(Q1)

(a) Target : $\pi(\theta_1, \theta_2) \propto \theta_1 \exp(-\theta_1\theta_2^2 - \theta_2^2 + 2\theta_2 - \theta_1)$

→ Find the full conditional densities for θ_1 and θ_2

$$\pi(\theta_1 | \theta_2) = \frac{\pi(\theta_1, \theta_2)}{\pi(\theta_2)}$$

$$\propto \pi(\theta_1, \theta_2)$$

$$\propto \theta_1 \exp(-\theta_1\theta_2^2 - \theta_2^2 + 2\theta_2 - \theta_1)$$

$$\propto \theta_1 \exp(-\theta_1(\theta_2^2 + 1))$$

$$\Rightarrow \theta_1 | \theta_2 \sim \text{Gamma}(1, \theta_2^2 + 1)$$

$$\pi(\theta_2 | \theta_1) \propto \pi(\theta_1, \theta_2)$$

$$\propto \exp(-\theta_1\theta_2^2 - \theta_2^2 + 2\theta_2)$$

$$\propto \exp\left(-(\theta_1 + 1)\theta_2^2 + 2\theta_2\right)$$

$$\propto \exp\left(-\frac{\left(\theta_2^2 - \frac{2\theta_2}{\theta_1+1}\right)^2}{2(\theta_1 + 1)}\right)$$

$$\propto \exp\left(-\frac{\left(\theta_2 - \frac{1}{\theta_1+1}\right)^2}{2(\theta_1+1)}\right)$$

$$\Rightarrow \theta_2 | \theta_1 \sim N\left(\frac{1}{\theta_1+1}, \theta_1 + 1\right)$$

∴ Our Gibbs sampler is:

1. Initialise with $\underline{\theta}^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)})$. Set $j=1$.

2. Obtain new values $\underline{\theta}^{(j)}$ from $\underline{\theta}^{(j-1)}$ via :

- $\theta_1^{(j)} \sim \text{Gamma}\left(1, \theta_2^{2(j-1)} + 1\right)$

- $\theta_2^{(j)} \sim N\left(\frac{1}{\theta_1^{(j)}+1}, \theta_1^{(j)} + 1\right)$

3. If $j = N$, stop. Otherwise $j \leftarrow j+1$, go to step 2.

(b) The transition density is given by our full conditional pdfs:

$$\pi(\theta_1 | \theta_2) = (\theta_2^2 + 1)\left(\theta_1 \exp\left(-(\theta_2^2+1)\theta_1\right)\right)$$

$$\pi(\theta_2 | \theta_1) = \frac{1}{\sqrt{2\pi(\theta_1+1)}} \exp\left(-\frac{1}{2}\left(\left(\frac{\theta_2 - \frac{1}{\theta_1+1}}{\sqrt{\theta_1+1}}\right)^2\right)\right)$$

Q2)

(a)

(i) Since we have a linear transformation of the MVN, we can apply a theorem here.

$$\Rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right)$$

$$\Sigma = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1+p & 1-p \\ 1+p & -1+p \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2p+2 & 0 \\ 0 & 2-2p \end{pmatrix} = \begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix}$$

Apologies for using 'p' as "rho"...!

$$\text{So, } \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix} \right)$$

(ii) Attached R code...

(iii) We can set p as 0.99 and still obtain good samples since we have independent variables ψ_1 and ψ_2
→ so each node (of MC) is updated independently from the last
→ ∴ The behavior is not affected (much) by p and we essentially have a regular sampler.

(b)

(i) We can see that if

$\tilde{\Theta}_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ and $\tilde{\Theta}_2 = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$

Then $\tilde{\Theta}_1 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(\theta_1 + \theta_2 + \theta_1 - \theta_2)\right) = \theta_1$

$\tilde{\Theta}_2 = \theta_2$

$\therefore$ We have the original distribution.

$$\begin{pmatrix} \tilde{\Theta}_1 \\ \tilde{\Theta}_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

(ii) We find that

$\theta_1 = \frac{\sqrt{2}}{2}(\psi_1 + \psi_2)$ and $\theta_2 = \frac{\sqrt{2}}{2}(\psi_1 - \psi_2)$

$\rightarrow$ see code attached [R].

(iii) Plotting $\tilde{\Theta}_1$ against $\tilde{\Theta}_2$ and θ_1 against θ_2, we don't see much difference in the shape of the plots for varying values of ρ.

However, when we look at the iteration plots of θ_1, θ_2 and $\tilde{\Theta}_1, \tilde{\Theta}_2$, we see that the mixing of $\tilde{\Theta}^{(1)} \dots \tilde{\Theta}^{(2)}$ is much better which is shown by the plot that is much denser and shows less variation.
$\rightarrow$ the sample averages converge much quicker.

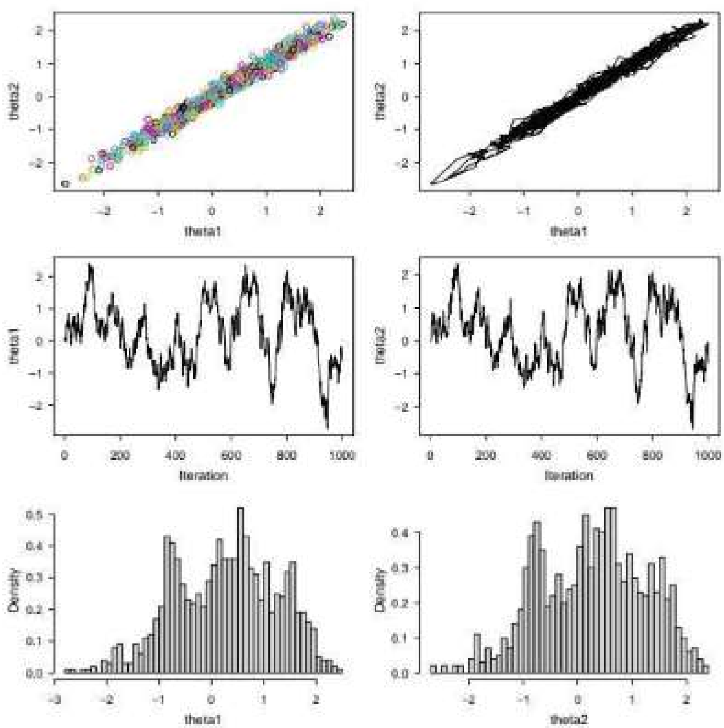
In addition, the histograms for $\underline{\tilde{\Theta}}$ appear more Normal than they do for $\underline{\Theta}$.

November 24, 2022

The results below are generated from an R script.

```
### Formative Assignment 3 - Gibbs Sampling Implementation
# gibbs_example: Code from Example 2.3.1
# gibbs_a: Q2a(ii) Gibbs sampler
# gibbs_b: Q2b(ii) Gibbs sampler

gibbs_example=function(N,rho)
{
  mat=matrix(ncol=2,nrow=N)
  th1=0
  th2=0
  mat[1, ]=c(th1,th2)
  for (i in 2:N)
  {
    th1=rnorm(1,rho*th2,sqrt(1-rho^2))
    th2=rnorm(1,rho*th1,sqrt(1-rho^2))
    mat[i, ]=c(th1,th2)
  }
  return(mat)
}

gibbs_a=function(N,rho)
{
  mat=matrix(ncol=2,nrow=N)
  phi1=0
  phi2=0
  mat[1, ]=c(phi1,phi2)
  for (i in 2:N)
  {
    phi1=rnorm(1,0,sqrt(1+rho))
    phi2=rnorm(1,0,sqrt(1-rho))
    mat[i, ]=c(phi1,phi2)
  }
  return(mat)
}

gibbs_b=function(N,rho)
{
  mat=matrix(ncol=2,nrow=N)
  phi1=0
  phi2=0
  mat[1, ]=c(phi1,phi2)
  for (i in 2:N)
  {
```

```r
    phi1=rnorm(1,0,sqrt(1+rho))
    phi2=rnorm(1,0,sqrt(1-rho))
    # now obtain th1 and th2 from phi1 and phi2
    # using change of variable transformation
    th1 = (1/2)*sqrt(2)*(phi1+phi2)
    th2 = (1/2)*sqrt(2)*(phi1-phi2)
    mat[i, ]=c(th1,th2)
  }
  return(mat)
}


# modify this line to produce different plots
# for comparison (eg. out = gibbs_a(1000, .99))
out=gibbs_example(1000,.99)

par(mfrow=c(3,2))
plot(out,col=1:1000,xlab="theta1",ylab="theta2")
plot(out,type="l",xlab="theta1",ylab="theta2")
plot(ts(out[,1]),xlab="Iteration",ylab="theta1")
plot(ts(out[,2]),xlab="Iteration",ylab="theta2")
hist(out[,1],40,freq=FALSE,main="",xlab="theta1")
hist(out[,2],40,freq=FALSE,main="",xlab="theta2")
```



The R session information (including the OS info, R version and all packages used):

```r
sessionInfo()

## R version 4.2.2 (2022-10-31 ucrt)
## Platform: x86_64-w64-mingw32/x64 (64-bit)
```