07.11.24 don't want to fix central char

Nadya $G = PGL_2(F)$, F p-adic, $q = |\mathcal{O}/p|$

LLC $\left\{ \begin{array}{l} \text{sm. irreps} \\ \text{of } G(F) \end{array} \right\} \overset{LLC}{\longleftrightarrow} \{ \phi : W_F \to SL_2(\mathbb{C}) \}$

$\forall$ π of G, χ of $GL_1(F)$

$\gamma(\pi, \chi, s)$, $\frac{Q}{P}(q^{-s})$ $\gamma(\phi, \phi_\chi, s)$

$L(\pi, \chi, s)$, $P^{-1}(q^{-s}) \longleftrightarrow$

$\varepsilon(\pi, \chi, s)$, $a q^{-ns}$

P, Q polys

<u>Goal.</u> $\Pi = \bigotimes_{\text{rest.}} \pi_v$ of $G(\mathbb{A})$.

$L(\pi, \chi, s) = \prod_v L(\pi_v, \chi_v, s)$

$\Pi \hookrightarrow A(G(F)\backslash G(\mathbb{A})) \iff L(\pi, \chi, s)$ is merom. $\forall \chi$.

+ Functional equation

$L(\tilde{\pi}, \chi^{-1}, 1-s)\, \varepsilon(\pi, \chi, s) = L(\pi, \chi, s)$

$G \supset B = N \cdot A$ $K = G(\mathcal{O})$

$A = \{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} : a \in F^\times \}$ $N = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \}$

$K \supset K_1 \supset K_2 \supset \dots$ $K_n = G \cap \begin{pmatrix} 1+p^n\mathcal{O} & p^n\mathcal{O} \\ p^n\mathcal{O} & 1+p^n\mathcal{O} \end{pmatrix}$ open cpt

π adm $\iff \tilde{\pi}$ adm.

(π, V) rep

$\forall\ \tilde{v} \in \tilde{V}, \ v \in V$

$m_{\tilde{v},v}(g) = \langle \tilde{v}, \pi(g)(v) \rangle \in S(G).$

$\tilde{v} : V \longrightarrow S(G)$

$V \otimes \tilde{V} \longrightarrow S(G)$

① π is called <u>Supercuspidal</u> if $m_{\tilde{v},v}$ is cptly supp'ed.

② π $\cap$ is called discrete series if $m_{\tilde{v},v} \in L^2(G, dg)$

③ π $\cap$ is called tempered if $m_{\tilde{v},v} \in L^{2+\varepsilon}(G, dg)$ $\forall \varepsilon > 0$

(π, G, V), $\pi_N, A, V/V(N)\delta_B^{-\frac{1}{2}}$

$V(N) := \mathrm{Span}\{\pi(n)v - v\}$

$[\pi_N]_0 = \oplus \chi_i, \quad \chi_i = \underbrace{\chi_{0,i}}_{\text{unit}} |\cdot|^{s_i} \quad s_i \in \mathbb{C}.$

① $\iff \pi_N = 0$

② $\iff \forall \chi_i \in [\pi_N] \quad \mathrm{Re}(s_i) > 0$

③ $\iff \forall \chi_i \in [\pi_N] \quad \mathrm{Re}(s_i) \geq 0$

$\mathrm{Ind}_B^G \chi_s = \{f : G \to \mathbb{C} : f(n\, t(a)\, g) = |a|^{s+\frac{1}{2}} f(g)\}$

$t(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad [(\mathrm{Ind}_B^G \chi_s)_{ss}] = \chi_s \oplus \chi_{-s}$

$\mathrm{Ind}_B^G |\cdot|^{it}$ is tempered

$$0 \to St \to \mathrm{Ind}_B^G \chi_{\frac{1}{2}} \to \mathbb{C} \to 0$$

$J_N(\ )$ $\quad 0 \to \chi_{\frac{1}{2}} \to \chi_{\frac{1}{2}} \oplus \chi_{-\frac{1}{2}} \to \chi_{-\frac{1}{2}} \to 0$

χ called exponents.

<u>Kirillov model</u> (π, V) rep of G. $\quad N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\}$

$\pi|_N$ $\qquad$ V irrep.

$\mathrm{Rep}\, N = \{1, \psi\} \circlearrowleft A$

$\psi : N \to \mathbb{C}$

$\psi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \psi(x)$

$\psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\right) = \psi(ax)$.

<u>Ex</u>: : ① $V^N = 0$

② $\forall \psi$ $\quad \pi_{N,\psi} = V / (\pi(n)v - \psi(n)v)$

$\neq 0$

③ $\pi_{N,\psi} \simeq \pi_{N,\psi'}$

$\Rightarrow W \in \mathrm{Hom}_N(\pi, \mathbb{C}_\psi) \qquad W(nv) = \psi(n) W(v)$

<u>Thm</u>: $\mathrm{Hom}_N(\pi, \mathbb{C}_\psi)$ is 1-dim'l (Kirillov).

Fix W.

$$i_\pi : V \hookrightarrow S(F^\times)$$

$$i_\pi(v)(a) = W\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(v)\right)$$

The image is denoted $S_\pi(F^\times)$ is called Kirillov model of (π, V)

<u>Define</u>: $\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f\right)(x) = \psi(bx) f(ax)$

$G = B \cup BwB$.

use $\pi(g\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}g^{-1})(v) = \chi(g)\ldots$ $\quad \begin{pmatrix} 1+p^n x & \\ & 1\end{pmatrix} v = v$

<u>Claims</u>: (1) $\forall f \in S_\pi(F^\times)$, $\quad f(a) = 0 \quad \forall |a| >> 0$.

(2) $S_\pi(F^\times) \supset S_c(F^\times)$ (cpt support).

(3) $S_\pi(F^\times) \ni f \qquad (n(x)f - f)(y) = 0 \quad \forall y \ |y|$ small

$[S_\pi(F^\times)]_0 \simeq \pi_N \qquad \left[\psi(xy) - 1 = 0 \text{ for } y \text{ small enough}\right]$

germs at 0

$\Rightarrow \pi$ is s.c. $\Rightarrow S_\pi(F^\times) = S_c(F^\times)$.

<u>Eitan</u>

Relative Langlands Duality

Generalized W. models as instances of RLD.

<u>Reminder</u>

(i) Langlands program

G - red. alg. gp, F - local field.

$\mathrm{Rep}_F(G)$ sm. reps of $G(F)$.

$\downarrow$

Classify irreps of G

(1) Construction

(2) Exhaustion

<u>LLC</u>: $\mathrm{Irr}(G)$ bijects w. a certain space

of homoms

$${}^L G$$ - Langlands dual of, closely related to $\check{G}$.

$$\mathrm{Lan}_F(G) = \{ W_F' \to {}^L G_{\mathbb{C}} \} / \sim$$

"$\mathrm{Gal}(\bar{F}/F)$"

W_F - Weil group of F.

LLC	G	Rep th.	LR	LC	(HCG)
GLC	G	relative rep th.	GR	LC	(HCG)

(2)

Global Langlands Conjecture

k #-field $\qquad$ G - red gp $/k$

$\mathbb{A}$ = adele ring of k $\qquad$ $k = \prod_v k_v$

$$X_G = Z(\mathbb{A}) G(k) \backslash G(\mathbb{A})$$

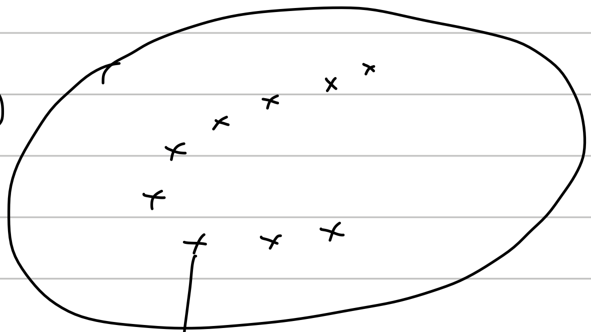
$\circlearrowleft$ $G(\mathbb{A})$

l.c.t.g $\qquad$ $L^2(X, \mu_X)$ "space of automorphic functions"

Spectral problem: describe the measure

$\hat{G}_{\mathbb{A}}$

$G(\mathbb{A}) = \prod G(k_v)$



list of rep $(\pi_v)_v$, π_v rep of $G(k_v)$

$$L^2(X, \mu_X) = \int_{\alpha \in I}^{\oplus} \pi_\alpha \, d\mu(\alpha)$$

discrete automorphic reps : occur w. pos. measure.

GLC Classify the irreps $\pi = \otimes' \pi_v$ of $G(\mathbb{A})$ that "occur" in $L^2(X_G, d\mu_X)$

$\uparrow$ aut. space

Using certain parameters similar to those required to describe the LLC $\pi = \otimes' \pi_v$

$$\pi_v \longleftrightarrow \phi_v : W'_{F_v} \longrightarrow {}^L G_\phi$$

Point of view of "relative rep theory".

Study reps of G w. embeddings into G spaces.

$$\pi \hookrightarrow F(Z) \xrightarrow{ev_{z_0}} \mathbb{C} \quad Z \quad G\text{-space}$$

Z is G-transitive

$$Z = G \cdot z_0 \longleftrightarrow G/H, \quad H = \mathrm{stab}_G(z_0)$$

$$\mathrm{Hom}_G(\pi, F(Z)) = \mathrm{Hom}_H(\pi, \mathbb{C})$$

Local th. G, H - group / F. local field.

Classify irreps of G s.t.

$\exists \ \ell : V_\pi \longrightarrow \mathbb{C}$ with

1. $\ell \neq 0$
2. ℓ is H-inv.

Q: Do it in terms of Langlands parameters.

<u>Global theory.</u> π aut. rep of $G(\mathbb{A})$

$v \in V_\pi \subset L^2(X_G, d\mu)$

$$X_G = Z_G(\mathbb{A}) G_K \backslash G_{\mathbb{A}} \supseteq C$$

$$\begin{tikzcd} H_{\mathbb{A}} \arrow[r] & Z_G(\mathbb{A}) G_K \backslash G_{\mathbb{A}} \arrow[d, "i_C"] \\ v \arrow[r, "\ell"'] & \mathbb{C} \end{tikzcd}$$

$\int i(v)|_C$

$H(\mathbb{A})$-inv

In many cases, P_H is related to $L(\pi, r, s)$

$L(\pi, r, s)$ – Dirichlet series

 ‖

$L(X, s)$

π: aut. rep of $G(\mathbb{A})$

$r : {}^L G \to GL_n(\mathbb{C})$

Hasse-Weil L-function?

$L(\pi, r, s) = \prod_v L_v(\pi_v, r_v, s)$ Dirichlet-like series

$\phi : W_F \to {}^L G$ Dual group of G/H.

14.11.24 $\quad G = PGL_2(F)$

$\{\text{Sm. irreps of } G\} \longleftrightarrow \{\phi : W_F' \to SL_2(\mathbb{C})\}$

"$Gal(\bar{F}/F)$"

Parameters

Preserves γ, L, ε-factors

$G = GL_1(F)$ $\chi : F^\times \to \mathbb{C}$

χ unramified $\iff \chi|_{\mathcal{O}^\times} = 1$

$\mathrm{cond}(\chi) = n \iff \chi|_{1+p^{n-1}} \neq 1, \quad \chi|_{1+p^n} = 1.$

$$Z(\phi, \chi, s) = \int_{F^\times} |x|^{-s} \chi(x) \phi(x)\, dx = (*)$$

$\phi \in C_c^\infty(F) \xrightarrow{\mathcal{F}} C_c^\infty(F)$ $\quad \psi : F \to \mathbb{C}, \quad d_\psi x$ of F

$\phi \longmapsto \hat{\phi}$

$$(*) = \sum_{n=-\infty}^{N} \int_{|x| = q^n} \underline{\qquad} \, d^\times x$$

Prop:

① $Z(\phi, \chi, s)$ for $\mathrm{Re}(s) >> 0$.

② It has merom. continuation w/ rat'l pole.

③ (rat'l of q^{-s})

$$Z(\hat{\phi}, \chi^\vee, 1-s) = \gamma(\chi, \psi, s) Z(\phi, \chi, s)$$

$\gamma(\chi, \psi, s)$ has fun'l eq'n.

$$L(\chi, s) = \gcd_{\phi \in S_c(F)} (Z(\phi, \chi, s))$$

$\dfrac{Z(\phi, \chi, s)}{L(\chi, s)}$ entire, and $= 1$ for som ϕ.

e.g. χ is unramified, det. by $\chi(\varpi)$

$$L(\chi, s) = \frac{1}{1 - \chi(\varpi) q^{-s}}$$

$$\frac{Z(\phi, \chi, s)}{L(\chi, s)} = \varepsilon(\chi, s, \psi) \frac{Z(\hat{\phi}, \chi^{-1}, 1-s)}{L(\chi^{-1}, 1-s)}, \quad \gamma(\chi, s, \psi) = \frac{\varepsilon(\chi, s) L(\chi, s)}{L(\chi^{-1}, 1-s)}$$

$\varepsilon(s) = a\, q^{a q}$ $\qquad \varepsilon(\chi, s)\,\varepsilon(\chi^{-1}, 1-s) = 1.$

$G = PGL_2$ $\quad \pi$ ∞-dim'l rep. $\quad PGL_2 \supseteq N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

$\mathrm{Hom}_N(\pi, \mathbb{C}_\psi) \neq 0$ $\quad$ 1-dim'l.

$V \hookrightarrow S^\infty(F^\times)$ $\qquad V \twoheadrightarrow \underbrace{S_\pi(F^\times)}$

$v \longmapsto f_v(a) = W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} v\right)$ $\qquad$ image of this map, it is

$W : V \to \mathbb{C}$ $\qquad W(n \cdot v) = \psi(n)\, W(v)$ $\qquad$ B-equiv.

$\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f\right)(x) = f(ax) \cdot \psi(bx)$ $\qquad [S_\pi(F^\times)]_0 \simeq \pi_N$

$G = B \cup BwB.$ $\qquad (\pi(w) f)(x) = \int_{F^\times} j_\pi(xy)\, f(y)\, d^\times y$

$Z(f, \chi, s) = \int_{F^\times} f(x)\, \chi^{-1}(x)\, |x|^{2s-1}\, d^\times x$ $\qquad f \in S_\pi(F^\times)$

Prop.

① $Z(\phi, \chi, s)$ $\downarrow$ for $\mathrm{Re}\, s >> 0.$

② it has at most 2 poles. (mero cnt w.) $\qquad \pi = \mathrm{Ind}_B^G\, \mu$

③ $Z(f, \chi, s) = \gamma(\pi, \psi, s)\, Z(\pi(w) f, \chi^{-1}, 1-s)$

④ π is s.c $\Rightarrow$ $Z(f, \chi, s)$ is entire.

$x >> 0$ $\qquad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix}$ $\qquad \begin{pmatrix} a & ax \\ 0 & 1 \end{pmatrix}$

$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) = W\left(\underbrace{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}} v\right) = W\left(\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right)$

$\begin{pmatrix} a & ax \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\quad = \underbrace{\psi(ax)}_{\neq 0} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right)$ $\qquad |a| >> 0$

$\Rightarrow W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right) = 0$

Cor: If π is s.c. $\Rightarrow L(\pi,\chi,s)=1$

If $\pi = \mathrm{Ind}_B^G \mu \Rightarrow L(\pi,\chi,s) = L(\mu\chi^{-1},s)L(\mu'^{-1}\chi^{-1},s)$

Godement - Jacquet for GL_2

$$Z(\phi,f,s) = \int_G \phi(g) f_\pi(g) |\det(g)|^s \, dg$$

matrix coeff

$G \subset M_{2\times 2}(F)$ $\qquad \phi : M_{2\times 2} \to \mathbb{C}$

14.11.24 Last time

RLLC - mult. prob.

$G, H \rightsquigarrow \dim \mathrm{Hom}_H(V,\mathbb{C})$

$(\pi,V) \longmapsto$ dual gp..

$(\pi,V,G) \xmapsto{\text{Langlands}} \phi_\pi : W_F \to {}^LG$

Say in terms of ϕ_π whether π is H-dist

$V \in \mathrm{Rep}(G)$

$(\pi,V,G) \xmapsto{\text{Lang.}} \phi_\pi : W_F \to {}^LG$

$G_X^\vee \hookrightarrow G^\vee$

$X = G/H$

In [SV],

① $i_X : G_X^\vee \times SL_2(\mathbb{C}) \to G^\vee$

② f. dim rep V_X of $G_X^\vee$.

Speculation

X-dist. reps of Arthur-type

$\longleftrightarrow$ ϕ^A_π that factor through i_X.

$\boxed{\text{L-function}}$ of X-dist. reps of A-type

to connect to "$L(V_X)$"

Beyond "Spherical pairs"

- Bessel model
- Fourier Jacobi models
- Howe's duality

$\Rightarrow$ Gen. Whit. models

Formal set up: Hypersph. space.

We move from $X = G/H$ to $M = T^*(X)$ $\supseteq$ Symp var

Aim of paper

general case

M

Hamiltonian

G-Spaces

$\cup\!|$

hyp. sph. var

$\uparrow$

$X = G/H$ sph. var.

To M attach a Hilb. space $\mathcal{H}_M$

Class. system $\longmapsto$ quant. version

Spectral problem: $G \curvearrowright M$

$$\downarrow$$

$$G \curvearrowright \mathcal{H}_M = \int_{\pi \in \hat{G}} \pi \, d\mu(\pi)$$

Exhibit $M^\vee$ with $G^\vee$ s.t. rep^n π can be described using $(M^\vee, G^\vee)$.

Hyp. vars are classified by

whit. induction

① $i : H \times SL_2 \longrightarrow G$, $H \subset Z_G(i(SL_2))$

② S fin. dim rep'n of H.

on this class there's a natural duality.

Spectral problem attached to (M, G) is answered using the geometry of $(M^\vee, G^\vee)$.

class of spaces for which the paper provides evidence.

<u>Datum</u>: $i : SL_2 \longrightarrow G$ $\quad H = Z_G(i(SL_2))$

$$\downarrow$$

$$M$$

$S = \{0\}$

$\phi_i : sl_2 \longrightarrow \mathfrak{g}$

$e = \phi_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} = \mathrm{Lie}(G)$ nilp el.

Jac. -Morozov: (e, f, h) sl_2-triple.

$\mathfrak{g}^e = \{ X \in \mathfrak{g} : [e, X] = 0 \}$ $\mathfrak{h} = \mathrm{Lie}(H)$

$M_e := ((f + \mathfrak{g}^e) \cap \mathfrak{h}^\perp) \leftarrow$ Hamiltonian space.

$M_e \xrightarrow[\text{quantization}]{}$ whittaker model attached to e Π_e

<u>Question</u>: classify reps embedding Π_e.