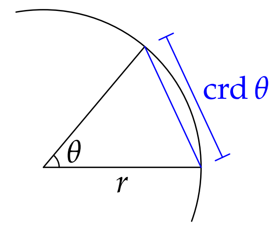
2 Trigonometric Functions and Polar Co-ordinates

In this chapter we review trigonometry and periodic functions and discuss their relation to polar co-ordinates. Some of this will be non-standard.

2.1 Definitions & Measuring Angles



Trigonometric functions date back at least 2000 years. Ancient mathematicians were interested in the relationship between the *chord* of a circle and the central angle, often for the purpose of astronomical measurement. It wasn't until 1595 that the term *trigonometry* (literally *triangle measure*) was coined, and the functions were considered as coming from triangles.

Here are several related definitions of sine, cosine and tangent based either on triangles or circles.

Definition 2.1. 1. (a) Given a right triangle with *hypotenuse* (longest side) 1 and angle θ, define $\sin\theta$ and $\cos\theta$ to be the side lengths *opposite* and *adjacent* to θ.

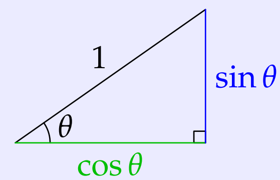
Define $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to be the slope of the hypotenuse.

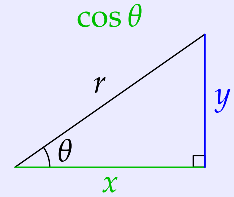
(b) Given a right triangle with angle θ, *hypotenuse* r, *adjacent* x and *opposite* y, define

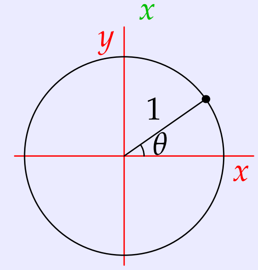
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

2. (a) $(\cos\theta, \sin\theta)$ are the co-ordinates of a point on the unit circle, where θ is its *polar angle* measured counter-clockwise from the positive x-axis. Provided $\cos\theta \neq 0$, also define $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

(b) Repeat the definition for a circle of radius r with co-ordinates $(r\cos\theta, r\sin\theta)$.







Discuss some of the advantages and weaknesses of these definitions:

- What prerequisites are you assuming in each case?
- Is it easier to think about *lengths* rather than ratios?
- Where do you need basic facts from Euclidean geometry such as *congruent/similar* triangles?
- Convince yourself that that the triangle definitions follow from the circle definitions. What is missing if you try to use the triangle definition to justify the circle version?
- If you were introducing trigonometry for the first time, what would you use?

If you've done sufficient calculus you might know of other definitions, for instance using power (Maclaurin) series. Plainly these are not suitable for grade-school, but have the great benefit of making the calculus relationship $\frac{d}{d\theta}\sin\theta = \cos\theta$ very simple. Establishing this using the triangle definition is a somewhat tricky!

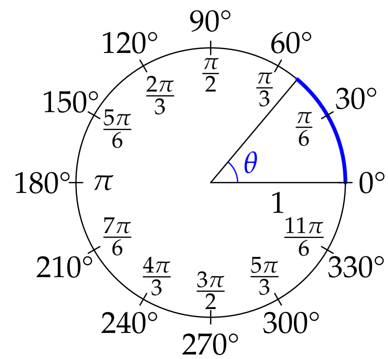
Measuring Angles

There are two standard ways to measure angles (to sensibly associate a *number* to each angle).

Degrees A full revolution has 360° and a right-angle 90°. Degree measure dates back to ancient Babylon 2–4000 years ago.[7]

Radians The radian measure of an angle is the length of the arc subtending the angle in a circle of radius 1. Since the circumference of a unit circle is 2π, we have the following identifications.

Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	n/a
180°	π	0	-1	0



In elementary mathematics, degrees are the most common way to measure angles. Do you know any other methods?

Exercises 2.1. 1. The identity $\cos^2\theta + \sin^2\theta = 1$ is the Pythagorean Theorem in disguise. Why?

2. The word *sine* is the result of a long list of translations and transliterations from an ancient Sanskrit term meaning *half-chord*. For the chord picture on page 21, how does the length of the chord $\operatorname{crd}\theta$ relate to modern trigonometric functions?

3. It is conventional not to state units when using radians since they are effectively a ratio and therefore *unitless*. Think this through: if the central angle in a circle of radius r is subtended by an arc with arc-length ℓ, what is the radian measure of the angle? What facts from Euclidean geometry justify this observation?

4. Explain how to get the values of sine and cosine in the above table.

   (*Hint: Draw some triangles and use Pythagoras*!)

5. Using the pictures, explain why we have the relations

$$\sin(\tfrac{\pi}{2} - \theta) = \cos\theta = \sin(\theta + \tfrac{\pi}{2}), \qquad \sin(-\theta) = -\sin\theta, \qquad \sin(\pi - \theta) = \sin\theta$$

   (*You cannot use multiple-angle formulæ for this!*)

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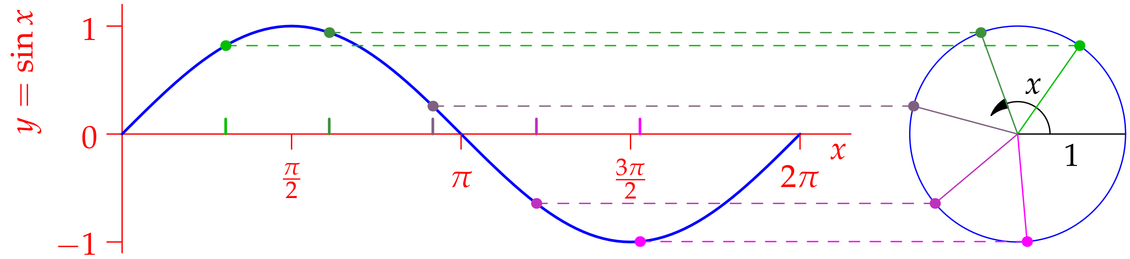
[7] It is not known why they chose 360, but it fits nicely with their *base-60* system of counting (decimals are base-10). The traditional subdivisions of a degree are also base-60. For instance, 34°12′45″ is 34 degrees, 12 (arc)minutes and 45 (arc)seconds; converted to decimal notation, this becomes

$$34°12'45'' = 34 + \frac{12}{60} + \frac{45}{60^2} = 34.2125°$$

The standard hour-minute-second measurement of time has the same origin.

2.2 Periodicity, Graphs & Inverses

One advantage of the circle definition is that it makes sketching the graphs of sine and cosine very easy. Simply draw axes next to a unit circle and transfer the heights across.

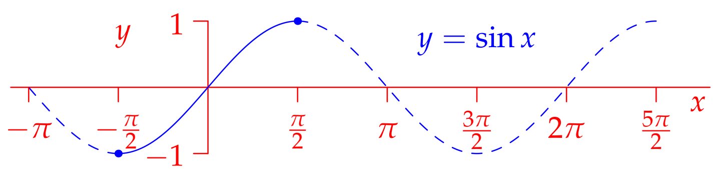


By Exercise 2.1.5, the graph of $\cos x = \sin(x + \frac{\pi}{2})$ is simply that of sine shifted $\frac{\pi}{2} = 90°$ to the left. Moreover, the circle definition allows us easily to extend trigonometric functions *periodically* since we can measure the polar angle by looping as many times round the origin as we like: for any integer n,

$$\sin(\theta + 2n\pi) = \sin\theta, \quad \cos(\theta + 2n\pi) = \cos\theta$$

Otherwise said, sine and cosine have period 2π radians (360°).

Sine and cosine are *non-invertible* unless we choose a domain on which they are 1–1.



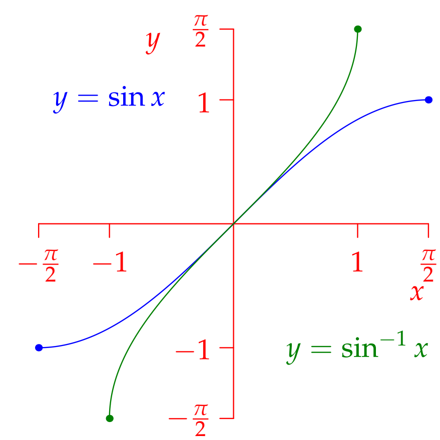
$f(x) = \sin x$ is 1–1 on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Inverse function $f^{-1}(x) = \arcsin x = \sin^{-1} x$

Domain $\mathrm{dom}(\arcsin) = [-1, 1] = \mathrm{range}(\sin)$

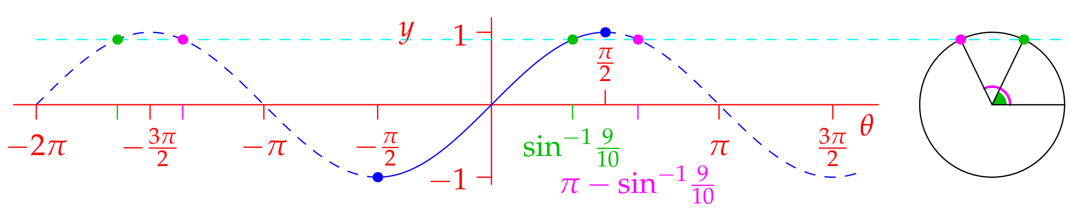
Range $\mathrm{range}(\arcsin) = [-\frac{\pi}{2}, \frac{\pi}{2}] = \mathrm{dom}(\sin)$

This is why your calculator always returns a value in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}] = [-90°, 90°]$ when you hit the $\sin^{-1}$ button.



Example 2.2. If you know the graphs, then symmetry and periodicity help you solve equations. For example, if $\sin\theta = \frac{9}{10}$ then all solutions are given by

$$\theta = \sin^{-1}\frac{9}{10} + 2\pi n \quad \text{or} \quad \pi - \sin^{-1}\frac{9}{10} + 2\pi n \qquad (n \text{ is any integer})$$



Alternatively, we could use the circle definition directly: $\sin\theta = \frac{9}{10}$ means we want angles θ corresponding to the intersections of the unit circle with the *horizontal line* $y = \frac{9}{10}$.

Periodic Models Trig functions find applications in modeling precisely because they are *periodic*. In general, a function has period T if

$$f(x+T) = f(x) \quad \text{for all } x$$

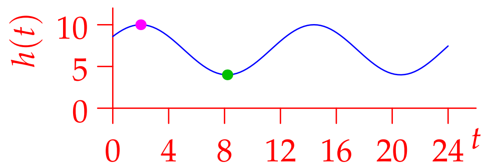
It is easy to find the period of the function $f(x) = \sin kx$ just by considering what we have to add to the input x to increase the argument kx of sine by 2π:

$$T = \frac{2\pi}{k} \implies f(x+T) = \sin(kx+2\pi) = \sin kx = f(x)$$

We may therefore obtain a simple periodic model regardless of what period is required.

Example 2.3. On a given day, high tide occurs at 2:00 with a water depth of 10 ft, whereas low tide occurs at 8:12 with a depth of 4 ft. We might model this using a periodic function with period $T = 2 \times 6\frac{12}{60} = \frac{62}{5}$ hours. For instance

$$h(t) = 7 + 3\cos\left(\frac{5\pi}{31}(t-4)\right)$$



where t is measured in hours from midnight might be suitable. In reality, tidal height is very close to being periodic, but the magnitude of the high and low tides are somewhat variable.

In fact *any* periodic function may be approximated using trigonometric functions. Indeed if $f(x)$ has period T and we define constants

$$a_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\frac{2\pi nx}{T}\,dx, \qquad b_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\frac{2\pi nx}{T}\,dx \tag{$*$}$$

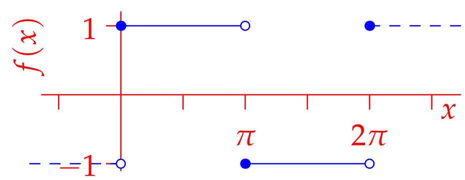
then

$$f(x) \approx \frac{a_0}{2} + a_1\cos\frac{2\pi x}{T} + b_1\sin\frac{2\pi x}{T} + a_2\cos\frac{4\pi x}{T} + b_2\sin\frac{4\pi x}{T} + \cdots \tag{$\dagger$}$$

This is the *Fourier series* of $f(x)$. It often takes only a small number of terms to obtain a very good approximation. Modern data-compression algorithms often employ Fourier series. Given a periodic function $f(x)$, one uses a computer to estimate (say) the first 100 Fourier coefficients $(*)$ and transmits these values to the receiver, who recovers an approximation to the original function using $(\dagger)$.

Example 2.4. A square-wave function with period $T = 2\pi$ is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \pi \\ -1 & \text{if } \pi \le x < 2\pi \end{cases}$$



extended periodically to the real line. With a little calculus, it is easily checked that the Fourier coefficients are

$$a_n = \int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0, \qquad b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Use a graphics tool to see how the first few terms of the series approximate the function.

Exercises 2.2. 1. $f(x) = \sin x$ is also 1–1 on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Sketch the graph of its corresponding inverse function.

2. Draw the graph for cosine and observe that it is invertible if we restrict the domain to the interval $[0, \pi]$. Draw the graph of $\cos^{-1}$.

3. Describe all solutions to the equation $\cos x = -0.2$.

4. Explain why the tangent function has period π; that is $\tan(\theta + n\pi) = \tan\theta$. What facts are we using about sine and cosine and why are they obvious from the definition?

5. Describe all solutions to the equation $\tan x = 5$.

6. Let $f(x) = \csc x = \frac{1}{\sin x}$ be the cosecant function. Describe a domain on which this function is 1–1 and sketch the graph of its inverse $y = f^{-1}(x)$.

7. Use a computer to sketch the curve

$$y = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x\right)$$

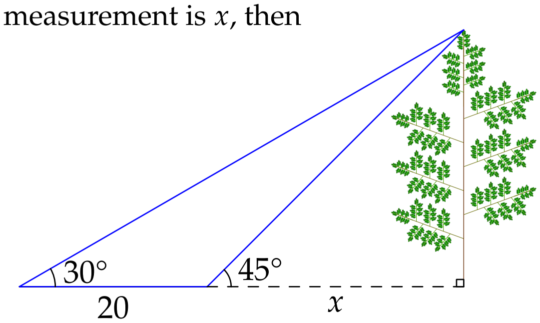
What simple periodic function do you think this is approximating?

2.3 Solving Triangles

Basic trigonometry often involves finding the edges and angles of a triangle given partial data.

Example 2.5. To find the height h of a tall tree, two angles of elevation $45°$ and $30°$ are measured a distance 20 ft apart along a straight line from the base of the trunk.

This is easily attacked by drawing a picture and observing that we have two right-triangles. If the (unknown) distance from the base of the tree to the nearer measurement is x, then



$$\frac{1}{\sqrt{3}} = \tan 30° = \frac{h}{x+20} \qquad 1 = \tan 45° = \frac{h}{x}$$

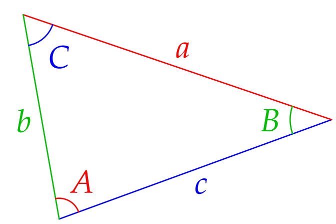
Substituting the second equation into the first returns

$$h = \frac{20}{\sqrt{3}-1} \approx 27.32\,\text{ft}$$

In fact there is enough data in the problem to recover everything about the original triangle.

- The second base angle is $(180° - 45° = 135°)$.
- The third (summit) angle is $180° - 30° - 135° = 15°$.
- Two applications of Pythagoras compute the remaining sides of the triangle

$$\sqrt{x^2+h^2} = \sqrt{2}h = \frac{20\sqrt{2}}{\sqrt{3}-1} \approx 38.64$$
$$\sqrt{h^2+(x+20)^2} = \sqrt{h^2+3h^2} = 2h = \frac{40}{\sqrt{3}-1} \approx 54.64$$

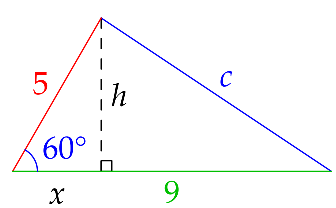


The example is just a disguised version of *solving a triangle*: computing all six sides and angles of a triangle given three of them. The Euclidean triangle congruence theorems tell us which combinations are sufficient to determine all the others. The example is the ASA congruence: angle-side-angle data (30°–20–135°) is enough to compute everything else about the triangle.

When in doubt, you can always attack basic trigonometry problems as we did in the example: create a right-triangle, then use the definitions of sin/cos/tan and/or Pythagoras.

Example 2.6. Given the SAS (side-angle-side) combination 5–60°–9, find the third side of the triangle.

The altitude h creates two right-triangles, from which



$$\begin{aligned} & h = 5\sin 60°, \quad x = 5\cos 60° \\ \implies & c^2 = (9-x)^2 + h^2 = 9^2 + (x^2+h^2) - 18x \\ & \quad = 9^2 + 5^2 - 18 \cdot 5\cos 60° = 61 \\ \implies & c = \sqrt{61} \approx 7.81 \end{aligned}$$

Since we now know c and $9 - x = \frac{13}{2}$ the remaining angles could also be easily found.

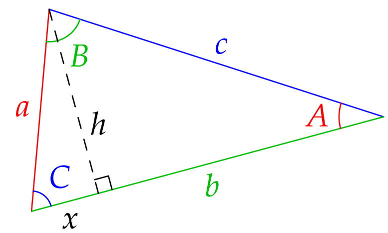
In elementary situations it is typically easier to have students drop the perpendicular as we've done. However, once comfortable with the method, it is helpful to have short-cuts which skip the need to work with the perpendicular at all.

Theorem 2.7 (Sine and Cosine Rules). *For any triangle,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

The cosine rule is just the Pythagorean Theorem with a correction for non-right triangles. Both rules follow straightforwardly by drawing an altitude as before!

Proof. Consider the picture. We have



$$h = a\sin C = c\sin A, \quad x = a\cos C, \quad b - x = c\cos A$$

The first equation rearranges to

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

Two applications of Pythagoras give the cosine rule

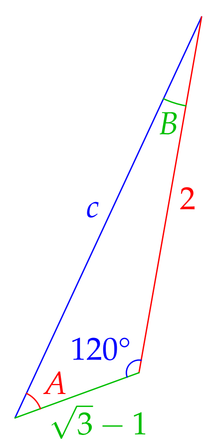
$$c^2 = h^2 + (b-x)^2 = h^2 + x^2 + b^2 - 2bx = a^2 + b^2 - 2ab\cos C$$

The remaining part of the sine rule and the other versions of the cosine rule are obtained by choosing other altitudes. ∎

Here are two examples where we use the rules instead of explicitly drawing an altitude.

Examples 2.8. 1. A triangle has sides 2 and $\sqrt{3}-1$, and the angle between them is 120°. Find the remaining sides and angles.

We apply the cosine rule with $a = 2$, $b = \sqrt{3}-1$ and $C = 120°$



$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 2^2 + (\sqrt{3}-1)^2 - 2\cdot 2(\sqrt{3}-1)\cos 120° \\ &= 4 + 3 + 1 - 2\sqrt{3} + 2(\sqrt{3}-1) = 6 \end{aligned}$$

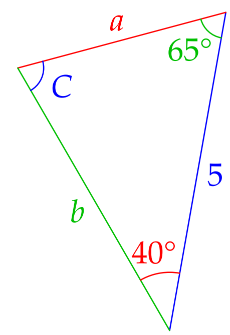
We have an opposite pair $(c, C) = (\sqrt{6}, 120°)$, so the sine rule may be used

$$\sin A = \frac{2}{\sqrt{6}}\sin 120° = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}} \implies A = 45°$$

We chose the acute angle since $A = 180° - B - C = 60° - B < 90°$.
The final angle is then $B = 180° - 45° - 120° = 15°$.

You could instead drop a perpendicular, say from the vertex A to the *extension* of the side of length 2. Think about why the perpendicular has to be *outside* the triangle...

2. A triangle has one side with length 5 and its two adjacent angles are 40° and 65°. Find the remaining data.



This time the initial data is ASA. Writing $c = 5$, $A = 40°$ and $B = 65°$, the remaining angle is plainly

$$C = 180° - 40° - 65° = 75°$$

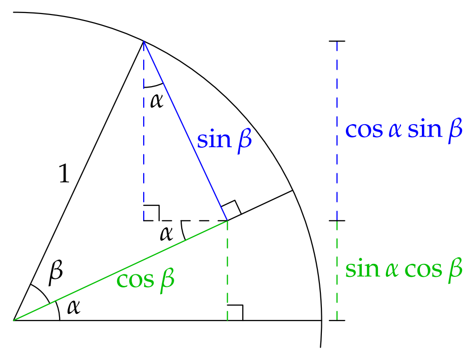
This gives us an opposite pair (c, C), so we can apply the sine rule

$$a = c\frac{\sin A}{\sin C} = 5\frac{\sin 40°}{\sin 75°} \approx 3.327$$

A second application yields

$$b = c\frac{\sin B}{\sin c} = 5\frac{\sin 65°}{\sin 75°} \approx 4.691$$

Multiple-angle Formulae



Also useful in the context of basic trigonometry is the ability to sum angles. The picture provides a simple justification of

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

at least when $0 < \alpha + \beta < \frac{\pi}{2}$. If you look carefully, you should be able to see how the same picture establishes

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Exercises 2.3. 1. Find the remaining angles in the triangle in Example 2.6; you'll need a calculator to convert to degrees.

2. The other Euclidean congruence theorems are SSS and SAA. Explain how to solve triangles using these minimal data in two ways:
   (a) By drawing an altitude.
   (b) Using the sine/cosine rules.

3. SSA isn't a triangle congruence theorem. For instance, there are *two* non-congruent triangles with data $a = 1$, $b = \sqrt{3}$ and $A = 30°$. Find them.

4. Use the multiple-angle formulae to derive the familiar expressions for $\sin 2\theta$ and $\cos 2\theta$.

5. Find the exact value of $\sin 105°$.

6. (a) Find an expression for $\tan(\alpha + \beta)$ purely in terms of $\tan\alpha$ and $\tan\beta$.
   (b) Two wooden wedges with slope $\frac{1}{4}$ are placed on top of each other to make a steeper slope. What is the gradient of the new slope?

2.4 Polar Co-ordinates

Definition 2.1 provides an alternative way to describe points in the plane. If θ is the polar angle of a point with Cartesian (rectangular) co-ordinates (x, y), then its polar-coordinates are precisely the values (r, θ) seen in the definition!
Computing $x = r\cos\theta$ and $y = r\sin\theta$ is easy given r and θ.

Example 2.9. A point with polar co-ordinates $(r, \theta) = (2, \frac{5\pi}{6})$ has Cartesian co-ordinates

$$(x, y) = \left(2\cos\tfrac{5\pi}{6}, 2\sin\tfrac{5\pi}{6}\right) = (-\sqrt{3}, 1)$$

Computing polar co-ordinates from Cartesian is harder, requiring some *visualization.*

1. Every point (x, y) has a unique radius $r = \sqrt{x^2 + y^2}$, but not polar angle. If θ is a polar angle, so is $\theta + 2\pi n$ for any integer $n \in \mathbb{Z}$. The origin $(x, y) = (0, 0)$ is even stranger; certainly $r = 0$, but *any* θ is a legitimate polar angle!

2. Whenever $x \neq 0$ (away from the y-axis),

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x}$$

however, this *doesn't* guarantee that $\theta = \tan^{-1}\frac{y}{x}$. Continuing the example shows us why...

Example (2.9, cont). If $(x, y) = (-\sqrt{3}, 1)$, then the radius is easy

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

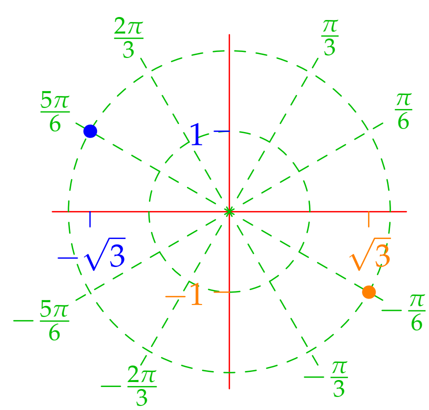
For the polar angle,

$$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right) \not\Longrightarrow \theta = -\frac{\pi}{6}$$

Arctan has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, so always returns an angle in quadrants 1 or 4. Our point is in the *second* quadrant ($x < 0 < y$) so we need to adjust, using the fact that tan is π-periodic:

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} = 150°$$

We could alternatively add any integer multiple of 2π.



The example wasn't too tricky since the polar angle was exactly computable. When you have to rely on a calculator, it is much easier to make a mistake.

Example 2.10. The point $(x, y) = (-8, -15)$ has polar co-ordinates (quadrant 3!)

$$r = \sqrt{8^2 + 15^2} = 17, \qquad \theta = \pi + \tan^{-1}\tfrac{15}{8} \approx 241.93°$$

We could summarize with formulæ describing precisely how to compute θ dependent on quadrant (the signs of x, y), though it is better to get in to the habit of drawing a picture!

Curves in Polar Co-ordinates

Polar co-ordinates are well-suited to describing curves that encircle the origin. Indeed circles centered at the origin with radius $a > 0$ have the very simple polar form $r = a$. Converting to rectangular co-ordinates recovers the the natural parametrization of a circle:

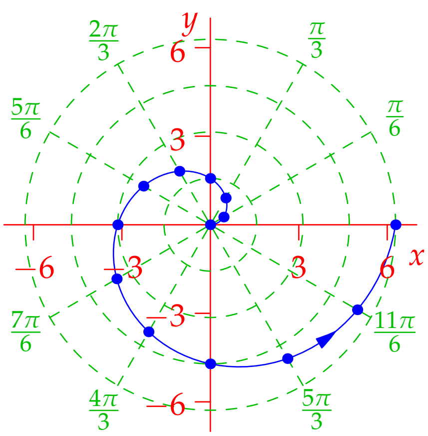
$$x(\theta) = a\cos\theta, \quad y(\theta) = a\sin\theta$$

This partly explains why mathematicians call sine and cosine *circular functions.*

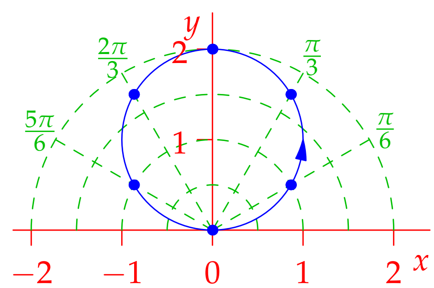
General polar graphs are harder to visualize, though the major reason is lack of familiarity. Have a little empathy: to graph *polar* functions, you'll likely have to follow the same approach as new students use to sketch *Cartesian* curves like $y = x^2$! Here are a couple of examples.

Examples 2.11. 1. The curve $r = \theta$ is relatively easy to plot since r increases at exactly the same rate as the angle; we therefore have a *spiral.*

To confirm this, plot several points (θ, θ); we've done for θ in multiples of $\frac{\pi}{6}$ (30°) from 0 to 2π. It is sensible to use 'polar graph paper' with concentric circles separated by (say) $\frac{\pi}{2} \approx 1.57$.



$r = \theta, \; 0 \le \theta \le 2\pi$



$r = 2\sin\theta, \; 0 \le \theta \le \pi$

θ	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{4\pi}{3}$	$\frac{5\pi}{6}$	π
$2\sin\theta$	1	1.73	1	1.73	1	0

2. The curve $r = 2\sin\theta$ is a little easier to work with since we know exact values for sine, assisted by $\sqrt{3} \approx 1.73$.

   This looks like a circle! To see this, multiply both sides by r and complete the square:

$$r^2 = 2r\sin\theta \implies x^2 + y^2 = 2y \implies x^2 + (y-1)^2 = 1$$

   describes the set of points with distance 1 from the point $(0, 1)$: a circle!

You should think about what happens in both examples if we extend the domain:

- What would $r = \theta$ look like if θ were allowed to be *negative*?
- What happens to $r = 2\sin\theta$ when $\theta > \pi$?

Exercises 2.4. 1. Convert the following points to polar co-ordinates.

(a) $(-5, 5)$ (b) $(3, -4)$ (c) $(-5\sqrt{3}, -15)$

(d) $(-1, \tan 3)$ (tricky—this is 3 radians!)

2. If $a > 0$, describe the curve with polar equation $r = 2a\cos\theta$.

   (*Be careful with* $\theta > \frac{\pi}{2}$ *since cosine goes negative...*)

3. The algebraic trickery in the last example sometimes bears fruit, though you have to be lucky! By multiplying both sides by $1 - \sin\theta$ and converting to rectangular co-ordinates, show that the polar function

$$r(\theta) = \frac{2a}{1 - \sin\theta}$$

   is a parabola in disguise and sketch it when $a = 1$. How does the graph depend on a?

4. Try to sketch the following curves.

   (a) $r = \theta(\theta - 4)$ (b) $r = (\theta - 1)^2 + 1$ (c) $r = (\theta - 1)^2 - 1$

   As well as plotting points directly, you should sketch the curve first on rectangular axes (e.g., (a) is $y = x(x - 4)$). What happens to (c) when $\theta = 1$?

   Once you've tried these, use a grapher to see if you're right, though see how close you can get without it!