PH101

Lecture 9

Review of Lagrange's equations from D'Alembert's Principle, Examples of Generalized Forces a way to deal with friction, and other non-conservative forces

D'Alembert's principle of virtual work

If virtual work done by the constraint forces is ($\vec{f}_c \cdot \delta\vec{r} = 0$) (from eq.-1),

$$\left(\vec{F}_e - m\ddot{\vec{r}}\right) \cdot \delta\vec{r} = 0$$

$\rightarrow$ **D'Alembert's principle of Virtual work**

Now, for a general system of N particles having virtual displacements, $\delta\vec{r}_1$, $\delta\vec{r}_2$,….., $\delta\vec{r}_N$,

$$\sum_{i=1}^{N}\left(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i\right) \cdot \delta\vec{r}_i = 0$$

$\vec{F}_{ie}$ →Applied force on i_{th} particle

Does not necessarily means that individual terms of the summation are zero as $\vec{r}_i$ are not independent, they are connected by constrain relation

Lagrange's equation from D'Alembert's principle

❑ **D'Alembert's principle,**

$$\sum_{i=1}^{N}\left(\vec{\boldsymbol{F}}_{ie} - m_i\ddot{\vec{\boldsymbol{r}}}_i\right)\cdot\delta\vec{\boldsymbol{r}}_i = \mathbf{0}$$

Constraint forces are out of the game!

Now, no need of additional subscript, we shall simply write $\vec{\boldsymbol{F}}_{\boldsymbol{i}}$ instead of $\vec{F}_{ie}$

But How to express this relation so that individual terms in the summation are zero?

Switch to generalized coordinate system as they are independent!

Let's take the 1st term

$$\sum_i \vec{F}_i\cdot\delta\vec{r}_i = \sum_i \vec{F}_i\cdot\sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left(\sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\right)\delta q_j = \sum_{j=1}^{n} Q_j\delta q_j$$

$$Q_j = \sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}$$

→ **Generalized force**

- ❑ Dimensions of Q_j **is not** always of force!
- ❑ Dimensions of $Q_j\delta q_j$ **is** always of work!

Lagrange's equation from D'Alembert's principle

- 2nd Term: $$\sum_i m_i\ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_i m_i\ddot{\vec{r}}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_{i,j} m_i\ddot{\vec{\boldsymbol{r}}}_i \cdot \frac{\boldsymbol{\partial \vec{r}_i}}{\boldsymbol{\partial q_j}}\delta q_j$$

- **Bit of rearrangement in derivatives**

$$\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

Time and coordinate derivative can be interchanged!

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)}$$

$$= \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\boldsymbol{\partial \dot{\vec{r}}_i}}{\boldsymbol{\partial \dot{q}_j}}\right) - \dot{\vec{r}}_i \cdot \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)$$

dot cancellation!

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\boldsymbol{\partial \dot{\vec{r}}_i}}{\boldsymbol{\partial \dot{q}_j}}$$

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{r}_i{}^2\right)\right\} - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i{}^2\right)$$

Lagrange's equation from D'Alembert's principle

❑ Thus 2nd term becomes

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i,j} m_i \left[\frac{d}{dt}\left\{\frac{d}{d\dot{q}_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right]\delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{q}_j}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^{\,2}\right)\right]\delta q_j$$

$$= \sum_j \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j}\right\}\delta q_j$$

The 1st term

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{j=1}^{n} Q_j \delta q_j$$

Lagrange's equation from D'Alembert's principle

❑ D'Alembert's principle in generalized coordinates becomes

$$\sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j = \sum_j Q_j \delta q_j$$

$$\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0$$

Well, we are very close to Lagrange's equation!

❑ Since generalized coordinates q_j are all independent each term in the summation is zero

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$-\left(\frac{\partial V_i}{\partial x_i}\hat{\imath} + \frac{\partial V_i}{\partial y_i}\hat{\jmath} + \frac{\partial V_i}{\partial z_i}\hat{k}\right).\left(\frac{\partial x_i}{\partial q_j}\hat{\imath} + \frac{\partial y_i}{\partial q_j}\hat{\jmath} + \frac{\partial z_i}{\partial q_j}\hat{k}\right)$$

$$= -\left(\frac{\partial V_i}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V_i}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V_i}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right)$$

❑ If all the forces are conservative, then $\vec{F}_i = -\vec{\nabla} V_i$

$$Q_j = \sum_i (-\vec{\nabla} V_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial V_i}{\partial q_j} = -\frac{\partial}{\partial q_j}\sum_i V_i = -\frac{\partial V}{\partial q_j}$$

Total potential

$$V = \sum_i V_i$$

Lagrange's equation from D'Alembert's principle

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial V}{\partial q_j}$$

- Assume that $\boldsymbol{V}$ **does not depend on** $\dot{\boldsymbol{q}}_{\boldsymbol{j}}$, then $\frac{\partial \boldsymbol{V}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{j}}} = \mathbf{0}$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T - V)\right\} - \frac{\partial (T - V)}{\partial q_j} = 0$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

Where,

$$\boldsymbol{L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - V(q_j, t)}$$

We have reached to Lagrange's equation from D'Alembert's principle.

Review of the steps we followed

- Started from Newton's law

$$m\ddot{\vec{r}} = \vec{F}_e + \vec{f}_c$$

- Taken dot product with virtual displacement to kick out constrain force from the game by using $\vec{f}_c \cdot \delta\vec{r} = 0$; Arrive at D'Alembert's principle $\left(\vec{\boldsymbol{F}}_{\boldsymbol{e}} - \boldsymbol{m}\ddot{\vec{\boldsymbol{r}}} \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}}\right) \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}} = \mathbf{0}$

- Extended D'Alembert's principle for a system of particles;

$$\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{N}} (\vec{\boldsymbol{F}}_{\boldsymbol{ie}} - \boldsymbol{m}_{\boldsymbol{i}}\ddot{\vec{\boldsymbol{r}}}_{\boldsymbol{i}}) \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}}_{\boldsymbol{i}} = \mathbf{0}$$

- Converted this expression in generalized coordinate system that *"every"* term of this summation is zero to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

This is a more general expression!

- Now, with the assumptions: i) Forces are conservative, $\vec{F}_i = -\vec{\nabla} V_i$, hence $Q_j = -\frac{\partial V}{\partial q_j}$ and ii) potential does not depend on $\dot{\boldsymbol{q}}_{\boldsymbol{j}}$, then $\frac{\partial V}{\partial \dot{q}_j} = 0$

We get back our Lagrange's eqn.,

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

Discussion on generalized force

- A system may experience both conservative, non-conservative forces

  i,e. $\vec{F}_i = \vec{F}_i^{\,c} + \vec{F}_i^{\,nc}$

- Hence generalized force for the system

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left(\vec{F}_i^{\,c} + \vec{F}_i^{\,nc}\right) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \vec{F}_i^{\,c} \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \sum_i \vec{F}_i^{\,nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$Q_j = Q_j^{\,c} + Q_j^{\,nc}$$

$$Q_j^{\,c} = \sum_i \vec{F}_i^{\,c} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

- Generalized force corresponding to conservative part

$$Q_j^{\,nc} = \sum_i \vec{F}_i^{\,nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

- Generalized force corresponding to non-conservative part

Lagrange's equation with both conservative and non-conservative force

- If system may experience both conservative, non-conservative forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j} = {Q_j}^c + {Q_j}^{nc}$$

- Generalized force corresponding to conservative force can be derived from potential ${Q_j}^c = -\frac{\partial V}{\partial q_j}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + {Q_j}^{nc}$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T - V)\right\} - \frac{\partial (T - V)}{\partial q_j} = {Q_j}^{nc}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_j} = {Q_j}^{nc}$$

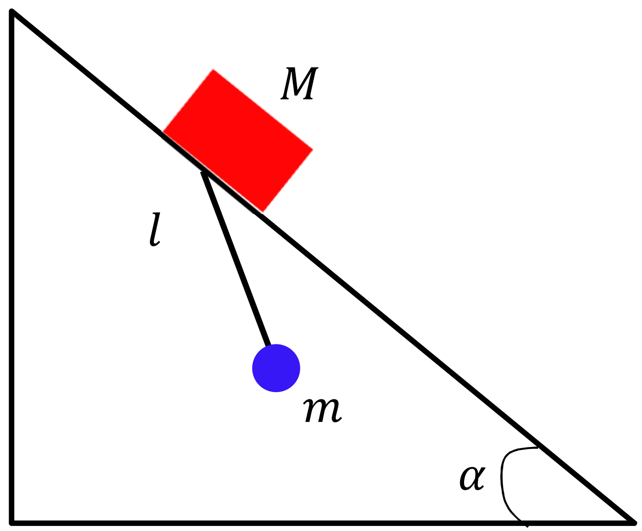
- Assume that $\boldsymbol{V}$ does not depend on $\dot{\boldsymbol{q}}_{\boldsymbol{j}}$, then $\frac{\boldsymbol{\partial V}}{\boldsymbol{\partial \dot{q}_j}} = \mathbf{0}$

$$L = T - V$$

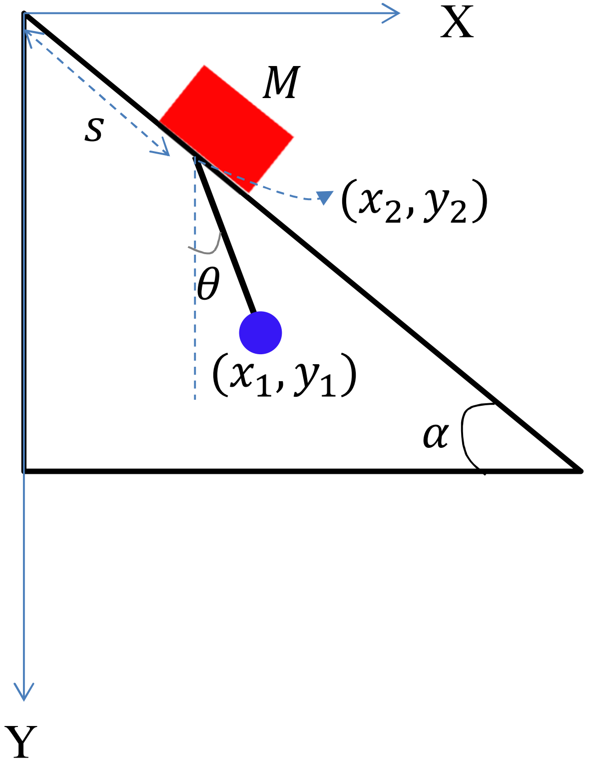
More on Lagrange's equations

Example-5

Example 5: A mass M slides down a frictionless plane inclined at angle α. A pendulum, with length l, and mass m, is attached to M. Find the equations of motion. For small oscillation



Example-5



Four constrains equations

$$z_1 = 0; \; z_2 = 0$$
$$y_2 = x_2 \tan \alpha$$
$$(y_2 - y_1)^2 + (x_2 - x_1)^2 = l^2$$

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

Two particles $N = 2, \;\; no.of \; constrains \; (k) = 4$

$$thus \; degrees \; of \; freedom = 3 \times 2 - 4 = 2$$

Hence number of generalized coordinates must be two.

's' and 'θ' can serve as generalized coordinates (they are independent nature)

Example-5 continued ….

Step-2: *Find out transformation relations*

$$x_2 = s\ \cos\alpha; y_2 = s\ \sin\alpha$$
$$x_1 = s\ \cos\alpha + l\sin\theta\ ; y_1 = s\ \sin\alpha + l\cos\theta$$

All the constrains relations have been included in the problem through these relationship

Step-3: *Write T and V in Cartesian*

$$T = \frac{1}{2}m\left(\dot{x}_1{}^2 + \dot{y}_1{}^2\right) + \frac{1}{2}M\left(\dot{x}_2{}^2 + \dot{y}_2{}^2\right)$$
$$V = -mgy_1 - Mgy_2$$

Step-4:Convert *T and V in generalized coordinate using transformation*

From transformation equation

$$\dot{x}_2 = \dot{s}\ \cos\alpha\ ;\ \dot{y}_2 = \dot{s}\sin\alpha$$
$$\dot{x}_1 = \dot{s}\ \cos\alpha + l\cos\theta\,\dot{\theta};$$
$$\dot{y}_1 = \dot{s}\sin\alpha - l\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m[\dot{s}^2 + l^2\dot{\theta}^2 + 2l\dot{s}\dot{\theta}\cos(\alpha+\theta)] + \frac{1}{2}M\dot{s}^2$$
$$V = -mg(s\sin\alpha + l\cos\theta) - Mgs\ \sin\alpha$$

Example-5 continued ….

Step-5: *Write down Lagrangian*

$$L = T - V$$

$$L = \frac{1}{2}m[\dot{s}^2 + l^2\dot{\theta}^2 + 2l\dot{s}\dot{\theta}\cos(\alpha+\theta)] + \frac{1}{2}M\dot{s}^2$$

$$+mg(s\sin\alpha + l\cos\theta) + Mgs\ \sin\alpha$$

Step-5: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \ \ and \ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

From 1st eqn

$$\frac{d}{dt}[m\dot{s} + ml\dot{\theta}\cos(\alpha+\theta) + M\dot{s}] - mg\sin\alpha - Mg\sin\alpha = 0$$

$$(m+M)\ddot{s} + ml\ddot{\theta}\ \cos(\alpha+\theta) + ml\dot{\theta}^2\sin(\alpha+\theta) - (m+M)g\sin\alpha = 0$$

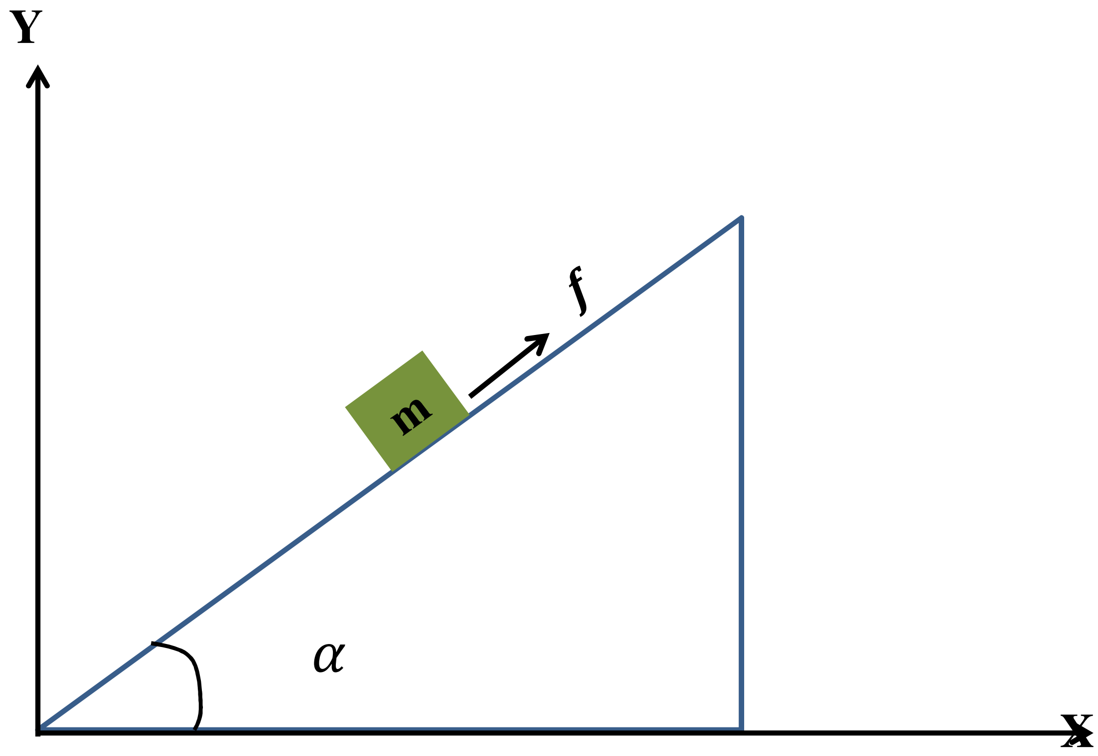
From 2nd eqn

$$\frac{d}{dt}[ml^2\dot{\theta} + ml\dot{s}\ \cos(\alpha+\theta)] + ml\dot{s}\dot{\theta}\sin(\alpha+\theta) + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + ml\ddot{s}\cos(\alpha+\theta) + mgl\sin\theta = 0$$

Problems with generalized force

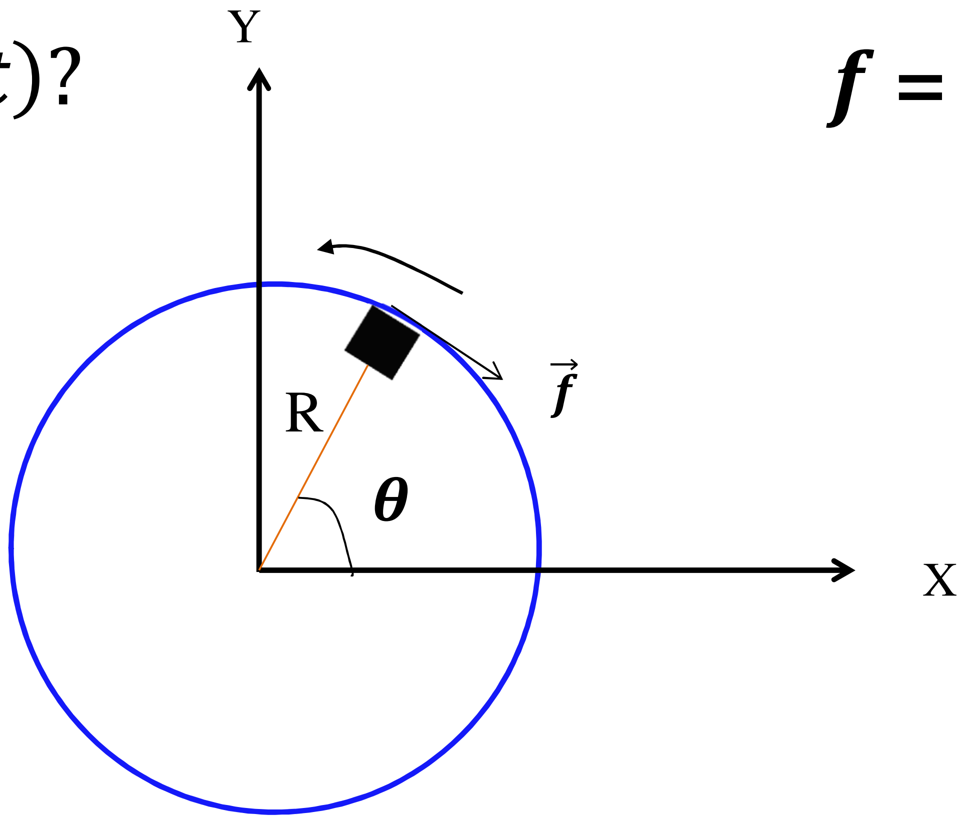
Example-6



Example-7; Ring & mass on horizontal plane

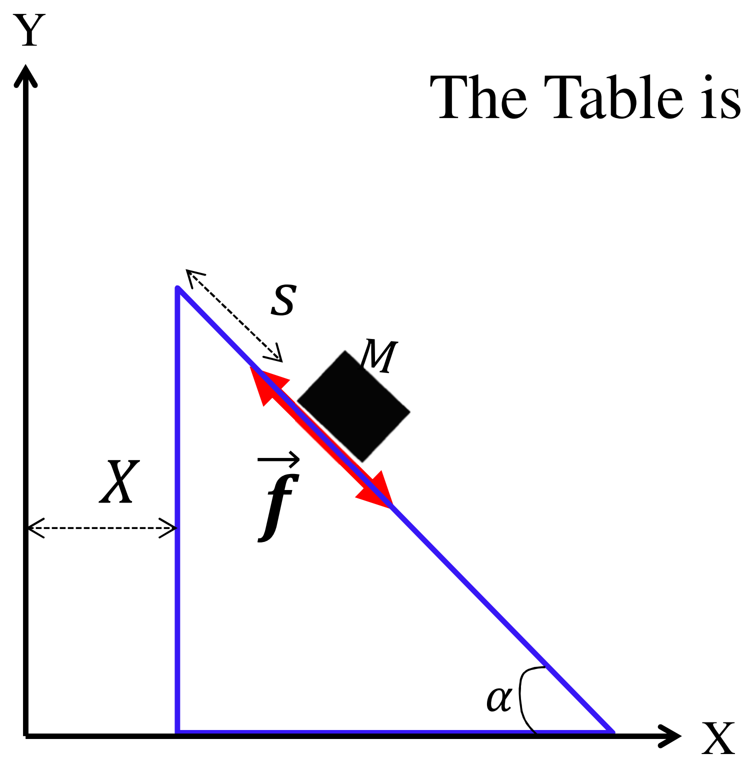
$\boldsymbol{Find\ \theta(t)}$?

$\boldsymbol{f = C\dot{\theta}^2}$



Example-8; Wedge & Block under friction, f

The Table is friction less!



Generalized coordinate (X, s)

QUESTIONS PLEASE