Digital Logic Design: a rigorous approach ©

Chapter 7: Asymptotics

Guy Even Moti Medina

School of Electrical Engineering Tel-Aviv Univ.

December 5, 2013

Book Homepage:
http://www.eng.tau.ac.il/~guy/Even-Medina

Order of Growth Rates

Consider the Fibonacci sequence $(0, 1, 1, 2, 3, 5, \ldots)$ where $g(n)$ is defined by

(i) Base case: $g(0) = 0$ and $g(1) = 1$.

(ii) Reduction rule: $g(n+2) = g(n+1) + g(n)$.

The exact value of $g(n)$, or an analytic equation for $g(n)$ is interesting, but sometimes, all we need to know is how "fast" does $g(n)$ grow?
Does it grow faster than $f(n) = n$, $f(n) = n^2$, $f(n) = 2^n$? We wish to capture the notion of "$g(n)$ does not grow faster than $f(n)$".

big-O, big-Omega, big-Theta

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq}$ denote two functions.

1. We say that $g(n) = O(f(n))$, if there exist constants $c \in \mathbb{R}$ and $N \in \mathbb{N}$, such that,
$$\forall n > N : g(n) \leq c \cdot f(n) \, .$$
2. We say that $g(n) = \Omega(f(n))$, if there exist constants $d \in \mathbb{R}^{\geq}$ and $N \in \mathbb{N}$, such that,
$$\forall n > N : g(n) \geq d \cdot f(n) \, .$$
3. We say that $g(n) = \Theta(f(n))$, if $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$.

Quick Observations

- If $g(n) = O(f(n))$, then $g(n)$ does not grow faster than $f(n)$.

Quick Observations

- If $g(n) = O(f(n))$, then $g(n)$ does not grow faster than $f(n)$.
- If $g(n) = \Omega(f(n))$, then $g(n)$ grows as least as fast as $f(n)$.

Quick Observations

- If $g(n) = O(f(n))$, then $g(n)$ does not grow faster than $f(n)$.
- If $g(n) = \Omega(f(n))$, then $g(n)$ grows as least as fast as $f(n)$.
- Finally, if $g(n) = \Theta(f(n))$, then $g(n)$ grows as fast as $f(n)$.

Quick Observations

- If $g(n) = O(f(n))$, then $g(n)$ does not grow faster than $f(n)$.
- If $g(n) = \Omega(f(n))$, then $g(n)$ grows as least as fast as $f(n)$.
- Finally, if $g(n) = \Theta(f(n))$, then $g(n)$ grows as fast as $f(n)$.
- $f(n) = O(g(n)) \quad \Leftrightarrow \quad g(n) = \Omega(f(n))$.

Examples - 1

1. The notation $f(n) = O(1)$ means that $\exists\ c\ \forall\ n : f(n) \leq c$.

Examples - 1

1. The notation $f(n) = O(1)$ means that $\exists\, c\ \forall\, n : f(n) \leq c$.
2. Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.

Examples - 1

1. The notation $f(n) = O(1)$ means that $\exists\ c\ \forall\ n : f(n) \leq c$.
2. Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.
3. Let $g(n) \triangleq n^2 + n + 1$. We claim that $g(n) = \Theta(n^2)$.

Examples - 1

1. The notation $f(n) = O(1)$ means that $\exists\ c\ \forall\ n : f(n) \leq c$.
2. Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.
3. Let $g(n) \triangleq n^2 + n + 1$. We claim that $g(n) = \Theta(n^2)$.
4. Let $g(n) = c \cdot f(n)$, then $g(n) = \Theta(f(n))$.

Examples - 1

1. The notation $f(n) = O(1)$ means that $\exists\ c\ \forall\ n : f(n) \leq c$.
2. Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.
3. Let $g(n) \triangleq n^2 + n + 1$. We claim that $g(n) = \Theta(n^2)$.
4. Let $g(n) = c \cdot f(n)$, then $g(n) = \Theta(f(n))$.
5. $\sum_{i=1}^{n} i = \Theta(n^2)$.

Examples - 1

1. The notation $f(n) = O(1)$ means that $\exists\ c\ \forall\ n : f(n) \leq c$.
2. Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.
3. Let $g(n) \triangleq n^2 + n + 1$. We claim that $g(n) = \Theta(n^2)$.
4. Let $g(n) = c \cdot f(n)$, then $g(n) = \Theta(f(n))$.
5. $\sum_{i=1}^{n} i = \Theta(n^2)$.
6. $\log(n) = O(n)$.

Examples - 2

1. If $q > 1$, then $\sum_{i=1}^{n} q^i = \Theta(q^n)$.

Examples - 2

1. If $q > 1$, then $\sum_{i=1}^{n} q^i = \Theta(q^n)$.
2. [transitivity] If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Examples - 2

1. If $q > 1$, then $\sum_{i=1}^{n} q^i = \Theta(q^n)$.
2. [transitivity] If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.
3. [addition,max] If $f(n), g(n) = O(h(n))$, then $f(n) + g(n), \max\{f(n), g(n)\} = O(h(n))$.

Examples - 2

1. If $q > 1$, then $\sum_{i=1}^{n} q^i = \Theta(q^n)$.
2. [transitivity] If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.
3. [addition,max] If $f(n), g(n) = O(h(n))$, then $f(n) + g(n), \max\{f(n), g(n)\} = O(h(n))$.
4. [addition,min] If $f(n), g(n) = \Omega(h(n))$, then $f(n) + g(n), \min\{f(n), g(n)\} = \Omega(h(n))$.

Examples - 2

1. If $q > 1$, then $\sum_{i=1}^{n} q^i = \Theta(q^n)$.
2. [transitivity] If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.
3. [addition,max] If $f(n), g(n) = O(h(n))$, then $f(n) + g(n), \max\{f(n), g(n)\} = O(h(n))$.
4. [addition,min] If $f(n), g(n) = \Omega(h(n))$, then $f(n) + g(n), \min\{f(n), g(n)\} = \Omega(h(n))$.
5. [asymmetry] $f(n) = O(g(n))$ does not imply that $g(n) = O(f(n))$.

In this section we deal with the problem of solving or bounding the rate of growth of functions $f : \mathbb{N}^+ \to \mathbb{R}$ that are defined recursively. We consider the typical cases that we will encounter later.

Recurrence 1

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & \text{if } n > 1. \end{cases} \tag{1}$$

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n)$.

What about $f(n) = n + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$?

Is it enough to solve for powers of 2?

In the following lemma we show that, under reasonable conditions, it suffices to consider powers of two when bounding the rate of growth.

Lemma

Assume that:

1. *The functions $f(n)$ and $g(n)$ are both monotonically nondecreasing.*
2. *The constant ρ satisfies, for every $k \in \mathbb{N}$,*

$$\rho \geq \frac{g(2^{k+1})}{g(2^k)}.$$

If $f(2^k) = O(g(2^k))$, then $f(n) = O(g(n))$.

What about big-Omega?

Lemma

Assume that:

1. *The functions $f(n)$ and $g(n)$ are both monotonically nondecreasing.*
2. *The constant ρ satisfies, for every $k \in \mathbb{N}$,*

$$\rho \leq \frac{g(2^{k+1})}{g(2^k)}.$$

If $f(2^k) = \Omega(g(2^k))$, then $f(n) = \Omega(g(n))$.

Recurrence 2.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + 2 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \tag{2}$$

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n \log n)$.

Recurrence 3.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + 3 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \tag{3}$$

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n^{\log_2 3})$.

hint: $f(2^k) = 3^{k+1} - 2^{k+1}$.

Example - 1

Consider the recurrence

$$f(n) \triangleq \begin{cases} c & \text{if } n = 1 \\ a \cdot n + b + f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & \text{if } n > 1, \end{cases} \tag{4}$$

where a, b, c are constants.

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n)$.

proof: $f(2^k) = 2a \cdot 2^k + b \cdot k + c - 2a$...

Example -2

Consider the recurrence

$$f(n) \triangleq \begin{cases} c & \text{if } n = 1 \\ a \cdot n + b + 2 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases} \tag{5}$$

where $a, b, c = O(1)$.

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n \log n)$.

proof: We claim that $f(2^k) = a \cdot k2^k + (b + c) \cdot 2^k - b$....

Example - 3

Consider the recurrence

$$F(k) \triangleq \begin{cases} 1 & \text{if } k = 0 \\ 2^k + 2 \cdot F(k-1) & \text{if } k > 0, \end{cases} \tag{6}$$

Lemma

$F(k) = (k+1) \cdot 2^k$.

Proof: Define $f(n) \triangleq F(\lceil \log_2 n \rceil)$. Observe that $f(2^x) \triangleq F(x)$....

Examples with floor and ceiling

1. $$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ 1 + f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases}$$

2. $$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + f(\lfloor \frac{n}{2} \rfloor) + f(\lceil \frac{n}{2} \rceil) & \text{if } n > 1, \end{cases}$$