

# Digital Logic Systems

## Chapter 7: Asymptotics

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## Order of Growth Rates

Consider the Fibonacci sequence  $(0, 1, 1, 2, 3, 5, \dots)$  where  $g(n)$  is defined by

- (i) Base case:  $g(0) = 0$  and  $g(1) = 1$ .
- (ii) Reduction rule:  $g(n + 2) = g(n + 1) + g(n)$ .

The exact value of  $g(n)$ , or an analytic equation for  $g(n)$  is interesting, but sometimes, all we need to know is how “fast” does  $g(n)$  grow?

Does it grow faster than  $f(n) = n$ ,  $f(n) = n^2$ ,  $f(n) = 2^n$ ? We wish to capture the notion of “ $g(n)$  does not grow faster than  $f(n)$ ”.

# big-O, big-Omega, big-Theta

## Definition

Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq}$  denote two functions.

- ① We say that  $g(n) = O(f(n))$ , if there exist constants  $c_1, c_2 \in \mathbb{R}^{\geq}$  such that, for every  $n \in \mathbb{N}$ ,

$$g(n) \leq c_1 \cdot f(n) + c_2.$$

- ② We say that  $g(n) = \Omega(f(n))$ , if there exist constants  $c_3, c_4 \in \mathbb{R}^{\geq}$  such that, for every  $n \in \mathbb{N}$ ,

$$g(n) \geq c_3 \cdot f(n) + c_4.$$

- ③ We say that  $g(n) = \Theta(f(n))$ , if  $g(n) = O(f(n))$  and  $g(n) = \Omega(f(n))$ .

If  $g(n) = O(f(n))$ , then  $g(n)$  does not grow faster than  $f(n)$ . If  $g(n) = \Omega(f(n))$ , then  $g(n)$  grows at least as fast as  $f(n)$ . Finally, if  $g(n) = \Theta(f(n))$ , then  $g(n)$  grows as fast as  $f(n)$ .

## Examples - 1

- ① The notation  $f(n) = O(1)$  means that  $\exists c \forall n : f(n) \leq c$ .
- ② Let  $g(n) \triangleq 2 \cdot n$ . We claim that  $g(n) = \Theta(n)$ .
- ③ Let  $g(n) \triangleq n^2 + n + 1$ . We claim that  $g(n) = \Theta(n^2)$ .
- ④ Let  $g(n) = c \cdot f(n)$ , then  $g(n) = \Theta(f(n))$ .
- ⑤  $\sum_{i=1}^n i = \Theta(n^2)$ .
- ⑥ If  $q > 1$ , then  $\sum_{i=1}^n q^i = \Theta(q^n)$ .

# Recurrence Equations

In this section we deal with the problem of solving or bounding the rate of growth of functions  $f : \mathbb{N}^+ \rightarrow \mathbb{R}$  that are defined recursively. We consider the typical cases that we will encounter later.

# Recurrence 1

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + f(\lceil \frac{n}{2} \rceil) & \text{if } n > 1. \end{cases} \quad (1)$$

## Lemma

*The rate of growth of the function  $f(n)$  is  $\Theta(n)$ .*

What about  $f(n) = n + f(\lfloor \frac{n}{2} \rfloor)$ ?

# Is it enough to solve for powers of 2?

In the following lemma we show that, under reasonable conditions, it suffices to consider powers of two when bounding the rate of growth.

## Lemma

Assume that:

- ① The functions  $f(n)$  and  $g(n)$  are both monotonically nondecreasing.
- ② The constant  $\rho$  satisfies, for every  $k \in \mathbb{N}$ ,

$$\rho \geq \frac{g(2^{k+1})}{g(2^k)}.$$

If  $f(2^k) = O(g(2^k))$ , then  $f(n) = O(g(n))$ .

# What about big-Omega?

## Lemma

Assume that:

- ① The functions  $f(n)$  and  $g(n)$  are both monotonically nondecreasing.
- ② The constant  $\rho$  satisfies, for every  $k \in \mathbb{N}$ ,

$$\rho \leq \frac{g(2^{k+1})}{g(2^k)}.$$

If  $f(2^k) = \Omega(g(2^k))$ , then  $f(n) = \Omega(g(n))$ .

## Recurrence 2.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + 2 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \quad (2)$$

### Lemma

*The rate of growth of the function  $f(n)$  is  $\Theta(n \log n)$ .*

## Recurrence 3.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + 3 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \quad (3)$$

### Lemma

The rate of growth of the function  $f(n)$  is  $\Theta(n^{\log_2 3})$ .

## Example - 1

Consider the recurrence

$$f(n) \triangleq \begin{cases} c & \text{if } n = 1 \\ a \cdot n + b + f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases} \quad (4)$$

where  $a, b, c$  are constants.

### Lemma

*The rate of growth of the function  $f(n)$  is  $\Theta(n)$ .*

proof:  $f(2^k) = 2a \cdot 2^k + b \cdot k + c - 2a\dots$

## Example -2

Consider the recurrence

$$f(n) \triangleq \begin{cases} c & \text{if } n = 1 \\ a \cdot n + b + 2 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases} \quad (5)$$

where  $a, b, c = O(1)$ .

### Lemma

*The rate of growth of the function  $f(n)$  is  $\Theta(n \log n)$ .*

proof: We claim that  $f(2^k) = a \cdot k2^k + (b + c) \cdot 2^k - b \dots$

## Example - 3

Consider the recurrence

$$F(k) \triangleq \begin{cases} 1 & \text{if } k = 0 \\ 2^k + 2 \cdot F(k-1) & \text{if } k > 0, \end{cases} \quad (6)$$

### Lemma

$$F(k) = (k+1) \cdot 2^k.$$

Proof: Define  $f(n) \triangleq F(\lceil \log_2 n \rceil)$ . Observe that  $f(2^x) \triangleq F(x) \dots$

## Examples with floor and ceiling

1

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ 1 + f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases}$$

2

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + f(\lfloor \frac{n}{2} \rfloor) + f(\lceil \frac{n}{2} \rceil) & \text{if } n > 1, \end{cases}$$