Linear Algebra

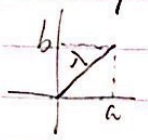
Symmetric Matrices / Real evalues / orthogonal evectors

$S^T = S$ real λ orthogonal x.

$A^T = -A$ pure imaginary λ, orthogonal x (complex)

$Q^TQ = I$ all $|\lambda| = 1$, orthogonal x (complex)

Complex numbers vectors matrices



$\lambda = a + ib$

$\sqrt{\lambda\bar{\lambda}} = \sqrt{a^2+b^2}$

$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\|x\|^2 = [1\ -i]\begin{bmatrix} 1 \\ i \end{bmatrix} = \bar{x}^T x$

$\bar{S}^T = S$ Hermitian matrix

$S = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

Orthogonal vectors $\bar{x}^T y = 0$

エルミート共役 $\overset{\text{Conjugate}}{{}^t\bar{A}}, A^\dagger$

$A^\dagger = A$, $A^\dagger = -A$ (反)

$Au = \lambda u$

$A^\dagger u = \lambda u$

$(A^\dagger u)^\dagger = (\lambda u)^\dagger$

$u^\dagger A = \bar{\lambda} u^\dagger$

$u^\dagger A u = \bar{\lambda} u^\dagger u$

$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ 实数 (real)

$-A^\dagger u = \lambda u$

$(-A^\dagger u)^\dagger = (\lambda u)^\dagger$

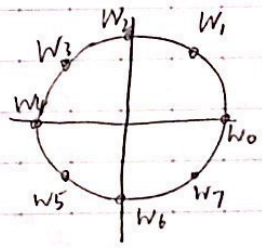
$-u^\dagger A = \bar{\lambda} u^\dagger$

$u^\dagger A u = -\bar{\lambda} u^\dagger u$

$\Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda$ 纯虚数 (pure imaginary)

Cyclic Matrix

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & & \ddots & & \vdots \\ a_2 & & & \ddots & a_1 \\ a_1 & a_2 & \cdots & & a_0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & 0 & \cdots & & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & E_{n-1} \\ 1 & 0 \end{bmatrix}$$



$\lambda = e^{i2k\pi/n}$ $(k = 0, 1, \cdots n-1)$

$n = 8$

$A_n = C_0 I + C_1 P^1 + C_2 P^2 + \cdots + C_{n-1} P^{n-1}$

eigenvectors:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W^1 & W^2 & & W^{n-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{n-1} & W^{2(n-1)} & & W^{(n-1)^2} \end{bmatrix}$$

$M = Q\Lambda Q^T$ normal matrix $\bar{M}^T M = M\bar{M}^T$

eigenvector of P will also be evector of A

$A = C_0 I + C_1 P^1 + \cdots + C_{n-1} P^{n-1}$

evectors of circulants = Columns of F_n

Matrix multiplication.

$$\underset{A_{m\times n}}{\begin{bmatrix} & | \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} — \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} & \end{bmatrix}}$$

$AB = \text{Sum of (Columns of } A) \times (\text{rows of } B)$

Block. $\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]$

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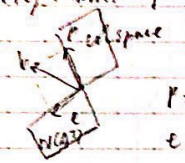
Proj. Matrix $P = A(A^TA)^{-1}A^T$



$p + e = b$

$e = (I-P)b$

Orthogonal $q_i^Tq_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

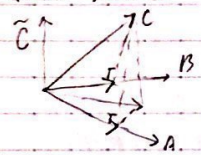
$Q = [q_1 \cdots q_n]$

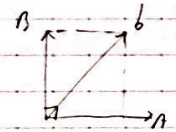
$Q^TQ = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} [q_1 \cdots q_n] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I \Rightarrow Q^T = Q^{-1}$

$P = Q(Q^TQ)^{-1}Q^T = QQ^T = I$ if Q is square

$Q^TQ\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb$

Gram-Schmidt





$B = b - p$

$= b - \frac{A^Tb}{A^TA}A$

$\tilde{C} = C - \frac{A^TC}{A^TA}A - \frac{B^TC}{B^TB}B$, $\tilde{C} \perp A$, $\tilde{C} \perp B$

$\begin{cases} y_1 = a_1 \\ y_2 = a_2 - \frac{(a_2^Ta_1)a_1}{(a_1^Ta_1)} \\ y_3 = a_3 - \frac{(a_3^Ta_1)a_1}{(a_1^Ta_1)} - \frac{(a_3^Ta_2)a_2}{(a_2^Ta_2)} \end{cases}$

$e_1 = \frac{y_1}{\|y_1\|}$ normalize

$e_2 = \frac{y_2}{\|y_2\|}$

$e_3 = \frac{y_3}{\|y_3\|}$

$q_1 = \frac{a_1}{\|a_1\|}$

$A_2 = a_2 - (a_2^Tq_1)q_1$, $q_2 = \frac{A_2}{\|A_2\|}$

$A_3 = a_3 - (a_3^Tq_1)q_1 - (a_3^Tq_2)q_2$

$q_3 = \frac{A_3}{\|A_3\|}$

Determinates $\det A = |A|$

① $\det I = 1$

② exchange rows, reverse sign of the det

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3.a $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3.b $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ linear for each row.

④ 2 equal rows $\rightarrow \det = 0$

⑤ Substract $l \times$ row i from k, DET DOESN'T CHANGE

⑥ Row of zeros $\rightarrow \det A = 0$

⑦ $\det U = \begin{vmatrix} d_1 & * & \cdots \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{vmatrix} = d_1 \cdot d_2 \cdots d_n$

⑧ $\det A = 0$, A singular if $\det A \neq 0$

⑨ $\det AB = (\det A)(\det B)$ A invertible

⑩ $\det A^T = \det A$

$\tilde{⑨}$: $\det(A^{-1}) = \frac{1}{\det A}$

$\det A^2 = (\det A)^2$

$\det 2A = 2^n \det A$

Cofactor of $a_{ij} = C_{ij}$

$\pm \det$ ($n-1$ matrix with row i, col j erased)

$+$ $i+j$ even, $-$ $i+j$ odd

cofactor formula

$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

Inverse $A^{-1} = \frac{1}{\det A}(A^*)$ $|\det A|$ = volume of box

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \frac{1}{\det A}C^T$ ← products of $n-1$ entries

C: matrix of cofactor.

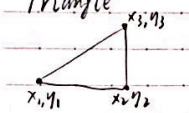
← products of n entries

$Ax = b$

$x = A^{-1}b$ Cramer's rule $x_1 = \frac{\det B_1}{\det A}$ B_j = A with column j replaced by b

A with column 1 replaced by b $= B_1 = [b \ \ n-1 \text{ columns of } A]$

Triangle



area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Eigenvalues & Eigenvectors.

$\det(A - \lambda I) = 0$ Trace $= \lambda_1 + \lambda_2 + \cdots + \lambda_n$ $\det A = \lambda_1\lambda_2\cdots\lambda_n$

$Ax = \lambda x$ λ: eigenvalue x: eigenvector

if A is singular, $\lambda = 0$ is evalue.

Diagonalize a matrix $S^{-1}AS = \Lambda$, Power of A/eqn $u_{k+1} = Au_k$.

Suppose n indep. eigenvectors of A, put them in the columns of S

$AS = A[x_1 \cdots x_n] = [\lambda_1x_1 \cdots \lambda_nx_n] = [x_1 \cdots x_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$

← diagonal matrix

$A = S\Lambda S^{-1}$ $A^k = S\Lambda^kS^{-1}$

Theorem: $A^k \rightarrow 0$ as $k \rightarrow \infty$ if all $|\lambda_i| < 1$

A is sure to have n indep. evectors if all λ's are different

$u_{k+1} = Au_k$, Start with u_0

$u_1 = Au_0$, $u_k = A^ku_0$

$u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n = Sc$

$Au_0 = c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n$

$u_{100} = A^{100}u_0 = c_1\lambda_1^{100}x_1 + \cdots + c_n\lambda_n^{100}x_n = \Lambda^{100}Sc$

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Ex: Fibonacci

$$\begin{cases} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{cases}$$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k \qquad \lambda_1, \lambda_2 \text{ are } A \text{ eigenvalues}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_1 \Rightarrow x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad \lambda_2 \Rightarrow x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \qquad c_1 \lambda_1^n \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \lambda_2^n \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

general solutions.

plug $c_1 \lambda_1^n \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \lambda_2^n \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ into $u_0 \Rightarrow c_1, c_2$

Differential Eqns $\frac{du}{dt} = Au$

* Exponential e^{At} of a matrix

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \Rightarrow \lambda_1, \lambda_2 \Downarrow x_1, x_2$$

General Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

Check $\frac{du}{dt} = Au$ plug in $e^{\lambda_1 t} x_1$ $\qquad \lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$

① Stability $u(t) \to 0$ / need $e^{\lambda t} \to 0$ / $Re\ \lambda < 0$ (Real part)

$$\left| e^{(-3+bi)t} \right| = e^{-3t} \qquad \left| e^{bit} \right| = 1$$

② Steady state $\lambda_1 = 0$, and other $Re\lambda < 0$

③ Blow up if any $Re\ \lambda > 0$

Matrix exponential $e^{At} = I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^n}{n!}$ $\qquad \left(e^x = \sum_0^\infty \frac{x^n}{n!} \right)$

$(I - At)^{-1} = I + At + (At)^2 + \cdots$ $\qquad \left(\frac{1}{1-x} = \sum_0^\infty x^n \right), \ |\lambda(At)| < 1$

$$e^{At} = I + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1} t^2}{2} + \cdots$$

$$= S e^{\Lambda t} S^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

Ex: $y'' + by' + ky = 0$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u$$

Markov Matrix

Steady state $\lambda = 1$

① All entries ≥ 0.

② All columns add to 1.

1. $\lambda = 1$ is one eigenvalue / eigenvector $x_1 \geq 0$
2. All other $|\lambda_i| < 1$

ex: $A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$

eigenvalue of A & eigenvalue of A^T are the same.

use Rank calculate λ. ex: $\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$ Rank = 1, $\lambda_1 = 0, 0$, $\mathbb{R}^3$, $\lambda_3 = 9$

trace of $(A+I) = (\lambda_1 + 1) + (\lambda_2 + 2) + \cdots$

$= \lambda_1 + \lambda_2 + \cdots + n$

a. Invertible : nonzero evalues

b. $\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right) \cdots \left(\frac{1}{\lambda_n}\right)$

Symmetric Matrix

① The evalues are REAL.

② The evectors are Perpendicular.

Usual: $A = S\Lambda S^{-1}$

Symm: $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$

REAL:

$Ax = \lambda x \Rightarrow A\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda}$

$(\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x)$ $\qquad \bar{x}^T A^T x = \bar{x}^T \bar{\lambda} x$

(length^2) as $A^T = A$, $Ax = \bar{\lambda} x \Rightarrow \lambda = \bar{\lambda}$ real

Positive Definite Matrix, $x^T A x > 0$

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ Symm

① $\lambda_1 > 0, \lambda_2 > 0$ $\qquad$ All $\lambda_i > 0$

② $a > 0$, $ac - b^2 > 0$ $\qquad$ All leading determinates > 0

③ pivots $a > 0$, $\frac{ac - b^2}{a} > 0$ All pivots in elimination > 0

④ $x^T A x > 0$ $\qquad$ Energy $x^T S x > 0$ (all $x \neq 0$)

⑤ $S = A^T A$ $\qquad$ independent cols in A

$A^T A$, $A_{m \times n}$ Rank n, Independent cols $\qquad x^T A^T A x = (Ax)^T Ax = \|Ax\| > 0$ (except for $x = 0$)

Similar Matrix $\qquad B = M^{-1} A M$

similar matrices have same λ's !! (Same evectors)

$Ax = \lambda x \qquad B = M^{-1} A M \qquad M B M^{-1} x = \lambda x$

$\underbrace{M^{-1} A M}_{B} M^{-1} x = \lambda M^{-1} x \qquad B M^{-1} x = \lambda M^{-1} x$

$\Rightarrow B$ & A has same λ.

$B M^{-1} x = \lambda M^{-1} x$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \text{ not similar } \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

BAD case. $\lambda_1 = \lambda_2 = 4$.

$M^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \leftarrow$ Jordan Form

$$J = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_N \end{bmatrix}$$

blocks = # eigenvectors

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Non-Diagonalizable Matrices : Power Vectors and Jordan Form

$\tilde{E}_\lambda = \{\vec{0}\} \cup \left\{ \begin{array}{l} \text{All power vectors} \\ \text{of } A \text{ w/ e-val } \lambda \end{array} \right\}$

$= \mathrm{Null}(A-\lambda I)^{ma(\lambda)}$

$\tilde{E}_\lambda$ contains eigenspace E_λ

Degree of p-vec w/ eval $\lambda \le ma(\lambda)$

$\mathrm{Dim}(\tilde{E}_\lambda) = ma(\lambda)$

Ex: $A = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow |A-\lambda I| = 0 \Rightarrow \lambda = \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 1 \\ \lambda_3 = 1 \end{cases}$ $\begin{bmatrix} 2 & & 0 \\ & 1 & 1 \\ 0 & & 1 \end{bmatrix}$ J-matrix

$\lambda_1 = 2,\ x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \lambda_2 = 1,\ x_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$(A-\lambda_2 I)x_2 = 0$

$\begin{cases} (A-\lambda_2 I)^2 x_3 = 0 \\ (A-\lambda_2 I)x_3 = x_2 \end{cases} \Rightarrow Ax_3 = \lambda_2 x_3 + x_2$

$AP = PJ$

$P(x_1, x_2, x_3) \Rightarrow AP = (Ax_1, Ax_2, Ax_3) = (\overset{\lambda=2}{2x_1}, \overset{\lambda=1}{x_2}, x_3 + x_2)$

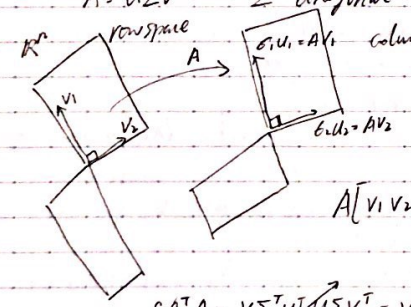
$Ax_3 = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1+x \\ 2+y \\ -1+z \end{bmatrix} \Rightarrow \begin{cases} y = 1+2x \\ y = 1+2x \\ z = -1-x \end{cases} \Rightarrow \underset{x_3}{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}$

set $x = 1$

$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}$ $(A = PJP^{-1})$

Singular Value Decomposition = SVD

$A = U\Sigma V^T$ Σ diagonal U, V orthogonal



$A[v_1\ v_2 \cdots v_r] = [u_1\ u_2 \cdots u_r]\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_r \end{bmatrix}$

$AV = U\Sigma$

$\begin{cases} A^TA = V\Sigma^T \cancel{U^T U} \Sigma V^T = V\Sigma^T\Sigma V^T \\ AA^T = U\Sigma V^T \cancel{V} \Sigma^T U^T = U\Sigma\Sigma^T U^T \end{cases}$

Ex: $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix},\ A^T = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$

$A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \Rightarrow \lambda_1 = 32,\ \lambda_2 = 18$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ free variable set 1

$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix}$

$AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \Rightarrow \tilde{\lambda}_1 = 32,\ \tilde{\lambda}_2 = 18,\ \tilde{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \tilde{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = U\Sigma V^T = \underset{U}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\underset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}}\underset{V^T}{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}} = \begin{bmatrix} -4 & 4 \\ -3 & 3 \end{bmatrix}$

Linear Transformation T $T(v+w) = T(w) + T(v)$, $T(cV) = cT(v)$

Pseudo-inverses $A^+b \to$ least square

$A^+ = V\Sigma^+U^T$, $\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$

real $A^TA = AA^T$, $Ax = 0 \Leftrightarrow A^Tx = 0$

$\|A^Tx\|^2 = (A^Tx)^T(A^Tx) = x^TAA^Tx$, $\because AA^T = A^TA$

$= x^TA^TAx = (Ax)^T(Ax) = 0$

$\Rightarrow A^Tx = 0$

$A^2x = 0 \Rightarrow A(Ax) = 0 \Rightarrow A^T(Ax) = 0$

$\Rightarrow x^TA^T(Ax) = 0 \Rightarrow (Ax)^T(Ax) = 0 \Rightarrow Ax = 0$

$\therefore A^2x = 0 \Leftrightarrow Ax = 0$

$\sum x_i A_{ij} x_j = x^TAx$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^TA^T$

共役転置行列 $A = (a_{ij})$ $A^* = \bar{A}^T = \bar{a}_{ji}$

$(cA)^* = \bar{c}A^*$, $(AB)^* = B^*A^*$

行列 $A = (a_{ij})$ (長さ) $\|A\| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$

内積 : $(x, y) = (y, x) = \begin{cases} x^Ty = a_1b_1 + a_2b_2 + \cdots + a_nb_n \text{ (real)} \\ x^T\bar{y} = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n \text{ (Im)} \end{cases}$

$(Ax, y) = (x, A^*y)$

$\|x\| = \sqrt{(x, x)} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

$|(x, y)| \le \|x\|\|y\|$, $\|x+y\| \le \|x\| + \|y\|$

$(x, y) = \|x\|\|y\|\cos\theta$

$A^{-1}A = AA^{-1} = I$

逆行列 A^{-1}存在 $\Rightarrow$ A正則 $(\det A \neq 0)$

$(AB)^{-1} = B^{-1}A^{-1}$ $(A^T)^{-1} = (A^{-1})^T$

$\begin{cases} \text{Symm. } A^T = A \\ \text{Semi-symm. } A^T = -A \\ \text{Orthogonal } A^TA = I \end{cases}$

A が直交行列 $\Leftrightarrow$ A が正則 $A^{-1} = A^T$

$A^* = \bar{A}^T = A$ エルミート行列

$A^* = \bar{A}^T = -A$ 歪エルミート

ユニタリー行列 $A^*A = E$

A が ユニタリー行列 $\Leftrightarrow$ A が正則 $A^* = A^{-1}$

正規行列 $AA^* = A^*A$ (複素正方行列)

扫描全能王 创建

迹 trace

$trA = SpA = \sum_{i=1}^{n} a_{ii}$

(i) $tr(\alpha A+\beta B) = \alpha trA + \beta trB$

(ii) $tr(AB) = tr(BA)$

(iii) $tr(P^{-1}AP) = tr(A)$ $(|P|\neq 0)$

$f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$

$f(A) = a_0A^k + a_1A^{k-1} + \cdots + a_{k-1}A + a_kE$

$h(x) = f(x)+g(x) \Rightarrow h(A) = f(A)+g(A)$

$h(x) = f(x)g(x) \Rightarrow h(A) = f(A)g(A)$

$f(P^{-1}AP) = P^{-1}f(A)P$ $(|P|\neq 0)$

$|x_{11}| = x_{11}$, $\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$

$|A^T| = |A|$

$|cA| = c^n|A|$ $|E| = 1$

$|AB| = |A||B|$

$\begin{vmatrix} A & B \\ O & D \end{vmatrix}$ $\begin{vmatrix} A & O \\ C & D \end{vmatrix} = |A||D|$

$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|$

余因子: $\Delta_{ij} = (-1)^{i+j}\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$ $\hat{i}$ $\hat{j}$ $(\hat{i},\hat{j}$: 行列删除的记号)

$\begin{cases} A^{-1} = \frac{1}{|A|}(\Delta_{ji}) \\ A^{-1} = \frac{1}{detA}A^* \end{cases}$ $|A^{-1}| = \frac{1}{|A|}$

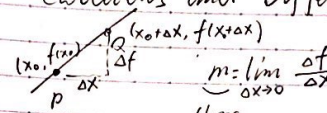
rank A = rank A^T

n阶的正方行列: (i) A正则 (ii) $|A|\neq 0$ (iii) rank $A = n$

$det(I+UV^T) = 1+U^TV$

$eig(AB) = eig(BA)$

Calculus and Differential Equation



$m = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}$

slope

$f'(x_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

Binomial Theorem

$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}x^0y^n$

Product Rule: $(UV)' = U'V + UV'$

Quotient Rule: $\left(\frac{U}{V}\right)' = \frac{U'V-UV'}{V^2}$, $V\neq 0$

Chain Rule: $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$ ex: $\frac{d}{dt}(\sin(10t)) = \cos 10t\cdot 10$

Implicit Differentiation Ex: $y^4+xy^2-2=0 \Rightarrow 4y^3y'+y^2+x\cdot 2y\cdot y' = 0$

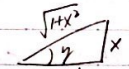
$\Rightarrow y' = \frac{-y^2}{4y^3+2xy}$

Inverse Function

$y = f(x)$, $g(y) = x$

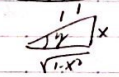
$g(f(x)) = x$

ex: $\tan y = x$



$\Rightarrow \cos y = \frac{1}{\sqrt{1+x^2}}$

ex: $\sin y = x$



$\Rightarrow \cos y = \sqrt{1-x^2}$

$\cos y\cdot y' = 1$

$\Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

Exponential and Log

$y = 2^x$ $\log_2 y = x \Rightarrow \frac{\ln y}{\ln 2} = x \Rightarrow \ln y = x\ln 2 \Rightarrow \frac{1}{y}y' = \ln 2$

$\Rightarrow y' = y\ln 2 = \ln 2\cdot 2^x$

Nature LOG $\frac{d}{dx}\ln x = \frac{1}{x}$ ex: $V = x^x$ $\ln V = x\ln x$

$\frac{1}{V}\cdot V' = \ln x + 1$

$\Rightarrow V' = x^x(\ln x+1)$

ex: $\lim_{n\to\infty}(1+\frac{1}{n})^n$ $\ln(1+\frac{1}{n})^n = n\ln(1+\frac{1}{n})$

$(\Delta x = \frac{1}{n}\to 0)$ $= \frac{1}{\Delta x}[\ln(1+\frac{1}{n}) - \ln 1]$

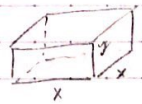
$\Rightarrow (1+\frac{1}{n})^n = e$ $= \frac{d}{dx}\ln x|_{x=1} = \frac{1}{x} = 1$

$f(x) \approx f(x_0) + f'(x_0)(x-x_0)|_{x\approx x_0}$

$\begin{cases} \sin x \approx x \\ \cos x \approx 1 \\ e^x \approx 1+x \end{cases}$ $x\approx 0$

Taylor series $\frac{f^n(0)x^n}{n!}$

Related Rates



V is fixed

$\begin{cases} V = x^2y \\ A = x^2+4xy \end{cases}$

implicit

$\frac{d}{dx}(V = x^2y) \Rightarrow 0 = 2x\cdot y + x^2y'$

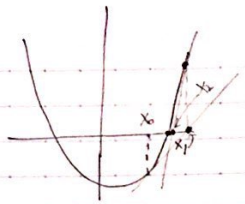
$\frac{d}{dx}A = 2x + 4y + 4xy' = 0$

$\Rightarrow x = 2y$

扫描全能王 创建

Newton's Method

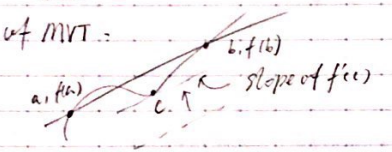
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



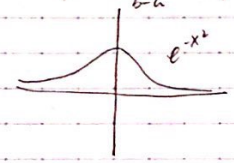
Mean Value Theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ ; for some } c,\ a<c<b$$

Proof of MVT:



$$\min f' \le \frac{f(b)-f(a)}{b-a} = f'(c) \le \max f' \text{ , min} \le \text{avg speed} \le \max$$

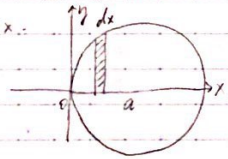


$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}F(x)$$

Volumn $\begin{cases} \#1.\ \text{Method of Disks} \\ \#2.\ \text{Method of shells} \end{cases}$

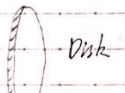
#1. Ex:



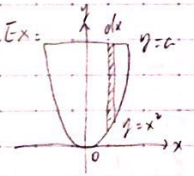
$(x-a)^2+y^2=a^2$

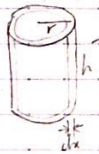
disk $dV = \pi y^2 dx$.

$$V = \int_0^{2a} \pi(2ax-x^2)dx = \frac{4}{3}\pi a^3$$



#2 Ex:

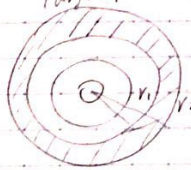


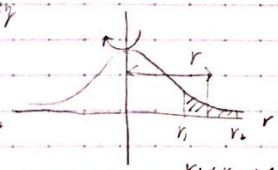


$\rightarrow V = 2\pi r \cdot h \cdot dx$

$$\int_0^{\sqrt{a}} 2\pi x(a-x^2)dx$$

Target Probability





PART: $\int_{r_1}^{r_2} (2\pi r)e^{-r^2}dr$

$= \pi(e^{-r_1^2} - e^{-r_2^2})$

$r_1 < r = x < r_2$ Whole: $\int_0^{\infty} (2\pi r)e^{-r^2}dr = \pi$

$\sec = \frac{1}{\cos}$ $\quad \tan = \frac{\sin}{\cos}$

$\csc = \frac{1}{\sin}$ $\quad \cot = \frac{\cos}{\sin}$

$\sec^2 = 1 + \tan^2$

$\tan' = \sec^2$

$\sec' = \sec\tan$

$\int \tan dx = \int \frac{\sin x}{\cos x}dx$, $u = \cos x \Rightarrow -\ln\cos x + C$

$\int \sec dx = \ln(\tan x + \sec x) + C$

$\sqrt{a^2-x^2}$ $\begin{cases} x = a\sin u \\ \text{or } x = a\sin\theta \end{cases}$

$\sqrt{a^2+x^2}$ $\quad x = a\tan\theta$

$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$

$$\frac{d}{dx}(\underbrace{\tan x + \sec x}_{u}) = \sec x(\sec x + \tan x)$$

$\Rightarrow u' = u\sec x$

$\Rightarrow \sec x = \frac{u'}{u} = (\ln u)'$

Partial Fractions

$\frac{P(x)}{Q(x)}$ = rational function

ex: $\frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

ex: $\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

ex: $\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

Integrate by part

$$\int uv'dx = uv - \int u'v dx$$

Reduction formula

$$\int (\ln x)^n dx$$

$$\int \underbrace{\ln x}_{(v'=1,\ u'=\frac{1}{x},\ v=x)} dx = x\ln x - x + C$$

$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\, dx$

$v'=1, u=(\ln x)^2$, $v=x$, $u' = 2\ln x \cdot \frac{1}{x}$

$= x(\ln x)^2 - 2(x\ln x - x) + C$

$$\int (\ln x)^n dx = F_n(x) \Rightarrow F_n(x) = x(\ln x)^n - nF_{n-1}(x)$$

ex: $\int \underbrace{x^n}_{u}\underbrace{e^x}_{v'}dx = x^n e^x - \int nx^{n-1}e^x dx$

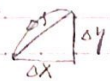
$\Rightarrow u' = nx^{n-1}$, $v = e^x$

$F_n(x) = \int x^n e^x dx$

$f_n(x) = x^n e^x$

$\Rightarrow F_n(x) = f_n(x) - nF_{n-1}(x)$

Arclength

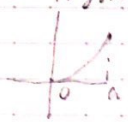


$\Delta s^2 = \Delta x^2 + \Delta y^2$

$ds = \sqrt{dx^2 + dy^2}$

$ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$

ex: length of parabola



$y = x^2$, $y' = 2x$

$ds = \sqrt{1+4x^2}\, dx$

$ds = \frac{1}{2}\sec^3\theta d\theta$

$\frac{1}{2}\underbrace{\sec\theta}_{u}\underbrace{\sec^2\theta d\theta}_{v'}$

扫描全能王 创建

$$\frac{1}{2}\int_0^a \sec^3\theta\, d\theta = \frac{1}{2}\int_0^a \underbrace{\sec\theta}_{u}\cdot\underbrace{\sec^2\theta}_{v'}\, d\theta \quad , u' = \tan\theta\sec\theta \ , \ v = \tan\theta$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^a - \int_0^a \tan^2\theta\sec\theta\, d\theta\right]$$

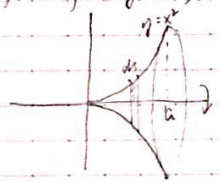
$$= \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^a - \int_0^a (\sec^2\theta - 1)\sec\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^a - \underbrace{\int_0^a \sec^3\theta}\, - \sec\theta\, d\theta\right]$$

$$\int_0^a \sec^3\theta\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^a + \int_0^a \sec\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta\Big|_0^a + \ln(\tan\theta + \sec\theta)\Big|_0^a\right]$$

Surface of $y = x^2$, x rotation



$$S = \int_0^a 2\pi x^2\sqrt{1+4x^2}\, dx \quad , x = \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$S = \int_0^a 2\pi\left(\frac{1}{2}\tan\theta\right)^2\cdot\sec\theta\cdot\frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{4}\pi\cdot\int_0^a \tan^2\theta\sec^3\theta\, d\theta$$

$$= \frac{\pi}{4}\cdot\int_0^a (\sec^2\theta - 1)\sec^3\theta\, d\theta$$

$\int \sec^n x\, dx$, n = odd ① partial Integral ② $\sec^2 x = \tan^2 x + 1$ ③ $\sec^{odd} x = \sec^{odd} x$

Surface Area of A SPHERE (radius a)

$$y = \sqrt{a^2 - x^2}$$

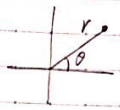
$$y' = \frac{1}{2}(a^2 - x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{a^2-x^2}}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{\frac{a^2}{a^2 - x^2}}$$

$$\int_{-a}^a 2\pi\sqrt{a^2-x^2}\cdot\sqrt{\frac{a^2}{a^2-x^2}}\, dx$$

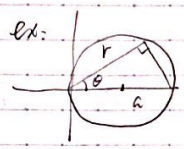
$$\int_{-a}^a 2\pi a\, dx = 4\pi a^2$$

Polar Coordinates



$x = r\cos\theta$
$y = r\sin\theta$

ex:



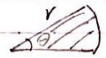
$(x-a)^2 + y^2 = a^2$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$

$r^2 - 2r\cos\theta\, a = 0$

$\Rightarrow r = 2\cos\theta\cdot a$



$$\begin{cases} A_\theta = \frac{\pi r^2}{2\pi}\cdot\theta = \frac{\theta r^2}{2} \\ dA = \frac{r^2}{2}\, d\theta \end{cases}$$

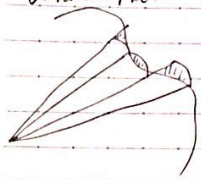
$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta\cdot a)^2}{2}\, d\theta$$

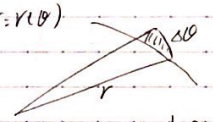
$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta\cdot a^2\, d\theta$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos 2\theta\, d\theta$$

$$= a^2\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin 2\theta}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = a^2\pi$$

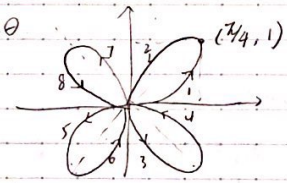
Variable Pie





$\Delta A \approx \frac{1}{2}r^2\Delta\theta$

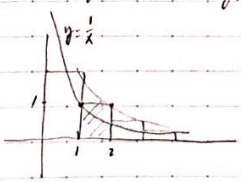
Ex: $r = \sin 2\theta$



θ	r
0	0
$\pi/4$	1
$\pi/2$	0
$3\pi/4$	-1
π	0

L'Hôpital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$



$$S_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1} + \frac{1}{N}$$

$$\ln N = \int_1^N \frac{1}{x}\, dx < S_N < \ln N + 1 \quad (N \to \infty)$$

$I := \int_1^N f(x)\, dx$ if f is decreasing function

$I + f(n) \le S \le I + f(1)$ (Reimann Sum)

else $I + f(1) \le S \le I + f(n)$

Stirling's formula

$$\ln(n!) = \ln(1\cdot 2\cdots(n-1)n) = \sum_{i=1}^n \ln(i)$$

$$S := \sum_{i=1}^n f(i) \ , \quad I := \int_1^n f(x)\, dx$$

$$\int \ln x\, dx = x\ln\left(\frac{x}{e}\right)_n$$

$$\Rightarrow n\ln\left(\frac{n}{e}\right) \le \sum_1 \ln i \le n\ln\left(\frac{n}{e}\right) + \ln n \Rightarrow \sum_{i=1}^n \ln i \approx n\ln\left(\frac{n}{e}\right) + \frac{\ln n}{2}$$

$$\Rightarrow n! \approx \sqrt{n}\left(\frac{n}{e}\right)^n$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

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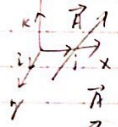
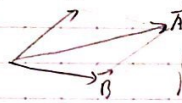
Multivariable Calculus

$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \langle a_1, a_2, a_3\rangle$



length of $\vec{A} = \langle 3,2,1\rangle = \sqrt{3^2+2^2+1^2} = \sqrt{14}$



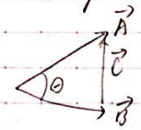
$\vec{A}+\vec{B} = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$

$\begin{cases}\vec{A}\langle a_1, a_2, a_3\rangle \\ \vec{B}\langle b_1, b_2, b_3\rangle\end{cases}$

Dot product

Def: $\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + \dots + a_nb_n$

Geometrically $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cdot\cos\theta$



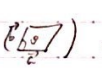
$\vec{C} = \vec{A} - \vec{B}$

$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$

$|\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})(\vec{A}-\vec{B})$

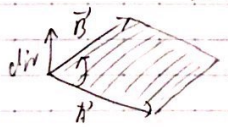
$= |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\vec{B}$

area ( ) $= |\vec{A}||\vec{B}|\sin\theta = \det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix} = a_1b_2 - a_2b_1$

volume () $\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}$

Cross product of 2 vectors

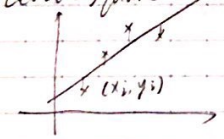
$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}\hat{k}$



$|\vec{A}\times\vec{B}|$ = area of parallelogram

$\text{dir}(\vec{A}\times\vec{B}) \perp$ to the plane of the para

Least-Square



find "best fit" line $y = ax+b$

Minimize $D(a,b) = \sum_{i=1}^{n}[y_i - (ax_i+b)]^2$

want $\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i+b))\cdot(-x_i) = 0$

$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - (ax_i+b))\cdot(-1) = 0$

Total Differential

$df = f_x dx + f_y dy + f_z dz$

$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$\frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt}$

Ex. $W = x^2y + z$, $x = t$, $y = e^t$, $z = \sin t$

$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt} + \frac{\partial W}{\partial z}\frac{dz}{dt}$

$= 2xy\cdot 1 + x^2\cdot e^t + 1\cdot\cos t$

$= 2t\cdot e^t + t^2e^t + \cos t$

$\begin{cases}\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}\end{cases}$

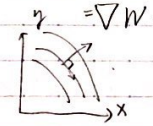
$f = f(x,y)$, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta\end{cases}$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cdot\cos\theta + f_y\sin\theta$

$W = W(x,y,z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$

$\frac{dW}{dt} = W_x\frac{dx}{dt} + W_y\frac{dy}{dt} + W_z\frac{dz}{dt}$

$= \nabla W\cdot\frac{d\vec{r}}{dt}$ $\left(\frac{d\vec{r}}{dt} = \langle\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle\right)$



Lagrange Multipliers

Minimize $f(x,y) = x^2+y^2$, subject to constraint $xy = 3$

$L(x,y;\lambda) = x^2+y^2+\lambda(xy-3)$

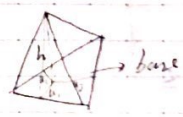
$\frac{\partial L}{\partial x} = 2x + \lambda y = 0$

$\begin{cases}\frac{\partial L}{\partial y} = 2y + \lambda x = 0 \Rightarrow \lambda = \pm 2 \Rightarrow x = y = \sqrt{3} \\ \frac{\partial L}{\partial \lambda} = xy - 3 = 0\end{cases}$

$\begin{bmatrix} 2 & \lambda \\ \lambda & 2\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix}$

M

$\det(M) = 0 \Rightarrow \lambda = \pm 2$



Ex.

min (max) Surface Area of pyramid with base and h

$V = \frac{1}{3}$ Area(base) · height

Side area $= \frac{1}{2}a_1\sqrt{a_1^2+h^2} + \frac{1}{2}a_2\sqrt{a_2^2+h^2}$

$+ \frac{1}{2}a_3\sqrt{a_3^2+h^2}$

$= f(a_1, a_2, a_3)$

Cut base into 3 $\Rightarrow$ Area base $= \frac{1}{2}a_1h_1 + \frac{1}{2}a_2h_2 + \frac{1}{2}a_3h_3 = g(h_1, h_2, h_3)$

$\nabla f = \lambda\nabla g$, $\frac{\partial f}{\partial h_1} = \frac{1}{2}a_1\frac{h_1}{\sqrt{a_1^2+h_1^2}} = \lambda\frac{1}{2}a_1 \Rightarrow \frac{h_1}{\sqrt{a_1^2+h_1^2}} = \lambda$

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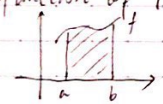
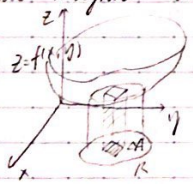
Double Integral

function of 1 variable



$\int_a^b f(x)dx$ = area below graph of f over $[a,b]$

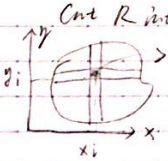
Double integral = volumn below graph $z=f(x,y)$



over a region R in x-y plane.

$\iint_R f(x,y)\,dA$

Def: Cut R into small pieces of area ΔA

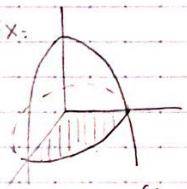


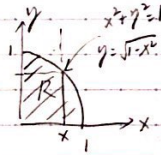
$\Delta A = \Delta x_i \cdot \Delta y_i$

$\left(\sum_i f(x_i,y_i)\Delta A_i\right)$

Take the limit as $\Delta A \to 0$, get $\iint$.

Ex:





$x^2+y^2=1$

$y=\sqrt{1-x^2}$

R = quarter-disk

$x^2+y^2 \le 1$

$x \ge 0,\ y \ge 0$

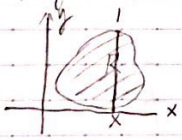
$\int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx$

$= \frac{\pi}{8}$

To compute $\iint_R f(x,y)dA$: take slices, let $S(x)$ area of slice by plane // yz-plane

Then volumn $= \int S(x)dx$

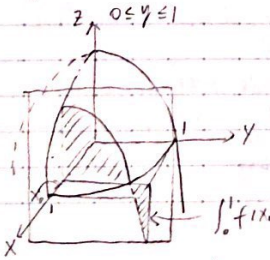
For given x, $S(x) = \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy$



$\iint_R f(x,y)dA = \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy\right] dx$

Ex1: $z = 1-x^2-y^2$

region: $0 \le x \le 1$, $0 \le y \le 1$



$\int_0^1 \int_0^1 1-x^2-y^2\,dy\,dx$

$\int_0^1 \left[y - x^2y - \frac{y^3}{3}\Big|_0^1\right] dx$

$\int_0^1 \left(1-x^2-\frac{1}{3}\right) dx = \frac{1}{3}$

$\int_0^1 f(x_0,y)\,dy$

Polar:

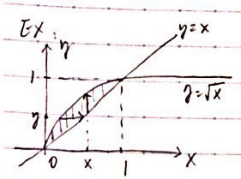
$x^2+y^2 \le 1$, $x \ge 0$, $y \ge 0$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\int_0^{\frac{\pi}{2}} \int_0^1 (1-r^2)r\,dr\,d\theta$

$= \frac{\pi}{8}$

Ex:

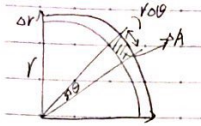


$\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx = \int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy$

$= \int_0^1 e^y - ye^y\,dy$

$= e^y\Big|_0^1 - \int_0^1 ye^y\,dy$

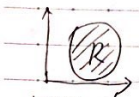
$= e-2$



$\Delta A \approx \Delta r \cdot r \cdot \Delta\theta$

$dA = r\,dr\,d\theta$

$\iint_{x^2+y^2\le 1} dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^1 r\,dr\,d\theta = \frac{\pi}{4}$



Area$(R) = \iint_R 1\,dA$

mass of a object with density δ = mass per unit area

$\Delta m = \delta \Delta A$

mass $= \iint_R \delta\,dA$

Changing variables in double $\iint$.

Jacobian: $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

① area element: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$

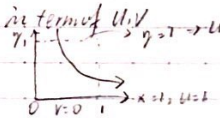
$du\,dv = x\,dx\,dy$

ex: $\int_0^1 \int_0^1 x^2y\,dx\,dy$ by changing to $\begin{cases} u = x \\ v = xy \end{cases}$

$\int_0^1 \int_v^1 v\,du\,dv$

② Integral in term of u, v

③ bounds



$\int_0^1 \int_v^1 v\,du\,dv$

u changes v = constant

xy = constant

Triple integrals $\iiint_R f\,dV$

volumn element $= dx\,dy\,dz$

Ex: region between paraboloids $\begin{cases} z = x^2+y^2 \\ z = 4-x^2-y^2 \end{cases}$

$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$

z bottom $< z$ top $\quad x^2+y^2 < 4-x^2-y^2$

$x^2+y^2 < 2$ disk of radius $\sqrt{2}$

Better use polar coords instead of x, y. $\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz\,dr\,d\theta$

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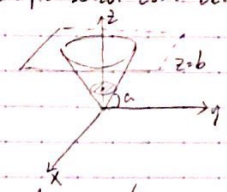
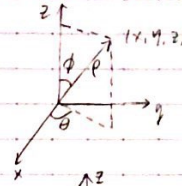
Ex. I_z of a solid cone between $z=ar$, $z=b$ ($\delta=1$)



$$I_z=\int_0^b\int_0^{2\pi}\int_0^{z/a} r^2\,r\,dr\,d\theta\,dz$$

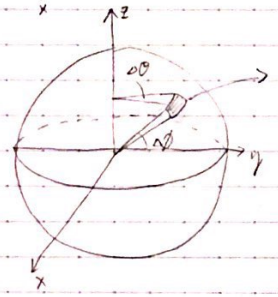
Spherical Coordinates

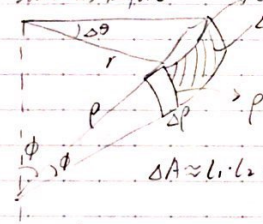


ρ = distance from origin

ϕ = angle down from +z axis $0\le\phi\le\pi$

θ Same as before





$l_1 = r\cdot\Delta\theta$, $r=\rho\cdot\sin\theta$

$\rho\,\Delta\phi = l_2$

$\Delta A \approx l_1\cdot l_2 \approx \rho\sin\phi\,\Delta\theta\cdot\rho\cdot\Delta\phi$

Surface Area (Sphere) $=\iint 1\,dA$

$$=\int_0^{\pi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\phi$$

$$=4\pi\rho^2$$

$dV=\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$

Volumn (sphere) $=\iiint 1\,dV$

$$=\int_0^{\rho}\int_0^{\pi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$$

$$=\frac{4}{3}\pi\rho^3$$

Differential Equations

First ORDER ODE'S

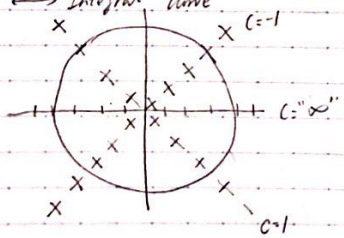
$y'=f(x,y)$ ⟷ Direction field

$y(x)$ Sol'n ⟷ Integral curve

$y'=\frac{-x}{y}$

$y'=c=-\frac{x}{y}$

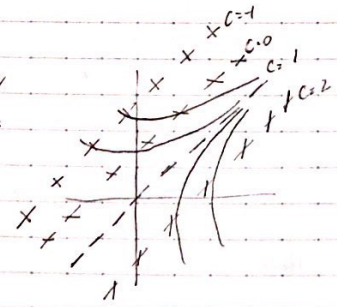
$y=\frac{-1}{c}x$



$y'=1+x-y$

isocline:

$y=1+x-c$

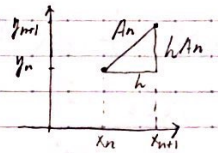


Euler's Method

$x_{n+1}=x_n+h$

$y_{n+1}=y_n+hA_n$

$A_n=f(x_n,y_n)$



First-Order LINEAR

$a(x)y'+b(x)y=c(x)$ ($c=0$ homogenous)

Standard linear form

$y'+p(x)y=q(x)$

method — Integrating factor $u(x)$

⓪ Standard linear form

① Calc $e^{\int p\,dx}$ int factor

② Multiply both sides by $e^{\int p\,dx}$

③ Integrate

Ex: $xy'-y=x^3$

$y'-\frac{1}{x}y=x^2$

$e^{\int -\frac{1}{x}dx}=\frac{1}{x}$

$(\frac{1}{x}y)'=x^2\cdot\frac{1}{x}$

$\frac{1}{x}y=\int x\,dx$

$\frac{1}{x}y=\frac{x^2}{2}+C$

$y=\frac{x^3}{2}+Cx$

Temp. $\frac{dT}{dt}+kT=kT_e$ IF: e^{kt}

$(e^{kt}T)'=e^{kt}kT_e$

$e^{kt}T=\int e^{kt}kT_e\,dt+C$

$$T=\underbrace{e^{-kt}\int_0^t kT_e(t)e^{kt}\,dt}_{\text{Steady state soln}}+Ce^{-kt}$$

Substitution $y'=f(x,y)$

scaling: $x_1=\frac{x}{a}$, $y_1=\frac{y}{b}$

Ex: $\frac{dT}{dt}=k(m^4-T^4)$

Set $T_1=\frac{T}{m}$

$m\frac{dT_1}{dt}=km^4(1-T_1^4)$

$\frac{dT_1}{dt}=k_1(1-T_1^4)$

Bernoulli Eqn

$y'=\frac{y}{x}-y^2 \Rightarrow \frac{y'}{y^2}-\frac{1}{xy}=-1$

set $V=\frac{1}{y}$, $V'=-\frac{1}{y^2}$

$-V'-\frac{1}{x}V=-1$

$V'+\frac{1}{x}V=1$

$(e^{\int\frac{1}{x}dx}V)'=x$

$xV=\frac{1}{2}x^2+C$

$V=\frac{1}{2}x+\frac{C}{x}$

$V=\frac{x^2+2C}{2x} \Rightarrow y=\frac{2x}{x^2+C_1}$

Bernoulli eqn

$y'=p(x)y+q(x)y^n$ ($n\neq0,\pm1$)

$\frac{y'}{y^n}=p(x)\frac{1}{y^{n-1}}+q(x)$, $V=\frac{1}{y^{n-1}}$

$V'=(1-n)\frac{1}{y^n}$

$\frac{V'}{1-n}=p(x)V+q(x)$

$dV=(1-n)\frac{1}{y^n}dy$

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Homogeneous ODE's

$y' = F(y/x)$

ex. $y' = \frac{x^2 y}{x^3 + y^3} = \frac{y/x}{1+(y/x)^3}$

set $\frac{y}{x} = z$, $dy = z'x + z$

$z'x + z = \frac{z}{1+z^3}$

ex: $y' = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$ set $\frac{y}{x} = z$

$z'x + z = \frac{1+z}{1-z}$

$\frac{dz}{dx} x = \frac{1+z^2}{1-z}$

$\int \frac{1-z}{1+z^2} dz = \int \frac{1}{x} dx$

set $z = \tan\theta$ $dz = \sec^2\theta \, d\theta$

$\Rightarrow \tan^{-1} z - \frac{1}{2}\ln(1+z^2) = \ln x + C$

$\tan^{-1}\frac{y}{x} = \ln\sqrt{x^2+y^2} + C$

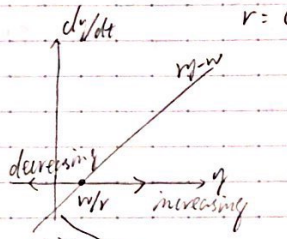
$\theta = \ln r + C$

$r = C_1 e^{\theta}$

Critical point

$\frac{dy}{dt} = ry - w$

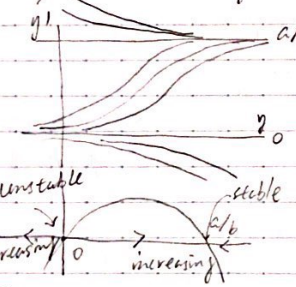
$\Rightarrow ry - w = 0$, $y = \frac{w}{r}$



Logistic Equation

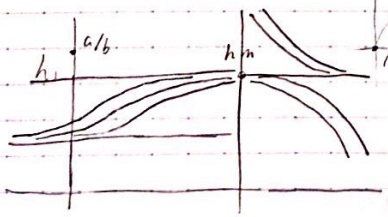
$\frac{dy}{dt} = ay - by^2$

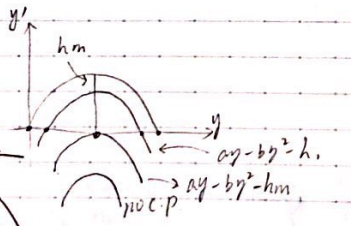
$\Rightarrow ay - by^2 = 0$, $y = 0$ or a/b



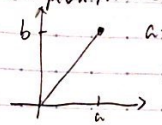
Logistic eqn with Harvesting

$\frac{dy}{dt} = ay - by^2 - h$





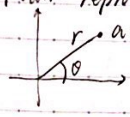
Complex Number



$a + ib = z$ $\begin{cases} z = a+bi \\ \bar{z} = a-bi \end{cases}$ $z\bar{z} = a^2 + b^2$

$i^2 = -1$

Polar repn



$e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{d}{dt} e^{it} = ie^{it}$

$e^{i0} = 1$

α complex number

$\frac{1}{\alpha} \cdot \alpha = 1$ $\arg(\frac{1}{\alpha}) + \arg(\alpha) = 0$

$\left|\frac{1}{\alpha}\right| = \frac{1}{|\alpha|}$ $\arg(\frac{1}{\alpha}) = -\arg(\alpha)$

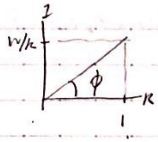
$\begin{cases} r = \sqrt{a^2+b^2}, \ \theta = \arg(\alpha) \ (a+bi) \\ \frac{1}{a+bi} \Rightarrow r = \frac{1}{\sqrt{a^2+b^2}} \quad \theta' = -\theta \end{cases}$

ex: $y' + ky = k\cos wt$

$(\tilde{y}e^{kt})' = ke^{iwt} \cdot e^{kt}$

$\tilde{y} e^{kt} = \int k \cdot e^{(iw+k)t} dt$

$= k \cdot \frac{e^{(iw+k)t}}{iw+k} + C$



$1 + i w/k$

$\tan\phi = \frac{w/k}{1}$

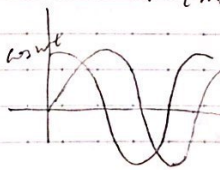
$\tilde{y} = \frac{k}{iw+k} e^{iwt} + Ce^{-kt} = \frac{1}{1+iw/k} e^{iwt} + Ce^{-kt}$

$y_{real}: \frac{1}{\sqrt{1+(w/k)^2}} e^{iwt} e^{-i\phi} = \frac{1}{\sqrt{1+w^2/k^2}} e^{i(wt-\phi)} = \frac{1}{\sqrt{1+(w/k)^2}} \cos(wt - \phi)$

Amplitude

phase lag

$\phi = \arctan(w/k)$



$\cos(wt - \phi)$

if $\phi = \pi/2$

$A\cos\theta + B\sin\theta = C\cos(\theta - \phi)$

$C = \sqrt{A^2+B^2}$, $\phi = \arctan\frac{B}{A} = \tan^{-1}\frac{B}{A}$

proof:

$(a-bi)(\cos\theta + i\sin\theta)$ real part

$= \sqrt{a^2+b^2} e^{-i\phi} e^{i\theta} = \sqrt{a^2+b^2} e^{i(\theta-\phi)}$

$= \sqrt{a^2+b^2} \cos(\theta - \phi)$

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Linear 2nd order ODE

$y'' + Ay' + By = 0$ ← homogenous (constant coeffs)

gen. soln $y = c_1 y_1 + c_2 y_2$

ex: $x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$

$$\begin{bmatrix} x'' \\ x' \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{c}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x' \\ x \end{bmatrix}$$

$\Rightarrow (-\frac{c}{m} - \lambda)(-\lambda) + \frac{k}{m} = 0$

$\Rightarrow \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

soln: $C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$

Case I real roots $r_1 \neq r_2$

gen. soln: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Case II complex roots $r = a \pm bi$

gen. soln: $y = e^{at}(C_1 \cos bt + C_2 \sin bt)$

Case III real roots $r_1 = r_2$

gen. soln: $(c_1 + t c_2) e^{rt}$

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$

$\cos t = \frac{e^{it} + e^{-it}}{2}$, $i \sin t = \frac{e^{it} - e^{-it}}{2}$

Inhomogeneous EQN

$y'' + P(x)y' + q(x)y = f(x)$

$y(x)$ response output, ↑ Input

Soln: $y_p + \underbrace{y_c}_{c_1 y_1 + c_2 y_2}$

ex: $y' + ky = q(t)$

$y = \underbrace{e^{-kt}\int q(t) e^{kt} dt}_{y_p} + \underbrace{Ce^{-kt}}_{y_c}$

$y_p = \frac{e^{\alpha x}}{p(\alpha)}$

ex: $y'' - y' + y = 10 e^{-x} \sin x$

$= \frac{10 e^{(i-1)x}}{(i-1)^2 - (i-1) + 2}$

$\tilde{y} = \frac{10 e^{(i-1)x}}{3 - 3i}$ (Im part)

$= \frac{10}{\sqrt{18}} e^{i\phi} e^{(i-1)x}$

$y = \frac{10}{\sqrt{18}} e^{-x} e^{i(x+\phi)} = \frac{10}{\sqrt{18}} e^{-x} \sin(x + \phi)$, $\phi = \arctan 1$

Exponential shift rule

$P(D) e^{ax} u(x) = e^{ax} P(D+a) u(x)$

(i) $y_p = \frac{e^{\alpha x}}{p(\alpha)}$

(ii) $y_p = \frac{x e^{\alpha x}}{p'(\alpha)}$ a simple root of $p(D)$

(iii) $y_p = \frac{x^2 e^{\alpha x}}{p''(\alpha)}$ double root of $p(D)$

Fourier series (t, x domain)

$y'' + ay' + by = f(t)$

$f(t) = C_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$

Input	Response
$b_n \sin nt$	$\longrightarrow b_n y_n^{(1)}(t)$
$a_n \cos nt$	$\longrightarrow b_n y_n^{(2)}(t)$
$f(t)$	$\longrightarrow \sum_1^{\infty} a_n \cos nt + b_n \sin nt + C_0$

$\int_{-\pi}^{\pi} \sin^2 nt \, dt = \pi$ $\quad \int_{-\pi}^{\pi} \frac{1 - \cos 2nt}{2} dt = \pi$

$\int_{-\pi}^{\pi} \cos^2 nt \, dt = \pi$

$\int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0$

$n \neq m$ proof:

$\sin nt$, $\cos mt$ $\quad \begin{cases} u'' + n^2 u = 0 \\ u'' + m^2 u = 0 \end{cases}$

$\int_{-\pi}^{\pi} u_n'' v_m dt = -n^2 \int_{-\pi}^{\pi} u_n v_m dt$

$\int_{-\pi}^{\pi} u_n v_m'' dt = -m^2 \int_{-\pi}^{\pi} u_n v_m dt$

$\int_{-\pi}^{\pi} u_n'' v_m dt = \underbrace{u_n' v_m \Big|_{-\pi}^{\pi}}_{0} - \int_{-\pi}^{\pi} u_n' v_m' dt$

① if $m \neq n \Rightarrow \int_{-\pi}^{\pi} u_n v_m dt = 0$

$m = n$,

$\int_{-\pi}^{\pi} \sin nt \cos nt \, dt = \int_{-\pi}^{\pi} \frac{\sin 2nt}{2} dt = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt$, $n = 1, 2, \ldots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt$, $n = 1, 2, \ldots$

$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) dt$, $n = 0$ $\quad \left[C_0 = \frac{a_0}{2}\right]$

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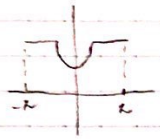
$f(t)$ even: $a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$ ($b_n = 0$)

$f(t)$ odd: $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$ ($a_n = 0$)

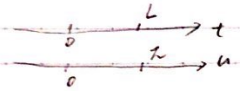


ex: $b_n = \frac{2}{\pi}\int_0^{\pi} t\sin nt\,dt$

$= \frac{2}{\pi}\left[-t\frac{\cos nt}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nt}{n}\,dt = \frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{\sin nt}{n^2}\right]_0^{\pi}$

$b_n = \frac{2}{n}(-1)^{n+1}$

period is $2L$



$\begin{cases} t = \frac{L}{\pi}u \\ u = \frac{\pi}{L}t \end{cases}$

Thm: If f is continuous function at t_0, then $f(t_0) = \sum$ of F.S. at t_0.

If f has a jump discontinuous at t_0 — F.S. converges

then $f(t_0)$ = sum of F.S. at t_0 = middle of the jump point.

Laplace Transform (t domain)

$\int_0^{\infty} f(t)e^{-st}\,dt = F(s)$

$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

$\mathcal{L}(cf) = c\mathcal{L}(f)$

$\int_0^{\infty} 1e^{-st}\,dt = \frac{1}{s}$ iff $s>0$ $\qquad 1 \rightsquigarrow \frac{1}{s}$

$\int_0^{\infty} e^{at}e^{-st}\,dt = \frac{1}{s-a}$ $\qquad e^{at} \rightsquigarrow \frac{1}{s-a}$

$\int_0^{\infty} e^{at}f(t)e^{-st}\,dt = F(s-a)$ $\qquad \mathcal{L}(e^{-at}f(t)) = F(s+a)$

First Shifting theorem

$\int_0^{\infty} \sin at\, e^{-st}\,dt = \frac{a}{s^2+a^2}$ $\qquad \sin at \rightsquigarrow \frac{a}{s^2+a^2}$

$\int_0^{\infty} \cos at\, e^{-st}\,dt = \frac{s}{s^2+a^2}$ $\qquad \cos at \rightsquigarrow \frac{s}{s^2+a^2}$

$t^n \rightsquigarrow \int_0^{\infty} t^n e^{-st}\,dt = \frac{n!}{s^{n+1}}$

$\int_0^{\infty} \sin at\, e^{-st}\,dt = \int_0^{\infty} e^{iat}e^{-st}\,dt = \int_0^{\infty} e^{(ia-s)t}\,dt = \left.\frac{e^{(ia-s)t}}{ia-s}\right|_0^{\infty}$

or $\sin at = \frac{e^{iat}-e^{-iat}}{2i} = \frac{\frac{1}{s-ia}-\frac{1}{s+ia}}{2i} = \frac{\cancel{2}ia}{s^2+a^2}\cdot\frac{1}{\cancel{2i}} = \frac{-(s+ia)}{s^2+a^2}\left.e^{(ia-s)t}\right|_0^{\infty}$

$= \frac{a}{s^2+a^2} = \frac{s+ia}{s^2+a^2}$ take Imaginary.

$= \frac{a}{s^2+a^2}$

IVP $y(t)$ Soln $\rightsquigarrow Y(s)$

$\boxed{\begin{array}{l} y''+Ay'+By = f(t) \\ y(0)=y_0,\ y'(0)=y_0' \end{array}}$ $\qquad \boxed{y = y(t)}$

$\downarrow \mathcal{L} \qquad\qquad \uparrow \mathcal{L}^{-1}$

$\boxed{\text{Alg eqn in } Y(s)} \xrightarrow[Y]{\text{solve for}} \boxed{Y = \frac{P(s)}{Q(s)}}$

$\mathcal{L}(f'(t)) = \int_0^{\infty} \underbrace{f'(t)}_{v'}\,\underbrace{e^{-st}}_{u}\,dt = e^{-st}f(t)\Big|_0^{\infty} - \int_0^{\infty} -se^{-st}f(t)\,dt$

$u' = -se^{-st}$ $\qquad = -f(0) + s\int_0^{\infty} f(t)e^{-st}\,dt = s\mathcal{L}(f(t)) - f(0)$

$v = f(t)$

$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0)$

$= s(s\mathcal{L}(f(t)) - f(0)) - f'(0)$

$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$

ex: $y''-y = e^{-t}$, $y(0)=1$, $y'(0)=0$

$\downarrow$

$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s+1}$

$s^2Y - s - Y = \frac{1}{s+1}$

$Y = \frac{s^2+s+1}{s+1}\cdot\frac{1}{(s+1)(s-1)} = \frac{s^2+s+1}{(s+1)^2(s-1)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s-1} = \frac{-1/2}{(s+1)^2} + \frac{1/4}{s+1} + \frac{3/4}{s-1}$

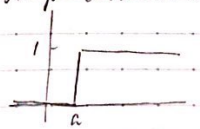
$y = -\frac{1}{2}t\cdot e^{-t} + \frac{1}{4}e^{-t} + \frac{3}{4}e^{t}$

Convolution

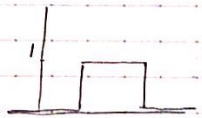
$F(s) = \int_0^{\infty} e^{-st}f(t)\,dt$ $\qquad f(t)*g(t) = \int_0^{t} f(x)g(t-x)\,dx$

$G(s) = \int_0^{\infty} e^{-st}g(t)\,dt$ $\qquad F(s)\cdot G(s) = \int_0^{\infty} e^{-st}(f*g)\,dt$

Jump discontinuous.



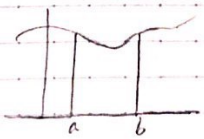
$u_a(t) = u(t-a)$



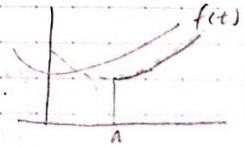
$u_{ab}(t)$ = "unit box"

$u_{ab}(t) = u_a(t) - u_b(t)$

$= u(t-a) - u(t-b)$



$u_{ab}(t)\cdot f(t)$



$u(t-a)f(t-a) \rightsquigarrow e^{-as}F(s)$

$u(t-a)f(t) \rightsquigarrow e^{-as}\mathcal{L}(f(t+a))$

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ex. $u(t-1)t^2 \xrightarrow{\mathcal{L}} e^{-1s}\mathcal{L}(t+1)^2 = e^{-1s}(t^2+2t+1) = e^{-1s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

$\mathcal{L}\, u(t-a)f(t-a) = \int_0^\infty u(t-a)f(t-a)e^{-st}dt$, $t-a=t_1$

$= \int_{-a}^\infty u(t_1)f(t_1)e^{-s(t_1+a)}dt_1$

$= e^{-as}\underbrace{\int_0^\infty u(t_1)f(t_1)e^{-st_1}dt_1}_{\mathcal{L}(f(t_1))}$

$\mathcal{L}^{-1}\left(\frac{1+e^{-2s}}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}+\frac{e^{-2s}}{s^2+1}\right) = \sin t + u(t-2)\sin(t-2)$

$f(t) = \begin{cases} \sin t & , 0\le t\le 2 \\ \sin t + (-\sin t) & , t\ge 2 \end{cases}$

Input unit impulse

$\frac{1}{h}u_{0h}(t) \xrightarrow{\mathcal{L}} \frac{1-e^{-hs}}{hs}$

$\downarrow \qquad \downarrow$ as $h\to 0$

$\delta(t) \longrightarrow 1$

$\delta(t) \longrightarrow 1 \qquad u(t)f(t)*\delta(t) \xrightarrow{\mathcal{L}} F(s)\cdot 1$

ex. $y''+y = A\delta(t-\frac{\pi}{2})$, $y(0)=1$, $y'(0)=0$

$s^2Y - sy(0) - y'(0) + Y = Ae^{-\frac{\pi}{2}s}$

$s^2Y - s + Y = Ae^{-\frac{\pi}{2}s}$

$Y = \frac{Ae^{-\frac{\pi}{2}s}+s}{s^2+1} \Rightarrow Au(t-\frac{\pi}{2})\sin(t-\frac{\pi}{2}) + \cos t = y$

$y = \begin{cases} \cos t & , 0\le t\le \frac{\pi}{2} \\ (1-A)\cos t & , t\ge \frac{\pi}{2} \end{cases}$

$y''+ay'+by = f(t)$, $y(0)=y'(0)=0$

$s^2Y + asY + bY = F(s)$

$Y = F(s)\cdot\underbrace{\frac{1}{s^2+as+b}}_{W(s)} \Rightarrow y = f(t)*W(t)$

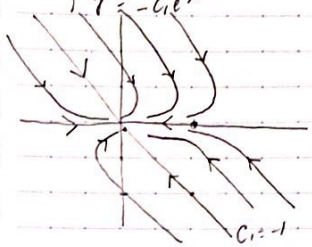
$Y = F(s)\cdot W(s)$ $\qquad = \int_0^t f(u)W(t-u)du$

$\begin{cases} x' = -x+by \\ y' = cx-3y \end{cases}$ $\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{bmatrix} -x & b \\ c & -3 \end{bmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \lambda^2 - tr(A)\lambda + \det A = 0$

ex: $\begin{matrix} x' = -x+2y \\ y' = -3y \end{matrix}$ $\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \lambda^2+4\lambda+3=0 \Rightarrow \begin{matrix} \lambda_1=-3 \\ \lambda_2=-1 \end{matrix}$

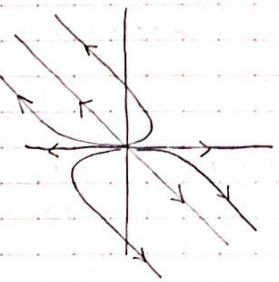
$\lambda_1=-3$, $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \lambda_2=-1$, $x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \underset{t\to\infty \text{ domain}}{C_1e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}} + \underset{t\to\infty \text{ domain}}{C_2e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\begin{cases} x = C_1e^{-3t}\cdot 1 + C_2e^{-t}\cdot 1 \\ y = -C_1e^{-3t} \end{cases}$



Nodal SINK
stable

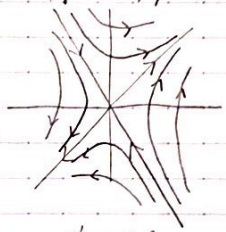
If A positive
Solv.



$C_1e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

source node
unstable

ex. $\begin{cases} x' = -x+3y \\ y' = 5x-3y \end{cases}$ $\quad A = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$ $\quad \begin{matrix} \lambda_1 = -6, & x_1 = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \\ \lambda_2 = 2, & x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{matrix}$ $\quad$ Gen Soln: $\begin{pmatrix} x \\ y \end{pmatrix} = C_1e^{-6t}\begin{pmatrix} -3 \\ 5 \end{pmatrix} + C_2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

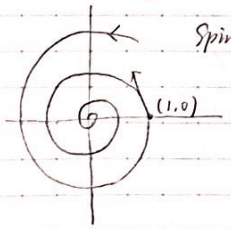


Saddle
unstable

A: Jacobian Matrix.

$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}$

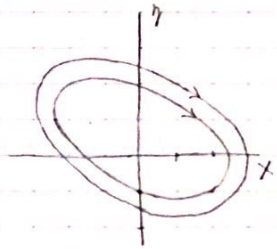
ex. $\begin{cases} x' = -x-y \\ y' = 2x-3y \end{cases}$ $\quad A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$ $\quad \begin{matrix} \lambda_1 = -2+i \\ \lambda_2 = -2-i \end{matrix}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = e^{-2t}\left(\overset{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}{C_1}\cos t + \overset{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{C_2}\sin t\right)$



Spiral sink

$x_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

$x_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$



ex: $\begin{cases} x' = x+2y \\ y' = -x-y \end{cases}$ $\quad A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ $\quad \begin{matrix} \lambda_1 = +i & x_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} \\ \lambda_2 = -i \end{matrix}$

Soln: $\begin{pmatrix} 1 \\ -1+i \end{pmatrix}e^{it} = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right](\cos t + i\sin t)$

real part $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix}$

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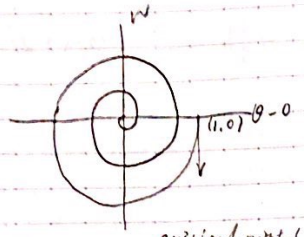
Autonomous

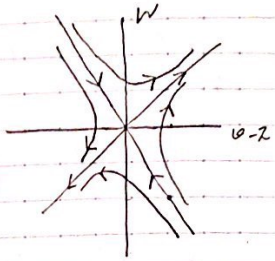
$\theta'' + c\theta' + k\sin\theta = 0$

$\begin{cases} \theta' = w \\ w' = -k\sin\theta - cw \end{cases}$

let $c=1$, $k=2$

$\begin{cases} \theta' = w \\ w' = -2\sin\theta - w \end{cases}$





critical point (0,0)

critical points: $w=0$, $-2\sin\theta - w = 0 \Rightarrow \theta = 0, \pm\pi, \pm 2\pi \cdots$

(i) at (0,0)

$\begin{cases} \theta' = w \\ w' = -2\theta - w \end{cases}$ $\underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}}_{A}$ Jmatrix $\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{-7}}{2}$

(ii) at (π,0)

$\begin{bmatrix} 0 & 1 \\ -2\cos\theta & -1 \end{bmatrix}$ $\underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A}$ $\lambda^2 + \lambda - 2 = 0$

$\begin{cases} \lambda_1 = -2 & x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ \lambda_2 = 1 & x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$

gen soln: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{cases} x' = y \\ y' = -x \end{cases}$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda = \pm i$ use (2) $x = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)(\cos t + i\sin t)$

real part: $c_1\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

Predator $x' = -ax + bxy$

pre $y' = cy - dxy$

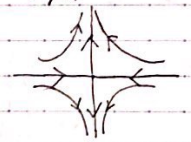
$A = \begin{bmatrix} -a+by & bx \\ -dy & c-dx \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1+y & x \\ -y & 1-x \end{bmatrix}$

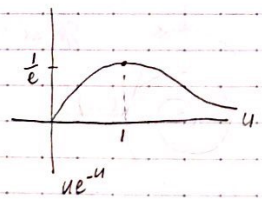
assume $a=b=c=d=1$

critical point: $\begin{cases} -x + xy = 0 \\ y - xy = 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}, \begin{cases} x=1 \\ y=1 \end{cases}$

(0,0): $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

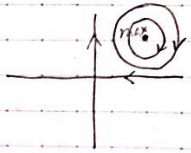
$\lambda = \pm 1$, $\lambda_1 = 1$, $x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\lambda_2 = -1$, $x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$





gen soln: $c_1 e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(1,1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda = \pm i$



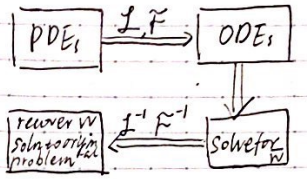
gen soln: $c_1\cos t + c_2\sin t$

$\begin{cases} x' = -x + xy \\ y' = y - xy \end{cases}$ $\frac{dy}{dx} = \frac{y(1-x)}{x(-1+y)}$ $\frac{y-1}{y}dy = \frac{1-x}{x}dx$ $\frac{x}{e^x} \cdot \frac{y}{e^y} = C$

$e^y \frac{1}{y} = x\frac{1}{e^x} \cdot C$

① Wave ② Heat ③ Laplace Eqns PDEs

- Separate Variable
- Fourier Transform (d'Alembert's)
- Laplace Transform



① Wave Eqn — Separate Variable

$u_{tt} = c^2 u_{xx}$ $u(0,t) = u(\pi,t) = 0$

$u(x,t) = X(x)T(t)$ $u(x,0) = \phi(x)$, $u_t(x,0) = \varphi(x)$

$T''(t)X(x) = c^2 X''(x)T(t)$ $u(x,t) = X(x)\,T(t)$

$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$ $\because$ $u(0,t) = X(0)\,T(t) = 0$, $u(\pi,t) = X(\pi)\cdot T(t) = 0$ $\Rightarrow \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$

$X''(x) + \lambda X(x) = 0$

$\sigma^2 + \lambda = 0 \Rightarrow \sigma = \pm\sqrt{-\lambda}$

(i) $\lambda < 0$, $X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ $\Rightarrow A = B = 0$

(ii) $\lambda = 0$, $X = Ax + B$ $\Rightarrow A = B = 0$

(iii) $\lambda > 0$, $\sigma = \pm i\sqrt{\lambda}$, $X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \Rightarrow A = 0$, $\sqrt{\lambda} = n$, $n = 1, 2, \cdots$

$X_n(x) = B_n \sin nx$

$T'' + \lambda c^2 T = 0 \Rightarrow T_n(t) = C_n\cos nct + D_n\sin nct$

$u_n(x,t) = X_n(x)\,T_n(t) = (a_n\cos nct + b_n\sin nct)\sin nx$

$u(x,t) = \sum^{n} (a_n\cos nct + b_n\sin nct)\sin nx$

$u(x,0) = \phi(x) = \sum_1^n a_n\sin nx \Rightarrow a_n = \frac{2}{\pi}\int_0^\pi \phi(x)\sin nx\,dx$

$\int_0^\pi \phi(x)\sin nx\,dx = \int_0^\pi a_n\sin nx\sin nx\,dx$

$\int_0^\pi \phi(x)\sin nx\,dx = \frac{1}{2}\pi a_n$

$u_t(x,0) = \varphi(x) = \sum_1^n b_n nc\sin nx \Rightarrow b_n = \frac{2}{\pi nc}\int_0^\pi \varphi(x)\sin nx\,dx$

$\int_0^\pi \varphi(x)\sin nx\,dx = \int_0^\pi nc\,b_n\sin nx\sin nx\,dx$

$= \frac{1}{2}\pi nc\,b_n$

$\Rightarrow u(x,t) = \sum_1^{n\to\infty}\left[\left(\frac{2}{\pi}\int_0^\pi \phi(x)\sin nx\,dx\right)\cos nct + \left(\frac{2}{\pi nc}\int_0^\pi \varphi(x)\sin nx\,dx\right)\sin nct\right]\sin nx$

① Using Initial Condition find A_n, B_n (Fourier Series)

② Boundary Condition $\Rightarrow \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$

③ $\lambda \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$ Analysis.

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Heat Eqn — Separate Variable

$u_t = c^2 u_{xx}$ $\quad u(0,t)=0 \quad u(x,0)=f(x)$

$u(x,t) = X(x)\cdot T(t)$ $\quad u(\pi,t)=0$

$X(x)T'(t) = c^2 X''(x)T(t)$

$\frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = -\lambda$ $\quad \because u(0,t)=0 \Rightarrow \begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases}$ $\quad u(\pi,t)=0$

$X''+\lambda X=0$

$\sigma^2+\lambda=0 \Rightarrow \sigma=\pm\sqrt{-\lambda}$

(i) $\lambda=0 \Rightarrow X(x)=Ax+B \Rightarrow A=B=0$

(ii) $\lambda>0 \Rightarrow \sigma=\pm i\sqrt{\lambda} \Rightarrow X(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x \Rightarrow \begin{matrix} A=0 \\ \sqrt{\lambda}=n \end{matrix}, n=1,2\ldots$

(iii) $\lambda<0 \Rightarrow X(x)=Ae^{\sqrt{-\lambda}x}+Be^{-\sqrt{-\lambda}x} \Rightarrow A=B=0$

$\therefore X_n(x)=B_n\sin nx$

$T'=-c^2\lambda T_n \Rightarrow T_n(t)=C_n e^{-n^2c^2t}$

$u(x,t)=\sum a_n \sin nx\, e^{-n^2c^2t}$

$u(x,0)=f(x) \Rightarrow \sum_1^n a_n\sin nx = f(x)$

$\int_0^\pi a_n \sin nx \sin nx\,dx = \int_0^\pi f(x)\sin nx\,dx$

$\frac{\pi}{2}a_n = \int_0^\pi f(x)\sin nx\,dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

$u(x,t)=\sum_1^\infty\left(\frac{2}{\pi}f(x)\sin nx\,dx\right)\sin nx\, e^{-n^2c^2t}, n=1,2\ldots$

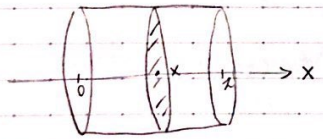
if $c^2=4$, $f(x)=\sin x-2\sin 3x$

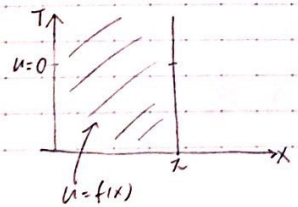
$a_n=\frac{2}{\pi}\int_0^\pi(\sin x-2\sin 3x)\sin nx\,dx, n=1,2\ldots$

$a_1=\frac{2}{\pi}\int_0^\pi \sin^2 x\,dx=1$

$a_3=\frac{2}{\pi}\int_0^\pi -2\sin^2 3x\,dx=-2$

$u(x,t)=\underbrace{a_1}_{1}\sin x\, e^{-4t}+\underbrace{a_3}_{-2}\sin 3x\, e^{-36t}$





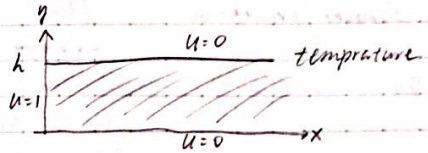
Laplace Eqn.

$u_{xx}+u_{yy}=0, \quad x>0, 0<y<h$

$u(x,0)=0, \quad u(x,h)=0, \quad x>0;$

$u(x,y)\to 0$ as $x\to\infty$, $0<y<h$

$u(0,y)=1, \quad 0<y<h.$



$u(x,y)=F(x)G(y)$

$F''(x)G(y)=-F(x)G''(y)$

$\frac{G''(y)}{G(y)}=-\frac{F''(x)}{F(x)}=-\lambda$ $\quad$ as. $u(x,0)=0, u(x,h)=0, \Rightarrow G(0)=0, G(h)=0$

(i) $\lambda>0$, $G''(y)+\lambda G(y)=0 \Rightarrow \sigma^2+\lambda=0 \Rightarrow \sigma=\pm\sqrt{-\lambda} \Rightarrow \sigma=\pm i\sqrt{\lambda}$

$G(y)=A\cos\sqrt{\lambda}y+B\sin\sqrt{\lambda}y \Rightarrow A=0, \sqrt{\lambda}=\frac{n\pi}{h}, n=1,2\ldots$

(ii) $\lambda<0$, $\sigma=\pm\sqrt{\lambda}$ $\quad G(y)=Ae^{\sqrt{\lambda}y}+Be^{-\sqrt{\lambda}y} \Rightarrow A,B=0$

(iii) $\lambda=0$, $\sigma=0$, $G(y)=Ay+B \Rightarrow A=B=0$

$G_n(y)=B_n\sin\frac{n\pi}{h}y \quad n=1,2\ldots$

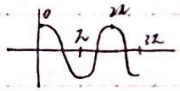
Also, $F''-\lambda F=0$, $\sigma=\pm\sqrt{\lambda}$ $\quad F_n(x)=C_n e^{\sqrt{\lambda}x}+D_n e^{-\sqrt{\lambda}x}$

$u(x,y)=\sum_1^n a_n\sin\frac{n\pi}{h}y\, e^{\frac{n\pi}{h}x}+b_n\sin\frac{n\pi}{h}y\, e^{-\frac{n\pi}{h}x}$ $\quad \int_0^h \sin\frac{n\pi}{h}y\,dy=\frac{h}{n}b_n$

$u(0,y)=1=\sum_1^n a_n\cancel{\sin\frac{n\pi}{h}y}+b_n\sin\frac{n\pi}{h}y \Rightarrow b_n=\frac{2}{h}\int_0^h\sin\frac{n\pi}{h}y\,dy$

$u(x,y)\to 0$ as $x\to\infty \Rightarrow a_n=0$

$u(x,y)=\sum_1^n b_n\sin\frac{n\pi}{h}y\, e^{-\frac{n\pi}{h}x}$



$b_n=\frac{2}{h}\int_0^h\sin\frac{n\pi}{h}y\,dy=-\frac{2}{n\pi}\cos\frac{n\pi}{h}y\Big|_0^h=-\frac{2}{n\pi}(\cos n\pi-1), n=1,2$

$b_n=\begin{cases}0 & \text{even}\\ \frac{4}{n\pi} & \text{odd}\end{cases}$

$u(x,y)=\sum_{odd}^\infty\frac{4}{n\pi}e^{-\frac{n\pi x}{h}}\sin\frac{n\pi y}{h}$

$u(x,y)=\frac{4}{\pi}\sum_1^\infty\frac{1}{2k-1}e^{-\frac{(2k-1)\pi x}{h}}\sin\left[\frac{(2k-1)\pi y}{h}\right]$

① $u(x,y)\to 0$ as $x\to\infty$

② $u(x,y)=\sum_1^\infty(\sim\sim)$

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(x domain)

Wave Eqn Fourier Transform PDEs $\longrightarrow$ ODEs

$u_{tt} = c^2 u_{xx}$ $\quad -\infty < x < \infty$, $t > 0$

$u(x,y) \longrightarrow \hat{u}(w,y)$ $\quad u(x,0) = g(x)$ $\quad -\infty < x < \infty$

$u_t(x,0) = 0$, $\quad -\infty < x < \infty$

$u(x,t) \to 0$, $u_x(x,t) \to 0$ as $|x| \to \infty$

$$F(f(x)) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$f(x) = F^{-1}(\hat{f}(w)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

$$F(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f'(x)}_{v'} \underbrace{e^{-iwx}}_{u} dx = \frac{1}{\sqrt{2\pi}} \left(\cancel{e^{-iwx} f(x) \Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f(x) (-iw) e^{-iwx} dx \right)$$

$u' = -iw e^{-iwx}$, $v = f(x)$ $\quad = iw \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = iw F(f(x))$

$F(f''(x)) = -w^2 F(f(x))$

$F(u(x,t)) = \hat{u}(w,t)$ $\qquad F(u_t) = \frac{\partial}{\partial t} \hat{u}(w,t)$

$F(u_x) = iw \hat{u}(w,t)$ $\qquad F(u_{tt}) = \frac{\partial^2}{\partial t^2} \hat{u}(w,t)$

$F(u_{xx}) = -w^2 \hat{u}(w,t)$

$-w^2 c^2 \hat{u}(w,t) = \frac{\partial^2}{\partial t} \hat{u}(w,t)$

$\Rightarrow \hat{u}'' + w^2 c^2 \hat{u} = 0$

$\sigma^2 = -w^2 c^2 \Rightarrow \sigma = \pm i w c$.

$\hat{u} = A \cos wct + B \sin wct$

$\hat{u}(x,0) = \hat{g}(w)$ $\qquad A = \hat{g}(w)$

$\hat{u}_t(x,0) = 0 \Rightarrow \hat{u}_t = -Awc \sin wct + \cancel{Bwc \cos wct}$ $\quad B = 0$

$\hat{u} = \hat{g}(w) \cos wct \xrightarrow{F^{-1}} u$

$\cos wct = \frac{e^{iwct} + e^{-iwct}}{2}$

$$u = F^{-1}(\hat{u}) = F^{-1}(\hat{g}(w) \cos(wt))$$

$$= F^{-1}\left(\hat{g}(w) \frac{e^{iwct} + e^{-iwct}}{2} \right)$$

$$= F^{-1}\left(\frac{\hat{g}(w) e^{iwct}}{2} + \frac{\hat{g}(w) e^{-iwct}}{2} \right)$$

$$= \frac{1}{2} (g(x+ct) + g(x-ct))$$

d'Alembert's soln

① Second shifting Theorem

$F(e^{iax} f(x)) = \hat{f}(w-a)$

$F^{-1}(\hat{f}(w-a)) = e^{iax} f(x)$

① First shifting Theorem of Fourier Transform

$\hat{f}(w) = F(f(x))$ $\qquad F(f(x-a)) = e^{-iaw} F(f(x))$

$F^{-1}(e^{-iaw} \hat{f}(w)) = f(x-a)$

Heat Eqn Fourier Transform

$u_t = c^2 u_{xx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = g(x)$ $\quad -\infty < x < \infty$

$u(x,t) \to 0$. $u_x(x,t) \to 0$ as $|x| \to \infty$

$\frac{\partial \hat{u}}{\partial t} = c^2(-w^2) \hat{u}$

$\hat{u}' + c^2 w^2 \hat{u} = 0$

$\Rightarrow \hat{u} = A e^{-c^2 w^2 t}$

$u(x,0) = g(x) \Rightarrow \hat{u}(w,0) = \hat{g}(w)$

$\Rightarrow A = \hat{g}(w)$

$\hat{u} = \hat{g}(w) e^{-c^2 w^2 t}$

$$u = F^{-1}(\hat{u}(w,t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\hat{g}(w) e^{-c^2 w^2 t}}_{\hat{f}(w,t)} e^{iwx} dw$$

$u(x,t) = \frac{1}{\sqrt{2\pi}} (f * g)(x,t)$

$g(x) = F^{-1}(\hat{g}(w))$

$\hat{f}(w,t) = e^{-c^2 w^2 t} \rightsquigarrow f(x,t)$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-c^2 w^2 t} e^{iwx} dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-c^2 w^2 t + iwx} dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-c^2 t \left(w^2 - \frac{iwx}{c^2 t}\right)} dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-c^2 t \left[\left(w - \frac{ix}{2c^2 t}\right)^2 + \frac{x^2}{4c^4 t^2}\right]} dw$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4c^2 t}} \int_{-\infty}^{\infty} e^{-c^2 t \left(w - \frac{ix}{2c^2 t}\right)^2} dw$$

$$\int_{-\infty}^{\infty} e^{-\left[c\sqrt{t}\left(w - \frac{ix}{2c^2 t}\right)\right]^2} dw$$

$c\sqrt{t}\left(w - \frac{ix}{2c^2 t}\right) = z$.

$c\sqrt{t}\, dw = dz$

$$c\sqrt{t} \int_{-\infty}^{\infty} e^{-z^2} dz = c\sqrt{t}\sqrt{\pi}$$

$$f(x,t) = \frac{1}{c\sqrt{2t}} e^{-\frac{x^2}{4c^2 t}}$$

$$u(x,t) \overset{?}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-p,t) g(p) dp$$

$$\overset{?}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(p) \frac{1}{c\sqrt{2t}} e^{-\frac{(x-p)^2}{4c^2 t}} dp$$

① $\hat{u} = \hat{g}(w) \hat{f}(w,t) \longrightarrow u$

$G(w) \cdot F(w,t) = F(f(x,t) * g(x))$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) * f(x,t) e^{-iwx} dx = G(w) \cdot F(w,t)$$

$\Downarrow F^{-1}$

$$g(x) * f(x,t) = \int_{-\infty}^{\infty} f(x-p,t) g(p) dp$$

Laplace / Fourier

$$F(s) \cdot G(s) = \int_0^{\infty} (f * g) e^{-st} dt$$

$$F(w) G(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g) e^{-iwx} dx$$

Convolution Theorem:

$F(f * g) = F(f) \cdot F(g)$ $\qquad f * g = F^{-1}(F(f) \cdot F(g))$

$F(f \cdot g) = F(f) * F(g)$ $\qquad f \cdot g = F^{-1}(F(f) * F(g))$

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Wave Laplace transform (t domain)

$u_{tt} = c^2 u_{xx}$ $\quad u(0,t) = g(t)$ $\quad \lim_{x\to\infty} u(x,t) = 0$ $\quad x,t \geq 0$

$u(x,0) = u_t(x,0) = 0$

$\mathcal{L}(u_{tt}) = \mathcal{L}(c^2 u_{xx})$

$\mathcal{L}(u(x,t)) = \hat{u}(x,s)$

$\mathcal{L}(u_t(x,t)) = s\hat{u}(x,s) - u(x,0)$

$\mathcal{L}(u_{tt}(x,t)) = s^2\hat{u}(x,s) - s\cancel{u(x,0)} - \cancel{u_t(x,0)}$

$s^2\hat{u}(x,s) = c^2 \frac{\partial}{\partial x^2}\hat{u}(x,s)$

$\hat{u}'' - \frac{s^2}{c^2}\hat{u} = 0$

$c^2 - (\frac{s}{c})^2 = 0 \Rightarrow c = \pm\frac{s}{c}$

$\hat{u}(x,s) = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x}$

$\hat{u}(0,s) = \hat{g}(s) \Rightarrow B = \hat{g}(s)$

$\lim_{x\to\infty} u(x,t) = 0 \Rightarrow A = 0$

$\therefore \hat{u}(x,s) = \hat{g}(s)e^{-\frac{s}{c}x} \Rightarrow u(x,t) = \mathcal{L}^{-1}(\hat{u}(x,s))$

$u(x,t) = u(t-\frac{s}{c})g(t-\frac{s}{c})$

$\mathcal{L}(f(t)) = F(s)$

$\mathcal{L}(f'(t)) = \int_0^\infty \underbrace{f'(t)}_{v'}\underbrace{e^{-st}}_{u}\,dt$ $\quad v = f(t)$, $u' = -se^{-st}$

$= e^{-st}f(t)\Big|_0^\infty - \int -se^{-st}f(t)\,dt$

$= -f(0) + s\mathcal{L}(f(t))$

$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0)$

$= s(s\mathcal{L}(f(t)) - f(0)) - f'(0)$

① First shifting Theorem (Laplace)

$\mathcal{L}(e^{-at}g(t)) = G(s+a)$

$\mathcal{L}^{-1}(G(s+a)) = e^{-at}(g(t))$

② Second shifting Theorem

$\mathcal{L}(u(t-a)g(t-a)) = e^{-at}G(s)$

$\mathcal{L}(u(t-a)g(t)) = e^{-as}G(s+a)$

Heat Laplace Transform

$u_t = c^2 u_{xx}$

$u(x,0) = 0 \quad x \geq 0$

$\lim_{x\to\infty} u(x,t) = 0, \quad \lim_{x\to\infty} u_x(x,t) = 0$

$u(0,t) = 1, \quad t \geq 0$

$\mathcal{L}(\hat{u}_t(x,s)) = \mathcal{L}(c^2\hat{u}_{xx}(x,s))$

$s\hat{u}(x,s) - \cancel{u(x,0)} = c^2\frac{\partial}{\partial x^2}\hat{u}(x,s)$

$\hat{u}'' - \frac{s}{c^2}\hat{u} = 0$

$\Rightarrow c^2 = \frac{s}{c^2} \Rightarrow c = \pm\sqrt{\frac{s}{c^2}} = \pm\frac{\sqrt{s}}{c}$

$\hat{u} = Ae^{\frac{\sqrt{s}}{c}x} + Be^{-\frac{\sqrt{s}}{c}x}$

$\hat{u}(x,s) = 0$ as $x \to \infty \Rightarrow A = 0$

$\hat{u}(0,s) = \frac{1}{s} = B$

$\Rightarrow \hat{u} = \frac{1}{s}e^{-\frac{\sqrt{s}}{c}x}$

$\mathrm{erf}\,x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \quad \mathcal{L}(1 - \mathrm{erf}(\frac{c}{2\sqrt{t}})) = e^{-x\sqrt{s}}/s$

PDEs via method of char eqn

solve $\quad au_x + bu_y = 0$

$(a,b)(u_x, u_y) = 0 = (a,b)\nabla u = 0$

$u = c_1$ $\quad$ direction of derivative $u \perp (a,b)$

$u = c_1$ $\quad$ u is constant along each line

$(x,y)\cdot(b,-a) = k$ and $bx - ay = k$ ← char eqn

Hence $u = f(bx-ay)$ general soln to $au_x + bu_y = 0$ for arbitrary f

ex: $2u_x + 3u_y = 0$

$u(x,y) = f(3x-2y)$

$3x - 2y = k$

ex: $u_t + 2xu_x = 0$

$f(2x\,dt - dx)$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2x} \Rightarrow x = ke^{2t}$

$\Rightarrow u(x,t) = f(k) = f(xe^{-2t})$

$u_t + 5u_x = 0, \quad u(x,0) = e^x$

$u(x,t) = f(x-5t) \Rightarrow u(x,t) = e^{x-5t}$

$u(x,0) = e^x = f(x)$

$u_t + u_x + 7u = 0, \quad u(x,0) = e^x$

$e^{7x}u_t + (ue^{7x})_x = 0$

$(ue^{7x})_t + (ue^{7x})_x = 0$

$\hat{u} = ue^{7x}$

$\hat{u} = f(x-t) \Rightarrow u(x,t) = e^{-7x}f(x-t)$

$u(x,0) = e^x = e^{-7x}f(x)$

$\Rightarrow f(x) = e^{8x}$

$u(x,t) = e^{-7x}f(x-t) = e^{-7x}e^{8(x-t)}$

$u(x,t) = e^{x-8t}$

Complex Analysis

$z = x + iy$

$w = f(z) = u(x,y) + iv(x,y)$

$x, y \in \mathbb{R}$ and $u(x,y), v(x,y)$ are real function

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Closure circle plan $\quad D = \{z : |z - z_0| \leq r\}$ $\quad$ every point a in D:

$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$

Cauchy's Integral formula

If γ is simple, closed contour and f is analytic on & inside γ

$\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!}f^n(z_0)$, $n \in (0,1,2,\dots)$

Ex: $\oint_{|z|=3} \frac{e^{iz}}{z+i}dz = \frac{2\pi i}{0!}e^{i(-i)} = 2\pi i e$

Ex: $\oint_{|z|=3} \frac{\sin(2z)}{3z-1}dz = \frac{\frac{1}{3}\sin(2\times\frac{1}{3})}{z - \frac{1}{3}} = \frac{2\pi i}{0!}\cdot\frac{1}{3}\sin(2\times\frac{1}{3}) = \frac{\pi i}{3}\sin$

Ex: $\oint_{|z|=3} \frac{z+2}{(z+1)(z+1)^2}dz = \oint_{|z|=3} \frac{\frac{z+2}{(z+1)^2}}{z-2}dz$

$= \frac{2\pi i}{0!}\left(\frac{z+2}{(z+1)^2}\right)\Big|_{z=2}$

$= \frac{2}{9}\pi i$

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$$\frac{d}{dx}\left((x^2y^4+\cos(y))=1\right)$$

$$x^2\cdot 4y^3\frac{dy}{dx}+2xy^4\frac{dy}{dx}+(-\sin y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2xy^4}{4x^2y^3-\sin y}$$

$$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy$$

$$=2xy^4\,dx+4x^2y^3\,dy+(-\sin y)\,dy$$

Total Differential

$$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy$$

$u(x,y)$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$J=\det\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r \Rightarrow dx\,dy=r\,dr\,d\theta$$

$$\begin{cases}x=r(\varphi)\cos\varphi\\ y=r(\varphi)\sin\varphi\end{cases}$$

$$\frac{dx}{d\varphi}=r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi$$

$$\frac{dy}{d\varphi}=r'(\varphi)\sin\varphi+r(\varphi)\cos\varphi$$

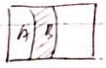
$$\Rightarrow \frac{dy}{dx}=\frac{r'(\varphi)\sin\varphi+r(\varphi)\cos\varphi}{r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi}$$

Probability and Statistics

$P(A)>0$

$P(\Omega)=1$

If $A\cap B=\phi$, then $P(A\cup B)=P(A)+P(B)$



If $A\subseteq B$, the $P(A)\le P(B)$

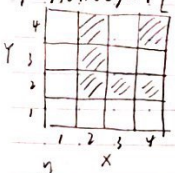
$B=A\cup(B\cap A^c)$ $\quad P(B)=P(A)+\underbrace{P(B\cap A^c)}_{\ge 0}\ge P(A)$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$P(A\cup B)\le P(A)+P(B)$

$P(A\cup B\cup C)=P(A)+P(A^c\cap B)+P(A^c\cap B^c\cap C)$

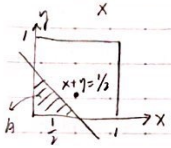
$P(A\cup B\cup C)=P[A\cup(B\cap A^c)\cup(C\cap A^c\cap B^c)]$



$P(X=1)=4\times 1/16=1/4$, $Z=\min(X,Y)$

$P(Z=4)=1/16$

$P(Z=2)=5/16$



$P(\{(x,y)\mid x+y\le 1/2\})=\frac{1}{2}\left(\frac{1}{2}\times\frac{1}{2}\right)=1/8$

$P(\{(0.5,0.3)\})=0$

$$S_n=a_1\left(\frac{1-q^n}{1-q}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right)=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{2^n}\right)=1$$

$$S_n=1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}$$

$$\frac{1}{2}S_n=\frac{1}{2}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}\xrightarrow{n\to\infty}0 \Rightarrow \left(1-\frac{1}{2}\right)S_n=1$$

$$S_n=\frac{1}{1-\frac{1}{2}}=2$$

$S\cup T$ $\quad x\in S\cup T\Leftrightarrow x\in S$ or $x\in T$

$S\cap T$ $\quad x\in S\cap T\Leftrightarrow x\in S$ and $x\in T$

properties:

$S\cup T=T\cup S$ $\qquad S\cup(T\cup V)=(S\cup T)\cup V$

$S\cap(T\cup V)=(S\cap T)\cup(S\cap V)$ $\qquad S\cup(T\cap V)=(S\cup T)\cap(S\cup V)$

$(S^c)^c=S$ $\qquad S\cap S^c=\phi$

$S\cup\Omega=\Omega$ $\qquad S\cap\Omega=S$

De Morgan's Laws

$(S\cap T)^c=S^c\cup T^c$ $\qquad \overline{(\bar{A}\bar{B})}=A+B$ $\qquad \left(\bigcap_n S_n\right)^c=\bigcup_n S_n^c$

$(S^c\cap T^c)^c=S\cup T$ $\qquad \overline{AB}=\bar{A}+\bar{B}$

$(\bar{A}+\bar{B})=A\cdot B$

$\overline{A+B}=\bar{A}\cdot\bar{B}$ $\qquad \left(\bigcup_n S_n\right)^c=\bigcap_n S_n^c$

Geometric Series

$$\sum_{i=0}^{\infty}\alpha^i=1+\alpha^1+\alpha^2+\cdots+\alpha^n+\cdots=\frac{1}{1-\alpha},\ |\alpha|<1$$

proof. $(1-\alpha)(1+\alpha+\cdots+\alpha^n)=1-\alpha^{n+1}$

$n\to\infty$ $\quad S_n=\frac{1}{1-\alpha},\ |\alpha|<1$

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The Bonferroni inequality

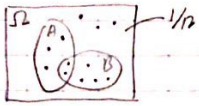
$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$ $\quad (A \cap B)^c = A^c \cup B^c$

$P((A_1 \cap A_2)^c) = P(A_1^c \cup A_2^c) \leq P(A_1^c) + P(A_2^c)$

$1 - P(A_1 \cap A_2) \leq 1 - P(A_1) + 1 - P(A_2)$

$\Rightarrow P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$

$\Rightarrow P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1)$



$P(A) = 5/12$

$P(B) = 6/12$

$P(A|B) = \frac{1}{3}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = 1$

$P(B|B) = \frac{P(B \cap B)}{P(B)} = 1$

If $A \cap C = \phi$, then $P(A \cup C|B) = P(A|B) + P(C|B)$

$= \frac{P((A \cup C) \cap B)}{P(B)} = \frac{P((A \cap B) \cup (C \cap B))}{P(B)} = P(A|B) \cup P(C|B)$

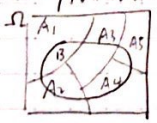
$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\Rightarrow P(A|B)P(B) = P(B|A)P(A)$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A^c \cap B \cap C^c) = P(A^c \cap B)\, P(C^c | A^c \cap B) = P(A^c) \cdot P(B|A^c)\, P(C^c | A^c \cap B)$

$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \prod_{i=2}^{n} P(A_i | A_1 \cap \cdots \cap A_{i-1})$

Total Probability Theorem



$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots$

$= P(A_1)P(B|A_1) + \cdots$

$P(B) = \sum_i P(A_i) P(B|A_i)$, $\sum_i P(A_i) = 1$

$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$ $\quad P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_i P(A_i) P(B|A_i)}$

Independent of two events

$P(B|A) = P(B)$

$P(A \cap B) = P(A)P(B|A) = P(A)\,P(B)$ ← Definition of independence

If A and B are independent, then A and B^c are independent

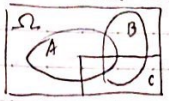
$P(A) = P(A \cap B) + P(A \cap B^c)$

$= P(A)P(B) + P(A \cap B^c)$

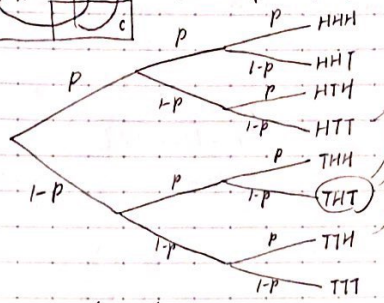
$P(A \cap B^c) = P(A)(1 - P(B))$

$= P(A)\,P(B^c)$

Conditional independent



$P(A \cap B | C) = P(A|C) \cap P(B|C)$



$P(THT) = (1-p)p(1-p) = (1-p)^2 p$

$P(1\ \text{Head}) = 3p(1-p)^2$

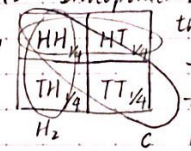
$P(\text{first toss is H} | 1\ \text{head}) = \frac{P(\text{first H \& 1 head})}{P(1\ \text{head})} = \frac{1}{3}$

$P(H_2|H_1) = p$

$P(H_2) = P(H_1)P(H_2|H_1) + P(T_1)P(H_2|T_1) = p^2 + (1-p)p = p$

Pairwise Independent Vs Independent

$\begin{cases} P(A_1 \cap A_2) = P(A_1)P(A_2) \\ P(A_1 \cap A_3) = P(A_1)P(A_3) \\ P(A_2 \cap A_3) = P(A_2)P(A_3) \end{cases}$



two independent fair coin tosses

H_1: first toss is H

H_2: second toss is H

$P(H_1) = P(H_2) = P = 1/2$

C: the two tosses had the same result $= \{HH, TT\}$

$\begin{cases} P(H_1 \cap C) = P(H_1 \cap H_2) = 1/4 \\ P(H_1)P(C) = 1/4 \end{cases} \Rightarrow \begin{cases} H_1, C \text{ indep} \\ H_2, C \text{ indep} \\ H_1, H_2 \text{ indep} \end{cases}$ pairwise indep.

$\begin{cases} P(H_1 \cap C \cap H_2) = 1/4 \\ P(H_1)P(C)P(H_2) = 1/8 \end{cases} \Rightarrow$ not independent

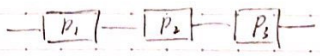
$P(C | H_1 \cap H_2) = 1 \neq P(C) = 1/2$

Independent:

$P(A_1 \cap A_2 \cap A_3)$

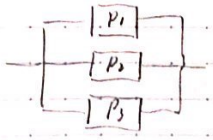
$= P(A_1)P(A_2)P(A_3)$

Reliability



probability that system is "up" $= P(U_1)P(U_2)P(U_3)$

$= P_1 P_2 P_3$

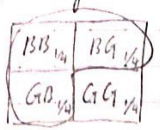


$P(\text{system up}) = P(U_1 \cup U_2 \cup U_3)$

$= 1 - P(F_1)P(F_2)P(F_3)$

$= 1 - (1-P_1)(1-P_2)(1-P_3)$

The king's sibling

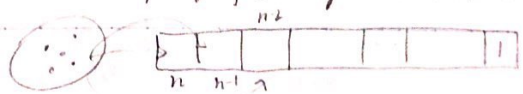


$P(\text{boy}) = P(\text{girl}) = 1/2$

$P(\text{female} | \text{king}) = 2/3$

Permutations

number of ways of ordering n elements



$n(n-1)(n-2) \cdots 1 = n!$

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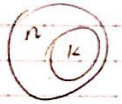
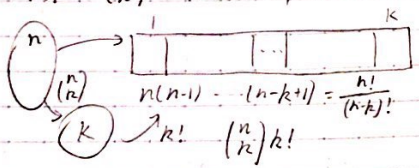
number of subsets of $\{1, 2, \dots n\} = 2^n$

Combinations



Def^n. $\binom{n}{k}$: number of k element subsets of a given n-element set.



$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ $\Rightarrow \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$k!$ $\binom{n}{k} k!$

$\binom{n}{n} = 1$

$\binom{n}{0} = 1$

$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$ all subsets

Binomial coefficient $\binom{n}{k}$ $\quad P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$

$P(k \text{ heads}) = p^k (1-p)^{n-k} \cdot (\#\, k\text{-head sequence})$

$\binom{n}{k}$

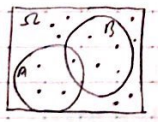
ex. Given 3 heads in 10 tosses

what is the probability that first two tosses were heads?

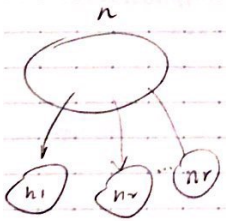
– event A: the first 2 tosses are head

– event B: 3 out of 10 tosses are heads

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(H_1 H_2 \text{ and } H \text{ in tosses } 3 \dots 10)}{P(B)} = \frac{p^2 \binom{8}{1} p (\cancel{1-p})^7}{\binom{10}{3} p^3 (\cancel{1-p})^7} = \frac{8}{\binom{10}{3}}$$



$$\frac{\#\text{ in } (A\cap B)}{\#\text{ in } B} = \frac{8}{\binom{10}{3}}$$



number of partitions $= \frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2 \dots n_r}$

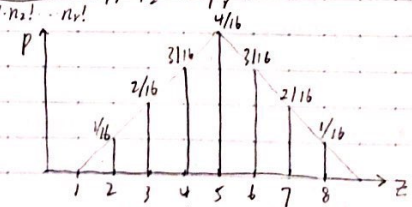
$\binom{n}{k} = \binom{n}{k, n-k}$

$P(\text{get type } (n_1, n_2 \dots n_r)) = \frac{n!}{n_1! \cdot n_2! \cdots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$

4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5
	1	2	3	4

Y $\quad X$

$Z = X + Y$



Convolution of X and Y

$S_n = a_1 \frac{1-q^n}{1-q}$, $S_n = \frac{n(a_1+a_n)}{2}$

$E[X] = \sum_x x\, p_X(x)$

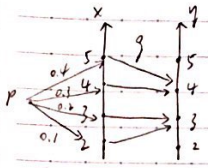
ex. Bernoulli r.v.

$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases}$

$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$

$\mathrm{Var}X = E[(X - E[X])^2] = p(1-p)^2 + (1-p)(0-p^2) = p(1-p)$

$\mathrm{Var}X = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$



$E[Y] = E[g(X)] = \sum_x g(x) p_X(x)$

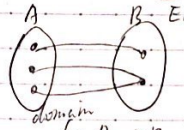
proof $\sum_y \sum_{x: g(x)=y} g(x) p_X(x) = \sum_y y \sum_{x: g(x)=y} p_X(x) = \sum_y y\, p_Y(y) = E[Y]$

linearity of Expectation $E[aX+b] = aE[X] + b$

$E[Y] = \sum_x g(x) p_X(x)$ $\quad \begin{cases} g(x) = ax+b \\ Y = g(x) \end{cases}$

$= \sum_x (ax+b) p_X(x)$

$= a \underbrace{\sum_x x\, p_X(x)}_{E[X]} + b \underbrace{\sum_x p_X(x)}_{1} = aE[X] + b$



$f: A \to B$

Variance – a measure of the spread of a PMF

$\mu = E[X]$

$\mathrm{Var}(X) = E[(X-\mu)^2]$

Standard deviation: $\sigma_X = \sqrt{\mathrm{Var}(X)}$

Properties of the variance $\quad \mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X) + 0$

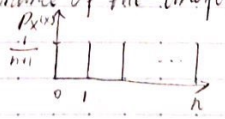
$\mu = E[X]$ $\quad$ let $Y = X+b$, $\nu = E[Y] = \mu + b$

$\mathrm{Var}(Y) = E[(Y-\nu)^2] = E[(X+b-\mu-b)^2] = E[(X-\mu)^2] = \mathrm{Var}(X)$

$\mu = E[X]$ $\quad$ let $Y = aX$, $\nu = E[Y] = a\mu$

$\mathrm{Var}(Y) = E[(Y-\nu)^2] = E[(aX - a\mu)^2] = E[(a^2(X-\mu)^2)] = a^2 \mathrm{Var}(X)$

Variance of the uniform



$\mathrm{Var}(X) = E[(X-\mu)^2]$

$= \frac{1}{n+1}\left[(0-\frac{n}{2})^2 + (1-\frac{n}{2})^2 + \dots + (n-\frac{n}{2})^2\right]$

$\mu = \frac{n}{2}$ $\quad$ or $E[X^2] - (E[X])^2$

$= \frac{1}{n+1}(0^2 + 1^2 + \dots + n^2) - (\frac{n}{2})^2$

$S_n = 0^2 + 1^2 + 2^2 + \dots + n^2$

0 1 5 14 30 55 ...

1 4 9 16 25

3 5 7 9

2 2 2

$\Rightarrow S_n = An^3 + Bn^2 + Cn + D$

$n=0 \Rightarrow S_n = 0 \Rightarrow D = 0$

$n=1$

$n=2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \\ 1/6 \end{bmatrix}$

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Conditional PMF and expectation

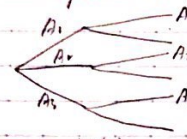
$p_X(x) = p(X=x)$ $\qquad$ $p_{X|A}(x) = p(X=x|A)$ $\quad$ assume $P(A)>0$

$\sum_x p_X(x) = 1$ $\qquad$ $\sum_x p_{X|A}(x) = 1$

$E[X] = \sum_x x\, p_X(x)$ $\qquad$ $E[X|A] = \sum_x x\, p_{X|A}(x)$

$E[g(X)] = \sum_x g(x)\, p_X(x)$ $\qquad$ $E[g(X)|A] = \sum_x g(x)\, p_{X|A}(x)$

Total expectation theorem



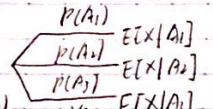
$p(B) = p(A_1)p(B|A_1) + \dots + p(A_n)p(B|A_n)$

$B = \{X=x\}$

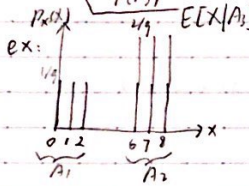
$p_X(x) = p(A_1)\, p_{X|A_1}(x) + \dots + p(A_n)\, p_{X|A_n}(x)$ $\quad$ for all x.

$\sum_x x\, p_X(x) = \underbrace{p(A_1)\sum_x x\, p_{X|A_1}(x)}_{} + \dots$

$E[X] = p(A_1)\cdot E[X|A_1] + \dots$



ex:



$p(A_1) = \frac{1}{3}$ $\qquad$ $E[X] = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 7 = 5$

$p(A_2) = \frac{2}{3}$

$E[X|A_1] = 1$

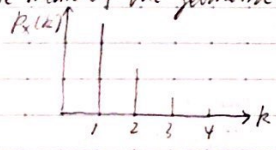
$E[X|A_2] = 7$

Geometric random variable

X: number of independent coin tosses until first head; $P(H) = p$

$p_X(k) = (1-p)^{k-1}p$, $k = 1, 2, \dots$ $\quad$ Memorylessness

The mean of the geometric $\quad E[X] = \sum_{k=1}^{\infty} k\, p_X(k) = \sum_{n=1}^{\infty} k(1-p)^{k-1}p$

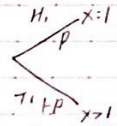


$E[X] = 1 + E[X-1]$

$= 1 + pE[X-1|X=1] + (1-p)E[X-1|X>1]$

$= 1 + (1-p)E[X]$

$E[X] = \frac{1}{p}$



Px, Py

$p_{X,Y}(x,y) = P(X=x \text{ and } Y=y)$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(1,3) = 2/20$

$p_X(4) = \frac{1}{20} + \frac{2}{20}$

$p_X(x) = \sum_y p_{X,Y}(x,y)$

$p_Y(y) = \sum_x p_{X,Y}(x,y)$

$Z = g(X,Y)$

PMF: $p_Z(z) = P(Z=z) = P(g(X,Y)=z) = \sum p_{X,Y}(x,y)$

Expected value rule: $E[g(X,Y)] = \sum_x\sum_y g(x,y)\, p_{X,Y}(x,y)$

The mean of Binomial $X \sim B(n,p)$ $\quad$ independent

$E[X] = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k}$

$= \sum_{k=0}^{n} k\,\frac{n(n-1)\cdots 1}{(n-k)!\,k!}p^k(1-p)^{n-k}$ $\qquad$ key point "k variable"

$= np\sum_{k=1}^{n}\frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!}p^{k-1}(1-p)^{(n-1)-(k-1)}$

$= np\sum_{l=0}^{n-1}\frac{(n-1)!}{((n-1)-l)!\,(l!)}p^{l}(1-p)^{n-1-l}$ (with $l = k-1$)

$= np\sum_{l=0}^{m}\frac{m!}{(m-l)!\,l!}p^{l}(1-p)^{m-l}$ (with $n-1 = m$)

$= np\sum_{l=0}^{m}\binom{m}{l}p^{l}(1-p)^{m-l} = np(p+1-p)^m = np$

Or: $X = X_1 + X_2 + \dots + X_n$, whereby all X_i are independent Bernoulli distributed.

$E[X_i] = p\cdot 1 + (1-p)\cdot 0 = p$, $\quad E[X] = E[X_1] + \dots + E[X_n] = np$

$Var(X) = Var(X_1) + Var(X_2) + \dots + Var(X_n) = np(1-p)$ $\quad$ as $Var(X_i) = p - p^2 = p(1-p)$

Conditional PMFs

$p_{X|A}(x|A) = p(X=x|A)$ $\qquad$ $p_{X|Y}(x|y) = p(X=x|Y=y) = \frac{p(X=x, Y=y)}{p(Y=y)}$

$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$ defined for y such that $p_Y(y)>0$ $\quad \sum_x p_{X|Y}(x|y) = 1$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X|Y}(1|2) = 0$, $\quad p_{X|Y}(2|2) = \frac{p(X=2, Y=2)}{p(Y=2)} = \frac{1/20}{5/20} = 1/5$

$p_{X|Y,Z}(x|y,z) = P(X=x \mid Y=y, Z=z)$ $\qquad$ $p_{X,Y|Z}(x,y|z) = p(X=x, Y=y|Z=z)$

$P(A\cap B\cap C) = P(A)\,P(B|A)\,P(C|A\cap B)$

Conditional Expectation

$E[X|A] = \sum_x x\, p_{X|A}(x)$ $\qquad$ $E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y)$

Total probability and expectation theorems

- $A_1, \dots, A_n$, Partition of Ω
- $p_X(x) = p(A_1)\,p_{X|A_1}(x) + \dots + p(A_n)\,p_{X|A_n}(x)$

  $p_X(x) = \sum_y p_Y(y)\,p_{X|Y}(x|y)$
- $E[X] = p(A_1)E[X|A_1] + \dots + P(A_n)E[X|A_n]$

  $E[X] = \sum_y p_Y(y)E[X|Y=y]$

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Independence and expectations / variance

If X, Y are independent: $E[XY] = E[X]E[Y]$

In general $Var(X+Y) \neq Var(X) + Var(Y)$

If X, Y are independent: $Var(X+Y) = Var(X) + Var(Y)$

ex. $Var(X-3Y) = Var(X) + Var(-3Y) = Var(X) + 9Var(Y)$

The inclusion-exclusion formula

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_i P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{n-1} P\left(\bigcap_{k=1}^{n} A_k\right)$$

Geometric PMF

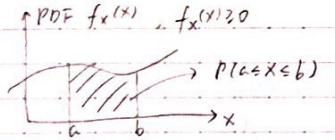
- Given $X > 1$, $X-1$ has the same geometric PMF (unconditional PMF of X)
- Given $X > 1$, conditional PMF of X: same as unconditional PMF of $X+1$

$E[X \mid X>1] = 1 + E[X]$ $\quad E[X^2 \mid X>1] = E[(X+1)^2]$ $\quad E[X] = 1/p$

$$\begin{aligned}
E[X^2] &= p(X=1)E[X^2 \mid X=1] + p(X>1)E[X^2 \mid X>1] \\
&= p + (1-p)E[(X+1)^2] \\
&= p + (1-p)E[X^2 + 2X + 1] \\
&= p + (1-p)[E[X^2] + 2E[X] + 1] \\
&= p + (1-p)(E[X^2] + 2/p + 1) = \frac{2}{p^2} - \frac{1}{p}
\end{aligned}$$

$\therefore Var(X) = E[X^2] - [E[X]]^2 = \frac{1}{p^2} - \frac{1}{p} = (1-p)/p$

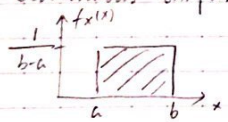
Def: A random variable is continuous if it can be described by a PDF.



$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$

$P(X=a) = 0$

Continuous uniform PDF



$$E[X] = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left.\frac{x^2}{2}\right|_a^b = \frac{(b^2-a^2)}{(b-a)2} = \frac{b+a}{2}$$

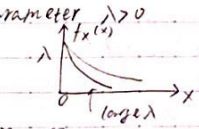
Expectation $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx = \int_{-\infty}^{\infty} \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx \Rightarrow \frac{(a-b)^2}{12}$$

Exponential random variable: parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$P(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda t}\,dt = e^{-\lambda a}$

$E[X] = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}}_{v'}\,dx = x \cdot (-e^{-\lambda x})\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x}\,dx = 1/\lambda$

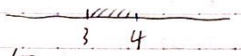
$E[X^2] = \int_0^{\infty} \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x}}_{v'}\,dx = x^2(-e^{-\lambda x})\Big|_0^{\infty} - \int_0^{\infty} 2x \cdot (-e^{-\lambda x})\,dx = \frac{2}{\lambda}\int_0^{\infty} x \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = 2/\lambda^2$

$Var(X) = E[X^2] - (E[X])^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$

CDF (Cumulative distribution function)

$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$

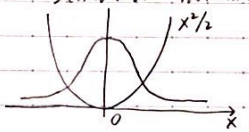
$P(X \leq 4) = P(X \leq 3) + P(3 < X \leq 4)$



$\frac{dF_X(x)}{dx} = f_X(x)$

Standard normal (Gaussian) random variables

- Standard normal $N(0,1)$: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$

$\frac{x}{\sqrt{2}} = u \Rightarrow dx = \sqrt{2}\,du = \sqrt{2}\underbrace{\int_{-\infty}^{\infty} e^{-u^2}\,du}_{\sqrt{\pi}} = \sqrt{2\pi}$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad N(\mu, \sigma^2)$

let $Y = aX + b$, $X \sim N(\mu, \sigma^2)$ Linear function of a normal random variables

$E[Y] = a\mu + b$

$Var(Y) = a^2 Var(X) = a^2\sigma^2$

$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$

$P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

Standardizing a random variable

let X have μ (mean) and $\sigma^2 > 0$ (variance)

let $Y = \frac{X-\mu}{\sigma}$, $E[Y] = 0$; $Var(Y) = \frac{1}{\sigma^2} Var(X) = 1$

ex: $X \sim N(6, 4)$

$P(2 \leq X \leq 8) = P\left(\frac{2-6}{2} \leq \frac{X-6}{2} \leq \frac{8-6}{2}\right)$

$P(-2 \leq Y \leq 1)$

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Conditional PDF, given an event

$p_X(x) = P(X=x)$ $\quad f_X(x)\cdot\delta = P(x \le X \le x+\delta)$

$p_{X|A}(x) = P(X=x|A)$ $\quad f_{X|A}(x)\cdot\delta = P(x \le X \le x+\delta | A)$

$P(X\in B) = \sum_{x\in B} p_X(x)$ $\quad P(X\in B) = \int_B f_X(x)dx$

$P(X\in B|A) = \sum_{x\in B} p_{X|A}(x)$ $\quad P(X\in B|A) = \int_B f_{X|A}(x)dx$

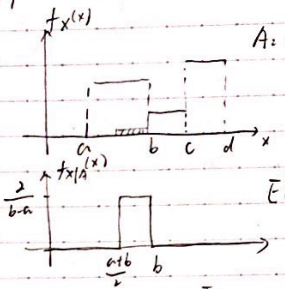
$\sum_x p_{X|A}(x) = 1$ $\quad \int f_{X|A}(x)dx = 1$

$E[X] = \int x f_X(x)dx$

$E[X|A] = \int x f_{X|A}(x)dx$

$E[g(X)|A] = \int g(x) f_{X|A}(x)dx$

ex.



$A: \frac{a+b}{2} \le x \le b$

$E[X|A]$

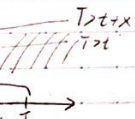
$E[X^2|A] = \int_{\frac{a+b}{2}}^{b} x^2 \left(\frac{2}{b-a}\right) dx$

Memorylessness of the exponential PDF

$P(T>x) = \int_x^\infty \lambda e^{-\lambda x}dx = e^{-\lambda x}$; for $x>0$

$P(X>x | T>t) = e^{-\lambda x}$, for $x>0$

$= \frac{P(T-t>x, T>t)}{P(T>t)} = \frac{P(T>t+x)}{P(T>t)} = \frac{\int_{t+x}^\infty \lambda e^{-\lambda t}dt}{\int_t^\infty \lambda e^{-\lambda t}dt} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$



Jointly Continuous r.v.'s and joint PDFs.

$p_{X,Y}(x,y) = P(X=x \text{ and } Y=y) \ge 0$ $\quad f_{X,Y}(x,y) \ge 0$

$P(X,Y\in B) = \sum\sum_{(x,y)\in B} p_{X,Y}(x,y)$ $\quad P((X,Y)\in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$

$\int_{-\infty}^\infty\int_{-\infty}^\infty f_{X,Y}(x,y) = 1$

$P(a\le X\le b, c\le Y\le d) = \int_c^d\int_a^b f_{X,Y}(x,y)dx\,dy$

$f_X(x) = \int f_{X,Y}(x,y)dy$

$f_Y(y) = \int f_{X,Y}(x,y)dx$

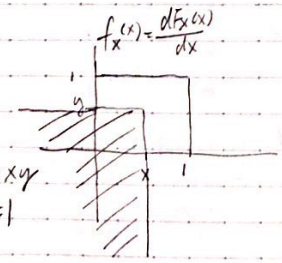
$E[g(X,Y)] = \iint g(x,y) f_{X,Y}(x,y)dx\,dy$

$F_{X,Y}(x,y) = P(X\le x, Y\le y) = \int_{-\infty}^y \left[\int_{-\infty}^x f_{X,Y}(s,t)ds\right]dt$

$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x\partial y}(x,y)$

$F_{X,Y}(x,y) = xy$

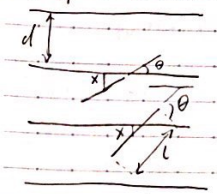
$f_{X,Y}(x,y) = 1$



The joint CDF

$F_X(x) = P(X\le x) = \int_{-\infty}^x f_X(t)dt$

Buffon's needle



model: X; Θ independent

X: $f_X(x) = \frac{2}{d}$, $0\le x\le \frac{d}{2}$

Θ: $f_\Theta(\theta) = \frac{2}{\pi}$, $0\le\theta\le\frac{\pi}{2}$ $\quad f_{X,\Theta}(x,\theta) = \frac{4}{d\pi}$

Intersect if $X \le \frac{l}{2}\sin\theta \cdot l$

$P(X\le \frac{l}{2}\sin\theta)$

$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{l}{2}\sin\theta} \frac{4}{d\pi}dx\,d\theta = \frac{4}{d\pi}\int_0^{\frac{\pi}{2}} \frac{l}{2}\sin\theta\,d\theta = \frac{2l}{d\pi}(-\cos\theta)\Big|_0^{\frac{\pi}{2}}$

$= \frac{2l}{d\pi}$

Conditional PDFs, given another r.v.

Def: $p_{X|Y}(x|y) = p(X=x|Y=y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$, if $p_Y(y)>0$

Def: $P(X\in A|Y=y) = \int_A f_{X|Y}(x|y)dx$

$\int_{-\infty}^\infty f_{X|Y}(x,y)dx = \frac{\int_{-\infty}^\infty f_{X,Y}(x,y)dx}{f_Y(y)} = 1$

Total probability and expectation theorems

$p_X(x) = \sum_y p_Y(y)p_{X|Y}(x|y)$ $\quad f_X(x) = \int_{-\infty}^\infty \underbrace{f_Y(y)f_{X|Y}(x|y)}_{f_{X,Y}(x,y)}dy$

$E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y)$ $\quad E[X|Y=y] = \int_{-\infty}^\infty x f_{X|Y}(x|y)dx$

$E[X] = \sum_y p_Y(y)E[X|Y=y]$ $\quad E[X] = \int_{-\infty}^\infty f_Y(y)E[X|Y=y]dy$

$E[g(X)|Y=y]$ $\quad = \int_{-\infty}^\infty f_Y(y)\int_{-\infty}^\infty x f_{X|Y}(x|y)dx\,dy$

$= \int_{-\infty}^\infty g(x) f_{X|Y}(x|y)dx$ $\quad = \int_{-\infty}^\infty\int_{-\infty}^\infty x f_Y(y) f_{X|Y}(x|y)dy\,dx$

$= \int_{-\infty}^\infty x f_X(x)dx = E[X]$

Independence

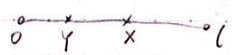
$p_{X,Y}(x,y) = p_X(x)\cdot p_Y(y)$, for all x,y

$f_{X,Y}(x,y) = f_X(x)f_Y(y)$, for all x and y

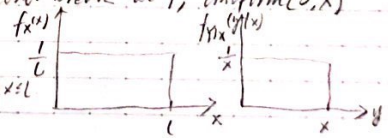
$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) \Rightarrow f_{X|Y}(x,y) = f_X(x)$, for all y with $f_Y(y)>0$ and all x

Stick breaking:

- first break at X, uniform$[0,l]$
- second break at Y, uniform$[0,X]$



$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y|x) = \frac{1}{lx}$, $0\le y\le x\le l$



$f_Y(y) = \int_Y^l f_{X,Y}(x,y)dx$

$= \int_Y^l \frac{1}{lx}dx = \frac{1}{l}\ln\left(\frac{l}{y}\right)$

$E[Y] = \int_0^l y\cdot\frac{1}{l}\ln\left(\frac{l}{y}\right)dy$

$E[Y] = \int_0^l \frac{1}{l}E[Y|X]dx$

$= \int_0^l \frac{1}{l}\cdot\frac{1}{2}x\cdot dx = \frac{1}{2}E[X] = \frac{l}{4}$

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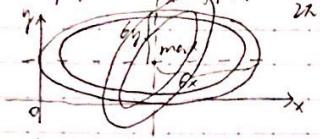
Independent standard normals

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\} = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(x^2+y^2)\right\}$$

Independent normal

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$



The Bayes Rule

$X \rightleftarrows Y$

unobserved — observed value

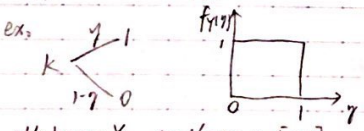
prior $P_X(\cdot)$ — model $P_{Y|X}(\cdot|\cdot)$

$$P_{X,Y}(x,y) = P_X(x)\,P_{Y|X}(y|x)$$
$$= P_Y(y)\,P_{X|Y}(x|y)$$
$$P_{X|Y}(x|y) = \frac{P_X(x)P_{Y|X}(y|x)}{P_Y(y)}$$
$$P_Y(y) = \sum_x P_X(x)P_{Y|X}(y|x)$$

"mix Bayes Rule"

$$P_{K|Y}(k|y) = \frac{P_K(k)f_{Y|K}(y|k)}{f_Y(y)}$$
$$f_{Y|K}(y|k) = \frac{f_Y(y)P_{K|Y}(k|y)}{P_K(k)}$$

ex:



- Unknown Y: uniform on $[0,1]$
- Distribution of Y given that $K=1$? $f_{Y|K}(y|1)$

$f_Y(y) = 1 \quad y\in[0,1]$

$\quad\quad 0$ otherwise

$P_{K|Y}(1|y) = y$

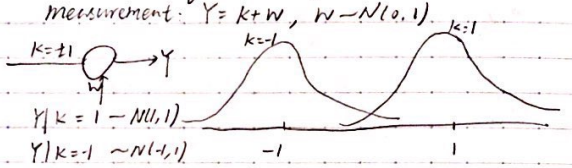
$P_K(1) = \int f_Y(y)P_{K|Y}(k|y)\,dy = \int_0^1 1\cdot y\,dy = \frac{1}{2}$

$\int f_{Y|K}(y|k)\,dy = \int \frac{f_Y(y)P_{K|Y}(k|y)}{P_K(k)}\,dy$

$\Rightarrow 1 = \int \cdots\, dy$

ex: unknown k: equally likely to be -1 or $+1$

measurement: $Y = k + W$, $W \sim N(0,1)$



$Y|k=1 \sim N(1,1)$

$Y|k=-1 \sim N(-1,1)$

Probability that $k=1$, given that Y?

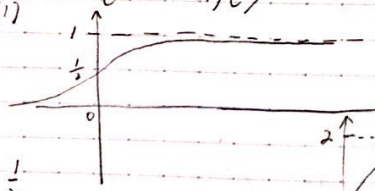
$P_{K|Y}(1|y) = ?$

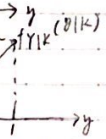
$$P_{K|Y}(k|y)\,f_Y(y) = f_{Y|K}(y|k)\cdot P_K(k)$$

$P_K(k) = \frac{1}{2}$, $f_{Y|K}(y|1) = \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{y-1}{\sqrt{2}}\right)^2}$

$$f_Y(y) = \sum f_{Y|K}(y|k)\cdot P_K(k) = \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{y-1}{\sqrt{2}}\right)^2} + \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{y+1}{\sqrt{2}}\right)^2}$$
$$= \frac{1}{2\sqrt{2\pi}}\left(e^{-\left(\frac{y-1}{\sqrt{2}}\right)^2} + e^{-\left(\frac{y+1}{\sqrt{2}}\right)^2}\right)$$

$$P_{K|Y}(1|y) = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}e^{-\frac{(y-1)^2}{2}}\cdot\cancel{\frac{1}{2}}}{\cancel{\frac{1}{2\sqrt{2\pi}}}\left(e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y+1)^2}{2}}\right)}$$
$$= \frac{\cancel{e^{-\frac{(y-1)^2}{2}}}}{\left(e^{-2(y-1)-2}+1\right)\cancel{e^{-\frac{(y-1)^2}{2}}}} = \frac{1}{e^{-2y}+1}$$





$$\Rightarrow f_{Y|K}(y|1) = \frac{f_Y(y)P_{K|Y}(k|y)}{P_K(k)} = \frac{1\cdot y}{\frac{1}{2}} = 2y \quad y\in[0,1]$$

Bayes Rule

$$f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|x)$$
$$= f_Y(y)f_{X|Y}(x|y)$$

$$f_{X|Y}(x|y) = \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)}$$
$$f_Y(y) = \int f_X(x)f_{Y|X}(y|x)\,dx$$

A linear function of a continuous r.v.

$Y = aX+b \quad P_Y(y) = P_X\left(\frac{y-b}{a}\right)$

$$F_Y(y) = P(Y\le y) = P(aX+b\le y)$$
$$= P\left(X\le\frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$, $P_Y(y) = P_X\left(\frac{y-b}{a}\right)$

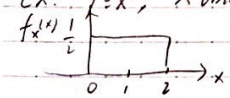
$X\sim N(\mu,\sigma^2) \quad Y = aX+b,\ a\neq 0$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}|a|\sigma}e^{-(y-b-a\mu)^2/2\sigma^2a^2}$$

$\Rightarrow \begin{cases}\mu_Y = a\mu+b\\ \sigma_Y^2 = a^2\sigma^2\end{cases}$

Ex: $Y = X^3$, X uniform on $[0,2]$

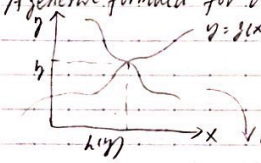


$f_Y(y)$? $\quad F_Y(y) = P(Y\le y) = P(X^3\le y)$

$= P(X\le\sqrt[3]{y})$

$$F_Y(y) = \int_0^{\sqrt[3]{y}}\frac{1}{2}\,dx = \frac{y^{1/3}}{2} \Rightarrow f_Y(y) = \frac{\partial F}{\partial y} = \frac{1}{6}y^{-2/3}$$

A general formula for the PDF of $Y = g(X)$ when g is monotonic



Assume g strictly increasing $x<x'$, $g(x)<g(x')$

$$F_Y(y) = P(Y\le y) = P(X\le h(y)) = F_X(h(y))$$
$$f_Y(y) = f_X(h(y))\left|\frac{dh}{dy}(y)\right|$$

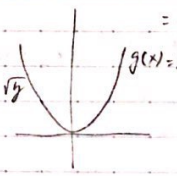
decreasing $\quad F_Y(y) = P(Y\le y) = P(X\ge h(y))$

$$= 1 - P(X\le h(y)) = 1 - F_X(h(y))$$
$$f_Y(y) = -f_X(h(y))\frac{dh}{dy}(y)$$
$$= f_X(h(y))\left|\frac{dh}{dy}(y)\right|$$

Ex: $Y = X^2$, X uniform on $[0,1]$

$$f_Y(y) = f_X(h(y))\left|\frac{dh}{dy}(y)\right|$$

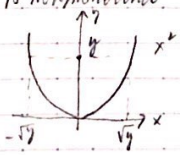
$y = x^2 \Rightarrow x = \sqrt{y}$, $h(y) = \sqrt{y}$



$f_Y(y) = 1\cdot\frac{1}{2\sqrt{y}}$

$0\le y\le 1$

A nonmonotonic example: $Y = X^2$



$$F_Y(y) = P(Y\le y) = P(X^2\le y) = P(|X|\le\sqrt{y}) = P(-\sqrt{y}\le X\le\sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$
$$f_Y(y) = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$$

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A function of multiple r.v.s $z = g(x, y)$

find CDF of z

let $z = Y/X$, X, Y independent, uniform on $[0,1]$

$F_Z(z) = P(Z \le z) = P(Y/X \le z)$

$P(Y \le Xz)$

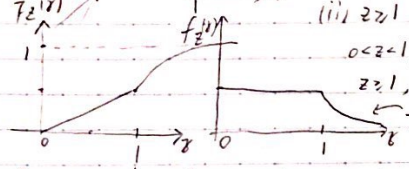
$\int_0^1 \int_0^{xz} dy\,dx = \frac{1}{2}z$

(i) $0 < z < 1$, $F_Z(z) = \frac{1}{2}z$

(ii) $z \ge 1$, $F_Z(z) = \frac{1}{2} + (1 - \frac{1}{z}) \cdot \frac{1}{2} = 1 - \frac{1}{2z}$

$0 < z < 1$, $f_Z(z) = \frac{1}{2}$

$z \ge 1$, $f_Z(z) = \frac{1}{2z^2}$



Sum of independent random variables

PMF / PDF of $X+Y$

$P_Z(z) = \sum_x P(X = x, Y = z - x)$

$= \sum_x P_X(x) P_Y(z - x)$

The distribution of $X+Y$: $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$ Convolution

$X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$

$Cov(X,Y) = E[(X - E[X]) \cdot (Y - E[Y])]$

$Cov(X,X) = Var(X)$

$Cov(X,Y) = E[X \cdot Y] - E[X]E[Y]$ if independent $Cov(X,Y) = 0$

$Cov(aX + b, Y) = E[(aX+b)Y] - aE[X,Y] + bE[Y] = a\,Cov(X,Y)$ assume 0 mean

$Cov(X, Y+Z) = E[X(Y+Z)] - E[XY] + E[XZ] = Cov(X,Y) + Cov(X,Z)$ assume 0 mean

The variance of a sum of random variables

$Var(X_1 + X_2) = E[(X_1 + X_2 - E[X_1 + X_2])^2]$

$= E[((X_1 - E[X_1]) + (X_2 - E[X_2]))^2]$

$= E[(X_1 - E[X_1])^2 + (X_2 - E[X_2])^2 + 2(X_1 - E[X_1])(X_2 - E[X_2])]$

$= Var(X_1) + Var(X_2) + \underbrace{2Cov(X_1, X_2)}_{\text{independent} = 0}$

$Var(X_1 + \cdots + X_n) = \sum_{i=1}^{n} Var(X_i) + \sum_{\{(i,j): i \ne j\}} Cov(X_i, X_j)$

$\rho(X,Y) = E\left[\frac{(X - E[X])}{\sigma_X} \cdot \frac{(Y - E[Y])}{\sigma_Y}\right] = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

Independent $\rho(X,Y) = 0$, $\rho(X,X) = \frac{Var(X)}{\sigma_X} = 1$

$E[X | Y = y] = \sum_x x P_{X|Y}(x|y) = g(y)$

$E[E[X|Y]] = E[X]$ $\quad E[g(Y)] = \sum_y g(y) P_Y(y) = \sum_y E[X | Y = y] P_Y(y) = E[X]$

The conditional variance as a random variable

$Var(X) = E[(X - E[X])^2]$

$Var(X | Y = y) = E[(X - E[X|Y=y])^2 | Y = y]$

Law of total variance $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

$Var(X) = E[X^2] - (E[X])^2$

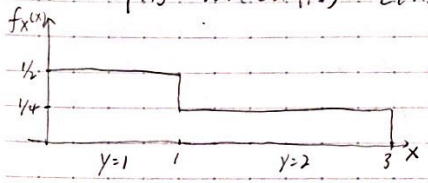
$E[Var(X|Y)] = E[E[X^2|Y=y] - (E[X|Y=y])^2]$

$= E[X^2] - E[(E[X|Y=y])^2]$

$Var(E[X|Y]) = E[(E[X|Y])^2] - (E[E[X|Y]])^2$

$= E[(E[X|Y])^2] - (E[X])^2$

$E[Var(X|Y)] + Var(E[X|Y]) = E[X^2] - (E[X])^2 = Var(X)$



$E[X|Y=1] = \frac{1}{2}$ $\quad P(X|Y=1) = \frac{1}{2}$, $P(X|Y=2) = \frac{1}{2}$

$E[X|Y=2] = 2$

$E[X] = E[E[X|Y]] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 2 = \frac{5}{4}$

Sum of a random number of independent r.v.'s

let $Y = X_1 + X_2 + \cdots + X_n$

$E[Y | N = n] = E[X_1 + \cdots + X_n | N = n] = E[X_1 + \cdots + X_n] = nE[X]$

Total Expectation theorem:

$E[Y] = \sum_n P_N(n) E[Y | N = n] = \sum_n P_N(n) nE[X] = E[N]E[X] = NE[X]$

Law of iterated expectations:

$E[Y] = E[E[Y|N]] = E[NE[X]] = E[N]E[X] \Rightarrow E[Y|N] = NE[X]$

Variance of sum of a random number of independent r.v.s

$Y = X_1 + \cdots + X_N$ $\quad Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$

- $E[Y|N] = NE[X]$ $\quad Var(Y) = E[N]Var(X) + (E[X])^2 Var(N)$

$Var(E[Y|N]) = Var(NE[X]) = (E[X])^2 Var(N)$

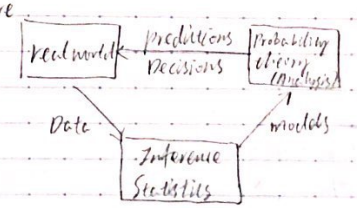
- $Var(Y | N = n) = Var(X_1 + \cdots + X_n | N = n) = Var(X_1 + \cdots + X_n) = nVar(X)$

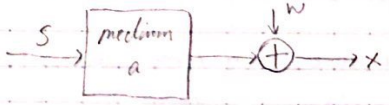
$Var(Y|N) = NVar(X)$

$E[Var(Y|N)] = E[NVar(X)] = E[N]Var(X)$

$E[g(Y)X | Y] = g(Y)E[X|Y]$

The big Picture





$X = aS + W$

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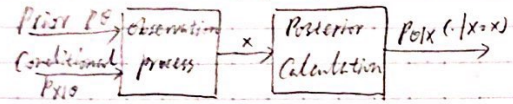
The Bayesian inference framework

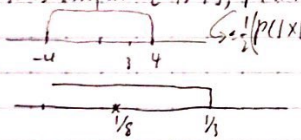
- Unknown θ
  - treated as a random variable
  - prior distribution p_Θ or f_Θ
- Observation X
  - observation model $p_{X|\Theta}$ or $f_{X|\Theta}$



Markov inequality: If $X \geq 0$ and $a > 0$, then $P(X \geq a) \leq \frac{E[X]}{a}$

$$E[X] = \int_0^\infty x f_X(x)\,dx \geq \int_a^\infty x f_X(x)\,dx \geq \int_a^\infty a f_X(x)\,dx = a\int_a^\infty f_X(x)\,dx = a\,P(X \geq a)$$

ex. X is uniform $[-4, 4]$; $P(X \geq 3)$



$\leq \frac{1}{2}\left(P(|X| \geq 3) \leq \frac{E[|X|]}{3} = \frac{2}{3}\right) \leq \frac{1}{3}$

Chebyshev inequality: $P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$

$$P(|X-\mu| \geq c) = P((X-\mu)^2 \geq c^2) \leq \frac{E[(X-\mu)^2]}{c^2} = \frac{\sigma^2}{c^2}$$

$$P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k\sigma^2} = \frac{1}{k^2}$$

ex. X is Exponential $(\lambda=1)$: $P(X \geq a) \leq \frac{1}{a}$ (markov)

$$P(X \geq a) = P(X-1 \geq a-1) \leq P(|X-1| \geq a-1) \leq \frac{1}{(a-1)^2} \sim \frac{1}{a^2}$$

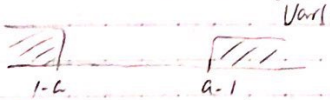
$f(x) = \lambda e^{-\lambda x}$, $E[X] = \int_0^\infty \underbrace{x}_{u}\underbrace{e^{-x}}_{v'}\,dx = -xe^{-x}\Big|_0^\infty - \int_0^\infty -e^{-x}\,dx$

$\lambda = 1$, $u' = 1$, $v = -e^{-x}$ $= 0 + \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$

$$Var(X) = E[X^2] - (E[X])^2 = \int_0^\infty x^2 e^{-x}\,dx - (1)^2$$

$\begin{cases} u' = 2x \\ v = -e^{-x} \end{cases}$ $\int_0^\infty \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v'}\,dx = \cancel{x^2(-e^{-x})\Big|_0^\infty} - \int_0^\infty 2x(-e^{-x})\,dx = 0 + 2\int_0^\infty x e^{-x}\,dx = 2$

$Var(X) = 2 - 1 = 1$



The weak law of large numbers (WLLN)

- $X_1, X_2, \ldots$ i.i.d. finite mean μ and variance σ^2

  Sample mean: $M_n = \frac{X_1 + \cdots + X_n}{n}$

- $E[M_n] = \frac{E[X_1 + \cdots + X_n]}{n} = \frac{\mu \cdot n}{n} = \mu$
- $Var(M_n) = \frac{Var(X_1 + \cdots + X_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

$$P(|M_n - \mu| \geq \varepsilon) \leq \frac{Var(M_n)}{\varepsilon^2} = \frac{\sigma^2/n}{\varepsilon^2} \xrightarrow[n\to\infty]{} 0 \text{ (fixed } \varepsilon > 0)$$

WLLN: For $\varepsilon > 0$, $P(|M_n - \mu| \geq \varepsilon) = P\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right) \to 0$, as $n \to \infty$

Sample mean M_n is unlikely to be far off from true mean μ

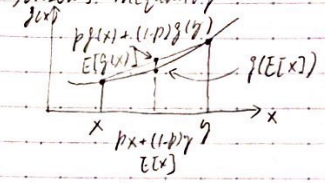
Convergence "in probability"

Define: A sequence Y_n converges in probability to a number a if: for any $\varepsilon > 0$ $\lim_{n\to\infty} P(|Y_n - a| \geq \varepsilon) = 0$

Chernoff bound $P(|M_n - \mu| \geq a) \leq e^{-nh(a)}$

Central limit theorem "$M_n \sim N(\mu, \frac{\sigma^2}{n})$"

Jensen's inequality



the central limit Theorem (CLT)

- WLLN: $\frac{X_1 + \cdots + X_n}{n} \longrightarrow E[X]$
- CLT: $X_1 + \cdots + X_n \approx$ normal
- $X_1, \ldots X_n$ i.i.d. finite mean μ and variance σ^2
- $S_n = X_1 + X_2 + \cdots + X_n$, $E[S_n] = nE[X] = n\mu$, $Var(S_n) = nVar(X) = n\sigma^2$
- $M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$, $E[M_n] = \mu$, $Var(M_n) = \frac{nVar(X)}{n^2} = \frac{\sigma^2}{n}$
- $\frac{S_n}{\sqrt{n}} = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$, $Var\left(\frac{S_n}{\sqrt{n}}\right) = \sigma^2$

  $Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $E[Z_n] = 0$, $Var(Z_n) = 1$

$S_n = X_1 + X_2 + \cdots + X_n$ $Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $Z \sim N(0,1)$

Central limit Theorem: For every z $\lim_{n\to\infty} P(Z_n \leq z) = P(Z \leq z)$

Normal approximation to the binomial

- X_i: independent, Bernoulli(p), $0 < p < 1$
- $S_n = X_1 + X_2 + \cdots + X_n$ Binomial(n, p)

  mean np, variance $np(1-p)$
- CDF of $\frac{S_n - np}{\sqrt{np(1-p)}} \longrightarrow$ standard normal

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The pollsters problem revisited

$X_i = \begin{cases} 1, & \text{if yes} \\ 0, & \text{if no} \end{cases}$ Binomial $E[X] = np$, $var(X) = np(1-p)$

Bernoulli $E[X] = p$, $Var(X_i) = p(1-p)$

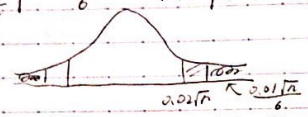
$M_n = (X_1 + X_2 + \cdots X_n)/n$ $E[M_n] = \frac{E[X_1 + \cdots X_n]}{n} = p$.

Would like "small error" e.g. $|M_n - p| < 0.01$ $Var(M_n) = Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{\sigma^2}{n}$

$P(|M_n - p| \geq 0.01) = P\left|\frac{M_n - p}{\frac{\sigma}{\sqrt{n}}}\right| \geq \frac{0.01\sqrt{n}}{\sigma} \leq P(|Z_n| \geq 0.02\sqrt{n}) = 2(1 - \Phi(0.02\sqrt{n})) = 0.05$

$\sigma = \sqrt{p(1-p)}$

$p = 1-p \Rightarrow p = \frac{1}{2}$



中心极限定理：大量相互独立随机变量的均值经适当标准化后依分布收敛于正态分布

Poisson approximation to binomial

- Interesting regime: large n, small p, moderate $\lambda = np$
- Number of arrivals S in n slots: $P_S(k) = \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \ldots, n$

$$P_S(k) = \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k}\left(1 - \frac{\lambda}{n}\right)^{n-k} = \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}}_{1} \cdot \frac{\lambda^k}{k!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{1}$$

$$\eta = \left(1 - \frac{\lambda}{n}\right)^n$$

$$\ln \eta = n \ln\left(1 - \frac{\lambda}{n}\right) \overset{n \to \infty}{=} \frac{\ln\left(1 - \frac{\lambda}{n}\right)}{\frac{1}{n}} \overset{n \to \infty}{=} \frac{\frac{1}{1 - \frac{\lambda}{n}} \cdot \left(+\frac{\lambda}{n^2}\right)}{-\frac{1}{n^2}} = \frac{-n\lambda}{n - \lambda} = \frac{-\lambda}{1} = -\lambda$$

"L'Hôspital"

$$\eta = e^{-\lambda} \Rightarrow P_S(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Discrete probability distribution

$$E[X] = \sum_1^\infty x P_S(x) = \sum_1^\infty x \cdot \frac{\lambda^x}{x!} e^{-\lambda} = \lambda \sum_1^\infty \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda}, \text{ let } Y = x - 1$$

$$= \lambda \underbrace{\sum_0^\infty \frac{\lambda^y}{y!} e^{-\lambda}}_{1} = \lambda.$$

$$Var(X) = E[X^2] - (E[X])^2 = \lambda$$

$$E[X(X-1)] = \sum_{x=2}^\infty x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2 \sum_{x=2}^\infty \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2$$

$E[X^2] - E[X] = \lambda^2$, $E[X] = \lambda \Rightarrow E[X^2] = \lambda^2 + \lambda$

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \quad k = 0, 1, 2 \ldots$$

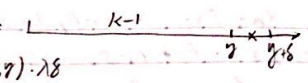
The time T_1 until the first arrival

Find the CDF: $P(T_1 \leq t) = 1 - P(T_1 > t) = 1 - P(0, t) = 1 - e^{-\lambda t}$

Exponential (λ)

$$\begin{cases} P = \lambda e^{-\lambda x}, & x \geq 0 \\ P = 0, & x < 0 \end{cases}$$

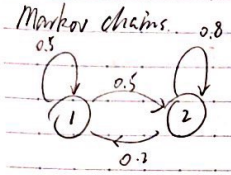
The time Y_k of the kth arrival:



$f_{Y_k}(y)\delta \approx P(y \leq Y_k \leq y + \delta) \approx P(k-1, y) \cdot \lambda\delta$

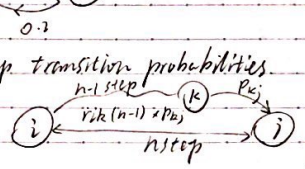
Erlang Distribution: $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$, $y \geq 0$

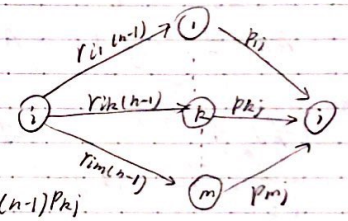
Markov chains



$$\begin{cases} r_{11}(n) = r_{11}(n-1) \cdot 0.5 + r_{12}(n-1) \cdot 0.2 \\ r_{12}(n) = 1 - r_{11}(n) \end{cases}$$

n-step transition probabilities $r_{ij}(0) = \begin{cases} 1, & \text{if } i = j \\ 0, & i \neq j \end{cases}$



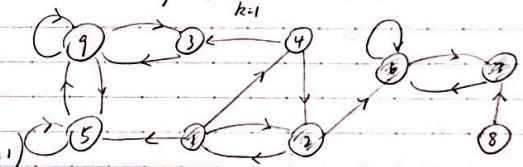


key recursion:

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1) p_{kj}$$

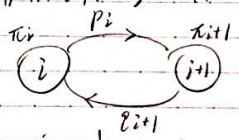
Multiplicative rule

$\mathbb{P}(B \cap C \cap D | A) = \mathbb{P}(B|A) \times \mathbb{P}(C|A \cap B) \times \mathbb{P}(D|A \cap B \cap C)$

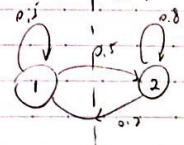


$P(X_1 = 2, X_2 = 6, X_3 = 7 | X_0 = 1)$

$= \mathbb{P}(X_1 = 2 | X_0 = 1) \times \mathbb{P}(X_2 = 6 | X_0 = 1, X_1 = 2) \times \mathbb{P}(X_3 = 7 | X_0 = 1, X_1 = 2, X_2 = 6)$



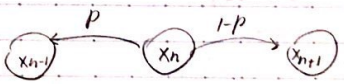
$\pi_i \cdot p_i = \pi_{i+1} \cdot q_{i+1}$



$\pi_1 \cdot 0.5 = \pi_2 \cdot 0.2$

$\Rightarrow \frac{\pi_1}{\pi_2} = \frac{2}{5}$, $\pi_1 = \frac{2}{7}$, $\pi_2 = \frac{5}{7}$

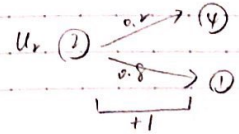
Random walk



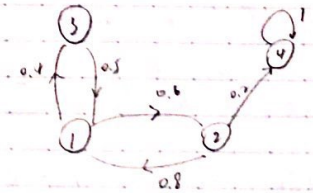
$X_n = (1-p) X_{n+1} + p X_{n-1}$

Expectation:

$E[X_n] = 1 + (1-p) E[X_{n+1}] + p E[X_{n-1}]$



$u_i = 1 + \sum_j p_{ij} u_j$



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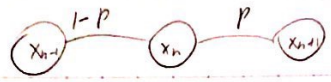


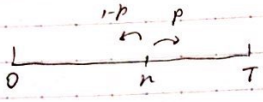
$X_n = pX_{n+1} + (1-p)X_{n-1}$

$p\theta^2 - \theta + (1-p) = 0$

$\theta = \pm \frac{1 \pm \sqrt{1-4p(1-p)}}{2p}$ $\quad \begin{cases} \theta_1 = 1 \\ \theta_2 = \frac{1-p}{p} \end{cases}$

(i) if $p \neq \frac{1}{2}$, then

$X_n = A\left(\frac{1-p}{p}\right)^n + B(1)^n$



Boundary condition

$0 = X_0 \Rightarrow A + B = 0$

$1 = X_T \Rightarrow A\left(\frac{1-p}{p}\right)^T + B = 1$

$A\left[\left(\frac{1-p}{p}\right)^T - 1\right] = 1 \Rightarrow A = \frac{1}{\left(\frac{1-p}{p}\right)^T - 1}$

$X_n = \frac{1}{\left(\frac{1-p}{p}\right)^T - 1}\left(\frac{1-p}{p}\right)^n - \frac{1}{\left(\frac{1-p}{p}\right)^T - 1}$

$= \frac{\left(\frac{1-p}{p}\right)^n - 1}{\left(\frac{1-p}{p}\right)^T - 1}$ if $p < \frac{1}{2}$, $\frac{1-p}{p} > 1$

$\leq \left(\frac{1-p}{p}\right)^{n-T} = \left(\frac{p}{1-p}\right)^m$

Thm: If $p < \frac{1}{2}$, then $Pr(\text{win \$m before lose \$n}) \leq \left(\frac{p}{1-p}\right)^m$

(ii) If $p = \frac{1}{2}$

$X_n = (An+B)\cdot(1)^n$

$X_0 = 0 \Rightarrow B = 0$

$X_T = 1 \Rightarrow A = \frac{1}{T}$ $\quad \Rightarrow X_n = \frac{1}{T}n = \frac{n}{n+m}$

Thm: if $p = 1/2$, then $Pr(\text{win \$m before lose \$n}) = \frac{n}{n+m}$

Def^n S = # steps until hit boundary.

Claim: $E_n \begin{cases} 0 & \text{if } n=0 \\ 0 & \text{if } n=T \\ 1 + pE_{n+1} + (1-p)E_{n-1} & \text{if } 0<n<T \end{cases}$

1. homog soln $\Rightarrow E_n = A\left(\frac{1-p}{p}\right)^n - B$ for $p \neq 1/2$

2 particular soln $\Rightarrow$ Guess $E_n = an + b$ $\Rightarrow \begin{cases} a = \frac{1}{1-2p} \\ b = 0 \end{cases}$

Gene: $E_n = A\left(\frac{1-p}{p}\right)^n - B + \frac{n}{1-2p}$

$n=0, E_n = 0 \Rightarrow A - B = 0$

$n=T, E_n = 0 \Rightarrow A\left(\frac{1-p}{p}\right)^T - B + \frac{T}{1-2p} = 0$

if $p = \frac{1}{2} \Rightarrow E_n = An + B - n^2$, $E_n = Tn - n^2 = nm$, $p = 1/2$

Moment-Generating Function

Def^n let W be a random variable. The moment-generating function (mgf) for W is denoted $M_W(t)$ and given by:

$$M_W(t) = E(e^{tW}) = \begin{cases} \sum_{k=0}^{\infty} e^{tk} p_W(k) & \text{if } W \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tw} f_W(w)\,dw, & \text{if } W \text{ continuous} \end{cases}$$

(i) Suppose the random variable X has a geometric pdf.

$p_X(k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$

$M_X(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tX}(1-p)^{k-1}p$

$= \sum_{k=1}^{\infty} e^{tk}(1-p)^k \frac{p}{1-p}$

$= \frac{p}{1-p}\sum_{k=1}^{\infty} e^{tk}(1-p)^k = \frac{p}{1-p}\sum_{k=1}^{\infty}[(1-p)e^t]^k$, $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, $0<r<1$

$M_X(t) = \frac{p}{1-p}\left[\sum_{k=0}^{\infty}[(1-p)e^t]^k - [(1-p)e^t]^0\right]$

$= \frac{p}{1-p}\left[\frac{1}{1-(1-p)e^t} - 1\right] = \frac{pe^t}{1-(1-p)e^t}$

(ii) Suppose that X is random variable with pdf.

$p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$, $k = 0, 1, 2$

Find $M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k(1-p)^{n-k}$ $\qquad (x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$

$= \sum_{k=0}^{n}\binom{n}{k}(e^t p)^k(1-p)^{n-k} = (1-p+pe^t)^n$

(iii) Suppose that Y has an exponential pdf. where $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$. Find $M_Y(t)$

$M_Y(t) = \int_0^{\infty} e^{ty}\lambda e^{-\lambda y}\,dy$

$= \lambda\int_0^{\infty} e^{(t-\lambda)y}\,dy = \lambda\left(\frac{1}{t-\lambda}e^{(t-\lambda)y}\Big|_0^{\infty}\right) = \frac{\lambda}{\lambda - t}$, $t < \lambda$

(4) Suppose Y normal distribution

$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$, $-\infty < y < \infty$

$M_Y(t) = \int_{-\infty}^{\infty} e^{ty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy$

$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\frac{2\sigma^2 ty - y^2 + 2\mu y - \mu^2}{2\sigma^2}}\,dy$ $\qquad e^{\frac{-(y^2 - (2\sigma^2 t + 2\mu)y + \mu^2)}{2\sigma^2}} = e^{-\frac{[(y-(\sigma^2 t + \mu))^2 - \sigma^4 t^2 - 2\mu t\sigma^2]}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi}\sigma}e^{\sigma^2 t^2/2 + \mu t}\int_{-\infty}^{\infty} e^{\frac{-[y-(\sigma^2 t + \mu)]^2}{2\sigma^2}}\,dy$

$= e^{\sigma^2 t^2/2 + \mu t}\left[\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\frac{-[y-(\sigma^2 t + \mu)]^2}{2\sigma^2}}\,dy\right] \Rightarrow M_Y(t) = e^{\sigma^2 t^2/2 + \mu t}$

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$M_Y(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ $\qquad M_Y(t) = E[e^{tX}]$

$M_Y'(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)\big|_{t=0} = \mu = E[X]$

$M_Y''(t) = \left[e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)^2 + e^{\mu t + \frac{\sigma^2 t^2}{2}} (\sigma^2) \right]\Big|_{t=0} = \sigma^2 + \mu^2 = E[X^2]$

$Var(X) = E[X^2] - (E[X])^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

Theorem: Suppose that Poisson events are occuring at the constant rate of λ per unit time. Let the random variable Y denoted the waiting time for the rth event. Then Y has Pdf $f_Y(y)$ where

$$f_Y(y) = \frac{\lambda^r}{(r-1)!} y^{r-1} e^{-\lambda y}, \quad y > 0$$

Proof: $F_Y(y) = P(Y \le y) = 1 - P(Y \ge y)$

$= 1 - P(\text{Fewer than } r \text{ events occur in } [0, y])$

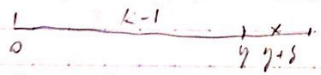
$= 1 - \sum_{k=0}^{r-1} e^{-\lambda y} \frac{(\lambda y)^k}{k!}$

Since the number of events that occur in the interval $[0, y]$ is a Poisson random variable with parameter λy.

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}\left[1 - \sum_{k=0}^{r-1} e^{-\lambda y} \frac{(\lambda y)^k}{k!}\right]$$

$$= \sum_{k=0}^{r-1} \lambda e^{-\lambda y} \frac{(\lambda y)^k}{k!} - \sum_{k=0}^{r-1} \lambda e^{-\lambda y} k \cdot \frac{(\lambda y)^{k-1}}{k!} \cdot \lambda$$

$$= \frac{\lambda^r}{(r-1)!} y^{r-1} e^{-\lambda y}$$



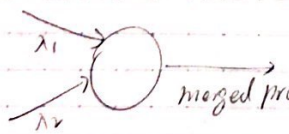
$$P = \begin{cases} 1 - \lambda\delta & k = 0 \\ \lambda\delta & k = 1 \end{cases}$$

$f_{Y_k}(y)\delta \approx P(y \le Y_k \le y + \delta) \approx P(k-1, y) \cdot \lambda\delta$

$$\frac{(\lambda y)^{k-1}}{(k-1)!} e^{-\lambda y} \cdot \lambda\delta$$

$$\Rightarrow f_{Y_k}(y) = \frac{\lambda (\lambda y)^{k-1}}{(k-1)!} e^{-\lambda y} = \frac{\lambda^k y^{k-1}}{(k-1)!} e^{-\lambda y}$$

Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr	Exponential	Geometric
Time to kth arrival	Erlang	Pascal
	POISSON	BERNOULLI



merged process: Poisson$(\lambda_1 + \lambda_2)$

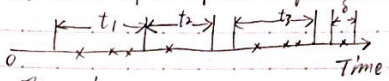
Bernoulli:
1. Discrete time: success probability p
2. Number of arrivals in n time slots: binomial pmf
3. Interarrival times: geometric pmf
4. Time to k arrivals: Pascal pmf
5. Memorylessness

$Y_k = T_1 + T_2 + \cdots + T_k$ independent geom(p)

$$P(Y_k = t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \quad t = k, k+1, \cdots \text{ Pascal pmf.}$$

Erlang in Poisson

Definition of the poisson process



- Time homogeneity

$P(k, \tau)$ = Prob. of k arrivals in interval of duration τ fixed $\sum_k P(k, \tau) = 1$

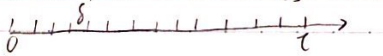
- Numbers of arrivals in disjoint time intervals are independent.
- Small interval probabilities:

For VERY small δ:

$$P(k, \delta) = \begin{cases} 1 - \lambda\delta, & \text{if } k = 0 \\ \lambda\delta, & \text{if } k = 1 \\ 0, & \text{if } k > 1 \end{cases}$$

$= \lambda$ "arrival rate"

$E[\#\text{ of arrivals in } [0, \delta]] = \lambda\delta \cdot 1 = \lambda\delta$



$n = \frac{\tau}{\delta}$ $\quad p = \lambda \cdot \delta$ $\quad np = \lambda\tau$

$$P(k \text{ arrivals}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda\tau}{n}\right)^k \left(1 - \frac{\lambda\tau}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \cdot \frac{(\lambda\tau)^k}{n^k} \underbrace{\left(1 - \frac{\lambda\tau}{n}\right)^n}_{e^{-\lambda\tau}} \underbrace{\left(1 - \frac{\lambda\tau}{n}\right)^{-k}}_{1}$$

$$= \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau}$$

$\left(1 - \frac{\lambda\tau}{n}\right)^n = y$

$\ln y = n \ln\left(1 - \frac{\lambda\tau}{n}\right)$

$$\frac{\ln\left(1 - \frac{\lambda\tau}{n}\right)}{1/n} \overset{LHS}{=} \frac{\frac{1}{1 - \frac{\lambda\tau}{n}} \cdot (-\lambda\tau) \cdot \frac{1}{n^2}}{-\frac{1}{n^2}} \overset{LHS}{=} -\lambda\tau \Rightarrow y = e^{-\lambda\tau}$$

$E[N_\tau] = \lambda\tau$, $\quad Var(N_\tau) = \lambda\tau$

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Poisson

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta, & \text{if } k=0 \\ \lambda\delta, & \text{if } k=1 \\ 0, & \text{if } k>1 \end{cases}$$

- N_t is a Poisson r.v. with parameter λt.

$$P(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k=0,1,\dots$$

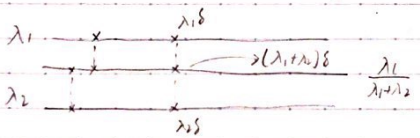
$E[N_t] = Var[N_t] = \lambda t$

- Interarrival times ($k=1$): exponential.

$f_{T_1}(t) = \lambda e^{-\lambda t}, \ t \geq 0, \quad E[T_1] = 1/\lambda$

- Time Y_k to kth arrival: Erlang(k).

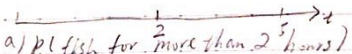
$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0.$$



ex. Poisson fishing

Assume: Poisson $\lambda = 0.6$/hour

- Fish for two hours
- if no catch, Continuone until first catch



a) P(fish for more than 2 hours) = $P(0,2) = e^{-0.6\cdot 2}$, $\int_2^\infty f_{T_1}(t)\,dt$

b) P(fish for more than two and less than 5 hours) = $P(0,2)(1-P(0,3))$, $\int_2^5 f_{T_1}(t)\,dt$

c) P(catch at least two fish) = $1 - P(0,2) - P(1,2)$ $\quad 0.6\cdot 2$

d) E[number of fish] = $\lambda\cdot t + P(0,2)\cdot 1$

e) E[future fishing time | fishing for four hours] $\frac{1}{\lambda}$ (exponential(λ))

f) E[total fishing time] = $2 + P(0,2)\cdot\frac{1}{\lambda}$

Light bulb example

- Each light bulb has independent exponential(λ) lifetime.
- Install three light bulbs. Find expected time until last light bulb dies out.

$E[\max\{X,Y,Z\}]$

merge, each exponential(λ)

$$\frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

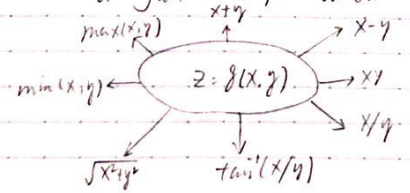
Leibniz's Rule

$\int_{a(x)}^{b(x)} f(x,t)\,dt$ where $-\infty < a(x), b(x) < \infty$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))\frac{d}{dx}b(x) - f(x,(a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\,dt$$

if $a(x)$ and $b(x)$ are constant

$$\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt$$

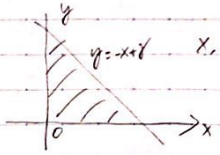


$F_Z(z) = P(Z \leq z) = P((X,Y) \leq z) = P[(X,Y) \in D_z]$

$Z = g(X,Y)$

$$= \iint_{x,y \in D_z} f_{XY}(x,y)\,dx\,dy$$

#1 $Z = X+Y$ Find $F_Z(z) \to f_Z(z)$



$x, y \geq 0$

$F_Z(z) = P(Z \leq z)$

$= P(X+Y \leq z)$

$= \iint_{x+y \leq z} f_{XY}(x,y)\,dx\,dy$

$= \int_0^z \int_0^{z-x} f_{XY}(x,y)\,dy\,dx$

$f_Z(z) = \int_0^z f_{XY}(x, z-x)\,dx$

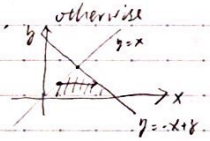
$f_X(x) = \lambda e^{-\lambda x}, \ f_Y(y) = \lambda e^{-\lambda y}$

$f_Z(z) = \int_0^z f_X(x)\cdot f_Y(z-x)\,dx$

$= \int_0^z \lambda e^{-\lambda x}\cdot\lambda e^{-\lambda(z-x)}\,dx$

$= \lambda^2 \int_0^z e^{-\lambda z}\,dx = \begin{cases} \lambda^2 z e^{-\lambda z}, & z \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Ex: $f_{XY}(x,y) = \begin{cases} e^{-x}, & x > y > 0 \\ 0, & \text{otherwise} \end{cases}$



$Z = X+Y$

$F_Z(z) = P(Z \leq z) = P(X+Y \leq z)$

$= \iint_{x+y \leq z} f_{XY}(x,y)\,dx\,dy$

$= \int_0^{\frac{z}{2}} \int_y^{z-y} f_{XY}(x,y)\,dx\,dy$

$f_Z(z) = \int_0^{\frac{z}{2}} f_{XY}(z-y, y)\,dy$

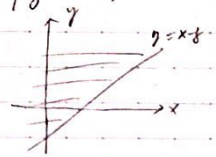
$= \int_0^{\frac{z}{2}} e^{-(z-y)}\,dy = \begin{cases} e^{-\frac{z}{2}} - e^{-z}, & x+y \leq z, \ x > y > 0 \\ 0, & \text{otherwise} \end{cases}$

#2. $Z = X-Y$

$F_Z(z) = P(Z \leq z) = P(X-Y \leq z)$

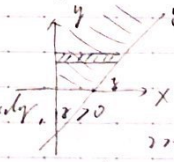
$= \iint_{x-y \leq z} f_{XY}(x,y)\,dx\,dy$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{y+z} f_{XY}(x,y)\,dx\,dy$

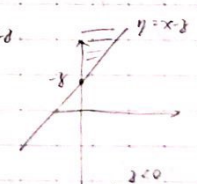


Ex. $x > 0, \ y > 0$.

$$F_Z(z) = P(X-Y \leq z) = \begin{cases} \int_0^\infty \int_0^{y+z} f_{XY}(x,y)\,dx\,dy, & z \geq 0 \\ \int_{-z}^\infty \int_0^{y+z} f_{XY}(x,y)\,dx\,dy, & z < 0 \end{cases}$$



(i)



(ii)

$$f_Z(z) = \begin{cases} \int_0^\infty f_{XY}(y+z, y)\,dy, & z \geq 0 \\ \int_{-z}^\infty f_{XY}(x,y)\,dy, & z < 0 \end{cases}$$

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$$= \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad x \geq 0$$

$$\int_0^\infty \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx = \frac{1}{2}$$

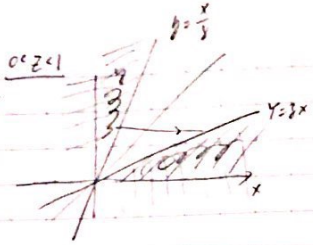
$$y = x\sin\theta, \quad dy = x\cos\theta\, d\theta$$

$$\int_0^{\pi/2} \cos\theta\, d\theta = 1$$

$$\int_0^{\pi/2} \cos\theta\, d\theta = \frac{1}{2}

$\times$ non-negative

$\Rightarrow$ $Z = \min(X,Y)/\max(X,Y) = \begin{cases} \frac{Y}{X}, & x>y \\ \frac{X}{Y}, & x<y \end{cases}$ $\quad 0<z<1$



$F_Z(z) = P(Z \le z) =$

$P(\frac{Y}{X} \le z, x>y) + P(\frac{X}{Y} \le z, x<y)$

$= 1 - \int_0^\infty \int_{yz}^{\frac{y}{z}} f_{XY}(x,y)\,dx\,dy.$

$f_Z(z) = -\int_0^\infty -\frac{y}{z^2} f_X(\frac{y}{z}) - y f_X(yz) z\,dy$, $X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$

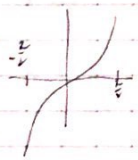
$\lambda e^{-\lambda x}$

$f_Z(z) = -\int_0^\infty -\frac{y}{z^2} \lambda e^{-\lambda \frac{y}{z}} \cdot \lambda e^{-\lambda y} - y\lambda e^{-\lambda yz} \lambda e^{-\lambda y} dy$

$= \int_0^\infty \frac{y}{z^2} \lambda e^{-\lambda(\frac{y}{z}+y)} + y\lambda e^{-\lambda(yz+y)} dy$

Ex. $z = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(\frac{Y}{X})$, X, Y, zero mean, independent Gaussian random variables with same var σ^2

$f_{Z\theta}(z,\theta) = |J(z,\theta)|\, f_{XY}(x,y) = \frac{1}{|J_{X,Y}|}$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \tan\theta = \frac{Y}{X}$

$\begin{cases} x_1 = r\cos\theta \\ y_1 = r\sin\theta \end{cases}$ / $\begin{cases} x_2 = -r\cos\theta \\ y_2 = -r\sin\theta \end{cases}$

$f_{Z\theta}(z,\theta) = z f_{X,Y}(x_1,y_1) + z f_{X,Y}(x_2,y_2)$

$f_{Z\theta}(z,\theta) = \frac{z}{2\sigma^2} e^{-z^2/2\sigma^2}$

$f_Z(z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_{Z\theta}(z,\theta)\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{z}{2\sigma^2} e^{-z^2/2\sigma^2} d\theta = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2}$

$f_\theta(\theta) = \int_0^\infty \frac{z}{2\sigma^2} e^{-z^2/2\sigma^2} dz = \frac{1}{2\sigma^2} \int_0^\infty z e^{-z^2/2\sigma^2} dz = \frac{1}{2} \int_0^\infty e^{-u} du = \frac{1}{2}, |\theta| < \frac{\pi}{2}$

$z^2/2\sigma^2 = u$, $\quad z/\sigma^2 \cdot dz = du$

$z/\sigma^2\, dz = du$

$f_Z(z) f_\theta(\theta) = \frac{z}{2\sigma^2} e^{-z^2/2\sigma^2} = f_{Z\theta}(z,\theta)$, independent

$f_Z(z) = f_\theta(h(z)) \left| \frac{dh(z)}{dz} \right|$

$f_{Z,\theta}(z,\theta) = |J_{z,\theta}|\, f_{XY}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial \theta} \end{vmatrix} f_{XY}(x,y)$

The method of Maximum Likelihood.

Def: let, $k_1, k_2, \dots k_n$ be a random sample of size n from the discrete pdf $P_X(k_i;\theta)$ where θ is an unknown parameter. The likelihood function $L(\theta)$, is the product of the pdf evaluated at the n k_i's. That is $L(\theta) = \prod_{i=1}^n P_X(k_i;\theta)$

If $y_1, y_2, \dots y_n$ is a random sample of size n from a continuous pdf $f_Y(y;\theta)$, where θ is an unknown parameter, the likelihood function is written: $L(\theta) = \prod_{i=1}^n f_Y(y_i;\theta)$

ex: $P_X(k;\lambda) = e^{-\lambda}\frac{\lambda^k}{k!}$ $k=0,1,2\dots$ Find the maximum likelihood λ

$L(\lambda) = e^{-\lambda}\frac{\lambda^{k_1}}{k_1!} e^{-\lambda}\frac{\lambda^{k_2}}{k_2!} \cdots e^{-\lambda}\frac{\lambda^{k_n}}{k_n!} = \prod_{i=1}^n e^{-\lambda}\frac{\lambda^{k_i}}{k_i!}$

$= e^{-n\lambda}\frac{\lambda^{\sum_{i=1}^n k_i}}{\prod_{i=1}^n k_i!}$

$\ln L(\lambda) = \ln(e^{-n\lambda}) + \ln(\lambda^{\sum_{i=1}^n k_i}) + \ln\left(\frac{1}{\prod_{i=1}^n k_i!}\right)$

$= -n\lambda + \ln(\lambda^{\sum_{i=1}^n k_i}) + \ln(\text{—})$

$\frac{d(\ln(\lambda))}{d\lambda} = -n + \frac{\sum_{i=1}^n k_i}{\lambda} = 0 \Rightarrow \sum_{i=1}^n k_i = n\lambda \Rightarrow \lambda_e = \frac{1}{n}\sum_{i=1}^n k_i = \bar{k}$

$\frac{d(\ln(\lambda^{\sum_{i=1}^n k_i}))}{d\lambda} = \frac{1}{\cancel{\lambda^{\sum_{i=1}^n k_i}}} \cdot \sum_{i=1}^n k_i \lambda^{\cancel{\sum_{i=1}^n k_i - 1}} = \frac{\sum_{i=1}^n k_i}{\lambda}$

ex: $f_Y(y;\theta) = \frac{1}{\theta} e^{-y/\theta}$, $0 \le y < \infty$, $0 < \theta < \infty$

$f_Z(z;\theta) = f_Z(x+y;\theta) = \int_0^t \frac{1}{\theta} e^{-y/\theta} \frac{1}{\theta} e^{-(t-y)/\theta} dy = \frac{1}{\theta^2}\int_0^t e^{-t/\theta} dy = \frac{1}{\theta^2} t e^{-t/\theta}$

$L(\theta) = \prod_{i=1}^n \frac{1}{\theta^2} z_i e^{-z_i/\theta}$

$= \frac{1}{\theta^{2n}}\left(\prod_{i=1}^n z_i\right) e^{-\sum_{i=1}^n z_i/\theta}$

$\ln L(\theta) = \ln\frac{1}{\theta^{2n}} + \ln\left(\prod_{i=1}^n z_i\right) + -\sum_{i=1}^n z_i/\theta$

$\frac{d(\ln L(\theta))}{d\theta} = (-\ln\theta^{2n})' + \sum_{i=1}^n z_i \cdot \frac{1}{\theta^2}$

$= -\frac{1}{\theta^{2n}} \cdot 2n \cdot \theta^{\cancel{2n-1}} + \frac{1}{\theta^2}\sum_{i=1}^n z_i$

$= -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n z_i = 0 \Rightarrow \theta_e = \frac{1}{2n}\sum_{i=1}^n z_i$

ex: $f_Y(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$, $-\infty < y < \infty$, $-\infty < \mu < \infty$, $\sigma^2 > 0$

$L(\mu,\sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}$

$= \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\frac{\sum_{i=1}^n (y_i-\mu)^2}{2\sigma^2}}$

$\ln L(\mu,\sigma^2) = \cancel{\ln\frac{1}{(\sqrt{2\pi}\sigma)^n}} + \left(-\frac{1}{2\sigma^2}\right)\sum_{i=1}^n (y_i-\mu)^2$

$\frac{\partial(\ln L(\mu,\sigma^2))}{\partial\mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^n 2(y_i-\mu)(-1)$

$= \frac{1}{2\sigma^2}\sum_{i=1}^n 2(y_i-\mu) = 0 \Rightarrow n\mu = \sum_{i=1}^n y_i \Rightarrow \mu = \frac{1}{n}\sum_{i=1}^n y_i$

$\frac{\partial(\ln(L(\mu,\sigma^2)))}{\partial\sigma^2} = (-1)\ln(\sqrt{2\pi}\sigma)^n - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2$

$= (-1)\ln(2\pi\sigma^2)^{\frac{n}{2}} - \text{—}$

$= -\frac{n}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n \text{—} = 0$

$n\sigma^2 = \sum_{i=1}^n (y_i-\mu)^2$

$= \sigma^2 = \frac{1}{n}\sum_{i=1}^n (y_i-\bar{y})^2$

$\mu = \bar{y}$

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Cauchy-Schwarz inequality. $|x \cdot y| \le \|x\|_2 \|y\|_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$$

Concave

$$f\left(\sum_{i=1}^{n} |x_i|\right) \le \sum_{i=1}^{n} (f|x_i|)$$

Proof: $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x_2\|$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2^2 = \sum_{i=1}^{N} |x_i|^2 \le \left(\sum_{i=1}^{N} |x_i|^2 + 2 * \sum_{i,j, i \ne j} |x_i||x_j|\right) = \|x\|_1^2$$

$$\|x\|_2^2 = \sum_{i=1}^{N} |x_i|^2 \le N * \max(|x_i|^2) = N\|x\|_\infty^2$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i| \cdot 1 \le \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} 1^2\right)^{1/2} = \sqrt{n}\|x\|_2$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

Matrix Norms

Definition: A matrix norm is a function $\mathbb{R}^{n \times n}$ to $\mathbb{R}$ which satisfies a certain number of properties. If $A \in \mathbb{R}^{n \times n}$, we symbolize its norm by $\|A\|$. The defining properties of a matrix norm are:

(i) $\|M\| \ge 0$ for all $M \in \mathbb{R}^{n \times n}$, and also $\|M\| = 0$ iff $M = 0$

(ii) $\|\alpha M\| = |\alpha| \|M\|$ for all $\alpha \in \mathbb{R}$

(iii) $\|M + N\| \le \|M\| + \|N\|$

(iv) $\|M \cdot N\| \le \|M\| \cdot \|N\|$

Definition: If $\|\cdot\|_*$ is a valid vector norm, its include matrix norm is define as

$$\|M\|_* = \max_{\substack{x \in \mathbb{R}^n \\ x \ne 0}} \left\{\frac{\|Mx\|_*}{\|x\|_*}\right\}$$

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