatrix} \implies \kappa = \left|\left|\mathbf{T}'\right|\right| = \frac{1}{4}\sqrt{\frac{1}{1+s}+\frac{1}{1-s}} = \frac{1}{2\sqrt{2}\sqrt{1-s^2}}$$

$$\mathbf{N} = \frac{1}{\kappa}\mathbf{T}' = \frac{2\sqrt{2}\sqrt{1-s^2}}{4}\begin{pmatrix} (1+s)^{-1/2} \\ 0 \\ (1-s)^{-1/2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-s} \\ 0 \\ \sqrt{1+s} \end{pmatrix}$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{1}{2}\begin{pmatrix} \sqrt{1+s} \\ -\sqrt{2} \\ -\sqrt{1-s} \end{pmatrix} \implies \tau = \mathbf{N}'\cdot\mathbf{B} = \frac{-1}{2\sqrt{2}\sqrt{1-s^2}}$$

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' & \mathbf{B}' \end{pmatrix} = \frac{\begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix}}{2\sqrt{2}\sqrt{1-s^2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The Frenet Frame in arbitrary parametrization

Since there are relatively few curves for which an *explicit* unit-speed parametrization can be found, we want to be able to find the Frenet frame for any biregular curve, regardless of parametrization. There is nothing stopping us from doing this already, if we're careful...

Example 1.35. Consider the exponential spiral $\mathbf{x}(t) = \left(e^t \cos t, e^t \sin t, e^t\right)$. We have

$$\mathbf{x}'(t) = e^t \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \\ 1 \end{pmatrix} \implies v(t) = \sqrt{3}e^t \implies \mathbf{T}(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \\ 1 \end{pmatrix}$$

Since $\mathbf{T}(t)$ is unit length, its derivative is perpendicular, thus

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix} \implies \mathbf{N}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix} \qquad \text{(make unit length)}$$

$$\implies \mathbf{B}(t) = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{6}} \begin{pmatrix} -\cos t + \sin t \\ -\sin t - \cos t \\ 2 \end{pmatrix}$$

It is tempting to think that the curvature should be $||\mathbf{T}'(t)|| = \sqrt{\frac{2}{3}}$, but this is not so. We need to use the chain rule:

$$\kappa = \left|\left| \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}(t) \right|\right| = \left|\left| \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t) \right|\right| = \frac{1}{v(t)} \left|\left|\mathbf{T}'(t)\right|\right| = \frac{\sqrt{2}}{3}e^{-t}$$

The torsion may be computed similarly

$$\tau = \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} \cdot \mathbf{B} = \frac{1}{v(t)}\mathbf{N}'(t) \cdot \mathbf{B}(t) = \frac{1}{3}e^{-t}$$

The proof of the general result is a (simple!) application of the chain-rule.

Corollary 1.36. *Let $\mathbf{x}(t)$ be a biregular spacecurve with arbitrary parametrization. The speed, curvature, torsion, Frenet frame, and structure equations are as follows.*

$$v(t) = ||\mathbf{x}'(t)|| \qquad \kappa(t) = \frac{||\mathbf{x}' \times \mathbf{x}''||}{v^3} \qquad \tau(t) = \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{v^6\kappa^2}$$

$$\mathbf{T}(t) = \frac{1}{v}\mathbf{x}' \qquad \mathbf{N}(t) = \frac{v\mathbf{x}'' - v'\mathbf{x}'}{v^3\kappa} \qquad \mathbf{B}(t) = \frac{\mathbf{x}' \times \mathbf{x}''}{v^3\kappa}$$

$$\begin{pmatrix} \mathbf{T}' & \mathbf{N}' & \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \mathbf{T} & \mathbf{N} & \mathbf{B} \end{pmatrix} \begin{pmatrix} 0 & -v\kappa & 0 \\ v\kappa & 0 & -v\tau \\ 0 & v\tau & 0 \end{pmatrix}$$

The curvature formula also holds if $\mathbf{x}(t)$ is merely regular.

Exercises 1.4. 1. Compute the curvature and torsion of the spiral $\mathbf{x}(t) = \big(e^t\cos t, e^t\sin t, e^t\big)$ directly using the expressions in Corollary 1.36.

2. A circular helix has the form $\mathbf{x}(t) = \big(r\cos t, r\sin t, ht\big)$, where $r > 0$ and h are constants. Find its Frenet frame and show that its curvature and torsion are given by

$$\kappa = \frac{r}{r^2+h^2}, \qquad \tau = \frac{h}{r^2+h^2}$$

3. Find the curvature and torsion of the curve $\mathbf{x}(t) = \big(t, t^2, t^3\big)$.

4. Given $\mathbf{x}(t) = \frac{1}{\sqrt{5}}\Big(\sqrt{1+t^2}, 2t, \ln\big(t+\sqrt{1+t^2}\big)\Big)$, find the Frenet frame, curvature and torsion.

5. Let $f(t) = \sqrt{2}\int_0^t \sqrt{1-e^{-2u}}\,\mathrm{d}u$, and define the curve $\mathbf{x}(t) = \frac{1}{\sqrt{2}}\big(e^{-t}\cos t, e^{-t}\sin t, f(t)\big)$, $t > 0$.

   (a) Verify that $\mathbf{x}(t)$ has unit speed.

   (b) Calculate the curvature of $\mathbf{x}$ and show that $\lim\limits_{t\to\infty}\kappa(t) = 0$.

6. Let a, b be positive constants and $\mathbf{x}(t) = \big(4a\cos^3 t, 4a\sin^3 t, 3b\cos 2t\big)$ where $0 < t < \frac{\pi}{2}$. Find the Frenet frame, curvature and torsion of $\mathbf{x}$.

7. Let $\mathbf{x} : I \to \mathbb{E}^3$ be a twice-differentiable regular curve. Prove the formula for κ in Corollary 1.36:

$$\kappa(t) = \frac{||\mathbf{x}' \times \mathbf{x}''||}{v^3}$$

   Hence conclude that $\kappa(t_0) = 0 \iff \mathbf{x}'(t_0)$ and $\mathbf{x}''(t_0)$ are parallel.
   (*Hint: let* $\mathbf{x}(t) = \mathbf{y}\big(s(t)\big)$ *where* $\mathbf{y}(s)$ *has unit speed*)

8. Prove as much as you can tolerate of Corollary 1.36.

9. Suppose $\mathbf{x} : I \to \mathbb{E}^3$ is a curve lying on the surface of the unit sphere ($||\mathbf{x}|| = 1$).

   (a) If $\mathbf{x}$ has unit speed, show that $\mathbf{x}'' \cdot \mathbf{x} = -1$.

   (b) Hence or otherwise, prove that the curvature of $\mathbf{x}$ is at least 1 everywhere.
   (*Hint:* $\mathbf{x}$ *and* $\mathbf{x}'$ *are orthonormal...*)

   (c) What happens if $\mathbf{x}$ lies on the surface of the sphere $||\mathbf{x}|| = r$ of radius $r > 0$?

   (d) (Hard) If a *unit-speed* curve lies on a sphere of radius r, show that

$$\tau^2(r^2\kappa^2 - 1) = (\kappa')^2$$

   (*Hint: compute the coefficients of* $\mathbf{x}$ *with respect to the Frenet frame*)

10. (Hard) Let $d(t) > 0$. Suppose $\mathbf{x}(t)$ and $\mathbf{y}(t) = \mathbf{x}(t) + d\mathbf{N}(t)$ are unit-speed curves such that the principal normal vector field $\mathbf{N}$ of $\mathbf{x}$ is the translate[a] of the binormal vector field $\hat{\mathbf{B}}$ of $\mathbf{y}$.

    Prove that the distance d between corresponding points of the curves is *constant*. Prove also that the curvature and torsion of $\mathbf{x}$ satisfy $2\kappa = d(\kappa^2 + \tau^2)$.

    (*Hint: Compute* $\hat{\mathbf{T}}$ *and take dot products with something useful...*)

[a] That is, the directional parts of $\mathbf{N}, \hat{\mathbf{B}}$ are identical: of course these are members of different tangent spaces.

1.5 The Fundamental Theorem of Biregular Spacecurves

Our goal for this section is to see that curvature and torsion determine a spacecurve uniquely up to rigid motions. We do this by recognizing the Frenet–Serret equations satisfied by the Frenet frame as a system of ordinary differential equations; provided sufficient initial conditions (starting point and orientation), the usual existence and uniqueness theorem for initial value problems can be invoked to show that there is a unique curve with this data.

As a precursor to this, we first consider how to interpret curvature and torsion, and how they change (or don't!) under rigid motions of a curve.

Theorem 1.37. 1. *A regular spacecurve has $\kappa \equiv 0$ if and only if it is a straight line.*

2. *A biregular spacecurve has $\tau \equiv 0$ if and only if it is contained in a fixed plane (the unmoving osculating plane of the curve).*

Proof. In both cases, we assume, without loss of generality, that $\mathbf{x}(s)$ is a unit-speed parametrization of our spacecurve.

1. $\kappa(s) = ||\mathbf{x}''(s)|| = 0 \iff \mathbf{x}''(s) = \mathbf{0}$. Thus $\mathbf{x}$ is a straight line.

2. ($\Leftarrow$) Suppose the curve lies in a fixed plane. Then $\mathbf{x}'$ and $\mathbf{x}''$ are parallel to this plane, whence $\mathbf{T}$ and $\mathbf{N}$ are also. But then $\mathbf{B}$ is a continuous unit vector orthogonal to the plane and is therefore *constant*. From the Frenet equations, $-\tau\mathbf{N} = \mathbf{B}' = \mathbf{0} \implies \tau \equiv 0$.

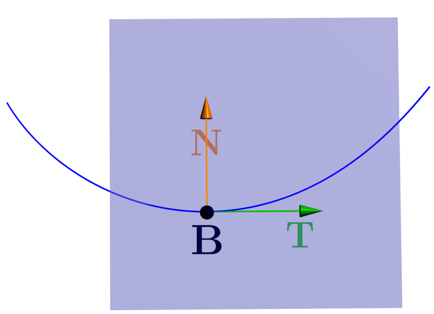
   ($\Rightarrow$) As above, if $\tau \equiv 0$, then $\mathbf{B}$ is constant. But then

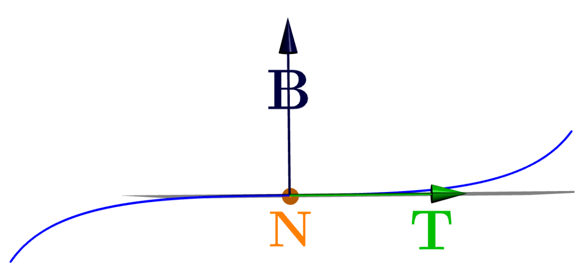
   $$(\mathbf{x} \cdot \mathbf{B})' = \mathbf{x}' \cdot \mathbf{B} + \mathbf{x} \cdot \mathbf{B}' = \mathbf{T} \cdot \mathbf{B} = 0$$

   from which $\mathbf{x} \cdot \mathbf{B}$ is constant. The curve therefore lies in a fixed plane perpendicular to $\mathbf{B}$. ■

Curvature measures the deviation of a curve from a straight line; its *bending*. Torsion measures how badly a curve fails to be planar; its *twisting*.

To visualize the difference, the pictures below show a segment of a standard helix. In the first we see the osculating plane; the non-zero curvature is clearly visible. In the second we look along the principal normal vector $\mathbf{N}$ and across the osculating plane; the positive torsion ($\tau = \frac{1}{2}$) indicates that the curve crosses the plane similarly to how the cubic function $y = x^3$ crosses the x-axis. The full 3D curve is linked via either picture.





Theorem 1.38. *Under an isometry $\hat{\mathbf{x}} := A\mathbf{x} + \mathbf{b}$ (recall Definition 1.26), the curvature and torsion of a biregular spacecurve transform as follows:*

Direct isometry/rigid motion: $\hat{\kappa} = \kappa, \quad \hat{\tau} = \tau.$

Indirect isometry: $\hat{\kappa} = \kappa, \quad \hat{\tau} = -\tau.$

Proof. Suppose $\mathbf{x}(s)$ has unit-speed. We relate the Frenet frame $(\hat{\mathbf{T}}\ \hat{\mathbf{N}}\ \hat{\mathbf{B}})$ of $\hat{\mathbf{x}}$ to the original.[10] Since orthogonal matrices preserve the dot product (Lemma 1.25), $\hat{\mathbf{x}}$ has unit-speed also:

$$\hat{\mathbf{x}}'(s) = A\mathbf{x}'(s) \implies \hat{v}(s) = \left|\left|\hat{\mathbf{x}}'(s)\right|\right| = \left|\left|\mathbf{x}'(s)\right|\right| = 1 \implies \hat{\mathbf{T}} = A\mathbf{T}$$

Moreover, since A is constant and both $\hat{\mathbf{N}}$ and $\mathbf{N}$ have unit length,

$$\frac{1}{\hat{\kappa}}\hat{\mathbf{N}} = \hat{\mathbf{T}}' = A\mathbf{T}' = \frac{1}{\kappa}A\mathbf{N} \implies \hat{\kappa} = \kappa \quad \text{and} \quad \hat{\mathbf{N}} = A\mathbf{N}$$

Curvature is therefore invariant under any isometry. Since A preserves angles, $A\mathbf{B}$ is perpendicular to both $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$, and so $A\mathbf{B} = \pm\hat{\mathbf{B}}$. Since the Frenet frame $E = (\mathbf{T}\ \mathbf{N}\ \mathbf{B})$ is a special orthogonal matrix, AE is also orthogonal, and moreover

$$\det AE = \det A \det E = \det A$$

We conclude that $\det A = 1 \iff AE = (A\mathbf{T}\ A\mathbf{N}\ A\mathbf{B}) = (\hat{\mathbf{T}}\ \hat{\mathbf{N}}\ A\mathbf{B})$ is *positively oriented*, whence

$$\hat{\mathbf{B}} = (\det A)A\mathbf{B} = \begin{cases} A\mathbf{B} & \text{if the isometry is direct,} \\ -A\mathbf{B} & \text{if the isometry is indirect.} \end{cases}$$

Finally, we compute the torsion:

$$\hat{\tau} = \hat{\mathbf{N}}' \cdot \hat{\mathbf{B}} = A\mathbf{N}' \cdot \big((\det A)(A\mathbf{B})\big) = (\det A)(A\mathbf{N}') \cdot (A\mathbf{B}) = (\det A)\mathbf{N}' \cdot \mathbf{B} = (\det A)\tau$$

■

Existence and Uniqueness of Solutions to ODEs

Our classification of spacecurves depends on the 'usual' existence and uniqueness result for ODEs. Here is a version suitable for our needs.

Theorem 1.39 (Existence/Uniqueness for Linear Equations (Picard, Lindelöf, etc.)). *Let $t_0 \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^n$ be given, and let $A(t) \in M_n(\mathbb{R})$ be a continuous matrix-valued function defined on an interval $|t - t_0| \le T$. Then the initial value problem*

$$\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} = A(t)\mathbf{E}, \qquad \mathbf{E}(t_0) = \mathbf{c}$$

has a unique solution $\mathbf{E} : [t_0 - T, t_0 + T] \to \mathbb{R}^n$.

[10] Recall Exercise 1.3.9: when we write $\hat{\mathbf{T}} = A\mathbf{T}$ we mean that the *directional parts* of the tangent vectors are thus related.

The rough idea of the proof is to define a sequence of functions

$$\mathbf{E}_0(t) := \mathbf{c}, \qquad \mathbf{E}_1(t) := \mathbf{c} + \int_{t_0}^t A(u)\mathbf{E}_0(u)\,\mathrm{d}u, \qquad \mathbf{E}_2(t) := \mathbf{c} + \int_{t_0}^t A(u)\mathbf{E}_1(u)\,\mathrm{d}u, \ \ldots$$

which are seen to converge to the required solution; this last step requires some ideas from topology/analysis and is beyond this course. A simple example should convince you of the approach.

Example 1.40. Given the initial value problem $\frac{\mathrm{d}E}{\mathrm{d}t} = 2tE$, $E(0) = 1$, we obtain

$$E_0(t) = 1, \quad E_1(t) = 1 + \int_0^t 2u\,\mathrm{d}u = 1 + t^2, \quad E_2(t) = 1 + \int_0^t 2u(1+u^2)\,\mathrm{d}u = 1 + t^2 + \frac{1}{2}t^4, \ldots$$

The *Picard iteration* builds up the correct solution as a power series

$$E(t) = e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots$$

Corollary 1.41. *Let $\mathcal{O}$ be an orthogonal matrix, $I = [t_0 - T, t_0 + T]$ an interval, and $W : I \to M_3(\mathbb{R})$ a matrix-valued function such that each $W(t)$ is* skew-symmetric. *Then:*

1. *There exists a unique solution $E : I \to \mathrm{O}_3(\mathbb{R})$ to the initial value problem*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = EW, \qquad E(t_0) = \mathcal{O}$$

2. *If $\det \mathcal{O} = 1$, then $E : I \to \mathrm{SO}_3(\mathbb{R})$.*

Proof.
1. The initial value problem consists of a system of nine linear first order ODEs in the entries of the 3×3 matrix E. We are therefore in the case of Picard's theorem where $\mathbf{E} : I \to \mathbb{R}^9$. There therefore exists a unique solution $E : I \to M_3(\mathbb{R})$. Now differentiate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(EE^T) &= E'E^T + E(E')^T = EWE^T + E(EW)^T = EWE^T + EW^TE^T \\ &= EWE^T + E(-W)E^T = 0 \qquad (W^T = -W!)\end{aligned}$$

   Thus EE^T is constant. However $E(t_0)E(t_0)^T = \mathrm{I}$ since $E(t_0) = \mathcal{O}$ is orthogonal. We conclude that $E(t)$ is always orthogonal.

2. Determinant is continuous (it is a polynomial!); E is differentiable, and so $\det E : I \to \mathbb{R}$ is continuous on an interval. But $\det E = \pm 1$ since E is orthogonal. It follows that $\det E$ is the constant 1.

■

For simple W, we might be able to state the solution using the matrix exponential; for instance

$$W \text{ constant} \implies E(t) = \mathcal{O}e^{tW}$$

This is of limited utility: the matrix exponential is rarely computable except as an infinite series, and the approach fails for general $W(t)$.

Corollary 1.42 (Fundamental theorem of biregular spacecurves).
Suppose we are given the following data:

- *Smooth functions* $\kappa > 0$ *and* τ *on an interval* $I = [t_0 - T, t_0 + T]$.
- *A position vector* $\mathbf{c} \in \mathbb{E}^3$ *and a positively oriented orthonormal basis* $(\mathbf{T}_0\ \mathbf{N}_0\ \mathbf{B}_0)$ *of* $T_{\mathbf{c}}\mathbb{E}^3$.

Then there exists a unique unit-speed biregular spacecurve $\mathbf{x} : I \to \mathbb{E}^3$ *with curvature* κ*, torsion* τ*, initial position* $\mathbf{x}(t_0) = \mathbf{c}$ *and Frenet frame* $E(t_0) = (\mathbf{T}_0\ \mathbf{N}_0\ \mathbf{B}_0)$ *at* $\mathbf{x}(t_0)$.

Proof. The structure equations $E' = EW$ put us in the situation of Corollary 1.41; there exists a unique solution $E = (\mathbf{T}\ \mathbf{N}\ \mathbf{B}) : I \to \mathrm{SO}_3(\mathbb{R})$. Integrate the unit tangent vector field to finish:

$$\mathbf{x}(t) = \mathbf{c} + \int_{t_0}^{t} \mathbf{T}(u)\, \mathrm{d}u$$

This is plainly the unique curve with the required initial conditions, curvature and torsion. ∎

Alternatively, a biregular curve is determined up to rigid motions by its curvature and torsion.

Corollary 1.43. *Given two biregular spacecurves with the same curvature and torsion functions, there exists a unique direct isometry transforming one to the other.*

Proof. Suppose $\mathbf{x}_1 : I \to \mathbb{E}^3$ and $\mathbf{x}_2 : I \to \mathbb{E}^3$ have Frenet frames E_1, E_2, and the same curvature and torsion functions. Choose some (any!) $t_0 \in I$. The required rigid motion $S : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ must satisfy the conditions at t_0, whence[11]

$$S\big(\mathbf{x}_1(t_0)\big) = \mathbf{x}_2(t_0) \quad \text{and} \quad AE_1(t_0) = E_2(t_0)$$

Plainly $A = E_2(t_0)\big(E_1(t_0)\big)^{-1}$ and $\mathbf{b} = \mathbf{x}_2(t_0) - A\mathbf{x}_1(t_0)$ provides the unique isometry S. Moreover $\det A = 1$ since both E_1 and E_2 do so also.

By Theorem 1.38, $\mathbf{x}_3 := S(\mathbf{x}_1)$ is a spacecurve with the *same* initial conditions (at t_0), curvature and torsion as $\mathbf{x}_2$. The Fundamental Theorem says that $\mathbf{x}_2 = \mathbf{x}_3 = S(\mathbf{x}_1)$. ∎

Compare what we've done to the standard acceleration/position problem, where *three scalar functions* $\mathbf{x}''(t) = \big(x''(t), y''(t), z''(t)\big)$ and *six scalar initial conditions* $\mathbf{x}(t_0)$ and $\mathbf{x}'(t_0)$ recover the motion by twice integrating.

The Fundamental Theorem tells us that a spacecurve is determined uniquely by *three scalar functions* $v(t), \kappa(t), \tau(t)$ and the *initial conditions* $\mathbf{x}(t_0), \mathbf{T}(t_0), \mathbf{N}(t_0)$, which also amount to *six* scalar constants.[12]

One benefit of our result is that, by standardizing $v(t) \equiv 1$ and ignoring rigid motions, we see that the *physical shape* of a curve depends only on *two* scalar functions $\kappa(t)$ and $\tau(t)$.

[11] As in Theorem 1.38, S acts on *position vectors* but Frenet frames consist of *tangent vectors* and thus only see A.

[12] You don't need explicitly to specify $\mathbf{B}(t_0) = \mathbf{T}(t_0) \times \mathbf{N}(t_0)$! The position $\mathbf{x}(t_0)$ requires three constants; $\mathbf{T}(t_0)$ needs two angles (spherical polar co-ordinates), and $\mathbf{N}(t_0)$ a single angle in the plane $(\mathbf{T}(t_0))^{\perp}$.

We finish this discussion with a quick application of the Fundamental Theorem.

Corollary 1.44. *Every biregular curve with κ and τ constant is a circular helix (circle if $\tau \equiv 0$).*

Proof. By (Exercise 1.4.2), the circular helix $\mathbf{x}(t) = \big(r\cos t, r\sin t, ht\big)$ has constant curvature $\kappa = \frac{r}{r^2+h^2}$ and torsion $\tau = \frac{h}{r^2+h^2}$.
Given constant κ, τ, it is a simple exercise to find suitable r, h. By Corollary 1.43, this is *only* such curve up to direct isometry (and constant speed re-parametrization). ∎

What changes in other dimensions?

For plane curves things are a little simpler. Here is a summary.

Assumptions $\mathbf{x} : I \to \mathbb{E}^2$ is regular; we don't need biregularity.

Frenet frame $\mathbf{T} := \frac{1}{v}\mathbf{x}'$ and $\mathbf{N} := J\mathbf{T}$ where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is rotation by 90° counter-clockwise; no differentiation is required to compute $\mathbf{N}$!

Curvature $\kappa = \frac{1}{v}\mathbf{T}' \cdot \mathbf{N}$ is *signed* as we saw in Section 1.2:

$\kappa > 0 \iff \mathbf{x}$ bends towards $\mathbf{N}$ $\qquad$ $\kappa < 0 \iff \mathbf{x}$ bends away from $\mathbf{N}$

Frenet–Serret equations In arbitrary parametrization $\begin{pmatrix} \mathbf{T}' & \mathbf{N}' \end{pmatrix} = \begin{pmatrix} \mathbf{T} & \mathbf{N} \end{pmatrix} \begin{pmatrix} 0 & -v\kappa \\ v\kappa & 0 \end{pmatrix}$

Isometries Direct isometries preserve curvature, indirect isometries change its sign.

Fundamental Theorem Given $\kappa(s)$, $\mathbf{x}(s_0) \in \mathbb{E}^2$ and $\mathbf{T}_0 \in T_{\mathbf{x}(s_0)}\mathbb{E}^2$, there exists a unique unit-speed curve with curvature $\kappa(s)$ and these initial data. In this case we can prove the Theorem in a more elementary fashion (Exercise 7).

We can also play this game in higher dimensions. Given a unit-speed curve $\mathbf{x} : I \to \mathbb{E}^n$ whose first $n-1$ derivatives at each point are linearly independent, we may apply a Gram-Schmidt orthogonalization process to obtain a moving frame $E = (\mathbf{e}_1 \ \cdots \ \mathbf{e}_n)$ and functions $\kappa_1, \ldots, \kappa_{n-1}$ (the generalized curvatures) satisfying the structure equations shown.

$$E' = E \begin{pmatrix} 0 & -\kappa_1 & 0 & \cdots & 0 & 0 \\ \kappa_1 & 0 & -\kappa_2 & & 0 & 0 \\ 0 & \kappa_2 & 0 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & -\kappa_{n-1} \\ 0 & 0 & 0 & \cdots & \kappa_{n-1} & 0 \end{pmatrix}$$

Conversely, the $n-1$ generalized curvatures determine the curve up to rigid motions.

Exercises 1.5. 1. Find an explicitly parametrized curve with constant curvature κ and torsion τ.

2. Reflection in the xy-plane $S(\mathbf{x}) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)\mathbf{x}$ is an indirect isometry. Explicitly compare the curvature and torsion of the standard helix $\mathbf{x}(t) = (\cos t, \sin t, t)$ with those of $S(\mathbf{x})$.

3. In the manner of Example 1.40, compute the Picard iteration process up to $\mathbf{E}_3(t)$ for the initial value problem

$$\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{E}, \quad \mathbf{E}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Verify that this comports with the correct solution $\mathbf{E}(t) = \left(\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix}\right)$to this system of ODEs.

4. Suppose f is a function such that $\mathbf{x}(t) = (\cos t, \sin t, f(t))$ lies in a fixed plane. Show that f satisfies the 3rd-order linear ODE $f'''(t) + f'(t) = 0$ and thus find all possible functions f.

   (*Hints: What is the torsion of a plane curve?*)

5. Assume that all principal normals of a biregular curve in $\mathbb{E}^3$ pass through a fixed point: $\exists\alpha(t)$ and a constant $\mathbf{n}$ such that $\mathbf{x}(t) + \alpha(t)\mathbf{N}(t) = \mathbf{n}$. Show that the curve is (part of) a *circle*.

6. Let $\mathbf{x} : I \to \mathbb{E}^2$ be a regular curve and let $\mathbf{y} = S(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ be a new curve resulting from a rigid motion. Prove that the curvatures of $\mathbf{x}$ and $\mathbf{y}$ are identical.

7. For regular curves in $\mathbb{E}^2$, the Fundamental Theorem is relatively simple to prove.

   (a) Suppose you are given a smooth function $\kappa : I \to \mathbb{R}$ on an interval I containing t_0, an initial position $\mathbf{x}(t_0) = (a, b)$ and an initial direction $\theta(t_0) = \theta_0$ (angle with positive x-axis).

   Use the Fundamental Theorem of Calculus to describe the unique unit-speed curve $\mathbf{x} : I \to \mathbb{E}^2$ with curvature κ and given initial data.

   (*Hints: use* $\theta(t) := \theta_0 + \int_{t_0}^{t} \kappa(u)\,\mathrm{d}u$ *to define* $\mathbf{T}(t)$ *and integrate! Your answer will contain definite integrals.*)

   (b) Suppose $\mathbf{x} : \mathbb{R} \to \mathbb{E}^2$ is unit-speed with $\kappa(t) = \frac{1}{1+t^2}$, $\mathbf{x}(0) = (0, 0)$, and $\mathbf{x}'(0) = (1, 0)$. Find $\mathbf{x}(t)$.

8. (Hard) A *cylindrical helix* is a curve $\mathbf{x}(t)$ whose unit tangent field $\mathbf{T}(t)$ makes a constant angle $\theta \in (0, \frac{\pi}{2})$ with a fixed vector $\mathbf{n}$.

   (a) If $\mathbf{x}(t) = (\cos t, \sin t, t)$ is the standard circular helix, describe a suitable vector $\mathbf{n}$.

   (b) Use the Frenet–Serret formulas to prove that a (unit-speed) non-planar curve is a cylindrical helix if and only if κ/τ is constant.

9. (Very hard) Suppose a moving frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ has structure equations where all three functions w_{12}, w_{13}, w_{23} are constant. Find the moving frame $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ where $\mathbf{f}_1 = \mathbf{e}_1$ such that $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ is the Frenet frame of a unit-speed circular helix. Calculate the curvature κ and torsion τ of this helix in terms of w_{12}, w_{13}, w_{23}. Can you find an orthogonal matrix A such that

$$A^{-1}\begin{pmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}?$$

1.6 Radii of curvature

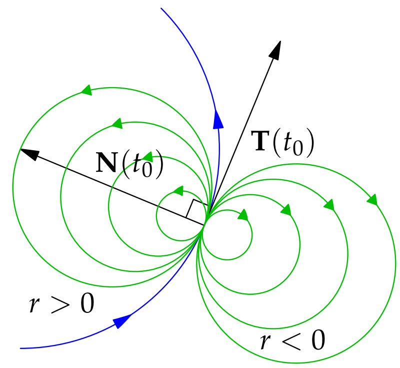
We have seen how curvature measures the deviation of a curve from a straight line and that the only planar curves with constant curvature κ are circles of radius $\frac{1}{\kappa}$. We could have started with this as our definition; at a given point, a curve has curvature κ if the circle which best approximates the curve has radius $\frac{1}{\kappa}$. Of course, we have to define what is meant by *best approximation.*

Definition 1.45. Unit-speed curves $\mathbf{x}, \mathbf{y}$ have n^{th} *order contact* at an intersection point $\mathbf{x}(t_0) = \mathbf{y}(s_0)$, if their first n derivatives agree there: $\mathbf{x}^{(j)}(t_0) = \mathbf{y}^{(j)}(s_0)$ for all $1 \leq j \leq n$.

Let $\mathbf{x}(t)$ be a unit-speed curve in $\mathbb{E}^2$, fix $r \neq 0$ and consider the unit-speed circle $\mathbf{c}(s)$ with (signed) radius r for which

$$\mathbf{c}(0) = \mathbf{x}(t_0) \quad \text{and} \quad \mathbf{c}'(0) = \mathbf{x}'(t_0)$$

We take $r > 0 \iff$ the circle lies on the same side of the curve as the principal normal vector $\mathbf{N}$.



The circle is straightforward to parametrize:

$$\mathbf{c}(s) = \underbrace{\mathbf{x}(t_0) + r\mathbf{N}(t_0)}_{\text{center}} + \underbrace{r\sin(s/r)\mathbf{T}(t_0) - r\cos(s/r)\mathbf{N}(t_0)}_{\text{rotation}}$$

Certainly this circle has 1st-order contact with the curve: $\mathbf{c}(0) = \mathbf{x}(t_0)$ and

$$\mathbf{c}'(s) = \cos(s/r)\mathbf{T}(t_0) + \sin(s/r)\mathbf{N}(t_0) \implies \mathbf{c}'(0) = \mathbf{T}(t_0) = \mathbf{x}'(t_0)$$

Moreover,

$$\mathbf{c}''(s) = -\frac{1}{r}\sin(s/r)\mathbf{T}(t_0) + \frac{1}{r}\cos(s/r)\mathbf{N}(t_0) \implies \mathbf{c}''(0) = \frac{1}{r}\mathbf{N}(t_0)$$

The circle has second-order contact with the curve if and only if

$$\mathbf{c}''(0) = \mathbf{x}''(t_0) \iff \frac{1}{r} = \kappa(t_0)$$

There is nothing stopping us from finding this circle for an arbitrary speed regular curve, since all we need is the curvature and the Frenet frame at the relevant point.

Definition 1.46. Let $\mathbf{x}(t)$ be a regular curve. At a point $\mathbf{x}(t_0)$ with non-zero curvature:

- The *radius of curvature* is $r = \frac{1}{\kappa(t_0)}$.
- The *center of curvature* is the point with position vector $\mathbf{x}(t_0) + r\mathbf{N}(t_0)$.
- The *osculating circle* is the radius r circle centered at the center of curvature. It has unit-speed parametrization

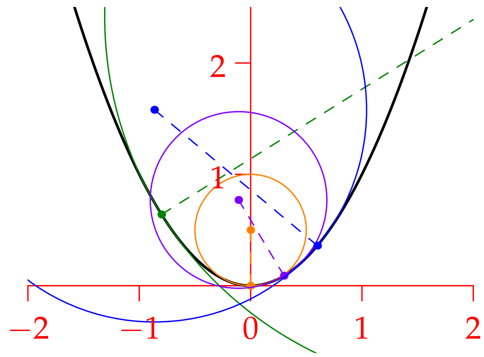
$$\mathbf{c}(s) = \mathbf{x}(t_0) + \frac{1}{\kappa(t_0)}\big(\sin(s/r)\mathbf{T}(t_0) + (1 - \cos(s/r))\mathbf{N}(t_0)\big)$$

Osculating means 'kissing.' If $\kappa(t_0) = 0$, some consider the tangent line to be an osculating circle with infinite radius!

Example 1.47. We find the osculating circles for the parabola $y = x^2$ parametrized in the obvious manner $\mathbf{x}(t) = (t, t^2)$. The relevant ingredients are

$$\mathbf{x}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \implies \mathbf{T}(t) = \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} 1 \\ 2t \end{pmatrix} \qquad \mathbf{N}(t) = \frac{1}{\sqrt{1+4t^2}} \begin{pmatrix} -2t \\ 1 \end{pmatrix}$$

$$\mathbf{x}''(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \kappa(t) = \frac{2}{(1+4t^2)^{3/2}}$$



The center of curvature when $t = t_0$ has position vector

$$\mathbf{x}(t_0) + \frac{1}{\kappa(t_0)}\mathbf{N}(t_0) = \begin{pmatrix} -4t_0^3 \\ \frac{1}{2} + 3t_0^2 \end{pmatrix}$$

Several osculating circle are drawn and their centers of curvature indicated.

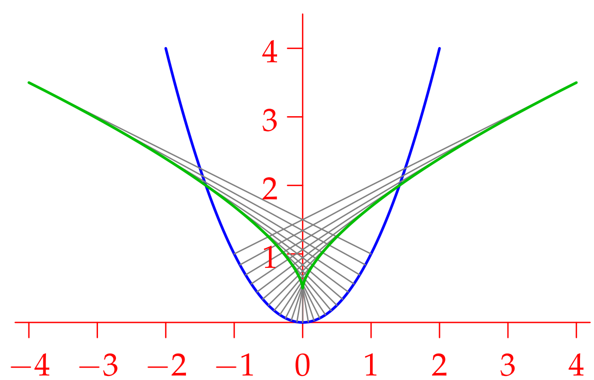
The above suggests that the centers of curvature form an interesting curve.

Definition 1.48. Let $\mathbf{x}(t)$ be a regular plane curve with non-zero curvature. The curve $\mathbf{e}(t)$ defined by the centers of curvature is the *evolute* of $\mathbf{x}(t)$:

$$\mathbf{e}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$$

Example (1.47 cont). The evolute of the parabola $\mathbf{x}(t) = (t, t^2)$ was found above:

$$\mathbf{e}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = \begin{pmatrix} -4t^3 \\ \frac{1}{2} + 3t^2 \end{pmatrix}$$



Alternatively, this is the graph $y = \frac{1}{2} + 3\left(\frac{x}{4}\right)^{2/3}$: notice that this isn't regular at $x = 0$.

The picture now animates to show the osculating circles and the construction of the evolute.

The gray lines are the *normal lines* to the parabola, and are also *tangent* to the evolute.

$$\mathbf{e}' = \mathbf{x}' - \frac{\kappa'}{\kappa^2}\mathbf{N} + \frac{1}{\kappa}(-v\kappa\mathbf{T}) = -\frac{\kappa'}{\kappa^2}\mathbf{N}$$

This last means that the evolute is a *focal curve* for the family of normal lines. The same equation shows that the evolute is regular precisely when $\kappa'(t) \neq 0$.

A related notion is the *involute*, which may be imagined by rolling a line along a curve and seeing what curve the end of the line traces out.

Definition 1.49. Suppose $\mathbf{x}(t)$ has unit speed. Its *involute* is the curve

$$\mathbf{i}(t) := \mathbf{x}(t) - t\mathbf{x}'(t) = \mathbf{x}(t) - t\mathbf{T}(t)$$

An involute depends crucially on its parametrization: it intersects its source curve when $t = 0$.

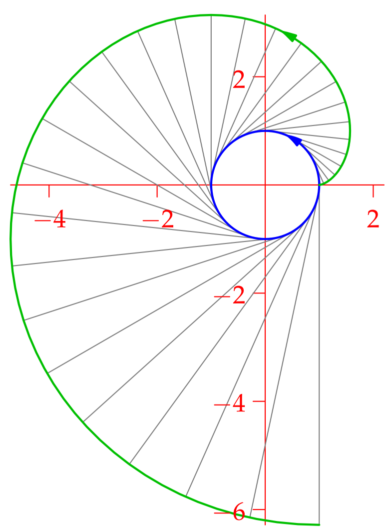
Examples 1.50. 1. The unit speed unit circle $\mathbf{x}(t) = \big(\cos t, \sin t\big)$. Its involute is therefore

$$\mathbf{i}(t) = \mathbf{x}(t) - t\mathbf{T}(t) = \begin{pmatrix} \cos t + t\sin t \\ \sin t - t\cos t \end{pmatrix}$$

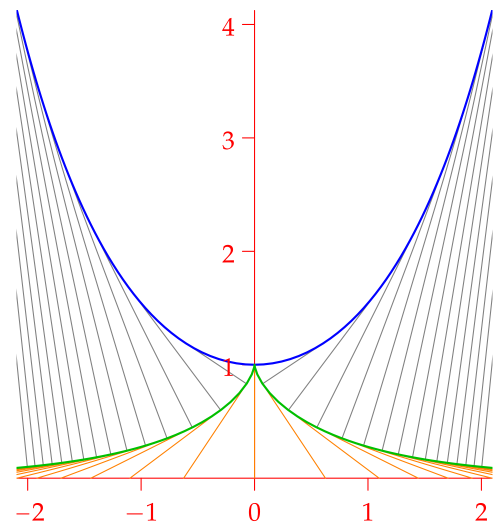
2. The involute of the unit speed *catenary* $\mathbf{x}(t) = \big(\sinh^{-1} t, \sqrt{1+t^2}\big)$ is the *tractrix*:

$$\mathbf{i}(t) = \begin{pmatrix} \sinh^{-1} t - t(1+t^2)^{-1/2} \\ (1+t^2)^{-1/2} \end{pmatrix}$$

This is the curve obtained when an object starting at the point $(0,1)$ is dragged (subjected to *traction*) by attaching a rope of length 1 to a vehicle moving along the x-axis.



Circle and involute (spiral)



Catenary and involute (tractrix)

Another way to visualize the involute of the catenary is to imagine attaching a weight at $(0,1)$ to a long string wrapped tightly along the catenary and then releasing the weight. Similarly, imagine a string is wound tightly around the circle and then uncoiled; the result is the involute.

Theorem 1.51. *The evolute of any involute is the original curve, except where $t = 0$ or $\kappa = 0$.*

We leave the argument as an exercise. The reverse process fails, as an observation of the parabola example should convince you: remember that an involute intersects its source curve at $t = 0$...

Exercises 1.6. 1. Find the center of curvature for the curve $\mathbf{x}(t) = (1 - t^{-1}, 1 + t)$ at $t = 1$.

2. Consider the ellipse $\mathbf{x}(t) = \big(a\cos t, b\sin t\big)$ where $a > b > 0$.

   (a) Compute the curvature of the ellipse.

   (b) Show that its evolute is the *astroid* $\mathbf{e}(t) = (a^2 - b^2)\left(\begin{smallmatrix} a^{-1}\cos^3 t \\ -b^{-1}\sin^3 t \end{smallmatrix}\right)$

   (c) The four-vertex theorem states that a simple closed plane curve with differentiable curvature has at least four points where $\kappa' = 0$. Show that the ellipse has precisely four.

3. Describe the involutes of a straight line.

   (*Hint: this is a trick question!*)

4. In Example 1.50.2 we constructed the tractrix as the involute of the catenary.

   (a) Use $\sinh^{-1} t = \ln\big(t + \sqrt{1+t^2}\big)$ to verify that $\mathbf{x}(t)$ has unit speed and thus confirm the derivation of $\mathbf{i}(t)$.

   (b) Compute the tangent line to the tractrix when $t > 0$ and show that this line cuts the x-axis a distance 1 from the curve, thus justifying the *traction* claim.

5. Suppose that the graph of a smooth function $y = f(x)$ passes horizontally through the origin: $f(0) = 0 = f'(0)$. Show that its Maclaurin series is

   $$f(x) \approx \frac{1}{2}\kappa(0)x^2 + \text{higher order terms}$$

   Use this to *quickly* state the curvature at $x = 0$ of the graph of $y = x^2(7x^2 - 29)$.

6. Let $\mathbf{x}(t)$ be unit speed with non-zero curvature κ and Frenet frame $\{\mathbf{T}, \mathbf{N}\}$. Moreover, let $\mathbf{i}(t) = \mathbf{x}(t) - t\mathbf{T}(t)$ be an involute and denote the speed and corresponding data for the involute $\hat{v}, \hat{\kappa}, \hat{\mathbf{T}}, \hat{\mathbf{N}}$. For simplicity, suppose $\kappa, t > 0$.

   (a) Compute the Frenet frame of $\mathbf{i}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$.

   (b) Show that $\hat{\kappa}(t) = \frac{1}{t}$.

   (c) Show that the evolute of $\mathbf{i}(t)$ is the original curve $\mathbf{x}(t)$.

7. We see how an involute of the evolute fails to recover the original curve.

   Let $\mathbf{x}(t)$ be regular with non-zero curvature, $\kappa'(t) \neq 0$, and evolute $\mathbf{e}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$. Since $\mathbf{e}(t)$ is regular, we may assume it is parametrized by arc-length.

   (a) If $\kappa' > 0$, explain why $\kappa' = \kappa^2$.

   (b) Show that the natural involute of the evolute is

   $$\mathbf{e}(t) - t\mathbf{e}'(t) = \mathbf{x}(t) + \frac{1}{\kappa(0)}\mathbf{N}(t)$$

   that is, the original curve shifted a constant distance $\frac{1}{\kappa(0)}$ in its normal direction.

   (*Hint: the ODE in part (a) is separable*)

   (c) Find the involute of the evolute of the parabola $\mathbf{x}(t) = (t, t^2)$.