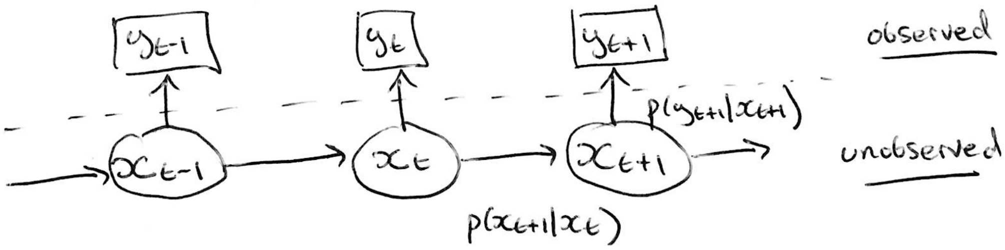
MATH3421 BCM 3

Topic 5 – Sequential Monte Carlo [Summary]

- Widely applicable in the area of hidden Markov models [consist of 2 discrete-time processes $\{X_t, t \geq 0\}$ and $\{Y_t \geq 0\}$, specified by $p(x_0)$, $p(x_t | x_{0:t-1}) = p(x_t | x_{t-1})$ and $p(y_t | x_t, y_{0:t-1}) = p(y_t | x_t)$.]

Picture :



- Wish to learn about x_t as observations arrive.
- Key density: $p(x_t | y_{0:t})$ [FILTERING DENSITY]
  $\hookrightarrow (y_0, y_1, \ldots, y_t)'$.
- Compute filtering densities recursively:

$t = 0$ $\quad \boxed{p(x_0 | y_0) \propto p(x_0) p(y_0 | x_0)}$

$t = 1, 2, \ldots$ $\quad \boxed{p(x_t | y_{0:t}) \propto p(y_t | x_t) \underbrace{\int p(x_{t-1} | y_{0:t-1}) p(x_t | x_{t-1}) dx_{t-1}}_{p(x_t | y_{0:t-1})}}$

Typically only know these up to proportionality.

$\rightarrow$ Generate unweighted samples using weighted resampling.

[Snapshot : weighted resampling for a target $f(x)$ and proposal $g(x)$

1. Simulate $x^{(i)} \sim g(\cdot)$, $i = 1, \ldots, N$ iid.
2. Construct + normalise weights, $w(x^{(i)}) \propto f(x^{(i)}) / g(x^{(i)})$,
   $\tilde{w}(x^{(i)}) = w(x^{(i)}) / \sum_{k=1}^{N} w(x^{(k)})$, $\quad i = 1, \ldots, N$
3. Resample N times (with replacement) from $\{x^{(1)}, \ldots, x^{(N)}\}$ using normalised weights as probs.]

<u>Output</u> of weighted resampling: an unweighted sample approx. distributed according to $f(x)$.

$\boxed{\text{Example: SV model}}$

$$Y_t \mid X_t = x_t \sim N(0, \exp\{x_t\} \times \kappa^2) \quad t=0,\dots,T$$
$$X_t \mid X_{t-1} = x_{t-1} \sim N(\phi x_{t-1}, \sigma^2) \quad , t=1,\dots,T$$
$$X_0 \sim N\left(0, \frac{\sigma^2}{1-\phi^2}\right) \qquad \sigma, \kappa > 0, \ \phi \in (0,1).$$

Suppose we have a sample $\{x_{t-1}^{(1)}, \dots, x_{t-1}^{(N)}\}$ approx. distributed according to $p(x_{t-1} \mid y_{0:t-1})$. observe y_t.

<u>Generic</u>	<u>Specific</u>
• 1 <u>propagate</u>: sample $x_t^{(i)} \sim \hat{p}(x_t \mid y_{0:t-1})$ by drawing $x_t^{(i)} \sim p(x_t \mid x_{t-1}^{(i)})$, $i=1,\dots,N$	• draw $x_t^{(i)} \sim N(\phi x_{t-1}^{(i)}, \sigma^2)$, $i=1,\dots,N$.
• 2 <u>weight</u>: construct $w_t(x_t^{(i)}) = p(y_t \mid x_t^{(i)})$, and $\tilde{w}_t(x_t^{(i)}) = w_t(x_t^{(i)}) / \sum_{k=1}^{N} w_t(x_t^{(k)})$, $i=1,\dots,N$.	• $w_t(x_t^{(i)}) = N(y_t ; 0, \kappa^2 \exp\{x_t^{(i)}\})$ Normalise.
3. Resample N times (with replacement) from $\{x_t^{(1)}, \dots, x_t^{(N)}\}$ using the normalised weights as probs.	• See LHS.

<u>Output</u>: sample $\{x_t^{(1)}, \dots, x_t^{(N)}\}$ approx. dist^d as $p(x_t \mid y_{0:t})$

$\boxed{\text{NB}}$: if step 1 is replaced with draws of $x_t^{(i)} \sim g(x_t \mid x_{t-1}^{(i)})$, the weight becomes

$$w_t(x_t^{(i)}) = \frac{p(y_t \mid x_t^{(i)})\, p(x_t^{(i)} \mid x_{t-1}^{(i)})}{g(x_t^{(i)} \mid x_{t-1}^{(i)})}.$$