Question 1 Suppose A is a complex $n \times n$ matrix. Show that the following are equivalent:

(a) The rows of A form an orthonormal basis in C^n.

(b) $AA^* = I$.

(c) $\|Ax\| = \|x\|$ for all $x \in C^n$.

Question 2 Suppose $A : V \to W$ is a linear map between two inner product spaces. Show that the nullspace of A^* is exactly the perpendicular complement of the range of A.

Question 3 Prove the Fredholm Alternative: Suppose $A : V \to W$ is a linear map between two inner product spaces. Let $b \in W$. Then either

(a) $Ax = b$ for some $x \in V$ or

(b) There is $w \in W$ with $A^*w = 0$ and $< b, w > \neq 0$.

Question 4 Use the Fredholm Alternative and the Fundamental Theorem of Algebra to prove the existence and uniqueness of polynomial interpolation: given $n+1$ distinct real numbers $x_0, x_1, \ldots x_n$ and $n+1$ complex numbers $f_j, f_1, \ldots f_n$, there exists a unique degree-n polynomial $P(x) = p_0 + p_1 x + \cdots + p_n x^n$ such that $P(x_j) = f_j$ for $0 \leq j \leq n$.

Question 5 Prove that a projection P on an inner product space is an orthogonal projection if and only if $P^* = P$.

Question 6 (a) Let

$$K_t(x) = \frac{t}{\pi(t^2 + x^2)}$$

for $t > 0$ and $x \in R$. Use the Dominated Convergence Theorem to show that

$$\int_{-\infty}^{\infty} K_t(x-y) f(y) \, \mathrm{d}y \to f(x)$$

as $t \to 0$, for all bounded continuous functions f.

(b) Use (a) to evaluate

$$\int_{-\infty}^{\infty} K_t(x-y) \, \mathrm{d}y$$

Question 7 Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-|x-y|/t}}{2t} f(y)\,\mathrm{d}y \to f(x)$$

as $t \to 0$, for all bounded continuous functions f.