Finance 361

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1 Introduction to Investments

An investment is the current commitment of money or other resource in the expectation of receiving future benefits.

2 Investors, Assets and Markets

3 Real world frictions and investment Styles

4 Risk and Return, Expectations and Pricing

4.1 Pricing risk

The argument for risk being priced stems from financial economics and only requires two basic assumptions; people have insatiable appetite for wealth and the law of diminishing marginal returns.

4.2 Utility Functions

Economically, happiness is measured in utility. Utility is a function of wealth $U(W)$, utility is increasing in wealth $U'(W) > 0$ and utility increases in wealth at a decreasing rate $U''(W) < 0$. Thus, we say utility is *concave* in wealth.

What this percolates to is the notion that the utility of certain wealth is always higher than the utility of the average of uncertain wealth; or, using this framework, people are risk averse. Ergo;

$$U(E[W]) > E[U(W)]$$

Where W is a vector of possible final wealth outcomes, and E[W] is the average of all final wealth outcomes. Finding $max(E[U(W)])$ is done by calculating the final wealth in each state of the world, then calculate the utility of this final wealth in each state of the world, then calculate the expectation over these utilities; read: weighted average according to respective probabilities.

4.2.1 Example

Bob has logarithmic utility over final wealth W_1, given by $U(W_1) = ln(W_1)$. Bob has initial wealth of $W_0 = \$1000$. Bob can invest at the risk free rate of 5% or invest \$1000 in a risky project with a 40% probability of making \$850 and 60% chance of making \$1350. What should Bob do?

Option 1;

$$\begin{aligned} E[U(W_1)] &= E[U(1000 \times (1 + 5\%))] \\ &= E[U(1050)] \\ &= E[ln(1050)] \\ &= 6.957 \end{aligned}$$

Option 2;

$$\begin{aligned} E[U(W_1)] &= 0.4U(W_1)0.6U(W_1) \\ &= 0.4 \times ln(850) + 0.6 \times ln(1350) \\ &= 7.023 \end{aligned}$$

Given that $U(O_2) > U(O_1)$, Bob will invest in the risky project.

4.3 Risk Aversion

By taking the second derivative of a person's utility function with respect to wealth. If the second derivative is negative, then they are risk averse. If the second derivative is equal to zero, then they are risk neutral. If the second derivative is greater than zero then utility is increasing in wealth at an increasing rate, making them risk seeking. This means that they will put a higher valuation on the opportunity than other risk-averse people.

Also, utilities may not be compared across people if they have different utility functions.

4.4 Derivative Hints

$$\frac{d}{dx}ln(x) = \frac{1}{x} \qquad\qquad \frac{d}{dx}e^x = e^x$$

4.5 Monte Carlo Simulation

Mathematical technique that generates random variables for modelling risk or uncertainty of a certain system. Practically, it is a method of simulating outcomes under a set of parameters, namely a probability distribution, average and variance. In finance, it is used to account for randomness in forecasting models. The process is;

- Generate a uniformly random number between 0 and 1. This represents a probability corresponding to our cumulative distribution function. Excel: `=RAND()`
- Evaluate the random outcome (return) associated with this probability given the specified distribution. Excel: `=NORM.INV(probability, mean, standard_dev)`
- Calculate payoff with simulated return.
- Repeat for 1000 trials and take the mean across all of them.

5 Markowitz

The basic insight of Markowitz builds on the accepted notion that, in considering a security, an investor should consider both risk and expected return. Additionally, portfolio risk was quantifiable by the variance of returns, giving rise to the volatility. Combined, this means that an investor should trade off portfolio expected return against portfolio variance.

In mean-variance analysing combinations of security portfolio, the investor seeks portfolio combinations from the **Pareto Optimal** expected return; where one improves one desirable aspect without decreasing any other desirable aspect.

5.1 Mean-Variance Analysis

The key assumptions are that investors have quadratic utility ($U(W) = W^2$), or investors have mean-variance preferences, or investor utility is approximated by a quadratic approximation. There too is the assumption that investors know the expected returns and the covariances of pairwise assets. This means that;

$$r_p = \sum_{i=1}^{n} w_i r_i \qquad\qquad \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \cdot COV[r_i, r_j]$$

This suggests that any porfolio constructed with less than perfect correlation will provide diversification benefits. This also implies that in a large, well diversified portfolio, idiosyncratic risk is diversified away, leaving only the systematic risk which is inherent to the market; yielding that only market risk should earn a risk premium.

$$\sigma_p^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w}$$

Where $\mathbf{w}$ is a vector of weights, σ is the variance, covariance matrix and $\mathbf{w}'$ is the transpose weight vector.

5.2 Portfolio Optimisation

The Markowitz optimal portfolio–or tangency portfolio–is one that maximises the Sharpe Ratio. The Sharpe ratio relates the returns of the investment to the risk, where the standard deviation of returns is used as a proxy for this. Thus, it is the average return earned in excess of the risk free rate, per unit volatility.

$$\begin{aligned} SR &= \frac{E[\mathbf{r}] - r_f}{\sigma_p} \\ &= \frac{\mathbf{w}'\mathbf{r} - r_f}{\sqrt{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}}} \end{aligned} \qquad\qquad \sigma_p = SD[\mathbf{r}]$$

So to thereby solve for the optimal portfolio, $\mathbf{w}$ must be selected where the Sharpe Ratio is maximised. Using $\mu = \mathbf{r} - r_f$, the Markowitz Optimal portfolio is;

$$\begin{aligned}\mathbf{w}^\star &= \frac{\mathbf{\Sigma^{-1}}\mu}{\mathbf{1'\Sigma^{-1}}\mu} \\ &= \frac{\mathbf{\Sigma^{-1}}\mu}{\mathbf{SUM(\Sigma^{-1}}\mu)}\end{aligned}$$

Though, variance may too be minimised with no consideration for the optimal returns of the portfolio.

$$\begin{aligned}\mathbf{w}^\star_{\mathbf{minvar}} &= \frac{\mathbf{\Sigma^{-1}1}}{\mathbf{1'\Sigma^{-1}1}} \\ &= \frac{\mathbf{\Sigma^{-1}1}}{\mathbf{SUM(\Sigma^{-1}1)}}\end{aligned}$$

5.3 Variance-Covariance Matrix

Estimation for this may be done by finding historical returns for n stocks, demeaning them (subtract the mean from each return of the stock), and the remaining vector is $\mathbf{R}$. From that, $\mathbf{\Sigma} = (\mathbf{R'R})/T$, where T is the number of periods. If you are using monthly data, this can be converted to annual frequency by multiplying by 12.

Note that *correlation* is different to *covariance*. They are however related thusly;

$$\sigma_{x,y} = \rho_{x,y}\sigma_x\sigma_y$$

Read as; "Covariance of x,y is equal to the product of the correlation coefficient and their respective standard deviations."

6 Black-Litterman

The motivation for Black-Litterman Optimisation was that under Markowitz, the optimisation strategy, devoid of any real world constraints, can yield impractical to extreme portfolio weightings. It is based on the assumption that the market is the optimal portfolio, then solving for the implied expected returns and covariances. The expected returns are then adjusted to incorporate the investors opinions in any mispricing, and then re-optimise using the Markowitz approach.

7 Ordinary Least Squares Regression

This process minimised the sum of least squared errors; where the error is the defect between the fitted value and the actual value of the dependent variable.

7.1 T-Statistic

The ratio of the departure of the estimated value of a parameter from its hypothesised value to its standard error. An absolute t-stat greater than or equal to 1.96 implies a p-value of $\leq 5\%$. Or less than a 5% chance that the true coefficient could be zero. Calculated as (Estimated coefficient)/standard error, or more formally;

$$t_{\hat{\beta}} = \frac{\hat{\beta} - \beta_0}{s.e(\hat{\beta})}$$

7.2 R-Square Measure

Roughly, this is the percentage variation in the dependent variable that is explained by the independent variables.

$$s.e = \frac{SD[\beta]}{n^2_{obs.}}$$

7.3 OLS Portfolio Optimisation

Regress the excess returns of n stocks on a constant = 1, *without an intercept.* Ergo;

$$1 = \sum_{i=0}^{n} [\beta_i r_i] + \epsilon$$

8 CAPM, APT and Factor Pricing Models

8.1 CAPM

This theory is emergent from the work of *Markowitz*; that model satisfies the demands of an individual investor, CAPM is the model that shows what happens if everybody does this.

Central to this is that everyone has identical beliefs, everyone optimises according to Markowitz and that no one has sufficient wealth to move the market alone. The assumptions are;

Markets:

- Not subject to transaction costs (frictionless markets)
- Not influenced by individual investors (perfectly competitive)
- Contains a risk free asset which investors can invest in or borrow

Investors:

- Have access to every asset (integrated market)
- Are mean-variance optimisers
- Have identical beliefs (homogenous)
- Have the same investment horizon

Assets:

- Are infinitely divisible
- Are tradable
- Can be shorted indefinitely

Market returns are calculated by the basic formula;

$$E[r_i] = r_f + \beta_i E[r_m - r_f] \qquad \beta_i = \frac{COV[r_m, r_i]}{VAR[r_m]}$$

Risk premium on the market portfolio is proportional to average risk aversion times the market variance.

$$COV[x, y] = E[(x - E[x])(y - E[y])] \qquad VAR[x] = COV[x, x] = E[(x - E[x])^2] = E[x^2] - E[x]^2$$

8.2 Beta Estimation

Under CAPM, the beta is supposed to be a forward looking estimate of covariance of risky returns with the market, scaled by the variance of the market return. However, we primarily estimate the beta from historical returns. To make this assumption, you must argue that the past conditions imply the future conditions. This can be because of regulated industry, low rate of technological innovation, single dominant industry leader, stable economy, etc.

$$r_{i,t} - r_{f,t} = \widehat{\alpha}_i + \widehat{\beta}_i (r_{m,t} - r_{f,t}) + \epsilon_{i,t}$$

Where i is a particular asset, and t is a particular point in time.

FINISH ALL OF THIS DETAIL

8.3 Arbitrage Pricing Theory

Contrasted with CAPM, arbitrage pricing theory is in effect a framework for a model in which returns are driven by a set of (presumed) uncorrelated risk factors. Realised excess returns are driven by a linear n-factor model, plus idiosyncratic noise. Lambda represents the sensitivity to the factor, which could be anything, not necessarily the market.

$$r_i - r_f = \sum_{j=0}^{n}(\beta_{i,j}\lambda_i) + \epsilon_i$$
$$E[r_i] - r_f = \sum_{j=0}^{n}(\beta_{i,j}\lambda_i)$$

Fama and French 3-factor model was one that was proposed that encapsulates SMB which is the return to a portfolio that is long small marketcap stocks and short big marketcap stocks, $MKTRF$ is the return on the market less the risk free rate (market excess return) and HML is the return to a portfolio that is long high book-to-market value stocks and short low book-to-value stocks.

9 Return Predictability

Return predictability is the study of various methods for predicting the future returns of financial assets.

9.1 Testing for Return Predictability

In the event that you suspect some variabl x which predicts future returns, this can be because; x may be correlated with a systemic priced risk factor, or there are other reasons why the market is not perfectly efficient at eliminating excess returns associated with x. This can be explained through institutional arrangments (e.g short selling constraints), market micro-structure (e.g bid-ask bounce) and behavioural biases (e.g overconfidence).

To test for these characteristics, the dataset has time as horizontal series and a cross section of individual stocks on the vertical series. This means that each variable is observed for each stock per time period. Observe the characteristic x for each time period and then sort this data set on the *prior* month x to ensure that the relationship is predictive.

$$x_{t-1} \implies r_t$$

For each month, the stocks are grouped into portfolios that are sorted on x. This set is now divided into 5 equal portfolios, quintiles. The quintiles then have the mean return within each quintile for all the months.

The *factor portfolio* of x is formed by going long on the edge of the portfolios and going short on the other edge portfolio, which is usually chosen such that the average return is positive, creating what is known as the hedge or zero-cost portfolio. The constructed portfolio is known as '5M1', quintile 5 minus quintile 1, and corresponds to the dynamic portfolio that you would construct to exploit the information characteristic x.

10 Behavioural Finance and Limits to Arbitrage

The argument is that if the market is affected by behavioural biases, it is not sufficient to prove that many or even most people suffer from behavioural biases; you have to either; show that *everyone* is biased, **or** show that most people are biased an that those who are not biased cannot exploit the resulting mispricing due to the limits of arbitrage.

This means that behavioural finance relies upon the limits of arbitrage and consistent, widespread psychological biases in decision making.

10.1 Limits to Arbitrage

The basis of arbitrage limitation is based upon the notion that there should not be any arbitrage opportunities available in the market. This is because that if there is an opportunity, it will be identified and depleted as soon as it appears. Common limits–Idiosyncratic volatility, agency problems, funding constraints, liquidity, capital.

10.2 Types of Bias

Information Processing;

- Forecasting Errors
- Overconfidence
- Conservatism
- Sample size neglect and representativeness

Behavioural;

- Framing
- Mental accounting
- Regret avoidance

10.3 Framing

According to the Invariance Principle, it should not matter how a question is framed so long as the actual outcomes and probabilities in the question are the same. Empirically, however, it does matter. Additionally, it is said that "Cognitive sophistication does note attenuate the bias blind spot."

11 Valuation

Every asset, financial as well as real, has a value. The key to successful investing in and managing these assets lies in understanding not only what the value is but also the sources of the value.

It is noted that even at the end of the most careful and detailed valuation, there will be uncertainty about the final numbers, coloured as they are by the assumptions that we make about the future of the company and the economy.

11.1 Discounted Cash Flow Valuation

This method operates under the assumption that value is the summation of the present value of all cash flows. This looks to find the 'intrinsic' value. It depends upon expected cash flows and discount rates, so is easiest to apply when cash flows are positive and can be reliably estimated and sensible risk proxies exist from which to estimate discount rates.

The caveats around DCF as a valuation method are when the firm is struggling as a result of either operational or financial distress. In these cases, having values that may tend to be negative, raises the question of whether there is any material value in negative equity. Other sources or error are for cyclical firms, where they are very sensitive to assumptions in the economic cycle.

$$V_0^{Firm} = \sum_{t=1}^{n}[\frac{FCFF_t}{(1+WACC)^t}] + \frac{TV_n}{(1+WACC)^n}$$

We take the $FCFF_t$ to be;

$$FCFF = EBIT \cdot (1-t) - GROSSCAPEX + D\&A - \Delta NWC$$
$$FCFE = EBIT \cdot (1-t) - InterestExp. - GROSSCAPEX + D\&A - \Delta NWC + (newdebt - debtrepaid)$$
$$TV_n = \frac{FCFF_{n+1}}{WACC_{stable} - g_{stable}}$$

Where $EBIT$ is Revenue - Expenses, t is the *marginal* tax rate; $CAPEX$ is Purchase of PPE - Disposals of PPE + Purchase of intangibles, $D\&A$ is Depreciation and amortisation; ΔNWC is the change in NWC from period $t-1$ to t. Look in *Working Capital* subsection to calculate.

11.2 Relative Valuation

Relies on comparable companies which are close substitutes for the firm that is to be valued. There is an underlying presumption that the market is pricing these firms correctly.

12 Cost of Capital

Weighted average summation of cost of equity and cost of debt.

$$WACC = \frac{D}{D+E} \cdot (1-t)r_d + \frac{E}{D+E} \cdot r_e$$
$$r_e = r_f + \beta \cdot MRP$$

Calculating the beta of a private company is done by building a β from the bottom up. This is done by identifying the businesses that the company is engaged in, finding comparable companies for each, obtaining those regression beta and financial leverage ratios, calculating the unlevered beta of the comparable group, determining the debt/equity ratio of your company and then calculating the levered β of the company. However, it is noted that regressing is perhaps a poor analogue for market sensitivity given that in absolute terms the standard error is quite high for the majority of these calculations which suggests we should consider regression beta estimates with caution.

Some services such as Bloomberg may calculate β which could be used in calculations. This however must be considered carefully for they provide an 'adjusted' β, which is roughly given by;

$$\beta_{adj} = \frac{2}{3}\beta_{raw} + \frac{1}{3} \cdot 1$$
$$\lim_{t \to \infty} \beta_{raw} \rightsquigarrow 1$$

Which is based on the empirical notion that in perpetuity, companies move towards the average beta as they become more diversified and their client base gets larger and thus the beta is a better indicator of *future* risk.

There are further issues around using historical beta estimates for small companies. Smaller companies often have less trading and overall liquidity in the stock which mean that large changes in the stock price may occur when trades do take place. Using a stock index as the 'market' may be disingenuous as indices are often dominated by large companies which does not accurately depict the "market portfolio" proposed by CAPM.

Importance must be placed upon what the fundamental determinant of a CAPM beta is; type of business, operating leverage and finacial leverage. The mechanics of constructing a beta can be used per sector of the business and then taking the weighted average using the contributions of those individual sectoral beta. A multi-factor model may also be used if there are additional factors such as labour income that may not be diversified away and there is a need for a proxy to this risk.

$$\beta_L = \beta_U[1 + (1-t)\frac{D}{E}] - \beta_D(1-t) \cdot \frac{D}{E} \qquad \beta_U = \frac{E}{(1-t)D+E}\beta_L + \frac{(1-t)D}{(1-t)D+E}\beta_D$$

Note that again, t is the marginal tax rate. In assessing the debt/equity ratio, this can be done by either;

- Using the industry average
- A *reasonable* arbitrary number
- Target debt/equity ratio

12.1 Cost of Debt

The cost of debt, or required rate of return on debt is the summation of the risk free rate, default risk, less the tax shield advantage of debt. After tax cost of debt = pre tax cost of debt x (1 - t). Bank loans can be valued at the current commercial rates. Newly issued bonds may be valued at the coupon rate if the bond was issued at par/face value. Old bonds that trade; use the yield to maturity.

12.2 Synthetic Ratings

$$ICR = \frac{EBIT}{\text{Interest Expense}}$$

13 Cash Flows, Growth, Terminal Value

13.1 Investment to Support Growth

To achieve growth, you must invest in long term assets (capex) and in current assets which are additions to your working capital. There must be consistency around estimates of growth and investment;

High Growth: Implies a high level of investment which in turn means a reduction in cash flows in the short run.

Low Growth: Implies little to no investment, but means more cash now. This also limits growth of cash in the long run.

13.2 Capex

Gross capex is the total expenditure on capital assets;

Capex = Purchase of PPE - Disposals of PPE + Purchase of intangibles

Conceptually, we can make the distinction between Maintenance capex, spending needed to maintain existing assets; and expansion capex, spending needed to create new assets. Note that for calculations involving FCFF, use **Net Capex**, which is Gross Capex minus D&A.

13.3 Working Capital

Defined as Current Assets - Current Liabilities. However, exclude any interest bearing debt from liabilities, and also exclude cash and equivalents from current assets. More succinctly, $OCA - OCL$, where $OCA = CA - \text{Cash and Equiv.}$ and $OCL = CL - \text{Borrowings}$

13.4 Growth

The assumed growth rate can make massive differences in valuation. To estimate growth rates you may; extrapolate past growth rates, rely on equity analysts, estimate based on elements such as capex or working capital investment.

Estimating the historical growth can be done by means of arithmetic or geometric means. Where arithmetic is simply the average of past annual growth rates, geometric is the constant growth rate that is applied to produce a certain growth in an attribute.

$$g_a = \frac{1}{n}\sum_{i=0}^{n} x_i \qquad\qquad g_g = (\frac{Level_t}{Level_{t-n}})^{-n} - 1$$

Geometric is preferred given that it is less sensitive to volatile annual growth rates.

14 FCFE and FCFF Valuation

15 Bonds

Debt packaged as a tradeable security. The cash flows of a bond are the interest and principle and are paid as coupons periodically, and in a lump sump at maturity, respectively.

$$C = c \times F \times \Delta$$

Where C is the coupon paid on each interest payment date, F is the face value, c is the annual coupon rate in % and Δ is the length of the coupon period in years (1 is annual, 0.5 is semi-annual, etc.).

15.1 Bond Pricing with Constant Discount Rate

As a consequence of the constant discount rate in bonds, we may introduce the concept of a discount factor d, evaluated at time t. The present value of some cash flow at time t is thereby given by;

$$PV(CF_t) = CF_t \cdot d_t \qquad\qquad d_t = \frac{1}{(1+r)^t}$$

15.2 Bond Formulae

$$B_0 = PV(\text{Principal}) + PV(\text{Interest})$$

Using K, which is the set of all interest payment dates, in years from today;

$$\begin{aligned}
K &= \{\Delta, 2\Delta, ..., T-\Delta, T\} \\
B_0 &= \frac{F}{(1+r)^T} + \sum_{t\in K} \frac{C}{(1+r)^t} \\
&= [F \cdot d_t] + \sum_{t\in K} [C \cdot d_t] \\
&= \mathbf{C}'\mathbf{d} + F \cdot d_T
\end{aligned}$$

Where the last example utilises matrices as $\mathbf{C}$, is a column vector of coupons and $\mathbf{d}$ is a column vector of discount factors.

Paul makes significant mention of a short-cut method which can be used if the *discount rate is constant* which is likening it to an annuity where the coupon payments go from $t = \Delta \to t = T$.

This is allegedly **really useful** "when you have to value a bond with 60 coupon payments in an exam".

$$\begin{aligned}
s &= (1+r)^{\Delta} - 1 \\
B_0 &= \frac{F}{(1+r)^T} + \frac{C}{s}\left[1 - \frac{1}{(1+s)^{T/\Delta}}\right]
\end{aligned}$$

s in this case is the periodic discount rate that corresponds to the annual discount rate r.

15.3 Accrued Interest

When bonds are transacted, there is sometimes a residual in 'accrued interest', they rarely are traded on the date of issuance of a coupon so the seller often requires compensation for holding the bond over the period.

The 'clean' price is the present value of the bond cash flows that exclude the next coupon to be paid. The 'dirty' price is inclusive of the time value of coupon interest that is accrued, but not earned for they do not receive the coupon payment.

15.4 Yield to Maturity

The yield to maturity (henceforth, "YTM") is the internal rate of return (IRR) of an investment in the bond assuming the bond is held to maturity (and does not default). YTM is the discount rate that makes the bond price equal to the market price. Some y, such that;

$$M_0 = \frac{F}{(1+y)^T} + \sum_{t\in K} \frac{C}{(1+y)^t}$$

15.5 Zero Coupon Bonds

A bond that does not pay coupons ($c = 0$), therfore, the only cash flow is the return of the face value at maturity. Even if there are no zero coupon bonds traded, one can still create a synthetic zero by forming a portfolio that is long and short various bonds such that the cash flows are net zero except on one date.

15.6 Floating Rate Notes

A bond that is linked to some index, often LIBOR, or equivalent. For instance, a bond with quarterly coupons may be quoted as $F \cdot$ (3-month LIBOR + margin), or more concisely; "TNZ 3mLibor + 0.65% 15/3/2018"

16 Term Structure

Different time periods have different discount rates and describe this term structure, or 'curves' of interest as it is also known. Pricing bonds using curves changes very little; we merely replace r with r_t, making special note that the annuity shortcut formula will no longer work as the discount rate is not constant.

$$B_0 = \frac{F}{(1+r_T)^T} + \sum_{t\in K} \frac{C}{(1+r_t)^t} \qquad \{r_t | t \in K\}$$

16.1 Forward Rates

Forward rates are a calculated rate which simulate current expectations of future bond interest rates. This calculation is based of the assumption of equivalence between holding bonds over different periods and allows you to determine future interest rates. For example, an investor can buy a one-year bond and hold it for the year, or he can buy a six-month bond, and then at the end of the sixth months, buy another six-month bond. Under these two scenarios, the investor knows the interest rates for both the one-year bond and the first six-month bond. The forward rate is the predicted rate on the second six month bond that shows the investor that they would earn the same under either scenario. This is based on the assumption that any given period of time does not imply rates are equal across it.

16.1.1 Notation

The point at which we are discounting *from* $t = t_2$, the point we are discounting *to* $t = t_1$ and the point at which we observe the discount rate t. Usually this implies that $t < t_1 < t_2$; that we discount back in time.

r_{t,t_1,t_2} means the rate we use at time t to discount a cash flow occurring at t_2 back to t_1. So, $r_{0,1,2}$ is the discount rate for discounting a cash flow occurring two years from now, to one year from now, as of the present time.

$$(1 + r_{0,0,t})^t = (1 + r_{0,0,1})(1 + r_{0,1,2})...(1 + r_{0,t-2,t-1})(1 + r_{0,t-1,t})$$

$$r_{0,t-\Delta,t} = \left(\frac{(1 + r_{0,0,t})^t}{(1 + r_{0,0,t-\Delta})^{t-\Delta}}\right)^{\frac{1}{\Delta}}$$

16.1.2 Example

Where the rate is known from period 0-2, and 0-1;

$$(1 + r_{0,0,2})^2 = (1 + r_{0,0,1})(1 + r_{0,1,2})$$

Which means that we can solve for $r_{0,1,2}$ thusly;

$$r_{0,1,2} = \frac{(1 + r_{0,0,2})^2}{(1 + r_{0,0,1})} - 1$$

16.1.3 Example

Given the annualised returns from the table below, what is the annualised forward rate between $T = 1.0$ and $T = 3.5$?

T (years)	0.5	1	1.5	2	2.5	3	3.5
r (annualised)	1.1	1.2	1.5	1.7	1.8	2	2.1

We are looking for $r_{0,1.0,3.5}$;

$$\begin{aligned}(1 + r_{0,0,3.5})^{3.5} &= (1 + r_{0,0,1.0})(1 + r_{0,1.0,3.5})^{2.5}\\ r_{0,1.0,3.5} &= \left[\frac{(1 + r_{0,0,3.5})^{3.5}}{(1 + r_{0,0,1.0})}\right]^{\frac{1}{2.5}} - 1\\ &= 2.4622\%\end{aligned}$$

16.2 Credit Ratings

Credit ratings give an assessment on expected loss in the event of default. Loss = Promised cash flows - actual cash flows.

$$\begin{aligned} EL &= \text{Expected Loss Rate} \\ &= \frac{E[\text{Loss}]}{E[CF|\text{no default}]} \\ &= \frac{E[CF|\text{no default}] - E[CF]}{E[CF|\text{no default}]} \end{aligned}$$

$$\begin{aligned} LGD &= \text{Loss Given Default rate} \\ &= E[Loss|\text{default occurs}] \end{aligned}$$

$$\begin{aligned} PD &= \text{Probability of Default} \\ &= P(\text{Default occurs}) \end{aligned}$$

$$EL = PD \cdot LGD$$

There is also the concept of rating transition probabilities; the probability of a security moving to a new rating from the current rating.

16.3 Measuring Credit Risk

Ratings are not market rates; they are merely an opinion. There are methods of measuring the market credit risk; namely, z-Spread and CDS-spread.

16.3.1 z-Spread

A single number z that is added to each risk free discount rate r_t such that the value of discounted cash flows using $r_t + z$ is equal to the market price of the risky bond

$$\begin{aligned} M_0 &= \frac{F}{1 + (r_T + z))^T} + \sum_{t \in K} \frac{C}{(1 + (r_t + z))^t} \\ d_t &= (1 + r_t + z)^{-t} \end{aligned}$$

z-Spread is found by looking at comparable bonds with same rating, maturity, industry, etc. It can be interpreted as the additional annual yield you receive for taking on additional credit risk.

16.3.2 Credit Default Swaps

Akin to buying insurance to cover the default risk of a bond. The protection buyer pays a periodic premium to the protection seller, if the bond defaults before maturity, the buyer hands over the bond to the seller and the seller pays the buyer the face value of the bond.

The premium that is paid is usually called the *CDS Spread* and is quoted in basis points per year. Annual dollar premium = FV x CDS spread. Given that CDS is the cost of insuring against credit risk, it can be interpreted as a measure of credit risk.

16.4 Determine Risk Free Rate from Zero Coupon Bond

This can be done by looking at the price of a safe zero coupon bond across different maturities.

$$Z_0 = \frac{F}{(1 + r_t)^t} \qquad\qquad r_t = \left(\frac{F}{Z_0}\right)^{\frac{1}{t}} - 1$$

16.4.1 Example

You are interested in buying a new fixed rate semi-annual coupon bond. Coupon rate = 4% p.a, maturity of exactly 2 years and face value of $100. Based upon the credit analysis you performed, the bond should trade at a z-spread of 125 basis points. You want to convert this to a price. To get started you download the prices of a few zero coupon bonds issued by the US Federal Reserve that are considered perfectly safe. What is pthe price of the fixed coupon bond on offer?

Maturity (t, in years)	Price of zero Z(t), in $
0.50	99.7509
1.00	98.7654
1.50	97.7915
2.00	96.1169
2.50	94.5892
3.00	92.8599

Then using the formula above to calculate the risk-free rate for each of the maturities of the zero coupon bond. For example, $r_{0.5} = (\frac{100}{99.7509})^{\frac{1}{0.5}} - 1 = 0.5\%$. From this we are able to directly calculate the discount rate of the bond by adding the Z-spread factor (1.25%/125 basis points) thusly, $r_t = r_t^f + 1.25\%$.

t	Risk Free Rates	Discount Rates
0.5	0.50	1.75
1.0	1.25	2.50
1.5	1.50	2.75
2.0	2.0	3.25

We then find the present values of all the cash flows by calculating the discount factor by using $d_0.5 = (1.0175^{0.5})^{-1} = 0.99136$, the present value by multiplying the discount factor by the cash flow, and then finding the summation of all of present values of these cash flows to yield the final bond price.

t	C	F	CF	r	Discount Factor	PV(CF)
0.5	2		2	1.75	0.9913634	1.982727
1	2		2	2.5	0.975609	1.951218
1.5	2		2	2.75	0.9601238	1.920248
2	2	100	102	3.25	0.9380368	95.67975
					Price	101.5339

17 Bond Portfolios and Risk Management

Bond price changes as yield changes. The Duration is the slope of this line. The slope/rate is not constant; it changes as the yield changes, which is *convexity*. The behaviour of the price of the bond as interest rates change is one way to characterise the interest rate risk of a bond.

17.1 Macaulay's Duration

Roughly interpretable as the average time of cash flows. The weighted average of cash flow time where weights are used for each time is equal to the present value of the cash flow at that time.

$$\begin{aligned} D_{Mac.} &= \sum_{t \in K} \left[t \cdot \frac{PV(CF_t)}{\sum_{s \in K} PV(CF_s)} \right] \\ &= \frac{1}{B_0} \sum_{t \in K} [t \cdot PV(CF_t)] \end{aligned}$$

Note that the denominator is the sum of the present value of all bond cash flows; the bond price, which gives rise to the second formula. It is, in effect, the point at which there is equal cash flows on either side.

17.2 Modified Duration

Arguably the more useful duration metric. The modified duration is a measure of the sensitivity of the bond to incremental changes in yield. It is useful for predicting the change in bond price for a given change in yield.

$$\begin{aligned} D &= \frac{D_{Mac.}}{(1+y)} \\ &= -\frac{1}{B_0} \frac{\partial B_0}{\partial y} \end{aligned}$$

Which gives a very rough approximation;

$$\Delta B_0 \approx -D \cdot B_0 \cdot \Delta y$$

Where in this case Δ is the traditional meaning of 'change in'. Obviously, this approximation gets increasingly inaccurate as Δy gets bigger.

17.3 DV01

Dollar value of a basis point. Given that 1 basis point is quite small, this approximation is pretty good in most practical situations.

$$DV01 \approx D \cdot B_0 \cdot 0.0001$$

17.4 Convexity

The slope of the duration plotted against yield. Bonds normally exhibit positive convexity and this is a good thing for bond holders as it implied that holding close to maturity means small increments in time hold large positive changes in the ytm; or, bonds with higher convexity will always have a higher price as interest rates rise or fall. Using the Taylor Approximation, we can establish;

$$\Delta B_0 \approx (-D \cdot \Delta y \cdot B_0) + (\frac{1}{2} \cdot C \cdot (\Delta y)^2 \cdot B_0)$$

Thus; changes in bond price are the sum of the change in price caused by the slope plus the additional change in price caused by the curvature. The first part of the above equation is the component of bond prices change that is driven by duration and the second part is the component driven by convexity.

17.5 Hedging

Aggregation of various types of risk and then manage or hedge them at the portfolio level. As a general rule, if you want to perfectly hedge n risks, you will need at least n hedging instruments, each with non-zero risk exposure to all n risks. There are two sources from which interest rate risk stems; duration and convexity. We use the Taylor approximation for the change in bond price to calculate the hedge, where ther objective of the hedge is;

Change in bond value due to duration = Change in hedge value due to duration

17.5.1 Example

With a bond portfolio $B_0^P = \$100m; D^P = 2.8; C^P = 40$ and a Swap contract $B_0^S = \$1m; D^S = 4.5; C^S = 15$, show how you could hedge your portfolio's interest rate exposure as measured by duration using the available hedging instruments. Change in value due to duration;

$$\approx -D \cdot \Delta y \cdot B_0$$

We can then solve using a simulteneous equation where we can find a constant number of swap contracts to reduce the interest rate risk.

$$\begin{aligned}
-D^P \cdot \Delta y \cdot B_0^P &= a(-D^S \cdot \Delta y \cdot B_0^S) \\
-2.8 \cdot \Delta y \cdot 100 &= a(-4.5 \cdot \Delta y \cdot 1) \\
-280 &= -4.5a \\
a &= 62.22
\end{aligned}$$

From this, we can see a long position in 62 swap contracts has the same interest rate risk exposure (in terms of duration) as the bond portfolio.

17.5.2 Example

For a bond portfolio $B_0^P = \$100m; D^P = 2.8; C^P = 40$, a swap contract $B_0^S = \$1m; D^S = 4.5; C^S = 15$ and a call option $B_0^C = \$1m; D^C = 2.0; C^C = 110$, how can you hedge the bond portfolio interest rate risk from both convexity and duration?

To solve for the duration;

$$-D^P \cdot \Delta y \cdot B_0^P = a(-D^S \cdot \Delta y \cdot B_0^S) + b(-D^C \cdot \Delta y \cdot B_0^C)$$

To solve for the convexity;

$$\frac{1}{2} \cdot C^P \cdot (\Delta y)^2 \cdot B_0^P = a(\frac{1}{2} \cdot C^S \cdot (\Delta y)^2 \cdot B_0^S) + b(\frac{1}{2} \cdot C^C \cdot (\Delta y)^2 \cdot B_0^C)$$

You then simplify these two equations by eliminating Δy from both equations, then get two equations in terms of a and b and solve these simultaneously. This yields, $a = 49.03$, $b = 29.68$.