07.11.24 don't want to fix central char

Nadya $G = PGL_2(F)$, F p-adic, $q = |\mathcal{O}/p|$

LLC $\left\{ \begin{array}{l} \text{sm. irreps} \\ \text{of } G(F) \end{array} \right\} \overset{LLC}{\longleftrightarrow} \{ \phi : W_F \to SL_2(\mathbb{C}) \}$

$\forall$ $\overset{\Psi}{\pi}$ of G, χ of $GL_1(F)$

$\gamma(\pi, \chi, s)$, $\frac{Q}{P}(q^{-s})$ $\gamma(\phi, \phi_\chi, s)$

$L(\pi, \chi, s)$, $P^{-1}(q^{-s}) \longleftrightarrow$

$\varepsilon(\pi, \chi, s)$, $a q^{-ns}$

P, Q polys

<u>Goal.</u> $\Pi = \bigotimes_{\text{rest.}} \pi_v$ of $G(\mathbb{A})$.

$L(\pi, \chi, s) = \prod_v L(\pi_v, \chi_v, s)$

$\Pi \hookrightarrow \mathcal{A}(G(F) \backslash G(\mathbb{A})) \iff L(\pi, \chi, s)$ is merom. $\forall \chi$.

+ Functional equation

$L(\tilde{\pi}, \chi^{-1}, 1-s)\, \varepsilon(\pi, \chi, s) = L(\pi, \chi, s)$

$G \supset B = N \cdot A$ $K = G(\mathcal{O})$

$A = \{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} : a \in F^\times \}$ $N = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \}$

$K \supset K_1 \supset K_2 \supset \dots$ $K_n = G \cap \begin{pmatrix} 1+p^n\mathcal{O} & p^n\mathcal{O} \\ p^n\mathcal{O} & 1+p^n\mathcal{O} \end{pmatrix}$ open cpt

π adm $\iff \tilde{\pi}$ adm.

(π, V) rep

$\forall\ \tilde{v} \in \tilde{V},\ v \in V$

$m_{\tilde{v},v}(g) = \langle \tilde{v}, \pi(g)(v) \rangle \in S(G)$.

$\tilde{v} : V \longrightarrow S(G)$

$V \otimes \tilde{V} \longrightarrow S(G)$

(1) π is called <u>Supercuspidal</u> if $m_{\tilde{v},v}$ is cptly supp'd.

(2) π is called discrete series if $m_{\tilde{v},v} \in L^2(G, dg)$

(3) π is called tempered if $m_{\tilde{v},v} \in L^{2+\varepsilon}(G, dg)\ \forall \varepsilon > 0$

(π, G, V), $\pi_N, A, V/V(N)\delta_B^{-\frac{1}{2}}$

$$V(N) := \operatorname{Span}\{\pi(n)v - v\}$$

$[\pi_N]_0 = \oplus \chi_i$, $\quad \chi_i = \underset{\uparrow\ \text{unit}}{\chi_{0,i}} |\cdot|^{s_i} \quad s_i \in \mathbb{C}$.

(1) $\iff \pi_N = 0$

(2) $\iff \forall \chi_i \in [\pi_N] \quad \operatorname{Re}(s_i) > 0$

(3) $\iff \forall \chi_i \in [\pi_N] \quad \operatorname{Re}(s_i) \geq 0$

$\operatorname{Ind}_B^G \chi_s = \{f : G \to \mathbb{C} : f(n\, t(a)\, g) = |a|^{s+\frac{1}{2}} f(g)\}$

$t(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad [(\operatorname{Ind}_B^G \chi_s)_{ss}] = \chi_s \oplus \chi_{-s}$

$\operatorname{Ind}_B^G |\cdot|^{it}$ is tempered

$$0 \to St \to \mathrm{Ind}_B^G \chi_{\frac{1}{2}} \to \mathbb{C} \to 0$$

$J_N(\ )$ $\quad 0 \to \chi_{\frac{1}{2}} \to \chi_{\frac{1}{2}} \oplus \chi_{-\frac{1}{2}} \to \chi_{-\frac{1}{2}} \to 0$

χ called exponents.

<u>Kirillov model</u> (π, V) rep of G. $\quad N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\}$

$\pi|_N$ $\quad V$ irrep.

Rep $N = \{1, \psi\} \circlearrowleft A$

$\psi : N \to \mathbb{C}$

$\psi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \psi(x)$

$\psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\right) = \psi(ax)$.

<u>Ex</u>: : ① $V^N = 0$

② $\forall \psi$ $\quad \pi_{N,\psi} = V / (\pi(n)v - \psi(n)v)$

$\neq 0$

③ $\pi_{N,\psi} \simeq \pi_{N,\psi'}$

$\Rightarrow W \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$ $\quad W(nv) = \psi(n) W(v)$

<u>Thm</u>: $\mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$ is 1-dim'l (Kirillov).

Fix W.

$$i_\pi : V \hookrightarrow S(F^\times)$$

$$i_\pi(v)(a) = W\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(v)\right)$$

The image is denoted $S_\pi(F^\times)$ is called Kirillov model of (π, V)

<u>Define</u>: $\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f\right)(x) = \psi(bx) f(ax)$

$G = B \cup BwB.$ use $\pi\left(g\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g^{-1}\right)(v) = \chi(g) \ldots$ $\begin{pmatrix} 1+p^n x & \\ & 1 \end{pmatrix} v = v$

Claims: ① $\forall f \in S_\pi(F^\times)$, $f(a) = 0$ $\forall |a| >> 0$.

② $S_\pi(F^\times) \supset S_c(F^\times)$ (cpt support).

③ $S_\pi(F^\times) \ni f$ $(n(x)f - f)(y) = 0$ $\forall y$ $|y|$ small

$[S_\pi(F^\times)]_0 \simeq \pi_N$ $\left[\psi(xy) - 1 = 0 \text{ for } y \text{ small enough} \right]$

germs at 0

$\Rightarrow$ π is s.c. $\Rightarrow$ $S_\pi(F^\times) = S_c(F^\times)$.

Eitan

Relative Langlands Duality

Generalized W. models as instances of RLD.

Reminder

(i) Langlands program

G - red. alg. gp, F - local field.

$Rep_F(G)$ sm. reps of $G(F)$.

classify irreps of G

① Construction

② Exhaustion

LLC: $Irr(G)$ bijects w. a certain space of homoms $\quad$ ${}^LG$ – Langlands dual of, closely related to $\check{G}$.

$$\mathrm{Lan}_F(G) = \{ W_F' \to {}^LG_{\mathbb{C}} \}/\sim$$

"$\mathrm{Gal}(\bar{F}/F)$"

W_F – Weil group of F.

LLC	G	Rep th.	LR	LC	(HCG)
GLC	G	relative rep th.	GR	LC	(HCG)

(2)

Global Langlands Conjecture

k #-field $\qquad$ G-red gp $/k$

$\mathbb{A}$ = adele ring of k $\qquad$ $k = \prod_v k_v$

$$X_G = Z(\mathbb{A})G(k)\backslash G(\mathbb{A})$$

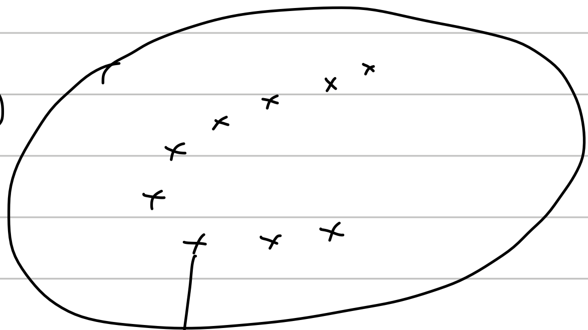
$\circlearrowleft$ $G(\mathbb{A})$

l.c.t.g $\qquad$ $L^2(X, \mu_X)$ "space of automorphic functions"

Spectral problem: describe the measure

$\hat{G}_{\mathbb{A}}$

$G(\mathbb{A}) = \prod G(k_v)$



list of rep $(\pi_v)_v$, π_v rep of $G(k_v)$

$$L^2(X, \mu_X) = \int_{\alpha \in I}^{\oplus} \Pi_\alpha \, d\mu(\alpha)$$

discrete automorphic reps: occur w. pos. measure.

GLC Classify the irreps $\pi = \otimes' \pi_v$ of $G(\mathbb{A})$ that "occur" in $L^2(X_G, d\mu_X)$

aut. space

Using certain parameters similar to those required to describe the LLC $\pi = \otimes' \pi_v$

$$\pi_v \longleftrightarrow \phi_v : W'_{F_v} \longrightarrow {}^L G_{\mathbb{C}}$$

Point of view of "relative rep theory".

Study reps of G w. embeddings into G spaces.

$$\pi \hookrightarrow F(Z) \xrightarrow{ev_{z_0}} \mathbb{C} \quad Z \quad G\text{-space}$$

Z is G-transitive

$$Z = G \cdot z_0 \longleftrightarrow G/H \ , \qquad H = \mathrm{stab}_G(z_0)$$

$$\mathrm{Hom}_G(\pi, F(Z)) = \mathrm{Hom}_H(\pi, \mathbb{C})$$

Local th. G, H - group / F. local field.

Classify irreps of G s.t.

$\exists \ l : V_\pi \longrightarrow \mathbb{C}$ with

① $l \neq 0$

② l is H-inv.

Q: Do it in terms of Langlands parameters.

Global theory. π aut. rep of $G(\mathbb{A})$

$v \in V_\pi \subset L^2(X_G, d\mu)$

$$\begin{tikzcd} H_{\mathbb{A}} \arrow[r] & Z_G(\mathbb{A})G_K \backslash G_{\mathbb{A}} \supseteq C \arrow[d, "i_C"] \\ & \mathbb{C} \end{tikzcd}$$

$v \xrightarrow{\ell} \mathbb{C}$ (ℓ: $H(\mathbb{A})$-inv), $\quad X_G = Z_G(\mathbb{A})G_K \backslash G_{\mathbb{A}}$, $\quad \int i(v)|_C$

In many cases, P_H is related to $L(\pi, r, s)$

$L(\pi, r, s)$ - Dirichlet series $\qquad$ π: aut. rep of $G(\mathbb{A})$

$= L(X, s)$

$r : {}^L G \to GL_n(\mathbb{C})$

Hasse-Weil L-function?

$L(\pi, r, s) = \prod_v L_v(\pi_v, r_v, s)$ Dirichlet-like series

$\phi : W_F \to {}^L G$ $\qquad$ Dual group of G/H.

14.11.24 $\qquad G = PGL_2(F)$

$\{\text{Sm. irreps of } G\} \longleftrightarrow \{\phi : W_F' \to SL_2(\mathbb{C})\}$ ("$Gal(\bar{F}/F)$")

Parameters

Preserves γ, L, ε-factors

$G = GL_1(F)$ $\quad \chi : F^\times \to \mathbb{C}$

χ unramified $\Leftrightarrow \chi|_{\mathcal{O}^\times} = 1$

$\mathrm{cond}(\chi) = n \Leftrightarrow \chi|_{1+p^{n-1}} \neq 1, \quad \chi|_{1+p^n} = 1.$

$$Z(\phi, \chi, s) = \int_{F^\times} |x|^{-s} \chi(x) \phi(x) \, dx = (*)$$

$\phi \in C_c^\infty(F) \xrightarrow{\mathcal{F}} C_c^\infty(F) \qquad \psi : F \to \mathbb{C}, \quad d_\psi x$ of F

$\phi \longmapsto \hat{\phi}$

$$(*) = \sum_{n=-\infty}^{N} \int_{|x| = q^n} \underline{\qquad} \, d^\times x$$

Prop:

① $Z(\phi, \chi, s)$ for $\mathrm{Re}(s) >> 0$.

② It has merom. continuation w/ rat'l pole.

(rat'l of q^{-s})

③

$$Z(\hat{\phi}, \chi^\vee, 1-s) = \gamma(\chi, \psi, s) Z(\phi, \chi, s)$$

$\gamma(\chi, \psi, s)$ has fun'l eq'n.

$$L(\chi, s) = \gcd_{\phi \in S_c(F)} (Z(\phi, \chi, s))$$

$\dfrac{Z(\phi, \chi, s)}{L(\chi, s)}$ entire, and $= 1$ for som ϕ.

e.g. χ is unramified, det. by $\chi(\varpi)$

$$L(\chi, s) = \frac{1}{1 - \chi(\varpi) q^{-s}}$$

$$\frac{Z(\phi, \chi, s)}{L(\chi, s)} = \varepsilon(\chi, s, \psi) \frac{Z(\hat{\phi}, \chi^{-1}, 1-s)}{L(\chi^{-1}, 1-s)}, \quad \gamma(\chi, s, \psi) = \frac{\varepsilon(\chi, s) L(\chi, s)}{L(\chi^{-1}, 1-s)}$$

$\varepsilon(s) = a\, q^{a q}$ $\varepsilon(\chi, s)\, \varepsilon(\chi^{-1}, 1-s) = 1.$

$G = PGL_2$ π ∞-dim'l rep. $PGL_2 \supseteq N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$\mathrm{Hom}_N(\pi, \mathbb{C}_\psi) \neq 0$ 1-dim'l.

$V \hookrightarrow S^\infty(F^\times)$ $V \twoheadrightarrow \underbrace{S_\pi(F^\times)}$ image of this map, it is B-equiv.

$v \longmapsto f_v(a) = W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} v\right)$

$W : V \to \mathbb{C}$ $W(n \cdot v) = \psi(n)\, W(v)$, $n \in N$

$\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f\right)(x) = f(ax) \cdot \psi(bx)$ $[S_\pi(F^\times)]_0 \simeq \pi_N$

$G = B \cup BwB.$ $(\pi(w) f)(x) = \int_{F^\times} j_\pi(xy)\, f(y)\, d^\times y$

$Z(f, \chi, s) = \int_{F^\times} f(x)\, \chi^{-1}(x)\, |x|^{2s-1}\, d^\times x$ $f \in S_\pi(F^\times)$

Prop.

① $Z(\phi, \chi, s)$ $\downarrow$ for $\mathrm{Re}\, s >> 0$.

② it has at most 2 poles. (mero cnt w.) $\pi = \mathrm{Ind}_B^G\, M$

③ $Z(f, \chi, s) = \gamma(\pi, \psi, s)\, Z(\pi(w) f, \chi^{-1}, 1-s)$

④ π is s.c $\Rightarrow$ $Z(f, \chi, s)$ is entire.

$x >> 0$ $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} a & ax \\ 0 & 1 \end{pmatrix}$

$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) = W\left(\underbrace{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}} v\right) = W\left(\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right)$

$\begin{pmatrix} a & ax \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $= \underbrace{\psi(ax)}_{\neq 0} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right)$ $|a| >> 0$

$\Rightarrow W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) = 0$

Cor: If π is s.c. $\Rightarrow L(\pi,\chi,s)=1$

If $\pi=\mathrm{Ind}_B^G \mu \Rightarrow L(\pi,\chi,s)=L(\mu\chi^{-1},s)L(\mu^{-1}\chi^{-1},s)$

Godement - Jacquet for GL_2

$$Z(\phi,f,s)=\int_G \phi(g) f_\pi(g)\,|\det(g)|^s\,dg$$

matrix coeff

$G\subset M_{2\times 2}(F)$ $\qquad \phi: M_{2\times 2}\to\mathbb{C}$

14.11.24 Last time

RLLC - mult. prob.

$G,H \rightsquigarrow \dim \mathrm{Hom}_H(V,\mathbb{C})$

$(\pi,V)\longmapsto$ dual gp..

$(\pi,V,G)\xrightarrow{\text{Langlands}}\phi_\pi: W_F\to {}^LG$

Say in terms of ϕ_π whether π is H-dist

$V\in \mathrm{Rep}(G)$

$(\pi,V,G)\xrightarrow{\text{Lang.}}\phi_\pi: W_F\to {}^LG$

$G_X^\vee\hookrightarrow G^\vee$

$X=G/H$

In [SV],

① $i_X: G_X^\vee\times SL_2(\mathbb{C})\to G^\vee$

② f. dim rep V_X of $G_X^\vee$.

Speculation

X-dist. reps of Arthur-type

$\longleftrightarrow$ ϕ_π^A that factor through i_X.

L-function of X-dist. reps of A-type

to connect to "$L(V_X)$"

Beyond "Spherical pairs"

- Bessel model
- Fourier Jacobi models
- Howe's duality

$\Rightarrow$ Gen. Whit. models

Formal set up: Hypersph. space.

We move from $X = G/H$ to $M = T^*(X)$

$\wr$

Symp var

Aim of paper

general case

M

Hamiltonian

G-Spaces

$\cup$

hyp. sph. var

$\uparrow$

$X = G/H$ sph. var.

To M attach a Hilb. space $\mathcal{H}_M$

M ← Class. system $\longmapsto$ quant. version

Spectral problem: $G \curvearrowright M$

$$\downarrow$$

$$G \curvearrowright \mathcal{H}_M = \int_{\pi \in \hat{G}} \pi \, d\mu(\pi)$$

Exhibit $M^\vee$ with $G^\vee$ s.t. rep^n π can be described using $(M^\vee, G^\vee)$.

Hyp. vars are classified by

whit. induction

① $i : H \times SL_2 \longrightarrow G$, $H \subset Z_G(i(SL_2))$

② S fin. dim rep'n of H.

on this class there's a natural duality.

Spectral problem attached to (M, G) is answered using the geometry of $(M^\vee, G^\vee)$.

class of spaces for which the paper provides evidence.

<u>Datum</u>: $i : SL_2 \longrightarrow G$ $\qquad H = Z_G(i(SL_2))$

$$\downarrow$$

$$M \qquad\qquad S = \{0\}$$

$d_i : sl_2 \to \mathfrak{g}$

$e = d_i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} = Lie(G)$ nilp el.

Jac.-Morozov: (e, f, h) sl_2-triple.

$\mathfrak{g}^e = \{ X \in \mathfrak{g} : [e, X] = 0 \}$ $\quad \mathfrak{h} = Lie(H)$

$M_e := ((f + \mathfrak{g}^e) \cap \mathfrak{h}^\perp) \leftarrow$ Hamiltonian space.

$M_e \xrightarrow[\text{quantization}]{}$ whittaker model attached to e $\quad \Pi_e$

<u>Question</u>: classify reps embedding Π_e.

---

kirilov orbit method

אורביטים נילפוטנטיים:

1. JM, שקילות עד הצגה
2. הצגה, הצגה רית שקולות
3. ך הסתכל על המוניים וגל לוסב.
4. בניה של JM, Parab., even orbits
5. סופיות מספר מסלולים, מעל $\mathbb{C}$, מעל $\mathbb{Q}_p$.

M_χ, למה היא רדקיקית?

להסתכל על המומנט? היי קופץ קספר של Ginzburg.

21.11.24

Eitan Nilp orbits and gener. Whittaker models

$G = SL_2(\mathbb{Q}_p)$ $\quad U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Q}_p \right\}$

$\mathrm{Ind}_U^G(\psi)$ $\quad \psi : U \to \mathbb{C}^\times$

whitt span $\mathrm{ind}_U^G(\psi)$ $\quad \psi(u(x)) = \psi_0(x)$ $\quad \psi_0 : F \to \mathbb{C}^\times$

Reason to care: Any (almost) irrep of GL_2 / SL_2 admits such a model.

Shalika model.

Bessel model

F-J model.

$\vdots$

General setup: F n.a. loc. field of char 0.

$\psi : F \to \mathbb{C}^\times$ unitary char.

G - split, red. gp $\rightsquigarrow$ ind, Ind.

$\mathfrak{g} = \mathrm{Lie}(G)$, κ: bilinear form on $\mathfrak{g}$
AdG inv. non-deg.

$(e,f,h) \longrightarrow e \in \mathfrak{g}$ nilp elt.

JM$^+$: conj. classes of γ's $\longleftrightarrow$ conj. classes of nilp elts.

<u>Concept</u>: "e is an even nilp orbit".

<u>Step 1</u>: Fix $\gamma \rightsquigarrow$ reps of sl_2 on $\mathfrak{g}$

$= \bigoplus$ irreps.

$$\mathfrak{g}_j = \{ v \in \mathfrak{g} : ad(h)(v) = jv \}$$

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

$\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j$ — non-neg. evals of h.

$\mathfrak{u} = \bigoplus_{j > 0} \mathfrak{g}_j \quad \oplus \mathfrak{g}_0 \qquad \mathfrak{l} = \mathrm{Cent}_{\mathfrak{g}}(h)$

$P = L \ltimes U \qquad$ "$\mathfrak{g}_1$ is an enemy"

$$\mathfrak{u}^+ = \bigoplus_{j \geq 2} \mathfrak{g}_j$$

Say <u>e</u> or <u>γ</u> is even if $\mathfrak{g}_0 = 0$.

<u>Claim</u>: M_γ is reductive ($\mathrm{cent}_G(\gamma) = \{ g \in G : \underset{x \in \{e,f,h\}}{Ad(g)(x) = x}$

<u>Def</u>: The Wh. model assoc. to γ is

(assuming γ is even): $W_{\gamma,\psi} = \mathrm{ind}^G_{M_\gamma U}(\chi_\gamma)$

$\chi_{\gamma,\psi} : U^+ \longrightarrow \mathbb{C}$

$\chi_\gamma(\exp(u)) = \psi(\kappa(f,u)) \qquad u \in U^+ \qquad \chi_\gamma \not\equiv 1$

$(*)$

Claim: Formula (*) defines a character $\chi_\gamma : U^+ \to \mathbb{C}^*$

$W_{\gamma,\psi} = \mathrm{ind}_{M_\gamma U}^G(\chi_\gamma)$ small Wh. space

Notions. $\mathrm{Hom}_G(\pi, \mathrm{Ind}_{M_\gamma U}^G \chi_\gamma)$ $\left(= \mathrm{Hom}_G(\mathrm{ind}_{M_\gamma U}^G \chi_\gamma, \tilde{\pi})\right)$

gen. Wh. model $\simeq \mathrm{Hom}_{M_\gamma U}(\pi, \chi_\gamma)$

$\downarrow$

FR

Claim: If e is a reg. nilp. orbit, $\underline{\mathfrak{g}_1} = 0$

then χ_γ as a char of U is rather generic. $L \cdot \chi_\gamma \subseteq \hat{U}$.

$\hat{U} \simeq U/[U,U]$ aff. sp.

Issue 1: non-even case.

Issue 2: What to expect.

Claim: $\mathfrak{g}_1$ is even dim'l space.

$(W, w) \longmapsto H(W) = W \oplus F$ Heisenberg group.

W_ψ oscilator rep.

$\curvearrowleft$ $\widetilde{Sp}(W)$

Def: $W_\gamma = \mathrm{ind}_{M_\gamma U}^G(\rho \otimes w_\psi)$?

$k_1(v,w) = k([f,[v,w]])$ $v,w \in \mathfrak{g}_1$

Stone v. Neumann

$\exists!$ sm. irred. unit. rep of $H_\gamma = H(\mathfrak{g}_1, k_1)$

$Z(H_\gamma)$ acts by ψ.

<u>fact</u>: this form is inv. w.r.t M_γ.

M_γ admits a covering and w_ψ is a rep'n of $\tilde{M}_\gamma$.

V v.s. B bil. form

Sym anti-Sym

$$G = \mathrm{Aut}(V, B)$$

$$\gamma = \{e,f,h\} \subseteq \mathfrak{g} \quad \mathfrak{sl}_2 \quad \subset G \curvearrowright V$$

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{g} \curvearrowright V = \bigoplus_{j=1}^{l} V^{(j)}$$

decompose to $\mathfrak{sl}_2$ reps

$$V^{(j)} = \boxed{W_j} \otimes \boxed{V_j}$$

$\dim V_j = a_j$

$\dim W_j = j$

j even $\to$ anti-Sym form

j odd $\to$ Sym form

<u>claim</u>: $\exists B$ s.t.

$$(V, B) = \bigoplus (W_j \otimes V_j,\ A_j \otimes B_j)$$

$$M_\gamma = \prod_{j=1}^{l} \mathrm{aut}(V_j, B_j)$$

28.11.24

Assaf

Nilpotent orbits & gen. Whittaker models: tutorial.

<u>Outline</u>: ① Heisenberg group

② Recap gen. W. models.

③ <u>Exs</u>: sl_3, Sp_4.

① The Heisenberg group. Let F be a non-Arch. local field of char $=0$. Let ψ be a non-triv. additive character of F.

<u>Def</u>: Let $(W, \langle \ \rangle)$ a symp. v.s.

$H(W) = W \oplus F$ <u>Heisenberg group</u>

why?

$$(w_1, f_1)\cdot(w_2, f_2) = (w_1 + w_2, f_1 + f_2 + \tfrac{1}{2}\langle w_1, w_2\rangle)$$

$Z(H(W)) = F \quad (=\{(0,f) : f \in F\} \subseteq H(W))$

<u>Stone–von-Neumann thm</u>: up to isom. there is a unique sm. irrep of $H(W)$ denoted (ρ_ψ, S), with central char ψ. $H(W) = \begin{pmatrix} 1 & a & t \\ & 1 & b \\ & & 1 \end{pmatrix}$

The oscillator repⁿ. Pick Lagrangian s.space $Y \subseteq W$. $H(Y) = Y \oplus F$ max abelian (form is zero)

$$\psi_Y : H(Y) \to \mathbb{C}^\times, \quad \psi_Y(y,t) := \psi(t)$$

$H(W) \circlearrowright$ by right trans.

$$S_Y = \operatorname{ind}_{H(Y)}^{H(W)} \psi_Y = \left\{ f : H(W) \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ sm} \\ f(h_1 h) = \psi_Y(h_1) f(h) \\ \forall\, h_1 \in H(Y),\ h \in H(W) \end{array} \right\}$$

Ex: $\operatorname{ind}_{H(Y)}^{H(W)} \psi_Y = \operatorname{Ind}_{H(Y)}^{H(W)} \psi_Y$ (use equivariance)

- $f(h(0,t)) = f((0,t)h) = \psi_Y(0,t) f(h) = \psi(t) f(h)$.

Choose a Lagrangian $X \subset W$ s.t. $W = X \oplus Y$

$$\varphi : S_Y \to S(X) \qquad \varphi(f)(x) = f(x,0)$$

$$\rho_\psi(x+y,t)(\varphi(f))(x_0) = f((x_0,0)\underbrace{(x+y,t)}_{\text{right trans.}})$$

$$= f((x_0+x+y,\ t+\tfrac{1}{2}w(x_0,x+y)))$$

$$= f(\underbrace{(y,\ t+\tfrac{1}{2}\langle x_0,y\rangle - \tfrac{1}{2}\langle y, x+x_0\rangle)}_{\in H(Y)}(x+x_0,0))$$

$$= \psi(t+\tfrac{1}{2}\langle x_0,y\rangle + \tfrac{1}{2}\langle x+x_0,y\rangle) f(x+x_0,0)$$

$$= \psi(t+\langle x_0,y\rangle + \tfrac{1}{2}\langle x,y\rangle)\varphi(f)(x+x_0)$$

$(\rho_\psi(x,0)\varphi(f))(x_0) = \varphi(f)(x+x_0)$.

$(\rho_\psi(y,0)\varphi(f))(x_0) = \psi(\langle x_0,y\rangle)\varphi(f)(x_0)$.

$(\rho_\psi(0,t))\varphi(f)(x_0) = \psi(t)\varphi(f)(x_0)$.

Lie algebra: $[x,p] = ih$

2. Generalized Whittaker models

G alg. group. $\mathfrak{g}$ Lie alg.

nilp orbit $\rightsquigarrow (e,f,h)$. $\quad \mathfrak{g}_i = \{X \in \mathfrak{g} : [h,X] = iX\}$

$$\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i, \quad \mathfrak{u} = \bigoplus_{i \geq 1} \mathfrak{g}_i, \quad \mathfrak{u}^+ = \bigoplus_{i\geq 2} \mathfrak{g}_i.$$

A_n orbit is even if $\mathfrak{g}_1 = 0$, or $\mathfrak{n} = \mathfrak{n}^+$.

<u>3.2</u> Sp_4 $\quad J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ $\quad Sp_4 = \left\{ \begin{pmatrix} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & B \\ C & \begin{smallmatrix} -d & -c \\ -b & -a \end{smallmatrix} \end{pmatrix} \right\}$

$h_1 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}$, $\quad h_2 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$

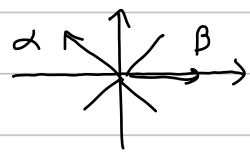
$E_{12} - E_{34}$, E_{23}, $E_{13} + E_{24}$, E_{14}.

$h = h_1$ or $h = h_2$

Choose $\Delta = \{\alpha, \beta\}$ $\quad [h, E_{12} - E_{34}] = \alpha(h)(E_{12} - E_{34})$

$\alpha(h_1) = -1$, $\beta(h_1) = 2$ $\quad [h, E_{23}] = \beta(h) E_{23}$.

$\alpha(h_2) = 1$ $\quad \beta(h_2) = 0$



<u>Prop</u>: nil. orbits of Sp_{2n} are in bij w/ partitions of n, such that every odd part has even multip.

<u>$n=4$</u>: $\quad$ reg. $\quad$ sreg $\quad$ min $\quad$ zero

$[4] - [2^2] - [21^2] - [1^4]$

$e_{[4]} = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix}$ $\quad e_{[2^2]} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ & 0 & & \\ \hline & & 0 & -1 \\ & & & 0 \end{array}\right)$ $\quad e_{[21^2]} = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \\ \hline & 0 \end{array}\right)$ $\quad e_{[1^4]} = 0$

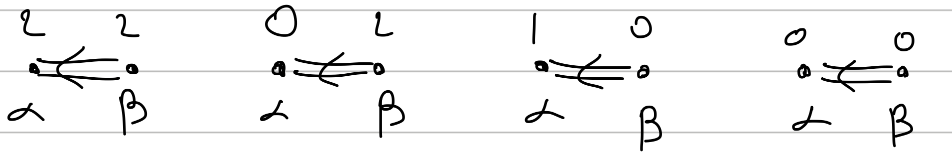
$h_{[4]} = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}$ $\quad h_{[2^2]} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ $\quad h_{[21^2]} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}$

<u>Def</u>: h is Δ-dominant if $\alpha(h) \geq 0$ $\forall \alpha \in \Delta$.

- $h_{[4]}$ is dominant $\quad \alpha(h_{[4]}) = 2$, $\beta(h_{[4]}) = 2$.
- $h_{[2^2]}$ <u>not</u> Δ-dominant: conj. to $h^+_{[2^2]} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, dominant. (using Weyl gp)

$\alpha(h^+_{[2^2]}) = 1$, $\beta(h^+_{[2^2]}) = 0$

Note: There is a unique s.s h' which is Δ-dominant and W-conj to h_λ.



$\mathcal{O}_{reg}$ $\mathcal{O}_{sreg}$ $\mathcal{O}_{min}$ $\mathcal{O}_{zero}$

$\mathfrak{sl}_3$ $(2,1)$ 1 -1 0

28.11.24

Hamiltonian spaces and quantization

Want a duality $G \curvearrowright M \longleftrightarrow G^\vee \curvearrowright M^\vee$

$M, M^\vee$ hyperspherical.

Def: M (sm. variety) has a symplectic structure if it has non-deg 2-form ω s.t. $d\omega = 0$.

Examples: (1) $2n$ dim'nal v. space.

(2) Cotangent bundles.

(3) Coadjoint orbits $\mathcal{O} \subseteq \mathfrak{g}^*$.

Def (Poisson brackets). Let (M, ω) be symplectic.

$\omega : T^*M \xrightarrow{\sim} TM$. For every function $\mathcal{O}(M)$

Section

we have $df \in T^*M$, ω gives $Z_f \in TM$.

For $f, g \in \mathcal{O}(M)$. $\{f, g\} := \omega(Z_f, Z_g) \in \mathcal{O}(M)$.

<u>Properties</u>: ① $(\mathcal{O}(M), \{\ \})$ is a Lie algebra

② $\{f, gh\} = \{f, g\}h + \{f, h\}g$.

<u>Def</u> (Hamiltonian)

(M, ω) symplectic and let $G \curvearrowright (M, \omega)$.

We have $\underset{\in \mathfrak{g}}{X} \longmapsto \left(\frac{d}{dt} \exp(tX).m\right)\Big|_{t=0}$

$\mathcal{O}(M) \longrightarrow$ Vector fields $= \mathrm{Vect}(M)$

$f \longmapsto Z_f$

<u>Definition</u>: $\mathrm{Vect}(M) = \Gamma(\mathcal{O}_{TM}, TM)$

$= \{Y : M \to TM : Y(m) \in T_m M\}$

M is called Hamiltonian if $\exists$ a G-equiv. map of Lie algebras $H : \mathfrak{g} \to \mathcal{O}(M)$ s.t. the following diagram commutes:

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, dashed, "H"'] \arrow[d] \\ \mathcal{O}(M) \arrow[r] & \mathrm{Vect}(M) \end{tikzcd}$$

$f \longmapsto Z_f$

We can define the moment map: (for a Hamiltonian G-space M)

$\mu : M \to \mathfrak{g}^* \; : \; \mu(m)(X) = H(X)(m).$

Examples: ① $Sp(W) \curvearrowright W$, $M = W$.

$m \in W$

$\mu(m)(X) = \frac{1}{2}\langle X \cdot m, m \rangle.$ $\quad X \in \mathfrak{sp}$, $m \in W$.

Can reconstruct $H(x)$: $(Sp(W), W)$ is Hamiltonian with this moment map.

(X sm. alg. var)

② $G \curvearrowright X$, $M = T^*X$, $m \in M$, $m = (x, \varphi)$, $x \in X$

$\varphi \in T^*_x X$. $\quad \mu(m)(A) = -\varphi\left(\left(\frac{d}{dt} \exp(At)x\right)\Big|_{t=0}\right)$ $\quad A \in \mathfrak{g}$

Quantization:

Idea: $G \curvearrowright (M, \omega)$ Hamiltonian.

$\quad\vdots$

$G \curvearrowright V$ unitary rep'n.

$\mathcal{O}(M) \dashrightarrow \mathrm{End}(V)$ some cts operators.

Ex: ① $Sp(W) \curvearrowright W$. Get Weil rep'n.

② $G \curvearrowright X$, $T^*X \rightsquigarrow L^2(X)$.

$\left(M \times^G_Y Z = \{(m, z) : p(m) = p(z)\} \big/ (gm, z) \sim (m, gz) \right)$

Operations: ① Symplectic reduction.

$M \times^G_{\mathfrak{g}^*} 0 = \{m : \mu(m) = 0\}/G = \mu^{-1}(0)/G$ Hamiltonian.

② Symplectic induction. $H \leq G$, S Hamiltonian w.r.t H

$S \times_{h^*}^{H} T^*G.$ $\dim S + 2\dim G - 2\dim H$

05.12.24

① Weil rep'n

② Example: Sp_4

Heis. gp: $H(W) = W \oplus F$

W - non-deg Symp. space

F - local field

ψ - additive char of

$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle)$

$S_\psi = \mathrm{ind}_{H(Y)}^{H(W)} \psi_Y \xrightarrow{f \mapsto \varphi(f)} S(X)$ $\varphi(f)(x) = f(x, 0)$

$(\psi|_Y = 1$

Polariz. of W, Lagrangian subspaces X, Y

$W = X \oplus Y.$

$Sp(W) \curvearrowright H(W)$, $g \in Sp(W)$, $h \in H(W)$

$h^g = (wg, t)$, $h = (w, t) \in H(W)$

- Stone-von-Neumann thm: $A(g) : \rho_\psi \to \rho_\psi^g$

$A_g : \rho_\psi \longrightarrow \rho_\psi^g$ $\rho_\psi^g(h) = \rho_\psi(h^g)$

$A(g_1)A(g_2) = A(g_1 g_2)\, c(g_1, g_2)$

by Stone-von-Neumann

$$A : Sp(W) \to GL(S)/\mathbb{C}^\times$$
$$g \longmapsto A(g)$$

$Mp(W)$ metaplectic

$$\begin{tikzcd} Mp(W) \arrow[r] \arrow[d] & GL(S) \arrow[d] \\ Mp(W) \arrow[r] & GL(S)/\mathbb{C}^\times \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}^\times \arrow[r] & Mp(W) \arrow[r] & Sp(W) \arrow[r] & 1 \\ 1 \arrow[r] & \mu_2 \arrow[r] \arrow[u, "i"] & Mp^{(2)}(W) \arrow[r] \arrow[u, "\alpha"] & Sp(W) \arrow[r] \arrow[u, equal] & 1 \end{tikzcd}$$

The Schrödinger model of the Weil rep'n

$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^\vee \end{pmatrix} : a \in GL(X) \right\}$ $\qquad P = MN$

$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in Hom(X, Y) \right\}$

$a^\vee \in GL(Y)$, s.t. $\quad \langle xa, ya^\vee \rangle = \langle x, y \rangle \quad \forall x \in X, \ y \in Y$

$\langle x_1, x_2 b \rangle = \langle x_2, x_1 b \rangle, \ x_1, x_2 \in X \qquad q_b : X \to F$

$(x \ y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (xa + yc, \ xb + yd) \qquad x \mapsto \langle x, xb \rangle$

$$A^0(g) : S_Y \to S_{Yg^{-1}}$$
$$(A^0(g)(f))(h) = f(h^g)$$

for $g \in P$, $A^0(g) : S_Y \to S_Y$

- $\left(A^0 \begin{pmatrix} a & 0 \\ 0 & a^\vee \end{pmatrix}\right) \varphi(f)(x) = \varphi(f)(xa)$

- $\left(A^{\circ}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right)\varphi(f)(x) = \psi\left(\frac{1}{2}q_b(x)\right)\varphi(f)(x)$

$w = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $\quad I_{Y_1,Y_2} : S_{Y_1} \to S_{Y_2}$

Y_1, Y_2 Lagrangian.

$$I_{Y_1,Y_2}(f)(h) = \int_{Y_1 \cap Y_2 \setminus Y_2} f((y,0)h)\, dy \qquad h \in H(W)$$

$f \in S_{Y_2}$

Claim: Y_1, Y_2 Lagrangian s. spaces of W. Then

- I_{Y_1,Y_2} is non-zero and $H(W)$-equiv.

$$A(g) = I_{Yg^{-1},Y} \circ A^{\circ}(g)$$
$$S_Y \to S_Y$$

- $(A(w)\varphi)(x) = \int_Y \psi\left(\left\langle \begin{pmatrix}0\\ x\end{pmatrix}^t, \begin{pmatrix}y\\ 0\end{pmatrix}^t \right\rangle\right)\varphi(y)\, dy$

$\varphi \in S(X)$ (use Fourier transform)

Gups.

Constructions in symp. geometry

1. Symp reduction.

Def Let $G \curvearrowright (M,\omega)$ be Hamiltonian then we have $\mu : M \to \mathfrak{g}^*$ $\quad M^G \times_{\mathfrak{g}^*} 0 = \mu^{-1}(0)/\!/G$.

Prop: If G acts freely on $\mu^{-1}(0)$ then $M^G \times_{\mathfrak{g}^*} 0$ is a symplectic variety. (G reductive)

Pf: $X = \mu^{-1}(0)$, $Y = \mu^{-1}(0)/\!\!/G$.

Y is a variety (follows from gen. results)

Want to show Y symplectic. For $y \in Y$ take $x \in X$ s.t. $\pi(x) = y$. We have $T_yY = T_xX/\mathfrak{g}$ → embed $\mathfrak{g} \hookrightarrow T_xX$

Define, $\omega_y(u,v) = \omega(\tilde{u},\tilde{v})$ where $\tilde{u},\tilde{v} \in T_xX$ are lifts of u,v. Need to check this is well defined.

For $A \in \mathfrak{g}$, need $\omega_x(\tilde{u}+A,\tilde{v}) \overset{?}{=} \omega_x(\tilde{u},\tilde{v})$

$\iff \omega_x(A,\tilde{v}) = 0 \iff \mu(x)(A) = 0$

Ex: ① Show that ω_Y is non-deg.

② Show $\mu(x)(A) = 0 \iff \omega_x(A,\tilde{v}) = 0$

2. Symp. induction

Let $H \leq G$, S be H-Hamiltonian with a map $\nu : S \to \mathfrak{h}^*$. Consider

$$\begin{tikzcd} T^*G \arrow[d, "\mu_G"'] \arrow[dd, bend left=60, "\mu_H"] \\ \mathfrak{g}^* \arrow[d] \\ \mathfrak{h}^* \end{tikzcd}$$

$S \times T^*G$ is $H \times G$ Hamiltonian with maps

$$(\nu - \mu_H, \mu_G) : S \times T^*G \to \mathfrak{h}^* \times \mathfrak{g}^*.$$

The induction is the reduction w.r.t the first coord. The rotation is

$$S \overset{H}{\underset{h^*}{\times}} T^*G := \left\{ (s, g, A) : \begin{matrix} s \in S \\ g \in G \end{matrix}, A \in g^* \left(\nu(s) - g^{-1} A|_h \, g \right) = 0 \right\} \Big/ H$$

This is a G-Hamiltonian variety.

Example: Cotangent bundle

$$T^*(G/H) = T^*G \overset{H}{\underset{h^*}{\times}} 0 .$$

Exer: $S = T^*(X)$

$$T^*(Y) \overset{H}{\underset{h^*}{\times}} T^*G \overset{?}{=} T^*(X \overset{H}{\times} G)$$

$$\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H = \mathrm{Ind}_K^G$$

12.12.24

Guy K. Bruhat Decomposition

Claim: $G/\bar{K}$ red.

- $W = N(T)/T \simeq \{ B \text{ Borel} : T \subset B \} \simeq B \times B / G \cong B \backslash G / B \cong W.$

(1) all B are conjugate

$(x, y) \mapsto x^{-1} y$

$G/H_1 \times G/H_2 \to H_1 \backslash G / H_2$

(2) $N_B(T) = T$

(3) T in B are conj in B.

(4) $N_G(B) = B$

(5) $B \cap B'$ contains a torus.

Fix both $T \subseteq B$.

$$N(T) \longrightarrow \{B \,;\, T \subseteq B,\ B \text{ Borel}\}$$
$$t \longmapsto tBt^{-1}\quad \text{(inj. is easy)}$$
$$\ker = T$$

<u>$T \subseteq B'$:</u> $\quad T \subseteq B \Rightarrow T^g \subseteq B^g = B'$ $\quad$ ($g \in G$ all Borels are conj./G)

$$\exists b' \in B' \quad (T^g)^{b'} = T \Rightarrow gb' \in N(T)$$

$$B' = (B')^{b'} = B^{gb'}$$

$$\begin{array}{c} (\overset{T}{\overset{\cap}{B}}, \overset{T}{\overset{\cap}{B'}}) \\ (\ )^g \downarrow \\ (B, B'') \end{array}$$

<u>Claim</u>: $w \in W$, $s \in W$ simple reflection

a) If $l(ws) > l(w)$ $\quad$ (or $l(ws) = l(w) + l(s)$)

Then $\quad BwB \cdot BsB = BwsB$

b) If $\quad l(ws) < l(w)$

$$BwB \cdot BsB = BwsB \cup BwB.$$

<u>Pf</u>: ⓐ $\supseteq$: clear.

Need $\quad wBs \subseteq BwsB \iff B \subseteq w^{-1}BwsBs^{-1}$.

$$B = T \cdot \prod_{\alpha \in R^+} U_\alpha \quad \leftarrow \text{root space}$$

$$B^w = T \prod_{w(\alpha) \in R^+} U_\alpha \leftarrow \text{condition ensures } s \text{ flips one root, which } w \text{ doesn't.}$$

<u>Claim</u>: Let $P \leq G$ be a parab. $B \in P$ Borel.

$$P = \bigcup_{w \in W_P} BwB, \qquad W_P = N_P(T)/T \subseteq W$$
$$(= W_L\,,\ L = P/R_u(P))$$

Claim P, Q Parabolics.

$$G \backslash (G/P \times G/Q) \simeq P \backslash G / Q \cong W_P \backslash W_G / W_Q = W^{P,Q}$$

$$W^{P,Q} = \left\{ w \in W : \begin{array}{ll} \alpha \in \Delta(P,T) & w(\alpha) \in R^+ \\ \beta \in \Delta(Q,T) & w^{-1}(\beta) \in R^+ \end{array} \right\}$$

$w \in W$, $PBwBQ$

$$PBwB = \bigcup_{w_1 \in W_P} Bw_1 B w B \subseteq \bigcup_{w_1 \in W_P} Bw_1 w B$$

Bw_1wB exhausts all possible candidates

$\supseteq$ clear!

$$\left(\bigcup_{w_1 \in W_P} Bw_1 w B\right)\left(\bigcup_{w_2 \in W_Q} Bw_2 B\right) = \bigcup_{\substack{w_1 \in W_P \\ w_2 \in W_Q}} Bw_1 w w_2 B$$

Claim: (a) If $w' \in W_P w W_Q$ is minimal w.r.t length $\Rightarrow w' \in W^{P,Q}$

(b) If $w \in W^{P,Q}$, $w' \in W_P w W_Q$, $\exists\, w_1 \in W_P$, $w_2 \in W_Q$ $w' = w_1 w w_2$

$l(w') = l(w_1) + l(w) + l(w_2)$.

$$\dim(BwB) = \dim B + l(w).$$

$$\dim(B) = rk(G) + R^+$$

$$\overline{BwB} = \bigcup_{w' \leq w} Bw'B$$

M is a Levi

$W_{G/P}$

$$\{P'\ |\ \begin{array}{l} P' = P^g \\ M \subseteq P' \end{array}\}$$

S_1

Claim:

If P is a maxl parab. Then $N(M)/M$ is of order 2.

<u>Example:</u> $G = GL_n(\mathbb{K})$

$M = GL_k \times GL_{n-k}$ $\qquad P = \begin{pmatrix} \lfloor & \\ 0 & \lfloor \end{pmatrix}$

$W = S_n$

$W_P = S_k \times S_{n-k}$ $\qquad W^{B,P}$

$$B \backslash G / P \cong S_1^n \backslash S_n / S_k \times S_{n-k} \cong \{ A \subseteq [n] : |A| = k \}$$

$$\Delta(P,T) = \Delta(G,T) \setminus \{ e_k - e_{k+1} \}$$

$$P \backslash G / P = S_k \times S_{n-k} \backslash S_n / S_k \times S_{n-k}$$

12.12.24

Guy S. <u>Hyperspherical varieties</u>

<u>Def:</u> Let $G \curvearrowright (M, \omega)$ be Hamiltonian (graded, w. $\mathbb{G}_m$-action which commutes with G).

$\mathcal{O}(M)$ graded

M is hyperspherical if the following conditions hold:

1. M is affine.
2. Multiplicity free: $(\mathcal{O}(M)^G, \{\ \})$ is commut.
3. $\mathrm{Stab}_G(x)$ is conn. for generic $x \in M$.
4. $\mu : M \to \mathfrak{g}^*$, $\mathrm{Im}(\mu) \cap \mathcal{N}_{\mathfrak{g}^*} \neq \emptyset$.
5. The grading is neutral.

Quantization:

$$M \rightsquigarrow G \curvearrowright V_M \quad \text{unitary}$$

$$\mathcal{O}(M) \longrightarrow \mathrm{End}(V_M)$$

$$\mathcal{O}(M)^G \longrightarrow \mathrm{End}_G(V_M) \quad \text{(reason for being mult. free)}$$

Example: (I) $M = W$, $G = Sp(W)$

Affine ✓

mult free : $\mathcal{O}(M)^G$ are const functions ✓

$\mathrm{Stab}_G(x)$ $\quad Sp(W) \cap \begin{pmatrix} 1 & \\ 0 & * \\ \vdots & \\ 1 & \end{pmatrix}$

$\mu : M \to \mathfrak{g}^*$, $\mathrm{Im}(\mu) = \{0\}$

(5) . . .

(II) $G \curvearrowright X$, $\quad M = T^* X \quad L^2(X)$

(1) Take X affine

(2) Thm (Knop): If X affine M is mult free iff X is spherical.

(3) Not true even for nice X.

$$X = SL_3 / SO_3$$

$$X = PGL_2 / N \qquad N = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right\}$$

$x \in T^* X$, $\quad \mathrm{Stab}_G(x) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

④ holds

If X spherical then (3) holds for T^*X iff $\mathrm{Stab}_B(x)$ for $x \in X$ in the open B-orbit is connected!