Digital Logic Systems

Recitation 1: Sets and Functions, Induction

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Sets

- Let $A \triangleq \{1, 2, 4, 8\}$ and $B \triangleq \{\text{pencil}, \text{pen}, \text{eraser}\}$.
- Examples of equal sets:
  (i) Order and repetitions do not affect the set, e.g., $\{1, 1, 1\} = \{1\}$ and $\{1, 2\} = \{2, 1\}$.
  (ii) $\{2, 4, 8, 1, 1, 2\} = A$,
  (iii) $\{1, 2, 44, 8\} \neq A$,
  (iv) $A \neq B$.
- The empty set is denoted by $\emptyset$. The set $\{\emptyset\}$ contains a single element which is the empty set. Therefore, $\emptyset \neq \{\emptyset\}$.

Sets (cont.)

- If $A \cap B = \emptyset$, then we say that A and B are *disjoint*. We say that the sets $A_1, \ldots, A_k$ are disjoint if $A_1 \cap \cdots \cap A_k = \emptyset$. We say that the sets $A_1, \ldots, A_k$ are *pairwise-disjoint* if for every $i \neq j$, the sets A_i and A_j are disjoint.
- Consider the three sets $\{1,2\}$, $\{2,3\}$ and $\{1,3\}$. Their intersection is empty, therefore, they are disjoint. However, the intersection of every pair of sets is nonempty, therefore, they are not pairwise disjoint.
- When A and B are disjoint, i.e., $A \cap B = \emptyset$, we denote their union by $A \uplus B$.
  1. $\{1,2\} \uplus \{4,8\} = A$,
  2. $\{1,2\} \cup A = A$.

Sets (cont.)

Lemma

For every sets A and B,

$$A \setminus B = A \cap \bar{B} .$$

Proof.

To prove this we show containment in both directions:

(i) We prove that $A \setminus B \subseteq A \cap \bar{B}$. Let $x \in A \setminus B$. By the definition of subtraction of sets, this means that $x \in A$ and $x \notin B$. By the definition of complement, $x \in \bar{B}$. By the definition of intersection, $x \in A \cap \bar{B}$, as required.

(ii) We prove that $A \cap \bar{B} \subseteq A \setminus B$. Let $x \in A \cap \bar{B}$. By the definition of intersection of sets, this means that $x \in A$ and $x \in \bar{B}$. By the definition of complement, $x \in \bar{B}$ implies that $x \notin B$. By the definition of subtraction, $x \in A \setminus B$, as required.

$\square$

Sets (cont.)

Lemma

For finite sets X and Y (regardless of their disjointness)

$$|X \times Y| = |X| \cdot |Y| .$$

Proof.

To be proven in Problem set 1. $\square$

Examples of compositions of functions.
Let $f(x) = 2x + 4$ and let $g(x) = x^2$, then

- $$\begin{aligned} f(g(x)) &= f(x^2) \\ &= 2(x^2) + 4 \\ &= 2x^2 + 4\,. \end{aligned}$$

- $$\begin{aligned} g(f(x)) &= g(2x + 4) \\ &= (2x + 4)^2 \\ &= (2x)^2 + 2 \cdot 2x \cdot 4 + 4^2 \\ &= 4x^2 + 16x + 16\,. \end{aligned}$$

Boolean Functions

- The *parity* function $p : \{0,1\}^n \rightarrow \{0,1\}$ is defined as follows.

$$p(b_1, \ldots, b_n) \triangleq \begin{cases} 1 & \text{if } \sum_{i=1}^n b_i \text{ is odd} \\ 0 & \text{if } \sum_{i=1}^n b_i \text{ is even.} \end{cases}$$

  For example: (i) $p(0,1,0,1,0) = 0$, (ii) $p(0,1,1,1,0) = 1$, (iii) for $n = 2$, the parity function is identical to the XOR function.

- The *majority* function $m : \{0,1\}^n \rightarrow \{0,1\}$ is defined as follows.

$$m(b_1, \ldots, b_n) = 1 \quad \text{if and only if} \quad \sum_{i=1}^n b_i > \frac{n}{2}.$$

  For example: (i) $m(0,1,0,1,0) = 0$, (ii) $m(0,1,1,1,0) = 1$, (iii) for $n = 2$, the majority function is identical to the AND function.

Boolean Functions (cont.)

- The *3-bit carry* function $c : \{0,1\}^3 \rightarrow \{0,1\}$ is defined as follows.

$$c(b_1, b_2, b_3) = 1 \quad \text{if and only if} \quad b_1 + b_2 + b_3 \geq 2\,.$$

For example: (i) $c(0,1,0) = 0$, (ii) $c(0,1,1) = 1$.

- The truth table of the 3-bit carry function is depicted in the following table.

b_1	b_2	b_3	$c(b_1, b_2, b_3)$
0	0	0	0
1	0	0	0
0	1	0	0
1	1	0	1
0	0	1	0
1	0	1	1
0	1	1	1
1	1	1	1

Definition

- The implication operator $\rightarrow(x, y)$ is defined by

$$x \rightarrow y \Leftrightarrow \bar{x} \vee y\,.$$

- The equivalence operator $\leftrightarrow(x, y)$ is defined by

$$x \leftrightarrow y \Leftrightarrow \neg(x \oplus y)\,.$$

Truth tables:

x	y	$\rightarrow(x, y)$
0	0	1
1	0	0
0	1	1
1	1	1

x	y	$\leftrightarrow(x, y)$
0	0	1
1	0	0
0	1	0
1	1	1

Commutative and Associative Binary Operations

The subtraction operation $- : \mathbb{R}^2 \rightarrow \mathbb{R}$ is neither associative nor commutative.

For example:

- $0 - (0 - 1) = 1$ but $(0 - 0) - 1 = -1$, and
- $1 - 2 = -1$ but $2 - 1 = 1$.

Pólya's proof that "all horses have the same color".
We obviously know that there are two horses with different colors, as depicted in the following figure.

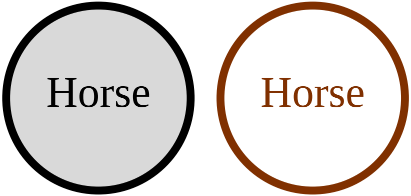

Figure: A counter example to the claim that all the (spherical) horses are the same color. To prove that a claim is not correct all we need is to supply a counter example.

- The “proof” is by induction on the number of horses, denoted by n.
- Thus, we wish to prove that in every set of n horses, all the horses have the same color.
- The induction basis, for $n = 1$, is trivial since in a set consisting of a single horse there is only one color.
- The induction hypothesis simply states that in every set of n horses, all horses have the same color.

Induction (cont.)

- The induction step. We need to prove that if the claim holds for n, then it also holds for $n+1$.
  1. Number the horses, i.e., $\{1, \ldots, n+1\}$.
  2. Consider two subsets of horses $A \triangleq \{1, \ldots, n\}$ and $B \triangleq \{2, \ldots, n+1\}$.
  3. By the induction hypothesis the horses in set A have the same color and the horses in set B also have the same color.
  4. Since $2 \in A \cap B \Rightarrow$ the horses in $A \cup B$ have the same color.
- We have "proved" the induction step, and the "theorem" follows.

Induction (cont.)

What is wrong with this proof?

- Note that, in the induction step, $A \cap B \neq \emptyset$ only if $n \geq 2$.
- However, the induction basis was proved only for $n = 1 \Rightarrow$ we did not prove the induction step for a set of 2 horses.
- A correct proof would have to extend the basis to $n = 2$, an impossible task.
- The take home advice is to make sure that the induction basis is proved for all the cases. In particular, never skip the induction basis even if you think that the claim is "easy" for small values of n.