Digital Logic Systems

Recitation 2: Induction & Recursion

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Application: One-to-one and Onto Functions

Definition

Let $f : A \to B$ denote a function from A to B.

1. The function f is one-to-one if $a \neq a'$ implies that $f(a) \neq f(a')$.
2. The function f is onto if, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.
3. The function f is a bijection if it is both onto and one-to-one.

- A one-to-one function is sometimes called an injective function (or an injection).
- A function that is onto is sometimes called a surjection.

Application: One-to-one and Onto Functions (Cont.)

Lemma

Let $f : A \to B$ denote a function. Let $A' \subseteq A$ and $B' \triangleq B \setminus \{f(a') \in B \mid a' \in A \setminus A'\}$. Let $f' : A' \to B'$ denote the restricted function defined by $f'(x) = f(x)$, for every $x \in A'$. If f is one-to-one, then f' is also one-to-one.

Proof.

On the whiteboard. $\square$

Application: One-to-one and Onto Functions (Cont.)

- Let $f : A \to B$ denote a function.
- Let $B' \subseteq B$.
- Let $f^{-1}(B')$ denote the set $\{a \in A \mid f(a) \in B'\}$.

Lemma

Let $f : A \to B$ denote a function. Let $B' \subseteq B$ and $A' \triangleq A \setminus f^{-1}(B \setminus B')$. Let $f' : A' \to B'$ denote the restricted function defined by $f'(x) = f(x)$, for every $x \in A'$. If f is onto, then f' is also onto.

Proof.

On the whiteboard. $\square$

Application: One-to-one and Onto Functions (Cont.)

The following two lemmas show how one-to-one and onto functions can be used to compare cardinalities of sets.

Lemma (2.5)

Let A and B denote two finite sets. If there exists a one-to-one function $f : A \to B$, then $|A| \leq |B|$.

Proof.

On the whiteboard. $\square$

Application: One-to-one and Onto Functions (Cont.)

Lemma (2.6)

Let A and B denote two finite sets. If there exists an onto function $f : A \to B$, then $|A| \geq |B|$.

Proof.

Similar to the previous proof. See the proof in the textbook. $\square$

Pigeonhole Principle

- By Lemma 2.5: If there exists a one-to-one function $f : A \rightarrow B$, then $|A| \leq |B|$.
- The contrapositive form of Lemma 2.5: if $|A| > |B|$, then every function $f : A \rightarrow B$ is not one-to-one.

We are now ready to formalize the Pigeonhole Principle, as follows.

The Pigeonhole Principle

Let $f : A \rightarrow \{1, \ldots, n\}$, and $|A| > n$, then f is not one-to-one, i.e., there are $a_1, a_2 \in A$; $a_1 \neq a_2$, such that $f(a_1) = f(a_2)$.