Q1)

(a) Target density, $\pi(x)$ $x \in \mathbb{R}$.

logs $\therefore$ $x[i-1]$ - y

$\longrightarrow$ $\dfrac{x[i-1]}{y}$ $\longrightarrow$ $\alpha = \dfrac{\quad}{\theta}$

Since log(runif(1)) < lalpha

$\implies$ runif(1) > alpha

$R = \dfrac{\pi(y)}{\pi(x)} = \dfrac{e^{-|y|}}{e^{-|x|}}$ since we inverse the log of absolute value function.

So $\pi(x) \propto e^{-|x|}$, and we know that $\int \pi(x)\,dx = 1$.

$$\therefore \int_{\mathbb{R}} k e^{-|x|}\,dx = 1$$

$$= \int_0^{\infty} k e^{-x}\,dx + \int_{-\infty}^{0} K e^{x}\,dx = 1$$

$$\left[-ke^{-x}\right]_0^{\infty} + \left[ke^{x}\right]_{-\infty}^{0} = 1$$

$$\left[0 + k\right] + \left[k\right] = 1$$

$$\Rightarrow k = 1/2$$

$$\therefore \pi(x) = \frac{1}{2} e^{-|x|}$$

(b) And the proposal $q(\theta^*|\theta)$ is simply a Normal distribution, with mean $\theta$, and s.d. sigma (default = 1)

$$q(\cdot) \sim N(\theta, \sigma)$$

$$\therefore q(\theta^*|\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\theta^* - \theta}{\sigma}\right)^2\right\}$$

Q2) Target: $f(\theta) = \sin\theta \quad 0 \le \theta \le \pi/2$

Proposal: $q(\theta^*) = \frac{8\theta^*}{\pi^2} \quad 0 \le \theta^* \le \pi/2$

(b)

(i) 1. Initialise w/ $\theta^{(0)} = \pi/4$, $j=1$

2. Propose $\theta^* \sim q(\cdot)$ $\left[ \frac{8\theta^*}{\pi^2} \right]$

3. Evaluate $\alpha(\theta^* \mid \theta^{(j-1)})$ where

$$\alpha(\theta^* \mid \theta) = \min\left\{1, \frac{\sin\theta^*}{\sin\theta} \times \frac{8\theta/\pi^2}{8\theta^*/\pi^2}\right\}$$

$$= \min\left\{1, \frac{\theta\sin\theta^*}{\theta^*\sin\theta}\right\}$$

4. With prob. $\alpha(\theta^* \mid \theta)$, put $\theta^{(j)} = \theta^*$, otherwise put $\theta^{(j)} = \theta^{(j-1)}$.

5. If $j=N$ stop, otherwise $j \leftarrow j+1$, go to step 2.

(ii) $$P(\text{move} \mid \theta) = \int_0^{\pi/2} \alpha(\phi \mid \theta)\, q(\phi \mid \theta)\, d\phi$$

$$= \int_0^{\pi/2} \min\left(1, \frac{\theta\sin\phi}{\phi\sin\theta}\right) \frac{8\phi}{\pi^2}\, d\phi$$

$\rightarrow$ When $\phi \ge \theta$, $\min = \frac{\theta\sin\phi}{\phi\sin\theta}$

$\rightarrow$ When $\phi < \theta$, $\min = 1$

$$P(\text{move} \mid \theta) = \int_0^{\pi/2} \frac{\theta \sin\phi}{\phi \sin\theta} \times \frac{8\phi}{\pi^2}\, d\phi + \int_0^{\theta} 1 \times \frac{8\phi}{\pi^2}\, d\phi$$

$$= \int_0^{\pi/2} \frac{8\theta \sin\phi}{\pi^2 \sin\theta}\, d\phi + \left[\frac{4\phi^2}{\pi^2}\right]_0^{\theta}$$

$$= \left[\frac{-8\theta\cos\phi}{\pi^2 \sin\theta}\right]_0^{\pi/2} + \frac{4\theta^2}{\pi^2}$$

$$= 0 - \left[\frac{-8\theta}{\pi^2 \sin\theta}\right] + \frac{4\theta^2}{\pi^2}$$

$$= \frac{8\theta + 4\theta^2 \sin\theta}{\pi^2 \sin\theta}$$

(iii) $$P(\text{move}) = \int_0^{\pi/2} P(\text{move} \mid \theta)\, f(\theta)\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{8\theta + 4\theta^2 \sin\theta}{\pi^2 \cancel{\sin\theta}}\right] \cancel{\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{8\theta + 4\theta^2 \sin\theta}{\pi^2}\, d\theta$$

$$= \frac{4}{\pi^2}\int_0^{\pi/2} 2\theta + \theta^2\sin\theta \, d\theta = \frac{4}{\pi^2}\left[ \left[\theta^2\right]_0^{\pi/2} + \left[-\theta^2\cos\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 2\theta\sin\theta \, d\theta \right]$$

$$= \frac{4}{\pi^2}\left[ \left(\frac{\pi}{4}\right) + 0 - \left[ \left[-2\theta\cos\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 2\sin\theta \, d\theta \right]\right]$$

$$= \frac{4}{\pi^2}\left[ \frac{\pi}{4} - \left( 0 - \left[-2\cos\theta\right]_0^{\pi/2} \right)\right]$$

$$= \frac{4}{\pi^2}\left[ \frac{\pi}{4} + (0+2)\right] = \frac{4}{\pi^2}\left(\frac{\pi}{4}+2\right)$$

$$= \frac{\cancel{4}\cancel{\pi}}{\cancel{4}\pi^{\cancel{2}}} + \frac{8}{\pi^2} = \frac{1}{\pi} + \frac{8}{\pi^2} = \frac{\pi+8}{\pi^2} \rightarrow > 1 \ ??$$

~~(iv) On average, e.g. from a few runs the empirical acceptance rate is higher for the uniform proposal.~~

(iv) We get $\approx 0.70$ for the sampler empirical acceptance rate, against our overall acceptance probability ; some minor error? Since it is greater than 1.

(v) On average, the empirical acceptance rate is higher for the uniform proposal, so we would prefer this.

Q3) If $\delta \leq 1$ then the distribution is bounded

$\Rightarrow$ Markov chain is irreducible.