Physics 5153 Classical Mechanics
D'Alembert's Principle and The Lagrangian

1 Introduction

The principle of virtual work provides a method of solving problems of static equilibrium without having to consider the constraint forces. This method requires that the system be varied through virtual displacements δq_j that are consistent with the constraints. From this point, the equations can be solved for whatever unknown applied forces exist. But this method does not apply directly to dynamical systems.

In this lecture, we will discuss the extension of the principle of virtual work to dynamical system. This extension is based on the work of D'Alembert, and is considered by many to be the most important development in the science of mechanics after Newton[1].

1.1 D'Alembert's Principle

D'Alembert's principle introduces the force of inertia $\vec{\mathbf{I}} = -m\vec{\mathbf{a}}$, thereby converting problems of dynamics to problems of statics

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \quad \Rightarrow \quad \vec{\mathbf{F}} - m\vec{\mathbf{a}} = 0 \quad \Rightarrow \quad \vec{\mathbf{F}} + \vec{\mathbf{I}} = 0 \tag{1}$$

where we show the transition from Newton to D'Alembert in this expression. The force $\vec{\mathbf{F}}$ is sometimes referred to as the real force, which I will do so in these lectures to distinguish it from the inertial force.

The requirement that the sum of all the forces at each particle be equal to zero is the necessary condition for static equilibrium. Since the principle of virtual work applies to systems in static equilibrium, we will apply it to this system of forces including the inertial force. The total work done by the forces in this system through an arbitray virtual displacement in Cartesian coordinates is

$$\delta W = \sum_i \left[\vec{\mathbf{F}}_i^a + \vec{\mathbf{F}}_i^c - m_i\ddot{\vec{\mathbf{r}}}_i\right] \cdot \delta\vec{\mathbf{r}}_i = 0 \tag{2}$$

where we have split the real forces into the applied and constraint forces. If the constraint forces are workless, and the virtual displacements reversible and consistent with the constraints, the total virtual work becomes

$$\delta W = \sum_i \left[\vec{\mathbf{F}}_i^a - m_i\ddot{\vec{\mathbf{r}}}_i\right] \cdot \delta\vec{\mathbf{r}}_i = 0 \tag{3}$$

This equation expands upon the principle of virtual work from static to dynamical system. Note, this equation applies to both rheonomic and scleronomic system, provided that the virtual displacements conform to the instantaneous constraint.

1.2 Example

As an example let's consider a wedge of mass M on a frictionless surface, with a block of mass m on the wedge (see Fig. 1). We will calculate the equations of motion for this system.

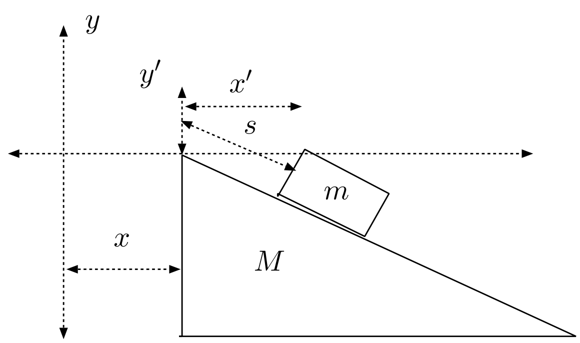

Figure 1: Block sliding down a frictionless incline, with inclide also free to slide frictionlessly on a flat surface.

Usng the coordinate system specified in Fig 1, the virtual work consistent with the constraints, in Cartesian coordinates is

$$\delta W = -mg\delta y' - m(\ddot{x} + \ddot{x}')(\delta x + \delta x') - M\ddot{x}\delta x - m\ddot{y}'\delta y' = 0 \tag{4}$$

Since the variables x' and y' are not independent of each other, because of the constraint of moving along the surface of the wedge, we apply the following transformation to take into account of the constraints

$$\left.\begin{aligned} x' &= s\cos\theta \\ y' &= -s\sin\theta \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} \delta x' = \delta s\cos\theta \\ \delta y' = -\delta s\sin\theta \\ \ddot{x}' = \ddot{s}\cos\theta \\ \ddot{y}' = -\ddot{s}\sin\theta \end{cases} \tag{5}$$

Applying this transformation and grouping like terms together, the virtual work becomes

$$[mg\sin\theta - m(\ddot{s} + \ddot{x}\cos\theta)]\,\delta s - [M\ddot{x} + m(\ddot{s}\cos\theta + \ddot{x})]\,\delta x = 0 \tag{6}$$

Notice at this point, we reduced the virtual work such that there are only two independent variations, which is the number of degrees of freedom: The wedge is constrained to move in one dimension, as is the block on the wedge. Since the variations are independent of each other and arbitrary, the terms in brackets must independently be equal to zero. Therefore, the equations of motion are

$$\begin{aligned} mg\sin\theta - m(\ddot{s} + \ddot{x}\cos\theta) &= 0 \\ M\ddot{x} + m(\ddot{s}\cos\theta + \ddot{x}) &= 0 \end{aligned} \tag{7}$$

1.3 Conservation of Energy

We return to the conservation of energy from the point of view of D'Alembert's principle. Let's start by considering the virtual work associated with a collection of particles in Cartesian coordinates

$$(\vec{\mathbf{F}}^a - \sum_i m_i\ddot{\vec{\mathbf{r}}}_i)\cdot\delta\vec{\mathbf{r}} \tag{8}$$

Next assume that the force can be written as the gradient of a scalar, the virtual work becomes

$$(\delta V + \sum_i m_i \ddot{\vec{\mathbf{r}}}_i) \cdot \delta \vec{\mathbf{r}} = 0 \tag{9}$$

The virtual displacement can be any arbitrary displacement that is consistent with the constraints. We will select it to be a real infinitesimal displacement

$$(dV + \sum_i m_i \vec{\mathbf{r}}_i) \cdot d\vec{\mathbf{r}} = 0 \tag{10}$$

The second term can be converted to a perfect differential of a scalar

$$\sum_i m_i \ddot{\vec{\mathbf{r}}}_i \cdot d\vec{\mathbf{r}} = \sum_i m_i \ddot{\vec{\mathbf{r}}}_i \cdot d\dot{\vec{\mathbf{r}}}\, dt = \sum_i \frac{d}{dt}\left(\frac{1}{2} m_i \dot{\vec{\mathbf{r}}}_i \cdot \dot{\vec{\mathbf{r}}}_i\right) dt = \sum_i \frac{dT_i}{dt}\, dt = dT \tag{11}$$

where T is the kinetic energy as previously defined. Based on this expression, the virtual work becomes

$$dV + dT = d(T + V) = 0 \quad \Rightarrow \quad T + V = E \tag{12}$$

Therefore, the sum of the kinetic and potential energy is a constant.

The question we must ask ourselves before using this result is, under what conditions does this hold? The first condition is that the force be derivable from a scalar potential. The second condition requires the virtual displacements be the same as the real displacement. This condition is satisfied if the the problem is time independent. That is the constraints and the potential are scleronomic (time independent).

1.4 The Lagrangian

We will now show the connection of the Lagrangian to D'Alembert's principle. Let's consider a system subject to a set of constraints

$$\sum (\vec{\mathbf{F}}_i^c + \vec{\mathbf{F}}_i^a - m\vec{\mathbf{a}}_i) = 0 \tag{13}$$

The virtual work is

$$\delta W = \sum (\vec{\mathbf{F}}_i^c + \vec{\mathbf{F}}_i^a - m\vec{\mathbf{a}}_i) \cdot \delta \vec{\mathbf{r}}_i = 0 \tag{14}$$

Since we are assuming the the constraints are workless, the constraint forces are removed from the equation

$$\delta W = \sum (\vec{\mathbf{F}}_i^a - m\vec{\mathbf{a}}_i) \cdot \delta \vec{\mathbf{r}}_i = 0 \tag{15}$$

Notice that all the information is still in this equation, the constraint are now in the virtual displacements.

Let's now transform the Eq. 15 to a set of generalized coordinates q_j, with the transformation being[1]

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}(q_j) \tag{16}$$

[1] At this point I will drop the subscript i and the summation. Everything that follows holds for a system of N particles unless otherwise stated

the velocity is

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \sum_j \frac{\partial \vec{\mathbf{r}}}{\partial q_j}\,\dot{q}_j + \frac{\partial \vec{\mathbf{r}}}{\partial t} \quad \Rightarrow \quad \frac{\partial \vec{\mathbf{v}}}{\partial \dot{q}_i} = \frac{\partial \vec{\mathbf{r}}}{\partial q_i} \tag{17}$$

where the second expression takes into account that the coordinate transformations do not depend explicitly on the generalized velocities. The virtual displacement is given by

$$\delta\vec{\mathbf{r}} = \sum_j \frac{\partial \vec{\mathbf{r}}}{\partial q_j}\,\delta q_j \tag{18}$$

From here we can write the virtual work associated with the applied forces as

$$\vec{\mathbf{F}} \cdot \delta\vec{\mathbf{r}} = \sum_j \vec{\mathbf{F}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j \tag{19}$$

where Q_j is the generalized force.

The force of inertia can also be written in generalized coordinates

$$m\ddot{\vec{\mathbf{r}}} \cdot \delta\vec{\mathbf{r}} = m\ddot{\vec{\mathbf{r}}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j}\,\delta q_j \tag{20}$$

The second time derivative of the Cartesian coordinates can be written in terms of first derivative, this allows some simplification

$$\ddot{\vec{\mathbf{r}}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{\mathbf{r}}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j}\right) - \dot{\vec{\mathbf{r}}} \cdot \frac{d}{dt}\left(\frac{\partial \vec{\mathbf{r}}}{\partial q_j}\right) \tag{21}$$

where the time derivative of the second term is

$$\frac{d}{dt}\left(\frac{\partial \vec{\mathbf{r}}}{\partial q_j}\right) = \frac{\partial \vec{\mathbf{v}}}{\partial q_j} \tag{22}$$

and from Eq. 17, we get

$$\frac{\partial \vec{\mathbf{v}}}{\partial \dot{q}_j} = \frac{\partial \vec{\mathbf{r}}}{\partial q_j} \tag{23}$$

Therefore, Eq. 21 can be written as

$$m\ddot{\vec{\mathbf{r}}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j} = \frac{d}{dt}\left(m\vec{\mathbf{v}} \cdot \frac{\partial \vec{\mathbf{v}}}{\partial \dot{q}_j}\right) - m\vec{\mathbf{v}} \cdot \frac{\partial \vec{\mathbf{v}}}{\partial q_j} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}mv^2\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}mv^2\right) \tag{24}$$

Substituting back into D'Alembert's principle in terms of generalized coordinates, we get

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0 \tag{25}$$

where $T \equiv \frac{1}{2}mv^2$ is the kinetic energy. If the force is derivable from a potential ($\vec{\mathbf{F}} = -\vec{\boldsymbol{\nabla}}V$), then the generalized force can be expressed as

$$Q_j = \vec{\mathbf{F}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j} = -\vec{\boldsymbol{\nabla}}V \cdot \frac{\partial \vec{\mathbf{r}}}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{26}$$

Therefore D'Alembert's principle becomes

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} \right] \delta q_j = 0 \tag{27}$$

if the constraints are holonomic, then the coefficients of the δq_j are independently equal to zero

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \tag{28}$$

Finally, since the potentials of this form are independent of the velocity, D'Alembert's principle can be put into the form

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{29}$$

where $L = T - V$.

1.5 Hamilton's Principle

In the previous section, we derived an expression that describes the motion at a point in space. In this section we derive an expression that describes the general properties of the motion through an integral relation. We will show that the D'Alembert's principle can be as the variation of an integral over time of a single scalar function.

Let's start by taking the time integral of the virtual work

$$\int_{t_0}^{t_1} \delta W dt = \int_{t_0}^{t_1} \sum_i \left[\vec{\mathbf{F}}_i^a - m_i \frac{d\vec{\mathbf{v}}_i}{dt} \right] \cdot \delta \vec{\mathbf{r}}_i dt = 0 \tag{30}$$

where we have assumed workless constraints. We take the virtual work associated with the applied force and add the assumption that it can be derived from a scalar potential

$$\int_{t_0}^{t_1} \sum_i \vec{\mathbf{F}}_i^a \cdot \delta \vec{\mathbf{r}}_i dt = -\int_{t_0}^{t_1} \delta V dt = -\delta \int_{t_0}^{t_1} V dt \tag{31}$$

Now we work to simplify the term with the inertial forces. This can be done throught an integration by parts

$$-\int_{t_0}^{t_1} m_i \frac{d\vec{\mathbf{v}}_i}{dt} \cdot \delta \vec{\mathbf{r}}_i dt = -\int_{t_0}^{t_1} m_i \frac{d}{dt} \left(\vec{\mathbf{v}}_i \cdot \delta \vec{\mathbf{r}} \right) dt + \int_{t_0}^{t_1} m_i \vec{\mathbf{v}}_i \cdot \frac{d}{dt}(\delta \vec{\mathbf{r}}_i) dt \tag{32}$$

where we work this out for a single term and then reintroduce the summation at the end. The first term on the right hand side is a total differential and therefore easily integrate

$$-\int_{t_0}^{t_1} m_i \frac{d}{dt} \left(\vec{\mathbf{v}}_i \cdot \delta \vec{\mathbf{r}} \right) dt = -\left[m_i \vec{\mathbf{v}}_i \cdot \delta \vec{\mathbf{r}}_i \right]_{t_0}^{t_1} \tag{33}$$

The second term on the right hand side can be written as follows

$$\int_{t_0}^{t_1} m_i \vec{\mathbf{v}}_i \cdot \frac{d}{dt}(\delta \vec{\mathbf{r}}_i) dt = \int_{t_0}^{t_1} m_i \vec{\mathbf{v}}_i \cdot (\delta \vec{\mathbf{v}}_i) dt \tag{34}$$

where we invert the variation with the time derivative. The next step is to use the variation of the square o the velocity to get the product of velocity and variation of the velocity

$$\int_{t_0}^{t_1} m_i\vec{\mathbf{v}}_i\cdot(\delta\vec{\mathbf{v}}_i)dt = \int_{t_0}^{t_1}\delta\left[\frac{1}{2}m_i\vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i\right]dt = \delta\int_{t_0}^{t_1}\left[\frac{1}{2}m_i\vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i\right]dt \tag{35}$$

Next we sum over all particles and combine all three pieces of the integral

$$-\delta\int_{t_0}^{t_1} V dt + \delta\int_{t_0}^{t_1}\sum_i\left[\frac{1}{2}m_i\vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i\right]dt - \left[m_i\vec{\mathbf{v}}_i\cdot\delta\vec{\mathbf{r}}_i\right]_{t_0}^{t_1} = 0 \tag{36}$$

The second term is the kinetic energy T. On the third term, we will impose the requirement that the variation on the endpoints be zero. Equation 36 becomes

$$\delta\int_{t_0}^{t_1}(T-V)dt = 0 \tag{37}$$

where L is the same function we found before, except that now we find it in terms a minimization principle[2]. The integral is defines the action

$$A = \int_{t_0}^{t_1} L dt \quad\Rightarrow\quad \delta A = 0 \tag{38}$$

Even thought this proceedure was carried out in rectangular coordinates, we could have transformed the equations through a point transformation to a new set of coordinates and carried through the proceedure in the new coordinates, and we would have found the same answer. Note that in this statement the coordinates are assumed to be independent, therefore the constraints must be holonomic in nature in order to reduce the number of degrees of freedom through substitutions. The nonholonomic problem will be discussed later.

We have the function L (Lagrangian) in two different equations. One is a differential equation that defines the dynamics at a single point, and the second is an integral equation that defines the global properties of the motion. The question that we must now answer is how are the two equations connected. For this we will need to learn something about the calculus of variations.

1.6 Example Lagrangian

Consider a bead constrained to move along a wire that makes an angle θ with respect to the upward vertical. The wire rotates about the vertical as shown in the Fig. 2 with an angular velocity ω. Gravity acts downward. We wish to determine the Lagrangian using an appropriate set of generalized coordinates.

Before we start setting up the Lagrangian, note that the wire does work on the bead, but the wire forms a workless constriant. The reason the constriant is workless, is that we take the instantaneous constraint and then apply a virtual displacement. To setup the problem, we start in Cartesian coordinates

$$L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - mgz \tag{39}$$

[2] Actually at this point we are finding a stationary point, either minimum or maximum.

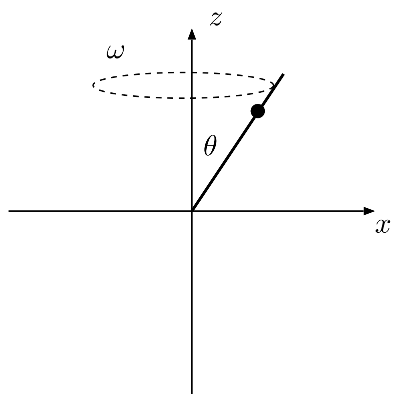

Figure 2: Bead on a rotating wire.

The constraints are given by

$$\left.\begin{aligned} x &= r\sin\theta\sin\omega t \\ y &= r\sin\theta\cos\omega t \\ z &= r\cos\theta \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= \dot{r}\sin\theta\sin\omega t + \omega r\sin\theta\cos\omega t \\ \dot{y} &= \dot{r}\sin\theta\cos\omega t - \omega r\sin\theta\sin\omega t \\ \dot{z} &= \dot{r}\cos\theta \end{aligned} \tag{40}$$

where r is along the wire. Working through the algebra, leads to

$$L = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2 \sin^2\theta) - mgr\cos\theta \tag{41}$$

References

[1] The Variational Principles of Mechanics, *C. Lanczos* pgs. xxi, xxix

[2] Classical Dynamics, *D.T. Greenwood* chap. 1