Beta-binomial conjugation

Prove $\text{Betabin}(\theta \mid a, b, N, m) = \text{Beta}(\theta \mid a', b')$

Bayesian

$$P(\theta \mid D) = \frac{P(D \mid \theta) \cdot P(\theta)}{P(D)}$$

Assume prior $P(\theta) = \text{Beta}(\theta \mid a, b)$

likelihood $P(m \mid \theta) = \text{Bin}(N, m) = \binom{N}{m} \theta^m (1-\theta)^{N-m}$

$$P(\theta \mid m) = \text{Betabin}(\theta \mid a, b, N, m) = \frac{P(m \mid \theta) \cdot P(\theta)}{P(m)}$$

$D = m$

$$P(\theta \mid m) = \frac{\binom{N}{m} \theta^m (1-\theta)^{N-m} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1}}{P(m)}$$

$$\text{Beta}(\theta \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)} \cdot \theta^{a-1} (1-\theta)^{b-1}$$

$$= \frac{1}{B(a, b)} \cdot \theta^{a-1} (1-\theta)^{b-1}$$

$$\because B(a,b) = \frac{\Gamma(a)+\Gamma(b)}{\Gamma(a+b)} = \int_0^1 \theta^{a-1}(1-\theta)^{b-1}\,d\theta$$

$$P(\theta \mid m) = \frac{C_m^N \theta^m (1-\theta)^{N-m} \cdot \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}}{P(m)}$$

$$= \frac{1}{B(a,b)} \cdot C_m^N \cdot \frac{\theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}}{P(m)}$$

$$\because P(m) = \int_0^1 P(m \mid \theta) \cdot P(\theta)\,d\theta$$

$$= \frac{1}{B(a,b)} \cdot C_m^N \cdot \underbrace{\int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}\,d\theta}_{B(m+a,\,N-m+b)}$$

$$P(\theta \mid m) = \frac{\cancel{\frac{1}{B(a,b)}} \cdot \cancel{C_m^N} \cdot \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}}{\cancel{\frac{1}{B(a,b)}} \cdot \cancel{C_m^N} \cdot B(m+a,\,N-m+b)}$$

$$= \frac{1}{B(m+a,\,N-m+b)} \cdot \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}$$

$$= \mathrm{Beta}(\theta \mid \underbrace{m+a}_{a'}, \underbrace{N-m+b}_{b'})$$

$$\therefore P(\theta \mid m) = \mathrm{BetaBin}(\theta \mid a, b, N, m)$$
$$= \mathrm{Beta}(\theta \mid m+a, N-m+b),\ a' = m+b,\ b' = N-m+b$$