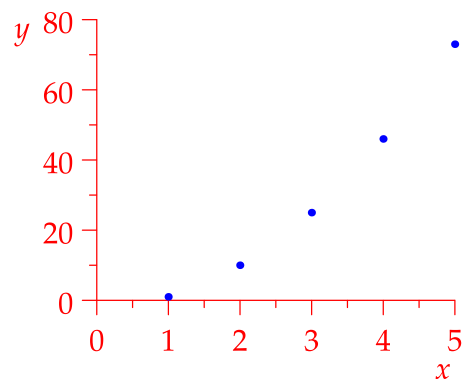
4 Sequences as Functions

We've seen many different types of function in this course and used them to model various situations. In practice, one is often faced with the opposite problem: given experimental data, what type of function should you try?

4.1 Polynomial Sequences: First, Second, and Higher Differences

To begin to answer this, first ask yourself, "What is a sequence?" Hopefully you have a decent intuitive idea already. More formally, a sequence a function whose domain is a set like the natural numbers, for example

$$f : \mathbb{N} \to \mathbb{R} : n \mapsto 3n^2 - 2$$



defines the sequence

$$\big(f(1), f(2), f(3), \ldots\big) = \big(1, 10, 25, 46, 73, \ldots\big)$$

This is indeed the intuitive idea of a function to many grade-school students: continuity and domains including fractions or even irrational numbers are more advanced concepts.

Suppose instead that all you have is a data set

x	1	2	3	4	5
y	1	10	25	46	73

perhaps arising from an experiment. Could you recover the original function $y = f(x)$ directly from this data? You could try plotting data points as we've done, though it is hard to decide directly from the plot whether we should try a quadratic model, some other power function/polynomial, or perhaps an exponential. Of course, the physical source of real-world data might also provide clues.

A more mathematical approach involves considering how data values *change*:

$$x \quad 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} 5$$

$$y \quad 1 \xrightarrow[+9]{} 10 \xrightarrow[+15]{} 25 \xrightarrow[+21]{} 46 \xrightarrow[+27]{} 73$$

$$+9 \xrightarrow[+6]{} +15 \xrightarrow[+6]{} +21 \xrightarrow[+6]{} +27$$

The first-differences in the x-values are constant whereas those for the y-values are increasing

$$(y_{n+1} - y_n) = (9, 15, 21, 27, \ldots)$$

You likely already notice the pattern: the sequence of first-differences is increasing *linearly* as the *arithmetic sequence*

$$y_{n+1} - y_n = 3 + 6n$$

To make this even clearer, note that the sequence of second-differences in the y-values is *constant* $(+6)$. These facts are huge clues that we expect a quadratic function.

But why? Well we can certainly check the following directly:

Linear Model If $f(n) = an + b$, then the sequence of first-differences is constant

$$f(n+1) - f(n) = a$$

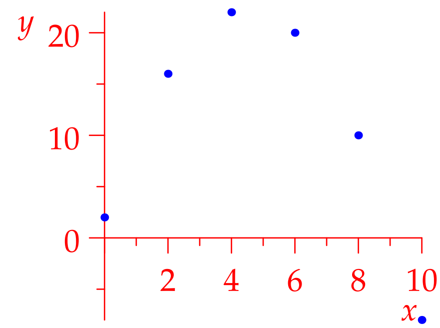
Quadratic Model If $f(n) = an^2 + bn + c$, then the sequence of first-differences is linear and the second-differences are constant:

$$g(n) := f(n+1) - f(n) = 2an + a + b, \qquad g(n+1) - g(n) = 2a$$

The relationship between these results and the *derivative(s)* of the original function $f(x)$ should feel intuitive: what happens if you differentiate a quadratic twice?

Example 4.1. You are given the following data set

x	0	2	4	6	8	10
y	2	16	22	20	10	-8



The x-values have constant first-differences while the y-values have constant second-differences

First-differences: $14, 6, -2, -10, -18$

Second-differences: $-8, -8, -8, -8$

We therefore suspect a quadratic model $y = f(n) = an^2 + bn + c$. Since the x-values are separated by 2 rather than 1, we should compute explicitly.[12]

First-differences: $f(n+2) - f(n) = a\big((n+2)^2 - n^2\big) + b\big((n+2) - n\big) = 4an + 4a + 2b$

Second-differences: $8a$

We conclude that $a = -1$, whence the first-differences have the form

$$f(n+2) - f(n) = -4n - 4 + 2b$$

Taking $n = 0$ quickly shows that

$$14 = -4 + 2b \implies b = 9 \implies f(n) = -n^2 + 9n + c = -n^2 + 9n + 2$$

where $c = 2$ was also found by considering $n = 0$.

There are at least two issues with our method:

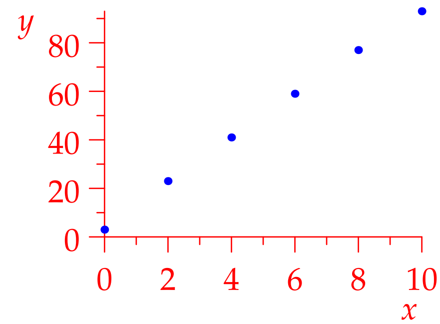
1. The question we're answering is, "Find a quadratic model satisfying given data." Think about why constant first-/second-differences cannot *guarantee* that a model must be linear/quadratic.

2. It is very unlikely that experimental data will fit such precise patterns: why not? However, if the differences are *close* to satisfying such patterns, then you should feel confident that a linear/quadratic model is a good choice.

[12] We could alternatively substitute $2x = z$ and use the above formulæ.

Example 4.2. Given the data set

x	0	2	4	6	8	10
y	3	23	41	59	77	93



with sequences of first- and second-differences

First-differences: 20, 18, 18, 18, 16

Second-differences: $-2, 0, 0, -2$

do you think a linear or quadratic model would be superior?

If you wanted a linear model, you'd likely be inclined to try $f(x) = 9x + b$ for some constant b. Here are two options:

1. $f(x) = 9x + 5$ fits the middle four data values perfectly, but as a predictor is too large at the endpoints: $f(0) = 5 > 3$ and $f(10) = 95 > 93$.

2. $f(x) = 9x + 5 - \frac{2}{3}$ doesn't pass through any of the data values but seems to reduce the net error to zero:

x	0	2	4	6	8	10
$f(x) - y$	$-\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$

$$\implies \sum_x f(x) - y = 0$$

Neither model is perfect, but then this is what you expect with real-world data!

Exercises 4.1.

1. For each data set, find a function $y = f(x)$ modelling the data.

(a)

x	2	4	6	8
y	-1	2	7	14

(b)

x	2	5	8	11	14
y	-6	-15	-6	21	66

(c)

x	0	6	9	15
y	3	15	21	33

(*Be careful with (c): the x-differences aren't constant!*)

2. Suppose a table of data values containing (x_0, y_0) has constant first-differences in both variables

$$\Delta x = x_{n+1} - x_n = a, \quad \Delta y = b$$

Find the equation of the linear function $y = f(x)$ through the data.

3. What relationship do you expect to find with the sequential differences of a cubic function $f(n) = an^3 + bn^2 + cn + d$? What about a degree-$m$ polynomial $f(n) = an^m + bn^{m-1} + \cdots$?

4. If $f(n) = an^2 + bn + c$ is a quadratic model for the data in Example 4.2 with constant second-differences -1, show that $a = -\frac{1}{8}$. What might be reasonable values for b, c?

5. (Hard) Suppose $f(x)$ is a twice-differentiable function and $h > 0$ is constant. Use the mean value theorem from calculus to explain the following.

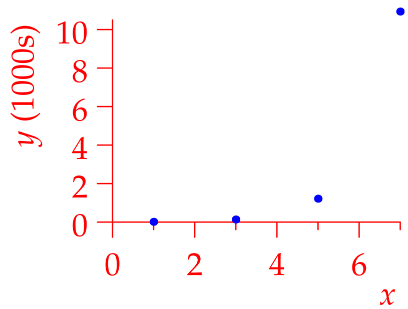
   (a) First-differences $f(x+h) - f(x)$ are proportional to $f'(\xi)$ for some $\xi \in (x, x+h)$.

   (b) Second-differences satisfy $\big(f(x+2h) - f(x+h)\big) - \big(f(x+h) - f(x)\big) = f''(\xi)h\alpha$ for some ξ between x and $x+h$ and some α. Why is it unlikely that α is constant?

4.2 Exponential, Logarithmic & Power Sequences

To observe relationships between data values, you might also have to consider *ratios* between successive terms or skip values.

Example 4.3. From a first glance at the given data, it is hard to decide whether an exponential or a quadratic (or higher degree polynomial) model is more suitable. If we try to apply the constant-difference method, we don't seem to get anything helpful:



$$\begin{array}{ccccccccc} & & +2 & & +2 & & +2 & & \\ x & 1 & \to & 3 & \to & 5 & \to & 7 & \\ y & 15 & \to & 135 & \to & 1215 & \to & 10935 & \\ & & +120 & & +1080 & & +9720 & & \\ & & & +960 & & +8640 & & & \end{array}$$

By the time we're looking at second-differences, any conclusion would be very weak since we only have two data values!

If instead we think about *ratios* of y-values, then a different pattern emerges:

$$\begin{array}{cccccccc} & & +2 & & +2 & & +2 & \\ x & 1 & \to & 3 & \to & 5 & \to & 7 \\ y & 15 & \to & 135 & \to & 1215 & \to & 10935 \\ & & \times 9 & & \times 9 & & \times 9 & \end{array}$$

The question remains: what type of function scales its output by 9 when 2 is added to its input: $f(x+2) = 9f(x)$? This is a function that converts *addition* to *multiplication*: an exponential! If we try $y = f(x) = ba^x$ for some constants a, b, then

$$f(x+2) = ba^{x+2} = ba^2b^x = a^2 f(x)$$

from which a suitable model is $y = 5 \cdot 3^x$.

We can see the pattern in the example more generally:

Exponential Model If $f(x) = ba^x$, then adding a constant to x results in

$$f(x+k) = ba^{x+k} = a^k f(x)$$

If x-values have constant differences ($+k$), then y-values will be related by a constant *ratio* ($\times a^k$). You might remember this as 'addition–product' or 'arithmetic–geometric.'

Such a simple pattern is often disguised:

- Complete data might not be given so you might have to skip some data values to see a pattern. For example, if our original data was

x	1	3	4	5	7
y	15	135	405	1215	10935

  then the x-values are not in a strictly arithmetic sequence.

- As in Example 4.2, real-world/experimental data will only *approximately* exhibit such patterns.

Example 4.4. A population of rabbits is measured every two months resulting in the data set

t	0	2	4	6	8	10
P	5	7	10	14	19	28

The data seems very close to being quadratic; consider the first and second sequences of P-differences

$$\Delta P = (2,3,4,5,9), \qquad \Delta\Delta P = (1,1,1,4)$$

However, the last difference doesn't fit the pattern. Instead, the fact that we expect an exponential model is buried in the experiment: the data is measuring population growth! We therefore instead consider the ratios of P-values:

$$\begin{array}{cccccccccccc} & & {\scriptstyle +2} & & {\scriptstyle +2} & & {\scriptstyle +2} & & {\scriptstyle +2} & & {\scriptstyle +2} & \\ t & 0 & \curvearrowright & 2 & \curvearrowright & 4 & \curvearrowright & 6 & \curvearrowright & 8 & \curvearrowright & 10 \\ P & 5 & \rightarrow & 7 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 19 & \rightarrow & 28 \\ & & {\scriptstyle \times 1.4} & & {\scriptstyle \times 1.43} & & {\scriptstyle \times 1.4} & & {\scriptstyle \times 1.36} & & {\scriptstyle \times 1.47} & \end{array}$$

The ratios are very close to being constant, whence an exponential model is suggested! To exactly match the first and last data values, we could take the model

$$P(t) \approx 5\left(\frac{28}{5}\right)^{\frac{t}{10}}$$

t	0	2	4	6	8	10
P	5	7.057	9.960	14.057	19.839	28

Only $P(19)$ doesn't match when we take rounding to the nearest integer into account.

We've seen that addition-addition corresponds to a linear model and that addition-multiplication to an exponential. There are two other natural combinations.

Logarithms These operate exactly as exponentials but in reverse. If $f(n) = \log_a x + b$, then *multiplying* x by a constant results in a constant *addition/subtraction* to y:

$$f(kx) = \log_a(kx) + b = \log_a k + \log_a x + b = \log_a k + f(x)$$

This could be summarized as 'product–addition.'

Power Functions If $f(x) = ax^m$, then multiplying x by a constant will do the same to y

$$f(kx) = a(kx)^m = ak^m x^m = k^m f(x)$$

We have a 'product–product' relationship between successive terms.

Examples 4.5. Find the patterns in the following data and suggest a model $y = f(x)$ in each case.

x	6	18	54	162
y	1	2	3	4

x	3	6	9	12
y	135	1080	3645	8640

The sequential approach in this chapter is a form of *discrete calculus*: using a pattern of *differences* to predict the original function is similar to how we use knowledge of a derivative $f'(x)$ to find $f(x)$.

Exercises 4.2. 1. Find the patterns in the following data sets and use them to find a model $y = f(x)$.

(a)

x	0	1	2	3	4
y	80	120	180	270	405

(b)

x	2	4	8	10
y	1	16	256	625

(c)

x	1	3	5	7	9
y	15	5	19	57	119

(d)

x	1	3	4	6
y	1	36	216	7776

(e)

x	20	60	180	540
y	2	4	6	8

(f)

x	2	6	54	486	4374
y	2	4	8	12	16

2. Take logarithms of the power relationship $y = ax^m$. What is the relationship between $\ln y$ and $\ln x$? Use this to give another reason why the inputs and outputs of power functions satisfy a 'product–product' relationship.

3. How does our analysis of exponential functions change if we add a constant to the model? That is, how might you recognize a sequence arising from a function $f(x) = ba^x + c$?

4. Suppose $f(5) = 12$ and $f(10) = 18$. Find the value of $f(20)$ supposing $f(x)$ is a:

   (a) Linear function;

   (b) Exponential function;

   (c) Power function.

   If $f(20) = 39$, which of the three *models* do you think would be more appropriate?