(Q1)

(a) For y_i,

$$f(y_i|\theta) = \frac{\theta^{y_i} e^{-\theta}}{y_i!} \quad \text{for } \theta > 0 \qquad (0 \text{ otherwise})$$

$$\Rightarrow l(\theta|\underline{y}) = \prod_{i=1}^{n} \frac{\theta^{y_i} \exp(-\theta)}{y_i!}$$

$$= \exp(-n\theta) \prod_{i=1}^{n} \frac{\theta^{y_i}}{y_i!}$$

$$= \exp(-n\theta) \frac{\theta^{T_1}}{\prod_{i=1}^{n} y_i!}, \quad \text{where } T_1 = \sum_{i=1}^{n} y_i$$

So, $f(\theta|\underline{y}) \propto f(\underline{y}|\theta) f(\theta)$

$$= \frac{e^{-n\theta}\theta^{T_1}}{\prod_{i=1}^{n} y_i!} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}$$

$$\propto \theta^{a+T_1-1} e^{-(n+b)\theta}$$

$\rightarrow$ Core of a Gamma$(a+T_1, b+n)$ distribution.

$\Rightarrow f(\theta|\underline{y}) \sim$ Gamma$(a+T_1, b+n)$

(b) $f(x_i|\theta) = \theta e^{-\theta x_i} \quad i=1,\dots,m$

$$f(\theta|\underline{x}) = \prod_{i=1}^{m} \theta \exp\{-\theta x_i\}$$

$$= \theta^m \exp\{-\theta \sum_{i=1}^{m} x_i\}$$

$$f(\theta|\underline{x}) \propto \theta^m \exp\{-\theta \sum_{i=1}^{m} x_i\} \times \theta^{a-1} \exp\{-b\theta\}$$

$$\propto \theta^{a+m-1} \exp\{-\theta(T_2+b)\}, \quad T_2 = \sum_{i=1}^{m} x_i$$

$$f(\theta|\underline{x}) \sim \text{Gamma}(a+m,\ b+T_2)$$

(c) m is defined as the total number of particle emissions so $m = T_1$ where $T_1 = \sum_{i=1}^{n} y_i$ since the y_i are the number of emissions for a given time-step.

And, n is defined as the total number of time-steps combined, so $n = T_2$, where $T_2 = \sum_{i=1}^{m} x_i$ since the x_i are the time steps between each emission.

Hence, $\theta | Y \sim \text{Gamma}(a+T_1,\ b+n)$ and
$\theta | X \sim \text{Gamma}(a+m,\ b+T_2)$ give the same information about θ.

(d) prior mode: $\dfrac{a-1}{b}$, for $a>1$.

$\Rightarrow$ posterior mode: $\dfrac{a+T_1-1}{b+n} = \dfrac{a+n\bar{y}-1}{b+n}$

Q2) $f(\underline{x}) = \int \pi(\theta) f(\underline{x}|\theta)\, d\theta$

$= E_\theta[f(\underline{x}|\theta)]$ likelihood

$\approx \frac{1}{n} \sum_{i=1}^{n} f(\underline{x}|\theta_i), \quad \theta_i \sim \pi(\theta)$

$\therefore$ We would simulate from the prior distribution to estimate the normalising constant.

Q3) ~~[illegible]~~ Let $g(\theta) = \pi(\theta)$, prior, be the proposal density function.

Then $f(\theta) = \pi(\theta|x)$, posterior.

$$E_f[X] = E_g\left[X \frac{f(X)}{g(X)}\right] \approx \frac{1}{n} \sum_{i=1}^{n} X_i \frac{f(X_i)}{g(X_i)},$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i W_i, \quad \text{so} \quad W_i = \frac{f(X_i)}{g(X_i)}$$

where $X_i \sim \pi(\theta)$ for $i=1,\dots n$.

And, ~~[illegible]~~ if $f(\theta) = \frac{\pi(\theta|x)}{Z}$, where Z is the unknown normalising constant,

then $E_f[X] = \sum_x x \cdot \frac{h(x)}{Z} = \sum_x x \cdot \frac{h(x)}{f(x) Z} f(x)$

$$\approx \frac{1}{Z} \frac{1}{n} \sum_{i=1}^{n} X_i \frac{h(X_i)}{f(X_i)}$$

$$\therefore Z = \sum_x h(x) \approx \frac{1}{n} \sum_{i=1}^{n} \frac{h(X_i)}{f(X_i)}$$

$$\hat{Z} = \bar{w}.$$

Q4) $\pi(x)\, p(y|x) = \pi(y)\, p(x|y) \quad \forall\, x, y \in \mathcal{X}$.

$$\int_{\mathcal{X}} \pi(x)\, p(y|x)\, dx = \int_{\mathcal{X}} \pi(y)\, p(x|y)\, dx$$

$$\pi(y) \overbrace{\int_{\mathcal{X}} p(x|y)\, dx}^{=1} = \int_{\mathcal{X}} \pi(x)\, p(y|x)\, dx$$

$$\pi(y) = \int_{\mathcal{X}} \pi(x)\, p(y|x)\, dx$$

$\rightarrow$ def^{n} of stationary distribution (analogous to $\pi = \pi P$).

Q5)

(a) $E[X_{n+1}] = E[X_n]$ in equilibrium.

$$X_{n+1} = \alpha X_n + \varepsilon_{n+1}, \quad \text{with } \varepsilon_{n+1} \sim N(0, \sigma^2) \text{ i.i.d.}$$
$$|\alpha| < 1$$

$$= \alpha(\alpha X_{n-1} + \varepsilon_n) + \varepsilon_{n+1}$$
$$= \alpha^2(\alpha X_{n-2} + \varepsilon_{n-1}) + \alpha\varepsilon_n + \varepsilon_{n+1}$$
$$= \dots$$
$$= \sum_{i=0}^{\infty} \alpha^i \varepsilon_{n-i}$$

$$\therefore E[X_{n+1}] = E\left[\sum_{i=0}^{\infty} \alpha^i \varepsilon_{n-i}\right]$$
$$= \sum_{i=0}^{\infty} \alpha^i E[\varepsilon_{n-i}]$$
$$= \sum_{i=0}^{\infty} \alpha^i \times 0 = 0$$

$$Var[X_{n+1}] = Var\left[\sum_{i=0}^{\infty} \alpha^i \varepsilon_{n-i}\right]$$
$$= \sum_{i=0}^{\infty} \alpha^{2i} Var[\varepsilon_{n-i}]$$
$$= \sum_{i=0}^{\infty} \alpha^{2i} \cdot \sigma^2 \quad \longrightarrow \frac{a}{1-r}$$
$$= \frac{\sigma^2}{1-\alpha^2}$$

(b) $\pi(\cdot) \sim N\left(0, \frac{\sigma^2}{1-\alpha^2}\right)$, $|\alpha| < 1$

$$p(y|x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{y-\alpha x}{\sigma}\right)^2\right\}$$

$$\Rightarrow \pi(x)\, p(y|x) = \frac{1}{\sigma\sqrt{2\pi/1-\alpha^2}} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma/\sqrt{1-\alpha^2}}\right)^2\right) \cdot \exp\left(-\frac{1}{2}\left(\frac{y-\alpha x}{\sigma}\right)^2\right)$$

$$= M \cdot \exp\left(-\frac{1}{2}\left(\frac{x^2(1-\alpha^2) + (y-\alpha x)^2}{\sigma^2}\right)\right)$$

$$= M \cdot \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2-2\alpha xy}{\sigma^2}\right)\right)$$

$$= M \cdot \exp\left(-\frac{1}{2}\left(\frac{(x-\alpha y)^2 + y^2 - \alpha^2 y^2}{\sigma^2}\right)\right)$$

$$= M \cdot \exp\left(-\frac{1}{2}\left(\frac{x-\alpha y}{\sigma}\right)^2\right) \cdot \exp\left(-\frac{1}{2}\left(\frac{y^2(1-\alpha^2)}{\sigma^2}\right)\right)$$

$$= M \cdot \exp\left(-\frac{1}{2}\left(\frac{x-\alpha y}{\sigma}\right)^2\right) \cdot \exp\left(-\frac{1}{2}\left(\frac{y}{\sigma/\sqrt{1-\alpha^2}}\right)^2\right)$$

$$= p(x|y)\, \pi(y)$$

$\therefore$ detailed balance holds.

Q6)

(a) $\mathcal{E}_{n+1} \sim Exp(\lambda)$

?

$\Rightarrow \quad X_{n+1} | (X_n = x) \sim Exp(\lambda + \alpha)$

So $p(y|x) = (\lambda+\alpha)e^{-(\lambda+\alpha)x}$

(b) $X_0 = 1$, $P(X_{n+1} | X_0 = 1)$ $\quad n = 0, \dots, N$.

- For i in $0, \dots, N$
  $X_{i+1} = \alpha X_i + \mathcal{E}_{i+1}$

incorrect?

where the X_i are simulated from the distribution in (a) with $p(y|x)$.