Constraints & Their Classification:

Object motion is restricted or prescribed in such a way that its coordinates and/or velocity components must satisfy some prescribed relation at every instant of time.

These relations can be either equations or inequalities.

Ex: Billiard ball of radius R moving on billiard board of length & breadth, $2a$ & $2b$ respectively, must satisfy

$$-a+R \leq x \leq a-R \;;\quad -b+R \leq y \leq b-R \;,\quad z=R.$$

This is a set of one equation and two inequalities which billiard ball must satisfy at all instant of time.

Most physical realizations ~~involve~~ of constraints involve surfaces of other bodies; above example of billiard which is in 2D. Similarly a train running on a railroad or a simple pendulum in a vertical plane are examples of 1D constraints.

What about carrom?

Properties of Constraint forces:

1> Elastic in nature and appear at the surface of contact. They arise because motion defined by extrenally applied forces are hindered by the contact.

2> They are so strong that they barely allow the body under casideration to even deviate slightly from a prescribed path or surface. This prescribed path or surface is called the "Constraint". The scalar equations that describe or prescribe the surface of constraint are called "constraint equation".

3> The sole ~~purpose~~ effect of constraint force is to keep constraint relation satisfied.

Classification of constraints:

Four ways to classify →
- Time dependent or not.
- Integrable algebric relation or not.
- Conservative or Dissipative
- ~~Algebric Equation~~
- Algebric Equation or Inequalities.

A "constraint" is

1. Either Scleronomic : Constraint relation do not explicilty depend on time.

   or Rheonomic : Constraint relation explicity depend on time.

and

2. Either Holonomic : Constraint relation are independent of velocities or can be made independent of velocities ~~$(\dot{x}, \dot{y}, \dot{z})$~~ $(\dot{x}, \dot{y}, \dot{z})$

   or non Holonomic : Constraint relations are not holonomic.

and

3. Either Conservative : total mechanical ~~work~~ energy of the system is conserved while performing the constraint motion. Constraint motion do not do any work.

   or Dissipative : Constraint forces do work and total mechanical energy is not conserved.

4. ~~Bilateral~~ Either Bilaterd : Constraint relations are in form of equalities

   or ~~Biasit~~ Unilateral : Constraint relations are in form of inequalities.

Properties of Constraints:

1> Just by looking at constraints it may be possible to determine the type qualification for the classes 1, 2 & 4, but the determination of the type qualification of class 3 depends on whether constraints relation or doing any work or not.

2> It may so happen that the constraint relation contains velocities but can be integrated with respect to time so that the resulting relation is made free of velocities. In such cases the constraint is holonomic. E.g.

$$(yz - 2x + y)\dot{x} + (xz - 2y + x)\dot{y} + xy\dot{z} = 0$$

can be integrated to

$$\underbrace{(1+z)xy = x^2 + y^2 + c}_{\text{Holonomic.}}$$

3> The general form of unilateral constraints can be written as

$$f(\underset{\text{Position}}{\underline{r}_1, \underline{r}_2, \underline{r}_3, \ldots}, \underset{\text{Velocity}}{\dot{\underline{r}}_1, \dot{\underline{r}}_2, \dot{\underline{r}}_3, \ldots}) \geq 0$$

i^{th} particle of system in motion

Whenever state of motion of the system is such that

$f = 0 \Rightarrow$ the constraint is "taut".

Unilateral constraint
- $f = 0$ "taut" $\Rightarrow$ Behaves like bilateral constraint
- $f \neq 0$, $f > 0$ $\Rightarrow$ constraint is not taut and the motion occurs as if there are no constraints.

4> Forces of Constraints:

Second Newton's law of motion is a complete law of nature.

[Observed accelaration of an object ≃ Totale forces acting on the object] True for inertial frames.

Newton did not provide a way to specify the total forces.

Total forces = Sum (Superposition) of many forces.

→ Obvious forces ⇒ Universally recognized laws of forces

such as → Newton's law of gravitation
→ Coulomb's law of electrostatics
etc

Do not depend on the nature of constraint relations

Externally Applied forces

→ Rest of the forces coming from the constraint of surfaces or curves.

Forces of Constraint.

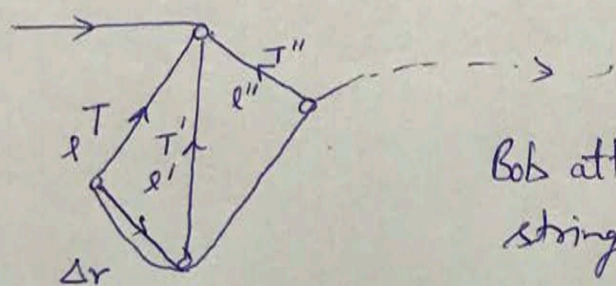
Unfortunately, Newton has not given any prescription for calculating these forces.
Hence in absence of total forces, Newton's second law in its differential form can not even be formulated let alone finding a solution

5> Work done by the constraint forces:

Usually the constraint forces act in direction $\perp$ to the surface of constraints at every point on it, while motion of object is $\parallel$ to surface at every point. $\Rightarrow$ Work done by constraint is zero.

Exception 1> Frictional force due to sliding.

Exception 2> Rheonomic Constraint in this case ~~work done~~ constraint force need not act $\perp$ to the displacement.



Bob attached with variable string length $l(t)$.

$$\underline{T} \cdot \Delta \underline{r} \neq 0$$

In case of simple pendulum $\underline{T} \cdot \Delta \underline{r} = 0$

Generally speaking rheonomic constraints are dissipative.

Examples of Constraints:

1 Rigid Body:

$$|\underline{r}_i - \underline{r}_k| = \text{Constant.}$$

~~$i, k \in [1, N]$~~

$$i, k = 1, \cdots N$$

Holonomic, ~~Scle~~ Scleronomic
Bilateral.

We will prove that the constraint is conservative.

$$|\underline{r}_i - \underline{r}_k|^2 = \text{Constant.}$$

Differential

$$(\underline{r}_i - \underline{r}_k) \cdot \Delta(\underline{r}_i - \underline{r}_k) = 0$$

Internal force of constraint on i^{th} particle due to the k^{th} particle be $\underline{F}_{ik}$. Using third law of Newton

$$\underline{F}_{ik} = -\underline{F}_{ki}$$

Workdone

$$\underbrace{\underline{F}_{ik} \cdot \Delta\underline{r}_i}_{\text{Possible Displacement.}} = -\underline{F}_{ki}\,\Delta\underline{r}_i$$

Total work done by the system

$$\Delta W = \sum_{\substack{i,k \\ i \neq k}} \underline{F}_{ik} \cdot \Delta\underline{r}_i = \sum_{\substack{i,k \\ k>i}} (\underline{F}_{ik} \cdot \Delta\underline{r}_i + \underline{F}_{ki} \cdot \Delta\underline{r}_k)$$

$$= \sum_{\substack{i,k \\ k>i}} \underline{F}_{ik} \cdot \Delta(\underline{r}_i - \underline{r}_k)$$

We know $\underline{F}_{ik}$ is an internal force between the particle i & k.
$\underline{F}_{ik}$ will act parallel to line joining i & k.

$$\underline{F}_{ik} = C_{ik}(\underline{r}_i - \underline{r}_k)$$

$$\Rightarrow \quad \Delta W = \sum_{\substack{i,k \\ k>i}} C_{ik}(\underline{r}_i - \underline{r}_k) \cdot \Delta(r_i - r_k) = 0$$

Scleronomic, Holonomic, Bilateral & Conservative.